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Abstract

We consider testing the null hypothesis of no spatial autocorrelation against the alternative of first order spatial autoregression. A Wald test statistic has good first-order asymptotic properties, but these may not be relevant in small or moderate-sized samples, especially as (depending on properties of the spatial weight matrix) the usual parametric rate of convergence may not be attained. We thus develop tests with more accurate size properties, by means of Edgeworth expansions and the bootstrap. The finite-sample performance of the tests is examined in Monte Carlo simulations.

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1 Introduction

The modelling and analysis of spatially correlated data can pose significant complications and difficulties. Correlation across spatial data is typically a possibility, due to competition, spillovers, aggregation and other circumstances. Such correlation might be anticipated in observable variables or in the unobserved disturbances in an econometric model, or both. In, for example, a linear regression model with exogenous regressors, if only the regressors are spatially correlated the usual rules for large sample inference (based on least squares) are unaffected. However, if also the disturbances are spatially correlated then though least squares estimates of the regression coefficients are likely to retain their consistency property, their asymptotic variance matrix reflects the correlation. This matrix needs to be consistently estimated in order to carry out statistical inference, and its estimation (whether parametric or nonparametric) offers greater challenges than when time series data are involved, due to the lack of ordering in spatial data, as well as possible irregular spacing or lack of reliable information on locations. In addition least squares estimates are rendered asymptotically inefficient by spatial correlation, and developing generalized least squares estimates is similarly beset by ambiguities.

A sensible first step in data analysis is therefore to investigate whether or not there is evidence of spatial correlation, by carrying out a statistical test of the null hypothesis of no spatial correlation. Many such asymptotically valid tests are potentially available, so one might focus on ones that are likely to have reasonable power against anticipated alternatives. This requires specifying a parametric model for the spatial correlation. A widely applicable and popular model is the (first-order) spatial autoregression (SAR). For simplicity we stress the case of zero mean observable data; we shall also allow in some of the paper for an unknown intercept but our work can also be extended to test for lack of spatial correlation in unobservable disturbances in more general models, such as regressions. Given the $n \times 1$ vector of observations $y = (y_1, ..., y_n)'$, the prime denoting transposition, the SAR model is

$$y = \lambda W y + \epsilon, \tag{1.1}$$

where $\epsilon = (\epsilon_1, ..., \epsilon_n)'$ consists of unobservable, uncorrelated random variables with zero mean and unknown variance σ^2, λ is an unknown scalar, and W is an $n \times n$ user-specified "weight" matrix, having (i, j)-th element w_{ij}, where $w_{ii} = 0$ for all i and (in order to identify λ) normalization restrictions satisfied. Such restrictions imply that in general each element w_{ij} changes with n as n increases, implying that W, and thus y, form triangular arrays (i.e. $W = W_n = (w_{ijn})$, $y = y_n = (y_{in})$) but we suppress reference to the n subscript. The element w_{ij} can be regarded as a (scaled) inverse economic distance between locations i and j, where symmetry of W is not necessarily imposed. Thus knowledge of actual locations is not required, extending the applicability of the model beyond situations when they are known, and entailing simpler modelling than is typically possible when one attempts to incorporate locations of irregularly spaced geographical observations.

The null hypothesis of interest is

$$H_0 : \lambda = 0, \tag{1.2}$$

whence the y_i are uncorrelated (and homoscedastic). An obvious statistic for testing (1.2) is the Wald statistic based on the least squares estimate $\hat{\lambda}$ of λ, which is given by

$$\hat{\lambda} = \frac{y'Wy}{y'W'Wy}. \tag{1.3}$$

Due to the dependence between right-hand side observables and disturbances in (1.1), $\hat{\lambda}$ is inconsistent for λ, as discussed by Lee (2002). However, $\hat{\lambda}$ does converge in probability to zero when $\lambda = 0$, so a test statistic for (1.2) based on $\hat{\lambda}$ might be expected to be asymptotically valid. In particular, under (1.1), (1.2) and regularity conditions a central limit theorem for independent non-identically distributed random variables gives

$$\left[tr\left(WW'\right) / \left\{ tr\left(W^2 + WW'\right) \right\}^{1/2} \right] \hat{\lambda} \to_d N(0,1). \tag{1.4}$$

Since the square-bracketed norming factor can be directly computed, asymptotically valid Wald tests against one-sided ($\lambda > 0$ or $\lambda < 0$) or two-sided ($\lambda \neq 0$) hypotheses are readily carried out.

The accuracy of such tests is dependent on the magnitude of n, and the normal approximation might not be expected to be good for smallish n. Moreover, under conditions described later and as shown by Lee (2004) for the Gaussian maximum likelihood estimate of λ, the rate of convergence in (1.4) can be less than the usual parametric rate $n^{1/2}$, depending on the assumptions imposed on W as n increases. In particular if $w_{ij} = O(1/h)$ is imposed, where the positive sequence $h = h_n$ can increase no faster than n, the rate is $(n/h)^{1/2}$, which increases more slowly than $n^{1/2}$ unless h remains bounded. This outcome renders the usefulness of the Wald test based on (1.4) more dubious than in standard parametric situations.

The present paper attempts to remedy these concerns by developing refined tests, which can be expected to perform better in moderate-sized samples. Formal Edgeworth expansions are established in the following section for both $\hat{\lambda}$ and also for the least squares estimate of λ when (1.1) is extended to include an unknown intercept. In Section 3 we deduce corrected critical values and as an alternative, corrected (asymptotically normal) test statistics. In each case the critical values are more accurate than ones based on the first-order normal approximation implied by (1.4). Both one-sided and two-sided tests are considered. Section 4 examines finite-sample performance of our tests in Monte Carlo simulations, comparing also with the simple uncorrected test and tests based on the bootstrap, which (see e.g. Singh (1981) or Hall (1992a)) might be expected to achieve our Edgeworth correction. Proofs are left to an appendix.

Our results are fairly straightforwardly extendable to situations in which y represents unobservable disturbances in regression models, and in which the intercept model we consider is extended to include explanatory variables, but as the topic of higher-order approximations in spatial econometrics is relatively new, we focus here on the most basic, classical settings.

2 Edgeworth expansions for the least squares estimate

The present section develops a (third-order) formal Edgeworth expansion for $\hat{\lambda}$ in (1.3) under the null hypothesis of no spatial correlation (1.2). We introduce first some further definitions and assumptions.

Assumption 1 *The ϵ_i are independent normal random variables with mean zero and unknown variance σ^2.*

Normality is an unnecessarily strong condition for the first-order result (1.4), but it provides some motivation for stressing a quadratic form objective function and is familiar in higher order asymptotic theory. Edgeworth expansions and resulting test statistics are otherwise complicated by the presence of cumulants of ϵ_i. Assumption 1 implies that under (1.2) the y_i are spatially independent.

For a real matrix A, let $||A||$ be the spectral norm of A (i.e. the square root of the largest eigenvalue of $A'A$) and let $||A||_\infty$ be the maximum absolute row sums norm of A (i.e. $||A||_\infty = \max_i \sum_j |a_{ij}|$, in which a_{ij} is the (i,j)th element of A and i and j vary respectively across all rows and columns of A). Let K be a finite generic constant.

Assumption 2

(i) *For all n, $w_{ii} = 0$, $i = 1, \ldots.., n$.*

(ii) *For all sufficiently large n, W is uniformly bounded in row and column sums in absolute value, i.e. $||W||_\infty + ||W'||_\infty \leq K$*

(iii) *For all sufficiently large n, uniformly in $i, j = 1, \ldots, n$, $w_{ij} = O(1/h)$, where $h = h_n$ is a positive sequence bounded away from zero for all n such that $h/n \to 0$ as $n \to \infty$.*

Parts (i) and (ii) of Assumption 2 are standard conditions on W imposed in the literature. In particular, part (ii) was introduced by Kelejian and Prucha (1998) to keep spatial correlation manageable. Commonly in practical applications W is symmetric with non-negative elements and row normalized, such that $\Sigma_{j=1}^{n} w_{ij} = 1$ for all i, in which case Assumption 2(ii) is automatically satisfied. Part (iii) covers two cases which have rather different implications for our results: either h is bounded (when in (1.4) $\hat{\lambda}$ enjoys a parametric $n^{1/2}$ rate of convergence), or h is divergent (when $\hat{\lambda}$ has a slower than parametric, $(n/h)^{1/2}$, rate).

By way of illustration consider (see Case (1991)),

$$W_n = I_r \otimes B_m, \qquad B_m = \frac{1}{m-1}(l_m l_m' - I_m), \tag{2.1}$$

where I_s is the $s \times s$ identity matrix, l_m is the $m \times 1$ vector of 1's, and $\otimes$ denotes Kronecher product. Here W is symmetric with non-negative elements and row normalized, $n = mr$. Parts (i) and (ii) of Assumption 2 are satisfied, and $h \sim m$, where "$\sim$" throughout indicates that the ratio of left and right sides converges to a finite, nonzero constant. Thus in the bounded h case only $r \to \infty$ as $n \to \infty$, whereas in the divergent h case $m \to \infty$ and $r \to \infty$.

