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Annals of Operations Research

ISSN 0254-5330

Ann Oper Res
DOI 10.1007/s10479-012-1185-3

Springer

Ann Oper Res
DOI 10.1007/s10479-012-1185-3

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Abstract This research is concerned with developing repair and training strategies for stand-by equipment which maximise the time until the equipment is unable to respond when it is needed. Equipment can only be used if it is in an operable state and the users have had sufficient recent training on it. Thus it is necessary to decide when to maintain/repair the equipment and when to use the equipment for training. Both actions mean the equipment is not readily available for use in an emergency. We develop discrete time Markov decision process formulations of this problem in order to investigate the form of the optimal policies which maximise the expected survival time until a catastrophic event when an emergency occurs and the equipment cannot respond. We also calculate the solution in a number of numerical examples.

Keywords Maintenance and repair · Training action · Markov decision processes · Stand-by equipment

1 Introduction

It is an honour to present this paper in a celebration of the work of Professor Derman. Not only was he one of the first to use Markov Decision Processes to model the problems of deteriorating and failing systems (Derman and Sacks 1960) but his book (Derman 1970) is one of the classic texts which introduced LT and many other operations researchers to Markov Decision Processes. Moreover the paper by Derman et al. (1984) introduced the idea of maximising the expected time until a catastrophic event in the context of replacement and maintenance problems. We use that measure in this paper albeit to measure the effectiveness of a stand-by system rather than one in continuous operation. Derman was one of the first to recognise that the way operators and repairmen are scheduled in their interaction with a deteriorating system can have a major impact on the system's performance (Katehakis and Derman 1984). Whereas that paper concentrated on which order a repairman should

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Published online: 19 July 2012

undertake repairs, this paper looks at how frequently and when an operator should be trained on a stand-by system so that he is able to use it correctly in an emergency.

Stand-by equipment is only brought into operation when there is a vital need for it, for example, a hospital emergency power supply system, or military equipment. We call the times when there is a vital need for a stand-by unit, initiating events and if the unit is not able to respond to an initiating event then it is deemed to be a catastrophic event.

This research is concerned with developing repair and training strategies which maximise the time until a catastrophic event for stand-by systems in an uncertain environment. Equipment can only be used if it is in an operable state and if its users have had sufficient recent training with it. Thus as well as repairing and maintaining the equipment, it is necessary to train users. This is particularly clear in the military context where soldiers are constantly trained to operate the equipment satisfactorily under all conditions. However, a problem with training is that it increases the wear and tear of the stand-by unit even though it enhances the operator's ability to respond well to an initiating event. Another problem in the military context is that the training may be done away from where the equipment may be needed and so there is not time to move it between the training area and the front line say. In this research we look at the interaction between the need for training and the need to service the equipment. We develop discrete time Markov decision process formulations of the problem in order to investigate the form of the optimal action policies which maximise the expected survival time until a catastrophic event. The reason for focusing on the expected survival time rather than on cost is because we assume that the cost is immeasurably high if the system fails to respond when required.

The literature on maintenance, repair and replacement policies for deteriorating equipment started with the work of Derman and Sacks (1960), Derman (1963) and Barlow and Proschan (1965). It has continued to be an active research area with many applications as is shown in the surveys and bibliographies of McCall (1965), Pierskalla and Voelker (1976), Sherif and Smith (1981), Monahan (1982), Thomas (1986), Valdez-Flores and Feldman (1989), Dekker (1996), Wang (2002) and Sarkar et al. (2011). Most of the literature concentrates on policies which minimise the average or discounted cost criteria. The idea of using a catastrophic event criterion to overcome the problem that failure will result in unquantifiably large costs was suggested first by Derman et al. (1984) in the case of continuously operated systems and by Thomas et al. (1987) in the case of stand-by systems. Other instances were considered by Kim and Thomas (2006, 2012). In all these cases the background environment and hence the probability of an initiating event is either fixed or follows a random Markovian process. Other authors such as Çinlar (1984), Çinlar and Özekici (1987), Çinlar et al. (1989), Shaked and Shanthikumar (1989), Lefèvre and Milhaud (1990) and Özekici (1995, 1996) have looked at maintenance in a random environment but for continuously operating units. In these cases the changes in the environment age the equipment at different rates, but do not affect when it is needed. Wortman and Klutke (1994), Klutke et al. (1996), Yang and Klutke (2000a, 2000b) and Kiessler et al. (2002) study protective systems, such as circuit breakers, alarms, and protective relays, as well as stand-by systems, with non-self-announcing failures where the rate of deterioration is governed by a random environment. Kim and Thomas (2006), on the other hand, allow the deterioration of the equipment to be independent of the environment, but the environment affects the need for the equipment. The change in the environment in this case was random, but in Kim and Thomas (2012) the changes were controlled by an opponent. Lam (1995, 2003) studies an optimal maintenance model for a stand-by system focusing on its availability and its reliability. None of these papers address the issue of how the training of the operators affects the readiness of the unit which is considered in this paper. The current paper brings into publication results which were first disseminated in a discussion paper (Kim and Thomas 2004).

We develop a Markov decision process model with random loss of expertise by the operators in Sect. 2. Given some standard conditions on the way the equipment deteriorates, one is able to prove results about the form of the optimal policy. Numerical examples of these results are presented in Sect. 3. In Sect. 4 we examine a modified model where the effect of training diminishes exponentially over time and again we examine when one should train, and when one should repair as a function of the environmental situation, the training level and the state of the equipment. A numerical example of this situation is given in Sect. 5 and conclusions drawn in Sect. 6. Throughout we motivate our models by reference to the military situation, but the models are equally appropriate when the environment is weather-based (generator usage), congestion based (traffic response unit) or time of the week, month or year based (emergency services).

2 Training model with random loss of expertise

2.1 Introduction

In the model, there are several environmental situations which are graded from very dangerous to completely peaceful. Each environmental situation has its own probability of an initiating event occurring which increases as the situation gets more dangerous. There are three dynamical factors in the model—the quality of the stand-by system, the environment in which it operates and the training level of the system's operators. There are three actions available at each period—do nothing, repair, and train. At the end of this section we look at the special case in which we only consider do nothing and training. This corresponds to equipment which cannot be repaired though we do not consider the problem of when to replace such equipment.

Our first task is to define the notation we will need for such modelling

2.2 Nomenclature and terminology

$i = 1, \ldots, N$: the quality states of the stand-by system

- P_{ij} quality state transition probability if no action
- $\tilde{P}_{ij}$ quality state transition probabilities when training is being undertaken

$m = 1, \ldots, M$: environmental states

- S_{mu} environmental state transition probability
- b_m probability of initiating event occurring in environmental state m where we assume $b_m > 0$ for all

$k = 1, \ldots, L$: training levels of the operators

- Q_{kp} training level transition probabilities if no current training occurring (i.e. when either unit is being repaired or no action is being taken)
- $\tilde{Q}_{kp}$ transition probabilities of changes in training level caused by one period of training
- d_k probability operators with training level k do not deal satisfactorily with an initiating event

f probability equipment cannot be moved from training area to deal with initiating event

2.2.1 *Possible stand-by unit quality state, i*

Regular inspection of the stand-by unit gives information on its operational quality. This categorisation of the equipment into various states after inspection is fairly common in the military context with states defined as new, excellent, operable, and failed. We assume that the stand-by unit has N different unit quality states, i.e. $1, 2, \dots, N$ where state 1 means that the stand-by unit is like new. The state $N - 1$ means that it is in a poor but still operable state, while in state N, it is in a "down" condition which means that it will not work.

