COMPEL: The International Journal for Computation and Mathematics in Electrical and Electronic Engineering

# Emerald Article: Geometric formulation of edge and nodal finite element equations in electromagnetics

Andrzej Demenko, Jan. K. Sykulski

# Geometric formulation of edge and nodal finite element equations in electromagnetics

Andrzej Demenko
*Poznań University of Technology, Poznań, Poland, and*
Jan. K. Sykulski
*University of Southampton, Southampton, UK and Technical University of Łódź, Łódź, Poland*

**Abstract**

**Purpose** – The purpose of this paper is to emphasise the analogies between variational and network formulations using geometrical forms, with the purpose of developing alternative but otherwise equivalent derivations of the finite element (FE) method.

**Design/methodology/approach** – FE equations for electromagnetic fields are examined, in particular nodal elements using scalar potential formulation and edge elements for vector potential formulation.

**Findings** – It is shown how the equations usually obtained via variational approach may be more conveniently derived using integral methods, employing a geometrical description of the interpolating functions of edge and facet elements. Moreover, the resultant equations describe the equivalent multi-branch circuit models.

**Originality/value** – The approach proposed in the paper explores the analogy of the FE formulation to loop or nodal magnetic or electric networks and has been shown to be very beneficial in teaching, especially to students well familiar with circuit methods. The presented methods are also helpful when formulating classical network models. Finally, for the first time, the geometrical forms of edge and facet element functions have been demonstrated.

**Keywords** Eddy currents, Electromagnetic fields, Electrical engineering education, Finite element method, Finite integration technique, Magnetic circuits

**Paper type** Research paper

## I. Introduction

Finite element equations are commonly derived using a variational approach, including weak forms (the Galerkin weighted residual method) and/or a strong formulation via a functional (the Rayleigh-Ritz method). These equations have various geometrical interpretations (Trevisan and Kettunen, 2004) and may be explained using the language of circuit theory by considering the nodal or loop descriptions of equivalent magnetic or electric circuits (Demenko and Sykulski, 2002; Ren and Qu, 2010). The classical equivalent circuits, however, arise from integral formulations. In this work we will demonstrate that by applying appropriate geometrical forms to the interpolating functions of an edge or a facet element the finite element equations may also be derived via integral methods. We will show that by applying approximate integration in the finite element formulation for a mesh with rectangular parallelepiped elements classical expressions for magnetic and electric networks emerge. Both magnetic and electric fields are considered, while for the electric fields conducting and displacement currents may be present.

COMPEL: The International Journal for Computation and Mathematics in Electrical and Electronic Engineering
Vol. 31 No. 5, 2012
pp. 1347-1357

0332-1649
DOI 10.1108/03321641211246392

## II. Geometrical representation of interpolating functions of finite elements

Consider the interpolating functions of an eight-node, six-facet element (Figure 1(a)), which allow for the field to be determined at an arbitrary point $P$ within the element. These functions are related to geometrical forms of eight hexahedra $v_i$ defined by drawing straight lines through the point $P$ as shown in Figure 1. It is well known that the ratio of the volume of the $i$th hexahedron to the element volume $V_e$ is related to the $i$th interpolating function of the nodal element, but it is rarely appreciated that the facets and edges of the volume $v_i$ inside the element represent interpolating functions of the associated edge or facet elements. For example, the interpolating function $\boldsymbol{w}_{e4,8}$ of the edge element for the edge $P_4P_8$ expresses the ratio of the facet vector $\boldsymbol{s}_{4,8}$ to the volume $V_e$ (Figure 1(a)), while the ratio $\boldsymbol{r}_i/V_e$ describes the interpolating function $\boldsymbol{w}_{fi}$ of the facet element for the facet $\boldsymbol{S}_i$. In a similar manner the interpolating functions for triangular prisms and pentahedron elements may be expressed. As an example, for the pentahedron of Figure 1(b), the expressions for the interpolating functions of the edge element for the edges $P_4P_5$ and $P_2P_5$ take the form $\boldsymbol{w}_{e4,5} = \boldsymbol{s}_{4,5}/(2V_e)$ and $\boldsymbol{w}_{e2,5} = \boldsymbol{s}_{2,5}/V_e$, respectively, while the interpolating functions of the facet element for the facets $\boldsymbol{S}_3$ and $\boldsymbol{S}_4$ are given by $w_{f3} = \boldsymbol{r}_3/(2V_e)$ and $w_{f4} = \boldsymbol{r}_4/V_e$, respectively.

By analysing the relevant integrals, where the geometrical forms provided are integrands, it may be easily inferred that the volume integral in $V_e$ of the product of $\boldsymbol{w}_{ei,j}$ and the current density vector, or flux density vector, represents current, or flux, associated with the region next to $P_iP_j$. At the same time the volume integral of the product of $\boldsymbol{w}_{fi}$ and the magnetic field strength represents the average value of voltage.

