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# Coarse Geometry and Groups

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A thesis submitted for the degree of
Doctor of Philosophy

April, 2012

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES
SCHOOL OF MATHEMATICS

Doctor of Philosophy

# COARSE GEOMETRY AND GROUPS

by Anastasia Khukhro

The central idea of coarse geometry is to focus on the properties of metric spaces which survive under deformations that change distances in a controlled way. These large scale properties, although too coarse to determine what happens locally, are nevertheless often able to capture the most important information about the structure of a space or a group. The relevant notions from coarse geometry and group theory are described in the beginning of this thesis.

An overview of the cohomological characterisation of property A of Brodzki, Niblo and Wright is given, together with a proof that the cohomology theories used to detect property A are coarse invariants. The cohomological characterisation is used alongside a symmetrisation result for functions defining property A to give a new direct, more geometric proof that expanders do not have property A, making the connection between the two properties explicit. This is based on the observation that both the expander condition and property A can be expressed in terms of a coboundary operator which measures the size of the (co)boundary of a set of vertices.

The rest of the thesis is devoted to the study of box spaces, including a description of the connections between analytic properties of groups and coarse geometric properties of box spaces. The construction of Arzhantseva, Guentner and Spakula of a box space of a finitely generated free group which coarsely embeds into Hilbert is the first example of a bounded geometry metric space which coarsely embeds into Hilbert space but does not have property A. This example is generalised here to box spaces of a large class of groups via a stability result for box spaces.

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# Author's declaration

I, Anastasia Khukhro, declare that the thesis entitled *Coarse geometry and groups* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: *Box spaces, group extensions and coarse embeddings into Hilbert space*, A. Khukhro, Journal of Functional Analysis 263, 2012; *Expanders and property A*, A. Khukhro and N. Wright, Algebraic & Geometric Topology 12, 2012.

Signed..............................................................

Date.................................................................

## Acknowledgements

Thank you to my Ph.D. supervisor Graham Niblo for his time and his guidance, as well as for giving me the opportunity to benefit from his experience. Thank you to Jacek Brodzki for his enthusiasm and support. Thank you to Nick Wright for many enjoyable mathematical conversations. Thanks to Ashot Minasyan for selflessly giving up his time to pursue many enlightening discussions with me and fellow graduate students.

I would like to thank everyone at the Department of Mathematics for providing such a warm and friendly environment during my time at Southampton. It is a wonderful coincidence that Building 54 is filled with such lovely people.

I would like to thank all the graduate students and postdocs, past and present, and my friends at Southampton. The intersection of these two sets is almost equal to their union. In particular, thank you to Aditi, Amin, Alex B., Alex S., Carl, Chris, George, Gio, Glenn, Joe, Kannan, Laura, Martin F.-S., Martin F., Michal, Michele, Paco, Raffaele, Ramesh, Rich, Rob, Rosie, Ruben, Sarah, Simon, Theresa, Tom H., Tom N., Wajid, Yago, Yuyen.

Thank you to David, Jenny, Katherine, Lina, Andrew, Hiroki, Sam and John for their continuing friendship and support.

Thanks also to all the people I've met at conferences and workshops, for their contagious passion for the subject.

I would like to thank my parents for their unconditional love and support, and for being my inspiration.

# Introduction

The *Baum–Connes conjecture* was first introduced by Paul Baum and Alain Connes in the 1980s, and forms part of Connes' non-commutative geometry programme. It serves as a unifying theme for this thesis, since many of the notions and results considered here arose from attempts to prove the conjecture for various classes of groups.

The conjecture identifies two objects associated to a countable group $\Gamma$, one analytic, and the other topological. Specifically, the conjecture says that the analytical assembly map

$$\mu_i^\Gamma : RK_i^\Gamma(\underline{\mathrm{E}}\Gamma) \longrightarrow K_i(C_r^*\Gamma)$$

($i = 0, 1$) between the $\Gamma$-equivariant K-homology with $\Gamma$-compact supports of the classifying space $\underline{E}\Gamma$ for proper actions of $\Gamma$ and the K-theory of the reduced C*-algebra of $\Gamma$ is an isomorphism. Often, injectivity and surjectivity of the above maps are considered separately.

Remarkably, this conjecture implies several other conjectures in topology, geometry and functional analysis, notably the Novikov conjecture, the Gromov–Lawson–Rosenberg conjecture (these follow from the rational injectivity of $\mu_i^\Gamma$, see [BCH] and [Ros] respectively), and the Kadison–Kaplansky conjecture (which follows from the surjectivity of $\mu_0^\Gamma$, see [BCH]).

Many analytic and geometric properties of groups have been studied extensively due to their usefulness towards the conjecture. The class of *amenable* groups, first introduced by von Neumann in 1929 [Neu], was found to lie inside the larger class of groups admitting a metrically proper isometric action on an affine Hilbert space for which the conjecture was proved in [HK]. Such groups are called *a-(T)-menable*, a term coined by Gromov in [Gr93], reflecting the fact that a-(T)-menability is a strong negation of *property (T)*.

Property (T) was first introduced by Kazhdan in 1967 [Kazh] as a tool for studying semi-simple Lie groups and their discrete subgroups. Lafforgue was the first to give examples of groups with property (T) for which the Baum–Connes conjecture holds in [Laf], although there are groups with property (T) for which the conjecture is still unknown, for example $SL(n, \mathbb{Z})$ for $n \geq 3$. A weaker version of (T) called *property* $(\tau)$ was introduced by Lubotzky and captures some of the important aspects of property (T) ([LZ]).

The properties mentioned above not only have connections to the Baum–Connes conjecture, but are also of interest in their own right.

The conjecture has a coarse geometric analogue for metric spaces, namely the coarse Baum–Connes conjecture, which relates a coarse homology theory of the space with the K-theory of a C*-algebra associated to the space. This conjecture has spawned various coarse counterparts of the group properties mentioned above, such as *property A* and *coarse embeddability into Hilbert space*. The study of such coarse properties, *coarse geometry*, is effective because it is often the large-scale properties of spaces that have important implications for topology, functional analysis and group theory.

Sometimes it is illuminating to study the interaction of the coarse and the algebraic properties of a group. One can view a finitely generated group as a metric space by constructing its *Cayley graph*. This is a graph, depending on the choice of finite generating set of the group, whose vertices are the elements of the group. Two elements are joined by an edge if one can get from one element to the other by multiplying by a generator on the right. The resulting graph metric is invariant under the left multiplication action of the group, and different choices of generating set yield *quasi-isometric* graphs. The Cayley graph allows us to "see" what our group looks like, and deduce things about its algebraic structure from this picture. The coarse geometric properties we can look at include the rate of growth of balls, the number of *ends* or the curvature of our graph.

The first glimpse of what can be achieved with this viewpoint is Gromov's beautiful polynomial growth theorem [Gr81], which states that the size of the balls in the Cayley graph of a group is bounded by a polynomial function of the radius precisely when the group is *virtually nilpotent*, i.e. has a nilpotent subgroup of finite index. The fact that this purely algebraic notion can be translated into the language of geometry is truly remarkable, and has led to a thriving and fruitful area of research, *geometric group theory*, which aims to exploit the correspondence between algebra and geometry further.

A group is said to be *residually finite* if for each non-trivial element of the group, there is a finite quotient such that the image of this element remains non-trivial. If a finitely generated group is residually finite, we can construct another associated geometric object called a *box space*. A box space is a metrised disjoint union of a sequence of Cayley graphs of finite quotients of the group which in a sense approximate the Cayley graph of the group. This definition was first formalised in [Roe], where some of the existing results on box spaces were collected. The coarse geometric properties of this object allow us to study the analytic properties of the group.

Notably, Guentner showed that a finitely generated residually finite group is amenable if and only if its box space has property A. This result gave rise to one of the only examples of metric spaces without property A. Box spaces were also used by Arzhantseva, Guentner and Spakula in the first construction of a metric space with *bounded geometry* which coarsely embeds into Hilbert space, but does not have property A [AGS].

## Overview

The original material in this thesis has been submitted for publication in the form of the following two papers:

[Khu] A. Khukhro, *Box spaces, group extensions and coarse embeddings into Hilbert space*, Journal of Functional Analysis 263, 2012;

[KW] A. Khukhro and N. Wright, *Expanders and property A*, Algebraic & Geometric Topology 12, 2012.

In Chapter 1 we give definitions of the group-theoretic properties that will be relevant to us. We list some equivalent characterisations of these properties, and discuss connections between them.

Chapter 2 contains an introduction to coarse geometric properties of metric spaces. We describe the cohomological characterisation of property A of Brodzki, Niblo and Wright and prove that the cohomology theories used to detect property A are coarse invariants.

In Chapter 3, we prove the results of the paper [KW] which is joint work with Nick Wright. We describe a symmetrisation result for functions defining property A, and use this together with the cohomological characterisation to give a new direct, more geometric proof that expanders do not have property A, making the connection between the two properties explicit. This is based on the observation that both the expander condition and property A can be expressed in terms of a coboundary operator which measures the size of the (co)boundary of a set of vertices.

Chapter 4 contains an introduction to box spaces, in which we describe the beautiful connections between analysis and coarse geometry. We give an overview of the construction of Arzhantseva, Guentner and Spakula of a bounded geometry metric space which coarsely embeds into Hilbert space but does not have property A. We then generalise this elegant example, using a stability result for box spaces, to a large class of groups. This material is contained in the paper [Khu].

# Chapter 1

# Analytic properties of groups

## 1.1 Amenability

The strong form of the Banach-Tarski paradox is the following statement: *if $A$ and $B$ are any two bounded subsets of $\mathbb{R}^3$ with non-empty interior, then there is a partition of $A$ into finitely many subsets which can be reassembled to give $B$.* The proof relies on two things: the axiom of choice, and the fact that the free group on two generators, $\mathbb{F}_2$, lacks a property called *amenability* (for an exposition, see [Rund]).

**Definition 1.1** (von Neumann, [Neu])**.** Let $G$ be a discrete group. We say $G$ is *amenable* if there exists a finitely-additive left-invariant probability measure on $G$.

If a group $G$ acts on $X \neq \emptyset$, then $E \subseteq X$ is called *$G$-paradoxical* if there exist pairwise disjoint subsets $A_1, ..., A_n, B_1, ..., B_m \subseteq E$ along with elements

$g_1, ..., g_n, h_1, ..., h_m \in G$ such that

$$E = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{i=1}^{m} h_i B_i.$$

A deep theorem of Tarski [Tar], which uses the axiom of choice, tells us that a discrete group $\Gamma$ is amenable if and only if $\Gamma$ does not have such a paradoxical decomposition (when acting on itself by left translation). We can now prove that $\mathbb{F}_2$ is indeed not amenable. Consider subsets of the form $W(x) = \{w \in \mathbb{F}_2 : w \text{ starts with } x\}$. Then

$$\mathbb{F}_2 = W(a) \cup aW(a^{-1}) = W(b) \cup bW(b^{-1})$$

with $W(a), W(a^{-1}), W(b), W(b^{-1})$ pairwise disjoint, giving a paradoxical decomposition.

While the original definition of von Neumann was motivated by the above, there exist many equivalent characterisations of amenability. We list some of them below.

**Theorem 1.2.** *The following are equivalent for a discrete countable group $G$:*

1. *for all $R, \varepsilon > 0$, there exists $\phi \in \ell^1(G)$, finitely supported, such that $\|\phi\|_1 = 1$ and such that for all $g \in B(e, R)$, $\|g\phi - \phi\|_1 \leq \varepsilon$, where $g\phi(h) = \phi(g^{-1}h)$;*

2. *(Kesten's Criterion [Kest]) the probability $P(n)$ that a uniform random walk on the Cayley graph of $G$ (with respect to some finite generating set) returns to its starting point after $2n$ steps does not decay exponentially fast;*

3. *G admits a Følner sequence, i.e., a sequence $F_i$ of finite subsets of G such that*

$$\lim_{i\to\infty} \frac{|gF_i \Delta F_i|}{|F_i|} = 0 \ , \ \forall g \in G;$$

4. *there exists a G-invariant mean $\mu$ on $\ell^\infty(G)$.*

By using the Følner criterion above we can now easily see that $\mathbb{Z}$ is amenable, defining $F_i$ to be $\{-i, ..., 0, ..., i\}$. Abelian and, more generally, solvable groups are amenable. For our purposes, we can think of amenable groups as generalisations of finite groups, since the existence of an invariant mean will allow us to average functions on groups. If a group $G$ is finite, given a function $f : G \longrightarrow \mathbb{C}$, we can average it over the group by defining the function

$$g \mapsto \frac{1}{|G|} \sum_{g \in G} f(g).$$

In the case when the group $G$ is infinite but amenable, the role of the average of a bounded function $f \in \ell^\infty(G)$ is played by $\mu(f)$.

Amenability is preserved by quotients, extensions and subgroups. Thus, any discrete group containing the free group on two generators cannot be amenable. The converse is not true in general, for instance, the Tarski monster is non-amenable but contains no $\mathbb{F}_2$ subgroup. It does however hold for groups which satisfy a Tits alternative, such as the original Tits alternative for linear groups, below.

**Theorem 1.3** (Tits Alternative, [Tits]). *Let $L$ be a Lie group with finitely many connected components, and let $K \leqslant L$ be a finitely generated subgroup. Then $K$ must either*

- *contain a free group of rank 2, or*

- *be virtually soluble (i.e., have a soluble subgroup of finite index).*

The following is a notion of when two metric spaces are almost the same, up to some controlled deformation.

**Definition 1.4.** Let $(X_1, d_1)$ and $(X_2, d_2)$ be metric spaces. A map $f : X_1 \longrightarrow X_2$ is called a *quasi-isometric embedding* if there exist constants $\lambda \geq 1$ and $\varepsilon \geq 0$ such that for all $x, y \in X_1$, we have

$$\frac{1}{\lambda} d_1(x, y) - \varepsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \varepsilon.$$

If, in addition, there exists a constant $C \geq 0$ such that every point of $X_2$ lies in the $C$-neighbourhood of the image of $f$, then we say that $f$ is a *quasi-isometry.* When such a map exists, we say $X_1$ and $X_2$ are *quasi-isometric.*

Amenability is also preserved by quasi-isometries. From this and the above, we can see that amenability is a very stable property.

The Baum–Connes conjecture holds for amenable groups and, more generally, groups with the *Haagerup property* [HK].

## 1.2 The Haagerup property

A function $\psi : G \longrightarrow \mathbb{R}_+$ on a second countable, locally compact group $G$ is said to be *conditionally negative definite* if for all $n$-tuples $(g_1, g_2, ..., g_n) \in G^n$ and all real scalars $(\lambda_1, \lambda_2, ..., \lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i = 0$, we have

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi(g_i^{-1} g_j) \leq 0.$$

The function $\psi$ is said to be *proper* if $\lim_{g\to\infty}\psi(g) = \infty$, where $g \to \infty$ means that $g$ eventually leaves every compact subset of $G$.

Given a locally compact group $G$, a *strongly continuous unitary representation* $(\pi, \mathcal{H})$ of $G$ on a Hilbert space $\mathcal{H}$ is a homomorphism

$$\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$$

from $G$ into the unitary group of $\mathcal{H}$ such that the function $g \mapsto \pi(g)v$ is norm-continuous for each $v \in \mathcal{H}$.

In characterising the Haagerup property, we will use the *Fell topology* on the set of equivalence classes of all continuous unitary representations of the group $G$, denoted $\tilde{G}$. Let $(\pi, \mathcal{H})$ be a representation in $\tilde{G}$. For a compact subset $K \subset G$, $\varepsilon > 0$ and $v \in \mathcal{H}$ with $\|v\| = 1$, define the neighbourhood $W(K, \varepsilon, v)$ to be those representations $(\pi', \mathcal{H}')$ in $\tilde{G}$ for which there exists $v' \in \mathcal{H}'$ with $\|v'\| = 1$ such that for all $g \in K$,

$$|\langle v, \pi(g)v\rangle - \langle v', \pi'(g)v'\rangle| < \varepsilon.$$

Given two representations $(\pi, \mathcal{H}), (\pi', \mathcal{H}') \in \tilde{G}$, we say that $\pi$ *weakly contains* $\pi'$ and write $\pi' \prec \pi$ if $\pi'$ is contained in the closure of $\pi$ in the Fell topology.

We give the definition of the Haagerup property and equivalent characterisations as the following theorem.

**Theorem 1.5.** *A second countable, locally compact group $G$ is said to have the* Haagerup property *if any of the following equivalent conditions hold:*

1. *there exists a continuous function $\phi : G \longrightarrow \mathbb{R}_+$ which is conditionally negative definite and proper;*

2. *there exists a sequence* $(\phi_n)_{n\in\mathbb{N}}$ *of continuous normalised positive definite functions on G vanishing at infinity on G and converging to 1 uniformly on compact subsets of G;*

3. *there exists a strongly continuous unitary representation of G, whose matrix coefficients vanish at infinity on G, which weakly contains the trivial representation;*

4. *there exists a continuous isometric action of G on an affine Hilbert space, which is metrically proper (i.e., for all bounded subsets B of* $\mathcal{H}$*, the set* $\{g \in G : g(B) \cap B \neq \emptyset\}$ *is relatively compact in G).*

The equivalence of (1) and (2) is due to Akemann and Walter [AW], (1) and (3) is due to Jolissaint [Jol], and (1) and (4) is due to Bekka, Cherix and Valette [BCV]. The proofs are also collected in [CJV].

Amenable groups all have the Haagerup property, as do groups which admit a proper action on a CAT(0) cube complex (or a space with walls). The Haagerup property is preserved by subgroups, but differs from amenability in that it is not in general preserved by quotients and extensions. Some partial results hold, for instance, the Haagerup property is preserved by extensions with amenable quotient [Jol]. A comprehensive discussion can be found in [CJV].

The Haagerup property is often referred to as *a-(T)-menability*, a term originally coined by Gromov for the last two characterisations in the list above. The term a-(T)-menability emphasises that this property is a strong negation of *Kazhdan's property (T)*, which we will now discuss.

## 1.3 Properties (T) and ($\tau$)

We give four equivalent conditions for a group to have property (T), which correspond to the different characterisations of the Haagerup property above.

**Theorem 1.6.** *A second countable, locally compact group $G$ is said to have* Kazhdan's property (T) *if any of the following equivalent conditions hold:*

1. *every continuous conditionally negative definite function on $G$ is bounded;*
2. *every sequence of continuous normalised positive definite functions on $G$ converging to 1 uniformly on compact subsets of $G$ converges to 1 uniformly on the whole group $G$;*
3. *any representation of $G$ weakly containing the trivial representation also contains it strongly (i.e. the representation has non-zero $G$-fixed vectors);*
4. *every continuous isometric action of $G$ on an affine Hilbert space has a fixed point.*

See [HV] for the proofs of the equivalences.

Another way of phrasing characterisation (3) uses the Fell topology, defined above. Let $\tilde{G}_0 \subset \tilde{G}$ denote the subset of equivalence classes of continuous unitary representations of $G$ which do not have invariant vectors. Then the group $G$ has property (T) if and only if the trivial representation is isolated away from $\tilde{G}_0$ in $\tilde{G}$, i.e. $1_G$ is not in the closure of $\tilde{G}_0$.

A group which has both the Haagerup property and property (T) has to be compact. In fact, there is a more general notion called *relative property (T)*, which also provides an obstruction to the Haagerup property.

**Theorem 1.7.** *Let $H$ be a closed subgroup of a compactly generated, locally compact group $G$. The pair $(G, H)$ is said to have* relative property (T) *if any of the following equivalent conditions hold:*

1. *every continuous conditionally negative definite function on $G$ is bounded on $H$;*
2. *every sequence of continuous normalised positive definite functions on $G$ converging to 1 uniformly on compact subsets of $G$ converges to 1 uniformly on $H$;*
3. *any representation of $G$ weakly containing the trivial representation has non-zero $H$-fixed vectors;*
4. *every continuous isometric action of $G$ on an affine Hilbert space has a $H$-fixed point.*

The presence of a non-compact subgroup $H < G$ such that $(G, H)$ has relative property (T) prevents $G$ from having the Haagerup property. In fact, this is the only obstruction to the Haagerup property for a connected Lie group $G$ [CJV]. It was asked in [CJV] whether this dichotomy holds in general, but a counterexample was later given by de Cornulier in his Ph.D. thesis [Cor], who introduced the more general concept of property (T) relative to subsets.

