Modeling Infrastructure Interdependencies, Resiliency and Sustainability

Tri-Dung Nguyen, Ximing Cai, Yanfeng Ouyang and Mashor Housh*
University of Illinois at Urbana-Champaign

Abstract

The three key concepts of interdependency, resiliency and sustainability of a complex system have appeared in a number of studies and in various contexts. Nevertheless, little has been done to define and analyze them, especially the latter two, in a unified quantitative framework for engineering infrastructures. In this paper, we propose overarching mathematical modeling frameworks to quantify these three key concepts in the context of complex infrastructure systems with multiple interdependent subsystems (i.e., the system of systems). We show how interdependencies between subsystems can affect the resiliency and sustainability of the entire system. We provide a case study in the context of biofuel development and use different dynamical models to demonstrate these concepts.

Keywords: Systems, complex, critical, infrastructure, interdependencies, resiliency, sustainability, Markov jump linear systems.

1 Introduction

Recent catastrophic events, such as the World Trade Center Disaster, Hurricane Katrina, and Northridge earthquake have alerted infrastructure designers and policy makers about the interdependencies between critical infrastructures in complex systems [23, 25, 22, 18]. Since then, many researchers have pointed out the importance of resilient and sustainable system designs. The terms of resiliency and sustainability have been used widely in many different contexts and, in most of the cases, these key words were defined qualitatively and separately. In fact, these terms have been used so extensively that their meanings are often inconsistent or even contradicting each other across different fields. There is also a lack of consistent analytical modeling frameworks that are suitable for designing resilient and sustainable engineering infrastructure systems. In view of such gaps, our paper attempts to provide not only a consistent set of definitions for these terms but also an underlying mathematical framework to analyze them. This framework allows us to provide interesting insights into the system behavior, particularly on how interdependencies between subsystems might change the system resiliency and sustainability.

With the primary goal of providing a mathematical framework for defining and quantifying interdependencies, resiliency and sustainability (IRS), our approach is different from those of most existing studies. First, we recognize that the resiliency and sustainability of a complex system of subsystems are directly affected by the interdependencies between the subsystems. This motivates us to study these three key concepts in one unified framework. Second, our work presents analytical tools to study the system at both the design and operational stages. The mathematical framework provides us with an understanding about the system's expected behavior as well as ways to mitigate consequences from extreme events before actual implementation. This feature distinguishes our paper from existing works such as [6, 25], which mainly focus on empirical case studies through the use of existing systems and the analysis of past events.

*Correspondence to: Ximing Cai, Department of Civil and Environmental Engineering, University of Illinois at Urbana-Champaign. E-mail: xmcai@uiuc.edu. Tel: (217) 333-4935

As a motivating example, we consider the complex infrastructure system that supports the rapid development of biofuel. This system consists of interdependent subsystems underlying biomass production, biofuel refinement and water supply. Figure 1 shows the overlaying interdependencies between these subsystems. The nation's increasing demand on biofuel as a source of clean energy exaggerates vulnerabilities of these critical supporting infrastructures. For example, biomass production and biofuel refinement will consume a large amount of water as well as affecting water quality. Therefore, studying the interdependencies, resiliency and sustainability of this complex system will provide us with 1) a better understanding of the expected dynamical behavior of the entire system, 2) guidelines for appropriate system design to improve system resiliency and sustainability, and 3) quantification of cascading effects from possible subsystem failures as well as strategies to mitigate these effects.

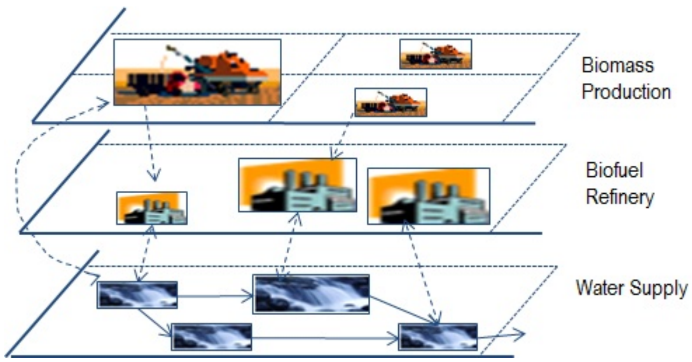


Fig. 1: Overlaying interdependent subsystems generates the complex "system of subsystems" for biofuel development.

For the ease of demonstration, we simplify our numerical example by aggregating all the farms into one biofuel production subsystem, all the refineries into one refinery subsystem and the entire water supply into one water subsystem. The purpose of this simple example is to illustrate how the key concepts of interdependencies, resiliency and sustainability can be modeled and quantified in a unified mathematical framework. It shall be noted, nevertheless, that the proposed framework can be extended to more complex systems and the same IRS metrics can be quantified. For example, it is possible to consider a more complex system with a spatial dimension within each subsystem, such as one with an underlying river network, and multiple refineries and farms. In that case the system would have a clearer and more realistic geography, but in this explorative study we keep the system relatively simple for the sake of illustrating the mathematical framework and generating insights.

Our paper is structured as follows: Section 2 starts with the system dynamics and the mathematical definition of interdependencies. Then, Sections 3 and 4 respectively present the conceptual development, definitions, and measures for resiliency and sustainability. Section 5 demonstrates how the key concepts of interdependencies, resiliency and sustainability can be modeled and quantified by using different dynamical system models such as a linear system with sudden losses and a Markov jump linear system. Section 6 presents the biofuel development case study and demonstrates how the mathematical framework can be applied to the complex system to draw insights. Section 7 concludes our paper.

2 Complex Systems of Subsystems & Interdependencies

2.1 Notations

We use the following notations:

- n: Number of subsystems.
- $\boldsymbol{s}_i(t), \boldsymbol{u}_i(t)$: State and control variables of subsystem i at time t.

- $\boldsymbol{s}(t), \boldsymbol{u}(t)$: State and control variables of the system at time t.
- $\mathcal{I}_{ij}(t)$: Dependency of subsystem i on subsystem j at time t.
- $\mathcal{I}(t)$: Interdependency matrix at time t.
- $\mathcal{R}$: Resiliency of the systems (there are different aspects of resiliency, which are denoted as $\mathcal{R}_{op}$ for operational resiliency, $\mathcal{R}_{rt}$ for recovery time resiliency, $\mathcal{R}_{rs}$ for recovery speed resiliency, and $\mathcal{R}_{ad}$ for adaptivity resiliency).
- κ: Sustainability of the system.
- $\mathcal{F}$: Set of possible failure scenarios under consideration.
- K: Number of failure scenarios.
- $\boldsymbol{p}$: Probability of failures.
- $\boldsymbol{\epsilon}, \boldsymbol{L}$: Randomness of the system ($\boldsymbol{\epsilon}$ is often the white noise while $\boldsymbol{L}$ is often a sudden jump).
- m: Number of modes of operations of the Markov jump linear system.
- $\boldsymbol{\pi}, \boldsymbol{\theta}$: Transition probability and the steady state probability of the Markov jump linear system.

We use bold symbols for vectors and matrices. We use i, j, k as general subscripts or indices, t, τ, T as time quantities or indices, and B, R, W, Q as subscript indices of the subsystems in the biofuel case study.

2.2 Complex System Dynamics

Consider a complex system of n subsystems $A_1, A_2, \ldots, A_n$, which form a network with n nodes. The interaction between these subsystems are represented by directed links in the network. For example, if A_i is directly dependent on A_j, then there is a directed link ($A_j \to A_i$) from subsystem A_j to subsystem A_i. Let $N\{i\}$ be the set of indices of the neighboring subsystems that have directed links to subsystem A_i.

Let $\boldsymbol{s}_i(t)$, $\boldsymbol{u}_i(t)$, and $\boldsymbol{\epsilon}_i(t)$ be the vectors of state variables, control (decision) variables, and randomness of subsystem A_i at time t. Suppose the dynamics of subsystem A_i is represented by the following dynamical equation:

$$\boldsymbol{s}_i(t+1) \quad = \quad h_i^t\left(\boldsymbol{s}_i(t), \boldsymbol{u}_i(t), \boldsymbol{s}_{N\{i\}}(t), \boldsymbol{\epsilon}_i(t)\right),$$

where $\boldsymbol{s}_{N\{i\}}(t)$ is the state of the neighboring subsystems $A_{N\{i\}}$ at time t. Let $d_i = dim(\boldsymbol{s}_i(t))$ and $k_i = dim(\boldsymbol{u}_i(t))$ be the dimensions of $\boldsymbol{s}_i(t)$ and $\boldsymbol{u}_i(t)$, respectively. Here, h_i^t are some known functions (e.g. linear functions with known parameters in as shown in subsections 5.1 and 6.1).

For example, in the case of biofuel development, we consider three main subsystems: biomass production, biofuel refinery and water supply, which are denoted as B, R and W respectively. Figure 1 shows the interactions among these subsystems. The biomass production subsystem takes water from the water supply subsystem to produce biomass, and then it sends the biomass output to the refinery subsystem. The refinery subsystem takes biomass and water from the biomass production and the water supply subsystems and the produced biofuel is then transshipped to the consumers. The biomass production and biofuel refinery subsystems also affect water quality through fertilization and waste water discharges. Based on these interdependencies, the system dynamics for this system can be represented as follows:

$$\begin{aligned}
\boldsymbol{s}_B(t+1) &= h_B^t\left(\boldsymbol{s}_B(t), \boldsymbol{u}_B(t), \boldsymbol{s}_W(t), \boldsymbol{\epsilon}_B(t)\right), \\
\boldsymbol{s}_R(t+1) &= h_R^t\left(\boldsymbol{s}_R(t), \boldsymbol{u}_R(t), \boldsymbol{s}_B(t), \boldsymbol{s}_W(t), \boldsymbol{\epsilon}_R(t)\right), \\
\boldsymbol{s}_W(t+1) &= h_W^t\left(\boldsymbol{s}_W(t), \boldsymbol{u}_W(t), \boldsymbol{s}_R(t), \boldsymbol{\epsilon}_W(t)\right).
\end{aligned}$$

The detailed formulations of $\boldsymbol{s}_B, \boldsymbol{s}_R$ and $\boldsymbol{s}_W$ are presented in the case study in section (6).

2.3 Interdependencies in Systems of Subsystems

There is a rich literature about interdependencies in critical infrastructures in the wake of catastrophic events in the last two decades. Qualitatively, Rinaldi et al. [22] categorize interdependencies into the following four types: a) *physical interdependencies*, b) *cyber interdependencies*, c) *geographic interdependencies*, and d) *logical interdependencies*.

Quantitatively, Zimmerman [25] defines the dependency index D_i of infrastructure i as the empirical effect ratio between the number of times that infrastructure i fails causing others to fail (C_i), and the number of times that infrastructure i fails due to the failure of other infrastructures (E_i), i.e. $D_i = C_i/E_i$. Zimmerman and Restrepo [26] use another metric to define the dependencies of infrastructures on electricity. They use the ratio between the affected duration T_i of infrastructure i and the electric outage duration T_e to model the infrastructure dependency on electricity, i.e. $D_i = T_i/T_e$.

Casalicchio and Galli [5] generalize the work by Zimmerman and Restrepo [25, 26] to categorize interdependencies into shape metrics, core metrics and sector specific metric depending on the information content and decision support. The shape metrics are defined similarly to that in [25, 26] while the core metrics go into the component level of the infrastructure.

