# Derivations for 'A Unary Error Correction Code for the Near-Capacity Joint Source and Channel Coding of Symbol Values from an Infinite Set'

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## I. UEC CODING RATE WHEN THE SYMBOL VALUES OBEY A GEOMETRIC DISTRIBUTION

Substituting $H[p] = p\log_2(1/p)$ and [1, (1)] into [1, (11)] yields

$$R = \frac{1}{ln}\sum_{x\in\mathbb{N}_1} P(x)\log_2\left(\frac{1}{p_1(1-p_1)^{x-1}}\right). \tag{1}$$

Respecifying the logarithmic terms of (1) in a more convenient form gives

$$\begin{aligned} R &= \frac{1}{ln}\sum_{x\in\mathbb{N}_1} P(x)\left[\log_2\left(\frac{1}{p_1}\right) + (x-1)\log_2\left(\frac{1}{1-p_1}\right)\right] \\ &= \frac{1}{ln}\log_2\left(\frac{1}{p_1}\right)\sum_{x\in\mathbb{N}_1} P(x) + \frac{1}{ln}\log_2\left(\frac{1}{1-p_1}\right)\left[\sum_{x\in\mathbb{N}_1} P(x)x - \sum_{x\in\mathbb{N}_1} P(x)\right]. \end{aligned} \tag{2}$$

Substituting $\sum_{x\in\mathbb{N}_1} P(x) = 1$ and $\sum_{x\in\mathbb{N}_1} P(x)x = l$ into (2) yields

$$R = \frac{1}{ln}\log_2\left(\frac{1}{p_1}\right) + \frac{1}{ln}\log_2\left(\frac{1}{1-p_1}\right)[l-1]. \tag{3}$$

Substituting $l = 1/p_1$ from [1, (6)] into (3) gives

$$R = \frac{1}{n}p_1\log_2\left(\frac{1}{p_1}\right) + \frac{1}{n}(1-p_1)\log_2\left(\frac{1}{1-p_1}\right). \tag{4}$$

Finally, the expression $H[p] = p\log_2(1/p)$ can be applied to (4) in order to obtain [1, (12)].

## II. Conditional transition probabilities

Substituting [1, (8)] into [1, (9)] gives

$$P(m|m') = \begin{cases} \dfrac{\left[1-\sum_{x=1}^{\lceil \frac{m'}{2} \rceil} P(x)\right]}{\left[1-\sum_{x=1}^{\lceil \frac{m'}{2} \rceil} P(x)\right]+P(x)\Big|_{x=\lceil \frac{m'}{2} \rceil}} & \text{if } m' \in \{1,2,3,\ldots,r-2\},\ m = m'+2 \\ \dfrac{P(x)\Big|_{x=\lceil \frac{m'}{2} \rceil}}{\left[1-\sum_{x=1}^{\lceil \frac{m'}{2} \rceil} P(x)\right]+P(x)\Big|_{x=\lceil \frac{m'}{2} \rceil}} & \text{if } m' \in \{1,2,3,\ldots,r-2\},\ m = 1+\mathrm{odd}(m') \\ \dfrac{\left[1-\sum_{x=1}^{\frac{r}{2}-1} P(x)\right]}{\left[1-\sum_{x=1}^{\frac{r}{2}-1} P(x)\right]+\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} P(x)\left(x-\frac{r}{2}\right)\right]} & \text{if } m' \in \{r-1,r\},\ m = 1+\mathrm{odd}(m') \\ \dfrac{\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} P(x)\left(x-\frac{r}{2}\right)\right]}{\left[1-\sum_{x=1}^{\frac{r}{2}-1} P(x)\right]+\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} P(x)\left(x-\frac{r}{2}\right)\right]} & \text{if } m' \in \{r-1,r\},\ m = m' \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Combining the terms in the denominators of (5) yields