There are groups with property (T) which are known to satisfy the Baum–Connes conjecture, such as cocompact lattices in the Lie groups $Sp(n, 1)$. However, it is not known whether the conjecture holds for the groups $SL(n, \mathbb{Z})$ where $n \geq 3$.

Kazhdan first introduced property (T) as a way of proving that certain lattices of locally compact groups are finitely generated. In fact, "most" groups have property (T), as it was shown by Żuk in [Zuk] that generic presentations of certain density (a control on the number of relations) have (T). Żuk proves this result using a clever sufficient condition for property (T), with the help of the following finite graph defined using the presentation of the group.

Let $\Gamma$ be a group which is finitely generated by the set $S$, where $S = S^{-1}$ and $e \notin S$. Define the graph $L(S)$ to have vertices $\mathcal{V}(L(S)) = \{s : s \in S\}$ and edges $\mathcal{E}(L(S)) = \{(s,t) : s, t, s^{-1}t \in S\}$. We can always ensure this graph is connected, for instance by replacing $S$ by $S \cup S^2 \backslash e$. Consider the discrete Laplace operator $\Delta$ on square-summable functions on the vertices of $L(S)$:

$$\Delta(f)(s) = f(s) - \frac{1}{\deg(s)} \sum_{(s,t) \in \mathcal{E}(L(S))} f(t),$$

for $s \in L(S)$ and $f \in \ell^2(L(S))$. $\Delta$ is a non-negative self-adjoint operator on $\ell^2(L(S))$ and zero is an eigenvalue of $\Delta$. Let $\lambda_1$ be the smallest non-zero eigenvalue of $\Delta$ acting on $\ell^2(L(S))$. Żuk proves the following sufficient condition for (T).

**Theorem 1.8** ([Zuk]). *Let $\Gamma$ be a group which is finitely generated by a symmetric set $S$ not containing the identity, such that the graph $L(S)$ is connected. If $\lambda_1 > \frac{1}{2}$, then $\Gamma$ has property (T).*

Finite quotients of groups with property (T) were used by Margulis [Mar] to give the first explicit examples of *expander graphs*, highly connected graphs of bounded valency, which will be discussed in Chapter 3.

There is a property weaker than property (T) which guarantees that certain finite quotients are expanders. Let $\Gamma$ be a finitely generated group, with a family of finite index subgroups $\mathcal{L} = \{N_i\}$. Let $R(\mathcal{L})$ be the subset of $\tilde{\Gamma}$ of representations which appear as a subrepresentation of the action of $\Gamma$ on $\ell^2(\Gamma/N_i)$ for some $N_i \in \mathcal{L}$, and let $R_0(\mathcal{L}) = R(\mathcal{L}) \cap \tilde{\Gamma}_0$.

**Definition 1.9.** $\Gamma$ has *property* $(\tau)$ *with respect to the family* $\mathcal{L}$ if the trivial representation is isolated from the set $R_0(\mathcal{L})$ in $\tilde{\Gamma}$.

We simply say $\Gamma$ has *property* $(\tau)$ if $\Gamma$ has property $(\tau)$ with respect to the family of *all* finite index subgroups. For more on $(\tau)$, see [LZ].

We will explain the construction of expanders from such groups later, when we elaborate on the connections between these analytic properties of groups, and the geometric properties of the next chapter.

# Chapter 2

# Coarse properties of metric spaces

When studying metric spaces, the main idea of coarse geometry is to focus on the properties which survive under deformations which change distances in some controlled way. These large scale properties, although too coarse to determine what happens locally, are nevertheless often able to capture the most important information about the structure of the space.

In the context of coarse geometry, coarse embedding (sometimes also referred to as uniform embedding) is the natural notion of inclusion of one space into another.

**Definition 2.1.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. $X$ is *coarsely embeddable* in $Y$ if there is a map $F : X \longrightarrow Y$ such that there exist non-decreasing functions $\rho_\pm : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\lim_{t\to\infty} \rho_\pm(t) = \infty$ and

$$\rho_-(d_X(x, x')) \leq d_Y(F(x), F(x')) \leq \rho_+(d_X(x, x'))$$

for all $x, x' \in X$.

It is clear from the definition that a quasi-isometric embedding is stronger than a coarse embedding. We consider two spaces to be the same when there exists a coarse embedding of one into the other which is almost a bijection, in the following sense.

**Definition 2.2.** Two metric spaces $(X, d_X)$ and $(Y, d_Y)$ are *coarsely equivalent* if there exists a coarse embedding $F : X \longrightarrow Y$ and a constant $C > 0$ such that for each $y \in Y$, there is $x \in X$ with $d_Y(y, F(x)) \leq C$.

Note that coarse equivalences between finitely generated groups are actually quasi-isometries. This is true in the more general case of quasi-geodesic spaces, see [NY].

So, in the world of coarse geometry, all finite spaces are as interesting as a single point, and the integers are coarsely equivalent to the real line. We will often restrict ourselves to looking at metric spaces which satisfy a uniform local finiteness condition, namely *bounded geometry.*

**Definition 2.3.** A metric space $X$ has *bounded geometry* if for all $R > 0$ there is $N_R$ such that for each $x \in X$, $|B(x, R)| \leq N_R$.

The properties we consider here are just some of the many properties which are stable under coarse equivalence.

## 2.1 Property A

In [Yu], Yu introduced the notion of *property A*, a non-equivariant analogue of amenability for metric spaces. This definition was motivated by the question of whether a given metric space admits a uniform embedding into Hilbert space, and the result of Bekka, Cherix and Valette which states that every amenable group admits a proper isometric action on Hilbert space [BCV].

**Definition 2.4** ([Yu])**.** A uniformly discrete metric space $(X, d)$ is said to have property A if for all $R, \varepsilon > 0$ there exists a family of non-empty subsets $\{A_x\}_{x \in X}$ of $X \times \mathbb{N}$ such that

- for all $x, y$ in $X$ with $d(x, y) < R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$,
- there exists $S$ such that for all $x$ in $X$ and $(y, n)$ in $A_x$ we have $d(x, y) \leq S$.

The reason we are interested in such spaces is that the coarse Baum–Connes conjecture holds for discrete metric spaces with bounded geometry which admit a uniform embedding into Hilbert space, and property A guarantees such an embedding [Yu]. The class of finitely-generated groups that have property A with respect to the word length metric strictly contains the class of finitely-generated amenable groups, and also includes Gromov hyperbolic groups. As one would hope, property A is a quasi-isometry invariant, and thus is independent of the choice of generating set in the case of Cayley graphs.

Just as with amenability, there are many characterisations of property A. The above definition is similar to Følner's criterion, and is geometric in flavour. To phrase property A in the language of functional analysis, we need to define the *uniform Roe algebra*.

**Definition 2.5.** Let $X$ be a metric space. Given a linear operator $T$ on $\ell^2(X)$, it is said to have *finite propagation* if there exists $R > 0$ such that

$\langle T\xi_x, \xi_y \rangle = 0$ for all $x, y \in X$ with $d(x,y) > R$ (here $\xi_x$ denotes the characteristic function of $x$).

The *uniform Roe algebra* of a metric space $X$ is the closure in the operator norm of the set of finite propagation operators on $\ell^2(X)$.

A C*-algebra $A$ is *nuclear* if for every C*-algebra $X$, the C*-completion of the tensor product $A \otimes X$ is unique.

**Theorem 2.6** ([Roe]). *A bounded geometry metric space $X$ has property A if and only if its uniform Roe algebra $C_u^*(X)$ is nuclear.*

In their paper [BNW], Brodzki, Niblo and Wright answer a question of Higson, who asked whether there exists a cohomological characterisation of property A. Their cohomology theory is analogous to the bounded cohomology Ringrose and Johnson used to detect amenability for a locally compact group, and introduces the notion of an asymptotically invariant mean on a discrete metric space to mimic the invariant mean which characterises amenability for a group.

We will use the following equivalent characterisation of property A for spaces of bounded geometry from [Tu].

**Theorem 2.7** ([Tu]). *A discrete metric space $(X, d)$ with bounded geometry has* property A *if and only if for every $R > 0$ and $\varepsilon > 0$ there exists an $S > 0$ and a function $\phi : X \longrightarrow \ell^2(X)$ such that $\|\phi(x)\| = 1$ for all $x \in X$ and such that for all $x_1, x_2 \in X$:*

(1) *if $d(x_1, x_2) \leq R$ then $|1 - \langle \phi(x_1), \phi(x_2) \rangle| \leq \varepsilon$, and*

(2) $\operatorname{Supp} \phi(x) \subset B_S(x)$ *for all $x \in X$.*

There are only a few known examples of spaces without property A. One example follows from a result of Guentner (Proposition 4.5), which deals with *box spaces*, certain metric spaces which can be associated to finitely generated residually finite groups. These are discussed in the final chapter of this thesis. In particular, Guentner's result states that non-amenable, finitely generated, residually finite groups give rise to box spaces without property A. Another class of examples of spaces without A is *expander sequences*, which will be discussed later.

Willett has studied metrised disjoint unions $\sqcup X_n$ of finite graphs $X_n$ which are at least 3-valent such that $\sqcup X_n$ has bounded geometry and is of *large girth*. In [Wil], he proves that if $\lim_{n\to\infty} \operatorname{girth}(X_n) = \infty$, then the disjoint union $\sqcup X_n$ does not have property A, giving a new class of examples.

Property A is preserved by subspaces and coarse equivalences. In the case of groups, it is preserved by many constructions including group extensions [DG03] and free products [CDGY]. Note that in contrast to amenability, property A is not preserved by quotients. This follows from the fact that although free groups have property A, Gromov has constructed finitely generated groups which contain coarsely embedded expander sequences and thus do not have property A ([Gr03], [AD]).

## 2.2 Coarse embeddability into Hilbert space

As we mentioned in the previous section, property A is a condition which guarantees coarse embeddability into Hilbert space.

**Definition 2.8.** Let $(X, d_X)$ be a metric spaces. $X$ is *coarsely embeddable into Hilbert space* if there exists a Hilbert space $\mathcal{H}$ and a map $F : X \longrightarrow$

$\mathcal{H}$ such that there exist non-decreasing functions $\rho_\pm : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\lim_{t\to\infty} \rho_\pm(t) = \infty$ and

$$\rho_-(d_X(x,x')) \leq \|F(x) - F(x')\|_{\mathcal{H}} \leq \rho_+(d_X(x,x'))$$

for all $x, x' \in X$.

In what follows, we will say that a metric space is *embeddable* if it embeds coarsely into Hilbert space.

Property A is designed to provide us with a coarse embedding in Hilbert space, and so it is natural to ask whether the converse holds. A counterexample was first given by Nowak [Now07], although his disjoint union of $n$-dimensional cubes $\{0,1\}^n$ over all $n \in \mathbb{N}$ is not of bounded geometry. The question of whether there exist bounded geometry metric spaces without property A which coarsely embed into Hilbert space has recently been answered affirmatively by Arzhantseva, Guentner and Spakula [AGS]. Their elegant example is a carefully chosen *box space* of the free group on two generators.

We will mostly use the following characterisation.

**Theorem 2.9** ([DG03])**.** *A discrete metric space $(X,d)$ with bounded geometry is coarsely embeddable into Hilbert space if and only if for every $R > 0$ and $\varepsilon > 0$ there exists a Hilbert space $\mathcal{H}$ and a map $\phi : X \longrightarrow \mathcal{H}$ such that $\|\phi(x)\| = 1$ for all $x \in X$ and such that for all $x_1, x_2 \in X$:*

(1) *if $d(x_1,x_2) \leq R$ then $|1 - \langle \phi(x_1), \phi(x_2)\rangle| \leq \varepsilon$, and*

(2) $\lim_{S\to\infty} \sup\{|\langle \phi(x_1), \phi(x_2)\rangle| : d(x_1,x_2) \geq S\} = 0.$

The second condition on the map $\phi$ is a weakening of the corresponding condition for property A. Thus, using this characterisation, it is immediately clear that property A implies coarse embeddability into Hilbert space. In Chapter 3, will use the following result of Nowak, who proved that being coarsely embeddable in $\ell^2$ is equivalent to being coarsely embeddable in $\ell^1$.

**Theorem 2.10** ([Now06]). *Let $X$ be a separable metric space. Then $X$ is coarsely embeddable in $\ell^2$ if and only if it is coarsely embeddable in $\ell^1$.*

Coarse embeddability into Hilbert space is preserved by subspaces, coarse equivalences, and in the case of groups, free products [CDGY]. The extensions result [DG03] differs from that for property A, because one needs the stronger assumption that the quotient group has property A, in order for coarse embeddability to be preserved. This result can be compared to its equivariant version, namely, that the Haagerup property is passed to extensions with amenable quotients. For a thorough review of permanence properties of many coarse geometric notions, see [Gue].

One of the few known examples of spaces which do not embed uniformly in Hilbert space are *expanding graphs*, discussed later. These do not have property A. Gromov [Gr03] has since described groups which contain a uniformly embedded expander sequence, and hence fail to embed uniformly in Hilbert space, thus providing a counterexample to the Baum–Connes conjecture with coefficients (a stronger version of the standard Baum–Connes conjecture). This construction was also studied in [AD].

The question of what prevents a (not necessarily bounded geometry) metric space from being uniformly embeddable in Hilbert space has been resolved by Tessera [Tes], who showed that a space does not uniformly embed in

Hilbert space if and only if it contains a uniformly embedded sequence of "generalised expanders". However, for spaces of bounded geometry the only known obstruction to uniform embeddability in Hilbert space is the presence of a uniformly embedded expander, and the problem of whether this is indeed the only possible obstruction remains unsolved.

## 2.3 A cohomological characterisation of property A

We now give an outline of the cohomological characterisation of property A by Brodzki, Niblo and Wright [BNW].

In [BNW], the authors make use of the following equivalent definition of property A, given in terms of *Reiter sequences*.

**Definition 2.11.** A metric space $(X, d_X)$ has property A if for each $x \in X$ and each $n \in \mathbb{N}$ there is an element $f_n(x) \in Prob(X)$ and a sequence $S_n$ such that $\text{supp}(f_n(x)) \subseteq B_{S_n}(x)$ and for any $R \geq 0$, $\|f_n(x_1) - f_n(x_0)\|_1 \to 0$ as $n \to \infty$ uniformly on the set $\{(x_0, x_1) | d_X(x_0, x_1) \leq R\}$.

Let $X$ be a metric space. We will build a bicomplex, the cohomology of which will be analogous to group cohomology, and will detect property A. Augmentation of the vertical differential will lead to an analogue of bounded cohomology, which will vanish for spaces with property A. Since we no longer have the structure of a group, equivariance will be replaced by a controlled support condition. For this, we need additional structure on the module of coefficients in the form of a "support"-style function.

Given a metric space $X$, an $X$-*module* is a triple $\mathcal{V} = (V, \|\cdot\|_V, \mathrm{Supp})$, where $V$ is a Banach space with norm $\|\cdot\|$ and $\mathrm{Supp} : V \longrightarrow \mathcal{P}(X)$ satisfies:

- $\mathrm{Supp}(v) = \emptyset$ if $v = 0$;
- $\mathrm{Supp}(v + w) \subseteq \mathrm{Supp}(v) \cup \mathrm{Supp}(w)$ for all $v, w \in V$;
- $\mathrm{Supp}(\lambda v) = \mathrm{Supp}(v)$ for all $v \in V$ and all $\lambda \neq 0$;
- if a sequence $v_n$ converges to $v$, then $\mathrm{Supp}(v) \subseteq \bigcup_n \mathrm{Supp}(v_n)$.

Let $X$ be a metric space with metric $d_X$ and let $\mathcal{V} = (V, \|\cdot\|_V, \mathrm{Supp})$ be an $X$-module. In [BNW], the additional assumption that there is a group acting by isometries on $X$ is included, so that one may detect amenability in the case when $X$ is the Cayley graph of a group. Here we will omit this for simplicity.

First, some notation. The coordinates of an element $(\mathbf{x}, \mathbf{y}) \in X^{p+1} \times X^{q+1}$ will be written as $((x_0, ..., x_p), (y_0, ..., y_q))$. We will write $\Delta_R^{p+1}$ for the set

$$\{\mathbf{x} \in X^{p+1} | d_X(u, v) \leq R \ \ \forall u, v \in \{x_0, ..., x_p\}\}$$

and $\Delta_R^{p+1,q+1}$ for the set

$$\{(\mathbf{x}, \mathbf{y}) \in X^{p+1} \times X^{q+1} | d_X(u, v) \leq R \ \ \forall u, v \in \{x_0, ..., x_p, y_0, ..., y_q\}\}.$$

If $\phi$ is a function from $X^{p+1} \times X^{q+1}$ to $V$, for each $R > 0$ we can define a semi-norm $\|\phi\|_R = \sup \|\phi(\mathbf{x}, \mathbf{y})\|_V$, where the supremum is taken over $\mathbf{x} \in \Delta_R^{p+1}$ and $\mathbf{y} \in X^{q+1}$.

We now define the notion of *controlled support*, which for us will play the role of equivariance.

**Definition 2.12.** A function $\phi$ is said to have controlled support if for every $R > 0$ there is an $S > 0$ such that if $(\mathbf{x}, \mathbf{y}) \in \Delta_R^{p+1,q+1}$, then $\mathrm{Supp}(\phi(\mathbf{x}, \mathbf{y}))$ is contained in $B_S(x_i)$ and $B_S(y_i)$ for all $i$.

We now consider the following bicomplex

$$\begin{array}{cccccccc}
 & D\big\uparrow & & D\big\uparrow & & D\big\uparrow & \\
 & \mathcal{E}^{2,0} & \xrightarrow{\;d\;} & \mathcal{E}^{2,1} & \xrightarrow{\;d\;} & \mathcal{E}^{2,2} & \xrightarrow{\;d\;} \\
p\big\uparrow & D\big\uparrow & & D\big\uparrow & & D\big\uparrow & \\
 & \mathcal{E}^{1,0} & \xrightarrow{\;d\;} & \mathcal{E}^{1,1} & \xrightarrow{\;d\;} & \mathcal{E}^{1,2} & \xrightarrow{\;d\;} \\
 & D\big\uparrow & & D\big\uparrow & & D\big\uparrow & \\
 & \mathcal{E}^{0,0} & \xrightarrow{\;d\;} & \mathcal{E}^{0,1} & \xrightarrow{\;d\;} & \mathcal{E}^{0,2} & \xrightarrow{\;d\;} \\
 & & & & \xrightarrow[q]{} & & 
\end{array}$$

where the $\mathcal{E}^{p,q}$ are defined as follows,

$\mathcal{E}^{p,q}(X, \mathcal{V}) = \{\phi : X^{p+1} \times X^{q+1} \to V \mid \phi \text{ has controlled support, } \|\phi\|_R < \infty\ \forall R > 0\}.$

The differential $D : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q}$ is given by

$$D\phi((x_0, ..., x_{p+1}), \mathbf{y}) = \sum_{i=0}^{p+1} (-1)^i \phi((x_0, ..., \hat{x}_i, ..., x_{p+1}), \mathbf{y})$$

and $d : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q+1}$ is given by

$$d\phi(\mathbf{x}, (y_0, ..., y_{q+1})) = \sum_{i=0}^{q+1} (-1)^{i+p} \phi(\mathbf{x}, (y_0, ..., \hat{y}_i, ..., y_{q+1})).$$

The maps $D$ and $d$ are well-defined anti-commuting differentials, and are continuous with respect to the topology induced by the semi-norms $\|\cdot\|_R$. We can consider an augmentation of the rows, $\mathcal{E}^{p,-1}(X, \mathcal{V})$, since the differential $d$ still makes sense for $q = -1$. Each augmented row $(\mathcal{E}^{p,*}, d)$ is then exact: for all $p \geq 0$ there is a continuous splitting $s : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q-1}$ defined by

$$s\phi((x_0, ..., x_p), (y_0, ..., y_{q-1})) = (-1)^p \phi((x_0, ..., x_p), (x_0, y_0, ..., y_{q-1}))$$

such that $(ds + sd)\phi = \phi$ for $\phi \in \mathcal{E}^{p,q}, p \geq 0$, and $sd\phi = \phi$ for $\phi \in \mathcal{E}^{p,-1}$.

From the exactness of the augmented rows, we can deduce that the cohomology of the totalisation of the bicomplex, $H^*_{\mathcal{E}}(X, \mathcal{V})$ is isomorphic to the cohomology of the cochain complex $(\mathcal{E}^{*,-1}, D)$. In order to detect property A, we will now describe two completions of the bicomplex which will yield more sensitive cohomology theories.