In 2D systems, where the $z$ component of the field is absent, e.g. $H_z = 0$, and the other components are functions of $x$ and $y$ only, the element shown in Figure 1(b) is reduced to a triangle with vertices $P_1P_2P_3$. The four-sided facets of the pentahedron then "collapse" to the sides of a triangle, e.g. the facet $\boldsymbol{S}_3$ becomes the edge $P_2P_3$. The edges $P_iP_{i+3}$ ($i = 1, 2, 3$) of a prism, parallel to the $z$-axis, are represented in 2D by nodes $P_i$. The functions of the edge element for these edges $P_iP_{i+3}$ become in 2D the functions of the nodal element for the nodes $P_i$, while the functions of the facet element for the edges of the prism become similar to the functions of the edge element for the edge of the triangular facets. This similarity arises due to the specific properties of the function describing the pentahedron when the point $P$ lies on the triangular surface of the facet, and thus its $z$ coordinate of point $P$ equals zero. The consequence of this similarity is the fact that the scalar and vector potential formulations in 2D are similar.

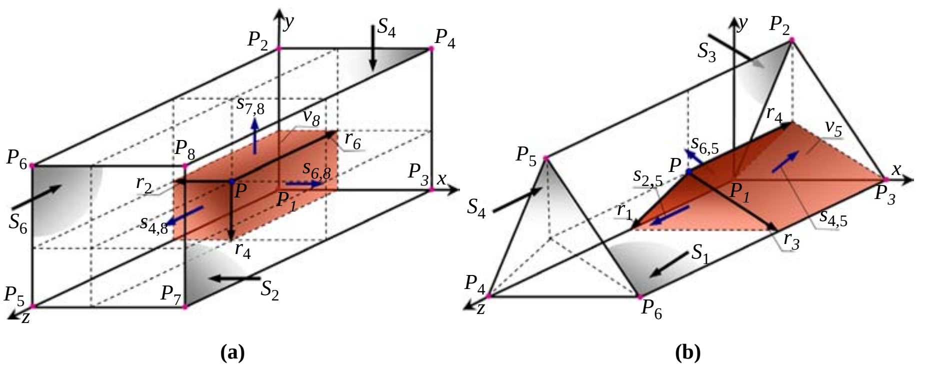


**Figure 1.**
Geometric representation of interpolation functions of hexahedron (a) and nine-edge pentahedron (b)

## III. Integral formulations of the finite element equations

We consider both vector potential ($\boldsymbol{A}$ for the magnetic field and $\boldsymbol{T}$ for the electric field) and scalar potential formulations ($\Omega$ for the magnetic field and $V$ for the electric field). It was noted in Demenko and Sykulski (2002) that the finite element equations formulated for nodal elements and scalar potentials are related to nodal equations of an edge network (EN) with branches assigned to element edges. The equations describing the edge values of vector potentials represent loop (mesh) equations of a facet network (FN) with nodes positioned in element centres and branches passing through the facets. Figure 2 shows the edge and facet models of single hexahedron and pentahedron elements. By appropriate connections between elements we create the network model of the discretised volume. In the case of the EN we make parallel connections between branches associated with common element edges. The FN, on the other hand, involves connecting in series the branches of the facet models of elements with a common facet. We first consider the equations for the edge model, that is the scalar potential formulation using nodal elements.

## IV. Integral formulations for scalar potentials and nodal elements

A single nodal equation of the EN results from the summation of currents or fluxes in the branches having common nodes. The currents $i_{i,j}$ and fluxes $\phi_{i,j}$ in the branch $P_iP_j$ associated with the element edge are described by:

$$i_{i,j} = \int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{J}\,\mathrm{d}v, \quad \phi_{i,j} = \int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{B}\,\mathrm{d}v. \tag{1a, b}$$

Next the constitutive equations are imposed $\boldsymbol{J} = \boldsymbol{\sigma}\boldsymbol{E} + \mathrm{d}(\boldsymbol{\varepsilon}\boldsymbol{E})/\mathrm{d}t$ and $\boldsymbol{B} = \boldsymbol{\mu}\boldsymbol{H}$ and the $\boldsymbol{E}$ and $\boldsymbol{H}$ vectors are expressed in terms of the functions of the edge element, hence:

$$i_{i,j} = \sum_{p,q} u_{Ep,q} \int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{\gamma}\boldsymbol{w}_{ep,q}\mathrm{d}v, \quad \phi_{i,j} = \sum_{p,q} u_{Hp,q} \int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{\mu}\boldsymbol{w}_{ep,q}\mathrm{d}v. \tag{2a, b}$$

In the above equations the summation includes all edges $P_pP_q$ of the element, where $\gamma = \boldsymbol{\sigma} + \mathrm{p}\boldsymbol{\varepsilon}$ ($\mathrm{p} = \mathrm{d}/\mathrm{d}t$), and $u_{Ep,q}$, $u_{Hp,q}$ are the edge values of vectors $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively. These values represent the voltages on the elements of the equivalent element model (Figure 2), $u_{Hp,q}$ is the voltage across the permeance and $u_{Ep,q}$ the voltage