Now define

$$t_{ij} = \frac{h}{n} tr(W^i W'^j), \quad i \geq 0, \quad j \geq 0, \quad i + j \geq 1, \tag{2.2}$$

$$t = \frac{h}{n} tr((WW')^2). \tag{2.3}$$

Under Assumption 2 all t_{ij} in (2.2) and t are $O(1)$ (because, for any real A such that $||A||_\infty \leq K$, we have $tr(AW) = O(n/h)$). To ensure the leading terms of the expansion in the theorem below are well defined, we introduce

Assumption 3

$$\underline{lim}_{n\to\infty} \frac{h}{n}(t_{20} + t_{11}) > 0. \tag{2.4}$$

By the Cauchy inequality, Assumption 3 implies $\underline{lim}_{n\to\infty} h t_{11}/n > 0$, and the two conditions are equivalent when W is symmetric or when its elements are all non-negative. Assumption 3 is automatically satisfied under (2.1). It follows from Assumptions 2 and 3 that in (1.4) the norming factor

$$\frac{tr(WW')}{(tr(W^2) + WW')^{1/2}} = \frac{t_{11}}{(t_{20} + t_{11})^{1/2}} \left(\frac{n}{h}\right)^{1/2} \sim \left(\frac{n}{h}\right)^{1/2}. \tag{2.5}$$

Now define

$$a = \frac{t_{11}}{(t_{20} + t_{11})^{1/2}}, \quad b = \frac{t_{21}}{(t_{20} + t_{11})^{1/2} t_{11}}, \quad c = \frac{2t_{30} + 6t_{21}}{(t_{20} + t_{11})^{3/2}}, \tag{2.6}$$

$$d = \frac{t}{t_{11}^2}, \quad e = \frac{12(t_{31} + t_{22})}{(t_{20} + t_{11}) t_{11}}, \quad f = \frac{6t_{40} + 24t_{31} + 6t_{22} + 12t}{(t_{20} + t_{11})^2}, \quad g = \frac{1}{t_{20} + t_{11}} \tag{2.7}$$

and

$$U(\zeta) = 2b\zeta^2 - \frac{c}{6}H_2(\zeta), \tag{2.8}$$

$$V(\zeta) = \frac{1}{6}(e-6bc)\zeta H_2(\zeta) - (d-6b^2)\zeta^3 - \frac{1}{24}fH_3(\zeta) + \frac{1}{3}bc\zeta^2 H_3(\zeta) - 2b^2\zeta^5, \tag{2.9}$$

where $H_j(\zeta)$ is the jth Hermite polynomial, such that

$$H_2(\zeta) = \zeta^2 - 1 \quad H_3(\zeta) = \zeta^3 - 3\zeta. \tag{2.10}$$

Thus $U(\zeta)$ is an even, generally non-homogeneous, quadratic function of ζ, while $V(\zeta)$ is an odd, generally non-homogeneous, polynomial in ζ of degree 5.

Write $\Phi(\zeta) = Pr(Z \leq \zeta)$ for a standard normal random variable Z, and $\phi(\zeta)$ for the probability density function (pdf) of Z. Let $F(\zeta) = P\left((n/h)^{1/2}a\hat{\lambda} \leq \zeta\right)$.

Theorem 1 *Let (1.1) and Assumptions 1-3 hold. Under H_0 in (1.2), for any real ζ, $F(\zeta)$ admits the third order formal Edgeworth expansion*

$$F(\zeta) = \Phi(\zeta) + U(\zeta)\phi(\zeta)\left(\frac{h}{n}\right)^{1/2} + V(\zeta)\phi(\zeta)\frac{h}{n} + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{2.11}$$

where

$$U(\zeta) = O(1), \quad V(\zeta) = O(1), \tag{2.12}$$

as $n \to \infty$.

Generally, $U(\zeta)$ and $V(\zeta)$ are non-zero, whence there are leading correction terms of exact orders $(h/n)^{1/2}$ and h/n, and both terms are known functions of ζ.

A corresponding result to Theorem 1 is available for the pure SAR model with unknown intercept, i.e.

$$y = \mu l + \lambda W y + \epsilon, \tag{2.13}$$

where μ is an unknown scalar and $l = l_n$. The least squares estimate of λ in (2.13) is

$$\tilde{\lambda} = \frac{y'W'Py}{y'W'PWy}, \tag{2.14}$$

where $P = I_n - l(l'l)^{-1}l'$. Under (1.2), the same kind of regularity conditions and the additional

Assumption 4 *For all n, $\Sigma_{j=1}^n w_{ij} = 1$, $i = 1, ..., n$,*

$\tilde{\lambda}$ has the same first-order limit distribution as $\hat{\lambda}$, so (1.4) holds with $\hat{\lambda}$ replaced by $\tilde{\lambda}$.

However the second- and higher-order limit distributions differ. In case Assumption 4 is not satisfied also the first-order limit distribution of $\tilde{\lambda}$ under (1.2) differs from that of $\hat{\lambda}$ and, in particular, $\tilde{\lambda}$ converges to the true value at the standard $n^{1/2}$ rate whether h is bounded or divergent as $n \to \infty$. Since the main goal of this paper is to provide refined tests when the rate of convergence might be slower than the parametric rate $n^{1/2}$, the case of model (2.13) when W is not row-normalized is not considered here.

Define

$$\tilde{U}(\zeta) = U(\zeta) + g^{1/2} \tag{2.15}$$

and

$$\tilde{V}(\zeta) = V(\zeta) + \left\{ \frac{g}{2}(1+p) + 2bg^{1/2} - \frac{g^4}{2} \right\} \zeta - 2bg^2\zeta^3 + \frac{cg^{1/2}}{6} H_3(\zeta), \tag{2.16}$$

where

$$p = l'WW'l/n. \tag{2.17}$$

(When W is symmetric Assumption 4 implies $p = 1$). Let $\tilde{F}(\zeta) = P((n/h)^{1/2} a\tilde{\lambda} \leq \zeta)$.

Theorem 2 *Let (2.13) and Assumptions 1-4 hold. Under H_0 in (1.2), for any real ζ, $\tilde{F}(\zeta)$ admits the third order formal Edgeworth expansion*

$$\tilde{F}(\zeta) = \Phi(\zeta) + \tilde{U}(\zeta)\phi(\zeta)\left(\frac{h}{n}\right)^{1/2} + \tilde{V}(\zeta)\phi(\zeta)\frac{h}{n} + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{2.18}$$

where

$$\tilde{U}(\zeta) = O(1), \quad \tilde{V}(\zeta) = O(1), \tag{2.19}$$

as $n \to \infty$.

The second- and third-order correction terms are again generally non-zero, and of orders $(h/n)^{1/2}$ and h/n respectively. Notice that $\tilde{U}(\zeta) > U(\zeta)$, so the second-order approximate distribution function (df) of $\tilde{\lambda}$ is greater than that of $\hat{\lambda}$. The Edgeworth approximation in (2.18) is unaffected by μ (and the approximations in both (2.11) and (2.18) are unaffected by σ^2). Consequently results can be similarly obtained when there is a more general linear regression component than in (2.13), at least when regressors are non-stochastic or strictly exogenous. Indeed, similar techniques will yield approximations with respect to the model $y - \mu l = \lambda W(y - \mu l) + \epsilon$, or more general linear regression models with SAR disturbances.

Finally, it is worth stressing that Theorems 1 and 2 hold not only under Assumption 1, but also for the class of spherically symmetric distributed disturbances (e.g. Hillier (2001) or Forchini (2002)). Specifically, let $w = \epsilon(\epsilon'\epsilon)^{-1/2}$, where ϵ satisfies Assumption 1. Thus, w is uniformly distributed on the unit sphere in $\Re^n$. It can be shown that the distributions of both $\epsilon'W\epsilon/\epsilon'W'W\epsilon$ and $\epsilon'W'P\epsilon/\epsilon'W'PW\epsilon$ are the same as those of $w'Ww/w'W'Ww$ and $w'W'Pw/w'W'PWw$, respectively. Hence Theorems 1 and 2 hold for scale-mixtures of normals and, more generally, under a spherically symmetric distribution for ϵ, since any random vector within such class would imply w being uniformly distributed on the unit sphere in $\Re^n$.

3 Improved tests for no spatial correlation

We consider first tests of the null hypothesis (1.2) against the alternative

$$H_1 : \quad \lambda > 0 \tag{3.1}$$

in the no-intercept model (1.1).

For $\alpha \in (0, 1)$ (for example $\alpha = 0.05$ or $\alpha = 0.01$) define the normal critical value z_α such that $1 - \alpha = \Phi(z_\alpha)$. Write $q = (n/h)^{1/2} a\hat{\lambda}$. On the basis of (1.4) a test that rejects (1.2) against (3.1) when

$$q > z_\alpha \tag{3.2}$$

has approximate size α. Theorem 1 readily yields more accurate tests that are simple to calculate because the coefficients of $U(\zeta)$ and $V(\zeta)$ are known, W being chosen by the practitioner.