2.2.2 *The quality state transition probability matrix (QSTPM), P_{ij}*

When the stand-by unit is in quality state i at the current stage, there is a probability, P_{ij} that it will be in state j at the next period where $i, j = 1, 2, \dots, N$ and

$$\sum_{j=1}^{N} P_{ij} = 1 \quad \text{where } i = 1, \dots, N$$

We assume that the QSTPM is ergodic so there is a positive probability that it will move to the down state. We also assume it satisfies a first order stochastic ordering condition $\sum_{j<k} P_{ij} \geq \sum_{j<k} P_{(i+1)j}$ so that the better the current state the more likely it is to continue in a satisfactory condition. This stochastic ordering assumption on the dynamics of the state of the equipment has been widely used in the literature from the start (Barlow and Proschan 1965). It means that equipment in a high quality state is more likely to be in a high quality state next period than that which is currently in a lower quality state but there is a chance the ordering might be reversed. We assume $P_{NN} = 1$ so once the stand-by unit reaches the down state N, it remains down until either it is repaired, or a catastrophic event occurs.

2.2.3 *Possible environmental situation, m*

Assume there are M different environmental states, $1, 2, \dots, M - 1, M$. Environmental state 1 reflects the most peaceful environment in which there is the smallest probability, b_1 of an initiating event occurring. Environmental state M is the most dangerous state with the highest probability, b_M of an initiating event occurring. We assume b_m is non-decreasing in the index of the environmental state m and $0 < b_m \leq 1$. In the military situation, the forces and their equipment are put in varying states of readiness, such as the US DEFCON level, or the UK, black/red/amber grading, corresponding to the environmental situation. Thus these states of readiness can be taken as surrogates for the environmental state.

2.2.4 *Environment situation transition probability matrix (ESTPM), S_{mu}*

The dynamics of the environmental situation is also described by a Markov chain with Environment Situation Transition Probability Matrix (ESTPM), S_{mu}. If the environmental situation is m, $1 \leq m \leq M$ in the current stage, this changes to another environmental situation u, $1 \leq u \leq M$ with probability S_{mu} at the next stage, where

$$\sum_{u=1}^{M} S_{mu} = 1, \quad \text{with } m \text{ and } u = 1, \dots, M$$

We assume the ESTPM also satisfies a first order stochastic ordering property, which reflects the fact that dangerous situations are more likely to stay dangerous than peaceful ones. So $\sum_{u=1}^{k} S_{mu} \geq \sum_{u=1}^{k} S_{m+1,u}$ for any $m = 1, 2, \dots, M - 1$ and for any k. The data for estimating these transition probabilities can be obtained by historical analysis.

2.2.5 The possible actions

Recall that in each period three possible actions can be applied to the stand-by system—do nothing, repair or use for training. The "do nothing" action means neither repair/maintenance nor training is undertaken. The "repair" action can be a maintenance action, if the unit is still operable, but is a true repair in state N. It takes R time periods. This action is not perfect in that there is a probability r_i the unit will be in quality state i at the end of the "repair" where $\sum_{i=1}^{N} r_i = 1$. This assumes that the state at the end of the repair is independent of the state before the repair which seems a reasonable assumption for major repairs corresponding to rebuilding or replacing the equipment. Otherwise one has to deal with the situation where there are a number of minor repairs that can be undertaken each for different states of the equipment but that would require a much more detailed description of the state of the system than the one used here. If an initiating event occurs during a repair period, the stand-by unit cannot respond to it, and so a catastrophic event occurs automatically.

2.2.6 Training level, k

The operator of the stand-by unit has L different training levels, i.e. $1, 2, \ldots, L$ where training level 1 is the highest training level of training and level L is the lowest level of training. If there is no training at the moment, the training level k goes to p at the next time stage with probability of Q_{kp} which is the training level transition probability matrix (TLTPM). We assume the Q_{kp} satisfy the stochastic first order condition defined earlier so that those with higher training levels at the start of a period are also likely to have higher one at the end of the period. This first order condition allows the cases of no spontaneous improvement without training, i.e. $Q_{kp} = 0$ if $p < k$, and also the case of a deterministic decrease in operator performance $Q_{k,k+1} = 1$. We take the basic time unit in the model to be the length of a training exercise. Training may not result in a return to the highest training level and the chance that someone entering training with training level k will leave will level p has a probability $\tilde{Q}_{kp}$ where $p = 1, \ldots, L$. This allows for the training to be counterproductive in that one might have a positive probability that training moves one from level k to level p, $p > k$. Again we assume $\tilde{Q}_{kp}$ satisfies the first order stochastic condition so that those with higher levels going into training are more likely to have higher levels coming out. This seems reasonable in that training builds on the existing training level but there is a random nature to the success of the training. The level at the end of the training could be measured through a test or an exercise. Training also causes wear and tear on the equipment to a different extent than when it is not being used. Hence there is a transition probability matrix for the stand-by unit quality state variation caused by the training which is called wear and tear transition probability matrix (WTTPM), $\tilde{P}_{ij}$. If $j < i$, $\tilde{P}_{ij} = 0$. We assume that the WTTPM also satisfies a first order stochastic ordering condition so that $\sum_{j<k} \tilde{P}_{ij} \geq \sum_{j<k} \tilde{P}_{(i+1)j}$ for any k. We assume that training causes more wear and tear on the system than if there is no use of the system and so require $\sum_{j\leq k} P_{ij} \geq \sum_{j\leq k} \tilde{P}_{ij}$ where k is an arbitrary quality state. If the unit is being repaired, no training is possible.

2.2.7 Catastrophic event

If an initiating event occurs either when the stand-by unit is down (in state N) or being repaired, it becomes a catastrophic event. To allow for the possibility that training could be aborted when an initiating event occurs, but only if the equipment is close to where it is needed, we say that if training is occurring there is a probability $(1 - f)$ that the equipment can be brought to where the initiating event has occurred. Finally when an initiating event occurs it is not enough for the equipment to be operating, but the training must be of a sufficient quality if there is not to be a catastrophic outcome. We assume that if the operators have a training level of k, they cannot successfully respond to an initiating event with probability d_k (where d_k increases with k) even if the stand-by unit is working. This reflects the idea that the training levels are more subjective than the state of the equipment. With equipment it is either in a state where it will work or it is not. With operator training one can never be sure that the training level means the operator will deal correctly with the equipment in an initiating event, only, that the more the training, the more likely the operator is able to perform satisfactorily.

