NONLINEAR SYSTEM ANALYSIS

by Volterra and Hermite functional expansions

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Preface

The Volterra series representation appeared in systems engineering just after the Second World War and has since been widely used for system modelling The number of papers using it has grown correspondingly although there are still few books devoted to its theory. The present publication is intended as an initial version of a book treating the subject from a mathematical perspective giving importance to clear derivation of results and their mathematical justification. Particular attention is given to convergence of Volterra series arising from solution of forced nonlinear differential equations.

For Gaussian inputs there is, alongside the system representation by Volterra series, also the representation by Hermite functional series. The book derives the Hermite representation for general Gaussian inputs including white noise input as a special case. The white noise case coincides with the Wiener G-functional expansion. The Hermite functional view consequently gives more generality than treatments using the Wiener theory and the emphasis on using the properties of Hermite polynomials conforms more to the mathematical literature. The approach also has a certain priority over Wiener's G-functional expansion and is related to the equivalent expansion of Itô which also preceded it.

This book originates from an undergraduate course on nonlinear systems given many years ago at Eindhoven Technical High School, now Eindhoven University of Technology. It has been completed in the present form while the writer has been a visitor in the signal processing group in the Institute of Sound and Vibration Research, University of Southampton.

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Chapter 1

VOLTERRA SERIES

1.1 Analytic functionals

It is a common idea in mathematical analysis to regard a function as being approximated by a histogram of its values as shown below for a function x(t) of time t.

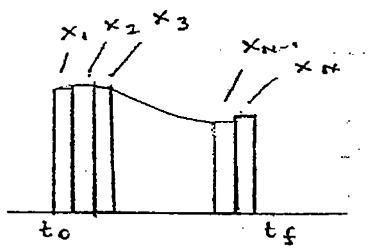

Fig. Discrete approximation to a function by a vector

The approximation is represented by the vector

$$x = [x_1, ..., x_N]$$

taken to be the values $x_1, ..., x_N$ of the function at N values equally spaced within a time interval $t_0 \le t \le t_f$ where t_0 is initial time and t_f final time (denoted by $[t_0, t_f]$). A real valued function

$$f(x) = f(x_1, ..., x_N)$$

may be consequently be regarded as approximating a function depending on another function x(t) defined over $[t_0, t_f]$. It may be written more directly as

$$f(x) = f(x(t), t \,\varepsilon\, [t_0, t_f])$$

Such a function depending on another function is called a *functional.* The idea of creating a theory of functionals on the lines of the normal theory of functions originated with the Italian mathematician Vito Volterra (1864 – 1940)

Volterra considered the analytic case, i.e. when the functional relation has a power series representation. In the discrete approximation this would take the form

$$y = f^{(0)} + \sum_{i_1=1}^{N} f^{(1)}{}_{i_1} x_{i_1} + \frac{1}{2!} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} f^{(2)}{}_{i_1 i_2} x_{i_1} x_{i_2} + \frac{1}{3!} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} f^{(3)}{}_{i_1 i_2 i_3} x_{i_1} x_{i_2} x_{i_3} +$$

$$= f^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} f^{(n)}{}_{i_1,\dots,i_n}\, x_{i_1} \dots x_{i_n} \right\}$$

Here the coefficient arrays $f^{(n)}{}_{i_1,\dots,i_n}$ correspond in the MacLaurin expansion to multidimensional partial derivatives of f evaluated at zero:

$$f^{(n)}{}_{i_1,\dots,i_n} = \frac{\partial^n f}{\partial x_{i_1} \dots \partial x_{i_n}}(0,\dots,0) \qquad i_1, \dots, i_n = 1, \dots, n$$

The coefficient arrays $f^{(n)}{}_{i_1,\dots,i_n}$ are completely symmetrical in the suffices i, i.e. they are the same for all permutations of $i_1,\dots,i_n$.

When interpreted in continuous time, the power series expansion leads naturally to the standard form for the Volterra series of the type originally proposed by Volterra*

$$y = f^{(0)} + \int_{t_0}^{t_f} f^{(1)}(t)\, x(t) + \frac{1}{2!} \int_{t_0}^{t_f} \int_{t_0}^{t_f} f^{(2)}(t_1, t_2)\, x(t_1)\, x(t_1)\, dt_1\, dt_2 +$$

$$+ \frac{1}{3!} \int_{t_0}^{t_f} \int_{t_0}^{t_f} \int_{t_0}^{t_f} f^{(3)}(t_1, t_2, t_3)\, x(t_1)\, x(t_2)\, x(t_3)\, dt_1\, dt_2\, dt_3 + \dots$$

$$= f^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^{t_f} \cdots \int_{t_0}^{t_f} f^{(n)}(t_1, \dots t_n)\, x(t_1) \dots x(t_n)\, dt_1 \dots dt_n$$

The functions $f^{(n)}(t_1, \dots t_n)$ are completely symmetrical in the variables $t_1, \dots, t_n$.

The theory of analytic functionals is more complex than that of ordinary analytic functions and Volterra's expansion is not the only possible type. There is for example Lévy's functional

$$\sum_{r_1 + \quad + r_s = n} \int \int \dots \int x^{r_1}(t_1) \dots x^{r_n}(t_n)\, a^{(n)}(t_1, \dots t_n)\, dt_1 \dots dt_n$$

There can also be an analytic function φ ( ) of derivatives:

$$\varphi\, (x(t), \dot{x}(t), \ddot{x}(t), \quad)$$

The Volterra functional though is of basic importance and in the modified form in which it is used in system analysis finds many applications, particularly to the solution of forced differential equations.

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* See e.g Volterra's well known and much quoted book 'The Theory of Functionals' (Dover reprint 1955). Note that this original form is not the same as what is nowadays called Volterra series which Volterra did once describe though it was not his main interest. Lévy also wrote on nonlinear functional analysis and was the first to use the term 'functional'. Other early writers were Gâteaux and Fréchet.

1.2 Analytic functional operators:

Closely related to the analytical functional is the analytic functional operator. Consider a function relation between two sets of variables $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ having the form

$$
\begin{aligned}
y_1 &= f_1(x_1, \ldots, x_N) \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
y_N &= f_N(x_1, \ldots, x_N)
\end{aligned}
$$

which can equivalently be written as a function relation

$$y = f(x)$$

between vectors

$$x = [x_1, \ldots, x_N], \; y = [y_1, \ldots, y_N]$$

If the variables x_i, y_i, $i = 1, \ldots, N$ are regarded as giving a discrete representation of functions $x(t)$, $y(t)$ on an interval $t_0 \le t \le t_f$, the function f may be regarded as giving a transformation between these functions (see diagram)

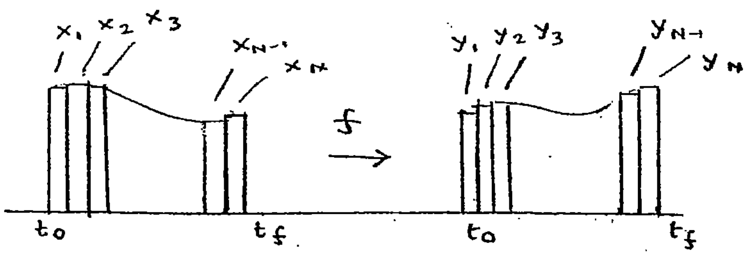

Fig. A function between vectors regarded as a transformation between functions.

When the functions are analytic there will be a multidimensional power series representation

$$y_k = f^{(0)}{}_k + \sum_{i_1=1}^{N} f^{(1)}{}_{k;i_1} x_{i_1} + \frac{1}{2!} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} f^{(2)}{}_{k;i_1 i_2} x_{i_1} x_{i_2}$$

$$+ \frac{1}{3!} \sum_{i_1=1}^{N} \sum_{i_1=1}^{N} \sum_{i_1=1}^{N} f^{(3)}{}_{k;i_1 i_2 i_3} x_{i_1} x_{i_2} x_{i_3} +$$

$$= \sum_{n=0}^{N} \frac{1}{n!} \left\{ \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} f^{(n)}{}_{k;\, i_1 \ldots i_n} x_{i_1} \ldots x_{i_n} \right\}$$

The value $n = 0$ corresponds to the initial constant term $f^{(0)}$. By adjustment of the y scale, this constant term can be assumed zero which is common practice.

In the limiting case, as the number of values on the fixed interval tends to infinity, the function relation becomes

$$y = f(x)$$

where now x and y are themselves functions:

$$x = (x(t), t \varepsilon [t_0, t_f]), \qquad y = (y(t), t \varepsilon [t_0, t_f])$$

these being thought of as infinite dimensional vectors (see the following diagram)

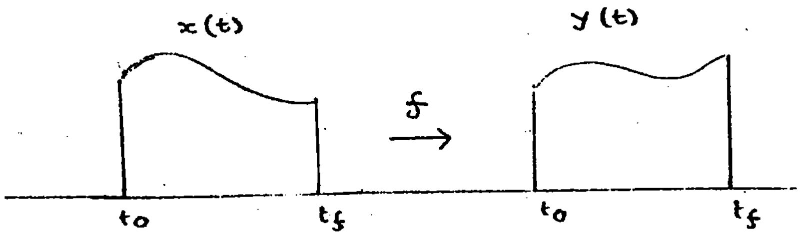

Fig. A function (operator) relating two time functions

Such a function relating two other functions will be called a *(functional) operator.* If the relation between x and y had been written

$$y_k = f_k(x_1, ..., x_n), \quad k = 1, ..., n$$

the corresponding continuous-time relation should be written as

$$y(t) = f(t, x(t); \ t \varepsilon [t_0, t_f])$$

defining y(t) as a functional of x at time t, t now taking the place of the indices. The expansion corresponding to that of Volterra is

$$y(t) = f^{(0)}(t) + \int_{t_0}^{t_f} f^{(1)}(t; t_1)\, x(t_1)\, dt_1 + \frac{1}{2!}\int_{t_0}^{t_f}\int_{t_0}^{t_f} f^{(2)}(t; t_1, t_2)\, x(t_1)\, x(t_1)\, dt_1\, dt_2 +$$

$$+ \frac{1}{3!}\int_{t_0}^{t_f}\int_{t_0}^{t_f}\int_{t_0}^{t_f} f^{(3)}(t; t_1, t_2, t_3)\, x(t_1)\, x(t_2)\, x(t_3)\, dt_1\, dt_2\, dt_3 + ...$$

$$= f^{(0)}(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^{t_f} ... \int_{t_0}^{t_f} f^{(n)}(t; t_1, ... t_n)\, x(t_1) ... x(t_n)\, dt_1 ... dt_n$$

1.3 Volterra series in systems analysis

The main theme of this book is the analysis of physical systems described in the manner of the well known 'block diagram' of the figure below.

$$\rightarrow \text{input} \rightarrow \boxed{\quad \text{system} \quad} \rightarrow \text{output} \rightarrow$$

Fig The block diagram

The block diagram describes a completely general system (often referred to as a 'black box'), in terms of its response (output) caused by a stimulus or disturbance (input). This description is the basis of the discipline of systems analysis which has developed out of engineering practice in the last 60 years. It originated from network analysis in electrical engineering. The input and output variables x and y could be any variables but most frequently they are scalar or finite dimensional vector quantities If they are both scalar then the system is commonly referred to as single-input, single-output. This will be assumed to be so for the present.