Consider the spaces $\mathcal{E}^{p,q}(X, \mathcal{V})$ of the bicomplex. The first completion that we will construct is similar to the classical completion obtained by taking the quotient of the space of all Cauchy sequences in $\mathcal{E}^{p,q}$ by the space of those sequences which converge to 0. By "Cauchy sequences", we mean the sequences which are Cauchy with respect to each of the semi-norms $\|\cdot\|_R$.

**Definition 2.13.** The quotient completion of $\mathcal{E}^{p,q}$, denoted by $\mathcal{E}^{p,q}_Q$, is the quotient of the space of *bounded* sequences in $\mathcal{E}^{p,q}$ by the space of sequences in $\mathcal{E}^{p,q}$ which converge to 0.

The second completion we will look at is a standard construction from functional analysis, although it is less apparent what the elements of this completion look like.

**Definition 2.14.** The weak* completion of $\mathcal{E}^{p,q}$, denoted by $\mathcal{E}^{p,q}_W$, is the double dual of $\mathcal{E}^{p,q}$.

Both of the above completions contain the classical completion described above. Let $H^*_Q(X, \mathcal{V})$ and $H^*_W(X, \mathcal{V})$ denote the cohomologies of the cochain complexes $(\mathcal{E}^{*,-1}_Q(X, \mathcal{V}), D)$ and $(\mathcal{E}^{*,-1}_W(X, \mathcal{V}), D)$ respectively. Note that these are isomorphic to the cohomologies of the totalisation of the corresponding bicomplexes.

When trying to detect amenability of a group $G$, it is not enough to look at the cohomology of the cochain complex whose $k$-dimensional cochains are bounded functions from $G^{k+1}$ to $\mathbb{C}$, as this cocomplex is exact. The $G$-invariant part however is not necessarily exact, and if the group $G$ is amenable, an averaging procedure using the invariant mean will provide us with an equivariant splitting. Thus the vanishing of the cohomology characterises amenability for a group.

In our situation, the notion of invariance is replaced by *asymptotic invariance*. We will define asymptotically invariant subcomplexes of $\mathcal{E}_Q$ and $\mathcal{E}_W$, to which the splitting of the horizontal differential $d$ will not restrict. Property A provides a way of asymptotically averaging the splitting so that just as in the group case, the cohomologies of the asymptotically invariant subcomplexes vanish.

We will state the following definitions and results only for the quotient completion, bearing in mind that all the statements also hold for the weak* completion.

**Definition 2.15.** An element $\phi \in \mathcal{E}^{0,q}$ is called asymptotically invariant if $D\phi = 0$ in $\mathcal{E}^{1,q}$.

**Proposition 2.16.** *The set of asymptotically invariant elements $\mathcal{E}^q_{QA}$ together with the differential $d$ is a cochain complex.*

Denote the cohomology of the complex $\mathcal{E}^*_{QA}(X, \mathcal{V})$ by $H^*_{QA}(X, \mathcal{V})$. Brodzki, Niblo and Wright characterise property A as the vanishing of this cohomology, and also as the vanishing of a specific class in $H^1_{QA}(X, \ell^1_0(X))$, namely the class $[\mathcal{J}^{0,1}_Q]$ of $H^1_{QA}(X, \ell^1_0(X))$ corresponding to the *Johnson element* in $\mathcal{E}^{0,1}(X, \ell^1_0(X))$ defined by $\mathcal{J}^{0,1}(x, (y_0, y_1)) = \delta_{y_1} - \delta_{y_0}$.

If we take the coefficient module to be $\ell^1(X)$, a characterisation of property A can be given in terms of the existence of an *asymptotically invariant mean.* To define this, we require the notion of morphisms of $X$-modules.

**Definition 2.17.** Given two $X$-modules $\mathcal{U} = (U, \|\cdot\|_U, \mathrm{Supp}_U)$ and $\mathcal{V} = (V, \|\cdot\|_V, \mathrm{Supp}_V)$, an *$X$-morphism* from $\mathcal{U}$ to $\mathcal{V}$ is a bounded linear map $\psi : U \longrightarrow V$ for which there exists $S \geq 0$ such that for all $u \in U$, $\mathrm{Supp}_V(\psi(u)) \subset B_S(\mathrm{Supp}_U(u))$.

The conditions on the supports in the above definition allow us to see that an $X$-morphism $\psi : \mathcal{U} \longrightarrow \mathcal{V}$ induces a continuous linear map $\psi_* : \mathcal{E}^{p,q}(X, \mathcal{U}) \longrightarrow \mathcal{E}^{p,q}(X, \mathcal{V})$ which commutes with both differentials, and can be extended to a map of the completions.

Consider the following short exact sequence:

$$0 \longrightarrow \ell^1_0(X) \xrightarrow{\iota} \ell^1(X) \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

where $\iota$ is the natural inclusion map, and $\pi$ is the summation map. Here $\ell^1_0(X)$ and $\ell^1(X)$ are given the usual support functions and on $\mathbb{C}$, we take the empty support function, i.e. $\mathrm{Supp}(\lambda) = \emptyset$ for all $\lambda \in \mathbb{C}$. The map $\pi$ induces a map

$$\pi_* : \mathcal{E}^{0,-1}(X, \ell^1(X)) \longrightarrow \mathcal{E}^{0,-1}(X, \mathbb{C})$$

which can be extended to both completions.

When the space $X = G$ is a group (recall that the cohomology theories can, with additional assumptions, be defined equivariantly to account for a group action on $X$; see [BNW]), it is amenable precisely when there is an element $\mu \in \mathcal{E}^{0,-1}_W(G, \ell^1(G))$ such that $D\mu = 0$ and $\pi_*(\mu) = \mathbf{1}_W$, since such a $\mu$ is an invariant mean for $G$. This is the motivation behind the concept of an asymptotically invariant mean.

**Definition 2.18.** An *asymptotically invariant mean* for a metric space $X$ is an element $\mu \in \mathcal{E}_W^{0,-1}(X, \ell^1(X))$ such that $D\mu = 0$ and $\pi_*(\mu) = \mathbf{1}_W$.

Let $\mathbf{1}_Q$ denote the constant function 1 on $X$, viewed as a 0-cocycle in $\mathcal{E}_Q^{0,-1}(X, \mathbb{C})$, and let $\pi_* \colon H_Q^0(X, \ell^1(X)) \to H_Q^0(X, \mathbb{C})$ be the map on cohomology induced by the summation map $\pi \colon \ell^1(X) \to \mathbb{C}$. Then the space $X$ has property A if and only if the class $[\mathbf{1}_Q] \in H_Q^0(X, \mathbb{C})$ is in the image of the map $\pi_*$.

We summarize the results below.

**Theorem 2.19** (Theorem 7.3, [BNW])**.** *Let $X$ be a discrete metric space. The following are equivalent:*

- $H_{QA}^q(X, \mathcal{V}) = 0$ *for all* $q \geq 1$ *and all* $X$*-modules* $\mathcal{V}$*;*
- $[\mathcal{J}_Q^{0,1}] = 0$ *in* $H_{QA}^1(X, \ell_0^1(X))$*;*
- $[\mathbf{1}_Q] \in Im(\pi_*)$ *in* $H_Q^0(X, \mathbb{C})$*;*
- $X$ *admits an asymptotically invariant mean;*
- $X$ *has property A.*

## 2.4 Coarse invariance

We will now show that the cohomology theories defined above are coarse invariants. For this purpose, we will use an equivalent definition of coarse equivalence, in terms of *coarse maps* which are almost inverse to each other.

**Definition 2.20.** Given two metric spaces $(X, d_X)$ and $(Y, d_Y)$, a *coarse map* between them is a metrically proper map $f : X \longrightarrow Y$ for which there exists a function $\rho_+ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\lim_{t\to\infty} \rho_+(t) = \infty$ such that

$$d_Y(f(x_1), f(x_2)) \leq \rho_+(d_X(x_1, x_2))$$

for all $x_1, x_2 \in X$.

Two metric spaces $X$ and $Y$ are coarsely equivalent if there exist coarse maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ and a constant $\varepsilon$ such that $g \circ f : X \longrightarrow X$ and $f \circ g : Y \longrightarrow Y$ do not map any point further than $\varepsilon$ from itself, i.e. the compositions $g \circ f$ and $f \circ g$ are close to the identity maps on $X$ and $Y$ respectively.

**Proposition 2.21.** *Let $(X, d_X)$ and $(Y, d_Y)$ be coarsely equivalent metric spaces, both equipped with the trivial action. We have coarse maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ and a constant $\varepsilon$ such that $g \circ f : X \longrightarrow X$ and $f \circ g : Y \longrightarrow Y$ do not map any point further than $\varepsilon$ from itself. Let $\mathcal{V}$ be the $X$-module $(V, \|\cdot\|_V, \mathrm{Supp})$. The coarse equivalence induces isomorphisms in cohomology, namely $H^*_{\mathcal{E}}(X, \mathcal{V}) \cong H^*_{\mathcal{E}}(Y, g^*\mathcal{V})$, $H^*_Q(X, \mathcal{V}) \cong H^*_Q(Y, g^*\mathcal{V})$ and $H^*_W(X, \mathcal{V}) \cong H^*_W(Y, g^*\mathcal{V})$.*

*Proof.* We will first give the induced maps between the cochain complexes $(\mathcal{E}^{*,-1}(X, \mathcal{V}), D)$ and $(\mathcal{E}^{*,-1}(Y, g^*\mathcal{V}), D)$. Given $\phi : X^{p+1} \longrightarrow V$ in $\mathcal{E}^{p,-1}(X, \mathcal{V})$, define $g^*\phi$ by $g^*\phi(\mathbf{y}) = \phi(g(\mathbf{y}))$. Algebraically, the module $g^*V$ is the same as our original $X$-module $V$, but we will impose a different support function, $\mathrm{Supp}_Y(g^*(v)) := N_\varepsilon(g^{-1} \mathrm{Supp}(v))$ where $N_\varepsilon$ denotes the $\varepsilon$-neighbourhood. It is clear that this support function satisfies the required axioms. We will write $g^*\mathcal{V}$ to mean the $Y$-module $(V, \|\cdot\|_V, \mathrm{Supp}_Y)$.

To see that the image $g^*\phi$ of $\phi \in \mathcal{E}^{p,-1}(X, \mathcal{V})$ has controlled support, take $\mathbf{y} \in \Delta_R^{p+1} \subset Y^{p+1}$. Then $\operatorname{Supp}_Y(g^*\phi(\mathbf{y})) = N_\varepsilon(g^{-1}\operatorname{Supp}(\phi(g(\mathbf{y}))))$ and $g(\mathbf{y}) \in \Delta_{R'}^{p+1} \subset X$ for some $R'$ as $g$ is a coarse map. Since $\phi$ has controlled support, there exists an $S$ such that $\operatorname{Supp}(\phi(g(\mathbf{y})))$ is contained in $B_S(g(y_i))$ for all $i$. Thus, taking $y \in N_\varepsilon(g^{-1}\operatorname{Supp}(\phi(g(\mathbf{y}))))$, we see that

$$\begin{aligned} d_Y(y_i, y) &\leq d_Y(y_i, f(g(y_i))) + d_Y(f(g(y_i)), f(g(g^{-1}(x)))) \\ &\quad + d_Y(f(g(g^{-1}(x))), g^{-1}(x)) + d_Y(g^{-1}(x), y) \\ &\leq \varepsilon + S' + \varepsilon + \varepsilon \end{aligned}$$

where $x \in \operatorname{Supp}(\phi(g(\mathbf{y})))$. Here, $S'$ is a constant which depends only on $S$ (since $f$ is a coarse map), so $N_\varepsilon(g^{-1}\operatorname{Supp}(\phi(g(\mathbf{y})))) \subset B_{S'+3\varepsilon}(y_i)$ and hence $g^*\phi$ has controlled support. The seminorms $\|g^*\phi\|_R$ are clearly finite for all $R$, so $g^*$ is indeed a map from $\mathcal{E}^{p,-1}(X, \mathcal{V})$ to $\mathcal{E}^{p,-1}(Y, g^*\mathcal{V})$. Similarly, $f^*$ maps $\mathcal{E}^{p,-1}(Y, g^*\mathcal{V})$ to $\mathcal{E}^{p,-1}(X, f^*g^*\mathcal{V})$.

Since $g \circ f$ is close to the identity map on $X$, $\{g \circ f, \mathrm{id}_X\} : X \times \{0,1\} \longrightarrow X$ is a coarse map, and showing that $g \circ f$ and $\mathrm{id}_X$ are chain homotopic is reduced to showing that the inclusion maps $i_0, i_1 : X \longrightarrow X \times \{0,1\}$ are chain homotopic.

We will choose the support function on $\mathrm{id}_X^* V$ to be the same as the support function on $(g \circ f)^*V$, namely $\operatorname{Supp}_X(\mathrm{id}_X^*(v)) := N_\varepsilon(f^{-1}N_\varepsilon(g^{-1}\operatorname{Supp}(v)))$. The support function on $\{g \circ f, \mathrm{id}_X\}^*V$ will be given by

$$\operatorname{Supp}_{X\times\{0,1\}}(\{g \circ f, \mathrm{id}_X\}^*(v)) := \operatorname{Supp}((g \circ f)^*v) \times \{0\} \cup \operatorname{Supp}(\mathrm{id}_X^* v) \times \{1\}$$

and the support functions on $i_0^*\{g \circ f, \mathrm{id}_X\}^*V$ and $i_1^*\{g \circ f, \mathrm{id}_X\}^*V$ evaluated at $i_0^*(w)$ and $i_1^*(w)$ respectively will be given by the first coordinate projection of $\operatorname{Supp}_{X\times\{0,1\}}(w)$, for $w \in \{g \circ f, \mathrm{id}_X\}^*V$. We can check that the support functions we defined on $i_0^*\{g \circ f, \mathrm{id}_X\}^*V$ and $i_1^*\{g \circ f, \mathrm{id}_X\}^*V$ coincide with those on $(g \circ f)^*V$ and $\mathrm{id}_X^* V$ respectively.

To see that the maps $i_0, i_1$ are chain homotopic, we write down the chain homotopy

$$h : \mathcal{E}^{p,-1}(X \times \{0,1\}, \{g \circ f, \mathrm{id}_X\}^*\mathcal{V}) \longrightarrow \mathcal{E}^{p-1,-1}(X, f^*g^*\mathcal{V})$$

$$h\phi(x_0, x_1, ..., x_{p-1}) = \sum_{j=0}^{p-1} (-1)^j \phi((x_0, 0), ..., (x_j, 0), (x_j, 1), ..., (x_{p-1}, 1))$$

with $\mathrm{Supp}_X(h\phi(x_0, x_1, ..., x_{p-1}))$ given by the first coordinate projection of $\mathrm{Supp}_{X\times\{0,1\}}(\sum_{j=0}^{p-1} (-1)^j \phi((x_0, 0), ..., (x_j, 0), (x_j, 1), ..., (x_{p-1}, 1)))$.

Computation shows that $Dh + hD = i_1^* - i_0^*$.

We now have the following situation, since $(g \circ f)^* = f^*g^*$:

$$\mathcal{E}^{*,-1}(X, \mathcal{V}) \xrightarrow{g^*} \mathcal{E}^{*,-1}(Y, g^*\mathcal{V}) \xrightarrow{f^*} \mathcal{E}^{*,-1}(X, f^*g^*\mathcal{V}).$$

We know that the composition $(g \circ f)^*$ is chain homotopic to $\mathrm{id}_X^*$. However, it thickens the support of $v \in V$ to $N_\varepsilon(f^{-1}N_\varepsilon(g^{-1}\,\mathrm{Supp}(v)))$. As we showed above, if $\phi \in \mathcal{E}^{p,-1}(X, \mathcal{V})$ has controlled support, then so are $g^*\phi$ and $f^*g^*\phi$. The converse also holds, since for any $v \in V$, $\mathrm{Supp}(v) \subseteq N_\varepsilon(f^{-1}N_\varepsilon(g^{-1}\,\mathrm{Supp}(v)))$. All of the arguments above also hold for $(f \circ g)^*$. Hence the maps $f^*$ and $g^*$ are inverse to one another, and their composition is the identity. Since these maps are also continuous, they extend to the completions $\mathcal{E}_Q$ and $\mathcal{E}_W$, and so we have the induced isomorphisms $H^*_{\mathcal{E}}(X, \mathcal{V}) \cong H^*_{\mathcal{E}}(Y, g^*\mathcal{V})$, $H^*_Q(X, \mathcal{V}) \cong H^*_Q(Y, g^*\mathcal{V})$ and $H^*_W(X, \mathcal{V}) \cong H^*_W(Y, g^*\mathcal{V})$. □

**Corollary 2.22.** *The asymptotically invariant cohomologies* $H^*_{QA}(X, \mathcal{V})$ *and* $H^*_{WA}(X, \mathcal{V})$ *are isomorphic to* $H^*_{QA}(Y, g^*\mathcal{V})$ *and* $H^*_{WA}(Y, g^*\mathcal{V})$ *respectively.*

*Proof.* Define $g^* : \mathcal{E}_Q^{0,q}(X,\mathcal{V}) \longrightarrow \mathcal{E}_Q^{0,q}(Y,g^*\mathcal{V})$ by $g^*\phi(x,\mathbf{y}) = \phi(g(x,\mathbf{y}))$. We need only check that the restriction of this map to the asymptotically invariant elements is well-defined. Take $\phi \in \mathcal{E}_Q^{0,q}(X,\mathcal{V})$ such that $D\phi = 0$ in $\mathcal{E}_Q^{1,q}(X,\mathcal{V})$. Then

$$\begin{aligned} Dg^*\phi(x_0,x_1,\mathbf{y}) &= g^*\phi(x_1,\mathbf{y}) - g^*\phi(x_0,\mathbf{y}) \\ &= \phi(g(x_1,\mathbf{y})) - \phi(g(x_0,\mathbf{y})) \\ &= D\phi(g(x_0),g(x_1),g(\mathbf{y})) \\ &= 0 \end{aligned}$$

and so the restriction of $g^*$ maps $\mathcal{E}_{QA}^q(X,\mathcal{V})$ to $\mathcal{E}_{QA}^q(Y,g^*\mathcal{V})$. We deduce that $H_{QA}^*(X,\mathcal{V}) \cong H_{QA}^*(Y,g^*\mathcal{V})$ as in the proposition above. Exactly the same argument applies to the weak* completion, and so $H_{WA}^*(X,\mathcal{V}) \cong H_{WA}^*(Y,g^*\mathcal{V})$. □

**Remark 2.23.** Property A is a coarse invariant and so given two coarsely equivalent spaces $X$ and $Y$, we know that the Johnson elements $[\mathcal{J}_Q^{0,1}]_X \in H_{QA}^1(X,\ell_0^1(X))$ and $[\mathcal{J}_W^{0,1}]_X \in H_{WA}^1(X,\ell_0^1(X))$ vanish if and only if the Johnson elements $[\mathcal{J}_Q^{0,1}]_Y \in H_{QA}^1(Y,\ell_0^1(Y))$ and $[\mathcal{J}_W^{0,1}]_Y \in H_{WA}^1(Y,\ell_0^1(Y))$ respectively vanish, since the vanishing of the Johnson elements characterises property A [BNW]. However, we can see this without resorting to the coarse invariance of property A, using only cohomology. Theorems 16 and 20 of [BNW] tell us that the vanishing of the Johnson element is equivalent to the vanishing of cohomology in degree greater than 0 for all modules, for both the Q and the weak* completions. Thus, from the above proposition,

$$\begin{aligned} [\mathcal{J}_Q^{0,1}]_X \neq 0 &\iff \exists q \geq 1, \mathcal{V} : H_{QA}^q(X,\mathcal{V}) \neq 0 \\ &\iff \exists q \geq 1, \mathcal{V} : H_{QA}^q(Y,g^*\mathcal{V}) \neq 0 \\ &\iff [\mathcal{J}_Q^{0,1}]_Y \neq 0. \end{aligned}$$

The same argument holds for the weak* completion.

# Chapter 3

# Expanders

In this chapter, we first prove a symmetrisation result for one of the cohomological characterisations of property A from [BNW]. We then introduce expander graphs, and give a cohomological description of the expander property. Finally, we use the symmetrisation result, together with the cohomological characterisation of expanders to prove that expanders do not have property A.