2.3 Model

The state space of this model S has three factors which are the system quality state, training level, and environmental state, so

$$S = \{(i,k,m) \in S,\ i = 1,2,\dots,N,\ k = 1,\dots,L \text{ and } m = 1,\dots,M\}$$

where i, k and m are the system's quality state, training level and the environmental situation respectively. Using the standard Markov decision process formulation we describe the model in terms of the optimality equation which reflects the choice of best action in each state. Thus when the system is in quality state i, training level k and the environmental situation is in state m, let $V(i,k,m)$ be the maximum expected number of periods until a catastrophic event occurs. The optimality equation means $V(i,k,m)$ satisfies

$$V(i,k,m) = \max\{W_1(i,k,m),\ W_2(k,m),\ \delta_{iN} W_3(i,k,m)\}$$
$$\text{where } \delta_{iN} = 0 \text{ if } i = N \quad \text{and} \quad \delta_{iN} = 1 \text{ otherwise} \tag{1}$$

where $W_1(i,k,m)$ is the expected number of periods until a catastrophic event occurs if nothing is done now; $W_2(k,m)$ is the expected number of periods until a catastrophic event if a repair is performed now; and $W_3(i,k,m)$ is the expected number of periods until a catastrophic event if training is selected now, provided of course the system is in an "up" state to allow training to occur. $W_1(i,k,m)$, $W_2(k,m)$, $W_3(i,k,m)$ satisfy

$$W_1(i,k,m) = (1 - b_m\phi_{ik})\left(1 + \sum_{j=1}^{N} P_{ij} \sum_{l=1}^{L} Q_{kl} \sum_{u=1}^{M} S_{mu} V(j,l,u)\right)$$
$$\text{where } \phi_{ik} = d_k \text{ if } i \neq N, \quad \phi_{iN} = 1 \text{ if } i = N \tag{2}$$

The first bracket is the probability the system will survive to the end of the current period, i.e. $1 - b_m d_k$ unless it is in quality state N when the initiating event occurs when the probability of surviving is $1 - b_m$. If it survives the current period then the expected number of periods it will survive is 1 plus the number it will survive in future given that is has moved to state (j,l,u) conditioned on the probability of that move and summed over all possible moves it can make. Similarly

$$W_2(k,m) = (1 - b_m)\left(1 + \sum_{j=1}^{R-1} \sum_{u=1}^{M} (\tilde{S}^j)_{mu}\right)$$
$$+ (1 - b_m)\left(\sum_{u=1}^{M} \sum_{l=1}^{L} \sum_{p=1}^{N} \sum_{v=1}^{M} (\tilde{S}^{R-1})_{mv} S_{vu} (Q^R)_{kp} r_j V(j,p,u)\right) \tag{3}$$

where $\tilde{S}_{mu} = S_{mu}(1 - b_u)$ and for any matrix X,

$$(X^R)_{ij} = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_{R-1}} X_{ii_1} X_{i_1 i_2} \cdots X_{i_{R-1} j} \text{ is its } R\text{-fold product}$$

This complicated expression reflects the fact that deciding to repair now means one is not in control of the system for another R periods. The first term reflects the expected number of these R periods the system will survive which is obtained by summing the probabilities of surviving 1, 2, 3, up to $R-1$ periods. Thus $(1-b_m)$ is the probability of surviving the first period and the jth term in the summation is the probability of surviving the $(j+1)$st period of the R periods. The second expression in (3) is the probability of surviving the whole repair process and emerging in state (j, p, u) after the repair. The $\tilde{S}^{R-1}$ term gives the probability of there being no initiating event in the first $R-1$ periods and being in environmental state v at the start of the last period of the repair. S_{vu} is the chance of the environmental state moving to state u in the last period. Q^R_{kp} is the probability of training level being p at the end of the repair and r_j is the probability that the repair leaves the equipment in state j. These all multiply $V(j, p, u)$ the maximum expected number of periods until a catastrophic event. Lastly we have

$$W_3(i,k,m) = \{1 - b_m[f + (1-f)d_k]\}\left(1 + \sum_{j=1}^{N} \tilde{P}_{ij} \sum_{p=1}^{L} \tilde{Q}_{kp} \sum_{u=1}^{M} S_{mu} V(j,p,u)\right) \quad (4)$$

The first bracket reflects the chance of the system surviving a period of training where failure occurs when there is an initiating event during training and the system is either too far away or the operator's training lets them down to respond correctly. The second bracket says that in this case the number of survival periods is 1 plus the future expected number given the system leaves training in state (j, p, u).

Equations (2), (3), (4) can be solved by value iteration where the nth iterate satisfies

$$V^n(i,k,m) = \max\{W_1^n(i,k,m), W_2^n(k,m), \delta_{iN} W_3^n(i,k,m)\} \quad (5)$$

where

$$W_1^n(i,k,m) = (1 - b_m\phi_{ik})\left(1 + \sum_{j=1}^{N} P_{ij} \sum_{l=1}^{L} Q_{kl} \sum_{u=1}^{M} S_{mu} V^{n-1}(j,l,u)\right) \quad (6)$$

$$\begin{aligned} W_2^n(k,m) &= (1-b_m)\left(1 + \sum_{j=1}^{R-1}\sum_{u=1}^{M}(\tilde{S}^j)_{mu}\right) \\ &\quad + (1-b_m)\left(\sum_{u=1}^{M}\sum_{l=1}^{L}\sum_{j=1}^{N}\sum_{v=1}^{M}(\tilde{S}^{R-1})_{mv} S_{vu} (Q^R)_{kl} r_j V^{n-1}(j,l,u)\right) \end{aligned} \quad (7)$$

where $\tilde{S}_{mu} = S_{mu}(1-b_u)$

$$W_3^n(i,k,m) = \{1 - b_m[f + (1-f)d_k]\}\left(1 + \sum_{j=1}^{N} \tilde{P}_{ij} \sum_{p=1}^{L} \tilde{Q}_{kp} \sum_{u=1}^{M} S_{mu} V^{n-1}(j,p,u)\right) \quad (8)$$

If we define the terminal value, $V^0(i,k,m) = 0$, then $V^n(i,k,m)$ is a bounded increasing sequence of functions and so converges to the limit $V(i,k,m)$. Boundedness comes from the ergodicity of P_{ij}. This means there is a positive probability the equipment will reach state N in a finite number of periods, say T, if nothing is done. Since there is a positive probability there will be an initiating event when the equipment is in state N, there is a positive probability of a catastrophic event within T periods and so the expected number of periods until a catastrophe is bounded. Standard results from Markov decision processes (Putterman 1994) show that the limit function satisfies the optimality equations (1)–(4).