Haimes et al. [12] develope a method based on the Leontief input-output framework [15] to model the failure of interdependent systems. The method defines the operability of system as linear relation to input arguments of the system. For interdependent systems, the input arguments are outputs of other systems. The interdependencies between systems can thus be expressed a matrix of proportionality coefficients. The above method was applied to estimate economic losses resulting from electromagnetic attacks in Haimes et al. [11, 10]. Many studies utilized the idea of using a network of nodes and links to represent the interdependent systems [16, 8, 9, 20, 14]. The network representation facilitates the use different types of links to represent different categories of dependencies (physical, informational, geo-spatial, policy/procedural, and societal interdependencies). An extensive survey of the existing methods for modeling interdependencies can be found in Pederson et al. [21].

In this paper, the interdependencies between the subsystems are represented by the links connecting them. Under our notation, subsystem A_i is dependent on subsystems $A_{N\{i\}}$. Subsystems A_i and A_j are interdependent if A_i is dependent on A_j and A_j is also dependent on A_i. In the case of biofuel development, B is dependent on W; R is dependent on (B, W); and W is dependent on R. In addition, the pairs (W, B) and (W, R) are interdependent.

Definition 1. *The* ***first order (direct) dependency*** $\mathcal{I}_{ij}(t)$ *between subsystems* A_i *and* A_j *at time* t *is defined as follows:*

$$\mathcal{I}_{ij}(t) \quad = \quad \frac{\partial \boldsymbol{s}_i(t+1)}{\partial \boldsymbol{s}_j(t)} = \frac{\partial h_i^t}{\partial \boldsymbol{s}_j(t)}. \tag{1}$$

Here, $\mathcal{I}_{ij}(t) \in R^{d_i \times d_j}$. *The* ***interdependency matrix*** $\mathcal{I}(t)$ *is defined as a big matrix of size* $d \times d$ *that is composed of blocks* $\mathcal{I}_{ij}(t)$*, where* $d = \sum_{i=1}^n d_i$.

Equation (1) defines the dependency of subsystem A_i on subsystem A_j as the change in A_i in the next period given one unit change in A_j in the current period. For example, the dependency of biomass production on water supply could be the yield increase in the next period (e.g. next month) given one unit increase in the rainfall this period (e.g. this month).

Remarks:

- Subsystem A_i is often also dependent on itself and the dependency $\mathcal{I}_{ii}(t)$ is defined similarly as shown in equation (1) with $j = i$.
- $\mathcal{I}_{ij}(t)$ represents the first order dependency between subsystems i and j. If $\mathcal{I}_{ij}(t) = 0$, subsystem i is independent of subsystem j, i.e. there is no directed link from A_j to A_i in the system network.
- It is often the case that the interdependency matrix is asymmetric, i.e. it is possible to have $\mathcal{I}_{ij}(t) \neq \mathcal{I}_{ji}(t)'$ for some i, j where $'$ is the matrix transpose operator. We can even see the cases when $\mathcal{I}_{ij}(t) = 0$

while $\mathcal{I}_{ji}(t) \neq 0$. For example, in the case of biofuel development, we have $\mathcal{I}_{BR} = 0$ while $\mathcal{I}_{RB} \neq 0$ since the biofuel refinery subsystem needs input from the biomass production but not vice versa. (Notice that, in reality, biofuel production also affects the decision of the farmers and hence there is an indirect effect from the biofuel subsystem to the biomass production subsystem. However, for clarity, we assume there is no such indirect effect. Nevertheless, we take the cascading effects into consideration as we will show in the remark about cascading effect).

- The number of directed links (or dependency) in the system network is equal to the number of non-zero off-diagonal blocks $\mathcal{I}_{ij}$. If the system network is disconnected, i.e. we can divide it into two groups of subsystems that are disconnected from each other, the matrix $\mathcal{I}$ can be rearranged to form a block diagonal matrix. In this case, the system can be analyzed by studying the two groups separately.

- **Cascading Effect**: The interdependency matrix shown earlier represents the first-order dependencies. The subsystems might also be interdependent through cascading effects. Considering two subsystems A_i and A_j and supposing there is a direct path connecting them and the shortest path length is equal to $(q+1)$, a change in the state of A_j will only affect the state of A_i in at least $q+1$ periods later. We define this effect as the $(q+1)^{th}$ order dependency. We also have a recursive formulation for approximating the $(q+1)^{th}$ order dependencies as follows:

$$\begin{aligned}\mathcal{I}_{ij}^{q+1}(t) &= \frac{\partial \boldsymbol{s}_i(t+q+1)}{\partial \boldsymbol{s}_j(t)} \\ &= \sum_{k=1}^{n} \frac{\partial \boldsymbol{s}_i(t+q+1)}{\partial \boldsymbol{s}_k(t+1)} \frac{\partial \boldsymbol{s}_k(t+1)}{\partial \boldsymbol{s}_j(t)} \\ &= \sum_{k=1}^{n} \mathcal{I}_{ik}^{q}(t+1)\mathcal{I}_{kj}(t).\end{aligned}$$

If the shortest directed path between two subsystems A_i and A_j is through q links, then the (i,j) block of matrix $\mathcal{I}^q$ is non-zero. If there is no directed path between them, then the (i,j) block of matrix $\mathcal{I}^q$ is zero for all q.

3 Resiliency

3.1 Resiliency: Conceptual Development

The term resiliency has been used in many different contexts including cybernetics security and supply chain management. Snyder et al. in [24] model resiliency as the worst case behavior of the system under different possibilities of failures. Brown et al. in [2], [3] present attacker-defender models where a defender aims to find the best design of the system such that the system's worst performance under all the possible attacks is maximized. Resiliency with these models is often measured as the performance of the system under the worst scenarios and is calculated by solving the corresponding robust optimization problems.

In general, resiliency is related to the capability of the system to recover from shocks. According to O'Rourke in [19], resilience is defined in Webster's Unabridged Dictionary as "the ability to bounce or spring back into shape, position, etc., after being pressed or stretched." The National Infrastructure Advisory Council at the Department of Homeland Security [17] defines infrastructure resilience as "the ability to reduce the magnitude and/or duration of disruptive events. The effectiveness of a resilient infrastructure or enterprise depends upon its ability to *anticipate*, *absorb*, *adapt to*, and/or *rapidly recover* from a potentially disruptive event." Bruneau et al. in [6], [4], present the necessary conditions for resiliency in both physical and social systems as having the following four qualities: *robustness*, *redundancy*, *resourcefulness* and *rapidity*. The authors also quantify resiliency as the system average functionality during shock recovery which is measured as a function of the magnitude of losses and their probabilities.

In our paper, we also use the same resiliency definition as in Webster's Unabridged Dictionary and by the National Infrastructure Advisory Council. We also use the similar conceptual framework for defining resiliency as presented by Bruneau et al. in [6], [4]. However, for the purpose of quantification, we extend these definitions and measure resiliency using four different quantifiable criteria as shown in the following definition:

Definition 2. *The **resiliency** of an infrastructure system is its capability to get back to its operational boundary after being affected by disruptions. Four measures of resiliency include:*

1. *The worst/average level of **functionality** degradation, denoted as $\mathcal{R}_{op}$, in the case of failures.*
2. *The capability and the worst/average **recovery time** of the system once experiencing shocks (denoted as $\mathcal{R}_{rt}$).*
3. *The **recovery speed** (denoted as $\mathcal{R}_{rs}$).*
4. *The **adaptability** of the system, denoted as $\mathcal{R}_{ad}$, is its capability to stabilize to a stable state, which could be different from the current stable state, after shock.*

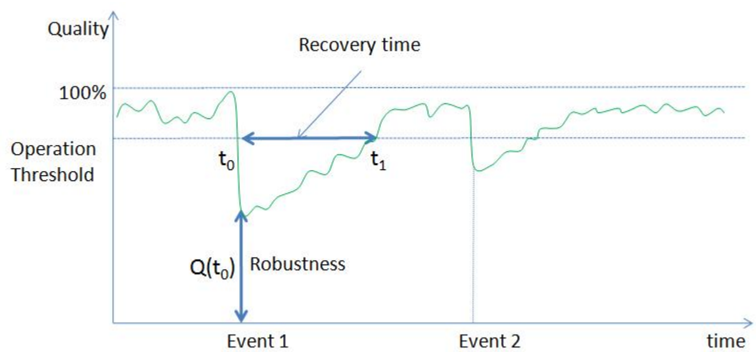


Fig. 2: Conceptual understanding of resiliency as adapted from Figure 1 in Bruneau et. al [6]

Notice that the first two dimensions of resiliency can be used to approximate the system average functionality during shock recovery. In addition, we extend Bruneau et al. framework in [6], [4] and define the system functionality $Q(t)$ as a function of the system states, i.e.

$$Q(t) \equiv Q(\boldsymbol{s}(t)) \ : \ \mathbb{R}^d \to [0, 1].$$

For example, the functionality of a water supply system could be a function of the water quality and water quantity. This extension allows us to achieve our main goal to study how interdependency between subsystems can affect the overall system resiliency as will be shown in Sections 5 and 6. Notice also that concepts such as robustness, redundancy, resourcefulness and rapidity are all captured in the system states $\boldsymbol{s}(t)$.

Figure 2 demonstrates the concept of resiliency which is adapted from Figure 1 in Bruneau et. al [6]. The vertical axis is for the system quality, which is a function of the system states. There is an operational threshold that defines the boundary above which the system is considered normal. Under different events, e.g. Event 1 in Figure 2, the system experiences a shock at time t_0 and its quality drops to a level $Q(t_0)$, which is below the operational threshold. We call $Q(t_0)$ the **robustness** of the system for that particular event. The higher the robustness, the better the system is in terms of avoiding failure consequences. The system then recovers gradually and comes back to the normal zone again at time t_1 and stays there until experiencing another shock at time t_2. We call the duration $t_1 - t_0$ to be the **recovery time**. The **speed of recovery** is defined as the ratio of the loss in system quality and the recovery time and can be approximated as $\frac{1-Q(t_0)}{t_1-t_0}$ for event 1. The system quality $Q(t)$ is often stochastic and the times t_0, t_1, t_2 as well as the values $Q(t_0), Q(t_1), Q(t_2)$ are often random.

Another aspect of resiliency concerns the system's **adaptability**, which represents the system's capability to return to a stable state after being shocked. That stable state could be the same or different from the stable state that the system would reach without shocks. In general, a system is resilient if it has high robustness, a small recovery time, a high speed of recovery, and is adaptable.

3.2 Resiliency: Quantification

We define a set $\mathcal{F}$ of the possible failure scenarios under consideration. We consider only the cases the set $\mathcal{F}$ is nonempty since, otherwise, we say the system is perfectly resilient. Our analysis applies for both discrete and continuous set of $\mathcal{F}$ but for simplification in notation, we only consider the case when $\mathcal{F}$ is discrete. Let $\boldsymbol{s}_j,\ j \in \mathcal{F}$ be the set of possible failure states and let p_j be the corresponding probability of occurrence.

According to Definition 2, we denote the functionality $\mathcal{R}_{op}$ and the recovery time $\mathcal{R}_{rt}$ as the two aspects of resiliency. Depending on the risk measure that we want to apply on the system, the resiliency can be taken as the expected functionality (or recovery time) or the worst case functionality (or recovery time) over all the possible random events within the set $\mathcal{F}$. In the context of supply chain management, we often care about the loss of the system's operability when a facility fails. We often use robust optimization to model these worst case behaviors.