$$P(m|m') = \begin{cases} \dfrac{1-\sum_{x=1}^{\lceil \frac{m'}{2} \rceil} P(x)}{1-\sum_{x=1}^{\lceil \frac{m'}{2} \rceil -1} P(x)} & \text{if } m' \in \{1,2,3,\ldots,r-2\},\ m = m'+2 \\ \dfrac{P(x)\Big|_{x=\lceil \frac{m'}{2} \rceil}}{1-\sum_{x=1}^{\lceil \frac{m'}{2} \rceil -1} P(x)} & \text{if } m' \in \{1,2,3,\ldots,r-2\},\ m = 1+\mathrm{odd}(m') \\ \dfrac{1-\sum_{x=1}^{\frac{r}{2}-1} P(x)}{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} P(x)\left(1+x-\frac{r}{2}\right)} & \text{if } m' \in \{r-1,r\},\ m = 1+\mathrm{odd}(m') \\ \dfrac{l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} P(x)\left(x-\frac{r}{2}\right)}{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} P(x)\left(1+x-\frac{r}{2}\right)} & \text{if } m' \in \{r-1,r\},\ m = m' \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

## III. Area beneath the inverted UEC EXIT curve

Substituting [1, (8)] and (6) into [1, (13)] gives

$$
\begin{aligned}
A &= \frac{1}{2ln}\sum_{m'=1}^{r-2}\left[1-\sum_{x=1}^{\left\lceil \frac{m'}{2}\right\rceil}P(x)\right]\log_2\left(\frac{1-\sum_{x=1}^{\left\lceil \frac{m'}{2}\right\rceil-1}P(x)}{1-\sum_{x=1}^{\left\lceil \frac{m'}{2}\right\rceil}P(x)}\right)\\
&+ \frac{1}{2ln}\sum_{m'=1}^{r-2}\left[P(x)\Big|_{x=\left\lceil \frac{m'}{2}\right\rceil}\right]\log_2\left(\frac{1-\sum_{x=1}^{\left\lceil \frac{m'}{2}\right\rceil-1}P(x)}{P(x)\Big|_{x=\left\lceil \frac{m'}{2}\right\rceil}}\right)\\
&+ \frac{1}{2ln}\sum_{m'=r-1}^{r}\left[1-\sum_{x=1}^{\frac{r}{2}-1}P(x)\right]\log_2\left(\frac{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)}{1-\sum_{x=1}^{\frac{r}{2}-1}P(x)}\right)\\
&+ \frac{1}{2ln}\sum_{m'=r-1}^{r}\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)\right]\log_2\left(\frac{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)}{l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)}\right).
\end{aligned}
\tag{7}
$$

The first two summations over $m'$ in (7) can be replaced with summations over $\check{x} = \lceil m/2 \rceil$ and multiplying them by two. Furthermore, since the terms within the third and fourth summations over $m'$ in (7) are independent of $m'$, they can be replaced with a multiplication by two. This yields

$$
\begin{aligned}
A &= \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}\left[1-\sum_{x=1}^{\check{x}}P(x)\right]\log_2\left(\frac{1-\sum_{x=1}^{\check{x}-1}P(x)}{1-\sum_{x=1}^{\check{x}}P(x)}\right)\\
&+ \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}P(\check{x})\log_2\left(\frac{1-\sum_{x=1}^{\check{x}-1}P(x)}{P(\check{x})}\right)\\
&+ \frac{1}{ln}\left[1-\sum_{x=1}^{\frac{r}{2}-1}P(x)\right]\log_2\left(\frac{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)}{1-\sum_{x=1}^{\frac{r}{2}-1}P(x)}\right)\\
&+ \frac{1}{ln}\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)\right]\log_2\left(\frac{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)}{l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)}\right).
\end{aligned}
\tag{8}
$$