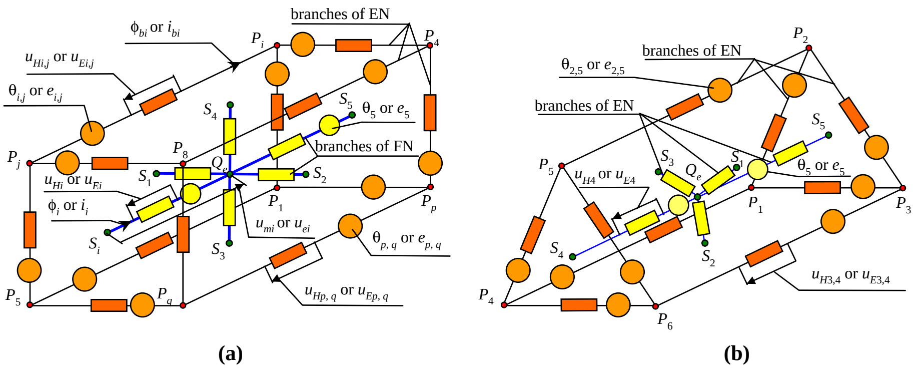


**Figure 2.**
Edge and facet models of hexahedron (a) and nine-edge pentahedron (b)

across the parallel connection of the capacitance and conductance. Relationship (2) suggests that in the equivalent models of the element there exist mutual couplings between branches, as in the expressions for current and flux in the branch $P_iP_j$ we have not only voltages in this branch but also in other branches of the element. Consequently, the mass matrices for nodal element method are non-diagonal.

By using the substitutions:

$$\mathrm{grad}V = \boldsymbol{E} + \mathrm{d}\boldsymbol{A}/\mathrm{d}t\boldsymbol{T}, \quad \mathrm{grad}\Omega = \boldsymbol{H} - \boldsymbol{T}, \tag{3a, b}$$

the edge values of vectors $\boldsymbol{H}$ and $\boldsymbol{E}$ may be expressed as:

$$u_{Ep,q} = V_q - V_p + e_{p,q}, \quad u_{Hp,q} = \Omega_q - \Omega_p + \theta_{p,q}, \tag{4a, b}$$

where $\Omega_q$, $\Omega_p$, $V_q$, $V_p$ are the nodal values of scalar potential for nodes $P_q$, $P_p$, and $\theta_{p,q}$ and $e_{p,q}$ are the edge values of $\boldsymbol{T}$ and $-\mathrm{d}\boldsymbol{A}$/dt, respectively, and represent the sources of magnetic and electric field. Substituting equations (4) to (2) yields:

$$\begin{aligned} i_{i,j} &= i_{si,j} + \sum_{p,q}(V_q - V_p)\int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{\gamma}\boldsymbol{w}_{ep,q}\mathrm{d}v, \\ \phi_{i,j} &= \phi_{si,j} + \sum_{p,q}(\Omega_q - \Omega_p)\int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{\mu}\boldsymbol{w}_{ep,q}\mathrm{d}v, \end{aligned} \tag{5a, b}$$

where $i_{si,j}$ is the current source and $\phi_{si,j}$ is the flux source related to branch $P_iP_j$:

$$i_{si,j} = \sum_{p,q} e_{p,q}\int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{\gamma}\boldsymbol{w}_{ep,q}\mathrm{d}v, \quad \phi_{si,j} = \sum_{p,q}\theta_{p,q}\int_{V_e} \boldsymbol{w}_{ei,j}\boldsymbol{\mu}\boldsymbol{w}_{ep,q}\mathrm{d}v. \tag{6a, b}$$

It has already been noted that a single equation of the nodal element formulation may be related to a nodal equation of the EN and is found by equalling to zero the sum of currents or fluxes in branches with a common node. For the node $P_i$ we may therefore write:

$$\sum_j i_{i,j} = 0, \quad \sum_j \phi_{i,j} = 0. \tag{7a, b}$$

where $j$ is the node index $P_j$ ($j = 1,2,\ldots,n$) of all $n$ branches $P_iP_j$ containing the node $P_i$. Substituting equations (5) into (7) and some further manipulation results in finite element formulation in terms of the scalar potential, which may be written as:

$$\begin{aligned} \sum_j\sum_{p,q}(V_q - V_p)\int_{V_{ej}} \boldsymbol{w}_{ei,j}\boldsymbol{\gamma}\boldsymbol{w}_{ep,q}\mathrm{d}v &= -\sum_j i_{si,j}, \\ \sum_j\sum_{p,q}(\Omega_q - \Omega_p)\int_{V_{ej}} \boldsymbol{w}_{ei,j}\boldsymbol{\mu}\boldsymbol{w}_{ep,q}\mathrm{d}v &= -\sum_j \theta_{si,j}, \end{aligned} \tag{8a, b}$$

where $V_{ej}$ is the volume of the element containing the edge $P_iP_j$.