Definition 3. *The average (or the worst)* ***operational resiliency*** $\mathcal{R}_{aop}$ *(or $\mathcal{R}_{wop}$) of a system under the set of failure events $\mathcal{F}$ is measured as the average (or the worst) system quality under those set of failure events, i.e.*

$$\begin{aligned}
\mathcal{R}_{aop} &= \mathbb{E}_{j\in\mathcal{F}}[Q(t) \mid \boldsymbol{s}(t) \equiv \boldsymbol{s}_j] = \sum_{j\in\mathcal{F}} p_j Q(\boldsymbol{s}_j), \\
\mathcal{R}_{wop} &= \min_{j\in\mathcal{F}}[Q(t) \mid \boldsymbol{s}(t) \equiv \boldsymbol{s}_j] = \min_{j\in\mathcal{F}} Q(\boldsymbol{s}_j).
\end{aligned}$$

Definition 4. *The average (or the worst)* ***recovery resiliency*** $\mathcal{R}_{art}$ *(or $\mathcal{R}_{wrt}$) of a system under the set of failure events $\mathcal{F}$ is measured as the average (or the worst) system recovery time under those set of failure events, i.e.*

$$\begin{aligned}
\mathcal{R}_{art} &= \mathbb{E}_{j\in\mathcal{F}}[\tau_j] = \sum_{j\in\mathcal{F}} p_j \tau_j, \\
\mathcal{R}_{wrt} &= \max_{j\in\mathcal{F}}[\tau_j],
\end{aligned}$$

where $\tau_j = [argmin\ \tau \mid Q(t+\tau) \geq 1-\delta,\ \boldsymbol{s}(t) \equiv \boldsymbol{s}_j]$ is the recovery time under failure scenario j and $(1-\delta)$ is the operational threshold of the system. For example, we can set $(1-\delta) = 90\%$ to specify that the system is only recovered if its functionality is at least 90% of its fully functional level.

Definition 5. *The average (or the worst)* ***speed of recovery*** $\mathcal{R}_{ars}$ *(or $\mathcal{R}_{wrt}$) of a system under the set of failure events $\mathcal{F}$ is measured as the average (or the worst) speed of recovery under those set of failure events, i.e.*

$$\begin{aligned}
\mathcal{R}_{ars} &= \mathbb{E}_{j\in\mathcal{F}}[\upsilon_j] = \sum_{j\in\mathcal{F}} p_j \upsilon_j, \\
\mathcal{R}_{wrs} &= \max_{j\in\mathcal{F}}[\upsilon_j],
\end{aligned}$$

where $\upsilon_j = \frac{1-Q_j}{\tau_j}$ is the recovery speed under failure scenario j and Q_j, τ_j are the quality of the system and the recovery time after experiencing failure scenario j.

Another concept that is related to the recoverability of a system is **adaptability** and is defined as follows:

Definition 6. *The **adaptability** $\mathcal{R}_{ad}$ of a system under the set of failure events $\mathcal{F}$ is measured as the capability to stabilize itself (probably to another stable state) after experiencing any shock among those set of events $\mathcal{F}$, i.e.*

$$\mathcal{R}_{ad} = \begin{cases} 1, & \textit{if the system can recover to a stable state,} \\ 0, & \textit{otherwise} \end{cases} .$$

There is a rich literature for studying the stability of dynamical systems, especially on linear systems and Markov jump linear systems. We can also show that the concepts of interdependency, resiliency, controllability, and stability are interrelated. For brevity, we do not show these results in this paper but they are available on request.

Remark: Among the four types of resiliencies defined, $\mathcal{R}_{op}$ and $\mathcal{R}_{rt}$ are the most popular ones and the other types can be approximated by using these two factors. For example, the recovery speed can be approximated as $\frac{1-\mathcal{R}_{op}}{\mathcal{R}_{rt}}$ and the adaptability can be derived by checking if the recovery time is greater than an acceptable threshold. In different systems, we might care more about one type of resiliency than others. For example, in a cyber security network, we care more about $\mathcal{R}_{op}$ but less about $\mathcal{R}_{rt}$ since it is assumed that once the system experiences some extreme events (such as link breakdown), the process for reassignment can be done in a reasonable time using some predetermined recovery protocol. In this case, we only care about the loss of operationality due to failure. In a hospital operation, we care about both the functionality $\mathcal{R}_{op}$ and the recovery time $\mathcal{R}_{rt}$. In general, different systems use different utility functions for these two types of resiliencies. We can define the overall system resiliency as $\mathcal{R} = U(\mathcal{R}_{op}, \mathcal{R}_{rt})$, which is a utility function over $\mathcal{R}_{op}$ and $\mathcal{R}_{rt}$.

Mitigating failure affect: When an extreme event occurs, the response variable $\boldsymbol{u}$ is set to improve the system resiliency and to mitigate the effects. For example, to minimize the recovery time, the following robust optimization can be solved for an optimal response $\boldsymbol{u}$:

$$\begin{aligned} \min_{\boldsymbol{u}} \quad & \tau \\ s.t. \quad & \boldsymbol{s}(0) = \text{extreme case}, \\ & \boldsymbol{s}(t+1) = h(\boldsymbol{s}(t), \boldsymbol{u}(t), \boldsymbol{\epsilon}(t)),\ \forall\ t, \\ & \|\boldsymbol{s}(\tau) - \boldsymbol{\mu}\| \le \epsilon, \\ & \boldsymbol{u}(t) \in B(\boldsymbol{s}(t)), \end{aligned}$$

where $\boldsymbol{\mu}$ is a desirable state (often a stable state of the system), $\boldsymbol{\mu} \pm \epsilon$ is a small acceptance range around $\boldsymbol{\mu}$ to which we want to drive the system state, and $B(\boldsymbol{s}(t))$ is the feasible region of the control $\boldsymbol{u}(t)$. Notice that the preceding model cannot be solved in its current form since the unknown variable τ cannot be used for indexing $\boldsymbol{s}(\tau)$. Instead, we can use a bisection search with some initial guess for τ and then solve the feasibility problem. We then reduce τ if the problem is feasible and increase τ if otherwise.

4 Sustainability

4.1 Sustainability: Conceptual Development

Sustainability can have different meanings in different contexts. In infrastructure systems, sustainability often means the durability, reliability, quality and maintainability of the system. Figure 3 demonstrates the concept of sustainability. The vertical axis is for the system quality, which is a function of the system states. The system is sustainable if its quality does not consistently degrade through time.

Definition 7. *System **sustainability** is its long-term capability to use its limited resources effectively to maintain its functionality and to endure stresses.*

The first part of this definition is concerned with the long-term functionality of the system. This aspect is related to the concept of stability. The second aspect is about the system's capability to bear with stresses. This aspect is related to the concept of resiliency that we have defined in subsection 3.1.

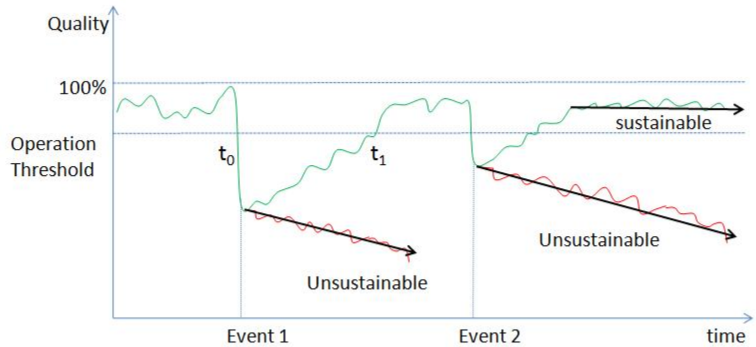


Fig. 3: Conceptual understanding of sustainability

4.2 Sustainability: Quantification

Definition 8. *The sustainability level κ of a system is defined as:*

$$\kappa = \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T} Q(t)\right].$$

Here, $Q(t)$ is the system quality at time t, which is a function of the system state variables. The sustainability is defined as the expected long-term average quality of the system. In order to calculate κ, we need to know the exact formulation for $Q(t)$. For demonstration purpose in the next sections, we choose the quality function to be $Q(t) = \boldsymbol{\omega}'\boldsymbol{s}(t)$ where $\boldsymbol{\omega}$ is a weighting vector on the system state variables $\boldsymbol{s}(t)$ that depends on the relative importance of these state variables. The weight vector $\boldsymbol{\omega}$ allows us put appropriate weights to the state variables when taking sustainability measures. For example, in the biofuel system, the water quality sustainability can be computed by setting $\omega_Q = 1$ while $\omega_j = 0,\ \forall j \neq wq$. Introducing the weight vector $\boldsymbol{\omega}$ allows us to consider different levels of sustainability measures (e.g. local sustainability or global sustainability). Notice that we might need to use some normalization operation on $\boldsymbol{\omega}'\boldsymbol{s}(t)$ to make sure $Q(t) \in [0, 1]$. However, for clarity, we assume this requirement is relaxed.

5 Demonstrating IRS Quantifications using Different Dynamical Systems

This section provides the measures and properties of infrastructure system interdependency, resiliency and sustainability in the framework of different dynamical system models including a stochastic linear system with sudden losses and a Markov jump linear system.

5.1 Stochastic Linear Systems with Sudden Losses (LSwSL)

Consider a simplified model where the dynamics of subsystem A_i are represented by a simple linear model as follows:

$$\begin{aligned} \boldsymbol{s}_i(t+1) &= \boldsymbol{M}_{ii}\boldsymbol{s}_i(t) + \boldsymbol{N}_{ii}\boldsymbol{u}_i(t) + \sum_{j\in N\{i\}} \boldsymbol{M}_{ij}\boldsymbol{s}_j(t) \\ &\quad + \boldsymbol{\epsilon}_i(t) + \boldsymbol{L}_i(t),\ \forall i, \end{aligned}$$

where $\boldsymbol{\epsilon}_i(t)$ is white noise and $\boldsymbol{L}_i(t)$ represents the sudden losses (jumps) of the subsystem. Suppose the losses follow a discrete distribution with $(K+1)$ scenarios, i.e.

$$\boldsymbol{L}_i(t) = \begin{cases} 0, & \text{w.p. } p_0, \\ \boldsymbol{L}_{ij}, & \text{w.p. } p_j, \quad \forall j = 1,\dots,K, \end{cases} \quad, \ \forall i = 1,\dots,n.$$

Let's define:

$$\boldsymbol{s}(t) = \left[\boldsymbol{s}_1(t), \ldots, \boldsymbol{s}_n(t)\right]', \quad \boldsymbol{u}(t) = \left[\boldsymbol{u}_1(t), \ldots, \boldsymbol{u}_n(t)\right]',$$
$$\boldsymbol{\epsilon}(t) = \left[\boldsymbol{\epsilon}_1(t), \ldots, \boldsymbol{\epsilon}_n(t)\right]', \quad \boldsymbol{L}(t) = \left[\boldsymbol{L}_1(t), \ldots, \boldsymbol{L}_n(t)\right]',$$
$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \cdots & \boldsymbol{M}_{1n} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{n1} & \cdots & \boldsymbol{M}_{nn} \end{bmatrix}, \ \boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{N}_{nn} \end{bmatrix}.$$

We have the system dynamics:

$$\boldsymbol{s}(t+1) = \boldsymbol{M}\boldsymbol{s}(t) + \boldsymbol{N}\boldsymbol{u}(t) + \boldsymbol{\epsilon}(t) + \boldsymbol{L}(t).$$

To demonstrate the LSwSL model, let us consider the following simple example:

$$\begin{cases} \boldsymbol{s}(t+1) = 0.4\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t) + \boldsymbol{L}(t) \\ \boldsymbol{L}_i(t) = \begin{cases} 0, & \text{w.p.} \ \ 0.96, \\ -0.15, & \text{w.p.} \ \ 0.01, \\ -0.2, & \text{w.p.} \ \ 0.01, \\ -0.25, & \text{w.p.} \ \ 0.01, \\ -0.3, & \text{w.p.} \ \ 0.01, \end{cases} \\ \boldsymbol{\epsilon}(t) \ : \ \text{Uniform on } [-0.01, 0.01] \\ Q(t) = max(0, 1 - |\boldsymbol{s}(t)|). \end{cases} \tag{2}$$

Figure 4 shows a sample path of the quality of the system using the LSwSL model shown in (2). We can see the sample path looks very similar to the conceptual picture shown in Figure 2.