The principal assumption implicit in the block diagram description is that the input/output relation between x and y is causal - which may be stated:

The causality condition: The value of the output at any time t is dependent only on previous and present values of the input.

There are two aspects here. Firstly it is affirmed that the output can be uniquely determined by the input and secondly that there is no dependence on future values. If this is so then the output y(t) at any time is a functional of the input at previous times i.e. of x(t'), $t' \leq t$ where t is time t now. A *Volterra system* is one for which this functional relation can be represented in the form

$$y(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{t_0}^{t} \int_{t_0}^{t} \ldots \int_{t_0}^{t} h_n(t; t_1, t_2, \ldots, t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

where t_0 represents an initial start-up time which may be minus infinity.

The realizability condition: The kernels h_n must satisfy the

$$h_n(t; t_1, t_2, \ldots, t_n) = 0 \quad \text{unless} \quad t > t_1,\ t_2, \ldots, t_n$$

In view of this we may write

$$y(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{t_0}^{\infty} \int_{t_0}^{\infty} \ldots \int_{t_0}^{\infty} h_n(t; t_1, t_2, \ldots, t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

Time-invariant systems: In the most important and common case, the input-output relationship is unchanging with time. In this case, a delay (or advance) in the time of occurrence of the input produces the corresponding delay (or advance) in the time of occurrence of the output, the form of the output remaining the same. This can only be so if the kernels h_n depend on time differences, the expansion then taking the form

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^{t} \ldots\ldots \int_{t_0}^{t} h_n(t - t_1, \ldots, t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

Most frequently such systems are considered to be in steady-state operation with start-up time at minus infinity the expansion then taking the form

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{t} \ldots \int_{-\infty}^{t} h_n(t - t_1, \ldots, t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

The realizability condition for time invariant systems becomes:

$$h_n(t - t_1, \ldots, t - t_n) = 0 \text{ unless } t > t_1, t_2, \ldots, t_n$$

This may also be written

$$h_n(\tau_1, \ldots, \tau_n) = 0 \text{ unless } \tau_1 > 0, \ldots, \tau_n > 0.$$

so the input/output relation can also be written

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{0}^{\infty} \ldots\ldots \int_{0}^{\infty} h_n(\tau_1, \ldots, \tau_n)\, x(t - \tau_1) \ldots x(t - \tau_n)\, d\tau_1 \ldots d\tau_n$$

Taking into account the realizability condition all the integrals can also for convenience be written over the doubly infinite range:

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \ldots\ldots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n)\, x(t - \tau_1) \ldots x(t - \tau_n)\, d\tau_1 \ldots d\tau_n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(t - t_1, \ldots, t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

1.4 Examples of Volterra systems :

Cascading: the LNL system: One way in which Volterra systems arise is through the combination of linear filters with amplitude-distorting nonlinearities. Consider the cascade combination of two time-invariant linear filters with transfer functions H(s), K(s) with an amplitude distortion described by the analytic function f(.) arranged as shown*

$$\xrightarrow{x} \boxed{H(p)} \xrightarrow{u} \boxed{f(.)} \xrightarrow{v} \boxed{K(p)} \xrightarrow{y}$$

Fig: The LNL cascade described by a Volterra series.

The indicated system variables satisfy the equations

$$u(t) = \int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt'$$

$$v(t) = f(u(t))$$

$$y(t) = \int_{-\infty}^{\infty} k(t - t')\, v(t')\, dt'$$

where the impulse response functions k(.), h(.) correspond to K(.), H(.) Suppose that f(.) has power series representation

$$v = \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)} u^n$$

Then, on writing

$$u(t)^n = \left\{\int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt'\right\}^n = \int_{-\infty}^{\infty} \ldots\ldots \int_{-\infty}^{\infty} h(t - t_1) \ldots h(t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

it is seen, by substitution, that the input-output relation of the cascade is the Volterra series

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g_n(t - t_1, \ldots, t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

with nth kernel

$$g_n(t - t_1, \ldots, t - t_n) = f^{(n)} \int_{-\infty}^{\infty} k(t - t')\, h(t' - t_1) \ldots h(t' - t_n)\, dt'$$

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* A distinction will be made between the complex variable s of the Laplace transform and the Heaviside operator p (=d/dt). In the literature this distinction is not usual, the two being confused.

Solution of a forced nonlinear differential equation: The solution of many forced nonlinear differential equations can be expressed as Volterra series. As a simple example, consider:

$$\frac{dy}{dt} + \alpha y - \varepsilon y^2 = x(t)$$

where x(t) is a forcing function (input) and y(t) the response (output). α and ε are constants ε being assumed small and α positive. When $\varepsilon = 0$ the equation is linear

$$\frac{dy}{dt} + \alpha y = x(t)$$

On the infinite time interval this has steady-state solution

$$y(t) = \int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt'$$

where h(), the impulse response which is zero for negative values and for positive values

$$h(\tau) = e^{-\alpha\tau} \qquad \tau > 0,$$

When ε has a small nonzero value the solution may be found by successive approximation the linear solution being taken as a first approximation. To do this the equation is written as

$$\frac{dy}{dt} + \alpha y = x(t) + \varepsilon y^2$$

It is then converted to an integral equation over the infinite time interval

$$y(t) = \int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt' + \varepsilon \int_{-\infty}^{\infty} h(t - t')\, y^2(t')\, dt'$$

The linear term on the right is the first approximation. Substituting it into the quadratic term on the right side gives the second approximation as the first two terms of a Volterra series

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt' + \varepsilon \int_{-\infty}^{\infty} h(t - t') \left\{ \int_{-\infty}^{\infty} h(t' - t'')\, x(t'')\, dt'' \right\}^2 dt' \\ &= \int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt' + \varepsilon \int_{-\infty}^{\infty} h(t - t') \left\{ \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h(t' - t_1)\, h(t' - t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2 \right\} dt' \\ &= \int_{-\infty}^{\infty} h^{(1)}(t - t')\, x(t')\, dt' + \frac{1}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^{(2)}(t - t_1, t - t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2 \end{aligned}$$

where

$$h^{(1)}(t - t') = h^{(1)}(t - t'), \quad h^{(2)}(t - t_1, t - t_2) = 2\varepsilon \int_{-\infty}^{\infty} h(t - t')\, h(t' - t_1)\, h(t' - t_2)\, dt',$$

Higher order approximations may be found in the same way. The method here is a simple example of what will be called the reversion method of solving forced nonlinear differential equations which will be developed more fully later.

1.5 Symmetrization of kernels

In the Volterra expansion the kernels may, without loss of generality, be considered to be symmetric functions of their variables, i.e. to satisfy the condition

$$h_n(\tau_1, \tau_1, .., \tau_n) = h_n(\tau_{\sigma 1}, \tau_{\sigma 2}, .., \tau_{\sigma n})$$

where $\sigma_1, \sigma_2, .., \sigma_n$ is any permutation of the integers 1, 2, .., n. The reason for this is that any kernel which initially is unsymmetrical may be replaced by an equivalent symmetrical kernel. without changing the value E.g. for n = 2,

$$\int \int h_2(\tau_1, \tau_2)\, x(t - \tau_1)\, x(t - \tau_2)\, d\tau_1\, d\tau_2 = \int \int h^{(2)}(\tau_1, \tau_2)\, x(t - \tau_1)\, x(t - \tau_2)\, d\tau_1\, d\tau_2$$

where $h^{(2)}(\tau_1, \tau_2)$ is the symmetrical kernel corresponding to $h_2(\tau_1, \tau_2)$.

$$h^{(2)}(\tau_1, \tau_2) = \tfrac{1}{2} \{h_2(\tau_1, \tau_2) + h_2(\tau_2, \tau_1)\}$$

In the case of 3 variables, the symmetric kernel will be

$$h^{(3)}(\tau_1, \tau_2, \tau_3) = 1/3!\ \{h_3(\tau_1, \tau_2, \tau_3) + h_3(\tau_3, \tau_1, \tau_2) + h_3(\tau_2, \tau_3, \tau_1) + h_3(\tau_3, \tau_2, \tau_1) + h_3(\tau_1, \tau_3, \tau_2) + h_3(\tau_2, \tau_1, \tau_3)\}$$

In general for nth order kernels, the symmetrized kernel $h^{(n)}(\tau_1, ..., \tau_n)$ for $h_n(\tau_1, ..., \tau_n)$ is:

$$h^{(n)}(\tau_1, ..., \tau_n) = 1/n! \sum_{\text{perms } \sigma_1, ..., \sigma_n} h_n(\tau_{\sigma 1}, ..., \tau_{\sigma n})$$

the sum on the right-hand side being over all n! permutations $\sigma_1, \sigma_2, ..., \sigma_n$ of 1, 2, ... n.
Then

$$\int ... \int h_n(\tau_1, ..., \tau_n)\, x(t - \tau_1) ... x(t - \tau_n)\, d\tau_1 ... d\tau_n =$$

$$\int ... \int h^{(n)}(\tau_1, ..., \tau_n)\, x(t - \tau_1) ... x(t - \tau_n)\, d\tau_1 ... d\tau_n$$

It will be shown later that symmetrical kernels uniquely determine the Volterra series So it is possible to equate corresponding kernels in two identically equal Volterra series having symmetrical kernels.

Often it is convenient to use unsymmetrical forms of kernels to abbreviate formulae. Then from equality of two Volterra expansions can only be deduced equivalence of the corresponding kernels, two kernels being defined equivalent if they have the same symmetrized form.