## V. Integral formulations for vector potentials and edge elements

The derivations of the last section referred to the final element equations formulated in terms of the scalar potentials. The substitutions equations (3) are also used in the vector potential formulations. A volume integral then needs to be considered of the products of the edge element functions and relevant terms in equations (3). Both sides of equations (3) are multiplied by a function $\boldsymbol{w}_{fi}$ of the facet element for the $i$th facet and the resultant expressions are integrated over the element volume, which ultimately leads to:

$$\begin{aligned}\int_{V_e} \boldsymbol{w}_{fi}\mathrm{grad}\Omega \mathrm{d}v &= \int_{V_e} \boldsymbol{w}_{fi}\boldsymbol{H}\mathrm{d}v - \int_{V_e} \boldsymbol{w}_{fi}\boldsymbol{T}\mathrm{d}v,\\ \int_{V_e} \boldsymbol{w}_{fi}\mathrm{grad}V\mathrm{d}v &= \int_{V_e} \boldsymbol{w}_{fi}\boldsymbol{E}\mathrm{d}v + \int_{V_e} \boldsymbol{w}_{fi}(\mathrm{d}\boldsymbol{A}/\mathrm{d}t)\mathrm{d}v.\end{aligned} \tag{9a, b}$$

It can be shown that:

$$\int_{V_e} \boldsymbol{w}_{fi}\mathrm{grad}\Omega\mathrm{d}v = \Omega_{Si} - \Omega_Q, \quad \int_{V_e} \boldsymbol{w}_{fi}\mathrm{grad}V\mathrm{d}v = V_{Si} - V_Q, \tag{10a, b}$$

where $\Omega_{Si}$ and $V_{Si}$ are average values of potentials $\Omega$ and $V$ for the $i$th facet, respectively, assigned in Figure 2 to the node $S_i$, while $\Omega_Q$ and $V_Q$ denote average values of potentials $\Omega$ and $V$ in the element, associated with the node $Q_e$. Through using equations (10), relationship (9) may be written as:

$$\Omega_{Si} - \Omega_Q = u_{mi} = u_{Hi} - \theta_i, \quad V_{Si} - V_Q = u_{ei} = u_{Ei} - e_i, \tag{11a, b}$$

where:

$$u_{Hi} = \int_{V_e} \boldsymbol{w}_{fi}\boldsymbol{H}\mathrm{d}v, \quad u_{Ei} = \int_{V_e} \boldsymbol{w}_{fi}\boldsymbol{E}\mathrm{d}v, \tag{12a, b}$$

$$\theta_i = \int_{V_e} \boldsymbol{w}_{fi}\boldsymbol{T}\mathrm{d}v, \quad e_i = -\mathrm{d}\left(\int_{V_e} \boldsymbol{w}_{fi}\boldsymbol{A}\mathrm{d}v\right)/\mathrm{d}t. \tag{13a, b}$$

Equations (11) describe inter-nodal voltages $u_{mi}$, $u_{ei}$ for the branch $Q_eS_i$ of the FN (Figure 2). The terms $u_{Hi}$ and $u_{Ei}$ represent voltages across the reluctance and across the impedance of the given branch, respectively, whereas $\theta_I$ and $e_i$ are the branch magnetomotive force (*mmf*) and the electromotive force (*emf*).

Analysing the geometrical form describing the function of the facet element reveals that the integrals in equations (13), describing the *mmf* and *emf*, may be viewed as the edge values of the potentials $\boldsymbol{T}$ and $\boldsymbol{A}$ for the edge described by a vector $\boldsymbol{r}_i$ which starts at the centre $S_i$ of the facet and ends in the middle $Q_e$ of the element. The mentioned edge values may be treated as loop currents and fluxes in a mesh of the EN, associated with the facet $\boldsymbol{S}_i$. For example, in the model of Figure 2(a) such a loop is made up of the branches with nodes $P_jP_8P_qP_5$.

When formulating the expressions for voltages $u_{Hi}$ and $u_{Ei}$, across the reluctance and impedance, the constitutive equations are used $\boldsymbol{H} = \boldsymbol{\mu}^{-1}\boldsymbol{B}$ and $\boldsymbol{E} = \boldsymbol{\gamma}^{-1}\boldsymbol{J}$, whereas to describe the flux density $\boldsymbol{B}$ and current density $\boldsymbol{J}$ the functions of the facet element are used. Substituting yields:

$$u_{Hi} = \sum_q \phi_q \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{\mu}^{-1} \boldsymbol{w}_{fq} \mathrm{d}v, \quad u_{Ei} = \sum_q i_q \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{\gamma}^{-1} \boldsymbol{w}_{fq} \mathrm{d}v \tag{14a, b}$$

The summation above refers to all the facets of the element, the integrals (under the summation) describe equivalent reluctances and impedances, respectively, while $\phi_q$ and $i_q$ are the facet values of the flux density $\boldsymbol{B}$ and current density $\boldsymbol{J}$ for the $q$th facet, respectively. Expressions (14) suggest that in the facet model of the element couplings between branches may exist, as indeed was the case for the edge model. Consequently, the mass matrices for edge element equations are also not diagonal, as is the case for the nodal elements.