Define the exact critical value w_α such that $1 - \alpha = F(w_\alpha)$, so a test that rejects when $q > w_\alpha$ has exact size α. Also introduce the Edgeworth corrected critical value

$$u_\alpha = z_\alpha - \left(\frac{h}{n}\right)^{1/2} U(z_\alpha). \tag{3.3}$$

Corollary 1 *Let (1.1) and Assumptions 1-3 hold. Under H_0 in (1.2), as $n \to \infty$*

$$w_\alpha = z_\alpha + O\left(\left(\frac{h}{n}\right)^{1/2}\right) \tag{3.4}$$

$$= u_\alpha + O\left(\frac{h}{n}\right). \tag{3.5}$$

Corollary 1 follows follows immediately from Theorem 1. From Corollary 1, the test that rejects (1.2) against (3.1) when

$$q > u_\alpha \tag{3.6}$$

is more accurate than (3.2). Of course when the alternative of interest is $\lambda < 0$, the same conclusion can be drawn for the tests which reject when $q < -z_\alpha$, $q < -u_\alpha$, respectively.

Instead of correcting critical values we can derive from Theorem 1 a corrected test statistic that can be compared with z_α. Introduce the polynomial

$$G(\zeta) = \zeta + \left(\frac{h}{n}\right)^{1/2} U(\zeta) + \frac{h}{n}\frac{1}{3}\left(2b - \frac{c}{6}\right)^2 \zeta^3. \tag{3.7}$$

which has known coefficients (see Yanagihara et al. (2005)). Since $G(\zeta)$ has derivative $(1+\zeta(2b-c/6)(h/n)^{1/2})^2 > 0$, it is monotonically increasing. Thus $F(\zeta) = P(G(q) \leq G(\zeta))$ and we invert the expansion in Theorem 1 to obtain

Corollary 2 *Let (1.1) and Assumptions 1-3 hold. Under H_0, as $n \to \infty$*

$$P(G(q) > z_\alpha) = \alpha + O\left(\frac{h}{n}\right). \tag{3.8}$$

Thus the test that rejects when

$$G(q) > z_\alpha \tag{3.9}$$

has size that differs from α by smaller order than the size of (3.2).

Still more accurate tests can be deduced from Theorem 1 by employing also the third-order correction factor $V(\zeta)$, but the above tests have the advantage of simplicity. The V term, however, is especially relevant in deriving improved tests against the two-sided alternative hypothesis

$$H_0: \quad \lambda \neq 0. \tag{3.10}$$

Because $U(\zeta)$ is an even function it follows from Theorem 1 that

$$P(|q| \leq \zeta) = 2\Phi(\zeta) - 1 + 2\frac{h}{n}V(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.11}$$

Thence define the Edgeworth-corrected critical value for a two-sided test,

$$v_{\alpha/2} = z_{\alpha/2} - \frac{n}{h}V(z_{\alpha/2}), \tag{3.12}$$

noting that the approximate size-α two-sided test based on (1.4) rejects H_0 against (3.10) when

$$|q| > z_{\alpha/2}. \tag{3.13}$$

Also, define $s_{\alpha/2}$ such that $P(|q| \leq s_{\alpha/2}) = 1 - \alpha$.

Corollary 3 *Let (1.1) and Assumptions 1-3 hold. Under H_0, as $n \to \infty$*

$$s_{\alpha/2} = z_{\alpha/2} + O\left(\frac{h}{n}\right) \tag{3.14}$$

$$= v_{\alpha/2} + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.15}$$

Thus rejecting (1.2) against (3.10) when

$$|q| > v_{\alpha/2} \tag{3.16}$$

rather than (3.13) reduces the error to $O((h/n)^{3/2})$. In fact, Theorem 1 can be established to fourth-order, with fourth-order term that is even in ζ, and error $O((h/n)^2)$, so the error in (3.15) can be improved to $O((h/n)^2)$.

As with the one-sided alternative (3.1), a corrected test statistic that can be compared with $z_{\alpha/2}$ can be derived from Theorem 1. Define (Yanagihara et al. (2005))

$$L(\zeta) = \zeta + \frac{h}{n}V(\zeta) + \left(\frac{h}{n}\right)^2 \frac{1}{4}\left(L_1^2\zeta + \frac{L_2^2\zeta^5}{5} + \frac{L_3^2\zeta^9}{9} + \frac{2}{3}L_1L_2\zeta^3 + \frac{2}{5}L_1L_3\zeta^5 + \frac{2}{7}L_2L_3\zeta^7\right), \tag{3.17}$$

where $L_1 = -\frac{1}{6}(e-6bc)+\frac{1}{8}f$, $L_2 = \frac{1}{2}(e-6bc)-3(d-6b^2)-\frac{1}{8}f-3bc$ and $L_3 = \frac{5}{3}bc-10b^2$, so $L(\zeta)$ is a degree-7 polynomial in ζ with known coefficients. It is readily checked that $V(\zeta)$ has derivative $L_1 + L_2\zeta^2 + L_3\zeta^4$, where $L(\zeta)$ has derivative $(1+(h/n)(L_1 + L_2\zeta^2 + L_3\zeta^4)/2)^2 > 0$ and is thus monotonically increasing. Therefore, from (3.11), we obtain

Corollary 4 *Let (1.1) and Assumptions 1-3 hold. Under H_0, as $n \to \infty$*

$$P(L(|q|) > z_{\alpha/2}) = \alpha + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.18}$$

The transformation in (3.17) and Corollary 4 follow from (3.11) using a minor modification of Theorem 2 of Yanagihara et al. (2005). From the latter result, we conclude that the test that rejects H_0 against (3.10) when

$$L(|q|) > z_{\alpha/2} \tag{3.19}$$

has size which is closer to α than (3.13).

Improved tests can be similarly derived from Theorem 2 for the intercept model in (2.13). We first consider tests of H_0 in (1.2) against (3.1). Let $\tilde{q} = (n/h)^{1/2}a\tilde{\lambda}$. A standard test based on first order asymptotic theory rejects (1.2) against (3.1) at approximate level α when

$$\tilde{q} > z_\alpha. \tag{3.20}$$

Define the exact and Edgeworth-corrected critical values $\tilde{w}_\alpha$, such that $1-\alpha = \tilde{F}(\tilde{w}_\alpha)$, and $\tilde{u}_\alpha = z_\alpha - \tilde{U}(z_\alpha)(h/n)^{1/2} = u_\alpha - g^{1/2}(h/n)^{1/2}$, respectively.

Similarly to Corollaries 1 and 2, from Theorem 2 we deduce

Corollary 5 *Let (2.13) and Assumptions 1-4 hold. Under H_0 in (1.2), as $n \to \infty$*

$$\tilde{w}_\alpha = z_\alpha + O\left(\left(\frac{h}{n}\right)^{1/2}\right) \tag{3.21}$$

$$= \tilde{u}_\alpha + O\left(\frac{h}{n}\right). \tag{3.22}$$

Notice that $\tilde{u}_\alpha < u_\alpha$ for any α, so that the second-order corrected critical value is lower for the intercept model.

Let

$$\tilde{G}(\zeta) = \zeta + \left(\frac{h}{n}\right)^{1/2} \tilde{U}(\zeta) + \frac{h}{n}\frac{1}{3}\left(2b - \frac{c}{6}\right)^2 \zeta^3 = G(\zeta) + \left(\frac{h}{n}\right)^{1/2} g^{1/2}. \tag{3.23}$$

Corollary 6 *Let (2.13) and Assumptions 1-4 hold. Under H_0 in (1.2), as $n \to \infty$*

$$P(\tilde{G}(\tilde{q}) > z_\alpha) = \alpha + O\left(\frac{h}{n}\right). \tag{3.24}$$

Thus, tests that reject (1.2) against (3.1) when either

$$\tilde{q} > \tilde{u}_\alpha \tag{3.25}$$

or

$$\tilde{G}(\tilde{q}) > z_\alpha, \tag{3.26}$$

are more accurate than (3.20).

Also, from Theorem 2 improved tests of (1.2) against (3.10) can be deduced. From (2.18), since $\tilde{U}(\zeta)$ is an even function we obtain,

$$P(|\tilde{q}| \leq \zeta) = 2\Phi(\zeta) - 1 + 2\frac{h}{n}\tilde{V}(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.27}$$

Define $\tilde{s}_{\alpha/2}$ such that $P(|\tilde{q}| \leq \tilde{s}_{\alpha/2}) = 1 - \alpha$ and $\tilde{v}_{\alpha/2} = z_{\alpha/2} - (n/h)\tilde{V}(z_{\alpha/2})$. A standard, approximate size α, two-sided test rejects (1.2) against (3.10) when

$$|\tilde{q}| > z_{\alpha/2}. \tag{3.28}$$

From (3.27) we deduce

Corollary 7 *Let (2.13) and Assumptions 1-4 hold. Under H_0, as $n \to \infty$*

$$\tilde{s}_{\alpha/2} = z_{\alpha/2} + O\left(\frac{h}{n}\right) \tag{3.29}$$

$$= \tilde{v}_{\alpha/2} + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.30}$$

Finally, define

$$\tilde{L}(\zeta) = \zeta + \frac{h}{n}\tilde{V}(\zeta) + \left(\frac{h}{n}\right)^2 \frac{1}{4}\left(\tilde{L}_1^2\zeta + \frac{\tilde{L}_2^2\zeta^5}{5} + \frac{L_3^2\zeta^9}{9} + \frac{2}{3}\tilde{L}_1\tilde{L}_2\zeta^3 + \frac{2}{5}\tilde{L}_1 L_3\zeta^5 + \frac{2}{7}\tilde{L}_2 L_3\zeta^7\right), \tag{3.31}$$

where $\tilde{L}_1 = L_1 + \frac{g}{2}(1+p) + 2bg^{1/2} - \frac{g^4}{2} - \frac{cg^{1/2}}{2}$, $\tilde{L}_2 = L_2 - 6bg^{1/2} + \frac{cg^{1/2}}{2}$.