1.6 Triangular form:

When the kernels $h_n()$, $n = 1, 2, \ldots$ of a Volterra series are symmetrical it is possible to write the expansion in the form

$$y(t) = \sum_{n=1}^{\infty} \int\int_{0 \le \tau_1 \le \tau_2 \le \ldots \le \tau_n \le t} \ldots\ldots \int h_n(\tau_1, \ldots, \tau_n)\, x(t - \tau_1) \ldots x(t - \tau_n)\, d\tau_1 \ldots d\tau_n$$

This is called the triangular form of the Volterra expansion since the variables are restricted to a generalized triangular region. Note that the n! no longer appears in this representation. The triangular form may alternatively be written as

$$y(t) = \sum_{n=1}^{\infty} \int\int_{t_1 \le t_2 \le \ldots, \le t_n \le t} \ldots \int h_n(t - t_1, \ldots, t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1\, dt_2 \ldots dt_n$$

As an example consider the second order kernel occuring in the previous solution of a differential equation

$$\int_{-\infty}^{\infty} h(t - t')\, h(t' - t_1)\, h(t' - t_2)\, dt'$$

This is zero if either $t_1 > t$ or $t_2 > t$ since h vanishes for negative values. So either $t_1 \le t_2 \le t$ or $t_2 \le t_1 \le t$ must be assumed. The resulting value is

(a) $(1/\alpha)\, e^{-\alpha(t - t_1)} [1 - e^{-\alpha(t - t_2)}] \quad (t_1 \le t_2 \le t)$

(b) $(1/\alpha)\, e^{-\alpha(t - t_2)} [1 - e^{-\alpha(t - t_1)}] \quad (t_2 \le t_1 \le t)$

These two values define a symmetrical kernel $h^{(2)}(\tau_1, \tau_2)$ and the quadratic Volterra term written

$$\int\int_{t_1 \le t_2 \le t} h^{(2)}(t - t_1, t - t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2$$

Chapter 2

CONVERGENCE AND STABILITY

2.1 Application of majorants to convergence of Volterra series.

The *uniform norm* of a function x(t) over a time interval $t \varepsilon [t_0, t_f]$ is defined* as

$$\| x \| = \max_{t \varepsilon [t_0, t_f]} |x(t)|$$

Normally the time interval will be the infinite range $(-\infty, \infty)$. Correspondingly, uniform norm for a Volterra series with kernels $h^{(n)}(..)$ will be defined as

$$\|h_n\| = \int_0^\infty \int_0^\infty \dots \int_0^\infty | h^{(n)}(\tau_1,\dots,\tau_n)| \, d\tau_1 \dots d\tau_n$$

The definition of norms makes it possible to discuss convergence of Volterra series by setting bounds on the terms of the series, use being made of the inequality

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty | h^{(n)}(\tau_1,\dots,\tau_n)| \, |x(t-\tau_1)| \dots |x(t-\tau_n)| \, d\tau_1 \dots d\tau_n \le \| h_n \| \, \| x \|^n$$

Using this inequality the convergence of Volterra series can then be discussed using the method of *majorant series*. A majorant series is defined as any power series relating positive scalar variables X, Y of the form

$$Y = \sum_{n=1}^{\infty} \frac{1}{n!} H_n X^n$$

where the coefficients H_n satisfy the $|h_n\| \le H_n$, $n = 1, 2, \dots$ If $| x(t) | \le X$ each term of the Volterra series is bounded by the corresponding term of the majorant series. So the convergence of the majorant series for a given value of X implies the convergence of the Volterra series (uniformly and absolutely) when $\max \| x(t) \| \le X$.**

Note that in the case of time-varying systems over any range $[t_0, t_f]$, the norm can be defined as

$$\|h_n\| = \max_{t \varepsilon [t_0, t_f]} \int_{t_0}^{t_f} \dots \int_{t_0}^{t_f} |h^{(n)}(t; t_1, t_2, \dots, t_n)| \, dt_1 \dots dt_n$$

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* The logical distinction between max and sup (and between min and inf) may be safely ignored because in applications all functions will certainly be piecewise continuous

** The functions x(t) should belong to a space of functions satisfying the condition of completeness i.e. a Banach space.

2.2 Use of majorant series to determine truncation error

Suppose that the Volterra series

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(t - t_1,\dots, t - t_n)\, x(t_1) \dots x(t_n)\, dt_1 \dots dt_n$$

is truncated to its first N terms giving an approximation

$$y(t) = \sum_{n=1}^{N} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(t - t_1,\dots, t - t_n)\, x(t_1) \dots x(t_n)\, dt_1 \dots dt_n$$

If $|x(t)| \leq X$, $t \; \varepsilon \; (-\infty, \infty)$ the truncation error is clearly bounded by the inequality

$$|y(t) - y_N(t)| \leq \sum_{n=N+1}^{\infty} \frac{1}{n!} H_n X^n = H(X) - \sum_{n=1}^{N} \frac{1}{n!} H_n X^n$$

In many cases the right-hand side may be computed numerically and so an upper bound to the truncation error obtained. If H(X) is known graphically, a bound on the truncation error is shown as the deviation between H(X) and its approximating Nth order polynomial. e.g. with N=1 an upper bound for the error committed by assuming the system linear is shown by the deviation of the graph of H(X) from its tangent as shown below.

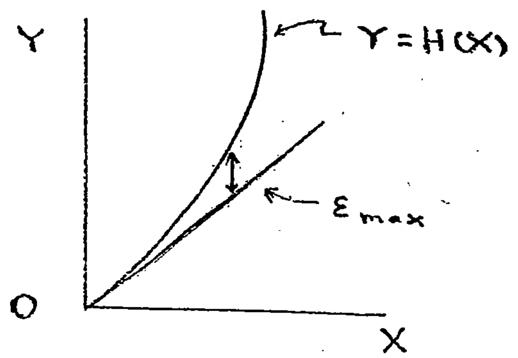

Fig: Showing maximum error for linearity assumption

2.3 Majorant series and bounded-input, bounded-output stability.

Suppose input and output are related by a Volterra series

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(t - t_1, \ldots, t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

and suppose there is a majorant series defining a function H(X):

$$H(X) = \sum_{n=1}^{\infty} \frac{1}{n!} H_n X^n$$

The power series for H(X) will be convergent for values X below a radius of convergence, X* say, which may be finite or infinite. If for all t

$$| x(t)| \leq X < X^*$$

then, as illustrated in the diagram below, the output will satisfy for all t

$$| y(t)| \leq Y = H(X)$$

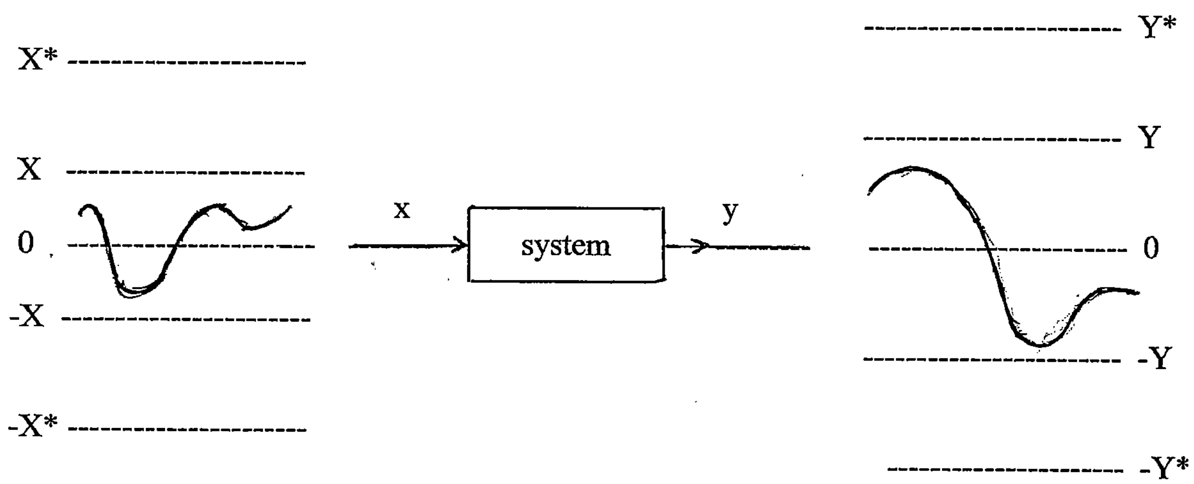

Fig. **Bounded-input, bounded-output (BIBO) system stability in a limited region.**

The system is then remaining stable under all input disturbances within this bound. Stability in this sense is called *bounded-input bounded- output stability* (BIBO stability) Although X* may be infinite commonly it is finite when there is only stability over a range of sufficiently bounded inputs as in the diagram. Such is usually the case with the solutions of forced differential equations.

The relation between input and output is conveniently shown by constructing the graph of the function H(X). The graph must curve upward and if X* is finite the value Y* being either finite or infinite. The graph goes to a vertical tangent there.

2.4 The reversion method for solving forced nonlinear differential equations

The method applies typically to differential equations having the form

$$L(p)\, y + g(y) = x(t)$$

where L(p) is a stable linear polynomial operator in p = d/dt:

$$L(p) = a_0 p^m + a_1 p^{m-1} + \dots + a_m$$

and the function g(y) is assumed analytic and to contain the nonlinear terms it having been assumed that any linear term has been absorbed into L(p).

$$g(y) = g_2 \frac{y^2}{2!} + g_3 \frac{y^3}{3!} + \dots = \sum_{n=2}^{\infty} g_n \frac{y^n}{n!}$$

Since the operator L(p) is stable, the linearized form of the equation i.e.

$$L(p)\, y = x(t)$$

will have a solution on the infinite time interval of

$$y(t) = \int_{-\infty}^{\infty} h(t - t_1)\, x(t_1)\, dt_1$$

where h(.) is the impulse response function which corresponds to the operator $L(p)^{-1}$.

In the nonlinear case, the left side of the differential equation has the form of a power series in y with L(p) as initial linear term. Following the method of reversion of series (appendix 1) we are led to write the equation as

$$y(t) = L(p)^{-1} x(t) - L(p)^{-1} \left\{ g_2 \frac{y(t)^2}{2!} + g_3 \frac{y(t)^3}{3!} + \dots \right\}$$

and solve for y(t) by successive approximation. This in the steady state can be reinterpreted as an equivalent nonlinear integral equation of Volterra type (see footnote)

$$y(t) = \int_{-\infty}^{\infty} h(t - t_1)\, x(t_1)\, dt_1 - \int_{-\infty}^{\infty} h(t - t_1)\, g(t_1)\, dt_1$$

Note: The integral equation is of a type considered by Lalesco in 1912 who called it a Volterra integral equation. See Lalesco: Introduction à la théorie des équations intégrales, Paris 1912 and also Volterra: Leçons sur les équations intégrales, Paris 1913 p.90.

For brevity it will be convenient to omit integration limits which are for convenience always taken over the doubly infinite range. The integral equation may be solved by the method of successive approximations starting with the linear approximation given above. Substitution of this into the quadratic term on the right hand side of the equation results in the second order approximation

$$
\begin{aligned}
y(t) &= \int h(t-t_1)\, x(t_1)\, dt_1 - \frac{g_2}{2!}\int h(t-t')\left\{\int h(t'-t_1)\, x(t_1) dt_1\right\}^2 dt' \\
&= \int h(t-t_1)\, x(t_1)\, dt_1 - \frac{g_2}{2!}\int h(t-t')\left\{\int\int h(t'-t_1)\, h(t'-t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2\right\} dt' \\
&= \int h(t-t_1)\, x(t_1)\, dt_1 + \frac{1}{2!}\int\int h_2(t-t_1, t-t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2
\end{aligned}
$$

where

$$h_2(t-t_1, t-t_2) = -g_2 \int h(t-t')\, h(t'-t_1)\, h(t'-t_2)\, dt'$$

Then the third order approximaton results from resubstitution of these terms into the right hand side of the integral equation. The third order kernel is found as

$$
\begin{aligned}
h_3(t-t_1, t-t_2, t-t_3) &= -3g_2 \int h(t-t')\, h(t'-t_1)\, h_2(t'-t_2, t'-t_3)\, dt' \\
&\quad - g_3 \int h(t-t')\, h(t'-t_1)\, h(t'-t_2)\, h(t'-t_3)\, dt' \\
&= 3g_2^2 \int\int h(t-t')\, h(t'-t_1)\, h(t'-t'')\, h(t''-t_2)\, h(t''-t_3)\, dt'\, dt'' \\
&\quad - g_3 \int h(t-t')\, h(t'-t_1)\, h(t'-t_2)\, h(t'-t_3)\, dt'
\end{aligned}
$$

This process may be continued to find further terms of the Volterra series. The higher order kernels soon become very complicated and can better be represented using either multidimensional transforms or by the multilinear representation described in the following two chapters.