Lemma 2.1 *$V(i,k,m)$ is a*

(a) *non-increasing function of i,*
(b) *non-increasing function of k,*
(c) *non-increasing function of m.*

Proof The proofs use induction hypothesis on n in $V^n(i,k,m)$ and then the result (Puterman 1994) that $V(i,k,m)$ is the limit of the value iteration functions $V^n(i,k,m)$. Consider (a) and define $V^0(i,k,m)=0$, then the property holds trivially for $n=0$. So assume $V^{n-1}(i,k,m)$ is non-increasing in i. This together with the stochastic ordering condition of QSTPM and WTTPM implies

$$\sum_{j=1}^{N} P_{ij}\sum_{p=1}^{L} Q_{kp}\sum_{u=1}^{M} S_{mu}V^{n-1}(j,p,u) > \sum_{j=1}^{N} P_{(i+1)j}\sum_{p=1}^{L} Q_{kp}\sum_{u=1}^{M} S_{mu}V^{n-1}(j,p,u) \quad \text{and}$$
$$\sum_{j=1}^{N} \tilde{P}_{ij}\sum_{p=1}^{L} \tilde{Q}_{kp}\sum_{u=1}^{M} S_{mu}V^{n-1}(j,p,u) > \sum_{j=1}^{N} \tilde{P}_{(i+1)j}\sum_{p=1}^{L} \tilde{Q}_{kp}\sum_{u=1}^{M} S_{mu}V^{n-1}(j,p,u)$$

We can conclude that $W_1^n(i,k,m) \geq W_1^n(i+1,k,m)$ and $W_n^3(i,k,m) \geq W_n^3(i+1,k,m)$. Since $W_2^n(k,m)$ is independent of i, it follows from (5) that $V^n(i,k,m) \geq V^n(i+1,k,m)$. Hence the result holds for $V^n(i,k,m)$ and by convergence the results hold in the limit for $V(i,k,m)$. The proofs of (b) and (c) follow in a similar way. □

Theorem 2.1 (a) *In state (i,k,m), the stand-by unit is repaired provided $i \geq i*(k,m)$.*
(b) *If the stand-by unit is down $($in state $N)$, it must be repaired, so $N \geq i*(k,m)$.*

Proof The proof of (a) follows because $V(i,k,m)$ is a non-increasing function in i and the stochastic dominance of P_{ij}, $\tilde{P}_{ij}$, Q_{kp}, and $\tilde{Q}_{kp}$ ensures the non-increasing property carries through to $\sum_j\sum_l Q_{kp}P_{ij}V(j,p,m)$ and $\sum_j\sum_l \tilde{Q}_{kp}\tilde{P}_{ij}V(j,p,m)$. Hence $W_1(i,k,m)$ and $W_3(i,k,m)$ are non-increasing in i. Since $W_2(k,m)$ is independent of i, once $W_2(k,m) \geq W_3(i,k,m)$ and $W_2(k,m) \geq W_1(i,k,m)$, then the same inequalities must hold for larger i. So one repairs if $i \geq i*(k,m)$.

To prove (b) we extend the definition of the value function so that $V(-i,k,m)$ is the expected number of periods until a catastrophic event if there are i more periods before the repair is finished, the training level is k and the environmental state is m. An induction argument will prove that $V(-i,k,m) < V(-i+1,k,m) < \cdots < V(1,k,m)$. If in state (N,k,m) the optimal policy, π^* was to do nothing for t periods and then start the repair, then the policy π which starts the repair immediately gives a higher expected number of periods until a catastrophic event as follows. If $t<R$, this is true because in the first t periods both policies have the same chance of a catastrophic event occurring but after t period π^* is in state $(-R,k',m')$ while π is in state $(-R+t,k',m')$ which has a higher expected number of periods until a catastrophic event. If $t>R$, then after R periods π^* is in state (N,k',m') while π is in state $(1,k',m')$ and $(1,k',m')$ has a higher V value than (N,k',m'). Thus π^* cannot be the optimal policy and so one must repair as soon as the equipment reaches state N. □

Whereas when the equipment is in its worst state, Theorem 2.1 says one needs to repair it immediately, if the training levels are at their worst, it is not always the case that one should train. Theorem 2.2 shows that under certain conditions one only trains in this worst training level if the equipment is in quite good conditions. Otherwise one needs to repair the equipment before improving the training. The conditions are as follows.

Theorem 2.2 *If $d_L = 1$ and $P_{ij} = \tilde{P}_{ij}$ $\forall_{ij}$, then in state (i, L, m) there exists a function $i^*(m)$ so that one trains if $i < i^*(m)$ and repairs if $i \geq i^*(m)$.*

Proof It is enough to show $W_1(i, L, m) \leq \max\{W_2(L, m), W_3(i, L, m)\}$ since then the property that $W_3(i, L, m)$ is non-increasing in i and the fact that $W_2(L, m)$ is independent of i gives the rest of the result. Since $P_{ij} = \tilde{P}_{ij}$ the non-increasing property of $V(i, k, m)$ in k, gives

$$\begin{aligned} &W_3(i, L, m) - W_1(i, L, m) \\ &\quad = (1 - b_m)\left(\sum_{j=1}^{N} \tilde{P}_{ij} \sum_{p=1}^{L} \tilde{Q}_{Lp} \sum_{u=1}^{M} S_{mu} V(j, p, u) - \sum_{j=1}^{N} P_{ij} \sum_{u=1}^{M} S_{mu} V(j, L, u)\right) \\ &\quad = (1 - b_m)\left(\sum_{j=1}^{N} P_{ij} \sum_{p=1}^{L} \tilde{Q}_{Lp} \sum_{u=1}^{M} S_{mu}\big(V(j, p, u) - V(j, L, u)\big)\right) \geq 0 \end{aligned}$$

□

This theorem does not prove that there are some equipment states where one will definitely train if the training level is at its lowest level. If we add the conditions that the state of the equipment after repair is no better than the state of the system after it was in state i (if $i = 1$ this is when it was as good as new) and that operators cannot improve from the worst training level without training, then we get a stronger result.

Theorem 2.3 *In the case when $d_L = 1$, $Q_{LL} = 1$, $P_{ij} = \tilde{P}_{ij}$, all i, j and $P_{i\bullet}$ stochastically dominates $\mathbf{r}_\bullet$, then one always trains in state $i = 1$, $k = L$.*

Proof If the training level is L with $d_L = 1$, (2), (3), (4) can be rewritten as

$$W_1(i, L, m) = (1 - b_m)\left(1 + \sum_{j=1}^{N} P_{ij} \sum_{u=1}^{M} S_{mu} V(j, L, u)\right)$$

$$\begin{aligned} W_2(L, m) &= (1 - b_m)\left(1 + \sum_{j=1}^{R-1} \sum_{u=1}^{M} (\tilde{S}^j)_{mu}\right) \\ &\quad + (1 - b_m)\left(\sum_{u=1}^{M} \sum_{v=1}^{M} \sum_{j=1}^{N} (\tilde{S}^{R-1})_{mv} S_{vu} r_j V(j, L, u)\right) \end{aligned}$$

$$W_3(1, L, m) = (1 - b_m)\left(1 + \sum_{j=1}^{N} \tilde{P}_{1j} \sum_{p=1}^{L} \tilde{Q}_{Lp} \sum_{u=1}^{M} S_{mu} V(j, p, u)\right)$$

Comparing doing nothing, W_1, with training, W_3, the last inequality in Theorem 2.2 still holds and so $W_3(i, L, m) \geq W_1(i, L, m)$.