When setting up the equations for the edge element method the facet quantities are expressed in terms of the edge values. The following relationships hold for the $q$th facet:

$$\phi_q = \sum_{r,j} \varphi_{r,j}, \quad i_q = \sum_{r,j} i_{or,j}, \tag{15a, b}$$

where $\varphi_{r,j}$ and $i_{or,j}$ are the edge values of $\boldsymbol{A}$ and $\boldsymbol{T}$, respectively, for the edge $P_rP_j$ of the facet $\boldsymbol{S}_q$. For example, after applying equation (15a) to the facet $\boldsymbol{S}_1$ of Figure 1(b) we find $\phi_1 = \varphi_{2,3} + \varphi_{3,6} + \varphi_{6,5} + \varphi_{5,2}$. The edge values of the potentials $\boldsymbol{A}$ and $\boldsymbol{T}$ represent the fluxes and currents in loops around the edge, that is in the loops of the FN. Therefore equations (15) expresses the flux/current in the $q$th branch of the FN in terms of fluxes/currents in loops around element edges. It should also be noted that the symbols used here are consistent with the descriptions of values in equivalent magnetic and electric networks; for example, the edge value of $\boldsymbol{T}$ represents the mesh current in the loop around the element edges, consequently a symbol similar to the symbol of current has been applied.

The equation of the edge element method for the edge $P_iP_j$ is found by summing up the voltages $u_{mi}$ and $u_{ei}$ for all branches $Q_eS_i$ around that edge, that is for those branches for which the node $S_i$ is related to the facet $\boldsymbol{S}_i$ with the edge $P_iP_j$. The resultant sums are then equated to zero:

$$\sum_i u_{mi} = 0, \quad \sum_i u_{ei} = 0 \tag{16a, b}$$

Incorporating equations (11) to (16) into the above results in:

$$\begin{aligned} \sum_i \sum_q \sum_{r,j} \varphi_{r,j} \int_{V_{ei}} \boldsymbol{w}_{fi} \boldsymbol{\mu}^{-1} \boldsymbol{w}_{fq} \mathrm{d}v &= \sum_i \theta_i, \\ \sum_i \sum_q \sum_{r,j} i_{or,j} \int_{V_{ei}} \boldsymbol{w}_{fi} \boldsymbol{\gamma}^{-1} \boldsymbol{w}_{fq} \mathrm{d}v &= \sum_i e_i. \end{aligned} \tag{17a, b}$$

The right-hand sides of these equations represent the resultant loop *mmf* $\theta_{or,j}$ and loop *emf* $e_{or,j}$.

These resultant *mmf* and *emf* may be established:

(a) from edge values of $\boldsymbol{T}$ and $\mathrm{d}\boldsymbol{A}/\mathrm{d}t$; or

(b) facet values of $\boldsymbol{J}$ and $\mathrm{d}\boldsymbol{B}/\mathrm{d}t$.

In the former case the potentials $\boldsymbol{T}$ and $\boldsymbol{A}$ in equations (13) are expressed in terms of their edge values. After substitution we find:

$$\theta_i = \sum_{r,j} i_{or,j} \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{w}_{er,j} \mathrm{d}v, \quad e_i = -\mathrm{d}\left( \sum_{r,j} \varphi_{r,j} \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{w}_{er,j} \mathrm{d}v \right) / \mathrm{d}t. \tag{18a, b}$$

The integral in equations (18) is dimensionless and may be treated as the weight parameter $\vartheta_{r,j}^{(i)}$:

$$\vartheta_{r,j}^{(i)} = \int_{V_e} \boldsymbol{w}_{fi} \boldsymbol{w}_{er,j} \mathrm{d}v, \tag{19}$$

This parameter defines the weight of the loop current/flux around the edge $P_rP_j$ (in the loop of the FN) in the expression describing the current/flux in the loop of the EN associated with the facet $\boldsymbol{S}_i$. The parameter $\vartheta_{r,j}^{(i)}$ is also used in the above mentioned approach (b) where the loop *mmf*s and *emf*s are established on the basis of the facet values, leading to:

$$\theta_{or,j} = \sum_q \vartheta_{r,j}^{(q)} i_q, \quad e_{or,j} = -\mathrm{d}\left( \sum_q \vartheta_{r,j}^{(q)} \phi_q \right) / \mathrm{d}t \tag{20a, b}$$