In the case when the function g(y) is of odd order so that

$$g(y) = \frac{g_3\, y^3}{3!} + \frac{g_3\, y^5}{5!} + \ldots$$

the quadratic kernel is zero and the third order kernel simplifies to

$$h_3(t-t_1, t-t_2, t-t_3) = -g_3 \int h(t-t')\, h(t'-t_1)\, h(t'-t_2)\, h(t'-t_3)\, dt'$$

The fifth order kernel is found recursively as

$$
\begin{aligned}
h_5(t-t_1, t-t_2, t-t_3, t-t_4, t-t_5) &= -3\, g_3 \int h(t-t')\, h(t'-t_1)\, h(t'-t_2)\, h_3(t'-t_3)\, dt' \\
&\quad - g_5 \int h(t-t')\, h(t'-t_1)\, h(t'-t_2)\, h(t'-t_3)\, h(t'-t_4)\, h(t'-t_5)\, dt'
\end{aligned}
$$

2.5 Use of a comparison equation to prove convergence of the Volterra series solution

To discuss convergence there is set up an algebraic *comparison equation* relating positive auxiliary variables X, Y having the same form as the nonlinear integral equation :

$$Y = HX + HG(Y)$$

H is assumed to exceed the integral of |h()| which will be finite because of the stability of L(p)

$$H \geq \int_0^{\infty} |h(\tau)|\, d\tau$$

G(Y) is assumed a majorant to g(y) having a series expansion

$$G(Y) = G_2 \frac{Y^2}{2!} + G_3 \frac{Y^3}{3!} + \ldots = \sum_{n=2}^{\infty} G_n \frac{Y^n}{n!} \qquad G_n \geq |g_n|,\ n = 2, 3, \ldots$$

The comparison equation may be solved by successive approximation in the same way as the integral equation and at each stage the solution of the comparison equation majorizes the Volterra approximation to the solution of the differential equation. The convergence of the Volterra solution consequently follows from convergence of the series solution of the comparison equation. This series solution is however just that found for reversion of a power series (see appendix 1) and a region of convergence may be easily determined from the graph of the comparison equation which takes the form shown below.

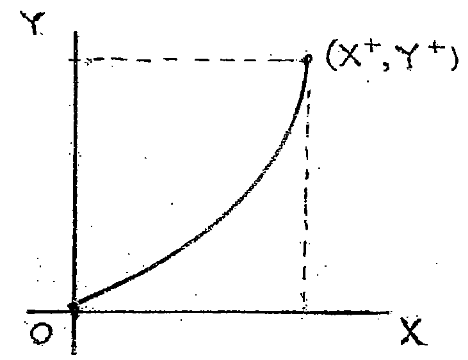

Fig. Graph of the comparison equation

The gradient is positive and steadily increasing becoming infinite at the point (X^+, Y^+) satisfying

$$Y^+ = HX^+ + HG(Y^+), \quad H\,G'(Y^+) = 1$$

This point will be called the *turning value*. Up to the turning value the contraction condition holds i.e. $H\,G'(Y) \leq 1$ that when $X < X^+$ the iterative solution by successive approximation will converge and so the Volterra series solution of the original equation will converge when max $|x(t)| < X^+$ with max $|y(t)| < Y^+$. Best estimates come from the minimum value of H.

Until the turning value, the lower branch of the graph is represents the majorant series solution of the comparison equation and can be used to find the accuracy of an approximation to the solution by a truncated sum of the Volterra solution.

2.6 Example

Let us reconsider the steady-state solution of the forced differential equation

$$\frac{dy(t)}{dt} + \alpha\, y(t) - \varepsilon\, y^2(t) = x(t), \qquad \alpha, \varepsilon > 0, \quad -\infty < t < \infty$$

found from the equivalent integral equation

$$y(t) = \int_0^\infty h(\tau)\, x(t-\tau)\, dt' + \varepsilon \int_0^\infty h(\tau)\; y^2(t-\tau)\, d\tau$$

$$h(\tau) = \exp(-\alpha\tau) \qquad \tau > 0$$

The comparison equation for the integral equation is

$$Y = HX + \varepsilon\, HY^2$$

X, Y being auxiliary positive variables and H is a positive constant chosen so that:

$$H \geq \int_0^\infty |h(\tau)|\, d\tau = \int_0^\infty \exp(-\alpha\tau)\, d\tau = \alpha^{-1}$$

The comparison equation can be solved by successive approximation in the same way as the integral equation resulting in a majorant series

$$Y = HX + \varepsilon\, H^3 X^2 + 2\, \varepsilon^2 H^5 X^3 + \ \ldots$$

The comparison equation graph is a sideways parabola with vertex $(1/4\varepsilon H^2,\ 1/2\varepsilon H)$. The majorant series represents it from the origin up to the vertex, the turning value.

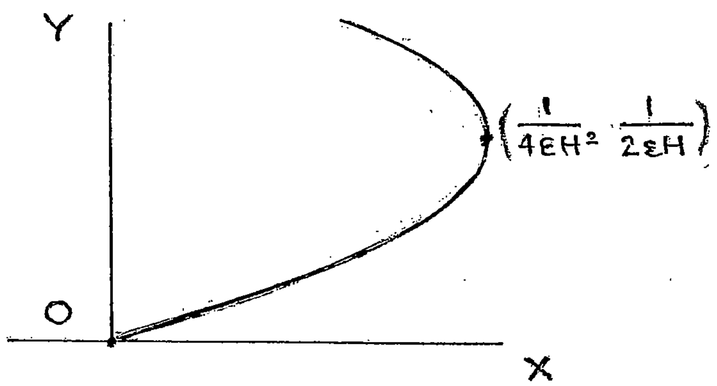

Fig. Graph of the comparison equation

By the preceding theory the Volterra series solution of the differential equation converges when max $|x(t)| < 1/4\varepsilon H^2$ and then max $|y(t)| < 1/2\varepsilon H$.

In this particular case the conclusion also follows since the solution for Y of the comparison equation can be written as the solution of a quadratic equation

$$Y = \frac{1 - \sqrt{(1 - 4\varepsilon H^2 X)}}{2\varepsilon H}$$

The series expansion of the square root here will converge if $4\varepsilon H^2 X < 1$ so giving $1/4\varepsilon H^2$ as radius of convergence of the majorant series. So the Volterra series solution of the original equation is convergent if $\max |x(t)| < 1/4\varepsilon H^2$ Also, since algebraically

$$Y = \frac{2HX}{1 + \sqrt{(1 - 4\varepsilon H^2 X)}} \leq 2HX < \frac{1}{2\varepsilon H}$$

the limit on y is $\max |y(t)| < 1/2\varepsilon H$. Choosing $H = a^{-1}$ now gives the best estimates

$$\max |x(t)| < \alpha^2/4\varepsilon, \quad \max |y(t)| < \alpha/2\varepsilon$$

Local stability and convergence: The validity of a Volterra series solution in only a limited region may be understood by the stability properties of the undisturbed equation

$$\frac{dy(t)}{dt} + \alpha\, y(t) - \varepsilon\, y^2(t) = 0 \qquad \alpha > 0, \varepsilon > 0, \ t \geq 0$$

This equation has two points of equilibrium at $y = 0$ and $y = \alpha/\varepsilon$ obtained by setting $dy/dt = 0$. The differential equation trajectories converge and diverge from these two equilibria as in the diagram below showing the equilibrium at $y = 0$ is stable and the equilibrium at $y = \alpha/\varepsilon$ is unstable.

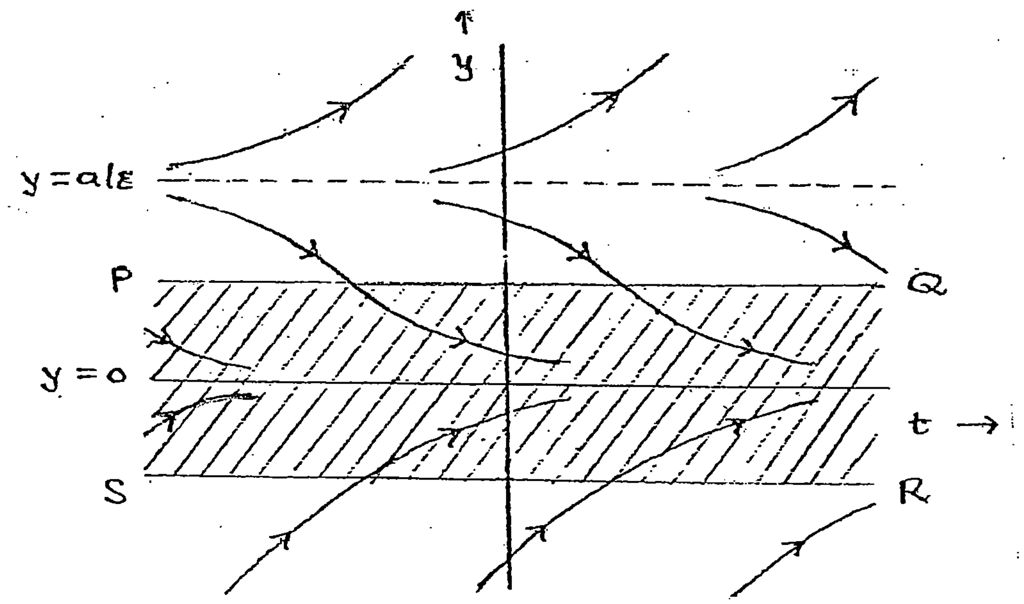

Fig. **Trajectory pattern illustrating local stability near equilibrium at y = 0**

If a small disturbance acts on the system when it is close to the equilibrium at $y = 0$ it will continue to remain in this neighbourhood. But a large disturbance might transfer the system into the region of instability. The above limits for the Volterra series solution restrict motion to the region PQRS.

2.7 Example: The forced damped Duffing equation

The forced under-damped Duffing equation for a hard spring is

$$L(p)y + \varepsilon y^3 = x(t) \quad \varepsilon > 0$$

where L(p) is the operator for the transfer function

$$L(p) = p^2 + 2\zeta\omega_0\, p + \omega_0^2 \quad \omega_0 > 0,\ 0 < \zeta < 1$$

The corresponding impulse response function h() is for $\tau > 0$

$$h(\tau) = \exp(-\zeta\omega_0\tau)\sin\omega_1\tau/\omega_1, \quad \omega_1 = \omega_0\sqrt{(1 - \zeta^2)}$$

The integral equation for the differential equation over the infinite interval is:

$$y(t) = \int_{-\infty}^{\infty} h(t-t')\, x(t')\, dt' - \varepsilon\int_{-\infty}^{\infty} h(t-t')\, y(t')^3\, dt', \qquad -\infty < t < \infty$$

The Volterra series solution comes from solving by successive approximation starting with the linear first approximation:

$$y(t) = \int_{-\infty}^{\infty} h(t-t')\, x(t')\, dt'$$

Substitution gives the third order approximation

$$y(t) = \int_{-\infty}^{\infty} h(t-t')\, x(t')\, dt' - \varepsilon\int_{-\infty}^{\infty} h(t-t')\, y(t')^3\, dt'$$

where the cubic term is written using the first approximation as

$$\int_{-\infty}^{\infty} h(t-t')\{\int_{-\infty}^{\infty} h(t'-t'')\, x(t'')\, dt''\}^3\, dt'$$

$$= \int_{-\infty}^{\infty} h(t-t')\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t'-t_1)\, h(t'-t_2)\, h(t'-t_3)\, x(t_1)\, x(t_2)\, x(t_3)\, dt_1\, dt_2\, dt_3\}dt'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_3(t-t_1, t-t_2, t-t_3)\, x(t_1)\, x(t_2)\, x(t_3)\, dt_1\, dt_2\, dt_3$$

the third order kernel here $h_3(.)$ being

$$h_3(t-t_1, t-t_2, t-t_3) = \int_{-\infty}^{\infty} h(t-t')\, h(t'-t_1)\, h(t'-t_2)\, h(t'-t_3)\, dt'$$

The third order approximation is then the cubic approximation to a Volterra series

$$y(t) = \int_{-\infty}^{\infty} h(t-t')x(t')dt' - \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_3(t-t_1, t-t_2, t-t_3)\, x(t_1)\, x(t_2)\, x(t_3)\, dt_1dt_2dt_3$$

Higher order approximations may be found found similarly.