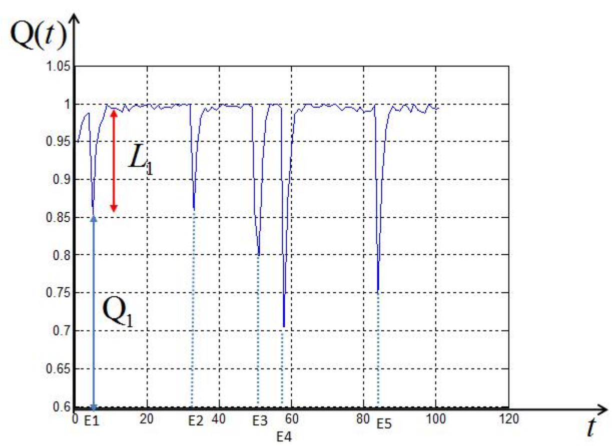


Fig. 4: Simulated quality of a linear system with sudden losses as shown in equation (2)

5.1.1 Interdependencies

The off-diagonal elements in matrix $\boldsymbol{M}$ represent the interdependencies between the subsystems. Notice that this LSwSL model is more general than the Liontief input/output infrastructure model by Haimes and Jiang in [12] since we include both the dynamics of the system, the control $\boldsymbol{u}$, and the randomness $\boldsymbol{\epsilon}$, $\boldsymbol{L}$. In this LSwSL model, we assume the interdependencies matrix is time invariant. In the next subsection, we will consider the case in which $\boldsymbol{M}$ changes over time.

5.1.2 Resiliency

The resiliency and sustainability of a system depend on the form of control $\boldsymbol{u}$. In order to provide concrete measures for resiliency and sustainability, let us consider the case of linear control $\boldsymbol{u}(t) = \boldsymbol{V}\boldsymbol{s}(t)$. The system dynamics can then be rewritten as:

$$\boldsymbol{s}(t+1) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t) + \boldsymbol{L}(t),$$

where $\boldsymbol{A} = \boldsymbol{M} + \boldsymbol{N}\boldsymbol{V}$.

Under failure event j, the system experiences a loss $\boldsymbol{L}_j$ to its state variables, and the system will expect a quality loss of $\boldsymbol{A}\boldsymbol{L}_j$ in the next period (in comparison to its quality if failure had not occurred). The expected quality loss in k period after that would be $\boldsymbol{A}^k\boldsymbol{L}_j$. The recover time $k_j(\delta)$ under event j is the time taken for the system to recover this loss and get back to a level δ from the original level, i.e.

$$k_j(\delta) = \min_k\{k \mid \|\boldsymbol{A}^k\boldsymbol{L}_j\| < \delta\}.$$

The average recovery time is: $\mathcal{R}_{art}(\delta) = \sum_{j=1}^{K} p_j k_j(\delta)$ and the worst recovery time is: $\mathcal{R}_{wrt}(\delta) = \max_j k_j(\delta)$.

The operational resiliency measure would depend on the form of the quality function $Q(\boldsymbol{s}(t))$. Suppose the quality of the system is defined as $Q(t) = \frac{\|\boldsymbol{s}(t)\|}{\max_\tau \|\boldsymbol{s}(\tau)\|}$, then the average operational resiliency is: $1 - \frac{\sum_{j=1}^{K} p_j\|\boldsymbol{L}_j\|}{\max_\tau \|\boldsymbol{s}(\tau)\|}$ and the worst operational resiliency is: $1 - \max_j \frac{\|\boldsymbol{L}_j\|}{\max_\tau \|\boldsymbol{s}(\tau)\|}$.

Let $\lambda(\boldsymbol{A})$ be the set of eigenvalues of $\boldsymbol{A}$ and let $\overline{\lambda}(\boldsymbol{A})$ and $\underline{\lambda}(\boldsymbol{A})$ be the maximum and minimum absolute eigenvalue of $\boldsymbol{A}$ correspondingly, i.e. $\overline{\lambda}(\boldsymbol{A}) = \max_j |\lambda_j(\boldsymbol{A})|$ and $\underline{\lambda}(\boldsymbol{A}) = \min_j |\lambda_j(\boldsymbol{A})|$. The speed of recovery can be approximated as the speed of loss reduction and is equal to $\frac{\|\boldsymbol{L}_j\|}{\|\boldsymbol{A}\boldsymbol{L}_j\|}$ which is bounded below and above by $\frac{1}{\sqrt{\overline{\lambda}(\boldsymbol{A}\boldsymbol{A}')}}$ and $\frac{1}{\sqrt{\underline{\lambda}(\boldsymbol{A}\boldsymbol{A}')}}$, i.e.

$$\frac{1}{\sqrt{\overline{\lambda}(\boldsymbol{A}\boldsymbol{A}')}} \leq \frac{\|\boldsymbol{L}_j\|}{\|\boldsymbol{A}\boldsymbol{L}_j\|} \leq \frac{1}{\sqrt{\underline{\lambda}(\boldsymbol{A}\boldsymbol{A}')}}.$$

This means the worst recovery speed is $\frac{1}{\sqrt{\overline{\lambda}(\boldsymbol{A}\boldsymbol{A}')}}$ which can be further approximated as $\frac{1}{\overline{\lambda}(\boldsymbol{A})}$. (The equality occurs when the control is adjusted such that $\boldsymbol{L}_j$ is the eigenvector of $\boldsymbol{A}$ that correspond to the eigenvalue $\overline{\lambda}(\boldsymbol{A})$).

The adaptability of the system is its capability to stabilize, which is determined by checking whether $\overline{\lambda}(\boldsymbol{A}) < 1$. Overall, we have the following measures:

$$\begin{cases} \mathcal{R}_{aop} = 1 - \frac{\sum_{j=1}^{K} p_j\|\boldsymbol{L}_j\|}{\max_\tau \|\boldsymbol{s}_\tau\|} \\ \mathcal{R}_{wop} = 1 - \max_j \frac{\|\boldsymbol{L}_j\|}{\max_\tau \|\boldsymbol{s}_\tau\|} \\ \mathcal{R}_{art}(\delta) = \sum_{j=1}^{K} p_j k_j(\delta) \\ \mathcal{R}_{wrt}(\delta) = \max_j k_j(\delta) \\ \mathcal{R}_{rs} = \frac{1}{\overline{\lambda}(\boldsymbol{A})} \\ \mathcal{R}_{ad} = \begin{cases} 1, & \text{if } \overline{\lambda}(\boldsymbol{A}) < 1, \\ 0, & \text{if } \overline{\lambda}(\boldsymbol{A}) \geq 1, \end{cases} \end{cases}.$$

5.1.3 Sustainability

Consider the following stochastic linear system with sudden jumps:

$$\boldsymbol{s}(t+1) = \boldsymbol{\mu} + \boldsymbol{A}(\boldsymbol{s}(t)) + \boldsymbol{\epsilon}(t) + \boldsymbol{L}(t), \tag{3}$$

The main steps for deriving the sustainability level is shown below (details can be found in the complementary document):

- **Case 1:** $\overline{\lambda}(\boldsymbol{A}) < 1$:

With the maximum absolute eigenvalue of $\boldsymbol{A}$ smaller than 1, i.e. $\overline{\lambda}(\boldsymbol{A}) < 1$, we have $det(\boldsymbol{I} - \boldsymbol{A}) \neq 0$ and hence there exist $\boldsymbol{\mu}_a = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{\mu}$. The system dynamics shown in equation (3) can be rewritten as follows:

$$\boldsymbol{s}(t+1) - \boldsymbol{\mu}_a = \boldsymbol{A}(\boldsymbol{s}(t) - \boldsymbol{\mu}_a) + \boldsymbol{\epsilon}(t) + \boldsymbol{L}(t).$$

We have:

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\epsilon}(t),\boldsymbol{L}(t)}[\boldsymbol{s}(t+1) - \boldsymbol{\mu}_a \mid \boldsymbol{s}(t)] &= \boldsymbol{A}(\boldsymbol{s}(t) - \boldsymbol{\mu}_a) + \sum_{j=1}^{K} p_j \boldsymbol{L}_j, \\ \Rightarrow \mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{L}}[\boldsymbol{s}(t+1) - \boldsymbol{\mu}_a] &= \boldsymbol{A}\mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{L}}[\boldsymbol{s}(t) - \boldsymbol{\mu}_a] + \bar{\boldsymbol{L}},\end{aligned}$$

where $\bar{\boldsymbol{L}} = \sum_{j=1}^{K} p_j \boldsymbol{L}_j$ is the average loss of the system through time and the expectation is taken for the randomness in $\boldsymbol{\epsilon}$ and $\boldsymbol{L}$. From now on, we will exclude $\boldsymbol{\epsilon}$ and $\boldsymbol{L}$ from the expectation operator for convenient notation.

Let $\bar{\boldsymbol{L}}_a = (\boldsymbol{I} - \boldsymbol{A})^{-1}\bar{\boldsymbol{L}}$. We call $\bar{\boldsymbol{L}}_a$ the adjusted average loss. We have:

$$\begin{aligned}\mathbb{E}[\boldsymbol{s}(t+1) - \boldsymbol{\mu}_a - \bar{\boldsymbol{L}}_a] &= \boldsymbol{A}\mathbb{E}[\boldsymbol{s}(t) - \boldsymbol{\mu}_a - \bar{\boldsymbol{L}}_a], \\ &= \boldsymbol{A}^{t+1}(\boldsymbol{s}(0) - \boldsymbol{\mu}_a - \bar{\boldsymbol{L}}_a), \\ \Rightarrow \sum_{t=0}^{T} \mathbb{E}[\boldsymbol{s}(t) - \boldsymbol{\mu}_a - \bar{\boldsymbol{L}}_a] &= \sum_{t=0}^{T} \boldsymbol{A}^t(\boldsymbol{s}(0) - \boldsymbol{\mu}_a - \bar{\boldsymbol{L}}_a), \\ \Rightarrow \mathbb{E}[\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{\omega}'\boldsymbol{s}(t)] &= \boldsymbol{\omega}'(\boldsymbol{\mu}_a + \bar{\boldsymbol{L}}_a) \\ &+ \frac{1}{T}\boldsymbol{\omega}'(\boldsymbol{I} - \boldsymbol{A}^{T+1})(\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}(0) - \boldsymbol{\mu}_a - \bar{\boldsymbol{L}}_a),\end{aligned}$$

Since $\overline{\lambda}(\boldsymbol{A}) < 1$, we have $\boldsymbol{A}^{T+1} \to 0$ as $T \to \infty$. Thus,

$$\lim_{T\to\infty} \frac{1}{T}\boldsymbol{\omega}'(\boldsymbol{I} - \boldsymbol{A}^{T+1})(\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}(0) - \boldsymbol{\mu}_a - \bar{\boldsymbol{L}}_a) = 0.$$

Hence,

$$\kappa = \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{\omega}'\boldsymbol{s}(t)\right] = \boldsymbol{\omega}'(\boldsymbol{\mu}_a + \bar{\boldsymbol{L}}_a) \tag{4}$$

- **Case 2**: $\overline{\lambda}(\boldsymbol{A}) \geq 1$:

Notice that, in this case, we might not be able to find $\boldsymbol{\mu}_a$ and $\bar{\boldsymbol{L}}_a$ that solves $(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{\mu}_a = \boldsymbol{\mu}$ and $(\boldsymbol{I} - \boldsymbol{A})\bar{\boldsymbol{L}}_a = \bar{\boldsymbol{L}}$ because $(\boldsymbol{I} - \boldsymbol{A})$ might be singular. In addition, the average $\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{\omega}'\boldsymbol{s}(t)$ would contain a weighted average of the elements in the series $\boldsymbol{A}^t,\ t = 0, \ldots, \infty$ which approaches infinity. Thus, the sustainability level would depend on the specific values of $\boldsymbol{\omega}, \boldsymbol{\mu}, \bar{\boldsymbol{L}}$ and $\boldsymbol{A}$. We consider here a regular case when $\boldsymbol{A}$ has its eigen-decomposition as follows: $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}$ where J is a diagonal matrix of eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{S}$ is the matrix formed by the eigenvectors of $\boldsymbol{A}$. Notice that this decomposition is only possible if the eigenvectors of $\boldsymbol{A}$ are linearly interdependent. In the general case, we can always use the Jordan decomposition to simplify the series $\boldsymbol{A}^t,\ t = 0, \ldots, \infty$. However, the results from using the Jordan decomposition is more complicated and we only consider the case when $\boldsymbol{A}$ has an eigendecomposition.