Corollary 8 *Let (2.13) and Assumptions 1-4 hold. Under H_0, as $n \to \infty$*

$$P(\tilde{L}(|\tilde{q}|) > z_{\alpha/2}) = \alpha + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.32}$$

From Corollaries 7 and 8, we conclude that the tests that reject H_0 against (3.10) when either

$$|\tilde{q}| > \tilde{v}_{\alpha/2} \tag{3.33}$$

or

$$\tilde{L}(|\tilde{q}|) > z_{\alpha/2} \tag{3.34}$$

have sizes closer to α than that obtained from (3.28).

Before concluding this section we should acknowledge that the distribution functions under (1.2) and Assumption 1 of both q and $\tilde{q}$ can also be evaluated numerically using the procedure introduced by Imhof (1961) (for implementation details see e.g. Lu and King (2002)). Exact critical values can then be numerically calculated. However, Imhof-type of implementations heavily rely on numerical solutions of highly non-linear equations and therefore might not be not fully reliable.

4 Bootstrap correction and simulation results

In this section we report and discuss a Monte Carlo investigation of the finite sample performance of the tests derived in Section 3 and of bootstrap tests, given that in many circumstances the bootstrap is known to achieve a first-order Edgeworth correction (see e.g. Singh (1981)). For the no-intercept model (1.1) the bootstrap test is as follows (e.g Paparoditis and Politis (2005)). We construct 199 $n \times 1$ vectors ϵ_j^*, whose elements are independently generated as $N(0, \hat{\sigma}^2)$, $j = 1, ..., 199$. The bootstrap test statistic is $q_j^* = (n/h)^{1/2} a \epsilon_j^{*\prime} W' \epsilon_j^* / \epsilon_j^{*\prime} W' W \epsilon_j^*$, $j = 1, ..., 199$, its $(1-\alpha)$th percentile being u_α^* which solves $\sum_{j=1}^{199} 1(q_j^* \leq u_\alpha^*)/199 \leq 1-\alpha$, where $1(.)$ indicates the indicator function. We reject (1.2) against the one-sided alternative (3.1) when

$$q > u_\alpha^*. \tag{4.1}$$

Defining the $(1-\alpha)$th percentile of $|q_j^*|$ as the value v_α^* solving $\sum_{j=1}^{199} 1(|q_j^*| \leq v_\alpha^*)/199 \leq 1-\alpha$, we reject (1.2) against the two-sided alternative (3.10) if

$$|q| > v_\alpha^*. \tag{4.2}$$

For the intercept model (2.13) we define $\tilde{q}_j^* = (n/h)^{1/2} a \tilde{\epsilon}_j^{*\prime} W' P \tilde{\epsilon}_j^* / \tilde{\epsilon}_j^{*\prime} W' P W \tilde{\epsilon}_j^*$, $j = 1, ..., 199$, where the components of each $\tilde{\epsilon}_j^*$ are independently generated from $N(0, \tilde{\sigma}^2)$ with $\tilde{\sigma}^2 = y' P y / n$. The $(1-\alpha)$th quantiles of $\tilde{q}_j^*$ and $|\tilde{q}_j^*|$, $\tilde{u}_\alpha^*$ and $\tilde{v}_\alpha^*$, solve $\sum_{j=1}^{199} 1(\tilde{q}_j^* \leq \tilde{u}_\alpha^*)/199 \leq 1-\alpha$, and $\sum_{j=1}^{199} 1(|\tilde{q}_j^*| \leq \tilde{v}_\alpha^*)/199 \leq 1-\alpha$, respectively. We reject (1.2) against (3.1) or (3.10) when

$$\tilde{q} > \tilde{u}_\alpha^* \tag{4.3}$$

or

$$|\tilde{q}| > \tilde{v}_\alpha^*, \tag{4.4}$$

respectively.

In the simulations we set $\sigma^2 = 1$ in Assumption 1, $\mu = 2$ in (2.13) and choose W as in (2.1), for various m and r. Recalling that orders of magnitudes in Theorems 1 and 2 are affected by whether h diverges or remains bounded as $n \to \infty$, we represent both cases by different choices of $m \sim h$. We choose $(m, r) = (8, 5)$, $(12, 8)$, $(18, 11)$, $(28, 14)$, i.e. $n = 40$, 96, 198, 392, to represent "divergent" h, and $(m, r) = (5, 8)$, $(5, 20)$, $(5, 40)$, $(5, 80)$, i.e. $n = 40$, 100, 200, 400 to represent "bounded" h. For each of these combinations we compute $\hat{\lambda}$ and $\tilde{\lambda}$ from the same realization of ϵ across 1000 replications. In all tests $\alpha = 0.05$.

Empirical sizes are displayed in Tables 1-8, in which "normal", "Edgeworth", "transformation" and "bootstrap" refer respectively to tests using the standard normal approximation, Edgeworth-corrected critical values, Edgeworth-corrected test statistic and bootstrap critical values, and the respective abbreviations N, E, T, B will be extensively used in the text.

(Tables 1 and 2 about here)

Tables 1 and 2 cover one-sided tests (3.2), (3.6), (3.9), (4.1) in the no-intercept model (1.1), when h is respectively "divergent" and "bounded". Test N is drastically under-sized for each n in both tables. The sizes for E are somewhat better, and improve as n increases, in particular for "divergent" h the discrepancy between empirical and nominal sizes is 18.2% lower relative to N, on average across sample size, while as n increases this discrepancy decreases by about 0.7% for N, but by 9.5% for E. Both T and B perform well for all n. Indeed, on average, when h is "divergent" empirical sizes for T and B are 80.4% and 85.4%, respectively, closer to 0.05 than those for N, with a similar pattern in Table 2. Tables 1 and 2 are consistent with Theorem 1 in which F converges to Φ at rate $n^{1/2}$ when h is bounded, but only at rate $(n/h)^{1/2}$ when h is divergent. Indeed, when h is "bounded", on average the difference between empirical and nominal size decreases by 6.8% as n increases for N, while this difference only decreases by 0.7% in case h is "divergent". Also, from Table 2, the average improvements offered by E, T and B over N are about 41%, 88% and 84%, respectively. Overall, T and B perform best.

(Tables 3 and 4 about here)

Tables 3 and 4 cover two-sided tests for the no-intercept model (1.1), namely (3.13), (3.16), (3.19) and (4.2). Again, N is very poor, though contrary to the one-sided test case the problem is now over-sizing, and E, T and B all offer notable improvements. Indeed, when h is "divergent" the difference between empirical and nominal sizes is reduced respectively on average across sample sizes by 87.4%, 59% and 94% for E, T and B relative to N, and by 86%, 59% and 95% when h is "bounded". In the tables B seems overall most accurate, followed by E.

(Tables 5 and 6 about here)

Tables 5 and 6 contain results for one-sided tests for the intercept model (2.13), the N, E, T and B tests being given in (3.20), (3.25), (3.26) and (4.3). The pattern is similar to that displayed in Tables 1 and 2. For "divergent" h, on average across sample sizes, empirical sizes for E, T and B are 12%, 65% and 89% closer to 5% than ones for N, with figures of 21.7%, 78.7% and 81% for "bounded" h. Overall, B performs best for "divergent" h, but it is difficult to choose between B and T when h is "bounded".

(Tables 7 and 8 about here)

Tables 7 and 8 correspondingly describe two-sided tests given in (3.28), (3.33), (3.34) and (4.4). The improvements on average across sample sizes offered by E, T and B over N are 58%, 27% and 87%, respectively, when h is "divergent", and 64%, 64% and 50%, respectively, when h is "bounded". For "divergent" h B again comes out top overall, followed by E, but for "bounded" h B is outperformed by both E and T.

(Figures 1 and 2 about here)

To illustrate the effect of the transformations $G(.)$ and $\tilde{G}(.)$ used in Section 3, in Figures 1 and 2 we plot the histograms with 100 bins of q and $G(q)$ (Figure 1) and of $\tilde{q}$ and $\tilde{G}(\tilde{q})$ (Figure 2) obtained from 1000 replications when $m = 28$ and $r = 14$. Both figures suggest that the densities of q and $\tilde{q}$ are very skewed to the left and that most of the skewness is removed by the transformations, as in Hall (1992b).

(Tables 9-12 about here)

In Tables 9-12 we assess power against a fixed alternative, i.e.

$$H_1 : \lambda = \bar{\lambda} > 0. \tag{4.5}$$

Tables 9 and 10 display the empirical power of one-sided tests in the no-intercept model (1.1) when h is "divergent" and "bounded" respectively, while Tables 11 and 12 correspondingly contain results for the intercept model (2.13). These are non-size-corrected tests. Exept for the smallest sample size when h is "divergent", even N performs well for the largest $\bar{\lambda} = 0.8$, as do all other tests in all settings. N also does comparably well to E, T and B when h is bounded and $\bar{\lambda} = 0.5$. But overall N is outperformed by the other tests, with T and B offering the greatest power.