Comparison equation: To find a region of convergence the comparison equation is:

$$Y = HX + |\varepsilon|\, H\, Y^3$$

Here H is greater or equal to the norm of the integral operation. The best convergence estimate is found by choosing equality:

$$H = \int_0^{\infty} |\, h(\tau)\, |\, d\tau \;=\; \coth\left(\frac{\pi}{2}\,\frac{\zeta}{\sqrt{(1-\zeta^2)}}\right)$$

The comparison equation has the graph shown below

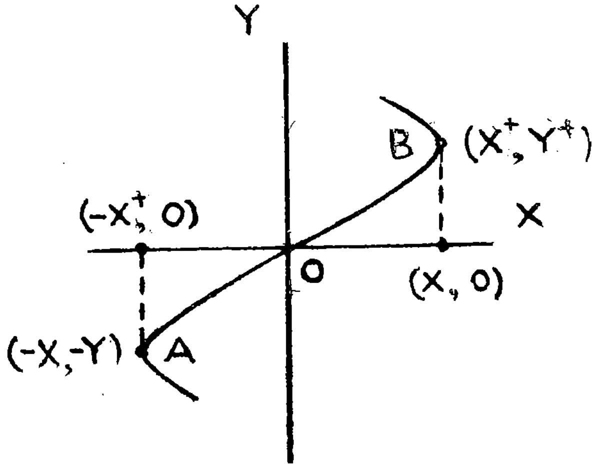

Fig: Graph of the comparison equation

The turning value is easily found as (X^+, Y^+) where:

$$X^+ = 2/\,|\varepsilon|^{1/2}\,(3H)^{3/2}\,, \qquad Y^+ = 1/(3|\varepsilon|H)^{1/2}$$

The contraction condition is $3|\varepsilon|\, H\, Y^2 < 1$ which is satisfied if $0 \le Y \le Y^+$. Both the comparison equation and the Volterra series will convergence within these limits In this case the comparison equation is a cubic equation which may be solved explicitly and the convergence range found from that (see the 1965 paper quoted below) The convergence limits agree in the two methods.

Ref: See the writer's paper in Intern. J. Control 1965, vol 1(3), 209-216. This was the first time the method of the comparison equation was used.

2.8 Example: The forced damped pendulum

The equation representing the angle response θ to an applied force f(t) near a stable equilibrium at angle θ = 0 is.

$$\ddot{\theta}(t) + 2\zeta\,\dot{\theta}(t) + \sin\theta(t) = f(t), \quad 0 < \zeta < 1$$

It is clear physically that if the forcing f(t) remains small the pendulum will perform small amplitude motions in the neighbourhood of the equilibrium position at θ = 0 so the system has local BIBO stability. It is also clear that it remains BIBO stable only when the maximum forcing amplitude remains below a critical value and when this critical value is exceeded, the pendulum may perform complete rotations and so no longer be BIBO stable.

The Volterra stability theory for this problem uses the steady-state solution from the nonlinear Volterra integral equation

$$\theta(t) = \int_{-\infty}^{\infty} h(t-t')\, f(t')\, dt' + \int_{-\infty}^{\infty} h(t-t')\, g(\theta(t'))\, dt' \qquad t\ \varepsilon\ (-\infty, \infty)$$

where

$$h(\tau) = e^{-\zeta\tau} \sin(\mu\tau), \quad \tau > 0 \quad \mu^2 = (1 - \zeta^2)$$

$$g(\theta) = \theta - \sin\theta = \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \ldots$$

The comparison equation for the integral equation is

$$\Theta = H F + H G(\Theta)$$

where

$$H > \int |h(\tau)|\, d\tau = \coth(\pi\zeta/\mu)$$

$$G(\Theta) = \operatorname{sh}\Theta - \Theta = \frac{\Theta^3}{3!} + \frac{\Theta^5}{5!} + \frac{\Theta^7}{7!} - \ldots$$

The series solution of the comparison equation is found as

$$\Theta = HF + \frac{1}{6} H^4F^3 + \frac{1}{12} H^6 (H + 0.1) F^8 + \ldots$$

By the general theory this series will be convergent when 0< F< F*, the coordinates (Θ*, F*) of the turning point on the comparison equation graph being determined by

$$\Theta^* = H F + H G(\Theta^*), \quad 1 = H G'(\Theta^*)$$

Reference: Math.Balkanica 1974

From these conditions are found

$$\Theta^* = \cosh^{[-1]}(1/H)$$

$$F^* = \frac{(1+H)}{H} \cosh^{[-1]}\frac{(1+H)}{H} - \frac{\sqrt{(1+2H+H^2)}}{H}$$

The diagram below illustrates the case $\zeta = 0.6$. The graph of the comparison equation is compared with the graph of the exact solution which is found by numerically integrating the equation for the special forcing

$$f(t) = F \operatorname{sgn}(d\theta(t)/dt)$$

it being clear physically that maximum response amplitude occurs when forcing is always in the direction of the motion.

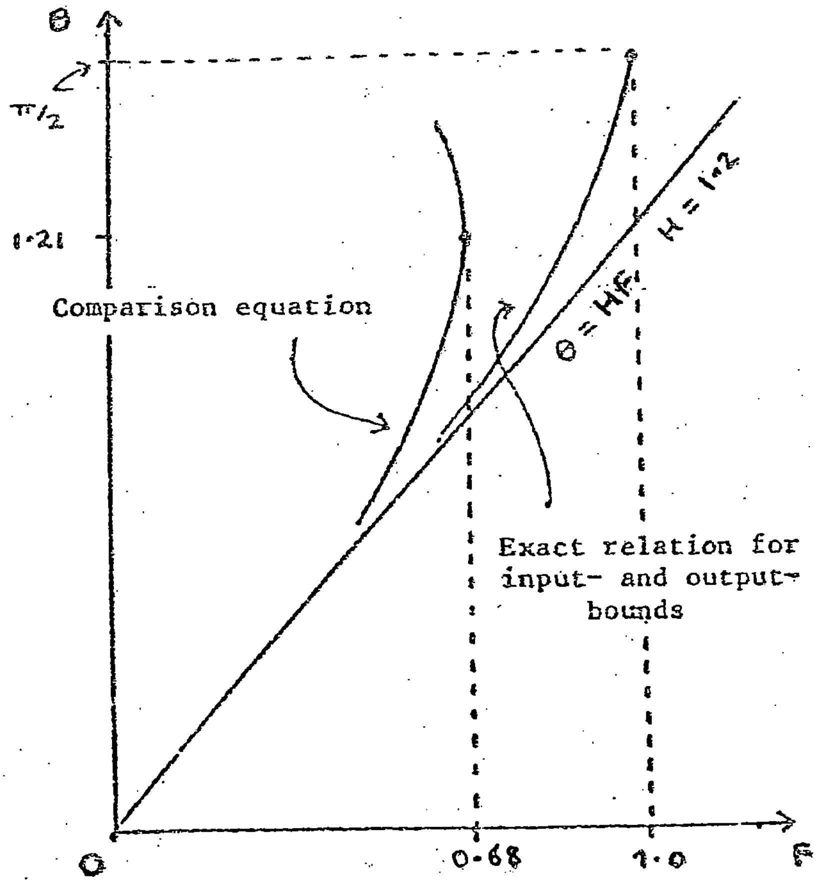

Fig. Illustrating input-output stability of a forced damped pendulum.

2.9 Example: An FM detector (phase-locked loop)

A signal cos ($\omega t + \theta$) is received and it is required to detect θ. The phase-locked loop does this by locking on to a locally generated signal cos ($\omega t + \theta_0$) using a feedback system. A signal proportional to sin (θ - θ_0) is generated by correlation of the received signal with the quadrature component sin ($\omega t + \theta_0$) of the locally generated signal. In the absence of noise, output phase θ_0 is driven towards θ according to the equation

$$d\theta_0/dt = K \sin(\theta - \theta_0)$$

This has an operational solution giving the first form of the feedback system shown below

$$\theta_0 = \frac{K}{p} \sin(\theta - \theta_0)$$

If the maximum value of $d\theta_0/dt$ remains small the error $e = \theta - \theta_0$ will remain small and the system will remain in the neighbourhood of the stable equilibrium at $e = 0$. The nonlinearity can then be neglected and the system behaves like a first order lag.

If the deviation between θ and θ_0 increases, nonlinearity causes loss of performance. This may be analysed by rewriting the equation as

$$de/dt + K \sin e = d\theta/dt$$

and then further transforming it to a form suitable for successive approximation

$$de/dt + K e = d\theta/dt + g(e)$$

$g(e)$ containing nonlinear terms $e - \sin e \approx e^3/6$. This gives the second feedback scheme below.

If the maximum value of $d\theta/dt$ increases the bounds of e increase but initially the system will remain BIBO stable. Finally, when $d\theta/dt$ exceeds a critical value, e may leave completely the neighbourhood of the equilibrium $e = 0$ and move near to another equilibrium e.g. at $e = 2\pi$. This is called 'cycle skipping'. It is analogous to a forced pendulum performing complete rotations.