We have:

$$\begin{aligned}
\mathbb{E}[\boldsymbol{s}(t+1)] &= \boldsymbol{A}\mathbb{E}[\boldsymbol{s}(t)] + \boldsymbol{\mu} + \bar{\boldsymbol{L}}, \\
&= \boldsymbol{A}^{t+1}\boldsymbol{s}(0) + \sum_{\tau=0}^{t} \boldsymbol{A}^{\tau}(\boldsymbol{\mu} + \bar{\boldsymbol{L}}), \\
\Rightarrow \mathbb{E}[\sum_{t=0}^{T} \boldsymbol{\omega}'\boldsymbol{s}(t+1)] &= \boldsymbol{\omega}' \sum_{t=0}^{T} \boldsymbol{A}^{t}\boldsymbol{s}(0) \\
&+ \boldsymbol{\omega}' \sum_{t=0}^{T-1} (T-t)\boldsymbol{A}^{t}(\boldsymbol{\mu} + \bar{\boldsymbol{L}}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\kappa &= \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{\omega}'\boldsymbol{s}(t)\right] \\
&= \lim_{T\to\infty} \left(\boldsymbol{\omega}'\boldsymbol{S}\left[\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{J}^{t}\right]\boldsymbol{S}^{-1}\boldsymbol{s}(0)\right. \\
&\quad \left. + \boldsymbol{\omega}'\boldsymbol{S}\left[\frac{1}{T}\sum_{t=0}^{T-1}(T-t)\boldsymbol{J}^{t}\right]\boldsymbol{S}^{-1}(\boldsymbol{\mu} + \bar{\boldsymbol{L}})\right).
\end{aligned}$$

Let $F = \left[\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{J}^{t}\right]$ and $H = \left[\frac{1}{T}\sum_{t=0}^{T-1}(T-t)\boldsymbol{J}^{t}\right]$. Then F and H are diagonal matrices with:

$$F_{jj} = \begin{cases} \frac{\lambda_j^{T+1}-1}{T(\lambda_j-1)}, \text{ if } \lambda_j \neq 1 \\ 1, \text{ otherwise.} \end{cases}$$

and,

$$H_{jj} = \begin{cases} \frac{\lambda_j^{T+1}-(T+2)\lambda_j+T+1}{T(\lambda_j-1)^2}, \text{ if } \lambda_j \neq 1 \\ \frac{T+1}{2}, \text{ otherwise.} \end{cases}$$

Let B, E and S be the sets of indices of the eigenvalues of $\boldsymbol{A}$ that are corresponding to $|\lambda(\boldsymbol{A})| > 1$, $|\lambda(\boldsymbol{A})| = 1$ and $|\lambda(\boldsymbol{A})| < 1$ respectively. Let $v_1 = \boldsymbol{S}'\boldsymbol{\omega}$, $v_2 = \boldsymbol{S}^{-1}\boldsymbol{s}(0)$, $v_3 = \boldsymbol{S}^{-1}(\boldsymbol{\mu} + \bar{\boldsymbol{L}})$

We have:

$$\begin{aligned}
\kappa &= \lim_{T\to\infty} \left(\boldsymbol{\omega}'\boldsymbol{S}\left[\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{J}^{t}\right]\boldsymbol{S}^{-1}\boldsymbol{s}(0)\right. \\
&\quad \left. + \boldsymbol{\omega}'\boldsymbol{S}\left[\frac{1}{T}\sum_{t=0}^{T-1}(T-t)\boldsymbol{J}^{t}\right]\boldsymbol{S}^{-1}(\boldsymbol{\mu} + \bar{\boldsymbol{L}})\right) \\
&= \lim_{T\to\infty} \left[\sum_{j=1}^{n} (v_{1j}v_{2j}F_{jj} + v_{1j}v_{3j}H_{jj})\right], \\
&= \begin{cases} \sum_{j\ni E} \frac{1}{1-\lambda_j} v_{1j}v_{3j} + \sum_{j\in E} v_{1j}v_{2j} \text{ if both} \\ \quad \text{(a) } v_{1j}v_{3j} = 0,\ \forall j \in E, \text{and} \\ \quad \text{(b) } (\lambda_j - 1)v_{1j}v_{2j} + v_{1j}v_{3j} = 0,\ \forall j \in B \\ \infty, \text{ otherwise} \end{cases} \qquad (5)
\end{aligned}$$

Notice that in the case $|\lambda(A)| < 1$, formulation (5) gives us:

$$\begin{aligned}\kappa &= \sum_j \frac{1}{1-\lambda_j} v_{1j} v_{3j} \\ &= \boldsymbol{\omega}' \boldsymbol{S}(\boldsymbol{I}-\boldsymbol{J})^{-1}\boldsymbol{S}^{-1}(\boldsymbol{\mu}+\bar{\boldsymbol{L}}) \\ &= \boldsymbol{\omega}'(\boldsymbol{\mu}_a + \bar{\boldsymbol{L}}_a),\end{aligned}$$

which is exactly what we found in equation (4) in **Case 1**. Notice that we still divide the derivation into two cases since the first case does not require matrix $\boldsymbol{A}$ to be eigendecomposable.

Notice also that the previous derivation can be extended to the case where the system dynamics follow some generic curve $g(t)$. For example, sustainable economic growth often contains a positive drift of the form $g(t) = \alpha + \beta t$. In these cases, the system dynamics is modeled as:

$$\boldsymbol{s}(t+1) - g(t+1) = \boldsymbol{A}(\boldsymbol{s}(t) - g(t)) + \boldsymbol{\epsilon}(t) + \boldsymbol{L}(t),$$

The sustainability level can also be derived using the same methodology. For example, for the case $\overline{\lambda}(\boldsymbol{A}) < 1$, we have:

$$\kappa = \lim_{T\to\infty} \mathbb{E}[\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{s}(t)] = \lim_{T\to\infty}\left(\frac{1}{T}\sum_{t=0}^{T} g(t)\right) + \bar{\boldsymbol{L}}_a.$$

5.2 Markov Jump Linear Systems

In the Markov jump linear systems (MJLS) model, there is a finite number of modes that the system can behave. Let denote these modes at time t as $r(t) \in \{0, 1, \cdots, m\}$. Notice that when $m = 0$, the MJLS model has only one mode of operation and it is equivalent to a linear model. The system dynamics at time t is:

$$\boldsymbol{s}(t+1) = \boldsymbol{\mu}_{r(t)} + \boldsymbol{M}_{r(t)}\boldsymbol{s}(t) + \boldsymbol{N}_{r(t)}\boldsymbol{u}(t) + \boldsymbol{\epsilon}(t).$$

Notice that we have excluded the term $\boldsymbol{L}(t)$ from the system dynamical equation for clarity. All the analysis still applies if we add this term.

The modes of operation follow a Markov transition mechanism with probability matrix $\boldsymbol{\pi}$, i.e.

$$P(r(t+1) = j \mid r(t) = i) = \pi_{ij}.$$

Let $\boldsymbol{\theta}$ be the steady state of the Markov chain, i.e. the probability of being in mode j is θ_j. For convenient characterization of the resiliency and sustainability, we also assume the control $\boldsymbol{u}$ is linear on the state variable, i.e. $\boldsymbol{u}(t) = \boldsymbol{V}_{r(t)}\boldsymbol{s}(t)$. This simplifies the MJLS as follows:

$$\boldsymbol{s}(t+1) = \boldsymbol{\mu}_{r(t)} + \boldsymbol{A}_{r(t)}\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t) + \boldsymbol{L}(t),$$

where $\boldsymbol{A}_{r(t)} = \boldsymbol{M}_{r(t)} + \boldsymbol{N}_{r(t)}\boldsymbol{V}_{r(t)}$.

The interdependency, resiliency and sustainability of the MJLS can be derived as follows:

5.2.1 Interdependencies

The off-diagonal elements of the matrix $\boldsymbol{M}_j$ contains the interdependencies of the system during mode j. Notice that the interdependency in the Markov jump linear system is different depending on the mode of the systems. This makes sense in complex infrastructure systems comprised of interdependent subsystems. For example, the interdependency is often higher during critical modes compared to that of the normal mode.

5.2.2 Resiliency

Let us define $\bar{\boldsymbol{\mu}}_j = (\boldsymbol{I} - \boldsymbol{A}_j)^{-1}\boldsymbol{\mu}_j$ as the steady state of the system if it only operates in mode j. Let mode 1 to be a good (functional) mode while other modes are considered failure modes. Under failure event j with $j = 1, \ldots, m$, the system experience a loss $\boldsymbol{L}_j = \bar{\boldsymbol{\mu}}_0 - \bar{\boldsymbol{\mu}}_j$.

The operational resiliency measure would depend on the form of the quality function $Q(\boldsymbol{s}(t))$. Supposing the quality of the system is defined as $Q(t) = \min\left(1, \frac{\|\boldsymbol{s}(t)\|}{\|\bar{\boldsymbol{\mu}}_0\|}\right)$, the average operational resiliency is: $\frac{\sum_{j=1}^{m}\theta_j\|\bar{\boldsymbol{\mu}}_j\|}{\|\bar{\boldsymbol{\mu}}_0\|}$ and the worst operational resiliency is: $\min_j \frac{\|\bar{\boldsymbol{\mu}}_j\|}{\|\bar{\boldsymbol{\mu}}_0\|}$.

The transition time $k_j(\delta)$ under event j is the time taken for the system to recover from the loss and return back to a level δ from the normal steady state level $\bar{\boldsymbol{\mu}}_0$ assuming the system experiences no further losses, i.e.

$$k_j(\delta) = \min_k\{k \mid \|\boldsymbol{A}_j^k\boldsymbol{L}_j\| < \delta\}.$$

Let N_j be the number of times the system is in the normal mode before it finally returns back to the operational zone. Then N_j follows a geometric distribution with probability of success being $q_j = \theta_0^{k_j}$, which is the probability of successfully returning back to the operational zone after experiencing shock scenario j. Thus, the expectation of N_j is $\bar{N}_j = q_j^{-1} = \theta_0^{-k_j}$.