## 3.1 Symmetrisation of property A

Let $X$ be a discrete bounded geometry metric space, and $\mathcal{V} = \{V, \|\cdot\|, \mathrm{supp}\}$ an $X$-module. Assuming the notation from previous sections, we will denote the column $\mathcal{E}^{p,-1}(X, \mathcal{V})$ of the bicomplex by $\mathcal{E}^p(X, \mathcal{V})$ in order to reduce notational complexity. Recall that the cohomology $H^*_Q(X, \mathcal{V})$ is the cohomology of the quotient completion of $\mathcal{E}^p(X, \mathcal{V})$.

Recall that $\mathbf{1}_Q$ denotes the constant function 1 on $X$, viewed as a 0-cocycle in $\mathcal{E}_Q^0(X,\mathbb{C})$, and $\pi_*\colon H_Q^0(X,\ell^1(X)) \to H_Q^0(X,\mathbb{C})$ is the map on cohomology induced by the summation map $\pi\colon \ell^1(X) \to \mathbb{C}$. By Theorem 2.19 (Theorem 7.3 of [BNW]), the space $X$ has property A if and only if the class $[\mathbf{1}_Q] \in H_Q^0(X,\mathbb{C})$ is in the image of the map $\pi_*$.

Here the module $\ell^1(X)$ is equipped with the usual support function, while all elements of $\mathbb{C}$ are defined to have empty support.

We now compare $\ell^1$ and $\ell^2$ coefficients. We define maps $\alpha\colon \ell^1(X) \to \ell^2(X)$ and $\beta\colon \ell^2(X) \to \ell^1(X)$ by

$$\alpha(\eta)(x) = \sqrt{|\eta(x)|} \text{ for } \eta \in \ell^1(X), \quad \beta(\xi)(x) = |\xi(x)|^2 \text{ for } \xi \in \ell^2(X).$$

Note that $\|\alpha(\eta)\|_{\ell^2}^2 = \|\eta\|_{\ell^1}$ and $\|\beta(\xi)\|_{\ell^1} = \|\xi\|_{\ell^2}^2$.

**Lemma 3.1.** *Let $\alpha, \beta$ be defined as above. Then the compositions with $\alpha$ and $\beta$, yield maps $\mathcal{E}^p(X,\ell^1(X)) \to \mathcal{E}^p(X,\ell^2(X))$ and $\mathcal{E}^p(X,\ell^2(X)) \to \mathcal{E}^p(X,\ell^1(X))$ which extend in the natural way to maps $\alpha_*, \beta_*$ on the quotient completions. Moreover these maps take 0-cocycles to 0-cocycles.*

*Proof.* The identity $\|\alpha(\eta)\|_{\ell^2}^2 = \|\eta\|_{\ell^1}$ shows that for $\phi_n$ a bounded sequence in $\mathcal{E}^p(X,\ell^1(X))$, the sequence $\alpha \circ \phi_n \in \mathcal{E}^p(X,\ell^2(X))$ is also bounded. Hence, as composition with $\alpha$ preserves supports, $\alpha \circ \phi_n$ defines an element in the quotient completion. We note that the inequalities

$$\left|\sqrt{|\eta(z)|} - \sqrt{|\eta'(z)|}\right| \leq \sqrt{\left||\eta(z)| - |\eta'(z)|\right|} \leq \sqrt{|\eta(z) - \eta'(z)|}$$

imply that $\|\alpha(\eta) - \alpha(\eta')\|_{\ell^2}^2 \leq \|\eta - \eta'\|_{\ell^1}$. It follows that if $\phi'_n$ is another bounded sequence in $\mathcal{E}^p(X,\ell^1(X))$ such that $\|\phi_n - \phi'_n\|_R \to 0$, then we have $\|\alpha \circ \phi_n - \alpha \circ \phi'_n\|_R \to 0$, and so the element of $\mathcal{E}_Q^p(X,\ell^2(X))$ obtained by composition with $\alpha$ is independent of the choice of representative of element of $\mathcal{E}^p_Q(X, \ell^1(X))$. Thus we have a well-defined map $\alpha_*\colon \mathcal{E}^p_Q(X, \ell^1(X)) \to \mathcal{E}^p_Q(X, \ell^2(X))$.

The estimate $\|\alpha(\eta) - \alpha(\eta')\|^2_{\ell^2} \leq \|\eta - \eta'\|_{\ell^1}$ also yields

$$\begin{aligned}
\|D\alpha(\phi_n)(x_0, x_1)\|^2_{\ell^2} &= \|\alpha(\phi_n(x_1)) - \alpha(\phi_n(x_0))\|^2_{\ell^2} \\
&\leq \|\phi_n(x_1) - \phi_n(x_0)\|_{\ell^1} \\
&= \|D\phi_n(x_0, x_1)\|_{\ell^1}
\end{aligned}$$

for $\phi_n$ a bounded sequence in $\mathcal{E}^0_Q(X, \ell^1(X))$. Hence $\alpha_*$ takes 0-cocycles to 0-cocycles.

The argument for $\beta_*$ is similar, using the identity $\|\beta(\xi)\|_{\ell^1} = \|\xi\|^2_{\ell^2}$ and the estimate $\|\beta(\xi) - \beta(\xi')\|_{\ell^1} \leq \|\xi - \xi'\|_{\ell^2} (\|\xi\|_{\ell^2} + \|\xi'\|_{\ell^2})$ which follows from

$$\big||\xi(x)|^2 - |\xi'(x)|^2\big| = \big||\xi(x)| - |\xi'(x)|\big|\big(|\xi(x)| + |\xi'(x)|\big) \leq \big|\xi(x) - \xi'(x)\big|\big(|\xi(x)| + |\xi'(x)|\big)$$

by the Cauchy-Schwartz inequality. □

We now prove a symmetrisation result. Note that we will omit norm subscripts where this does not cause confusion.

For an element $\phi$ of $\mathcal{E}^0_Q(X, \ell^1(X))$ or $\mathcal{E}^0_Q(X, \ell^2(X))$ we say $\phi$ is *symmetric* if it can be represented by a sequence $\phi_n$ such that $\phi(x)(z)$ is real and $\phi_n(x)(z) = \phi_n(z)(x)$ for all $x, z \in X$. We say that $\phi$ is *everywhere unital* if $\lim_{n\to\infty} \|\phi_n(x)\| = 1$ for all $x \in X$ (note that this limit is independent of the choice of representative sequence).

**Theorem 3.2.** *Let $X$ be a bounded geometry metric space. The following are equivalent:*

1. *X has property A;*

2. *There is a cocycle $\phi \in \mathcal{E}^0_Q(X, \ell^1(X))$ such that $\pi_*(\phi) = \mathbf{1}_Q$;*

3. *There is a symmetric cocycle $\phi \in \mathcal{E}^0_Q(X, \ell^1(X))$ such that $\pi_*(\phi) = \mathbf{1}_Q$;*

4. *There is a symmetric cocycle $\psi \in \mathcal{E}^0_Q(X, \ell^2(X))$ such that $\psi$ is everywhere unital.*

*Proof.* The equivalence of (1) and (2) is Theorem 7.2 of [BNW].

First we prove (2) $\implies$ (4). Suppose there exists a cocycle $\phi \in \mathcal{E}^0_Q(X, \ell^1(X))$ such that $\pi_*(\phi) = \mathbf{1}_Q$. We consider $\alpha_*\phi$. Choosing a representative sequence $\phi_n$ for $\phi$ we note that $\|\alpha(\phi_n(x))\|^2 = \|\phi_n(x)\| \geq 1$ for all $x$ since $\pi(\phi(x)) = 1$. Let $\theta_n(x) = \frac{1}{\|\alpha(\phi_n(x))\|}\alpha(\phi_n(x))$. We know that $\alpha_*\phi$ is a cocycle. The estimate

$$\begin{aligned}\left\|\frac{1}{\|\xi\|}\xi - \frac{1}{\|\xi'\|}\xi'\right\| &\leq \frac{\|\xi - \xi'\|}{\|\xi\|} + \left|\frac{1}{\|\xi\|} - \frac{1}{\|\xi'\|}\right| \|\xi'\| = \frac{\|\xi - \xi'\| + |\,\|\xi'\| - \|\xi\|\,|}{\|\xi\|} \\ &\leq 2\|\xi - \xi'\|\end{aligned}$$

for $\xi \in \ell^2(X)$ with $\|\xi\| \geq 1$, shows that $D\theta_n \to 0$, i.e. $\theta$ again determines a cocycle.

Consider the operators $T_n \colon \ell^2(X) \to \ell^2(X)$ defined by $(T_n\xi)(y) = \sum_{x\in X} \theta_n(x)(y)\xi(x)$. The support condition on $\theta_n$ provides an $S_n > 0$ such that $\theta_n(x)$ is supported in $B_{S_n}(x)$, and bounded geometry gives a bound $N_n$ on the size of these balls, hence the operators $T_n$ are bounded. The support condition also shows that these operators have finite propagation, and thus they are elements of the uniform Roe algebra of $X$ (see [Roe], section 4.4). Consider $T'_n = (T_n^*T_n)^{1/2}$. This lies in the uniform Roe algebra since $T_n$ does, and hence for each $n$ we can find another self-adjoint operator $T''_n$ with $T''_n$ of finite propagation and $\|T''_n - T'_n\| \to 0$ as $n \to \infty$.

Define $\psi_n(x) = T_n''(\delta_x)$. We note that for $\xi \in \ell^2(X)$ we have

$$\langle T_n\xi, T_n\xi\rangle = \langle T_n^*T_n\xi, \xi\rangle = \langle (T_n')^2\xi, \xi\rangle = \langle T_n'\xi, T_n'\xi\rangle$$

so $\|T_n\xi\| = \|T_n'\xi\|$ for all $\xi$. We have $\|T_n'(\delta_x)\| = \|T_n(\delta_x)\| = \|\theta_n(x)\| = 1$. Hence $\|\psi_n(x)\| = \|T_n''(\delta_x)\| \to 1$ as $n \to \infty$. Finite propagation of $T_n''$ provides the support condition for $\psi_n$ and so $\psi_n$ gives an everywhere unital element of $\mathcal{E}_Q^0(X, \ell^2(X))$. To see that $\psi$ is a cocycle note that $\|D\theta_n(x_0, x_1)\| = \|T_n(\delta_{x_1} - \delta_{x_0})\| = \|T_n'(\delta_{x_1} - \delta_{x_0})\|$ and $\|D\psi_n(x_0, x_1)\| = \|T_n''(\delta_{x_1} - \delta_{x_0})\|$. As $T_n'' - T_n' \to 0$, $D\theta_n \to 0$ implies $D\psi_n \to 0$.

As $T_n''$ is self-adjoint, we have $\psi_n(x)(z) = \langle T_n''\delta_x, \delta_z\rangle = \langle \delta_x, T_n''\delta_z\rangle = \overline{\psi_n(z)(x)}$. To make $\psi_n$ symmetric it therefore suffices to ensure that $\psi_n(x)(z)$ is real. For an operator $T\colon \ell^2(X) \to \ell^2(X)$, let $\overline{T}$ denote the operator defined by $\overline{T}\xi = \overline{T\overline{\xi}}$ where $\overline{\xi}$ denotes the entry-wise complex conjugate of $\xi$. As $\theta_n$ is real, it follows that $\overline{T_n} = T_n$, and so $\overline{T_n^*T_n} = \overline{T_n}^*\overline{T_n} = T_n^*T_n$, hence as $T_n^*T_n = T_n'^2$ we have $\overline{T_n'}^2 = \overline{T_n'^2} = T_n^*T_n$. Since the positive square-root $T_n'$ of $T_n^*T_n$ is unique we have $\overline{T_n'} = T_n'$. Without loss of generality we may assume that $\overline{T_n''} = T_n''$, since replacing $T_n''$ with its real part $\frac{1}{2}(T_n'' + \overline{T_n''})$ reduces the distance from $T_n'$. Hence we have $\psi_n(x)(z) = \langle T_n''\delta_x, \delta_z\rangle$ real, so we have proved (4).

(4) $\implies$ (3) is immediate from Lemma 3.1: given $\psi$, we take $\phi = \beta_*\psi$. Symmetry is preserved and as $\psi$ is everywhere unital, the same holds for $\phi$. So, as $\phi$ is non-negative, we have $\pi_*\phi = \mathbf{1}_Q$.

(3) $\implies$ (2) is trivial. $\square$

## 3.2 Expanders

Informally, an expander is a highly connected graph which is also sparse. Although expanders are interesting as purely graph-theoretic objects, their seemingly contradictory properties are also useful in coding theory, network design and computational group theory. They are also of theoretical interest as they provide counterexamples to the coarse Baum–Connes conjecture [HLS]. For more about expanders, see [Lub].

Let $\{\Gamma_i\}$ be a sequence of finite graphs with uniformly bounded valencies. Abusing notation, we will also denote the vertex set by $\Gamma_i$ and the edges by $E_i$. We take the edges to be directed, with an edge connecting $x$ to $y$ if and only if there is an edge connecting $y$ to $x$. The *Cheeger constant* of the graph $\Gamma_i$ is defined by $h(\Gamma_i) = \frac{1}{2} \inf \frac{|\partial F|}{|F|}$, where $F$ ranges over the non-empty subsets of $\Gamma_i$ such that $|F| \leq \frac{1}{2}|\Gamma_i|$ and $\partial F$ denotes the coboundary of $F$, i.e. the set of edges of $\Gamma_i$ with exactly one end point in $F$. Note that $\partial F$ is usually referred to as the boundary of $F$, however as the map goes from vertices to edges, homologically it is a coboundary. The factor of $\frac{1}{2}$ compensates for the doubling arising from the use of directed edges.

**Definition 3.3.** A finite graph $\Gamma$ is a $(k, \varepsilon)$-*expander* if each vertex of $\Gamma$ has valency at most $k$, and $h(\Gamma) \geq \varepsilon$.

A sequence of finite graphs $\{\Gamma_i\}$ is called an *expander sequence* if $|\Gamma_i| \to \infty$ and there exists $k, \varepsilon$ such that each $\Gamma_i$ is a $(k, \varepsilon)$-expander.

It is not obvious that such graphs exist. Their existence was first proved by Pinsker, in a non-constructive way. More precisely, consider the probability space of all $k$-regular graphs on $n$ marked vertices such that there is an equal probability of choosing any one of the graphs. Then for any $k \geq 3$, there exists an $\varepsilon_k > 0$ such that the probability that the Cheeger constant of a randomly chosen graph is greater than $\varepsilon_k$ tends to 1 as $n$ tends to infinity.

Margulis was the first to give explicit examples of expanders, using quotients of discrete property T groups [Mar]. We give a proof of this result in the next chapter (Proposition 4.6).

It is well-known that expander graphs do not uniformly embed into Hilbert space (see for example [Roe]). It follows that expanders cannot have property A. By using non-embeddability into Hilbert space, the existing proof of this fact obscures the relationship between these two notions. We give an outline of the existing proof in Proposition 3.5, following Lemma 3.4.

We will show that the expander condition can be rephrased in a cohomological way. In the next section, we will use this together with the cohomological characterisation of property A of [BNW] to give a more explicit, direct argument for why expanders cannot have property A, which does not use non-embeddability into Hilbert space.

Let $\Gamma$ be a finite graph and let $E$ denote its set of directed edges. We view $\mathbb{C}$ as the subspace of $\ell^1(\Gamma)$ consisting of constant functions, and write $\overline{f}$ for the class in $\ell^1(\Gamma)/\mathbb{C}$ represented by $f \in \ell^1(\Gamma)$. The norm on $\ell^1(\Gamma)/\mathbb{C}$ is the quotient norm defined by $\|\overline{f}\|_{\ell^1/\mathbb{C}} = \inf_{c\in\mathbb{C}} \|f + c\|_{\ell^1}$. We will write $\ell^1_0(E)$ for the subspace of $\ell^1(E)$ consisting of functions whose sum is zero. The norm on $\ell^1_0(E)$ is the usual $\ell_1$ norm. Define a coboundary map

$$d\colon \ell^1(\Gamma)/\mathbb{C} \longrightarrow \ell^1_0(E)$$

by $d\overline{f}(e) = f(e^+) - f(e^-)$ where $e^-$ is the starting vertex and $e^+$ is the end vertex of the directed edge $e$.

**Lemma 3.4.** *The Cheeger constant $h(\Gamma)$ is at least $\frac{\varepsilon}{2}$ if and only if $\|d\overline{f}\|_{\ell^1} \geq \varepsilon\|\overline{f}\|_{\ell^1/\mathbb{C}}$ for every $\overline{f} \in \ell^1(\Gamma)/\mathbb{C}$.*

*Proof.* Suppose $\|d\overline{f}\|_{\ell^1} \geq \varepsilon\|\overline{f}\|_{\ell^1/\mathbb{C}}$ for every $\overline{f} \in \ell^1(\Gamma)/\mathbb{C}$. Then in particular, for any subset $F \subset \Gamma$ such that $|F| \leq \frac{1}{2}|\Gamma|$ we have $\|d\overline{\chi_F}\|_{\ell^1} \geq \varepsilon\|\overline{\chi_F}\|_{\ell^1/\mathbb{C}}$, where $\chi_F$ denotes the characteristic function of $F$. It is clear that $\|d\overline{\chi_F}\|_1$ is equal to $|\partial F|$, the coboundary of the set $F$ (recall that we are taking our edges to be directed). Also, since $|F| \leq \frac{1}{2}|\Gamma|$, we have

$$\sum_{\gamma\in\Gamma} |\chi_F(\gamma) + c| = \sum_{\gamma\in F} |1 + c| + \sum_{\gamma\notin F} |c| \geq \sum_{\gamma\in F} 1 - \sum_{\gamma\in F} |c| + \sum_{\gamma\notin F} |c| \geq \sum_{\gamma\in F} 1.$$

From this, we can see that the infimum over $c \in \mathbb{C}$ of $\sum_{\gamma\in\Gamma} |\chi_F(\gamma) + c|$ is achieved when $c = 0$ and so we have $\|\overline{\chi_F}\|_{\ell^1/\mathbb{C}} = |F|$. Hence for every $F$ with $|F| \leq \frac{1}{2}|\Gamma|$, we have $|\partial F| \geq \varepsilon|F|$ and so $h(\Gamma) \geq \frac{\varepsilon}{2}$.

Suppose now that $h(\Gamma)$ is at least $\frac{\varepsilon}{2}$. Given $\overline{f} \in \ell^1(\Gamma)/\mathbb{C}$, pick an $f' \in \ell^1(\Gamma)$ which takes positive values on each element of $\Gamma$ and such that $\overline{f'} = \overline{f}$. We can write $f'$ as $\sum a_j\chi_{F_j}$ for some nested collection of subsets $F_1 \subset F_2 \subset ... \subset F_n$ of $\Gamma$ and coefficients $a_j > 0$. Now $\|d\overline{f}\|_{\ell^1} = \|d\overline{f'}\|_{\ell^1}$ is equal to $\sum a_j\|d\overline{\chi_{F_j}}\|_{\ell^1}$ since the $F_j$ are nested. Hence

$$\|d\overline{f}\|_{\ell^1} = \sum_j a_j\|d\overline{\chi_{F_j}}\|_{\ell^1} = \sum_j a_j|\partial F_j|.$$

Let $F_j^c$ denote the complement of $F_j$ in $\Gamma$. Since $h(\Gamma) \geq \frac{\varepsilon}{2}$, when $|F_j| \leq \frac{1}{2}|\Gamma|$ we have $|\partial F_j| \geq \varepsilon|F_j| = \|\overline{\chi_{F_j}}\|_{\ell^1/\mathbb{C}}$, while for $|F_j| > \frac{1}{2}|\Gamma|$ we have

$$|\partial F_j| = |\partial F_j^c| \geq \varepsilon|F_j^c| = \varepsilon\|\overline{\chi_{F_j^c}}\|_{\ell^1/\mathbb{C}} = \varepsilon\|\overline{1 - \chi_{F_j}}\|_{\ell^1/\mathbb{C}} = \varepsilon\|\overline{\chi_{F_j}}\|_{\ell^1/\mathbb{C}},$$

and so

$$\|d\overline{f}\|_{\ell^1} \geq \varepsilon\sum_j a_j\|\overline{\chi_{F_j}}\|_{\ell^1/\mathbb{C}} \geq \varepsilon\|\overline{\sum_j a_j\chi_{F_j}}\|_{\ell^1/\mathbb{C}} = \varepsilon\|\overline{f}\|_{\ell^1/\mathbb{C}}.$$

This completes the proof. □

The map $\ell^1(\Gamma) \to \ell^1_0(\Gamma)$ taking a function $f \in \ell^1(\Gamma)$ to $g = f - \frac{1}{|\Gamma|}\sum_{\beta\in\Gamma} f(\beta)$ has kernel $\mathbb{C}$, and hence induces an isomorphism from $\ell^1(\Gamma)/\mathbb{C}$ to $\ell^1_0(\Gamma)$. This map has norm at most 2 since

$$\|g\|_{\ell^1} = \sum_{\gamma\in\Gamma} |f(\gamma) - \frac{1}{|\Gamma|}\sum_{\beta\in\Gamma} f(\beta)| \leq \sum_{\gamma\in\Gamma} |f(\gamma)| + \sum_{\beta\in\Gamma} |f(\beta)| = 2\,\|f\|_{\ell^1}$$

while the inverse is given by the inclusion of $\ell^1_0(\Gamma)$ in $\ell^1(\Gamma)$ which has norm 1. Hence identifying $\ell^1(\Gamma)/\mathbb{C}$ with $\ell^1_0(\Gamma)$, the norms differ by a factor of at most 2.