A remark on consistency of standard and corrected tests is desirable. As previously mentioned, $\hat{\lambda}$ and $\tilde{\lambda}$ are inconsistent when λ is non-zero. Therefore, in case $plim\hat{\lambda} < \lambda \;\; (> \lambda)$ as $n \to \infty$ for $\lambda > 0$ $(\lambda < 0)$, it might be that under H_1, $plim\hat{\lambda} = 0$ as $n \to \infty$, with the same possibilities for $\tilde{\lambda}$. Then the standard and corrected tests would be inconsistent. For the special case of W in (2.1), the following theorem shows that the direction of inconsistency follows the sign of λ.

Theorem 3

(i) *Let model (1.1) hold. Under Assumption 1 and (2.1),* $\underset{n\to\infty}{plim}\;(\hat{\lambda} - \lambda)$ *is finite and has the same sign as* λ.

(ii) *Let model (2.13) hold. Under Assumption 1 and (2.1),* $\underset{n\to\infty}{plim}\;(\tilde{\lambda} - \lambda)$ *is finite and has the same sign as* λ.

The proof is in the Appendix. Assumption 1 could be relaxed here, but is retained for algebraic simplicity. Under (1.1), as $n \to \infty$ $plim\hat{\lambda} > \lambda$ $(< \lambda)$ as $n \to \infty$ when $\lambda > 0$ $(\lambda < 0)$ and hence, $P(q > z_\alpha | H_1) \to 1$, $P(q > u_\alpha | H_1) \to 1$ and $P(G(q) > z_\alpha | H_1) \to 1$. Similarly under (2.13), $P(\tilde{q} > z_\alpha | H_1) \to 1$, $P(\tilde{q} > \tilde{u}_\alpha | H_1) \to 1$ and $P(\tilde{G}(\tilde{q}) > z_\alpha | H_1) \to 1$ as $n \to \infty$. The direction of inconsistency could be computed similarly for other choices of W, although it might not always be possible to obtain closed form expressions.

Appendix

Proof of Theorem 1

Under H_0, $\hat{\lambda} = \epsilon' W' \epsilon / \epsilon' W' W \epsilon$ and thus $P(\hat{\lambda} \leq x) = P(\varsigma \leq 0)$, where $\varsigma = \epsilon'(C + C')\epsilon/2$, $C = W' - xW'W$ and x is any real number. We proceed much as in, e.g., Phillips (1977). Under Assumption 1, the characteristic function (cf) of ς is

$$\begin{aligned} E(e^{\frac{it}{2}\epsilon'(C+C')\epsilon}) &= \frac{1}{(2\pi)^{n/2}\sigma^n} \int_{\Re^n} e^{\frac{it}{2}\xi'(C+C')\xi} e^{-\frac{\xi'\xi}{2\sigma^2}} d\xi \\ &= \frac{1}{(2\pi)^{n/2}\sigma^n} \int_{\Re^n} e^{-\frac{1}{2\sigma^2}\xi'(I - it\sigma^2(C+C'))\xi} d\xi \\ &= det(I - it\sigma^2(C + C'))^{-1/2} = \prod_{j=1}^{n} (1 - it\sigma^2 \eta_j)^{-1/2}, \end{aligned} \tag{A.1}$$

where the η_j are eigenvalues of $C + C'$ and $det(A)$ denotes the determinant of a generic square matrix A. From (A.1) the cumulant generating function (cgf) of ς is

$$\begin{aligned} \psi(t) &= -\frac{1}{2}\sum_{j=1}^{n} ln(1 - it\sigma^2\eta_j) = \frac{1}{2}\sum_{j=1}^{n}\sum_{s=1}^{\infty} \frac{(it\sigma^2\eta_j)^s}{s} \\ &= \frac{1}{2}\sum_{s=1}^{\infty} \frac{(it\sigma^2)^s}{s} \sum_{j=1}^{n} \eta_j^s = \frac{1}{2}\sum_{s=1}^{\infty} \frac{(it\sigma^2)^s}{s} tr((C + C')^s). \end{aligned} \tag{A.2}$$

Denoting by κ_s the $s-$th cumulant of ς, from (A.2)

$$\kappa_1 = \sigma^2 tr(C), \tag{A.3}$$

$$\kappa_2 = \frac{\sigma^4}{2} tr((C + C')^2), \tag{A.4}$$

$$\kappa_s = \frac{\sigma^{2s} s!}{2} \frac{tr((C + C')^s)}{s}, s > 2. \tag{A.5}$$

Let $\varsigma^c = (\varsigma - \kappa_1)/\kappa_2^{1/2}$. The cgf of ς^c is

$$\psi^c(t) = -\frac{1}{2}t^2 + \sum_{s=3}^{\infty} \frac{\kappa_s^c (it)^s}{s!}, \tag{A.6}$$

where

$$\kappa_s^c = \frac{\kappa_s}{\kappa_2^{s/2}}, \tag{A.7}$$

so the cf of ς^c is

$$\begin{aligned}E(e^{it\varsigma^c}) &= e^{-\frac{1}{2}t^2}\exp\{\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!}\} \\ &= e^{-\frac{1}{2}t^2}\{1+\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!}+\frac{1}{2!}(\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!})^2+\frac{1}{3!}(\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!})^3+.....\} \\ &= e^{-\frac{1}{2}t^2}\{1+\frac{\kappa_3^c(it)^3}{3!}+\frac{\kappa_4^c(it)^4}{4!}+\frac{\kappa_5^c(it)^5}{5!}+\{\frac{\kappa_6^c}{6!}+\frac{(\kappa_3^c)^2}{(3!)^2}\}(it)^6+.....\}.\end{aligned} \tag{A.8}$$

Thus by Fourier inversion, formally

$$P(\varsigma^c \le z) = \int_{-\infty}^{z}\phi(z)dz + \frac{\kappa_3^c}{3!}\int_{-\infty}^{z}H_3(z)\phi(z)dz + \frac{\kappa_4^c}{4!}\int_{-\infty}^{z}H_4(z)\phi(z)dz + ... \quad . \tag{A.9}$$

Collecting the above results,

$$\begin{aligned}P(\hat{\lambda} \le x) &= P(\varsigma \le 0) = P(\varsigma^c\kappa_2^{1/2}+\kappa_1 \le 0) = P(\varsigma^c \le -\kappa_1^c) \\ &= \Phi(-\kappa_1^c) - \frac{\kappa_3^c}{3!}\Phi^{(3)}(-\kappa_1^c) + \frac{\kappa_4^c}{4!}\Phi^{(4)}(-\kappa_1') + ... \quad .\end{aligned} \tag{A.10}$$

From (A.3), (A.4) and (A.7),

$$\kappa_1^c = \frac{tr(C)}{(\frac{1}{2}tr((C+C')^2))^{1/2}}. \tag{A.11}$$

The numerator of κ_1^c is

$$tr(W) - xtr(WW') = -xtr(WW') = -\frac{n}{h}xt_{11}, \tag{A.12}$$

while its denominator is

$$\begin{aligned}(\frac{1}{2}tr(C+C')^2)^{1/2} &= (tr(W^2)+tr(WW')-4xtr(W^2W')+2x^2tr((WW')^2))^{1/2}. \\ &= \left(\frac{n}{h}\right)^{1/2}\left(t_{20}+t_{11}-4xt_{21}+2x^2t\right)^{1/2}.\end{aligned} \tag{A.13}$$

Thus

$$\kappa_1^c = \frac{-xt_{11}(n/h)^{1/2}}{(t_{20}+t_{11}-4xt_{21}+2x^2t)^{1/2}} = \frac{-xt_{11}(n/h)^{1/2}}{(t_{20}+t_{11})^{1/2}(1-\frac{4xt_{21}-2x^2t}{(t_{20}+t_{11})})^{1/2}}. \tag{A.14}$$