The length of each trial also follows a geometric distribution with a probability of success being $1 - \theta_0$, which is the probability of not returning to the normal mode. Thus, the trial expected length is $\bar{T} = (1-\theta_0)^{-1}$. The expected recovery time is therefore:

$$\mathcal{R}_{art}(\delta) = \sum_{j=1}^{m}\theta_j\bar{N}_j\bar{T}_j = \sum_{j=1}^{m}\theta_j\theta_0^{-k_j}(1-\theta_0)^{-1},$$

and the worst recovery time is: $\mathcal{R}_{wrt} = \theta_0^{-k_j}(1-\theta_0)^{-1}$.

The adaptability of the system is its capability to maintain a stable state. If we want to make sure the system is adaptable in every mode of operation, the adaptability can be quantified as:

$$\mathcal{R}_{ad} = \begin{cases} 1, & \text{if } \bar{\lambda}(\boldsymbol{A}) < 1,\ \forall j \\ 0, & \text{otherwise.} \end{cases}$$

5.2.3 Sustainability

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\epsilon}(t),r(t)}[\boldsymbol{s}(t+1) \mid \boldsymbol{s}(t)] &= \sum_{j=0}^{m}\theta_j\boldsymbol{\mu}_j + \sum_{j=0}^{m}\theta_j\boldsymbol{A}_j\boldsymbol{s}(t), \\
\Rightarrow \mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{L}}[\boldsymbol{s}(t+1)] &= \sum_{j=0}^{m}\theta_j\boldsymbol{\mu}_j + \sum_{j=0}^{m}\theta_j\boldsymbol{A}_j\mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{L}}\boldsymbol{s}(t), \\
&= \bar{\boldsymbol{A}}\mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{L}}[\boldsymbol{s}(t)] + \bar{\boldsymbol{\mu}},
\end{aligned}$$

where $\bar{\boldsymbol{A}} = \sum_{j=0}^{m}\theta_j\boldsymbol{A}_j$ is the average of $\boldsymbol{A}_{r(t)}$ under the randomness on $r(t)$ and $\bar{\boldsymbol{\mu}} = \sum_{j=0}^{m}\theta_j\boldsymbol{\mu}_j$. From now on, we will exclude $\boldsymbol{\epsilon}$ and $\boldsymbol{L}$ from the expectation operator for convenient notation.

We have:

$$\mathbb{E}[\boldsymbol{s}(t+1)] = \bar{\boldsymbol{A}}\mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{L}}[\boldsymbol{s}(t)] + \bar{\boldsymbol{\mu}},$$

From here, we can use the similar technique shown in subsection 5.1 to derive the sustainability level. Let λ_j be the eigenvalues of $\bar{\boldsymbol{A}}$ and let B, E and S be the sets of indices of the eigenvalues of $\bar{\boldsymbol{A}}$ that are corresponding to $\lambda(\bar{\boldsymbol{A}}) > 1$, $\lambda(\bar{\boldsymbol{A}}) = 1$ and $\lambda(\bar{\boldsymbol{A}}) < 1$, respectively. Suppose $\bar{\boldsymbol{A}}$ can be decomposed into $\bar{\boldsymbol{A}} = \boldsymbol{SJS}^{-1}$ where $\boldsymbol{J}$ is a diagonal matrix of eigenvalues of $\bar{\boldsymbol{A}}$ and $\boldsymbol{S}$ is the matrix formed by the eigenvectors of $\bar{\boldsymbol{A}}$. Let $v_1 = \boldsymbol{S}'\boldsymbol{\omega}$, $v_2 = \boldsymbol{S}^{-1}\boldsymbol{s}(0)$, $v_3 = \boldsymbol{S}^{-1}\bar{\boldsymbol{\mu}}$. We have the following result:

$$\kappa = \begin{cases} \sum_{j \ni E} \frac{1}{1-\lambda_j} v_{1j} v_{3j} + \sum_{j \in E} v_{1j} v_{2j} \text{ if both} \\ \quad \text{(a) } v_{1j} v_{3j} = 0, \ \forall j \in E, \text{and} \\ \quad \text{(b) } (\lambda_j - 1) v_{1j} v_{2j} + v_{1j} v_{3j} = 0, \ \forall j \in B \\ \infty, \ \text{otherwise,} \end{cases}$$

6 Case Study of Biofuel Development and Insights

6.1 Biofuel System

The system configured for this study consists of interdependent subsystems of biomass production (with a single feedstock such as Miscanthus), biofuel refinery and water supply considering both quantity and quality (of nitrate load). Figure 5 depicts the interdependencies of the system, showing the various physical interdependencies with a form of human interferences to natural processes (a-c) and supply-demand relationships (d) , and land use decision depending on water availability (e). Other interdependency relationships in the given system, such as the dependencies of land use on refinery production capacity and water quality regulation on land use, can also be included but we omit them in this example for simplicity. Let

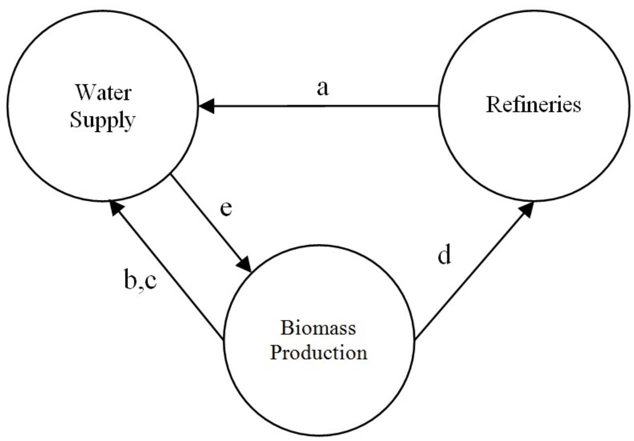


Fig. 5: Physical and functional interdependencies. The arrow direction $A_j \rightarrow A_i$ means system i depends on system j. The physical interdependencies include the effects of: (a) refinery water withdrawal on water quantity, (b) irrigation water withdrawal on water quantity, (c) land use pattern on water quality, (d) biomass production on biofuel production, and (e) water availability on land use.

$\boldsymbol{s}_B(t), \boldsymbol{s}_R(t), \boldsymbol{s}_W(t), \boldsymbol{s}_Q(t)$ be the state variables for biomass production area (in km^2), biofuel refinery capacity (in million m^3 of ethanol) , water supply quantity (in million m^3) and water supply quality (in nitrate mass) at year t, respectively. We also use the following vector notations: $\boldsymbol{s}(t) = \left[\boldsymbol{s}_B(t), \boldsymbol{s}_R(t), \boldsymbol{s}_W(t), \boldsymbol{s}_Q(t)\right]'$.

We assume that the changes in biomass production area $\boldsymbol{s}_B(t)$ is determined by the availability of the water supply availability $\boldsymbol{s}_W(t)$. The amount of water required per year in million m^3 for one km^2 of biomass production is $\alpha_1 = 0.423$.[1]. Then the maximum area of biomass that can be produced is $\boldsymbol{s}_W(t)/\alpha_1$, and the difference $\boldsymbol{s}_W(t)/\alpha_1 - \boldsymbol{s}_B(t)$ represents the maximum new biomass production area due to water availability. In observation of such an opportunity, the decision maker might choose to make a fractional increase in

[1] Source: David et al. [7]

biomass production, $\rho(\boldsymbol{s}_W(t)/\alpha_1 - \boldsymbol{s}_B(t))$, for some $\rho \in [0,1]$. The biomass area in the next period is then:

$$\begin{aligned} \boldsymbol{s}_B(t+1) &= \boldsymbol{s}_B(t) + \rho(\boldsymbol{s}_W(t)/\alpha_1 - \boldsymbol{s}_B(t)) + \boldsymbol{\epsilon}_B(t), \\ &= (1-\rho)\boldsymbol{s}_B(t) + \rho/\alpha_1 \boldsymbol{s}_W(t) + \boldsymbol{\epsilon}_B(t), \end{aligned}$$

where $\boldsymbol{\epsilon}_B(t)$ is the error term associated with factors other than the water supply availability. For our numerical demonstration, we choose $\rho = 0.5$. The refinery capacity is assumed to be proportional to the biomass yield:

$$\boldsymbol{s}_R(t+1) = \alpha_2 \boldsymbol{s}_B(t) + \boldsymbol{\epsilon}_B(t),$$

where $\alpha_2 = 0.0015$ is the yield of ethanol per unit area of biomass production (in million m^3 ethanol per km^2)[2]

The water supply quantity in the next period $\boldsymbol{s}_W(t+1)$ depends on crop water consumption and refinery water consumption as follows:

$$\boldsymbol{s}_W(t+1) = \mu_W - \alpha_1 \boldsymbol{s}_B(t) - \alpha_3 \boldsymbol{s}_R(t) + \alpha_4 \boldsymbol{s}_W(t) + \boldsymbol{\epsilon}_W(t),$$

where μ_W is the reservoir recharge (in million m^3), $\alpha_1 = 0.423$ is crop water consumption (in million m^3 per km^2), $\alpha_3 = 6$ is refinery water consumption (in million m^3 water per million m^3 ethanol production), $\alpha_4 \in [0,1]$ is the percentage of water left from the previous period. In our numerical test, we choose $\alpha_4 = 0.8$ which means 20% of the water in the reservoir is either used for other purposes or lost (e.g. due to evaporation).[3]

Let $\boldsymbol{s}_Q(t)$ be the state variable for the water quality which is represented by the nitrate mass (in tons). This state variable is dependent on the biomass production area as follows. Let $\alpha_6 = 0.6$ ($tons/km^2$) be the nitrate yield from land covered with Miscanthus and $\alpha_7 = 3$ (ton/km^2) be the nitrate yield for land covered by other crops (e.g. corn). Then the nitrate mass for the land covered by Miscanthus is $\alpha_6 \boldsymbol{s}_B(t)$ and that of the remaining land is $\alpha_7(L - \boldsymbol{s}_B(t))$ where L is the total area of land. We also assume $\alpha_5 \in [0,1]$ is the percentage of nitrate mass left from the previous period and we choose $\alpha_5 = 0.8$ for our numerical demonstration. The dynamic equation for the nitrate mass becomes:

$$\boldsymbol{s}_Q(t+1) = \alpha_5 \boldsymbol{s}_Q(t) + \alpha_6 \boldsymbol{s}_B(t) + \alpha_7(L - \boldsymbol{s}_B(t)) + \boldsymbol{\epsilon}_Q(t).$$

Combining all the above relationships together, we obtain the system dynamics: $\boldsymbol{s}(t+1) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{\mu} + \boldsymbol{\epsilon}(t)$ where the interdependency matrix $\boldsymbol{A}$ is:

$$\boldsymbol{A} = \begin{bmatrix} 1-\rho & 0 & \rho/\alpha_1 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ -\alpha_1 & -\alpha_3 & \alpha_4 & 0 \\ -(\alpha_7 - \alpha_6) & 0 & 0 & \alpha_5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 1.1820 & 0 \\ 0.0015 & 0 & 0 & 0 \\ -0.423 & -6 & 0.8 & 0 \\ -2.4 & 0 & 0 & 0.8 \end{bmatrix}.$$

The elements in the matrix $\boldsymbol{A}$ reflect the effects subsystems have on one another, and the off-diagonal elements highlight the interdependencies. For example, $a_{13} = 1.1820$ is the dependency of biomass production on water. Notice that $a_{12} = 0$ since there is no direct feedback link from the biofuel refinery subsystem to the biomass production subsystem.