Separating each logarithmic term in (8) into two parts yields

$$
\begin{aligned}
A \;=\;& \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}\left[1-\sum_{x=1}^{\check{x}}P(x)\right]\log_2\left(\frac{1}{1-\sum_{x=1}^{\check{x}}P(x)}\right)\\
-\;& \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}\left[1-\sum_{x=1}^{\check{x}}P(x)\right]\log_2\left(\frac{1}{1-\sum_{x=1}^{\check{x}-1}P(x)}\right)\\
+\;& \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}P(\check{x})\log_2\left(\frac{1}{P(\check{x})}\right)\\
-\;& \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}P(\check{x})\log_2\left(\frac{1}{1-\sum_{x=1}^{\check{x}-1}P(x)}\right)\\
+\;& \frac{1}{ln}\left[1-\sum_{x=1}^{\frac{r}{2}-1}P(x)\right]\log_2\left(\frac{1}{1-\sum_{x=1}^{\frac{r}{2}-1}P(x)}\right)\\
-\;& \frac{1}{ln}\left[1-\sum_{x=1}^{\frac{r}{2}-1}P(x)\right]\log_2\left(\frac{1}{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)}\right)\\
+\;& \frac{1}{ln}\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)\right]\log_2\left(\frac{1}{l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)}\right)\\
-\;& \frac{1}{ln}\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)\right]\log_2\left(\frac{1}{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)}\right).
\end{aligned}
\tag{9}
$$

Combining terms in (9) having the same logarithmic subterm yields

$$
\begin{aligned}
A \;=\;& \frac{1}{ln}\sum_{x=1}^{\frac{r}{2}-1}P(x)\log_2\left(\frac{1}{P(x)}\right)\\
+\;& \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}\left[1-\sum_{x=1}^{\check{x}}P(x)\right]\log_2\left(\frac{1}{1-\sum_{x=1}^{\check{x}}P(x)}\right)\\
-\;& \frac{1}{ln}\sum_{\check{x}=1}^{\frac{r}{2}-1}\left[1-\sum_{x=1}^{\check{x}-1}P(x)\right]\log_2\left(\frac{1}{1-\sum_{x=1}^{\check{x}-1}P(x)}\right)\\
+\;& \frac{1}{ln}\left[1-\sum_{x=1}^{\frac{r}{2}-1}P(x)\right]\log_2\left(\frac{1}{1-\sum_{x=1}^{\frac{r}{2}-1}P(x)}\right)\\
+\;& \frac{1}{ln}\left[l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)\right]\log_2\left(\frac{1}{l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(x-\frac{r}{2}\right)}\right)\\
-\;& \frac{1}{ln}\left[1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)\right]\log_2\left(\frac{1}{1+l-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1}P(x)\left(1+x-\frac{r}{2}\right)}\right).
\end{aligned}
\tag{10}
$$

In the first summation over $\breve{x}$ in (10), the special case where $\breve{x} = r/2 - 1$ can be removed and treated separately. Likewise, the special case where $\breve{x} = 1$ can be removed from the second summation over $\breve{x}$ in (10) and treated separately, yeilding

$$\begin{aligned} A &= \frac{1}{ln} \sum_{x=1}^{\frac{r}{2}-1} P(x) \log_2 \left( \frac{1}{P(x)} \right) \\ &+ \frac{1}{ln} \sum_{\breve{x}=1}^{\frac{r}{2}-2} \left[ 1 - \sum_{x=1}^{\breve{x}} P(x) \right] \log_2 \left( \frac{1}{1 - \sum_{x=1}^{\breve{x}} P(x)} \right) + \frac{1}{ln} \left[ 1 - \sum_{x=1}^{\frac{r}{2}-1} P(x) \right] \log_2 \left( \frac{1}{1 - \sum_{x=1}^{\frac{r}{2}-1} P(x)} \right) \\ &- \frac{1}{ln} \sum_{\breve{x}=2}^{\frac{r}{2}-1} \left[ 1 - \sum_{x=1}^{\breve{x}-1} P(x) \right] \log_2 \left( \frac{1}{1 - \sum_{x=1}^{\breve{x}-1} P(x)} \right) - \frac{1}{ln} \left[ 1 - \sum_{x=1}^{0} P(x) \right] \log_2 \left( \frac{1}{1 - \sum_{x=1}^{0} P(x)} \right) \\ &+ \frac{1}{ln} \left[ 1 - \sum_{x=1}^{\frac{r}{2}-1} P(x) \right] \log_2 \left( \frac{1}{1 - \sum_{x=1}^{\frac{r}{2}-1} P(x)} \right) \\ &+ \frac{1}{ln} \left[ l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( x - \frac{r}{2} \right) \right] \log_2 \left( \frac{1}{l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( x - \frac{r}{2} \right)} \right) \\ &- \frac{1}{ln} \left[ 1 + l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( 1 + x - \frac{r}{2} \right) \right] \log_2 \left( \frac{1}{1 + l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( 1 + x - \frac{r}{2} \right)} \right). \end{aligned} \tag{11}$$