For the proof that $W_3(1, L, m) \geq W_2(L, m)$, consider the case in which just for this current time period the repair only takes 1 time period. It is obvious that the expected survival time if the repair takes 1 time period, ${}^1W_2(k, m)$, is longer than or equal to the expected survival time if the repair takes R ($R \geq 1$) time periods, $W_2(k, m)$. Hence, ${}^1W_2(k, m) \geq W_2(k, m)$. Now compare training this period with repair this period, if the latter only takes one time period, so

$$\begin{aligned}
&W_3(i,L,m) - {}^1W_2(L,m)\\
&\quad = (1-b_m)\left(1+\sum_{j=1}^{N}\tilde{P}_{ij}\sum_{p=1}^{L}\tilde{Q}_{Lp}\sum_{u=1}^{M}S_{mu}V(j,p,u)\right)\\
&\qquad - (1-b_m)\left(1+\sum_{j=1}^{N}r_j\sum_{u=1}^{M}S_{mu}V(j,L,u)\right)\\
&\quad = (1-b_m)\left(\sum_{p=1}^{L}\tilde{Q}_{Lp}\sum_{u=1}^{M}S_{mu}\left(\sum_{j=1}^{N}\big(\tilde{P}_{ij}V(j,p,u)-r_jV(j,L,u)\big)\right)\right)\ge 0
\end{aligned}$$

where the last inequality follows since $(1-b_m)\ge 0$, and the second bracket is positive. This is true because with $V(i,k,m)$ being a non-increasing function in k and $\sum_{j=1}^{l}\tilde{P}_{ij} = \sum_{j=1}^{l}P_{ij}\ge\sum_{j=1}^{l}r_j$ (stochastic ordering) for any quality state l, the standard stochastic ordering property used in Lemma 2.1 ensures positivity. Hence,

$$W_3(1,L,m)\ge {}^1W_2(L,m)\ge W_2(k,m)$$

Therefore training is also better than repairing and the result is proved. □

If one considers equipment which is not repairable and so maintenance has no effect, then the only actions possible are training or doing nothing.

The optimality equation for $V(i,k,m)$ when $i<N$ in this case satisfies

$$V(i,k,m)=\max\big\{W_1(i,k,m),\,W_3(i,k,m)\big\} \tag{9}$$

where $W_1(i,k,m)$ and $W_3(i,k,m)$ are defined by (2) and (4).

In this case one can show that if the effect of the training is independent of the level going into the training ($\tilde{Q}_{kp}=\tilde{q}_p$) then if one decides to train in state (i,k,m) one should train in all states (i,k',m), $k'\ge k$.

Theorem 2.4 *In the non-repairable equipment special case with $\tilde{Q}_{kp}=\tilde{q}_p$, if one trains in state (i,k,m), one should train in all states (i,k',m), $k'\ge k$.*

Proof If training is optimal at training level k,

$$\begin{aligned}
W_3(i,k,m) &= \big[1-b_m\big(f+(1-f)d_k\big)\big]\left(1+\sum_{j=1}^{N}\tilde{P}_{ij}\sum_{p=1}^{L}\tilde{q}_p\sum_{u=1}^{M}S_{mu}V(j,p,u)\right)\\
&\ge W_1(i,k,m)=(1-b_md_k)\left(1+\sum_{j=1}^{N}P_{ij}\sum_{l=1}^{L}Q_{kl}\sum_{u=1}^{M}S_{mu}V(j,l,u)\right)
\end{aligned}$$

If we let

$$\begin{aligned}
&1-b_m\times\big(f+(1-f)d_k\big)=A(k)\\
&1+\sum_{j=1}^{N}\tilde{P}_{ij}\sum_{p=1}^{L}\tilde{q}_p\sum_{u=1}^{M}S_{mu}V(j,p,u)=B\\
&1-b_md_k=C(k)\\
&1+\sum_{j=1}^{N}P_{ij}\sum_{l=1}^{L}Q_{kl}\sum_{u=1}^{M}S_{mu}V(j,l,u)=D(k)
\end{aligned}$$

Table 1 Quality state TPM, P_{ij}

i	j 1	2	3	4	5	6	7	8	9	10
1	0.2	0.2	0.2	0.1	0.08	0.05	0.05	0.05	0.05	0.02
2	0	0.2	0.2	0.2	0.1	0.1	0.08	0.05	0.04	0.03
3	0	0	0.2	0.2	0.2	0.1	0.1	0.1	0.05	0.05
4	0	0	0	0.2	0.2	0.2	0.15	0.1	0.1	0.05
5	0	0	0	0	0.2	0.3	0.2	0.1	0.1	0.1
6	0	0	0	0	0	0.2	0.3	0.2	0.2	0.1
7	0	0	0	0	0	0	0.2	0.3	0.3	0.2
8	0	0	0	0	0	0	0	0.3	0.4	0.3
9	0	0	0	0	0	0	0	0	0.4	0.6
10	0	0	0	0	0	0	0	0	0	1

the above relation can be rewritten as

$$A(k)B \geq C(k)D(k) \quad \text{or} \quad B \geq \frac{C(k)}{A(k)}D(k)$$

Since $\frac{C(k)}{A(k)} = \frac{1-b_m d_k}{1-b_m(f+(1-f)d_k)}$ as $b_m > 0$, and $t > 0$, $\frac{C(k)}{A(k)}$ is a decreasing function of k from $\frac{1}{1-b_m f}$ at $k = 1$ (perfect training level with $d_1 = 0$) to 1 at $k = L$ (worst training level with $d_L = 1$).

With this and $D(k)$ being a decreasing function in k, we get the inequalities

$$B \geq \frac{C(k)}{A(k)}D(k) \geq \frac{C(k+1)}{A(k+1)}D(k) \geq \frac{C(k+1)}{A(k+1)}D(k+1)$$

Hence,

$$A(k+1)B \geq C(k+1)D(k+1) \quad \Rightarrow \quad W_3(i,k+1,m) \geq W_1(i,k+1,m)$$

So if training is optimal at (i,k,m), it is also optimal for (i,k',m), $k' \geq k$. □

3 Examples

The numerical examples are of two kinds. The first type show optimal policies that are typical for the problem. The second type show optimal policies where the relationship between doing nothing and training is not straightforward. These are counterexamples to conjectures that as the state of the equipment deteriorates one might move from doing nothing to training and then to repairing or from training to doing nothing to repairing.

Consider a problem with five environmental states, 1 (most peaceful environment), 2, 3, 4, 5 (most dangerous), two training levels $(1,2)$, 10 unit quality states, 1 (new), $2,\ldots,9,10$ (down) and with the Quality State Transition Probability Matrix (QSTPM) and Wear and Tear Transition Probability Matrix (WTTPM) given by Tables 1 and 2.

The transition between environmental situations is given in Table 3. The probability of an initiating event, b_m is $\{0.1, 0.2, 0.4, 0.6, 0.7\}$ for environmental situation 1 to 5.

We also assume that repair is not perfect but given by r_j in Table 4 and takes 1 time period $(R = 1)$.