**Proposition 3.5.** *An expander sequence $\{\Gamma_i\}$ cannot coarsely embed into Hilbert space.*

*Proof.* From Theorem 2.10, we know that coarse embedding into Hilbert space is equivalent to coarsely emedding in $\ell^1$. Thus, suppose that $F : \{\Gamma_i\} \longrightarrow \ell^1$ is a coarse emedding, meaning that each $\Gamma_i$ embeds into $\ell^1$ with the same coarse embedding functions $\rho_-$, $\rho_+$. Let $F_i$ denote the restriction of $F$ to $\Gamma_i$.

We know from Lemma 3.4 and the above remark that the function $f_i : \Gamma_i \longrightarrow \mathbb{C}$ defined by $f_i(\gamma) = \|F_i(\gamma)\|_{\ell^1} - \frac{1}{|\Gamma_i|}\sum_{\beta\in\Gamma_i} \|F_i(\beta)\|_{\ell^1}$, which lies in $\ell^1_0(\Gamma_i)$, satisfies

$$\|df_i\|_{\ell^1} \geq \varepsilon\|f_i\|_{\ell^1}$$

for some $\varepsilon > 0$, since $\{\Gamma_i\}$ is an expander sequence. We have

$$\varepsilon \sum_{\gamma\in\Gamma_i} \|F_i(\gamma)\|_{\ell^1} = \varepsilon\|f_i\|_{\ell^1} \leq \|df_i\|_{\ell^1} \leq \sum_{(\gamma,\gamma')\in E(\Gamma_i)} \|F_i(\gamma) - F_i(\gamma')\|_{\ell^1}.$$

Since $F$ is a coarse embedding, there is some $C > 0$ such that $\|F_i(\gamma) - F_i(\gamma')\|_{\ell^1} < C$ for all $(\gamma, \gamma') \in E(\Gamma_i)$. Thus, we have

$$\varepsilon \sum_{\gamma\in\Gamma_i} \|F_i(\gamma)\|_{\ell^1} \leq |\Gamma_i|C.$$

This implies that no more than $\frac{|\Gamma_i|}{2}$ elements $\gamma \in \Gamma_i$ can have $\|F_i(\gamma)\|_{\ell^1} > \frac{2C}{\varepsilon}$, for otherwise the inequality is violated. So, each $F_i$ maps at least $\frac{|\Gamma_i|}{2}$ elements of $\Gamma_i$ into the ball of radius $\frac{2C}{\varepsilon}$ in $\ell^1$, which contradicts the fact that $F$ was assumed to be a coarse embedding. $\square$

We now move on to the definition of the cohomology which detects expander sequences. Let $\{\Gamma_i\}_{i\in\mathbb{N}}$ be a sequence of graphs. We denote by $\prod_{i\in\mathbb{N}}^{\infty} \ell^1(\Gamma_i)$ the space of bounded elements of the direct product. That is, $\prod_{i\in\mathbb{N}}^{\infty} \ell^1(\Gamma_i)$ is the space of functions from $\coprod_i \Gamma_i$ to $\mathbb{C}$, such that the sup-$\ell^1$-norm

$$\|f\| = \sup_{i\in\mathbb{N}} \|f|_{\Gamma_i}\|_{\ell^1}$$

is finite. We define a summation map $\sigma_0\colon \prod_{i\in\mathbb{N}}^{\infty} \ell^1(\Gamma_i) \to \ell^\infty(\mathbb{N})$ by $\sigma_0(f)(i) = \sum_{x\in\Gamma_i} f(x)$. Similarly $\prod_{i\in\mathbb{N}}^{\infty} \ell^1(E_i)$ is the space of functions on $\coprod_i E_i$ with finite sup-$\ell^1$-norm, and we define $\sigma_1\colon \prod_{i\in\mathbb{N}}^{\infty} \ell^1(E_i) \to \ell^\infty(\mathbb{N})$ by $\sigma_1(f)(i) = \sum_{x\in E_i} f(x)$.

We define

$$C^0(\{\Gamma_i\}) = \ker(\sigma_0), \quad C^1(\{\Gamma_i\}) = \ker(\sigma_1).$$

Note that $C^0(\{\Gamma_i\})$ consists of functions whose restriction to each $\Gamma_i$ lies in $\ell^1_0(\Gamma_i)$, and $C^1(\{\Gamma_i\})$ consists of functions whose restriction to each $E_i$ is in $\ell^1_0(E_i)$. Hence combining the coboundary maps on each component yields a coboundary map $d\colon C^0(\{\Gamma_i\}) \to C^1(\{\Gamma_i\})$, and it is easy to see that this is bounded. In the spirit of [BNW], our cohomological description of the expander condition is given by completing this cochain complex.

**Definition 3.6** ([BNW, Def. 3.1])**.** The *quotient completion* of a pre-Fréchet space $V$ (a space equipped with a countable family of seminorms $\|\cdot\|_j$) is the space $V_Q = \ell^\infty(\mathbb{N}, V)/c_0(\mathbb{N}, V)$ of bounded sequences in $V$ modulo sequences vanishing at infinity.

For simplicity we suppose that the seminorms are monotonic, that is $\|\cdot\|_i \leq \|\cdot\|_j$ for $i < j$. We note the following useful property of this completion.

**Lemma 3.7.** *Let $T\colon V \to W$ be a bounded map from a normed spaced $V$ to a pre-Fréchet space $W$. Then $T$ is bounded below if and only if the induced map $T^Q\colon V_Q \to W_Q$ [BNW, Prop. 3.3] is injective.*

*Proof.* One direction is obvious: if $T$ is bounded below then $T^Q$ is also bounded below hence injective. For the converse suppose that $T$ is not bounded below. This means that for each seminorm $\|\cdot\|_{j,W}$ for $W$ and all $\varepsilon > 0$ there exists $v$ in $V$ with $\|Tv\|_{j,W} < \varepsilon \|v\|_V$. Hence we can find a sequence $v_n \in V$ with $\|v_n\|_V = 1$ and $\|Tv_n\|_{n,W} < \frac{1}{n}$. As the sequence $v_n$ is bounded, it determines an element $v$ of $V_Q$. Its image under $T^Q$ is given by the sequence $Tv_n$, and since for $n \geq j$ we have $\|Tv_n\|_{j,W} \leq \|Tv_n\|_{n,W} < \frac{1}{n}$, we have $Tv_n \in c_0(\mathbb{N}, W)$. Hence $T^Q v = 0$, so $T^Q$ is not injective. □

We remark that the lemma is not true in general if $V$ is a pre-Fréchet space. Whilst for $T$ not bounded below there still exists a sequence $v_n$ not tending to zero such that $Tv_n \to 0$, there may be no *bounded* sequence with these properties.

We now give our cohomological description of the expander condition. Let $C^p_Q(\{\Gamma_i\})$ denote the quotient completion of $C^p(\{\Gamma_i\})$ for $p = 0, 1$. The extension of the coboundary map $d$ to the completion we again denote by $d$.

**Definition 3.8.** The *Cheeger cohomology* of a sequence of graphs $\{\Gamma_i\}$, denoted $H^*_h(\{\Gamma_i\})$ is the cohomology of the cochain complex $(C^p_Q(\{\Gamma_i\}), d)$.

We remark that $C^p_Q(\{\Gamma_i\})$ is the kernel of the induced map $\sigma^Q_p$, since the quotient completion preserves exactness (cf. [BNW]).

**Theorem 3.9.** *Let $\{\Gamma_i\}_{i\in\mathbb{N}}$ be a sequence of finite graphs with bounded valency. Then $\{\Gamma_i\}$ is an expander sequence if and only if $H^0_h(\{\Gamma_i\})$ vanishes.*

*Proof.* Using Lemma 3.4 and the identification of $\ell^1(\Gamma_i)/\mathbb{C}$ with $\ell^1_0(\Gamma_i)$, the graphs $\{\Gamma_i\}$ form an expander sequence if and only if there exists $\varepsilon > 0$ such that for each graph $\Gamma_i$ the coboundary map $d\colon \ell^1_0(\Gamma_i) \to \ell^1_0(E_i)$ is $\varepsilon$-bounded below. The individual coboundary maps are bounded below by a common $\varepsilon$ if and only if the map $d\colon C^0(\{\Gamma_i\}) \to C^1(\{\Gamma_i\})$ is bounded below. By Lemma 3.7 this is equivalent to injectivity of the coboundary map $d\colon C^0_Q(\{\Gamma_i\}) \to C^1_Q(\{\Gamma_i\})$ on the completed complex. Hence the graphs $\{\Gamma_i\}$ form an expander sequence if and only if $H^0_h(\{\Gamma_i\}) = 0$. □

## 3.3 Expanders do not have property A

Let $\mathbf{\Gamma}$ be a disjoint union of graphs $\{\Gamma_i\}_{i\in\mathbb{N}}$ equipped with a proper metric such that the restriction to each component $\Gamma_i$ is the graph metric on $\Gamma_i$, and such that the distance between $\Gamma_i$ and its complement $\Gamma_i^c$ tends to infinity as $i \to \infty$. If $\mathbf{\Gamma}$ has property A then there is a cocycle $\phi \in \mathcal{E}^0_Q(\mathbf{\Gamma}, \ell^1(\mathbf{\Gamma}))$ with $\pi_*(\phi) = \mathbf{1}_Q$, while if $\{\Gamma_i\}$ is an expander sequence then $H^0_h(\{\Gamma_i\})$ is zero. We will show that these two cohomological conditions are contradictory. This implies that expanders cannot have property A.

**Theorem 3.10.** *Let $\mathbf{\Gamma}$ be a disjoint union of graphs $\Gamma_i$ with bounded valency, such that $d(\Gamma_i, \Gamma_i^c) \to \infty$ and $|\Gamma_i| \to \infty$ as $i \to \infty$. If there exists a cocycle $\phi \in \mathcal{E}^0_Q(\mathbf{\Gamma}, \ell^1(\mathbf{\Gamma}))$ such that $\pi_*(\phi) = \mathbf{1}_Q$ then $H^0_h(\{\Gamma_i\})$ is non-zero.*

*Proof.* Suppose there exists a cocycle $\phi \in \mathcal{E}^0_Q(\mathbf{\Gamma}, \ell^1(\mathbf{\Gamma}))$ such that $\pi_*(\phi) = \mathbf{1}_Q$. We will use this to construct a non-zero cocycle in $C^0_Q(\{\Gamma_i\})$ thus proving that $H^0_h(\{\Gamma_i\})$ is non-zero. By Theorem 3.2 we may assume that $\phi$ is a symmetric cocycle.

For each $n \in \mathbb{N}$ the controlled support condition provides an $S_n > 0$ such that for each $x \in \mathbf{\Gamma}$, the support of $\phi_n(x)$ lies in $B_{S_n}(x)$. As the distance between components tends to $\infty$, if $i$ is sufficiently large then the distance between $\Gamma_i$ and the other components of $\mathbf{\Gamma}$ exceeds $S_n$. Hence there exists $j_n$ such that if $i \geq j_n$ then $\phi_n(x)$ is supported in $\Gamma_i$ for all $x \in \Gamma_i$.

For each $i, n$, we choose a vertex $e^i_n \in \Gamma_i$ so that the infimum of $\sum\limits_{(x_0,x_1)\in E_i} |D\phi_n(x_0, x_1)(z)|$ over all $z \in \Gamma_i$ is realised at $z = e^i_n$, where $E_i$ denotes the set of edges of $\Gamma_i$. Note that the infimum is actually a minimum, since each $\Gamma_i$ is finite, and so such an $e^i_n$ exists. For $i \geq j_n$ we define $f^i_n \in \ell^1\Gamma_i$ by $f^i_n(x) = \phi_n(x)(e^i_n) - \frac{1}{|\Gamma_i|}$, and for $i < j_n$ we define $f^i_n$ to be 0. By symmetry of $\phi_n$, when $i \geq j_n$ we have

$$\sum_{x\in\Gamma_i} |f^i_n(x)| = \sum_{x\in\Gamma_i} |\phi_n(e^i_n)(x) - \frac{1}{|\Gamma_i|}| \leq \left\|\phi_n(e^i_n)\right\|_{\ell^1} + 1.$$

This is bounded in $i, n$, hence $f_n = (f^1_n, f^2_n, \dots)$ defines an element $f$ in the quotient completion of $\prod^{\infty}_{i\in\mathbb{N}} \ell^1(\Gamma_i)$. We will show that this is a non-zero cocycle in $C^0_Q(\{\Gamma_i\})$.

For $i < j_n$ we have $\sigma_0(f_n)(i) = \sum_{x\in\Gamma_i} f^i_n(x) = 0$, while for $i \geq j_n$ we have

$$\sum_{x\in\Gamma_i} f^i_n(x) = \sum_{x\in\Gamma_i}\Big(\phi_n(x)(e^i_n) - \frac{1}{|\Gamma_i|}\Big) = \sum_{x\in\Gamma_i}\Big(\phi_n(e^i_n)(x) - \frac{1}{|\Gamma_i|}\Big) = \pi_*(\phi_n)(e^i_n) - 1.$$

by symmetry of $\phi_n$. Since $\pi_*(\phi) = \mathbf{1}_Q$, the sequence $\pi_*(\phi_n)(e^i_n) - 1$ tends to zero (uniformly in $i$) as $n \to \infty$. Thus $\sigma^Q_0(f) = 0$, so $f$ is an element of $C^0_Q(\{\Gamma_i\})$.

Recalling that the valencies of the $\Gamma_i$ are uniformly bounded, we have a bound $N_n$ on the cardinality of the balls $B_{S_n}(e^i_n)$. As $\phi_n(e^1_n)(x) = 0$ outside

$B_{S_n}(e_n^i)$, when $i \geq j_n$ we have the following lower bound for the $\ell^1$-norm of $f_n^i$:

$$\|f_n^i\|_{\ell^1} \geq \sum_{x \in \Gamma_i \setminus B_{S_n}(e_n^i)} \frac{1}{|\Gamma_i|} \geq \frac{|\Gamma_i| - N_n}{|\Gamma_i|} = 1 - \frac{N_n}{|\Gamma_i|}.$$

Hence $\|f_n\| = \sup_{i \in \mathbb{N}} \|f_n^i\|_{\ell^1} \geq 1$ for all $n$. In particular $\|f_n\|$ does not tend to zero, so $f$ is a *non-zero* element of $C_Q^0(\{\Gamma_i\})$.

It remains to verify that $f$ is a cocycle. We apply the coboundary operator $d$ to $f_n^i$. This clearly vanishes when $i < j_n$, while for $i \geq j_n$ we have

$$df_n^i(x_0, x_1) = f_n^i(x_1) - f_n^i(x_0) = D\phi(x_0, x_1)(e_n^i).$$

Our choice of $e_n^i$ now comes into play. Let $k$ be an upper bound on the valency of the graphs, so that $|E_i|/|\Gamma_i| \leq k$ for all $i$. Then we have

$$\begin{aligned}
\|df_n^i\|_{\ell^1} &\leq \sum_{(x_0,x_1) \in E_i} |D\phi(x_0, x_1)(e_n^i)| \\
&= \frac{1}{|\Gamma_i|} \sum_{z \in \Gamma_i} \sum_{(x_0,x_1) \in E_i} |D\phi(x_0, x_1)(e_n^i)| \\
&\leq \frac{1}{|\Gamma_i|} \sum_{z \in \Gamma_i} \sum_{(x_0,x_1) \in E_i} |D\phi(x_0, x_1)(z)| \\
&\leq k\|D\phi_n\|_{R=1}
\end{aligned}$$

as $\sum_{z \in \Gamma_i} |D\phi(x_0, x_1)(z)| \leq \|D\phi_n\|_{R=1}$. This tends to zero as $n \to \infty$ since $\phi$ is a cocycle. Hence $df = 0$, so $f$ is a non-zero cocycle and $H_h^0(\{\Gamma_i\})$ is non-zero. □

Since property A is equivalent to existence of a cocycle $\phi \in \mathcal{E}_Q^0(X, \ell^1(X))$ such that $\pi_*(\phi) = \mathbf{1}_Q$, and a sequence of graphs is an expander if and only if $H_h^0(\{\Gamma_i\})$ vanishes we obtain the following immediate corollary to Theorem 3.10.

**Corollary 3.11.** *Let $\mathbf{\Gamma}$ be the disjoint union of an expander sequence, with metric as above. Then $\mathbf{\Gamma}$ does not have property A.*

# Chapter 4

# Box spaces

**Definition 4.1.** A group $G$ is *residually finite* if for every non-trivial element $g \in G$, there exists a finite quotient $F$ of $G$ such that the image of $g$ in $F$ remains non-trivial.

Having such a finite quotient for every non-trivial element allows us to study the group via a sequence of increasing finite groups. Instead of the Cayley graph of the whole group, we can build the Cayley graphs of the sequence of finite groups which approximate the Cayley graph of the whole group in a certain sense. A *box space* is a disjoint union of such finite Cayley graphs, metrised in a certain way which we will make precise in the first section.

In this chapter, we will give the motivation behind studying box spaces, namely, connections to group-theoretic properties. A section of the chapter will also be devoted to investigating basic properties of box spaces.

We will investigate how coarse embeddability of box spaces into Hilbert space behaves under group extensions. In particular, we prove a result which implies that a semidirect product of a finitely generated free group by a finitely generated residually finite amenable group has a box space which coarsely embeds into Hilbert space. This provides a new class of examples of metric spaces with bounded geometry which coarsely embed into Hilbert space but do not have property A, generalising the example of Arzhantseva, Guentner and Spakula, which we will also discuss in this chapter.

We will prove a technical result about extensions, which will be used in the following theorem.

**Theorem 4.2.** *Let $\Gamma = H \rtimes G$ be a finitely generated semidirect product of two residually finite groups $H$ and $G$. If $G$ is amenable and $H$ has a nested sequence of finite index characteristic subgroups with trivial intersection such that the corresponding box space embeds coarsely into Hilbert space, then $\Gamma$ has a box space which coarsely embeds into Hilbert space.*

In particular, the above result applies to *(finitely generated free)-by-cyclic* groups, a large class which includes the fundamental groups of certain 3-manifolds.

## 4.1 Preliminaries

We will now describe a geometric object which can be built from a residually finite group. There is a beautiful parallel between the coarse geometric properties of this object and the analytic properties of the group.

Let $G$ be a finitely generated residually finite group. Let $\{K_i\}$ be some collection of finite index subgroups of $G$, for which the intersection $\cap_{n\in\mathbb{N}} K_n$

is trivial. We will be particularly interested in the case when the $K_i$ are nested normal subgroups of $G$:

$$G = K_1 \triangleright K_2 \triangleright K_3 \triangleright K_4 \triangleright ...$$

Note that given a finitely generated residually finite group, we can always choose a collection $\{K_i\}$ of finite index subgroups with trivial intersection such that the $K_i$ are nested normal subgroups of $G$.

**Definition 4.3.** The *box space* of $G$ corresponding to $\{K_i\}$, denoted by $\square_{\{K_i\}}G$, is the disjoint union $\sqcup_i G/K_i$ of finite quotient groups of $G$, where each quotient is endowed with the metric induced by the image of the generating set of $G$, and the distances between the identity elements of two successive quotients are chosen to be greater than the maximum of their diameters.