In (11), the special case from the first summation over $\breve{x}$ is equal to another term in the expression, allowing them to be combined. Meanwhile, the special case from the second summation over $\breve{x}$ can be eliminated, since its logarithm has a value of zero. Finally, a transformation can be applied to one of the remaining summations over $\breve{x}$, so that these two summations cancel each other out. This leads to

$$\begin{aligned} A &= \frac{1}{ln} \sum_{x=1}^{\frac{r}{2}-1} P(x) \log_2 \left( \frac{1}{P(x)} \right) \\ &+ \frac{2}{ln} \left[ 1 - \sum_{x=1}^{\frac{r}{2}-1} P(x) \right] \log_2 \left( \frac{1}{1 - \sum_{x=1}^{\frac{r}{2}-1} P(x)} \right) \\ &+ \frac{1}{ln} \left[ l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( x - \frac{r}{2} \right) \right] \log_2 \left( \frac{1}{l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( x - \frac{r}{2} \right)} \right) \\ &- \frac{1}{ln} \left[ 1 + l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( 1 + x - \frac{r}{2} \right) \right] \log_2 \left( \frac{1}{1 + l - \frac{r}{2} - \sum_{x=1}^{\frac{r}{2}-1} P(x) \left( 1 + x - \frac{r}{2} \right)} \right). \end{aligned} \tag{12}$$

Finally, the expression $H[p] = p \log_2(1/p)$ can be applied to (12), in order to obtain [1, (14)].

## IV. Area beneath the inverted UEC EXIT curve when the symbol values obey a geometric distribution

Substituting [1, (1)] and $l = 1/p_1$ from [1, (6)] into (12) gives

$$
\begin{aligned}
A &= \frac{p_1}{n}\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\log_2\left(\frac{1}{p_1(1-p_1)^{x-1}}\right)\\
&+ \frac{2p_1}{n}\left[1-\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\right]\log_2\left(\frac{1}{1-\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}}\right)\\
&+ \frac{p_1}{n}\left[\frac{1}{p_1}-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\left(x-\frac{r}{2}\right)\right]\log_2\left(\frac{1}{\frac{1}{p_1}-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\left(x-\frac{r}{2}\right)}\right)\\
&- \frac{p_1}{n}\left[1+\frac{1}{p_1}-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\left(1+x-\frac{r}{2}\right)\right]\log_2\left(\frac{1}{1+\frac{1}{p_1}-\frac{r}{2}-\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\left(1+x-\frac{r}{2}\right)}\right).
\end{aligned}
\tag{13}
$$

The identities

$$
\begin{aligned}
\sum_{k=0}^{n} r^k &= \frac{r^{n+1}-1}{r-1}\\
\sum_{k=0}^{n} kr^k &= \frac{r(nr^{n+1}-(n+1)r^n+1)}{(r-1)^2}
\end{aligned}
$$

can be employed to obtain

$$\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1} = 1-(1-p_1)^{\frac{r}{2}-1} \tag{14}$$

$$\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\left(x-\frac{r}{2}\right) = \frac{1}{p_1}-\frac{r}{2}-\frac{1-p_1}{p_1}(1-p_1)^{\frac{r}{2}-1} \tag{15}$$

$$\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\left(1+x-\frac{r}{2}\right) = 1+\frac{1}{p_1}-\frac{r}{2}-\frac{1}{p_1}(1-p_1)^{\frac{r}{2}-1} \tag{16}$$

$$\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}(x-1) = \frac{1}{p_1}-1+\left(2-\frac{1}{p_1}-\frac{r}{2}\right)(1-p_1)^{\frac{r}{2}-1}. \tag{17}$$