6.2 Insights

6.2.1 Insight 1: Interdependency Affects Loss Propagation

Consider the biofuel system that experiences $K = 5$ scenarios of losses with the following distribution:

$$\boldsymbol{L}_i(t) = \begin{cases} 0, & \text{w.p. } p_0, \\ L_{ij}, & \text{w.p. } p_j, \quad \forall j = 1, \ldots, K, \end{cases}, \ \forall i = 1, \ldots, n.$$

[2] Miscanthus yield is 39 ton/ha and it typically continues yielding for fifteen to twenty years before replanting is required. For every one ton of Miscanthus the refinery receives, it produces approximately 0.3 ton of ethanol with a density of 0.003 ton/gallon. Thus, $\alpha_2 = \frac{39*0.3}{0.003} = 3900$ gallon/ha $= 0.0015 Mm^3/km^2$. The data is obtained from Heaton et al. [13].

[3] Source: Aden et al. [1].

where,

$$\boldsymbol{L} \;=\; \begin{bmatrix} -0.1 & 0 & 0 & 0 & -0.025 \\ 0 & -0.1 & 0 & 0 & -0.025 \\ 0 & 0 & -0.1 & 0 & -0.025 \\ 0 & 0 & 0 & -0.1 & -0.025 \end{bmatrix}.$$

Let the probability of the system experiencing the losses be $p_j = 0.01,\ \forall\, j = 1, \ldots, 5$, $p_0 = 1 - \sum_{j=1}^{5} p_j = 0.95$. This means the system endures losses 5% of the time and works normally 95% of the time.

Under the first loss scenario, the biomass production subsystem experiences a loss of magnitude $L_{11} = -0.1$ while the other subsystems remain intact, i.e. $L_{21} = L_{31} = L_{41} = 0$. The loss of the biomass production system in period t would result in a loss of $m_{21}L_{11}$ in the refinery system at period $t+1$. However, the reduction in biomass production would also mean the amount of water supply used by the biomass production system is reduced. This leads to an increase in the water quality and water quantity in period $t+1$. In period $t+2$, the increase in the water supply from period $t+1$ leads to the increase in the biomass production and the refinery subsystems. These increases would reduce the loss effect from period t and the chain of reactions between the subsystems continues until the system is stable at a new state that is equal to:

$$(\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{\mu} + (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{L}_{.1},$$

where $\boldsymbol{L}_{.1}$ is the system loss under scenario one. The system will be stable at this state until it experiences another loss. The long-term average (sustainability level)) of the system state is:

$$(\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{\mu} + (\boldsymbol{I} - \boldsymbol{A})^{-1}\bar{\boldsymbol{L}},$$

as we have derived in subsection 5.1 where $\bar{\boldsymbol{L}}$ is the system average loss.

It is interesting to analyze the overall system loss quantity $(\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{L}_{.j}$ for each scenario j.

The overall system losses in the five scenarios are:

$$(\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{L} = \begin{bmatrix} -0.0328 & 1.1618 & -0.1936 & 0.0000 & 0.9354 \\ -0.0000 & -0.0983 & -0.0003 & 0 & -0.0986 \\ 0.0707 & 0.4914 & -0.0819 & 0.0000 & 0.4803 \\ 0.3931 & -13.9411 & 2.3235 & -0.5000 & -11.7245 \end{bmatrix},$$

where each column corresponds to a scenario. For example, the overall system loss under the first scenario is $[-0.0328, 0, 0.0707, 0.3931]$ for the four system state variables of biomass production, biofuel refinery, water quantity, and water quality, respectively. It is interesting to notice that under the first scenario with a sudden loss in the biomass production subsystem, only the biomass production endures the loss of -0.0328, while the water supply has the benefits from this with the water quantity and water quality increase by 0.0707 and 0.3931 respectively. These facts result from the negative interdependencies between water supply and the other two subsystems.

It is interesting to notice the sign of the losses under different scenarios. Overall, we found:

- A loss in biomass production will negatively affect itself while benefiting the water supply.
- A loss in refinery subsystem will affect itself and the water quality but benefit other subsystems.
- A loss in water quantity will affect all the subsystems but not the water quality. This is because the losses in the biomass production benefits the water quality more than the negative effect caused by the loss in the water quantity. However, it should be noticed that these results only apply for our choice of parameters. The results could be different if we change the interdependencies factors.
- A loss in the water quantity only affects itself and leads to no changes in other subsystems. This is because the water quality only absorbs the changes from other system state variables (i.e. the water quality node has only incoming links in the biofuel network)

Notice also that the signs of the elements in the interdependencies matrix have strong effects on the signs of the system losses. For example, if all the interdependencies are non-negative, the loss on any subsystem will degrade all other subsystems to which it is connected.

6.2.2 Insight 2: Interdependency Affects Resiliency

The interdependencies matrix $\boldsymbol{A}$ decides how the system behaves while the maximum absolute eigenvalue $\overline{\lambda}(\boldsymbol{A})$ is one of the key measures of the system and it appears in most of the equations for the resiliency and the sustainability levels. Specifically, $\overline{\lambda}(\boldsymbol{A})$ is a factor of the recovery time and recovery speed. In addition, the sustainability level is only finite and well defined when $\overline{\lambda}(\boldsymbol{A}) \leq 1$. Since $\overline{\lambda}(\boldsymbol{A})$ is dependent on the magnitude of the interdependency factor in $\boldsymbol{A}$, the interdependency factors affect the resiliency and sustainability levels.

Consider the case when the interdependency matrix is changed from $\boldsymbol{A}$, as shown in subsection (6.1), to $\tilde{\boldsymbol{A}}_{ij}(\rho)$ by changing only the dependency of system i on j to a varying level ρ. In other words, $\tilde{\boldsymbol{A}}_{ij}(\rho)$ is exactly the same with $\boldsymbol{A}$ except for the (i, j) element, which is equal to ρ. Figure 6 shows how the recovery time changes when we vary the interdependency levels ρ for the pair $(i = 1, j = 3)$, which corresponds to the dependency of the biomass production on water supply.

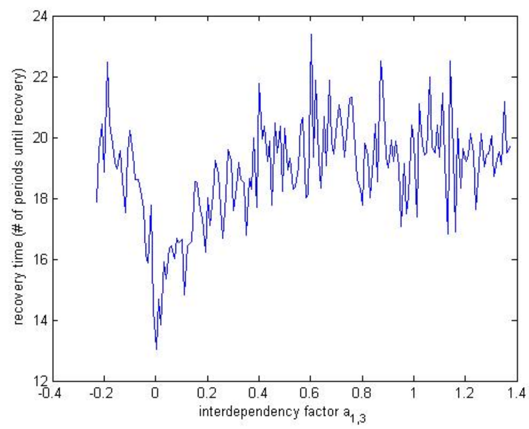


Fig. 6: Recovery time changes when varying the dependency factor a_{13}.

Figure 6 also shows that the required recovery times have V-shapes that increase when the magnitudes (i.e. the absolute values) of the dependency factors increase. Notice that, we only show the dependency factors in specific ranges. For example, the feasible ranges of ρ is $[-0.228, 1.372]$ which was found by finding the threshold of ρ such that the system is stable, i.e. $\overline{\lambda}(\tilde{\boldsymbol{A}}_{ij}(\rho)) \leq 1$.

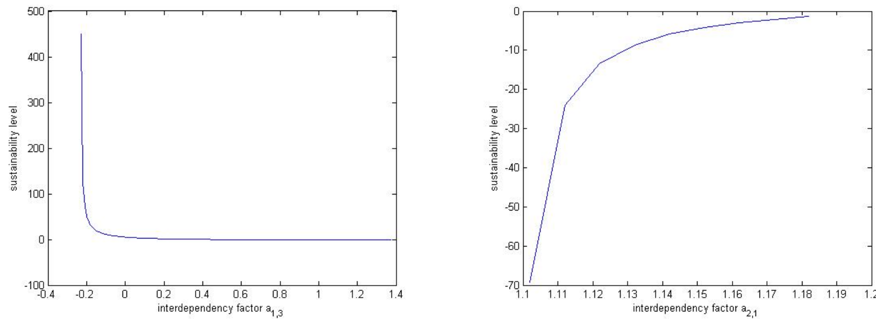
6.2.3 Insight 3: Interdependency Affects Sustainability

Consider again the simple linear system with the interdependency matrix as shown in subsection (6.1). Under the case when $\overline{\lambda}(\boldsymbol{A}) < 1$, the sustainability level under measure $\boldsymbol{\omega}$ is $\kappa = \boldsymbol{\omega}'(\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{\mu} + \bar{\boldsymbol{L}})$, which is a function of $\boldsymbol{\omega}, \boldsymbol{A}, \boldsymbol{\mu}$ and $\bar{\boldsymbol{L}}$. Once we change the interdependency factors, the sustainability level also changes. Similar to subsection 6.2.2, we consider the case when the interdependency matrix $\boldsymbol{A}$ is changed to $\tilde{\boldsymbol{A}}_{ij}(\rho)$ by changing only the dependency of system i on j to ρ. Figure 7 shows how the sustainability changes when we vary the dependency factors for the pairs $(i = 1, j = 3)$ and $(i = 2, j = 1)$.

Overall, we find that increasing the dependencies of water supply on biomass production, i.e. increasing the magnitudes of a_{31}, will reduce the system sustainability while increasing the dependency of the biofuel production on the biomass production increase the system sustainability.

Sensitivity analysis: It is interesting to analyze the change of the system sustainability when we change the dependency factors. Suppose $\boldsymbol{A}$ is adjusted to $\tilde{\boldsymbol{A}}_{ij}(\varrho) = \boldsymbol{A} + \varrho \boldsymbol{I}_{ij}$ where $\boldsymbol{I}_{ij}$ is a matrix of the same size with $\boldsymbol{A}$ and has also elements equal to zero except for the element at row i and column j which is equal to one. (Notice a slight difference between $\tilde{\boldsymbol{A}}_{ij}(\varrho)$ and $\tilde{\boldsymbol{A}}_{ij}(\rho)$). Let us define:

$$\kappa_{ij}(\varrho) = \boldsymbol{\omega}'(\boldsymbol{I} - \boldsymbol{A} - \varrho \boldsymbol{I}_{ij})^{-1}(\boldsymbol{\mu} + \bar{\boldsymbol{L}})$$



(a) Dependency factor: a_{13}

(b) Dependency factor: a_{21}

Fig. 7: System sustainability changes when changing the dependency factors.

Then we can take the derivative of $\kappa(\varrho)$ on ϱ to find out the sensitivity of the sustainability levels on the interdependency factor a_{ij}. We have:

$$\begin{aligned}\frac{\partial \kappa_{ij}(\varrho)}{\partial \varrho} &= \boldsymbol{\omega}'\left[(\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{I}_{ij})(\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{\mu}+\bar{\boldsymbol{L}})\right] \\ &= \left[\boldsymbol{\omega}'(\boldsymbol{I}-\boldsymbol{A})^{-1}\right]_i \left[(\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{\mu}+\bar{\boldsymbol{L}})\right]_j .\end{aligned}$$

Let us take the case when $\boldsymbol{\omega} = [0.25, 0.25, 0.25, 0.25]'$. We have the following sensitivity matrix $\Delta\kappa$ at $\boldsymbol{A}$ when we vary the elements in the interdependency matrix:

$$\begin{aligned}\Delta\kappa &= \left[\boldsymbol{\omega}'(\boldsymbol{I}-\boldsymbol{A})^{-1}\right]'\left[(\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{\mu}+\bar{\boldsymbol{L}})\right]' \\ &= \begin{bmatrix} -8.4273 & -0.0038 & -3.5721 & 36.5 \\ 242.1473 & 0.1098 & 102.6379 & -1049.1 \\ -40.0321 & -0.0181 & -16.9682 & 173.4 \\ 9.7749 & 0.0044 & 4.1432 & -42.3 \end{bmatrix},\end{aligned}$$

Interestingly, the signs of the elements of $\Delta\kappa$ show the directions of change in κ when we change $\boldsymbol{A}$. For example, the sustainability decreases when we increase a_{13}, a_{31} and increase when we increase a_{21}. We can see the trends shown in figure 7 match with the sign of the sustainable sensitivity matrix.