Choose

$$x = \left(\frac{h}{n}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}}{t_{11}}\zeta = (\frac{h}{n})^{1/2}a^{-1}\zeta, \tag{A.15}$$

where a was defined in (2.6). By Taylor expansion

$$\kappa_1^c = -\zeta\left(1 - \frac{4xt_{21} - 2x^2 t}{(t_{20}+t_{11})}\right)^{-1/2} = -\zeta - 2\left(\frac{h}{n}\right)^{1/2} \frac{t_{21}}{t_{11}(t_{20}+t_{11})^{1/2}}\zeta^2$$

$$+ \frac{h}{n}\frac{t}{t_{11}^2}\zeta^3 - 6\frac{h}{n}\left(\frac{t_{21}}{(t_{20}+t_{11})^{1/2}t_{11}}\right)^2\zeta^3 + O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$

$$= -\zeta - 2\left(\frac{h}{n}\right)^{1/2} b\zeta^2 + \frac{h}{n}d\zeta^3 - 6\frac{h}{n}b^2\zeta^3 + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{A.16}$$

where b and d were defined in (2.6) and (2.7). Then by Taylor expansion and using

$$(-d/dx)^j\Phi(x) = -H_{j-1}(x)\phi(x), \tag{A.17}$$

we have

$$\Phi(-\kappa_1^c) = \Phi\left(\zeta + 2\left(\frac{h}{n}\right)^{1/2} b\zeta^2 - \frac{h}{n}d\zeta^3 + 6\frac{h}{n}b^2\zeta^3 + O\left(\left(\frac{h}{n}\right)^{3/2}\right)\right)$$

$$= \Phi(\zeta) + \left(2\left(\frac{h}{n}\right)^{1/2} b\zeta^2 - \frac{h}{n}d\zeta^3 + 6\frac{h}{n}b^2\zeta^3\right)\phi(\zeta) + 2\frac{h}{n}b^2\zeta^4\Phi^{(2)}(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$

$$= \Phi(\zeta) + 2\left(\frac{h}{n}\right)^{1/2} b\zeta^2\phi(\zeta) + \frac{h}{n}\left(-d\zeta^3 + b^2(6\zeta^3 - 2\zeta^4 H_1(\zeta))\right)\phi(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$

$$= \Phi(\zeta) + 2\left(\frac{h}{n}\right)^{1/2} b\zeta^2\phi(\zeta) + \frac{h}{n}\left(-d\zeta^3 + b^2(6\zeta^3 - 2\zeta^5)\right)\phi(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{A.18}$$

Similarly,

$$\Phi^{(3)}(-\kappa_1^c) = \Phi^{(3)}(\zeta) + 2\left(\frac{h}{h}\right)^{1/2} b\zeta^2\Phi^{(4)}(\zeta) + O\left(\frac{h}{n}\right)$$

$$= \left(H_2(\zeta) - 2\left(\frac{h}{h}\right)^{1/2} b\zeta^2 H_3(\zeta)\right)\phi(\zeta) + O\left(\frac{h}{n}\right). \tag{A.19}$$

From (A.5), (A.7),

$$\kappa_3^c = \frac{tr((C+C')^3)}{(\frac{1}{2}tr((C+C')^2))^{3/2}}.$$

By standard algebra, for x defined in (A.15),

$$\frac{1}{2}tr((C+C')^2) = \frac{n}{h}\left(t_{20} + t_{11} - 4\left(\frac{h}{n}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}t_{21}}{t_{11}}\zeta + O\left(\frac{h}{n}\right)\right)$$

$$= \frac{n}{h}(t_{20}+t_{11}) - 4\left(\frac{n}{h}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}t_{21}}{t_{11}}\zeta + O(1), \tag{A.20}$$

$$tr((C+C')^3) = \frac{n}{h}\left(2t_{30}+6t_{21}-12\left(\frac{h}{n}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}(t_{31}+t_{22})}{t_{11}}\zeta + O\left(\frac{h}{n}\right)\right)$$
$$= \frac{n}{h}(2t_{30}+6t_{21}) - 12\left(\frac{n}{h}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}(t_{31}+t_{22})}{t_{11}}\zeta + O(1) \tag{A.21}$$

and thus

$$\kappa_3^c = \frac{\frac{n}{h}(2t_{30}+6t_{21}) - 12\left(\frac{n}{h}\right)^{1/2}(t_{20}+t_{11})^{1/2}(t_{31}+t_{22})t_{11}^{-1}\zeta + O(1)}{\left(\frac{n}{h}\right)^{3/2}(t_{20}+t_{11})^{3/2}\left(1-4\left(\frac{h}{n}\right)^{1/2}t_{21}t_{11}^{-1}(t_{20}+t_{11})^{-1/2}\zeta + O\left(\frac{h}{n}\right)\right)^{3/2}}$$
$$= \left(\left(\frac{h}{n}\right)^{1/2}\frac{2t_{30}+6t_{21}}{(t_{20}+t_{11})^{3/2}} - 12\frac{h}{n}\frac{t_{31}+t_{22}}{t_{11}(t_{20}+t_{11})}\zeta + O\left(\left(\frac{h}{n}\right)^{3/2}\right)\right)$$
$$\times\left(1+6\left(\frac{h}{n}\right)^{1/2}\frac{t_{21}}{t_{11}(t_{20}+t_{11})^{1/2}}\zeta + O\left(\frac{h}{n}\right)\right)$$
$$= \left(\frac{h}{n}\right)^{1/2}\frac{2t_{30}+6t_{21}}{(t_{20}+t_{11})^{3/2}} - 12\frac{h}{n}\frac{t_{31}+t_{22}}{t_{11}(t_{20}+t_{11})}\zeta + \frac{h}{n}\frac{6(2t_{30}+6t_{21})t_{21}}{(t_{20}+t_{11})^2 t_{11}}\zeta + O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$
$$= \left(\frac{h}{n}\right)^{1/2}c - \frac{h}{n}(e-6bc)\zeta + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{A.22}$$

where b, c and e were defined in (2.6) and (2.7).

Similarly,

$$3tr((C+C')^4) = \frac{n}{h}(6t_{40}+24t_{31}+12t+6t_{22}) + O\left(\left(\frac{n}{h}\right)^{1/2}\right) \tag{A.23}$$

and thus

$$\kappa_4^c = \frac{h}{n}\frac{6t_{40}+24t_{31}+12t+6t_{22}}{(t_{20}+t_{11})^2} + O\left(\left(\frac{h}{n}\right)^{3/2}\right) = \frac{h}{n}f + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{A.24}$$

where f was defined in (2.7).

Substituting (A.15), (A.18), (A.19), (A.22) and (A.24) in (A.10) and rearranging using (2.8) and (2.9) completes the proof.

Proof of Theorem 2

Under H_0 and by Assumption 2(i), $\hat{\lambda} = \epsilon' W' P\epsilon/\epsilon' W' PW\epsilon$. Proceeding as before, $P(\tilde{\lambda} \leq x) = P(\varsigma \leq 0)$, which can be written as the right side of (A.10), with $\varsigma = \epsilon'(C+C')\epsilon/2$ and

$$C = W'P(I - xW). \tag{A.25}$$

Derivation of the cumulants κ_j of ς is very similar to that in the proof of Theorem

1, and so is not described in detail. From (A.25), (2.2) and (2.17),

$$\kappa_1 = \sigma^2 tr(C) = -\sigma^2 \left(1 + xtr(W'W) - \frac{x}{n}(l'WW'l)\right) = -\sigma^2 \left(1 + x\frac{n}{h}t_{11} - xp\right). \tag{A.26}$$

Similarly, since

$$l'W^iW'^jl = O(n) \quad \text{for all} \quad i \geq 0, \quad j \geq 0, \tag{A.27}$$

$$\begin{aligned}
\kappa_2 &= \frac{\sigma^4}{2} tr((C + C')^2) \\
&= \sigma^4 \left(tr(W^2) + tr(W'W) - 1 - \frac{1}{n} l'W'Wl - 4x(tr(WW'W) + O(1)) + 2x^2(tr((W'W)^2) + O(1)) \right) \\
&= \sigma^4 \left(\frac{n}{h}(t_{20} + t_{11}) - 1 - p - 4x\left(\frac{n}{h}t_{21} + O(1)\right) + 2x^2\left(\frac{n}{h}t + O(1)\right)\right). \qquad (A.28)
\end{aligned}$$

Proceeding as in the proof of Theorem 1, the first centred cumulant of ς is

$$\kappa_1^c = \frac{-x\frac{n}{h}t_{11} - 1 + xp}{\left(\frac{n}{h}(t_{20} + t_{11})\right)^{1/2}} \left(1 - \frac{1 + p + 4x\left(\frac{n}{h}t_{21} + O(1)\right) - 2x^2\left(\frac{n}{h}t + O(1)\right)}{\frac{n}{h}(t_{20} + t_{11})}\right)^{-1/2}. \tag{A.29}$$

Setting x as in (A.15) and by Taylor expansion,

$$\begin{aligned}
\kappa_1^c &= -\left(\zeta + \frac{(h/n)^{1/2}}{(t_{20} + t_{11})^{1/2}} - \frac{h}{n}\frac{p}{t_{11}}\zeta\right) \\
&\times \left(1 + \left(\frac{h}{n}\right)^{1/2} \frac{2t_{21}}{t_{11}(t_{20} + t_{11})^{1/2}}\zeta + \frac{h}{n}\left(\frac{1}{2(t_{20} + t_{11})} + \frac{1}{2}\frac{p}{t_{20} + t_{11}} - \frac{t}{t_{11}^2}\zeta^2 + \frac{6t_{21}^2}{t_{11}^2(t_{20} + t_{11})}\zeta^2\right)\right) \\
&+ O\left(\left(\frac{h}{n}\right)^{3/2}\right) \\
&= -\left(\zeta + \left(\frac{h}{n}\right)^{1/2} g^{1/2} - \frac{h}{n}\frac{p}{t_{11}}\zeta\right)\left(1 + \left(\frac{h}{n}\right)^{1/2} 2b\zeta + \frac{h}{n}\left(\frac{g}{2} + \frac{g}{2}p - d\zeta^2 + 6b^2\zeta^2\right)\right) \\
&+ O\left(\left(\frac{h}{n}\right)^{3/2}\right) \\
&= -\zeta - \left(\frac{h}{n}\right)^{1/2}(2b\zeta^2 + g^{1/2}) - \frac{h}{n}\left(\frac{g}{2}\zeta + \frac{g}{2}p\zeta - d\zeta^3 + 6b^2\zeta^3 + 2bg^{1/2}\zeta\right) + O\left(\left(\frac{h}{n}\right)^{3/2}\right),
\end{aligned} \tag{A.30}$$

with b, d, g and p defined in (2.6), (2.7) and (2.17). Similarly, by standard algebra and using (A.27),

$$tr((C + C')^3) = \frac{n}{h}(2t_{30} + 6t_{21}) - 12\left(\frac{n}{h}\right)^{1/2} \frac{(t_{20} + t_{11})^{1/2}(t_{31} + t_{22})}{t_{11}}\zeta + O(1), \tag{A.31}$$

agreeing with the corresponding formula in the proof of Theorem 1, so that the third centred cumulant of ς, κ_3^c, is (A.22), whereas the fourth centred cumulant of ς, κ_4^c, is again (A.24).