We should think of the box space as a sequence of finite Cayley graphs which are "strung" onto a thread through the identity elements, so that the distance between two elements from two different Cayley graphs, $\alpha \in G/K_i$ and $\beta \in G/K_j$, is given by

$$d(\alpha,\beta) = d_i(\alpha,e_i)+d(e_i,e_j)+d_j(e_j,\beta) = d_i(\alpha,e_i)+\sum_{k=i}^{j-1} d(e_k,e_{k+1})+d_j(e_j,\beta),$$

where $e_i$ denotes the identity element of $K_i$, and $d_i$ denotes the metric on the quotient $G/K_i$. In particular, the distance $d(e_i, e_j)$ is given by $\sum_{k=i}^{j-1} d(e_k, e_{k+1})$.

Note that however we choose the distances between the quotients, we will obtain coarsely equivalent spaces as long as we choose the distances between two successive quotients to be greater than the maximum of their diameters. In fact, it is clear that the identity map will be a coarse equivalence between two box spaces with the same quotients, but different spacing between the quotients, as long as the distance between successive quotients tends to infinity.

The properties of $\Box_{\{K_i\}}G$ can vary greatly depending on the choice of $\{K_i\}$. For example, Arzhantseva, Guentner and Spakula's chosen sequence of subgroups gives a box space of $\mathbb{F}_2$ which coarsely embeds in Hilbert space. However, the full box space of $\mathbb{F}_2$ corresponding to the collection of *all* finite index normal subgroups does not, since the box space of $SL(3,\mathbb{Z})$, being a quotient of $\mathbb{F}_2$, coarsely embeds in the full box space $\Box_{\text{all}}\mathbb{F}_2$ of $\mathbb{F}_2$. Thus $\Box_{\text{all}}\mathbb{F}_2$ contains a coarsely embedded expander, for reasons explained below.

## 4.2 Links with group properties

An exposition of the following remarkable results linking the geometric properties of $\Box G$ to analytic properties of $G$ can be found in [Roe].

**Proposition 4.4.** *Let $G$ be a finitely generated residually finite group, and let $\{K_i\}$ be a nested collection of finite index normal subgroups with trivial intersection. If $\Box_{\{K_i\}}G$ is coarsely embeddable into Hilbert space, then $G$ has the Haagerup property.*

We give a proof of the following result, due to Guentner.

**Proposition 4.5** (Guentner)**.** *Let $G$ be a finitely generated residually finite group, and let $\{K_i\}$ be a nested collection of finite index normal subgroups with trivial intersection. Then $\Box_{\{K_i\}}G$ has property A if and only if $G$ is amenable.*

*Proof.* Suppose first that $G$ is amenable. Let us fix $R, \varepsilon > 0$, and aim to define a function as in Definition 2.11, namely, $f : \Box_{\{K_i\}}G \longrightarrow \ell^1(\Box_{\{K_i\}}G)$ such that $\|f(x)\|_1 = 1$ for all $x \in \Box_{\{K_i\}}G$, there exists an $S$ so that $\mathrm{supp}(f(x)) \subseteq B(x, S)$ for all $x \in \Box_{\{K_i\}}G$, and $\|f(x) - f(y)\|_1 \leq \varepsilon$ whenever $d(x, y) \leq R$.

Since $G$ is amenable, there exists $\phi \in \ell^1(G)$, finitely supported on a ball of radius $M$ about the identity, such that $\|\phi\|_1 = 1$ and such that for all $g \in B(e, R)$, $\|g\phi - \phi\|_1 \leq \varepsilon$, where $g\phi(h) = \phi(g^{-1}h)$.

There is some index $N$ such that for all $i \geq N$, the quotient map $\pi_i : G \longrightarrow G/K_i$ is an isometry on balls of radius $R + M$, and such that the distance between two successive quotients $G/K_i$ and $G/K_{i+1}$ is greater than $R$. Choose a cross-section of $\pi_i$, $\sigma_i : G/K_i \longrightarrow G$, such that $\sigma_i$ is an isometry on balls of radius $R + M$.

Define a function $f : \Box_{\{K_i\}}G \longrightarrow \ell^1(\Box_{\{K_i\}}G)$ in the following way. For $x \in G/K_i$ with $i \geq N$:

$$f(x)(y) = \phi(\sigma_i(x^{-1}y)) \text{ for } y \in G/K_i;$$
$$f(x)(y) = 0 \text{ otherwise.}$$

For $x \in G/K_i$ with $i < N$:

$$f(x)(e_1) = 1;$$
$$f(x)(y) = 0 \text{ otherwise.}$$

Here, $e_1$ is used to denote the identity element of the first quotient $G/K_1$.

We now check that the function satisfies the relevant conditions. First, it is easy to see that $\|f(x)\|_1 = 1$ for $x \in G/K_i$ with $i < N$. For $x \in G/K_i$ with

$i \geq N$ we have:

$$\begin{aligned} \|f(x)\|_1 &= \sum_{y \in G/K_i} |f(x)(y)| \\ &= \sum_{y \in G/K_i} |\phi(\sigma_i(x^{-1}y))| \\ &= \sum_{h \in G} |\phi(h)| = 1. \end{aligned}$$

The support of each $f(x)$ is clearly contained in the $M$-ball about $x$. For the last condition, suppose $d(x, y) \leq R$. Either both $x$ and $y$ lie in $\sqcup_{i=1}^{N-1} G/K_i$, whence $\|f(x) - f(y)\| = 0$, or both $x$ and $y$ lie in the same component $G/K_i$ for $i \geq N$. In this case, we have

$$\begin{aligned} \|f(x) - f(y)\| &= \sum_{z \in G/K_i} |f(x)(z) - f(y)(z)| \\ &= \sum_{z \in G/K_i} |f(x)(xz) - f(y)(xz)| \\ &= \sum_{z \in G/K_i} |\phi(\sigma_i(z)) - \phi(\sigma(y^{-1}xz))| \\ &= \sum_{h \in G} |\phi(h) - \sigma(x^{-1}y)\phi(h)| \\ &= \|\phi - \sigma(x^{-1}y)\phi\|_1 \leq \varepsilon, \end{aligned}$$

where the last inequality follows from the properties of $\phi$ and the fact that $d(x, y) \leq R$ implies that $\sigma(x^{-1}y) \in B(e, R)$.

For the converse, suppose that $\Box_{\{K_i\}} G$ has property A. This implies that the finite quotients of the box space have property A uniformly. More precisely, for each $R, \varepsilon > 0$ there exists an $S$ such that for all $i$, there is a map $f_i : G/K_i \longrightarrow \ell^1(G/K_i)$ such that $\|f_i(x)\|_1 = 1$ for all $x \in G/K_i$, $\mathrm{supp}(f_i(x)) \subseteq B(x, S)$ for all $x \in G/K_i$, and $\|f_i(x) - f_i(y)\|_1 \leq \varepsilon$ whenever $d(x, y) \leq R$.

Define a function $\phi_i \in \ell^1(G/K_i)$ by

$$\phi_i(y) = \frac{1}{|G/K_i|} \sum_{x \in G/K_i} |f_i(x)(xy)|.$$

One can easily see that $\|\phi_i\|_1 = 1$:

$$\begin{aligned}
\|\phi_i\|_1 &= \sum_{y \in G/K_i} \frac{1}{|G/K_i|} \sum_{x \in G/K_i} |f_i(x)(xy)| \\
&= \frac{1}{|G/K_i|} \sum_{x \in G/K_i} \sum_{y \in G/K_i} |f_i(x)(xy)| \\
&= \frac{1}{|G/K_i|} \sum_{x \in G/K_i} \|f_i(x)\|_1 \\
&= 1.
\end{aligned}$$

Also, for $z \in B_{G/K_i}(e, R)$, we have

$$\begin{aligned}
\|\phi_i - z\phi_i\|_1 &= \sum_{y \in G/K_i} |\phi(y) - \phi(z^{-1}y)| \\
&= \frac{1}{|G/K_i|} \sum_{y \in G/K_i} \left| \sum_{x \in G/K_i} \big(|f(x)(xy)| - |f(x)(xz^{-1}y)|\big) \right| \\
&\leq \frac{1}{|G/K_i|} \sum_{y \in G/K_i} \sum_{x \in G/K_i} \big||f(x)(xy)| - |f(x)(xz^{-1}y)|\big| \\
&\leq \frac{1}{|G/K_i|} \sum_{x \in G/K_i} \sum_{y \in G/K_i} \big|f(x)(xy) - f(x)(xz^{-1}y)\big| \\
&= \frac{1}{|G/K_i|} \sum_{x \in G/K_i} \|x^{-1}f_i(x) - zx^{-1}f_i(x)\|_1 \\
&\leq \varepsilon.
\end{aligned}$$

The support of the function $\phi_i$ lies inside the ball of radius $S$ about the identity in $G/K_i$. Now as above, there is some index $N$ such that for all $i \geq N$, the quotient map $\pi_i : G \longrightarrow G/K_i$ is an isometry on balls of radius $R + S$, and such that the distance between two successive quotients $G/K_i$

and $G/K_{i+1}$ is greater than $R$. Define $\phi \in \ell^1(G)$ by

$$\phi(g) = f_i(\pi_i(g)) \text{ for } g \in B_G(e, R+S);$$
$$\phi(g) = 0 \text{ otherwise.}$$

This functions has the required properties, namely, it is finitely supported on a ball of radius $S$ about the identity, is of norm 1, and for all $g \in B(e, R)$ we have $\|g\phi - \phi\|_1 \leq \varepsilon$, inherited from the property of $f_i$.

□

The argument for Proposition 4.4 is similar to the second implication above.

The first explicit construction of expander graphs by Margulis [Mar] can be rephrased in the following way, using box spaces.

**Proposition 4.6.** *Let $G$ be a finitely generated residually finite group with property (T), and equip it with the word length arising from a symmetric generating set $S$ which does not contain the identity. If $\{K_i\}$ is a sequence of nested finite index normal subgroups with trivial intersection, then $\square_{\{K_i\}}G$ is an expander.*

*Proof.* We will prove that each of the quotients $G/K_i$ is an expander, with Cheeger constant independent of $i$.

Consider $\ell^2(G/K_i)$. The group $G$ acts (transitively) on this Hilbert space via $(gf)(x) = f(g^{-1}x)$, so the only $G$-invariant functions in $\ell^2(G/K_i)$ are the constant functions. We have, identifying the constant functions on $G/K_i$ with $\mathbb{C}$,

$$\ell^2(G/K_i) = \ell^2_0(G/K_i) \oplus \mathbb{C}$$

as $G$-modules, where $\ell_0^2(G/K_i)$ is the subspace of zero-sum functions. So, $\ell_0^2(G/K_i)$ does not contain any $G$-invariant functions and thus does not contain almost invariant functions, since $G$ has property (T).

We can now use this to show that the Cheeger constant is bounded below, as in the proof of Lemma 3.4. Since $\ell_0^2(G/K_i)$ does not contain almost invariant functions, there exists an $\varepsilon > 0$ independent of $i$ such that for all $f \in \ell_0^2(G/K_i)$,

$$\|sf - f\| \geq \varepsilon \|f\|$$

for some $s \in S$. We will show that for each subset $F \subset G/K_i$ with $|F| \leq \frac{1}{2}|G/K_i|$, $|\partial F| \geq \frac{\varepsilon^2}{2}|F|$.

Given $F \subset G/K_i$ with $|F| \leq \frac{1}{2}|G/K_i|$, let $|F| = n$. Define a function $f$ as follows:

$$f(x) = -n \text{ for } x \in F^c;$$
$$f(x) = |G/K_i| - n \text{ for } x \in F.$$

Then $f$ is in $\ell_0^2(G/K_i)$:

$$\begin{aligned}\sum_{x \in G/K_i} f(x) &= -n|F^c| + (|G/K_i| - n)|F| \\ &= -n(|G/K_i| - |F|) + (|G/K_i| - n)|F| \\ &= -n(|G/K_i| - n) + (|G/K_i| - n)n \\ &= 0,\end{aligned}$$

and so $\|sf - f\| \geq \varepsilon \|f\|$ for some $s \in S$. Note that

$$\|f\|^2 = n^2(|G/K_i| - n) + (|G/K_i| - n)^2 n = n|G/K_i|(|G/K_i| - n).$$

Set $E_s(F)$ to be $\{(s^{-1}x, x) : x \in G/K_i\} \cap \partial F$, i.e. the set of edges of the form $(s^{-1}x, x)$ which occur in the boundary of the set $F$. We have

$$\|sf - f\|^2 = \sum_{x \in G/K_i} |sf(x) - f(x)|^2 = \sum_{x \in G/K_i} |f(s^{-1}x) - f(x)|^2.$$

For each $x$, the contribution to the sum is zero unless $(s^{-1}x, x) \in E_s(F)$, in which case the contribution is $|f(s^{-1}x) - f(x)|^2 = |-n - (|G/K_i| - n)|^2 = |G/K_i|^2$. Hence, $\|sf - f\|^2 = |G/K_i|^2 |E_s(F)|$.

We then have

$$\begin{aligned}|\partial F| \geq |E_s(F)| &= \frac{\|sf - f\|^2}{|G/K_i|^2} \\ &\geq \varepsilon^2 \frac{\|f\|^2}{|G/K_i|^2} \\ &= \varepsilon^2 \frac{n|G/K_i|(|G/K_i| - n)}{|G/K_i|^2} \\ &= \varepsilon^2 \frac{n(|G/K_i| - n)}{|G/K_i|} \\ &\geq \frac{\varepsilon^2}{2}|F|,\end{aligned}$$

so the proof is complete.

□

For instance, a box space of $SL(3, \mathbb{Z})$ is an expander.

We can see that there is a spectrum of analytic properties of groups which roughly corresponds to geometric properties of box spaces. To summarise, for a residually finite group $G$, and $\{K_i\}$ a sequence of nested finite index normal subgroups of $G$ with trivial intersection, we have

$$\begin{aligned} G \text{ amenable } &\iff \square_{\{K_i\}} G \text{ property A,} \\ G \text{ Haagerup } &\Longleftarrow \square_{\{K_i\}} G \text{ coarsely embeddable into Hilbert space,} \\ G \text{ property (T) } &\Longrightarrow G \text{ property } (\tau) \text{ w.r.t. } \{K_i\} \iff \square_{\{K_i\}} G \text{ expander.} \end{aligned}$$

The one-way implications above are known not to be reversible. For example, consider the group $SL(2,\mathbb{Z})$ with the sequence of *congruence subgroups*

$$N_m := \ker(SL(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{Z}/m\mathbb{Z})).$$

The box space of $SL(2,\mathbb{Z})$ with respect to this sequence of subgroups is an expander [Lub], although $SL(2,\mathbb{Z})$ has the Haagerup property (and therefore does not have property (T)).

As we will see later in the chapter, there does exist a box space of $SL(2,\mathbb{Z})$ which embeds coarsely into Hilbert space, since $SL(2,\mathbb{Z})$ contains a finitely generated free group as a finite index subgroup. One may be tempted to ask whether every residually finite Haagerup group has *some* choice of subgroups with respect to which the box space embeds into Hilbert space. However, this is false: there are examples of groups with the Haagerup property for which *every* choice of box space is an expander sequence. The group $SL(2,\mathbb{Z}[\frac{1}{p}])$ is such an example. This group has the *congruence subgroup property*, and property $(\tau)$.

## 4.3 The construction of Arzhantseva, Guentner and Spakula

The main result of this chapter is inspired by the construction of a box space of the free group on two generators which coarsely embeds into Hilbert space [AGS]. We will make use of this result repeatedly.

In [AGS], the authors define a sequence $\{N_i\}$ of normal subgroups of the free group $\mathbb{F}_2$ inductively: let $N_0 := \mathbb{F}_2$ and let $N_{i+1} := N_i^2$, where $N_i^2$ denotes the subgroup of $N_i$ generated by all the squares of the elements of $N_i$. Note that each $N_{i+1}$ is a *verbal* subgroup of the previous subgroup $N_i$, and is thus fully invariant and characteristic (in both $N_i$, and the whole of $\mathbb{F}_2$, since being fully invariant and characteristic are transitive properties). In fact, we note that each $N_i$ is a verbal subgroup of $\mathbb{F}_2$, since fully invariant and verbal are equivalent for free groups (see for example 2.3.1 of [Rob]). By a theorem of Levi (see Proposition 3.3 in Chapter 1 of [LS]), the intersection $\cap N_i$ of all the $N_i$ is trivial.

We now describe the construction of a $\mathbb{Z}/2\mathbb{Z}$-homology cover of any finite graph $X$.

Given a finite graph $X$, let $\rho$ be the surjective homomorphism

$$\rho : \pi_1(X) \longrightarrow \pi_1(X)/\pi_1(X)^2$$

where, as above, $\pi_1(X)^2$ denotes the subgroup of $\pi_1(X)$ generated by all the squares of the elements in $\pi_1(X)$. Note that $\pi_1(X)/\pi_1(X)^2$ is a direct sum of $\mathbb{Z}/2\mathbb{Z}$'s.

Denote the vertex set of $X$ by $V$ and the edge set by $E$. Choose a maximal tree $T \subset X$. The set of edges $\{e_1, e_2, ..., e_r\}$ which are not in the maximal tree $T$ correspond to the generators of $\pi_1(X)$, and so we can consider their image under the quotient map $\rho$. The $\mathbb{Z}/2\mathbb{Z}$-homology cover of $X$ is the finite graph $\widetilde{X}$ with vertex set given by

$$\widetilde{V} = V \times (\bigoplus_1^r \mathbb{Z}/2\mathbb{Z})$$

and edge set given by

$$\widetilde{E} = E \times (\bigoplus_1^r \mathbb{Z}/2\mathbb{Z}).$$

We now just need to specify the vertices which are connected by each edge in $\widetilde{E}$.

Given an edge $(e, k) \in \widetilde{E}$ (where $e \in E$ and $k \in \bigoplus_1^r \mathbb{Z}/2\mathbb{Z}$), let $v$ and $w$ be the vertices of $X$ connected by $e$. There are two cases: $e \in T$ and $e \notin T$. If $e \in T$, let $(e, k)$ connect the vertices $(v, k)$ and $(w, k)$. If $e \notin T$, let $(e, k)$ connect $(v, k)$ and $(w, \rho(e)k)$. The graph $\widetilde{X}$ defined in this way is the $\mathbb{Z}/2\mathbb{Z}$-*homology cover* of $X$.

In the situation above, the Cayley graph of $\mathbb{F}_2/N_{i+1}$ is the $\mathbb{Z}/2\mathbb{Z}$-homology cover of $\mathbb{F}_2/N_i$, corresponding to the quotient of the fundamental group of the Cayley graph of $\mathbb{F}_2/N_i$ (which is isomorphic to $N_i$) by the fundamental group of the Cayley graph of $\mathbb{F}_2/N_{i+1}$ (which is isomorphic to $N_{i+1}$).

In [AGS], Arzhantseva, Guentner and Spakula consider the general setting of a pair of graphs $(\widetilde{X}, X)$ where $X$ is a finite graph such that removing a single edge does not disconnect the graph (2-*connected*), and $\widetilde{X}$ is the $\mathbb{Z}/2\mathbb{Z}$-homology cover of $X$. They use $X$ to induce a *wall structure* on the cover $\widetilde{X}$.

Given any finite graph $X$, a *wall* $w$ in $X$ is a subset of the edges of $X$ such that removing these edges disconnects the graph into two connected components.

These components are called the *half spaces* associated to the wall $w$. A set of walls $W$ is called a *wall structure* if each edge is contained in exactly one wall. A wall structure $W$ gives rise to a pseudo-metric $d_W$ on $X$ as follows. For two vertices $x$ and $y$ in $X$, define $d_W(x, y)$ to be the number of walls in $W$ for which $x$ and $y$ lie in different half spaces.

The wall structure which Arzhantseva, Guentner and Spakula construct on the $\mathbb{Z}/2\mathbb{Z}$-homology cover $\widetilde{X}$ of $X$ is defined in the following way. Let $\pi : \widetilde{X} \longrightarrow X$ be the covering map, and write $w_e$ for the preimage $\pi^{-1}(e)$ of an edge $e$. Note that $w_e$ is a set of edges in $\widetilde{X}$. The wall structure $W$ on $\widetilde{X}$ is defined to be the set $\{w_e : e \in E(X)\}$. The fact that this really is a wall structure is proved in Lemma 3.3 of [AGS].

**Proposition 4.7** (Proposition 3.4, [AGS])**.** *Let $(\widetilde{X}, X)$ be a pair of finite graphs such that $X$ is 2-connected, and $\widetilde{X}$ is the $\mathbb{Z}/2\mathbb{Z}$-homology cover of $X$. Then the pseudo-metric $d_W$ on $\widetilde{X}$ induced by the wall structure from $X$ is less than or equal to the graph metric $d_{\widetilde{X}}$ on $\widetilde{X}$, i.e. $d_W(x, y) \leq d_{\widetilde{X}}(x, y)$ for all $x, y \in V(\widetilde{X})$.*

In particular, this proves that $d_W$ is a metric and not just a pseudo-metric on $\widetilde{X}$.