It is also interesting to break down the system sustainability level into two components, one due to $\boldsymbol{\mu}$ and another due to $\bar{\boldsymbol{L}}$ and analyze their corresponding sensitivity matrices. We have:

$$\begin{aligned}\Delta\kappa &= \left[\boldsymbol{\omega}'(\boldsymbol{I}-\boldsymbol{A})^{-1}\right]'\left[(\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{\mu}+\bar{\boldsymbol{L}})\right]' \\ &= \Delta\kappa(\boldsymbol{\mu}) + \Delta\kappa(\bar{\boldsymbol{L}}),\end{aligned}$$

where $\Delta\kappa(\boldsymbol{\mu})$ and $\Delta\kappa(\bar{\boldsymbol{L}})$ are the sensitivity matrices of $\boldsymbol{A}$ through $\boldsymbol{\mu}$ and $\bar{\boldsymbol{L}}$ correspondingly.

We have:

$$\begin{aligned}\Delta\kappa(\boldsymbol{\mu}) &= \left[\boldsymbol{\omega}'(\boldsymbol{I}-\boldsymbol{A})^{-1}\right]'\left[(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{\mu}\right]' \\ &= \begin{bmatrix} -8.3467 & -0.0123 & -3.5306 & 35.5 \\ 239.8302 & 0.3541 & 101.4482 & -1020 \\ -39.6490 & -0.0585 & -16.7715 & 168.6 \\ 9.6813 & 0.0143 & 4.0952 & -41.2 \end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}\Delta\kappa(\bar{\boldsymbol{L}}) &= \left[\boldsymbol{\omega}'(\boldsymbol{I}-\boldsymbol{A})^{-1}\right]'\left[(\boldsymbol{I}-\boldsymbol{A})^{-1}\bar{\boldsymbol{L}}\right]' \\ &= \begin{bmatrix} -0.0806 & 0.0085 & -0.0414 & 1.0108 \\ 2.3171 & -0.2443 & 1.1897 & -29.0443 \\ -0.3831 & 0.0404 & -0.1967 & 4.8016 \\ 0.0935 & -0.0099 & 0.0480 & -1.1724 \end{bmatrix},\end{aligned}$$

It can be seen very clearly that the sustainability level component of $\boldsymbol{\mu}$ is the predominant cause of the loss since our choice of the loss probabilities and magnitudes is quite small.

6.2.4 Insight 4: Sustainability Level is Dependent on the Scale of Study

From the formulation of the sustainability level (5), we can see very clearly the sustainability level of the system is dependent on the choice of the weight vector $\boldsymbol{\omega}$. We will show a more striking result that the system can be sustainable in some choices of $\boldsymbol{\omega}$ but unsustainable in others. This reflects the notion of local versus global sustainability in which a system might be sustainable at a local scale of study but not in a larger global scale.

Consider a slight change to the interdependency matrix of the linear system as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0.5 & 0 & 1.1820 & 0 \\ 0.0015 & 0 & 0 & 0 \\ -0.423 & -6 & 2 & 0 \\ -2.4 & 0 & 0 & 0.8 \end{bmatrix}.$$

Notice that we have changed a_{33} from 0.9 to 2. We deliberately make this simple change so that $\overline{\lambda}(\boldsymbol{A}) > 1$ and all the eigenvalue of $\boldsymbol{A}$ are real for a simple demonstration. In this case, $\lambda(\boldsymbol{A}) = \{0.8, 0.0069, 1.4855, 1.0214\}$ with four corresponding linear independent eigenvectors. Let $\boldsymbol{S}$ be the matrix formed by the eigenvectors and $\boldsymbol{J}$ is a diagonal matrix of eigenvalues of $\boldsymbol{A}$. We have: $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}$ and $\boldsymbol{A}^k = \boldsymbol{S}\boldsymbol{J}^k\boldsymbol{S}^{-1}$

Using the same method as shown in section 5.1, the sustainability level of the system is:

$$\kappa = \begin{cases} \boldsymbol{\omega}'(\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{\mu}+\bar{\boldsymbol{L}}), \\ \quad \text{if} \quad [\boldsymbol{\omega}'\boldsymbol{S}_{.j}] * \left[\boldsymbol{S}_{.j}^{-1}((\lambda_j - 1)\boldsymbol{s}(0) - \boldsymbol{\mu} - \bar{\boldsymbol{L}}_a)\right] = 0, \ \forall j \in \{3,4\} \\ \infty, \qquad \text{otherwise} \end{cases}$$

where $\boldsymbol{S}_{.3} = \{0.2677, 0.0003, 0.2232, -0.9373\}'$ and $\boldsymbol{S}_{.4} = \{0.0918, 0.0001, 0.0405, -0.9950\}'$ are the third and the fourth eigenvectors of $\boldsymbol{A}$.

Suppose we have chosen the initial state $\boldsymbol{s}(0)$ such that $\left[\boldsymbol{S}_{.j}^{-1}((\lambda_j - 1)\boldsymbol{s}(0) - \boldsymbol{\mu} - \bar{\boldsymbol{L}}_a)\right] \neq 0, \forall j \in \{3,4\}$. Then the sustainability level depends on whether $[\boldsymbol{\omega}'\boldsymbol{S}_{.j}] = 0 \ \forall j \in \{3,4\}$. The system sustainability level κ is finite under all sustainability measures $\boldsymbol{\omega}$ that satisfy:

$$\begin{cases} 0.2677\omega_1 + 0.0003\omega_2 + 0.2232\omega_3 - 0.9373\omega_4 = 0 \\ 0.0918\omega_1 + 0.0001\omega_2 + 0.0405\omega_3 - 0.995\omega_4 = 0 \end{cases}$$

The sustainability level could be $+\infty$ or $-\infty$ if either of these equalities does not hold. In an infrastructure system, there are many options and views for choosing $\boldsymbol{\omega}$. Some of these views consider the global scale of the system while other consider only the local scale. Depending on these views, the system might be considered as sustainable or not.

6.2.5 Insight 5: Transition Probability

Our discussion so far was for complex systems with fixed and fully known interdependencies between subsystems. However, in reality, complex systems might be ill-defined or we might only know the inter-relationships partially. In the case the interdependency matrix is not deterministic but follows a Markov stochastic process, then we can apply results from subsection 5.2 that makes use of the Markov jump linear systems to derive closed-form solution for the system resiliency and sustainability factors. Suppose $\boldsymbol{s}$ follows a Markov jump linear system with $m = 2$ modes of operations, a "good" (functional) mode and a "bad" (failure) mode. Let the dynamical equations for the good mode be:

$$\boldsymbol{s}(t+1) = \boldsymbol{\mu}_G + \boldsymbol{A}_G \boldsymbol{s}(t) + \boldsymbol{\epsilon}(t)$$

and bad mode be:

$$\boldsymbol{s}(t+1) = \boldsymbol{\mu}_B + \boldsymbol{A}_B \boldsymbol{s}(t) + \boldsymbol{\epsilon}(t)$$

Let the transition probability be: $\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$. Let θ_1 be the steady state probability of being the good mode and θ_2 be that of the bad mode. The steady state probability is $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ solves: $\Pi'\boldsymbol{\theta} = \boldsymbol{\theta}$ and $\theta_1 + \theta_2 = 1$. We will vary θ_2 and analyze the effects to the system resiliency and sustainability.

Consider the case when $\boldsymbol{\mu}_G = \boldsymbol{e}$ and $\boldsymbol{\mu}_B = 0.6\boldsymbol{e}$ where $\boldsymbol{e}'$ is an identity vector. Let the interdependency matrices be $\boldsymbol{A}_G = \boldsymbol{A}_B = \boldsymbol{A}$ as shown in subsection (6.1).

Figures 8 shows how varying θ_2 might affect the system resiliency and sustainability. The horizontal axis is for the steady state probability of being in the bad mode (i.e. θ_2). The left vertical axis is for the recovery time and the right vertical axis is for the sustainability level. We can see the recovery time increases while the sustainability level decreases when we increase the probability of being in the bad mode.

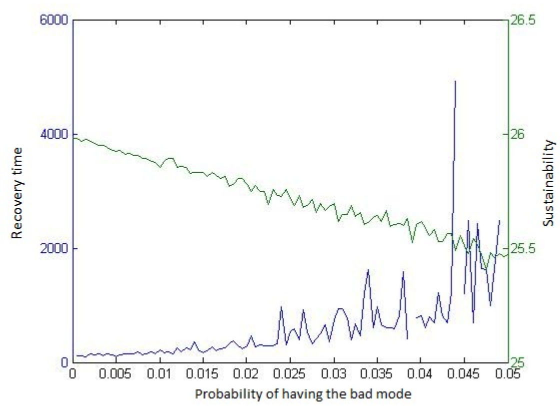


Fig. 8: System recovery time and sustainability change when changing the steady state probability of having the bad mode.

For complex systems with more generic dynamic processes, we might not be able to obtain the closed form solution for the system resiliency and sustainability. In that case, we can use Monte-Carlo simulation to simulate the possible value of the interdependency matrix and find out the distribution of the system resiliency and sustainability. From which, we can tell the expected system sustainability level or the worst recovery speed.

7 Conclusion

This paper presents a unified modeling framework to analyze infrastructure interdependency, resiliency and sustainability (IRS), each of which has been the topic of intensive discussion. We first develop mathematical models to quantify how interdependency among infrastructure subsystems affects resiliency and sustainability of the whole complex system. We then apply the models to an illustrative case study on biofuel development. The metrics and the analytical framework developed in this effort lay a foundation for improving the design of complex infrastructure systems in the sense that the interdependency may be adjusted to achieve a desirable level of system resiliency and sustainability. For example, in the biofuel case, the decision on land allocation for feedstock will affect nearby water supply; similarly, the decision on the locations for refinery facilities will affect the water supply. These design choices could ultimately be reflected in the interdependency matrix, which will in turn affect the resiliency and the sustainability of the overall system.

The main objectives of this research are (i) to provide a consistent set of metrics for engineering infrastructure IRS, and (ii) to provide insights on the interplay among these different system metrics. In order to achieve the second objective, we have made some simplifying assumptions in our numerical example; e.g., the system dynamics is either linear or Markov-jump linear, and we have omitted the spatial dimension in each of the subsystems (refinery, water source, and biomass farm). We shall note that we keep the system very simple for the sake of insight exploration but the framework can be extended to more complex systems, where the same IRS metrics can be quantified either in closed forms or through simulations. For example, if we consider a more detailed system (e.g., one with an underlying river network, multiple refineries and multiple farms), each river node, each farm and each refinery shall be treated as a separate subsystem, and they are linked together via spatial proximity. This extension would lead to a larger dynamical system with significantly more state variables, but the methodology presented in this paper is still applicable.

Many analytical results and interesting insights have been drawn from this study. For example, interdependency affects loss propagation, system resiliency and sustainability. Overall, we find that increasing the magnitudes of interdependencies increases the system recovery time (and hence reduces the recovery speed). Similarly, in general, increasing the magnitudes of interdependencies decreases the overall system sustainability. However, there are specific choices of the interdependency matrix and the loss distribution in which increasing interdependency magnitudes can improve system sustainability. In addition, the concept of sustainability is scale-dependent, as a system might be considered sustainable at one scale (e.g. local scale) but unsustainable at others (e.g. global scale).

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