The summation here refers to all facets $\boldsymbol{S}_q$ of elements sharing the common edge $P_rP_j$, while the parameter $\vartheta_{r,j}^{(i)}$ is a weight with which the current/flux passing through the loop of the EN – hence through the facet $\boldsymbol{S}_i$ – is taken when the current/flux passing through the loop of the FN (the loop around the edge $P_rP_j$) is calculated. As may be seen in Figure 1 the significant proportion of the vectors representing the functions of the facet element is perpendicular to the vectors representing the functions of the edge elements. Thus, the scalar product $\boldsymbol{w}_{fi} \boldsymbol{w}_{er,j}$ of such vectors equals zero and the resulting weight parameter is also zero. For the element of Figure 1(a) the only non-zero (and in fact equal to 1/8) weights are those which related to transformations between quantities related to facets and edges with parallel vectors.

All the component terms discussed above appearing in the equations for the nodal element method using scalar potential formulations and the edge element method using vector potentials formulations are tabulated and presented in Table I.

## VI. Approximate description of integrals in FE equations

The integrals describing coefficients of the finite element method may be easily established using accurate analytical methods, but only for regular multi-sided elements and linear materials. Irregular elements and non-linear material properties inevitably call for approximate numerical methods. The authors of the article recommend the following approximation:

$$\int_{V_e} f \mathrm{d}v = \frac{V_e}{n_n} \sum_{i=1}^{n_n} f(P_i), \tag{21}$$

where $n_n$ is the number of element nodes $P_i$, and $f(P_i)$ is the value of the function $f$ in the node $P_i$ (Demenko *et al.*, 2010). Application of this approximation results in significant simplification of the description of the coefficients of a mesh made up of parallelepiped elements. As an example, consider the coefficients of the equations

**Table I.**
Components of the FEM equations

| Potential, type of network | FE equations | FE equation components | Description of the integrands | FE coefficients | Substitutions, entries to RHS vector |
|---|---|---|---|---|---|
| $\Omega$, edge-magnetic | $\sum \phi_{i,j} = 0$ | $\phi_{i,j} = \int_{V_e} \boldsymbol{w}_{ei,j} \boldsymbol{B} \mathrm{d}v$ | $\boldsymbol{B} = \mu \boldsymbol{H} = \mu \sum \boldsymbol{w}_{ep,q} u_{Hp,q}$ | $\int_{V_e} \boldsymbol{w}_{ei,j} \mu \boldsymbol{w}_{ep,q} \mathrm{d}v$ | $u_{Hp,q} = \Omega_q - \Omega_p + \theta_{p,q}, \quad \sum \theta_{p,q}$ |
| $V$, edge-electric | $\sum i_{i,j} = 0$ | $i_{i,j} = \int_{V_e} \boldsymbol{w}_{ei,j} \boldsymbol{J} \mathrm{d}v$ | $\boldsymbol{J} = \nu \boldsymbol{E} = \gamma \sum \boldsymbol{w}_{ep,q} u_{Ep,q}$ | $\int_{V_e} \boldsymbol{w}_{ei,j} \gamma \boldsymbol{w}_{ep,q} \mathrm{d}v$ | $u_{Ep,q} = V_q - V_p + e_{p,q}, \quad \sum e_{p,q}$ |
| $\boldsymbol{A}$, facet-magnetic | $\sum u_{mi} = 0$ | $u_{mi} = \int_{V_e} \boldsymbol{w}_{fi} \mathrm{grad}\Omega \mathrm{d}v$ | $\mathrm{grad}\Omega = \nu \boldsymbol{B} - \boldsymbol{T} = \nu \sum w_{fq} \phi_q - \boldsymbol{T}$ | $\int_{V_e} \boldsymbol{w}_{fi} \nu \boldsymbol{w}_{fq} \mathrm{d}v$ | $\phi_q = \sum \varphi_{r,j}, \quad \sum \int_{V_e} w_{fi} \boldsymbol{T}$ |
| $\boldsymbol{T}$, facet-electric | $\sum u_{ei} = 0$ | $u_{ei} = \int_{V_e} \boldsymbol{w}_{fi} \mathrm{grad} V \mathrm{d}v$ | $\mathrm{grad} V = \rho \boldsymbol{J} + \mathrm{p}\boldsymbol{A} = \rho \sum w_{fq} i_q + p\boldsymbol{A}$ | $\int_{V_e} \boldsymbol{w}_{fi} \rho \mathbf{w}_{fq} \mathrm{d}v$ | $i_q = \sum i_{or,j}, \quad \sum \int_{V_e} w_{fi} \mathrm{p} \boldsymbol{A}$ |