Arzhantseva, Guentner and Spakula then show that for distances smaller than the *girth* of $X$ (i.e. the length of the smallest loop), the wall metric $d_W$ agrees with the graph metric $d_{\widetilde{X}}$ on $\widetilde{X}$.

**Proposition 4.8** (Proposition 3.11, [AGS])**.** *For $(\widetilde{X}, X)$ as above, we have for all $x, y \in V(\widetilde{X})$,*

$$d_W(x, y) < \text{girth}(X) \iff d_{\widetilde{X}}(x, y) < \text{girth}(X).$$

*If the above inequalities hold, then we have $d_W(x, y) = d_{\widetilde{X}}(x, y)$ for all $x, y \in V(\widetilde{X})$.*

In the case of $\mathbb{F}_2$, the box space $\Box_{\{N_i\}}\mathbb{F}_2$ with metric $d_W$ (defined to be the wall metric on each component and the distances between successive quotients taken to be larger than both of their diameters), is coarsely embeddable into Hilbert space since $d_W$ is an effective symmetric normalised negative type kernel on $\Box_{\{N_i\}}\mathbb{F}_2$, in the sense of [Roe], Chapter 11.

Note that the girth of the graphs $\mathbb{F}_2/N_i$ tends to infinity. This is because for all $R > 0$, there is an $i$ such that balls of radius $R$ in $\mathbb{F}_2/N_i$ are isometric to balls in $\mathbb{F}_2$, which is a tree. So, the length of the smallest loop in $\mathbb{F}_2/N_i$ will be at least $R$.

From this, the authors conclude that the metric $d$ on $\Box_{\{N_i\}}\mathbb{F}_2$ induced by the word metric of $\mathbb{F}_2$ with its natural generating set is coarsely equivalent to the metric $d_W$, and hence $(\Box_{\{N_i\}}\mathbb{F}_2, d)$ embeds coarsely into Hilbert space. We now make a remark about the generality of this construction, which follows directly from the results of [AGS].

**Remark 4.9.** Given a sequence $\{\widetilde{X}_i\}$ of graphs with increasing diameters such that each $\widetilde{X}_i$ is the $\mathbb{Z}/2\mathbb{Z}$-homology cover of a finite 2-connected graph $X_i$, one can induce a wall structure and hence a wall metric on each $\widetilde{X}_i$ from $X_i$. The disjoint union $\sqcup\widetilde{X}_i$, metrised using the wall metrics, is coarsely embeddable into Hilbert space. So provided that girth$(X_i)$ tends to infinity, $\sqcup\widetilde{X}_i$ metrised in the usual way using the natural graph metrics will be coarsely equivalent to $\sqcup\widetilde{X}_i$ with the wall metrics and will thus coarsely embed into Hilbert space.

## 4.4 Basic properties

Recall that given a box space, we obtain a coarsely equivalent space if we change the distances between the quotients, as long as the distances between two successive quotients are chosen to be greater than the maximum of their diameters. Thus, for a finitely generated residually finite group $H$ with a proper left-invariant metric $d$ and a nested sequence of finite index normal subgroups $\{K_i\}$ with trivial intersection, we will write $(\Box_{\{K_i\}}H, d')$ for the metric space obtained by taking the metric induced by $d$ on each finite quotient, with some valid choice of distances between quotients.

**Proposition 4.10.** *Suppose $d_1$ and $d_2$ are two proper left-invariant metrics on a finitely generated residually finite group $H$. Consider a box space $\Box_{\{K_i\}}H$ of $H$ with respect to some nested sequence of finite index normal subgroups $\{K_i\}$ with trivial intersection. The metric space $(\Box_{\{K_i\}}H, d_1')$ is coarsely equivalent to $(\Box_{\{K_i\}}H, d_2')$.*

*Proof.* By Proposition 2.3 of [DG03], the spaces $(H, d_1)$ and $(H, d_2)$ are coarsely equivalent. In other words, the identity mapping from $(H, d_1)$ to $(H, d_2)$ is a coarse embedding, so there exist non-decreasing functions $\rho_\pm : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\lim_{t\to\infty} \rho_\pm(t) = \infty$ such that

$$\rho_-(d_1(g,h)) \leq d_2(g,h) \leq \rho_+(d_1(g,h))$$

for all $g, h \in H$. We need to show that the identity mapping from $(\Box_{\{K_i\}}H, d_1')$ to $(\Box_{\{K_i\}}H, d_2')$ is a coarse embedding.

We can choose the distance between the identity elements of successive quotients to be the same in both spaces, ensuring that this distance is greater than the maximum of the diameters of the two quotients, with respect to both metrics $d'_1$ and $d'_2$.

If two elements $gK_i, hK_j \in \Box_{\{K_i\}}H$ lie in different quotients, we have

$$\begin{aligned} d'_1(gK_i, hK_j) &= d'_1(gK_i, eK_i) + d'_1(eK_i, eK_j) + d'_1(eK_j, hK_j) \\ &= d'_1(gK_i, eK_i) + d'_2(eK_i, eK_j) + d'_1(eK_j, hK_j) \\ &\leq 3d'_2(eK_i, eK_j) \leq 3d'_2(gK_i, hK_j), \end{aligned}$$

and vice versa. Thus we have

$$\frac{1}{3}d'_1(gK_i, hK_j) \leq d'_2(gK_i, hK_j) \leq 3d'_1(gK_i, hK_j).$$

When $gK_i$ and $hK_i$ lie in the same quotient $H/K_i$, choose $k_2$ in $H$ such that its image $k_2K_i$ in $H/K_i$ is equal to $g^{-1}hK_i$ and such that

$$\begin{aligned} d'_2(gK_i, hK_i) = d'_2(eK_i, g^{-1}hK_i) &:= \min_{b\in H}\{d_2(e, b) : bK_i = g^{-1}hK_i \in H/K_i\} \\ &= d_2(e, k_2), \end{aligned}$$

i.e. the above minimum is attained at $k_2 \in H$. Similarly, choose $k_1$ in $H$ such that $k_1K_i = g^{-1}hK_i$ and

$$\min_{b\in H}\{d_1(e, b) : bK_i = g^{-1}hK_i \in H/K_i\} = d_1(e, k_1).$$

We then have

$$\begin{aligned} \rho_-(d'_1(gK_i, hK_i)) = \rho_-(d'_1(eK_i, g^{-1}hK_i)) &= \rho_-(d'_1(eK_i, k_2K_i)) \\ &\leq \rho_-(d_1(e, k_2)) \leq d_2(e, k_2) \\ &= d'_2(eK_i, g^{-1}hK_i) = d'_2(gK_i, hK_i) \end{aligned}$$

and

$$\begin{aligned} d_2'(gK_i, hK_i) &= d_2'(eK_i, g^{-1}hK_i) = d_2'(eK_i, k_1K_i) \\ &\leq d_2(e, k_1) \leq \rho_+(d_1(e, k_1)) \\ &= \rho_+(d_1'(eK_i, g^{-1}hK_i)) = \rho_+(d_1'(gK_i, hK_i)). \end{aligned}$$

Hence, taking $\rho_-'(t) = \min\{\frac{1}{3}t, \rho_-(t)\}$ and $\rho_+'(t) = \max\{3t, \rho_+(t)\}$, we have

$$\rho_-'(d_1'(gK_i, hK_i)) \leq d_2'(gK_i, hK_i) \leq \rho_+'(d_1'(gK_i, hK_i)).$$

The functions $\rho_-'$ and $\rho_+'$ are non-decreasing, and $\lim_{t\to\infty} \rho_\pm'(t) = \infty$. This completes the proof. □

**Corollary 4.11.** *For a finitely generated residually finite group $H$, whether or not a given box space embeds coarsely into Hilbert space is independent of the choice of finite generating set.*

**Proposition 4.12.** *Let $G$ be a finitely generated residually finite group with a nested sequence $\{K_i\}$ of finite index normal subgroups with trivial intersection such that the corresponding box space $\Box_{\{K_i\}}G$ embeds coarsely into Hilbert space. Then any subgroup $H$ of $G$ has a coarsely embeddable box space.*

*Proof.* The sequence of subgroups formed by intersecting $H$ with the $K_i$ gives a sequence of finite index normal subgroups $N_i$ of $H$. Clearly the intersection of the $N_i$ is still trivial. Each quotient $H/N_i = H/H \cap K_i$ is isomorphic to $HK_i/K_i$, and is thus a subgroup of $G/K_i$. Thus, the box space $\Box_{\{N_i\}}H$ with the subspace metric induced from $\Box_{\{K_i\}}G$ coarsely embeds in $\Box_{\{K_i\}}G$, and hence coarsely embeds into Hilbert space. This metric is the metric induced on quotients of $H$ by the subspace metric on $H$ as a subgroup of $G$. All proper left-invariant metrics on $H$ are coarsely equivalent, so by Proposition 4.10, $\Box_{\{N_i\}}H$ with a metric induced by any proper left-invariant metric on $H$ also coarsely embeds into Hilbert space. □

In particular, if $H$ in the above proposition is a finitely generated subgroup of $G$, then its word metric is coarsely equivalent to the metric induced on $H$ by the word metric on $G$. Hence $\Box_{\{N_i\}}H$ with the metric induced by the word metric on $H$ coarsely embeds into Hilbert space.

**Corollary 4.13.** *For any finitely generated free group $\mathbb{F}_k$, there exists a nested sequence of finite index normal subgroups with trivial intersection such that the corresponding box space embeds coarsely into Hilbert space.*

*Proof.* For any finite $k$, $\mathbb{F}_k$ is a finite index subgroup of $\mathbb{F}_2$, which we know has an embeddable box space by the result of [AGS]. □

**Remark 4.14.** Note that we can find an embeddable box space of a finitely generated free group $\mathbb{F}_k$ in many ways, since using the inductively defined sequence of subgroups of [AGS] will also result in a coarsely embeddable box space (see Remark 4.9). In addition, we can view $\mathbb{F}_k$ as a subgroup of another finitely generated free group $\mathbb{F}_n$ (in many ways), and obtain the required sequence of subgroups by intersecting, as above. It would be interesting to know whether the box spaces obtained in this way are coarsely equivalent.

**Proposition 4.15.** *Let $H$ be a finitely generated residually finite group with a nested sequence $\{C_i\}$ of finite index characteristic subgroups with trivial intersection such that the corresponding box space $\Box_{\{C_i\}}H$ embeds coarsely into Hilbert space. Then any group $G$ containing $H$ as a finite index normal subgroup also has a coarsely embeddable box space.*

*Proof.* Each of the subgroups $C_i$ is normal in $G$. The box space $\Box_{\{C_i\}}G$ is coarsely equivalent to the box space $\Box_{\{C_i\}}H$, and thus embeds coarsely into Hilbert space. □

**Example 4.16.** There exists a nested sequence of finite index normal subgroups of $SL(2,\mathbb{Z})$ with trivial intersection such that the corresponding box space embeds coarsely into Hilbert space, since $SL(2,\mathbb{Z})$ contains a finitely generated free group as a finite index normal subgroup.

Conversely, one can obtain a box space of a finitely generated free group which is an expander by viewing it as a finite index subgroup of $SL(2,\mathbb{Z})$ as follows.

**Example 4.17.** By intersecting the congruence subgroups

$$N_m := \ker(SL(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{Z}/m\mathbb{Z}))$$

with the finite index finitely generated free group $\mathbb{F}_n$ in $SL(2,\mathbb{Z})$, we obtain a sequence of subgroups of this free group such that the corresponding box space is an expander sequence. This is because the quotients $\mathbb{F}_n/(\mathbb{F}_n \cap N_m) \cong \mathbb{F}_n N_m/N_m$ are uniformly coarsely equivalent to the quotients $SL(2,\mathbb{Z})/N_m$, which form an expander sequence. The uniformity of the coarse equivalence can be seen from the fact that the index of $\mathbb{F}_n N_m/N_m$ in $SL(2,\mathbb{Z})/N_m$ is always between 1 and the index of the free group $\mathbb{F}_n$ in $SL(2,\mathbb{Z})$, since we have

$$1 \leq |SL(2,\mathbb{Z})/N_m : \mathbb{F}_n N_m/N_m| = |SL(2,\mathbb{Z}) : \mathbb{F}_n N_m| \leq |SL(2,\mathbb{Z}) : \mathbb{F}_n|.$$

Thus the groups $SL(2,\mathbb{Z})$ and $\mathbb{F}_n$ have both an embeddable box space, and a non-embeddable box space.

## 4.5 Extensions

Let $\{1 \to H_i \to \Gamma_i \to G_i \to 1\}_{i\in\mathbb{N}}$ be a sequence of group extensions, where all groups involved are finite and such that the number of elements required to generate the groups $\Gamma_i$ is uniformly bounded across $i \in \mathbb{N}$. Consider the metric space $(\sqcup\Gamma_i, d_\Gamma)$ made from the disjoint union of the $\Gamma_i$ with their word metrics, with distances between two consecutive components defined to be greater than the largest of their diameters. Make the disjoint union $\sqcup H_i$ into a metric space by taking the metric induced by $d_\Gamma$. Call this metric $d_H$. Make $\sqcup G_i$ into a metric space by defining distance between two consecutive components $G_k$ and $G_{k+1}$ to be the same as the distance between the corresponding $\Gamma_k$ and $\Gamma_{k+1}$, and on each component $G_i$ taking the metric induced from $\Gamma_i$ by the quotient $\Gamma_i/H_i \cong G_i$. Call this metric $d_G$. Note that each of these spaces has bounded geometry thanks to the assumption on the generators of the $\Gamma_i$, above.

For each $i$ let $\pi_i$ be the projection $\pi_i : \Gamma_i \longrightarrow G_i$ and choose a set-theoretic cross-section $\sigma_i : G_i \longrightarrow \Gamma_i$ such that distances to the identity are preserved. Define a map $\eta_i : \Gamma_i \times G_i \longrightarrow H_i$ by

$$\eta_i(\gamma, g) = \sigma_i(g)^{-1}\gamma\sigma_i(\pi_i(\gamma)^{-1}g).$$

We will drop the indices $i$, as this should not cause any confusion.

We will make use of the following lemma. Fix an index $i$.

**Lemma 4.18** (Lemma 4.4, [DG03]). *Let $\gamma_1$ and $\gamma_2$ be elements of $\Gamma_i$, and $g$ an element of $G_i$. Then*

$$d_{\Gamma_i}(\gamma_1, \gamma_2) \leq d_{G_i}(g, \pi(\gamma_1)) + d_{G_i}(g, \pi(\gamma_2)) + d_{H_i}(\eta(\gamma_1, g), \eta(\gamma_2, g))$$
$$d_{H_i}(\eta(\gamma_1, g), \eta(\gamma_2, g)) \leq d_{G_i}(g, \pi(\gamma_1)) + d_{G_i}(g, \pi(\gamma_2)) + d_{\Gamma_i}(\gamma_1, \gamma_2)$$

*Proof.* Let the cross-section $\sigma_i : G_i \longrightarrow \Gamma_i$ be given by $g = \gamma H_i \mapsto \gamma h_\gamma$ where

$h_\gamma \in H_i$ such that the length $\ell(\gamma h_\gamma)$ in $\Gamma_i$ is minimal. First we note that

$$\begin{aligned}\eta(\gamma_1, \gamma H_i)^{-1}\eta(\gamma_2, \gamma H_i) &= \sigma_i(\gamma_1^{-1}\gamma H_i)^{-1}\gamma_1^{-1}\gamma_2\sigma_i(\gamma_2^{-1}\gamma H_i)\\ &= (\gamma_1^{-1}\gamma h_{\gamma_1^{-1}\gamma})^{-1}\gamma_1^{-1}\gamma_2\gamma_2^{-1}\gamma h_{\gamma_2^{-1}\gamma}\\ &= h_{\gamma_1^{-1}\gamma}^{-1}\gamma^{-1}\gamma_1\gamma_1^{-1}\gamma_2\gamma_2^{-1}\gamma h_{\gamma_2^{-1}\gamma}\\ &= h_{\gamma_1^{-1}\gamma}^{-1}h_{\gamma_2^{-1}\gamma}.\end{aligned}$$

Thus we have

$$\begin{aligned}&d_{G_i}(\pi(\gamma_1), \gamma H_i) + d_{G_i}(\pi(\gamma_2), \gamma H_i) + d_{H_i}(\eta(\gamma_1, \gamma H_i), \eta(\gamma_2, \gamma H_i))\\ &\geq \ell(\gamma_1^{-1}\gamma h_{\gamma_1^{-1}\gamma}) + \ell(\gamma_2^{-1}\gamma h_{\gamma_2^{-1}\gamma}) + \ell(h_{\gamma_1^{-1}\gamma}^{-1}h_{\gamma_2^{-1}\gamma})\\ &= d_{\Gamma_i}(\gamma^{-1}\gamma_1, h_{\gamma_1^{-1}\gamma}) + d_{\Gamma_i}(\gamma^{-1}\gamma_2, h_{\gamma_2^{-1}\gamma}) + d_{H_i}(h_{\gamma_1^{-1}\gamma}, h_{\gamma_2^{-1}\gamma})\\ &\geq d_{\Gamma_i}(\gamma_1, \gamma_2).\end{aligned}$$

The other inequality is proved in a similar fashion:

$$\begin{aligned}&d_{H_i}(\eta(\gamma_1, \gamma H_i), \eta(\gamma_2, \gamma H_i)) = d_{H_i}(h_{\gamma_1^{-1}\gamma}, h_{\gamma_2^{-1}\gamma})\\ &\leq d_{\Gamma_i}(\gamma^{-1}\gamma_1, h_{\gamma_1^{-1}\gamma}) + d_{\Gamma_i}(\gamma^{-1}\gamma_2, h_{\gamma_2^{-1}\gamma}) + d_{\Gamma_i}(\gamma_1, \gamma_2)\\ &\leq d_{G_i}(\pi(\gamma_1), \gamma H_i) + d_{G_i}(\pi(\gamma_2), \gamma H_i) + d_{\Gamma_i}(\gamma_1, \gamma_2).\end{aligned}$$

□

**Theorem 4.19.** *Let $\{1 \to H_i \to \Gamma_i \to G_i \to 1\}_{i\in\mathbb{N}}$ be as above, such that the diameters of the $\Gamma_i$ increase strictly with $i$. If the space $(\sqcup H_i, d_H)$ coarsely embeds into Hilbert space and the space $(\sqcup G_i, d_G)$ has property A, then the metric space $(\sqcup \Gamma_i, d_\Gamma)$ embeds coarsely into Hilbert space.*

*Proof.* We will check that the space $\sqcup\Gamma_i$ satisfies the following criterion for coarse embedding into Hilbert space given in Proposition 2.1 of [DG03]: if for each $\varepsilon > 0$ and $R > 0$ there exists a map $\varphi : \sqcup\Gamma_i \longrightarrow \ell^2(\sqcup G_i, \mathcal{H})$, $\mathcal{H}$ a real Hilbert space, such that $\|\varphi(\gamma)\| = 1$ for all $\gamma \in \sqcup\Gamma_i$ and such that for all $\gamma_1, \gamma_2 \in \sqcup\Gamma_i$,

($1_\Gamma$) if $d_\Gamma(\gamma_1, \gamma_2) \leq R$ then $|1 - \langle\varphi(\gamma_1), \varphi(\gamma_2)\rangle| < \varepsilon$, and

($2_\Gamma$) $\forall\delta > 0$ $\exists S > 0$ such that if $d_\Gamma(\gamma_1, \gamma_2) \geq S$, then $|\langle\varphi(\gamma_1), \varphi(\gamma_2)\rangle| < \delta$

then $(\sqcup\Gamma_i, d_\Gamma)$ coarsely embeds into Hilbert space.

Let $\varepsilon > 0$ and $R > 0$ be given. We know that the space $(\sqcup G_i, d_G)$ has property A, and so Proposition 2.6 of [DG03] tells us that there exists $\phi : \sqcup G_i \longrightarrow \ell^2(\sqcup G_i)$ and $S_G > 0$ such that $\|\phi(g)\| = 1$ for all $g \in \sqcup G_i$, and such that

($1_G$) for all $g_1, g_2 \in \sqcup G_i$, if $d_G(g_1, g_2) \leq R$ then $|1 - \langle\phi(g_1), \phi(g_2)\rangle| < \frac{\varepsilon}{2}$, and

($2_G$) $\operatorname{Supp}\phi(g) \subset B_{S_G}(g)$ for all $g \in \sqcup G_i$.