Substituting (14), (15) and (16) into (13) gives

$$
\begin{aligned}
A &= \frac{p_1}{n}\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\log_2\left(\frac{1}{p_1(1-p_1)^{x-1}}\right)\\
&+ \frac{2p_1}{n}\left[(1-p_1)^{\frac{r}{2}-1}\right]\log_2\left(\frac{1}{(1-p_1)^{\frac{r}{2}-1}}\right)\\
&+ \frac{p_1}{n}\left[\frac{1-p_1}{p_1}(1-p_1)^{\frac{r}{2}-1}\right]\log_2\left(\frac{1}{\frac{1-p_1}{p_1}(1-p_1)^{\frac{r}{2}-1}}\right)\\
&- \frac{p_1}{n}\left[\frac{1}{p_1}(1-p_1)^{\frac{r}{2}-1}\right]\log_2\left(\frac{1}{\frac{1}{p_1}(1-p_1)^{\frac{r}{2}-1}}\right).
\end{aligned}
\tag{18}
$$

Separating the logarithmic terms of (18) yields

$$\begin{aligned} A &= \frac{p_1}{n}\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\left[(x-1)\log_2\left(\frac{1}{1-p_1}\right)+\log_2\left(\frac{1}{p_1}\right)\right] \\ &+ \frac{2p_1}{n}(1-p_1)^{\frac{r}{2}-1}\left(\frac{r}{2}-1\right)\log_2\left(\frac{1}{1-p_1}\right) \\ &+ \frac{1-p_1}{n}(1-p_1)^{\frac{r}{2}-1}\left[\frac{r}{2}\log_2\left(\frac{1}{1-p_1}\right)-\log_2\left(\frac{1}{p_1}\right)\right] \\ &- \frac{1}{n}(1-p_1)^{\frac{r}{2}-1}\left[\left(\frac{r}{2}-1\right)\log_2\left(\frac{1}{1-p_1}\right)-\log_2\left(\frac{1}{p_1}\right)\right]. \end{aligned} \tag{19}$$

Regrouping the logarithmic terms of (19) gives

$$\begin{aligned} A &= \frac{1}{n}\log_2\left(\frac{1}{p_1}\right)\left[p_1\left[\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}\right]+p_1(1-p_1)^{\frac{r}{2}-1}\right] \\ &+ \frac{1}{n}\log_2\left(\frac{1}{1-p_1}\right)\left[p_1\left[\sum_{x=1}^{\frac{r}{2}-1} p_1(1-p_1)^{x-1}(x-1)\right]-p_1\left(2-\frac{1}{p_1}-\frac{r}{2}\right)(1-p_1)^{\frac{r}{2}-1}\right]. \end{aligned} \tag{20}$$

Substituting (14) and (17) into (20) yields

$$A = \frac{1}{n}p_1\log_2\left(\frac{1}{p_1}\right)+\frac{1}{n}(1-p_1)\log_2\left(\frac{1}{1-p_1}\right). \tag{21}$$

Finally, the expression $H[p] = p\log_2(1/p)$ can be applied to (21), in order to obtain [1, (15)].

## References

[1] R. G. Maunder, W. Zhang, T. Wang, and L. Hanzo, “A unary error correction code for the near-capacity joint source and channel coding of symbol values from an infinite set,” *IEEE. Trans. Commun.*, 2013. [Online]. Available: http://eprints.soton.ac.uk/341736/