Next,

$$\begin{aligned}\Phi(-\kappa_1^c) &= \Phi(\zeta) + \left(\frac{h}{n}\right)^{1/2}(2b\zeta^2 + g^{1/2})\phi(\zeta) + \frac{h}{n}\left(\frac{g}{2}\zeta + \frac{g}{2}p\zeta - d\zeta^3 + 6b^2\zeta^3 + 2bg^{1/2}\zeta\right)\phi(\zeta) \\ &+ \frac{1}{2}(2b\zeta^2 + g^{1/2})^2\Phi^{(2)}(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right) \\ &= \Phi(\zeta) + \left(\frac{h}{n}\right)^{1/2}(2b\zeta + g^{1/2})\phi(\zeta) \\ &+ \frac{h}{n}\left(\frac{g}{2}\zeta + \frac{g}{2}p\zeta - d\zeta^3 + 6b^2\zeta^3 + 2bg^{1/2}\zeta - \frac{1}{2}(2b\zeta^2 + g^{1/2})^2 H_1(\zeta)\right)\phi(\zeta) \\ &+ O\left(\left(\frac{h}{n}\right)^{3/2}\right) \qquad \text{(A.32)}\end{aligned}$$

and

$$\begin{aligned}\Phi^{(3)}(-\kappa_1^c) &= \Phi^{(3)}(\zeta) + \left(\frac{h}{h}\right)^{1/2}(2b\zeta^2 + g^{1/2})\Phi^{(4)}(\zeta) + O\left(\frac{h}{n}\right) \\ &= \left(H_2(\zeta) - \left(\frac{h}{h}\right)^{1/2}(2b\zeta^2 + g^{1/2})H_3(\zeta)\right)\phi(\zeta) + O\left(\frac{h}{n}\right). \qquad \text{(A.33)}\end{aligned}$$

Substituting (A.15), (A.22), (A.24), (A.32) and (A.33) in the right side of (A.10) complete the proof.

Proof of Theorem 3

(i) From (1.1), $y = S^{-1}(\lambda)\epsilon$, where $S(x) = I_n - xW$. Under (2.1), $S^{-1}(\lambda)$ exists for any $\lambda \in (-1, 1)$ and

$$S^{-1}(\lambda) = \sum_{i=0}^{\infty}(\lambda W)^i. \qquad \text{(A.34)}$$

From (A.34) $S^{-1}(\lambda)$ is symmetric, $S^{-1}(\lambda)W = WS^{-1}(\lambda)$ and $||S^{-1}(\lambda)||_\infty \leq K$.

For any $\lambda \in (-1, 1)$,

$$\hat{\lambda} - \lambda = \frac{y'W\epsilon}{y'W^2y} = \frac{h\epsilon' S^{-1}(\lambda)W\epsilon/n}{h\epsilon' S^{-1}(\lambda)W^2S^{-1}(\lambda)\epsilon/n}. \qquad \text{(A.35)}$$

As $n \to \infty$, the numerator of the RHS of (A.35) converges in probability to $lim(h/n)\sigma^2 tr(S^{-1}(\lambda)W)$ since $(h/n)(\epsilon' S^{-1}(\lambda)W\epsilon - \sigma^2 tr(S^{-1}(\lambda)W)) \to 0$ in second mean. Similarly, as $n \to \infty$, the denominator of the RHS of (A.35) converges in probability to $lim(h/n)\sigma^2 tr((S^{-1}(\lambda)W)^2)$. Thus

$$\hat{\lambda} - \lambda \xrightarrow{p} \frac{\underset{n\to\infty}{lim} \frac{h}{n} tr(S^{-1}(\lambda)W)}{\underset{n\to\infty}{lim} \frac{h}{n} tr((S^{-1}(\lambda)W)^2)}. \tag{A.36}$$

First we show that the RHS of (A.36) is finite. Since $||S^{-1}(\lambda)||_\infty \leq K$, $(h/n)tr(S^{-1}(\lambda)W) = O(1)$. The denominator in the RHS of (A.36) is non-negative and, by (A.34), $(h/n)tr((S^{-1}(\lambda)W)^2) \sim (h/n)tr(W^2)$, which is non-zero under (2.1). Hence, the RHS of (A.36) is finite and its sign depends on its numerator.

From (2.1) and (A.34),

$$tr(S^{-1}(\lambda)W) = tr(\sum_{i=0}^{\infty} \lambda^i tr(W^{i+1})) = r\sum_{i=0}^{\infty} \lambda^i tr(B_m^{i+1}). \tag{A.37}$$

Since B_m has one eigenvalue equal to 1 and the other $(m-1)$ equal to $-1/(m-1)$,

$$tr(B_m^{i+1}) = 1 + (m-1)\left(\frac{-1}{m-1}\right)^{i+1} \tag{A.38}$$

and hence, since $|\lambda| < 1$,

$$tr(S^{-1}(\lambda)W) = r\sum_{i=0}^{\infty} \lambda^i \left(1 - \left(\frac{-1}{m-1}\right)^i\right) = \frac{r}{1-\lambda} - \frac{r}{1+\frac{\lambda}{m-1}} = \frac{\lambda}{1-\lambda}\frac{rm}{m-1+\lambda}. \tag{A.39}$$

By substituting $h = m-1$ and $n = mr$ into (A.39),

$$\frac{h}{n} tr(S^{-1}(\lambda)W) = \frac{m-1}{mr}\frac{\lambda}{1-\lambda}\frac{rm}{m-1+\lambda} = \frac{\lambda}{1-\lambda}\frac{m-1}{m-1+\lambda}, \tag{A.40}$$

which, for all $\lambda \in (-1, 1)$, has the same sign of λ, whether m is divergent or bounded, for all $m > 1$.

(ii) Under (2.13),

$$\tilde{\lambda} - \lambda = \frac{y'WP\epsilon}{y'WPWy} = \frac{h\epsilon' S^{-1}(\lambda)WP\epsilon/n}{h\epsilon' S^{-1}(\lambda)WPWS^{-1}(\lambda)\epsilon/n}, \tag{A.41}$$

where $y = S^{-1}(\lambda)(\mu l + \epsilon)$ and since from (A.34) $l'S^{-1}(\lambda)WP = l'S^{-1}(\lambda)'W'P = 0$. Thus, similarly to (A.36),

$$\tilde{\lambda} - \lambda \xrightarrow{p} \frac{\underset{n\to\infty}{lim} \frac{h}{n} tr(S^{-1}(\lambda)WP)}{\underset{n\to\infty}{lim} \frac{h}{n} tr((S^{-1}(\lambda)W)^2 P)}. \tag{A.42}$$

The result in (ii) follows from the proof of part (i), after observing that, as $n \to \infty$, $lim(h/n)tr(S^{-1}(\lambda)WP) = lim(h/n)tr(S^{-1}(\lambda)W) + o(1)$ and $lim(h/n)tr((S^{-1}(\lambda)W)^2 P)$

$$= lim(h/n)tr((S^{-1}(\lambda)W)^2) + o(1).$$

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	$m = 8$ $r = 5$	$m = 12$ $r = 8$	$m = 18$ $r = 11$	$m = 28$ $r = 14$
normal	0	0	0.001	0.001
Edgeworth	0.004	0.008	0.010	0.016
transformation	0.036	0.038	0.040	0.047
bootstrap	0.039	0.061	0.053	0.054

Table 1: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.1) in no-intercept model (1.1) when h is "divergent".

	$m = 5$ $r = 8$	$m = 5$ $r = 20$	$m = 5$ $r = 40$	$m = 5$ $r = 80$
normal	0.001	0.001	0.001	0.011
Edgeworth	0.001	0.025	0.028	0.034
transformation	0.042	0.045	0.043	0.052
bootstrap	0.043	0.040	0.057	0.055

Table 2: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.1) in no-intercept model (1.1) when h is "bounded".