We will view $\phi$ as a function on $\sqcup G_i \times \sqcup G_i$. Since $\sqcup H_i$ is coarsely embeddable, there exists according to Proposition 2.1 of [DG03] a Hilbert space-valued map $\psi : \sqcup H_i \longrightarrow \mathcal{H}$ such that $\|\psi(h)\| = 1$ for all $h \in \sqcup H_i$ and such that for all $h_1, h_2 \in \sqcup H_i$,

($1_H$) if $d_H(h_1, h_2) \leq 2S_G + R$ then $|1 - \langle\psi(h_1), \psi(h_2)\rangle| < \frac{\varepsilon}{2}$, and

($2_H$) $\forall\delta > 0$ $\exists S_H > 0$ such that if $d_H(h_1, h_2) \geq S_H$, then $|\langle\psi(h_1), \psi(h_2)\rangle| < \delta$.

We will now define the map $\varphi : \sqcup\Gamma_i \longrightarrow \ell^2(\sqcup G_i, \mathcal{H})$. Let $N_R \in \mathbb{N}$ be such that for $i \geq N_R$, the distance between $\Gamma_i$ and $\Gamma_{i+1}$ is greater than $R$. For $\gamma \in \Gamma_i$ with $i \leq N_R$, let $\varphi(\gamma)$ be given by $\varphi(\gamma)(\pi(e_1)) = \psi(e_1)$ and $\varphi(\gamma)(g) = 0$ for $g \neq \pi(e_1)$, where $e_1$ denotes the identity element of $\Gamma_1$. For $\gamma \in \Gamma_i$ with $i > N_R$, define

$$\varphi(\gamma)(g) = \phi(\pi(\gamma), g)\psi(\eta(\gamma, g))$$

for $g \in G_i$ and

$$\varphi(\gamma)(g) = \phi(\pi(\gamma), g)\psi(e_i)$$

for $g \notin G_i$, where $e_i$ is the identity element of $H_i$.

We will first check that $\|\varphi(\gamma)\| = 1$ for all $\gamma \in \sqcup\Gamma_i$. For $\gamma \in \Gamma_i$ with $i \leq N_R$, we have

$$\|\varphi(\gamma)\| = \|\psi(e_1)\| = 1$$

and for $\gamma \in \Gamma_i$ with $i > N_R$, we have

$$\begin{aligned}
\|\varphi(\gamma)\|^2 &= \sum_{g \in G_i} |\phi(\pi(\gamma), g)|^2 \cdot \|\psi(\eta(\gamma, g))\|^2 + \sum_{g \notin G_i} |\phi(\pi(\gamma), g)|^2 \cdot \|\psi(e_i)\|^2 \\
&= \sum_{g \in G_i} |\phi(\pi(\gamma), g)|^2 + \sum_{g \notin G_i} |\phi(\pi(\gamma), g)|^2 \\
&= \|\phi(\pi(\gamma))\|^2 = 1.
\end{aligned}$$

Let us now check the remaining two conditions $(1_\Gamma)$ and $(2_\Gamma)$.

Take $\gamma_1$ and $\gamma_2$ in $\sqcup\Gamma_i$ such that $d_\Gamma(\gamma_1, \gamma_2) \leq R$. Then if both $\gamma_1$ and $\gamma_2$ lie in $\sqcup_{i=1}^{N_R}\Gamma_i$, we have $|1 - \langle\varphi(\gamma_1), \varphi(\gamma_2)\rangle| = |1 - \|\psi(e_1)\|^2| = |1 - 1| = 0$. The only other possibility is that they both lie in the same component, say $\Gamma_i$, for $i > N_R$. In this case we have

$$\begin{aligned}
|1 - \langle \varphi(\gamma_1), \varphi(\gamma_2) \rangle| = & \Big| 1 - \sum_{g \in G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) \langle \psi(\eta(\gamma_1, g)), \psi(\eta(\gamma_2, g)) \rangle \\
& - \sum_{g \notin G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) \langle \psi(e_i), \psi(e_i) \rangle \Big| \\
= & \Big| 1 - \sum_{g \in G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) \langle \psi(\eta(\gamma_1, g)), \psi(\eta(\gamma_2, g)) \rangle \\
& + \sum_{g \in G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) - \sum_{g \in G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) \\
& - \sum_{g \notin G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) \Big| \\
\leq & \Big| \sum_{g \in G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g)(1 - \langle \psi(\eta(\gamma_1, g)), \psi(\eta(\gamma_2, g)) \rangle) \Big| \\
+ & \Big| 1 - \langle \phi(\pi(\gamma_1)), \phi(\pi(\gamma_2)) \rangle \Big|
\end{aligned}$$

The quotient map $\pi$ is contractive and hence we have that $d_G(\pi(\gamma_1), \pi(\gamma_2)) \leq R$, so $(1_G)$ tells us that the second term is bounded by $\frac{\varepsilon}{2}$. By $(2_G)$, the sum in the first term ranges over $g \in B_{S_G}(\pi(\gamma_1)) \cap B_{S_G}(\pi(\gamma_2))$, and thus the first term can be bounded by

$$\sup\{|1 - \langle \psi(\eta(\gamma_1, g)), \psi(\eta(\gamma_2, g)) \rangle| : g \in B_{S_G}(\pi(\gamma_1)) \cap B_{S_G}(\pi(\gamma_2))\}.$$

By Lemma 4.18 above, for $g \in B_{S_G}(\pi(\gamma_1)) \cap B_{S_G}(\pi(\gamma_2))$ we have

$$d_H(\eta(\gamma_1, g), \eta(\gamma_2, g)) \leq 2S_G + R$$

and so by $(1_H)$, the supremum above is bounded by $\frac{\varepsilon}{2}$. This completes the proof of $(1_\Gamma)$.

For $(2_\Gamma)$, fix $\delta > 0$. Take the $S_H$ corresponding to $\frac{\delta}{3}$ as in $(2_H)$. The required $S$ will be given by $3S_G + 3S_H + M_\Gamma$, where

$$M_\Gamma = \max\{d_\Gamma(\gamma_1, \gamma_2) : \gamma_1, \gamma_2 \in \sqcup_{i=1}^{N_R} \Gamma_i\}.$$

If $d_\Gamma(\gamma_1,\gamma_2) \geq 3S_G + 3S_H + M_\Gamma$, then at most one of $\gamma_1, \gamma_2$ lies in $\sqcup_{i=1}^{N_R}\Gamma_i$. If one of them does, without loss of generality $\gamma_1 \in \sqcup_{i=1}^{N_R}\Gamma_i$, and $\gamma_2 \in \Gamma_j$ for some $j > N_R$. We then have

$$|\langle\varphi(\gamma_1),\varphi(\gamma_2)\rangle| = |\phi(\pi(\gamma_2),\pi(e_1))\,\langle\psi(e_1),\psi(e_j)\rangle|.$$

Now notice that $d_H(e_1,e_j) = d_\Gamma(e_1,e_j) \geq d_\Gamma(\gamma_1,e_j) \geq d_\Gamma(\gamma_1,\gamma_2) - d_\Gamma(\gamma_2,e_j)$, and

$$\begin{aligned} 2d_\Gamma(\gamma_2,e_j) &\leq \operatorname{diam}(\Gamma_j) + d_\Gamma(\gamma_2,e_j) \\ &\leq d_\Gamma(e_{j-1},e_j) + d_\Gamma(\gamma_2,e_j) \\ &\leq d_\Gamma(\gamma_1,\gamma_2), \end{aligned}$$

so $d_H(e_1,e_j)$ is greater than or equal to $d_\Gamma(\gamma_1,\gamma_2) - \frac{1}{2}d_\Gamma(\gamma_1,\gamma_2) \geq S_H$. We thus have $|\langle\varphi(\gamma_1),\varphi(\gamma_2)\rangle| < \delta$ by $(2_H)$.

If neither $\gamma_1$ nor $\gamma_2$ lie in $\sqcup_{i=1}^{N_R}\Gamma_i$, then there are two possibilities. If $\gamma_1 \in \Gamma_i$ and $\gamma_2 \in \Gamma_j$ for $N_R < i < j$, then we have

$$\begin{aligned} |\langle\varphi(\gamma_1),\varphi(\gamma_2)\rangle| \leq& \left|\sum_{g\in G_i}\phi(\pi(\gamma_1),g)\phi(\pi(\gamma_2),g)\,\langle\psi(\eta(\gamma_1,g)),\psi(e_j)\rangle\right| \\ &+\left|\sum_{g\in G_j}\phi(\pi(\gamma_1),g)\phi(\pi(\gamma_2),g)\,\langle\psi(e_i),\psi(\eta(\gamma_2,g))\rangle\right| \\ &+\left|\sum_{g\notin G_i,G_j}\phi(\pi(\gamma_1),g)\phi(\pi(\gamma_2),g)\,\langle\psi(e_i),\psi(e_j)\rangle\right|. \end{aligned}$$

The distances between each of the pairs of points $(\eta(\gamma_1,g),e_j)$, $(e_i,\eta(\gamma_2,g))$ and $(e_i,e_j)$ are greater than $d_\Gamma(e_i,e_j)$ and thus greater than $\frac{1}{3}d_\Gamma(\gamma_1,\gamma_2) \geq S_H$ (since $d_\Gamma(\gamma_1,\gamma_2) \leq d_\Gamma(\gamma_1,e_i)+d_\Gamma(e_i,e_j)+d_\Gamma(e_j,\gamma_2) \leq 3d_\Gamma(e_i,e_j)$), so by $(2_H)$ we have $|\langle\varphi(\gamma_1),\varphi(\gamma_2)\rangle| < \delta$ as required.

The second possibility is that $\gamma_1$ and $\gamma_2$ lie in the same component $\Gamma_i$, for

$i > N_R$. Then we have

$$| \langle \varphi(\gamma_1), \varphi(\gamma_2) \rangle | \leq \left| \sum_{g \in G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) \langle \psi(\eta(\gamma_1, g)), \psi(\eta(\gamma_2, g)) \rangle \right| + \left| \sum_{g \notin G_i} \phi(\pi(\gamma_1), g)\phi(\pi(\gamma_2), g) \langle \psi(e_i), \psi(e_i) \rangle \right|$$

The first term is bounded above by

$$\sup\{| \langle \psi(\eta(\gamma_1, g)), \psi(\eta(\gamma_2, g)) \rangle | : g \in B_{S_G}(\pi(\gamma_1)) \cap B_{S_G}(\pi(\gamma_2))\}.$$

So, from Lemma 4.18 above, we deduce that $d_H(\eta(\gamma_1, g), \eta(\gamma_2, g)) \geq d_\Gamma(\gamma_1, \gamma_2) - d_G(g, \pi(\gamma_1)) - d_G(g, \pi(\gamma_2)) \geq S_H$ for any $g \in B_{S_G}(\pi(\gamma_1)) \cap B_{S_G}(\pi(\gamma_2))$ and thus the first term is bounded by $\frac{\delta}{3}$.

Now $B_{S_G}(\pi(\gamma_1)) \cap B_{S_G}(\pi(\gamma_2)) \subset G_i$, since for any $\gamma$ in $\Gamma_i$ and $g$ in $G_k$, $k \neq i$, we have

$$d_G(g, \pi(\gamma)) \geq d_G(e_k, e_i) \geq \mathrm{diam}(\Gamma_i) \geq d_\Gamma(\gamma_1, \gamma_2) \geq S_G.$$

Thus, the set over which the second sum is taken is empty. This completes the proof of $(2_\Gamma)$, and hence the theorem.

□

Note that we do not require the diameters of the $H_i$ or the $G_i$ to increase.

## 4.6 Applications

In this section, we apply Theorem 4.19 in two different ways. First, we start with a residually finite group $\Gamma$, which is an extension of $H$ by $G$, and some sequence of nested finite index normal subgroups of $\Gamma$ with trivial intersection. We give a sufficient condition for the corresponding box space to embed coarsely into Hilbert space, in terms of the groups $G$ and $H$.

However, in practice it may be difficult to check whether these conditions hold for a given group. We therefore give a concrete application of Theorem 4.19 for semidirect products, where we can build a sequence of nested finite index normal subgroups of the semidirect product out of such sequences for the factors.

Consider first a sequence of extensions arising from an extension $1 \longrightarrow H \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ in the following way. Suppose $\Gamma$ is residually finite, and let $\{K_i\}$ be a sequence of nested finite index normal subgroups of $\Gamma$ with trivial intersection. Then $H \cap K_i$ is such a sequence for $H$, and we have the following sequence of extensions:

$$1 \longrightarrow H/H \cap K_i \longrightarrow \Gamma/K_i \longrightarrow \Gamma/HK_i \longrightarrow 1,$$

where the groups $\Gamma/HK_i$ can be seen as finite quotients of $G$ by $L_i := HK_i/H$. We can then apply our theorem to conclude that the box space $\Box_{\{K_i\}}\Gamma$ coarsely embeds into Hilbert space if the space $\sqcup G/L_i$ has property A and the box space $\Box_{\{H \cap K_i\}}H$ with the induced metric coarsely embeds into Hilbert space. Note that in general, $G$ will not be residually finite.

Define $L$ to be the intersection of the $L_i$.

**Proposition 4.20.** *Let* $1 \longrightarrow H \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ *be an extension as above. If* $G/L$ *is amenable and the box space* $\Box_{\{H \cap K_i\}}H$ *with the induced metric coarsely embeds into Hilbert space, then* $\Box_{\{K_i\}}\Gamma$ *embeds coarsely into Hilbert space.*

*Proof.* For each $i$, we have $G/L_i \cong (G/L)/(L_i/L)$. Note that the intersection $\cap L_i/L$ is trivial, and that the quotient $G/L$ is amenable. Hence, by Guentner's result, the box space $\Box_{\{L_i/L\}}G/L$ has property A and hence the space $\sqcup G/L_i$ does too. We can now apply Theorem 4.19 to conclude that $\Box_{\{K_i\}}\Gamma$ embeds coarsely into Hilbert space. $\Box$

We can now state a sufficient condition for a residually finite group to have the Haagerup property.

**Corollary 4.21.** *Let $\Gamma$ be a finitely generated residually finite group which is an extension of a group $H$ by a group $G$. Suppose there exists a sequence $\{K_i\}$ of nested normal finite index subgroups of $\Gamma$ such that the intersection $\cap K_i$ is trivial and $\Box_{\{H\cap K_i\}}H$ with the induced subspace metric coarsely embeds into Hilbert space. Let $L$ denote the intersection of the subgroups $HK_i/H$ of $G$. Then $\Gamma$ has the Haagerup property if $G/L$ is amenable.*

Let us remark here that the Haagerup property is known to be preserved under extensions with amenable quotients (see [CJV]).

Since it may be difficult to check that the space $\Box_{\{H\cap K_i\}}H$ is embeddable, we now give a more concrete application for semidirect products.

**Theorem 4.22.** *The semidirect product of two residually finite groups is also residually finite.*

The proof is implicitly contained in the proof of our main result, below.

**Theorem 4.23.** *Let $\Gamma$ be the semidirect product $H \rtimes G$ of two finitely generated residually finite groups $H$ and $G$ such that there is a nested sequence of finite index characteristic subgroups $\{N_i\}$ of $H$ with $\cap N_i = 1$, such that*

$\Box_{\{N_i\}}H$ *embeds coarsely into Hilbert space, and* $G$ *is amenable. Then* $\Gamma$ *has an embeddable box space.*

*Proof.* Enumerate the non-trivial elements of $\Gamma$, so that $\Gamma = \{e, \gamma_1, \gamma_2, \gamma_3, ...\}$. For each $\gamma_i$, we will find a normal finite index subgroup $K_i$ of $\Gamma$ such that the image of $\gamma_i$ is non-trivial in $\Gamma/K_i$ and such that the $K_i$ are nested. We will do this inductively. We essentially prove that $\Gamma$ is residually finite while making sure that the subgroups $K_i$ are built from the subgroups of $H$ and $G$ in a particular way.

First, take $\gamma_1 = (x, a)$, where $x \in H$ and $a \in G$. If $a$ is non-trivial, then there is some finite quotient $Q$ of $G$ in which the image of $a$ is still non-trivial. In this case, take $K_1$ to be the kernel of the surjection $\Gamma \longrightarrow Q$. If $a$ is trivial, then $x$ is non-trivial, and so we can take one of the characteristic subgroups $N_j$ of H such that the image of $x$ is non-trivial in the quotient $H/N_j$. Since $N_j$ is characteristic in $H$, it is normal in $\Gamma$ and so we have a quotient of $\Gamma$ which is isomorphic to $H/N_j \rtimes G$, in which the image of $\gamma_1$ is non-trivial. To get a finite quotient, we take the subgroup $A$ of $G$ which acts trivially on $H/N_j$. $A$ is normal in $H/N_j \rtimes G$, and the quotient $H/N_j \rtimes G/A$ is finite because each element of $G/A$ now acts as a non-trivial automorphism of $H/N_j$, and so $G/A$ is a subgroup of $\mathrm{Aut}(H/N_j)$, which is finite since $H/N_j$ is. Let $K_1$ be the kernel of the homomorphism $\Gamma \longrightarrow H/N_j \rtimes G/A$.

Now, suppose we have defined $K_1, K_2, ...$ up to $K_{i-1}$. Given $\gamma_i$, we want to find a normal finite index subgroup $K_i$ of $\Gamma$ such that the image of $\gamma_i$ is non-trivial in $\Gamma/K_i$ and such that $K_i \leq K_{i-1}$. Let $\gamma_i$ be given by $(h, b)$, where $h \in H$ and $b \in G$. Take the characteristic subgroup $N_k$ of $H$ such that $K_{i-1} \cap H \geq N_k$ and, additionally, such that the image of $h$ is non-trivial in $H/N_k$ if $h$ is non-trivial. Consider the quotient $H/N_k \rtimes G$. Now take the subgroup $B$ of $G$ which acts trivially on $H/N_k$, and take the intersection with $K_{i-1} \cap G$. Call this subgroup $K$. Now $K$ still acts trivially on $H/N_k$, and is thus a normal subgroup of $H/N_k \rtimes G$. The quotient $H/N_k \rtimes G/K$ is finite (since $K_{i-1} \cap G$ and $B$ are both of finite index in $G$, and hence so is their intersection), and $\gamma_i$ has a non-trivial image in this quotient. Thus, we can define $K_i$ to be the kernel of the map $\Gamma \longrightarrow H/N_k \rtimes G/K$, which lies inside $K_{i-1}$ by construction.

We can now apply Theorem 4.19. Note that we have obtained a sequence of nested normal finite index subgroups $\{K_i\}$ such that each quotient $\Gamma/K_i$ is an extension of a quotient $H/N_n$ by a finite quotient of $G$. The intersection of all the $K_i$ is trivial. The disjoint union of the quotients of $H$ coarsely embeds into Hilbert space, and the disjoint union of the quotients of $G$ has property A, since $G$ is amenable. The conditions of Theorem 4.19 are thus satisfied, so the box space $\square_{\{K_i\}}\Gamma$ coarsely embeds into Hilbert space. $\square$

**Remark 4.24.** We can apply the above to show that semidirect products of finitely generated free groups by residually finite amenable groups have an embeddable box space. This provides a new class of examples of spaces with bounded geometry which embed coarsely into Hilbert space but do not have property A, generalising the example of Arzhantseva, Guentner and Spakula.

**Corollary 4.25.** *Let $\Gamma$ be an extension of a finitely generated free group by a (finite or infinite) cyclic group. Then $\Gamma$ has an embeddable box space.*

*Proof.* Suppose first that $\Gamma$ is an extension of a finitely generated free group $\mathbb{F}_n$ by a finite cyclic group $C$. Take the sequence of subgroups of $\mathbb{F}_n$ defined inductively as in [AGS], $\{N_i\}$. These subgroups are characteristic in $\mathbb{F}_n$, and hence we can apply Proposition 4.15, which gives the required result.

Suppose now that $\Gamma$ is an extension of a finitely generated free group $\mathbb{F}_n$ by $\mathbb{Z}$. This extension splits, and so $\Gamma$ is isomorphic to $\mathbb{F}_n \rtimes \mathbb{Z}$. We can now apply Theorem 4.23 to conclude the proof. $\square$

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