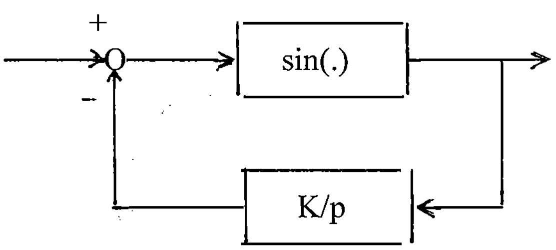

Fig. The phase-locked loop

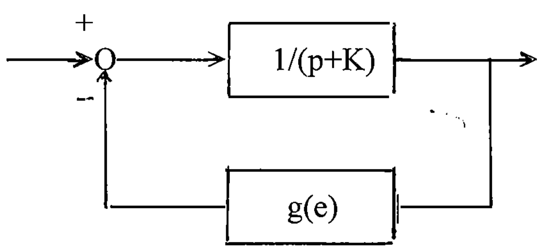

Fig. The transformed loop
$g(e) = e - \sin e = e^3/6$

Chapter 3

MULTIDIMENSIONAL SYSTEM TRANSFORMS

3.1 Multidimensional system transforms

Multidimensional transforms generalize the transfer function method from linear to analytic nonlinear systems. They arose from Wiener's (1942) use of multidimensional Fourier transforms in a special problem which then became generalized in early work on Volterra series (Deutsch 1955, Barrett 1955-57, Brilliant 1957, Zames 1959, George 1959)

Consider a time-invariant Volterra series corresponding to a physically realizable system

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} \dots \int_0^{\infty} h_n(\tau_1,\dots,\tau_n)\, x(t-\tau_1) \dots x(t-\tau_n)\, d\tau_1 \dots d\tau_n$$

The system transforms are the following multidimensional Laplace transforms:

$$H_1(s) = \int_0^{\infty} h_1(\tau_1) \exp -s\tau \, d\tau$$

$$H_2(s_1, s_2) = \int_0^{\infty} \int_0^{\infty} h_2(\tau_1, \tau_2) \exp -(s_1\tau_1+s_2\tau_2)\, d\tau_1\, d\tau_2$$

....

$$H_n(s_1,\dots, s_n) = \int_0^{\infty} \dots \int_0^{\infty} h_n(\tau_1,\dots, \tau_n) \exp -(s_1\tau_1+\dots+ s_n\tau_n)\, d\tau_1 \dots d\tau_n$$

If the kernels are completely symmetric these transforms will also be completely symmetric.

For physically realizable kernels the integrals here written from 0 to ∞ may also be written over the range $-\infty$ to ∞. The multidimensional Fourier transforms of the kernels can then be written in terms of system transforms:

$$H_1(i\omega) = \int_{-\infty}^{\infty} h_1(\tau_1) \exp -i\omega\tau \, d\tau$$

$$H_2(i\omega_1, i\omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(t_1, t_2) \exp -i(\omega_1\tau_1+\omega_2\tau_2)\, d\tau_1 d\tau_2$$

....

$$H_n(i\omega_1,\dots i\omega_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots, \tau_n) \exp -i(\omega_1\tau_1+\dots+ \omega_n\tau_n)\, d\tau_1 \dots d\tau_n$$

These Fourier transforms have an inversion formula

$$h_n(\tau_1,.., \tau_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(i\omega_1,..., i\omega_n) \exp i(\omega_1\tau_1+..+ \omega_n\tau_n)\, d\omega_1....d\omega_n$$

which for the Laplace transforms would correspond to a formula of the type

$$h_n(\tau_1,.., \tau_n) = \frac{1}{(2\pi i)^n} \int_C \int H_n(s_1,..., s_n) \exp (s_1\tau_1+..+ s_n\tau_n)\, ds_1....ds_n$$

Such integrals would be over suitable contours C in the n-dimensional complex space of $s_1, ..., s_n$ As insufficient is known about this kind of representation it has limited use but can sometimes be used as a guide in inverting certain types of integrals (e.g. by association of variables as described later.)

Stability condition: In linear systems analysis the stability condition is that the transfer function H(s) has all its infinities in the left half s-plane. A similar condition applies to multidimensional transforms of kernels satisfying the boundedness condition

$$\int_0^{\infty} \dots \int_0^{\infty} |h_n(\tau_1,..\tau_n)|\, d\tau_1....d\tau_n < \infty$$

On putting $s_1 = \sigma_1 + i\omega_1, ..., s_n = \sigma_n + i\omega_n$ and assuming $\sigma_1 \geq 0, ..., \sigma_n \geq 0$,

$$| H_n(s_1,..., s_n) | = \left| \int_0^{\infty} \int_0^{\infty} h_n(\tau_1,..\tau_n) \exp -(s_1\tau_1+..+ s_n\tau_n)\, d\tau_1....d\tau_n \right|$$

$$\leq \int_0^{\infty} \int_0^{\infty} | h_n(\tau_1, ..\tau_n)| \exp -(\sigma_1\tau_1+..+ \sigma_n\tau_n)\, d\tau_1....d\tau_n$$

$$\leq \int_0^{\infty} \int_0^{\infty} | h_n(\tau_1, ..\tau_n)|\, d\tau_1....d\tau_n < \infty$$

The function $H_n(s_1,..., s_n)$ is consequently uniformly bounded when Re $s_1 \geq 0$,..., Re $s_n \geq 0$. So any singularities must lie in a region where one at least of these inequalities fails to hold.

3.2 Use of delta functions for instantaneous operations:

Instantaneous operations are represented by delta functions. Basic is

$$x(t) = \int_{-\infty}^{\infty} \delta(t-t_1)\, x(t_1)\, dt_1$$

The operation of raising to the nth power leads to the nth kernel: e.g.

$$x^n(t) = \left(\int_{-\infty}^{\infty} \delta(t-t_1)\, x(t_1)\, dt_1\right)^n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta(t-t_1) \ldots \delta(t-t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

The corresponding n-dimensional kernel transform is

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta(\tau_1) \ldots \delta(\tau_n) \exp -\{s_1\tau_1 + \ldots + s_n\tau_n)\, d\tau_1 \ldots d\tau_n = 1$$

As a consequence, an analytic function

$$f(x(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n\, x^n(t)$$

may be considered as a Volterra series with n-dimensional transforms constant and equal to f_n, $n = 0, 1, 2, \ldots$

Derivatives are similarly defined e.g.

$$x'(t) = \int_{-\infty}^{\infty} \delta'(t-t_1)\, x(t_1)\, dt_1,$$

so that

$$x'(t)^n = \left(\int_{-\infty}^{\infty} \delta'(t-t_1)\, x(t_1)\, dt_1\right)^n = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta'(t-t_1) \ldots \delta'(t-t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

The transform is

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta'(\tau_1) \ldots \delta'(\tau_n) \exp -\{s_1\tau_1 + \ldots + s_n\tau_n)\, d\tau_1 \ldots d\tau_n = s_1\, s_2 \ldots s_n$$

and similarly with higher derivatives. The nth order transform of rth derivative is $s_1^r, \ldots, s_n^r$

3.3 Transforms for instantaneous nonlinearities combined with linear operators

(a) Linear + instantaneous nonlinear: The nth power $\{H(p)\, x(t)\}^n$ of a linear operation $H(p)$ acting on $x(t)$ has the nth order kernel given by

$$\{H(p)\, x(t)\}^n = \left(\int_{-\infty}^{\infty} h(t-t_1)\, x(t_1)\, dt_1\right)^n$$

$$= \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} h(t-t_1) \ldots h(t-t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

giving transform $H(s_1) \ldots, H(s_n)$. Consequently if, as in the diagram below

$$z = f(y(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n\, y^n(t) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n \{H(p)\, x(t)\}^n$$

the combination system has nth order kernel transform $f_n\, H(s_1) \ldots, H(s_n)$

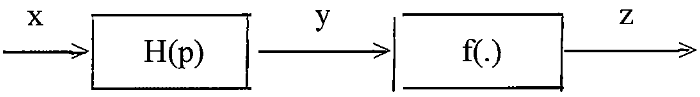

Fig. The LN system

(b) Instantaneous nonlinear + linear: Taking a linear operation $K(p)$ on an nth power $x(t)^n$ corresponds to an nth order kernel given by

$$K(p)\, x^n(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} .. .. \int_{-\infty}^{\infty} k(t-t')\, \delta(t'-t_1) \ldots \delta(t'-t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n\, dt'$$

which has kernel transform $K_n(s_1 + \quad + s_n)$ So the system shown with

$$z(t) = K(p)\, y(t) = K(p)\, f(x(t)) = \sum_{n=1}^{\infty} \frac{1}{n!} f_n\, K(p)\, x(t)^n$$

has nth order kernel transform $f_n\, K(s_1 + \ldots + s_n)$

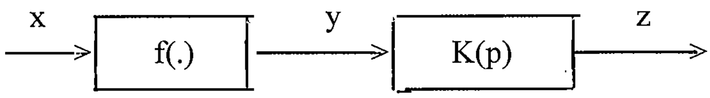

Fig. The NL system

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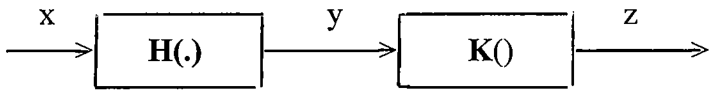
* The names 'Wiener model' and 'Hammerstein model' are commonly used to describe systems (a) and (b). While convenient the names are inappropriate as Wiener did not use this simple model and Hammerstein was concerned with boundary type problems arising in nonlinearly vibrating systems (Hammerstein: Acta Math. 1930)

3.4 Cascade formulae for multidimensional transforms

With analytic systems, cascading corresponds to the process of series substitution
Suppose there are two analytic time-invariant systems **H** and **K** in cascade giving a total system **G** of **KoH** as in the diagram.



-------------- **KoH(.)** = **G(.)**--------------

Fig. Cascaded systems

The input-output relations are assumed given by the Volterra series

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(t - t_1,\ldots,t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

$$z(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k_n(t - t_1,\ldots,t - t_n)\, y(t_1) \ldots y(t_n)\, dt_1 \ldots dt_n$$

By substitution of the first into the second there is found

$$z(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_n(t - t_1,\ldots,t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

The first 3 kernels of the resulting system **KoH** are:

$$g_1(t - t_1) = \int_{-\infty}^{\infty} k_1(t - t')\, h_1(t' - t_1)\, dt'$$

$$\begin{aligned} g_2(t - t_1, t - t_2) = &\int_{-\infty}^{\infty} k_1(t - t')\, h_2(t' - t_1, t' - t_2)\, dt' \\ &+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_2(t - t'_1, t - t'_2)\, h_1(t'_1 - t_1)\, h_1(t'_2 - t_2)\, dt'_1 dt'_2 \end{aligned}$$

$$g_3(t - t_1, t - t_2, t - t_3) = \int_{-\infty}^{\infty} k_1(t - t')\, h_2(t' - t_1, t' - t_2, t' - t_3)\, dt'$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2(t - t'_1, t - t'_2)\, h_1(t'_1 - t_1)\, h_1(t'_2 - t_2, t'_2 - t_3)\, dt'_1 dt'_2$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2(t - t'_1, t - t'_2)\, h_1(t'_1 - t_2)\, h_1(t'_2 - t_3, t'_2 - t_1)\, dt'_1 dt'_2$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2(t - t'_1, t - t'_2)\, h_1(t'_1 - t_3)\, h_1(t'_2 - t_1, t'_2 - t_2)\, dt'_1 dt'_2$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_3(t - t'_1, t - t'_2, t - t'_3)\, h_1(t'_1 - t_1)\, h_1(t'_2 - t_2)\, h_1(t'_3 - t_3) dt'_1\, dt'_2\, dt'_3$$

On taking Laplace transforms of the appropriate orders the following equations are found relating successive transforms.