Table 2 Wear and tear TPM, $\tilde{P}_{ij}$

i	j									
	1	2	3	4	5	6	7	8	9	10
1	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.05	0.05
2	0	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.1
3	0	0	0.1	0.1	0.2	0.2	0.1	0.1	0.1	0.1
4	0	0	0	0.1	0.1	0.2	0.2	0.2	0.1	0.1
5	0	0	0	0	0.1	0.1	0.2	0.2	0.2	0.2
6	0	0	0	0	0	0.1	0.2	0.3	0.2	0.2
7	0	0	0	0	0	0	0.1	0.3	0.3	0.3
8	0	0	0	0	0	0	0	0.2	0.4	0.4
9	0	0	0	0	0	0	0	0	0.3	0.7
10	0	0	0	0	0	0	0	0	0	1

Table 3 Environmental situation TPM, $S_{mm'}$

m	m'				
	1	2	3	4	5
1	0.4	0.3	0.2	0.05	0.05
2	0.2	0.4	0.23	0.1	0.07
3	0.1	0.2	0.4	0.2	0.1
4	0.05	0.15	0.2	0.3	0.3
5	0.05	0.1	0.15	0.2	0.5

Table 4 Repair TPM, r_j

J	1	2	3	4	5	6	7	8	9	10
r_j	0.2	0.2	0.1	0.1	0.1	0.1	0.08	0.05	0.05	0.02

The other values are $d_1 = 0$, $d_2 = 1$, $\tilde{Q}_{11} = \tilde{Q}_{21} = 1$, $\tilde{Q}_{12} = \tilde{Q}_{22} = 0$ for training levels 1 and 2. So in training level 2 the operators cannot successfully respond to an initiating event, while in level 1 they will respond successfully to all of them, provided the equipment works. Training will always move someone to level 1 no matter what level they are at. The probability, f, that the training is done so far away that one cannot respond to an initiating event is 0.7. We use different versions of this problem with different training transitions. For example 1 $Q_{kl} = \left(\begin{smallmatrix} 0.6 & 0.4 \\ 0 & 1 \end{smallmatrix}\right)$ and for example 2, $Q_{kl} = \left(\begin{smallmatrix} 0.01 & 0.99 \\ 0 & 1 \end{smallmatrix}\right)$. So training has a positive effect for $1/4 = 2.5$ periods in example 1 and $1/0.99 = 1.01$ periods in example 2.

The results for examples 1 and 2 are shown in Figs. 1a and 1b and Figs. 2a and 2b respectively. We know from Lemma 2.1 that the expected survival period until a catastrophic event is a non-increasing function in quality state i, training level k and environment situation m. Because the transition probability from training level 1 to training level 2 in example 2 is larger than in example 1, training occurs even when training is already at the top level in example 2 as we can see in Fig. 2a, but does not occur in that state in example 1. From

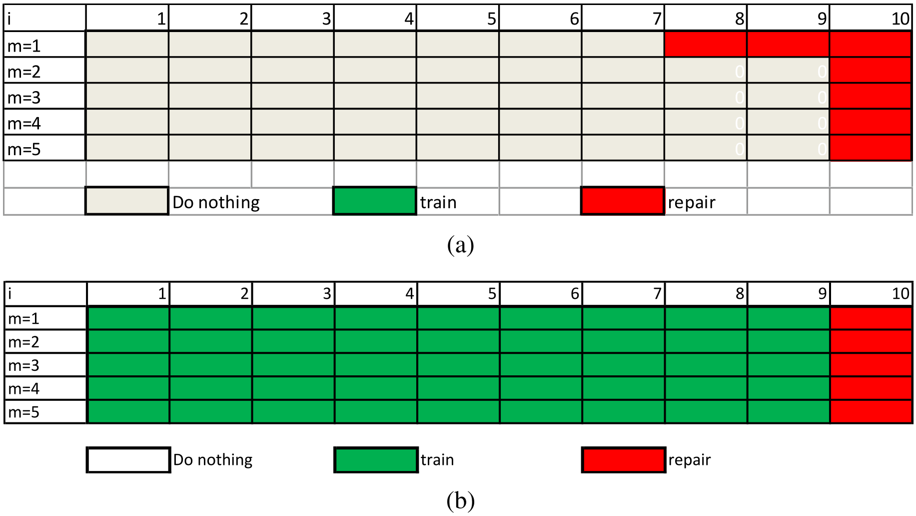

Fig. 1 Result of example 1, (**a**) training level 1, (**b**) training level 2

Figs. 1a and 1b one sees that in most states as one moves from training level 1 to training level 2, the policy changes from doing nothing to training but in the state (8, ., 1) one moves from repairing at training level 1 to training in training level 2.

In example 2, one trains in states $m = 1$, $1 \le i \le 8$ because it is very unlikely there is an initiating event then. It means the operator starts the next period in the top training level whereas doing nothing would make it very likely that the operator is in a poor training level for the next period. However one trains more in the poor training level state in example 1 than example 2. In Fig. 1b, training is optimal when $k = L$ (i.e. level 2 in example 1) and $i \ne N$. However, this is not always true as example 2 shows. From Fig. 2b, repair is optimal even though the training level is $k = 2$ (worst training level) and the quality state is in working condition ($i = 9$).

Example 3 is the case of $P_{ij} = \tilde{P}_{ij}$. This means that training does not cause any more wear and tear than doing nothing. In this example, $P_{i,i+1} = \tilde{P}_{i,i+1} = 1$ where $i = 1, \dots, 5$.

Otherwise, $P_{iN} = \tilde{P}_{iN} = 1$ where $i = 6, \dots, 10$. We use the $Q_{kk'}$ used in example 2. The other conditions are the same as in previous examples. The results for example 3 are shown in Figs. 3a and 3b. Training is always optimal when $k = 2$ (worst training level) and $i = 1$. For the worst training level ($k = 2$), as the quality state increases, the optimal action will change from training to repair provided $d_2 = 1$ and $P_{ij} = \tilde{P}_{ij}$. Environment state 2 in Fig. 3a shows that as the state of the equipment worsens one could move from doing nothing to training, back to doing nothing, train again and final do nothing if the equipment is really bad but not yet needing to be repaired. So there is no general "limit" results in the optimal policy so one trains if the quality of the equipment is above that limit and does nothing if it is below the limit. Theorem 2.1 proves this limit result holds for the repair action in terms of the quality of the equipment. Even for repair though, Kim and Thomas (2006) have examples in the no training case (corresponding to $L = 1$, $d_1 = 0$, $f = 0$) which show that one can change from not repairing to repairing and back to not repairing as the environmental state worsens.