**Notes:** $\phi_{i,j}$, $i_{i,j}$ are fluxes and currents associated with edges $P_iP_j$ of common node; $u_{Hp,q}$, $u_{Ep,q}$ are edge values $\boldsymbol{H}$ and $\boldsymbol{E}$ for edges $P_pP_q$; $u_{mi}$, $u_{ei}$ are voltages associated with facets $S_i$ of common edge; $\phi_q$, $i_q$ are facet values of $\boldsymbol{B}$ and $\boldsymbol{J}$ expressed by edge values of $\boldsymbol{A}$ and $\boldsymbol{T}$; $\varphi_{i,j}$, $i_{oi,j}$ are edge values of $\boldsymbol{A}$ and $\boldsymbol{T}$ for $P_iP_j$; $\gamma = \sigma + \mathrm{p}\varepsilon$; $\mathrm{p} = \mathrm{d}/\mathrm{d}t$; $\rho = 1/\gamma$

describing the magnetic field in a magnetically non-linear region. When a scalar potential formulation $\Omega$ is employed the equation coefficients represent the permeance:

$$\Lambda_{i,j}^{(p,q)} = \int_{V_e} \boldsymbol{w}_{ei,j} \mu \boldsymbol{w}_{ep,q} \mathrm{d}v, \tag{22}$$

whereas if a vector potential $\boldsymbol{A}$ is used they represent the reluctance:

$$R_i^{(q)} = \int_{V_e} \boldsymbol{w}_{fi} \nu \boldsymbol{w}_{fq} \mathrm{d}v, \tag{23}$$

where $\mu = \mu(\boldsymbol{H})$ and $\nu = \mu^{-1} = \nu(\boldsymbol{B})$.

Applying equation (21) yields:

$$\Lambda_{i,j}^{(i,j)} = \mu_{i,j} V_e / l_{i,j}^2, \quad \Lambda_{i,j}^{(p,q)} = 0 \quad \text{for } i,j \neq p,q, \tag{24a, b}$$

$$R_i^{(i)} = v_i V_e / S_i^2, \quad R_i^{(q)} = 0 \quad \text{for } i \neq q, \tag{25a, b}$$

where $l_{i,j}$ is the length of the edge $P_iP_j$, $S_i$ is the surface area of the facet $\boldsymbol{S}_i$, $\mu_{i,j}$ describes the magnetic permeability in the proximity of the edge $P_iP_j$, and $\nu_i$ is the reluctivity of the medium in the area close to the facet $S_i$. For a system with magnetic non-linear characteristics:

$$\mu_{i,j} = 0.5(\mu(\boldsymbol{H}_i) + \mu(\boldsymbol{H}_j)), \quad \nu_i = 0.25 \sum_{p=1}^{4} \nu(B_p), \tag{26a, b}$$

where $\boldsymbol{H}_i$ is the field intensity in the proximity of the node $P_i$ of the edge $P_iP_j$, and $\boldsymbol{B}_p$ the flux density in the $p$th corner $P_p$ of the facet $\boldsymbol{S}_i$. $\boldsymbol{H}_i$ may be found from the edge values of $\boldsymbol{H}$, while $\boldsymbol{B}_p$ from the facet values of $\boldsymbol{B}$. For example, for Figures 1(a) and 3 the value of $\boldsymbol{H}_i$ in the surrounding region close to $P_i$ and the flux density $\boldsymbol{B}_p$ in the proximity of $P_q$ may be calculated as:

$$\boldsymbol{H}_i = \mathbf{1}_x u_{Hi,p}/h_x + \mathbf{1}_y u_{H1,i}/h_y + \mathbf{1}_z u_{Hi,j}/h_z \tag{27}$$

$$\boldsymbol{B}_p = \mathbf{1}_x \phi_2/(-h_y h_z) + \mathbf{1}_y \phi_4/(-h_x h_z) + \mathbf{1}_z \phi_i/(h_x h_y) \tag{28}$$

When deriving the above equations it has been taken into account that the facet vectors are directed "into" the element, whereas the direction of the edges depends on the sequence of the indices in the description of the edge value (Figures 1(a) and 2(a)).

A closer inspection of equations (24) and (25) reveals that in the models following from application of the recommended approximating formula (21) there are no couplings between branches; hence the resulting mass matrix is diagonal and the coefficients express only self-permeances/reluctances. Expressions (24a) and (25a) describing these permeances/reluctances are identical to those obtained form a classical formulation using magnetic networks, whose parameters may be established via the tubes and slices method (Hammond and Sykulski, 1994). For a hexahedron the geometrical forms are as shown in Figure 3; it is taken into account that the permeance of the branch $P_iP_j$ of the EN is related to the "magnetic conductance" of a block of length $l_{i,j}$ and cross-section $0.25h_xh_y$ (Figure 3(a)). The reluctance in the branch $Q_eS_i$ of the FN, on the other hand, represents the "magnetic resistance" of a block of length $0.5l_{i,j}$ and cross-section $h_xh_y$.