$$G_1(s) = K_1(s)\, H_1(s)$$

$$G_2(s_1,s_2) = K_1(s_1+s_2)\, H_2(s_1, s_2) + K_2(s_1, s_2)\, H_1(s_1)\, H_1(s_2)$$

$$\begin{aligned} G_3(s_1, s_2, s_3) = {} & K_1(s_1 + s_2 + s_3)\, H_1(s_1, s_2, s_3) \\ & + K_2(s_1, s_2 + s_3)\, H_1(s_1)\, H_2(s_2, s_3) \\ & + K_2(s_2, s_3 + s_1)\, H_1(s_2)\, H_2(s_3, s_1) \\ & + K_2(s_3, s_1 + s_2)\, H_1(s_3)\, H_2(s_1, s_2) \\ & + K_3(s_1, s_2, s_3)\, H_1(s_1)\, H_1(s_3)\, H_1(s_3) \end{aligned}$$

Note a characteristic repetition with permutation of variables. Here there is a threefold repetition for K_2 resulting from 3 partitions of s_1, s_2, s_3 into groups of 1 + 2 For 4th order there are found 15 terms starting with

$$\begin{aligned} G_4(s_1, s_2, s_3, s_4) = {} & K_1(s_1 + s_2 + s_3 + s_4)\, H_3(s_1, s_2, s_3, s_4) \\ & + K_2(s_1, s_2 + s_3 + s_4)\, H_1(s_1)\, H_3(s_2, s_3, s_4) \\ & + K_2(s_2, s_3 + s_4 + s_1)\, H_1(s_2)\, H_3(s_3, s_4, s_1) \\ & + K_2(s_3, s_4 + s_1 + s_2)\, H_1(s_3)\, H_3(s_4, s_1, s_2) \\ & + K_2(s_4, s_1 + s_2 + s_3)\, H_1(s_4)\, H_3(s_1, s_2, s_3) \\ & + K_2(s_1 + s_2, s_3 + s_4)\, H_2(s_1, s_2)\, H_2(s_3, s_4) \\ & + K_2(s_1 + s_2, s_3 + s_4)\, H_2(s_1, s_2)\, H_2(s_3, s_4) \\ & + K_2(s_1 + s_2, s_3 + s_4)\, H_2(s_1, s_2)\, H_2(s_3, s_4) \\ & \quad + \text{etc} \end{aligned}$$

Here the partitions of the 4 variables s_1, s_2, s_3, s_4 into partitions1 + 3 and 2 + 2 result in 4 and 3 repetitions respectively. Altogether there are found 1+4+3+6 +1 = 15 terms. The number of terms clearly rapidly increases with order, e.g. the symmetrical 5th order form has 52 terms resulting from the 1+5+10+10 + 15 + 10 + 1 different partitions of 5.

A contracted way of writing the equations is to use an unsymmetrical form corresponding to a fixed order of the variables which is conveniently taken as ascending order. See the table on the next page

.

3.5 Cascade formulae in contracted notation

$$G_1(s) = K_1(s)\, H_1(s)$$

$$G_2(s_1,s_2) = K_1(s_1+s_2)\, H_2(s_1, s_2) + K_2(s_1, s_2)\, H_1(s_1)\, H_1(s_2)$$

$$\begin{aligned} G_3(s_1,s_2,s_3) = {} & K_1(s_1 + s_2 + s_3)\, H_3(s_1, s_2, s_3) \\ & + 3K_2 (s_1, s_2 + s_3)\, H_1(s_1)\, H_2(s_2, s_3) \\ & + K_3 (s_1, s_2, s_3)\, H_1(s_1)\, H_1(s_3)\, H_1(s_3) \end{aligned}$$

$$\begin{aligned} G_4(s_1, s_2, s_3, s_4) = {} & K_1 (s_1 + s_2 + s_3 + s_4)\, H_4(s_1, s_2, s_3, s_4) \\ & + 4K_2 (s_1, s_2 + s_3 + s_4)\, H_1(s_1)\, H_3 (s_2, s_3, s_4) \\ & + 3K_2 (s_1 + s_2, s_3 + s_4)\, H_2(s_1, s_2)\, H_2(s_3, s_4) \\ & + 6K_3(s_1 + s_2, s_3, s_4)\, H_2(s_1, s_2)\, H_1(s_3)\, H_1(s_4) \\ & + K_4 (s_1, s_2, s_3, s_4)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4) \end{aligned}$$

$$\begin{aligned} G_5(s_1, s_2, s_3, s_4, s_5) = {} & K_1 (s_1 + s_2 + s_3 + s_4 + s_5)\, H_5(s_1, s_2, s_3, s_4, s_5) \\ & + 5K_2 (s_1, s_2 + s_3 + s_4 + s_5)\, H_1(s_1)\, H_4 (s_2, s_3, s_4) \\ & + 10K_2 (s_1 + s_2, s_3 + s_4 + s_5)\, H_2(s_1, s_2)\, H_3 (s_3, s_4, s_5) \\ & + 10K_3 (s_1, s_2, s_3 + s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_3 (s_3, s_4, s_5) \\ & + 15K_2 (s_1, s_2 + s_3, s_4 + s_5)\, H_2(s_1)\, H_2(s_2, s_3)\, H_2(s_4, s_5) \\ & + 10K_4 (s_1, s_2, s_3, s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_2 (s_4, s_5) \\ & + K_5 (s_1, s_2, s_3, s_4, s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4)\, H_1(s_5) \end{aligned}$$

$$\begin{aligned} G_6(s_1, s_2, s_3, s_4, s_5, s_6) = {} & K_1(s_1 + s_2 + s_3 + s_4 + s_5 + s_6)\, H_6(s_1, s_2, s_3, s_4, s_5, s_6) \\ & + 6\, K_2(s_1, s_2 + s_3 + s_4 + s_5 + s_6)\, H_1(s_1)\, H_5(s_2, s_3, s_4, s_5, s_6) \\ & + 15\, K_2(s_1 + s_2, s_3 + s_4 + s_5 + s_6)\, H_2(s_1, s_2)\, H_5(s_3, s_4, s_5, s_6) \\ & + 10\, K_2(s_1 + s_2 + s_3, s_4 + s_5 + s_6)\, H_3(s_1, s_2, s_3)\, H_3(s_4, s_5, s_6) \\ & + 15\, K_3(s_1, s_2, s_3 + s_4 + s_5 + s_6)\, H_1(s_1)\, H_1(s_2)\, H_4(s_3, s_4, s_5, s_6) \\ & + 60\, K_3(s_1, s_2 + s_3, s_4 + s_5 + s_6)\, H_1(s_1)\, H_2(s_2, s_3)\, H_3(s_4, s_5, s_6) \\ & + 15\, K_3(s_1 + s_2, s_3 + s_4, s_5 + s_6)\, H_2(s_1, s_2)\, H_2(s_3, s_4)\, H_2(s_5, s_6) \\ & + 20\, K_4(s_1, s_2, s_3, s_4 + s_5 + s_6)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_3(s_4, s_5, s_6) \\ & + 45\, K_4(s_1, s_2, s_3 + s_4 , s_5 + s_6)\, H_1(s_1)\, H_1(s_2)\, H_2(s_3, s_4)\, H_2(s_5, s_6) \\ & + 15\, K_5 (s_1, s_2, s_3, s_4, s_5 + s_6)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4)\, H_2(s_5, s_6) \\ & + K_6(s_1, s_2, s_3, s_4, s_5, s_6)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4)\, H_1(s_5)\, H_1(s_6) \end{aligned}$$

Odd order systems

$$G_1(s) = K_1(s)\, H_1(s)$$

$$G_3(s_1, s_2, s_3) = K_1(s_1 + s_2 + s_3)\, H_3(s_1, s_2, s_3) + K_3 (s_1, s_2, s_3)\, H_1(s_1)\, H_1(s_3)\, H_1(s_3)$$

$$\begin{aligned} G_5(s_1, s_2, s_3, s_4, s_5) = {} & K_1(s_1 + s_2 + s_3 + s_4 + s_5)\, H_5(s_1, s_2, s_3, s_4, s_5) \\ & +10K_3 (s_1, s_2, s_3 + s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_3 (s_3, s_4, s_5) \\ & + K_5(s_1, s_2, s_3, s_4, s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4)\, H_1(s_5) \end{aligned}$$

3.6 Inverse systems

The mathematical definition of inverse extends to system analysis A system with operator **H(.)** is said to have an inverse system with operator **K(.)** if this system results in the identity when cascaded with the first system **H(.)**. viz

$$\xrightarrow{x} \boxed{\mathbf{H(.)}} \xrightarrow{y} \boxed{\mathbf{K(.)}} \xrightarrow{z = x}$$

-------------- **KoH(.) = I(.)**--------------

Fig. Inverse systems

Terminology is confusing because mathematically **H(.)** is the right inverse of **K(.)** although in the figure it lies to the left of **K(.)** - and similarly with the left inverse. So, taking definitions from the cascading relation of the figure, **H(.)** will be called the *pre-inverse* of **K(.)** and **K(.)** the *post-inverse* of **H(.)**.

Inversion with system transforms: Setting **KoH(.)** equal to unity gives the equations

$$1 = K_1(s)\, H_1(s)$$

$$0 = K_1(s_1 + s_2)\, H_2(s_1, s_2) + K_2(s_1, s_2)\, H_1(s_1)\, H_1(s_2)$$

$$\begin{aligned} 0 = {} & K_1(s_1 + s_2 + s_3)\, H_3(s_1, s_2, s_3) \\ & + 3K_2(s_1, s_2 + s_3)\, H_1(s_1)\, H_2(s_2, s_3) \\ & + K_3(s_1, s_2, s_3)\, H_1(s_1)\, H_1(s_3)\, H_1(s_3) \end{aligned}$$

$$\begin{aligned} 0 = {} & K_1(s_1 + s_2 + s_3 + s_4)\, H_4(s_1, s_2, s_3, s_4) \\ & + 4K_2(s_1, s_2 + s_3 + s_4)\, H_1(s_1)\, H_3(s_2, s_3, s_4) \\ & + 3K_2(s_1 + s_2, s_3 + s_4)\, H_2(s_1, s_2)\, H_2(s_3, s_4) \\ & + 6K_3(s_1 + s_2, s_3, s_4)\, H_2(s_1, s_2)\, H_1(s_3)\, H_1(s_4) \\ & + K_4(s_1, s_2, s_3, s_4)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4) \end{aligned}$$

$$\begin{aligned} 0 = {} & K_1(s_1 + s_2 + s_3 + s_4 + s_5)\, H_5(s_1, s_2, s_3, s_4, s_5) \\ & +5K_2(s_1, s_2 + s_3 + s_4 + s_5)\, H_1(s_1)\, H_4(s_2, s_3, s_4) \\ & +10K_2(s_1 + s_2, s_3 + s_4 + s_5)\, H_2(s_1, s_2)\, H_3(s_3, s_4, s_5) \\ & +10K_3(s_1, s_2, s_3 + s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_3(s_3, s_4, s_5) \\ & +15K_2(s_1, s_2 + s_3, s_4 + s_5)\, H_2(s_1)\, H_2(s_2, s_3)\, H_2(s_4, s_5) \\ & +10K_4(s_1, s_2, s_3, s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_2(s_4, s_5) \\ & + K_4(s_1, s_2, s_3, s_4, s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4)\, H_1(s_5) \end{aligned}$$

The pre-inverse: Solving for the kernels of **H** gives a recursive form for the kernels of the pre-inverse of **K.** This is the more applicable form of inverse as it is used for solving differential equations and for analysing feedback system.