Example 4 looks only at the do nothing or training problem where repair is not possible. We assume 5 training levels for this example and Table 5 gives the training level

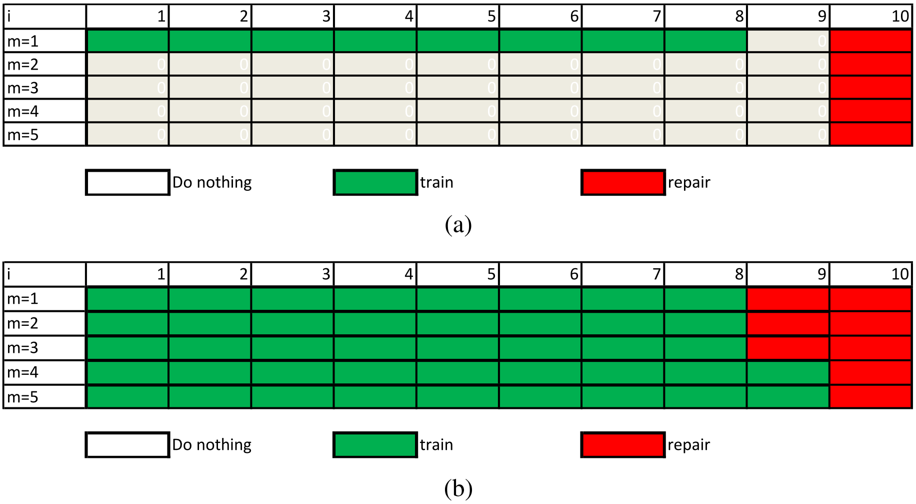

Fig. 2 Result of example 2, (**a**) training level 1, (**b**) training level 2

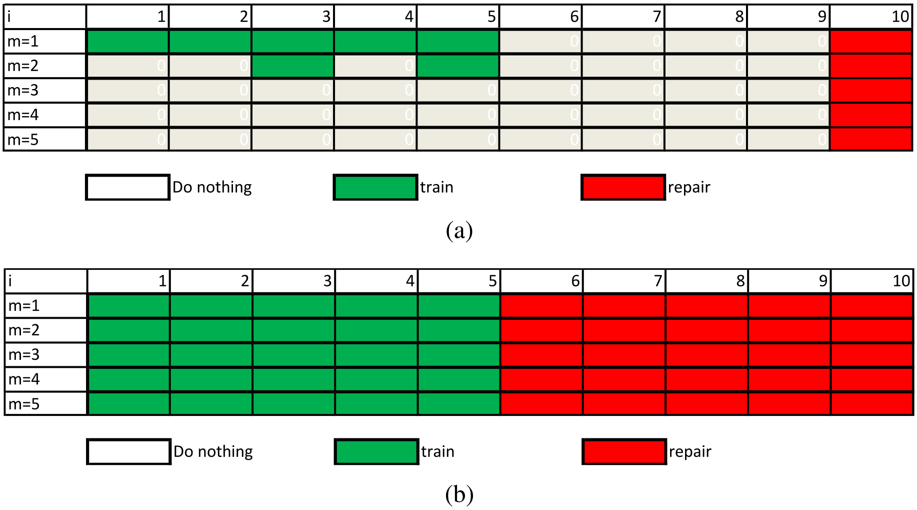

Fig. 3 Result of example 3, (**a**) training level 1, (**b**) training level 2

transition probability whereas the environmental state and quality state transitions stay as in example 2. The values of d_k and $\tilde{Q}_{pk} = \tilde{q}_k$ are $d_k = (0.0, 0.2, 0.5, 0.7, 1.0)$, and $\tilde{q}_k = (0.6, 0.2, 0.1, 0.05, 0.05)$ for training level $k = 1, 2, 3, 4, 5$ respectively. Figures 4a and 4b show the results of example 4 for environmental states 1 and 5. In this case once training is optimal at training level k, training is optimal for all $k' \geq k$.

These examples also show that there cannot be any general form of the optimal train or not train decision in terms of the state of the equipment. In Figs. 3a, 4a and 4b we have examples where one moves from training to doing nothing as the equipment state worsens and also examples where the change is the other way around. If the environmental state worsens then one tends to move from training to doing nothing.

Table 5 Training level TPM for example 4, $Q_{kk'}$

k	k'				
	1	2	3	4	5
1	0.3	0.3	0.2	0.1	0.1
2	0	0.3	0.3	0.2	0.2
3	0	0	0.4	0.3	0.3
4	0	0	0	0.4	0.6
5	0	0	0	0	1

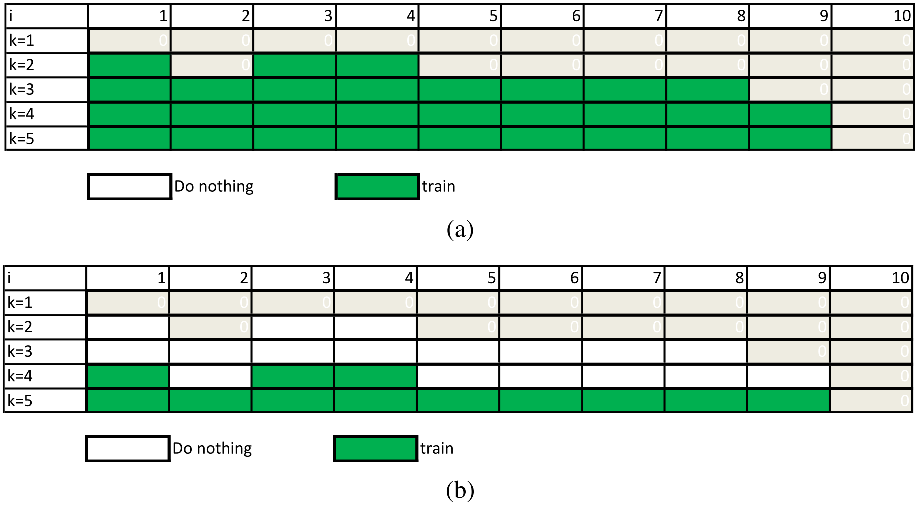

Fig. 4 Result of example 4, (**a**) environment state 1, (**b**) environment state 5

4 Training model with exponential loss of expertise

In this section we consider the problem where the expertise obtained by training on the equipment is gradually lost over time rather than subject to random changes as in Sect. 2. To do this we define an expertise index which shows how well trained the operator of the stand-by unit has been. An operator with higher values in this index is likely to perform better. Apart from the modification of an expertise index instead of a training level, the other conditions are the same as in Sect. 2.

To define the expertise index, we assume that if it is at training level $\tilde{T}$, $\tilde{T} \geq 0$ then

(a) when no training occurs at the next period, it moves to $\alpha\tilde{T}$, $0 \leq \alpha \leq 1$,
(b) when training occurs at the next period, it moves to $\alpha\tilde{T} + 1$.

So in a sense, expertise obtained through training dissipates geometrically (the equivalent of exponentially in discrete time) and each period of training adds 1 unit to the expertise level whatever it is. Thus training all the time gives us a level of $1+\alpha+\alpha^2+\alpha^3+\cdots=(1-\alpha)^{-1}$, while no training gives an expertise of 0. In order to make the index easy to understand, we multiply the above index by $(1-\alpha)$ to arrive at one where all values are between 0 and 1, and if we let $T=(1-\alpha)\tilde{T}$, training changes T into $\alpha T+(1-\alpha)$, while no training changes T into αT. If the expertise index is T, the probability the operator can not respond to satisfactorily to an initiating event is d_T where $0 \le d_T \le 1$, $d_{T'} < d_T$ if $T < T'$.