	$m = 8$ $r = 5$	$m = 12$ $r = 8$	$m = 18$ $r = 11$	$m = 28$ $r = 14$
normal	0.132	0.130	0.126	0.106
Edgeworth	0.062	0.058	0.060	0.057
transformation	0.105	0.088	0.073	0.060
bootstrap	0.048	0.044	0.045	0.047

Table 3: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.10) in no-intercept model (1.1) when h is "divergent".

	$m=5$ $r=8$	$m=5$ $r=20$	$m=5$ $r=40$	$m=5$ $r=80$
normal	0.096	0.078	0.068	0.061
Edgeworth	0.062	0.051	0.049	0.052
transformation	0.055	0.025	0.042	0.052
bootstrap	0.049	0.047	0.051	0.050

Table 4: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.10) in no-intercept model (1.1) when h is "bounded".

	$m=8$ $r=5$	$m=12$ $r=8$	$m=18$ $r=11$	$m=28$ $r=14$
normal	0	0	0.001	0.001
Edgeworth	0.003	0.005	0.007	0.010
transformation	0.076	0.068	0.064	0.061
bootstrap	0.040	0.048	0.047	0.046

Table 5: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.1) in intercept model (2.13) when h is "divergent".

	$m=5$ $r=8$	$m=5$ $r=20$	$m=5$ $r=40$	$m=5$ $r=80$
normal	0.002	0.005	0.020	0.024
Edgeworth	0.007	0.022	0.027	0.028
transformation	0.062	0.064	0.053	0.055
bootstrap	0.061	0.039	0.054	0.053

Table 6: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.1) in intercept model (2.13) when h is "bounded".

	$m=8$ $r=5$	$m=12$ $r=8$	$m=18$ $r=11$	$m=28$ $r=14$
normal	0.281	0.187	0.170	0.148
Edgeworth	0.127	0.123	0.104	0.084
transformation	0.220	0.168	0.140	0.107
bootstrap	0.080	0.070	0.062	0.062

Table 7: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.10) in intercept model (2.13) when h is "divergent".

	$m=5$ $r=8$	$m=5$ $r=20$	$m=5$ $r=40$	$m=5$ $r=80$
normal	0.156	0.082	0.063	0.062
Edgeworth	0.103	0.068	0.047	0.048
transformation	0.112	0.065	0.052	0.053
bootstrap	0.042	0.058	0.061	0.040

Table 8: Empirical sizes (nominal $\alpha = 0.05$) of tests of H_0 (1.2) against H_1 (3.10) in intercept model (2.13) when h is "bounded".

	$\bar{\lambda}$	$m=8$ $r=5$	$m=12$ $r=8$	$m=18$ $r=11$	$m=28$ $r=14$
normal	0.1	0	0	0.005	0.009
	0.5	0	0.335	0.673	0.854
	0.8	0.257	0.994	1	1
Edgeworth	0.1	0.001	0.008	0.013	0.019
	0.5	0.200	0.562	0.764	0.904
	0.8	0.957	0.998	1	1
transformation	0.1	0.059	0.087	0.129	0.130
	0.5	0.680	0.854	0.924	0.958
	0.8	0.986	0.999	1	1
bootstrap	0.1	0.111	0.119	0.155	0.164
	0.5	0.725	0.873	0.938	0.966
	0.8	0.996	1	1	1

Table 9: Empirical powers of tests of H_0 (1.2) against H_1 (4.5), with nominal size $\alpha = 0.05$ in no-intercept model (1.1) when h is "divergent".

	$\bar{\lambda}$	$m = 5$ $r = 8$	$m = 5$ $r = 20$	$m = 5$ $r = 40$	$m = 5$ $r = 80$
normal	0.1	0.010	0.083	0.187	0.363
	0.5	0.551	0.988	1	1
	0.8	0.999	1	1	1
Edgeworth	0.1	0.016	0.095	0.200	0.375
	0.5	0.676	0.992	1	1
	0.8	1	1	1	1
transformation	0.1	0.122	0.172	0.280	0.420
	0.5	0.858	0.993	1	1
	0.8	1	1	1	1
bootstrap	0.1	0.139	0.203	0.296	0.451
	0.5	0.888	0.992	1	1
	0.8	1	1	1	1

Table 10: Empirical powers of tests of H_0 (1.2) against H_1 (4.5), with nominal size $\alpha = 0.05$ in no-intercept model (1.1) when h is "bounded".

	$\bar{\lambda}$	$m = 8$ $r = 5$	$m = 12$ $r = 8$	$m = 18$ $r = 11$	$m = 28$ $r = 14$
normal	0.1	0	0	0.001	0.008
	0.5	0	0.243	0.627	0.802
	0.8	0.176	0.988	1	1
Edgeworth	0.1	0.002	0.004	0.006	0.013
	0.5	0.231	0.493	0.699	0.852
	0.8	0.924	0.991	1	1
transformation	0.1	0.146	0.147	0.172	0.169
	0.5	0.727	0.863	0.950	0.967
	0.8	0.991	1	1	1
bootstrap	0.1	0.095	0.121	0.133	0.167
	0.5	0.670	0.836	0.924	0.960
	0.8	0.988	0.999	1	1

Table 11: Empirical powers of tests of H_0 (1.2) against H_1 (4.5), with nominal size $\alpha = 0.05$ in intercept model (2.13) when h is "divergent".

	$\bar{\lambda}$	$m = 5$ $r = 8$	$m = 5$ $r = 20$	$m = 5$ $r = 40$	$m = 5$ $r = 80$
normal	0.1	0.004	0.061	0.161	0.316
	0.5	0.455	0.981	1	1
	0.8	0.992	1	1	1
Edgeworth	0.1	0.016	0.055	0.155	0.343
	0.5	0.597	0.981	1	1
	0.8	0.995	1	1	1
transformation	0.1	0.151	0.225	0.313	0.465
	0.5	0.869	0.998	1	1
	0.8	1	1	1	1
bootstrap	0.1	0.101	0.175	0.302	0.437
	0.5	0.858	0.995	1	1
	0.8	0.998	1	1	1

Table 12: Empirical powers of tests of H_0 (1.2) against H_1 (4.5), with nominal size $\alpha = 0.05$ in intercept model (2.13) when h is "bounded".

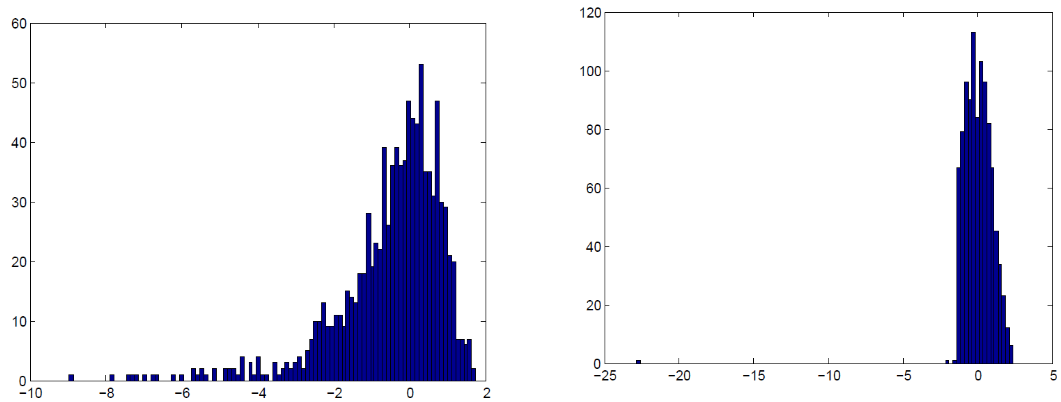

Figure 1: Histograms of q (left picture) and $G(q)$ (right picture) for 1000 replications, $m = 28$, $r = 14$

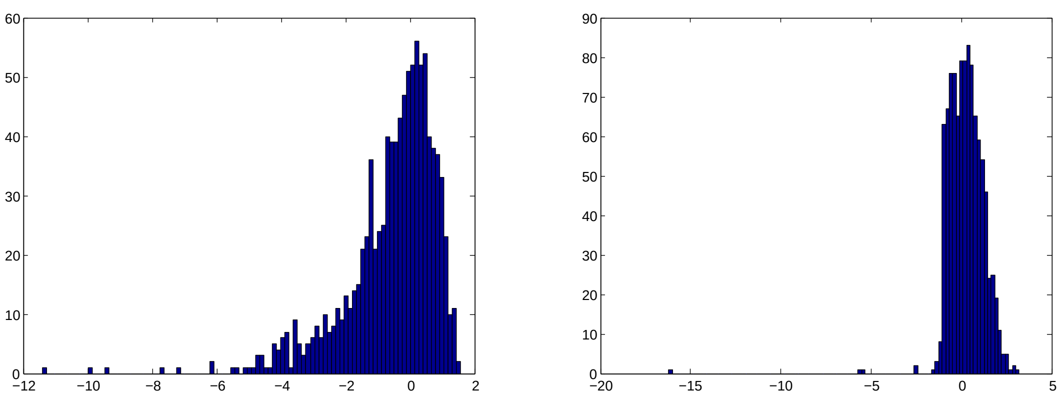

Figure 2: Histograms of $\tilde{q}$ (left picture) and $\tilde{G}(\tilde{q})$ (right picture) for 1000 replications, $m = 28$, $r = 14$