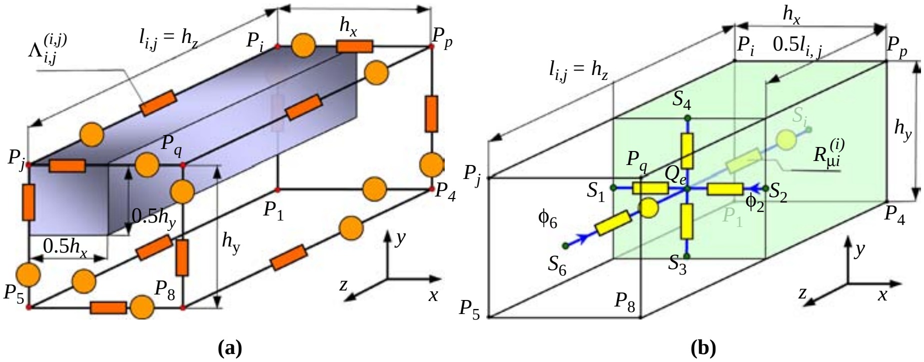


**Figure 3.**
Geometric forms representing permeance (a) and reluctance (b)

## VII. Conclusion

Finite element equations may be derived from equivalent circuit models without the need for variational formulation. By exploiting the geometrical properties of interpolating functions the relevant parameters may be established using integral methods. The presented approach is valid not only for hexahedra and pentahedra but also for tetrahedra (Trevisan and Kettunen, 2004) and mixed finite elements (Dular *et al.*, 1995). The proposed approach promises to be very beneficial in teaching, especially to students well familiar with circuit methods, to whom the analogy of the finite element formulation to loop or nodal magnetic or electric networks may be appealing and easier to understand. Thus, the teaching of computational electromagnetics may be seen as supplementing the circuit theory by the relevant information about the integral methods of calculating network model parameters, as argued in this paper. The presented methods are also helpful when formulating classical network models, such as described in Amrhein and Krein (2009).

## References

Amrhein, M. and Krein, P.T. (2009), "Force calculation in 3-D magnetic equivalent circuit networks with a Maxwell stress tensor", *IEEE Transactions on Energy Conversion*, Vol. 24 No. 3, pp. 587-93.

Demenko, A. and Sykulski, J.K. (2002), "Network equivalents of nodal and edge elements in electromagnetics", *IEEE Transactions on Magnetics*, Vol. 38 No. 2, pp. 1305-8.

Demenko, A., Sykulski, J.K. and Wojciechowski, R. (2010), "On the equivalence of finite element and finite integration formulations", *IEEE Transactions on Magnetics*, Vol. 46 No. 8, pp. 3169-72.

Dular, P., Nicolet, A., Genon, A. and Legros, W. (1995), "A discrete sequence associated with mixed finite elements and its gauge condition for vector potential", *IEEE Transactions on Magnetics*, Vol. 31 No. 3, pp. 1356-9.

Hammond, P. and Sykulski, J.K. (1994), *Engineering Electromagnetism, Physical Processes and Computation*, Oxford University Press, Oxford.

Ren, Z. and Qu, H. (2010), "Investigation of the complementarity of dual eddy current formulations on dual meshes", *IEEE Transactions on Magnetics*, Vol. 46 No. 8, pp. 3161-4.

Trevisan, F. and Kettunen, L. (2004), "Geometric interpretation of discrete approaches to solving magnetostatic", *IEEE Transactions on Magnetics*, Vol. 40 No. 2, pp. 361-5.

**About the authors**
Andrzej Demenko is Professor of Electrical Machines and Applied Electromagnetics at Poznan University of Technology, Poland. Currently he is Head of Division of Mechatronics and Electrical Machines and Chairman of the Committee on Electrical Engineering of the Polish Academy of Sciences (PAN). Since 1970 he has been employed in research and education. He has published five books and over 250 conference and journal papers on electromagnetics, electrical machines, numerical methods and applied physics. His personal research is development of field method in analysis and designing of electrical machines and drives. He is chairman of the Steering Committee of Symposiums on Electromagnetic Phenomena in Nonlinear Circuits, Editor-in-chief of *Archives of Electrical Engineering* (PAN) and member of the Steering Committees of several international conferences. Andrzej Demenko is the corresponding author and can be contacted at: andrzej.demenko@put.poznan.pl

Jan. K. Sykulski is Professor of Applied Electromagnetics at the University of Southampton, UK. His personal research is in development of fundamental methods of computational electromagnetics, power applications of high temperature superconductivity, simulation of coupled field systems and design and optimisation of electromechanical devices. He has published over 325 scientific papers and is founding Secretary of the International Compumag Society, Visiting Professor at universities in Canada, France, Italy, Poland and China, Editor of *IEEE Transactions on Magnetics*, Editor-in-chief of *COMPEL* (Emerald) and member of International Steering Committees of several international conferences. He is a Fellow of IEEE (USA), Fellow of the Institution of Engineering and Technology (IET), Fellow of the Institute of Physics (IoP), Fellow of the British Computer Society (BCS) and has an honorary title of Professor awarded by the President of Poland.