The state space of this model S then has three factors which are the unit quality state, training level, and environmental state, so

$$S = \{(i, T, m) \in S,\ i = 1, 2, \dots, N,\ 0 \le T \le 1 \text{ and } m = 1, 2, \dots, M\}$$

where i, T and m mean the unit quality state, expertise index and the environmental situation respectively. When the unit is in quality state i, expertise index T and the environmental situation is in state m, $V(i, T, m)$ is the maximum expected number of periods until a catastrophic event occurs.

$$\begin{aligned} V(i, T, m) &= \max\{W_1(i, T, m), W_2(T, m), \delta_{iN} W_3(i, T, m)\} \\ &\text{where } \delta_{iN} = 0 \text{ if } i = N; \quad \delta_{iN} = 1 \text{ otherwise} \end{aligned} \tag{10}$$

and

$$W_1(i, T, m) = (1 - b_m \phi_{iN}) \left(1 + \sum_{j=1}^{N} P_{ij} \sum_{u=1}^{M} S_{mu} V(j, \alpha T, u)\right)$$

$$\text{where } \phi_{iN} = d_T \text{ if } i \ne N, \quad \phi_{iN} = 1 \text{ if } i \ne N \tag{11}$$

$$\begin{aligned} W_2(T, m) &= (1 - b_m)\left(1 + \sum_{j=1}^{R-1} \sum_{u=1}^{M} (\tilde{S}^j)_{mu}\right) \\ &\quad + (1 - b_m)\left(\sum_{u=1}^{M} \sum_{j=1}^{N} \sum_{v=1}^{M} (\tilde{S}^{R-1})_{mv} S_{vu} r_j V(j, \alpha^R T, u)\right) \\ &\text{where } \tilde{S}_{mu} = S_{mu}(1 - b_u), \end{aligned} \tag{12}$$

$$\begin{aligned} W_3(i, T, m) &= \{1 - b_m[f + (1 - f) d_T]\} \\ &\quad \times \left(1 + \sum_{j=1}^{N} \tilde{P}_{ij} \sum_{u=1}^{M} S_{mu} V(j, \alpha T + (1 - \alpha), u)\right) \end{aligned} \tag{13}$$

Equations (10)–(13) can be solved using value iteration. The results of the previous section extend to this model and the proofs follow by induction on value iteration. The value iteration scheme satisfies Eqs. (10)–(13) with V^n, W^n on the L.H.S. and V^{n-1} on the R.H.S.

Lemma 4.1 *$V(i, T, m)$ is a*

(a) *non-increasing function of i,*
(b) *non-increasing function of T,*
(c) *non-increasing function of m.*

Proof As in Lemma 2.1, the proofs use induction hypothesis on n in $V^n(i, T, m)$ and then the result that $V(i, T, m)$ is limit of the value iteration functions $V^n(i, T, m)$. If we consider (a) and define $V^0(i, T, m) = 0$, then the property holds trivially for $n = 0$. So assume $V^{n-1}(i, T, m)$ is non-increasing in i. This together with the stochastic ordering condition of

Table 6 Repair TPM, E_r

j	1	2	3	4	5	6	7	8	9	10
r_j	0.1	0.1	0.2	0.2	0.1	0.1	0.08	0.05	0.05	0.02

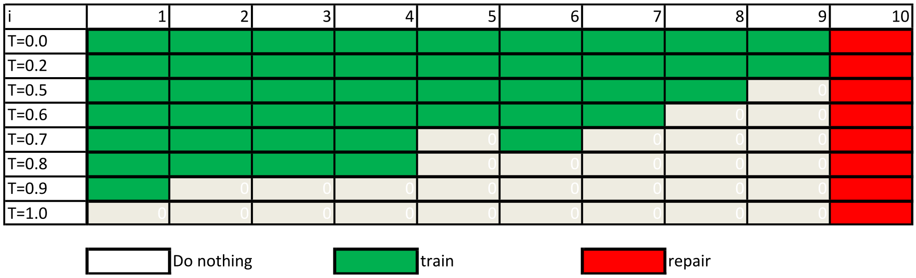

Fig. 5 Result of example 5, environment 3

P_{ij} and $\tilde{P}_{ij}$ implies

$$W_1^n(i, T, m) \geq W_1^n(i+1, T, m)$$
$$W_3^n(i, T, m) \geq W_3^n(i+1, T, m)$$

So it follows that

$$\begin{aligned} V^{n+1}(i, T, m) &= \max\left\{W_1^n(i, T, m),\, W_2^n(T, m),\, W_3^n(i, T, m)\right\} \\ &\geq \max\left\{W_1^n(i+1, T, m),\, W_2^n(T, m),\, W_3^n(i+1, T, m)\right\} \\ &= V^{n+1}(i+1, T, m) \end{aligned}$$

By convergence the results hold in the limit for $V(i, T, m)$. The proofs of (b) and (c) follow in a similar way. □

Theorem 4.1 *In state* N (*the down state*), *one should always repair.*

Proof The proof follows as (b) in Theorem 2.1. □

5 Example

In the example of this model, example 5, there are also 5 different environmental situation states and 10 different unit quality states. The probability of an initiating event, b_m, where $1 \leq m \leq 5$ is $(0.1, 0.2, 0.4, 0.6, 0.7)$. Repair takes 1 time period ($R = 1$) in this example. The transition matrices for quality state, environment situation are the same as in previous examples. The probability, f, that if the equipment is being used for training, it cannot even get to the required position let alone respond to an initiating event is 0.7. The discount factor for the expertness index, α is 0.6. The effect of the repairs has the following distributions in Table 6.

The results in Lemma 4.1 show that the expected survival period is a non-increasing function of the quality state and environmental situation and non-decreasing in the expertness index. The results in general show a pattern where as the quality state increases (gets worse) one initially trains, then does nothing, and then repairs. In Fig. 5, this is the case for T between 0.5 and 0.7. However at $T = 0.7$ this pattern is violated in that one trains and then does nothing, then trains again, then does nothing and finally repairs the equipment. So again there cannot be "clean" results about the form of the optimal policy in terms of the state of the equipment.

6 Conclusions

The models presented in this paper show that there is a strong interaction between the quality state of the stand-by unit, the general environment state, the training level of the operator and the decision on whether to repair or train. In both models, the expected survival time until a catastrophic situation decreases as the quality state of the equipment, the training levels and environment situation worsen. One always repairs when the unit is down. Also, once repair is optimal for a given quality state, repair is optimal for worse quality states. When there is no difference between QSTPM and WTTPM, and if the training level is at its lowest and is one where the operator can not respond satisfactorily to an initiating event, one always trains or repairs. If one adds the extra condition $\sum_{j<l} P_{ij} > \sum_{j<l} r_j$, one can show one always trains when the quality state is in state i or better. If the repair action is not available and the quality state is in a working condition, once training is optimal at a certain training level, one always trains at worse training levels.

In the training model with continuous loss of expertise, we find that one always repairs when the unit is down. If the unit is operating then generally as the expertise index increases, one moves from training to doing nothing, but this need not always be the case. Although there are some results on the form of the optimal policy, particularly on when to repair these are what one might expect. However the relationship between when to train and when to do nothing is complex. There are examples when either the state of the equipment or the environmental state deteriorates and one moves from doing nothing to training as well as moving from training to doing nothing.

It may be possible to extend these models to more than three dimensions using the above approaches so as to include other features that affect the likelihood of a stand-by system responding. However we feel that adding training is the most important extension of the standard model so as to make such models more realistic.

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