To solve for kernels H it is usually necessary for $K_1(s)$ to have a stable inverse. Assuming this condition satisfied the equations can be solved recursively as follows.

$$H_1(s) = K_1(s)^{-1}$$

$$H_2(s_1, s_2) = - K_1(s_1 + s_2)^{-1}\{K_2(s_1, s_2)\, H_1(s_1)\, H_1(s_2)\}$$

$$\begin{aligned} H_3(s_1, s_2, s_3) = - K_1(s_1 + s_2 + s_3)^{-1} \{ & \\ & 3K_2(s_1, s_2 + s_3)\, H_1(s_1)\, H_2(s_2, s_3) + \\ & K_3(s_1, s_2, s_3)\, H_1(s_1)\, H_1(s_3)\, H_1(s_3)\} \end{aligned}$$

$$\begin{aligned} H_4(s_1, s_2, s_3, s_4) = - K_1(s_1 + s_2 + s_3 + s_4)^{-1}\{ & \\ & 4K_2(s_1, s_2 + s_3 + s_4)\, H_1(s_1)\, H_3(s_2, s_3, s_4) + \\ & 3K_2(s_1 + s_2, s_3 + s_4)\, H_2(s_1, s_2)\, H_2(s_3, s_4) + \\ & 6K_3(s_1 + s_2, s_3, s_4)\, H_2(s_1, s_2)\, H_1(s_3)\, H_1(s_4) + \\ & K_4(s_1, s_2, s_3, s_4)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4)\} \end{aligned}$$

$$\begin{aligned} H_5(s_1, s_2, s_3, s_4, s_5) = - K_1(s_1 + s_2 + s_3 + s_4 + s_5)^{-1} \{ & \\ & 5K_2(s_1, s_2 + s_3 + s_4 + s_5)\, H_1(s_1)\, H_4(s_2, s_3, s_4) + \\ & 10K_2(s_1 + s_2, s_3 + s_4 + s_5)\, H_2(s_1, s_2)\, H_3(s_3, s_4, s_5) + \\ & 10K_3(s_1, s_2, s_3 + s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_3(s_3, s_4, s_5) + \\ & 15K_2(s_1, s_2 + s_3, s_4 + s_5)\, H_2(s_1)\, H_2(s_2, s_3)\, H_2(s_4, s_5) + \\ & 10K_4(s_1, s_2, s_3, s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_2(s_4, s_5) + \\ & K_5(s_1, s_2, s_3, s_4, s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_1(s_4)\, H_1(s_5)\} \end{aligned}$$

The post-inverse: Solving for kernels of **K** gives a recursive form for the kernels of the post-inverse of **H**. To solve the equations it is usually necessary for the linear kernel $H_1(s)$ to have a stable inverse

$$K_1(s) = H_1(s)^{-1}$$

$$K_2(s_1, s_2) = - H_1(s_1)^{-1}\, H_1(s_2)^{-1}\{K(s_1 + s_2)\, H_2(s_1, s_2)\}$$

$$\begin{aligned} K_3(s_1, s_2, s_3) = - H_1(s_1)^{-1}\, H_1(s_2)^{-1}\, H_1(s_3)^{-1}\{ & \\ & K(s_1 + s_2 + s_3)\, H_3(s_1, s_2, s_3) + \\ & 3K_2(s_1, s_2 + s_3)\, H_1(s_1)\, H_2(s_2, s_3)\} \end{aligned}$$

$$\begin{aligned} K_4(s_1, s_2, s_3, s_4) = - H_1(s_1)^{-1}\, H_1(s_2)^{-1}\, H_1(s_3)^{-1}\, H_1(s_4)^{-1}\{ & \\ & K_1(s_1 + s_2 + s_3 + s_4)\, H_4(s_1, s_2, s_3, s_4) \\ & + 4K_2(s_1, s_2 + s_3 + s_4)\, H_1(s_1)\, H_3(s_2, s_3, s_4) \\ & + 3K_2(s_1 + s_2, s_3 + s_4)\, H_2(s_1, s_2)\, H_2(s_3, s_4) \\ & + 6K_3(s_1 + s_2, s_3, s_4)\, H_2(s_1, s_2)\, H_1(s_3)\, H_1(s_4)\} \end{aligned}$$

$$\begin{aligned} K_5(s_1, s_2, s_3, s_4, s_5) = - H_1(s_1)^{-1}\, H_1(s_2)^{-1}\, H_1(s_3)^{-1}\, H_1(s_4)^{-1}\, H_1(s_5)^{-1}\{ & \\ & K_1(s_1 + s_2 + s_3 + s_4 + s_5)\, H_5(s_1, s_2, s_3, s_4, s_5) \\ & +5K_2(s_1, s_2 + s_3 + s_4 + s_5)\, H_1(s_1)\, H_4(s_2, s_3, s_4) \\ & +10K_2(s_1 + s_2, s_3 + s_4 + s_5)\, H_2(s_1, s_2)\, H_3(s_3, s_4, s_5) \\ & +10K_3(s_1, s_2, s_3 + s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_3(s_3, s_4, s_5) \\ & +15K_2(s_1, s_2 + s_3, s_4 + s_5)\, H_2(s_1)\, H_2(s_2, s_3)\, H_2(s_4, s_5) \\ & +10K_4(s_1, s_2, s_3, s_4 + s_5)\, H_1(s_1)\, H_1(s_2)\, H_1(s_3)\, H_2(s_4, s_5)\} \end{aligned}$$

3.8 Multidimensional transforms for a typical forced differential equation

Consider again the equation

$$L(p)\,y(t) \;+\; \frac{g_2}{2!}\,y(t)^2 \;+\; \frac{g_3}{3!}\,y(t)^3 + \quad = x(t)$$

i.e a power series with the first coefficient g_1 a stable linear operator L(p). In the cascade below it is convenient for the notation to call **K** the differential operator $K_1(s) = L(s)$, $K_n() = g_n \;\; n > 1$
The Volterra solution is then given by the left hand operator **H**.

$$\xrightarrow{x} \boxed{\mathbf{H(.)}} \xrightarrow{y} \boxed{\mathbf{K(.)}} \xrightarrow{x}$$

-------------- **KoH(.) = I(.)**--------------

Fig. Solution of differential equation having kernels K

The first three transforms of **H** are given in the unsymmetrical preinverse formulae by

$$\begin{aligned}
H_1(s) &= K_1(s)^{-1} \\
H_2(s_1, s_2) &= -\,K_1(s_1 + s_2)^{-1}\{K_2(s_1, s_2)\,H_1(s_1)\,H_1(s_2)\} \\
H_3(s_1, s_2, s_3) &= -\,K_1(s_1 + s_2 + s_3)^{-1}\,\{3K_2\,(s_1, s_2 + s_3)\,H_1(s_1)\,H_2(s_2, s_3) + \\
&\qquad\qquad K_3\,(s_1, s_2, s_3)\,H_1(s_1)\,H_1(s_2)\,H_1(s_3)\}
\end{aligned}$$

Substituting values for the K kernels gives the transforms of the first three kernels of the Volterra solution of the forced differential equation

$$\begin{aligned}
H_1(s) &= L(s)^{-1} = H(s) \text{ say} \\
H_2(s_1, s_2) &= -\,g_2\,H(s_1 + s_2)\,H(s_1)\,H(s_2) \\
H_3(s_1, s_2, s_3) &= -\,K(s_1 + s_2 + s_3)\,\{3g_2\;H_1(s_1)\,H_2(s_2, s_3) + g_3\;H_1(s_1)\,H_1(s_2)\,H_1(s_3)\}
\end{aligned}$$

The corresponding kernels have previously been derived as

$$h_1(t - t_1) \;=\; h(t - t_1)$$

$$h_2(t - t_1, t - t_2) = -\,g_2 \int h(t - t')\,h(t' - t_1)\,h(t' - t_2)\,dt'$$

$$\begin{aligned}
h_3(t - t_1, t - t_2, t - t_3) \;=\; &-3g_2 \int h(t' - t)\,h(t' - t_1)\,h_2(t' - t_2, t - t_3)\,dt' \\
&-\,g_3 \int h(t - t')\,h(t' - t_1)\,h(t' - t_2)\,h(t' - t_3)\,dt'
\end{aligned}$$

It is easily verified that multidimensional Laplace transform of these kernels gives the above values.

Substitution of the second order transform H_2 and second order kernel h_2 gives the full non-recursive form for the third order transform H_3 and kernel h_3. The kernels are then still unsymmetrical but can easily be symmetrized if necessary.

3.9 Feedback systems

The usual form of a feedback system is that shown in the diagram below. It is characterized by open-loop and closed-loop operators **G** and **H**.

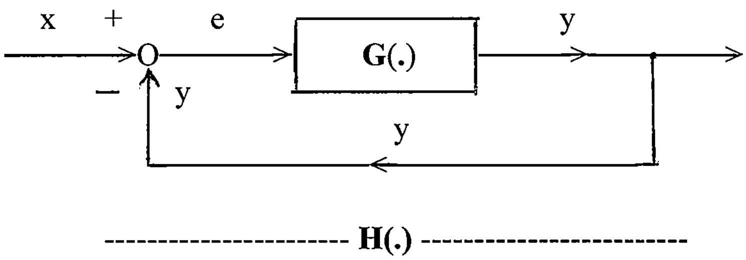


Fig: A typical feedback system

Starting from open-loop dynamics the system equations are:

$$y = \mathbf{G}(e)$$

$$e = x - y$$

From these follow

$$x = e + \mathbf{G}(e) = (\mathbf{I} + \mathbf{G})(e)$$

Stability condition: To proceed further the existence of the pre-inverse $(\mathbf{I} + \mathbf{G})^{[-1]}$ must be assumed. In the linear case this assumption corresponds to the Nyquist stability condition and so it may therefore be considered to be the nonlinear analogue of the stability condition. When satisfied the last equation may be inverted to give.

$$e = (\mathbf{I} + \mathbf{G})^{[-1]}(x) = \mathbf{U}(x)$$

The operator **U**() defined by this equation is called the *return difference operator*. Substitution gives

$$y = x - \mathbf{U}(x) = (\mathbf{I} - \mathbf{U})\,(x)$$

This gives *input-output operator* **H**() as

$$\mathbf{H} = \mathbf{I} - \mathbf{U}$$

There follows the relation between the basic operators **U, H,G** describing the system:

$$\mathbf{U} = \mathbf{I} - \mathbf{H} = (\mathbf{I} + \mathbf{G})^{[-1]}$$

Relation between transforms: If the system is analytic and the transforms of the forward loop operator are $G_1(s)$, $G_2(s_1, s_2)$, $G_3(s_1, s_2, s_3)$, ... etc then the existence of the pre-inverse to **I** + **G** depends on existence of the inverse $(1 + G_1(s))^{-1}$ of the linear term $1 + G_1(s)$ so $G_1(s)$ must be Nyquist stable. The relation between system transforms is then

$$U_1(s) = (1 + G_1(s))^{-1}$$

$$U_2(s_1, s_2) = -(1 + G_1(s_1 + s_2))^{-1} U_1(s_1) U_1(s_2)$$

$$U_3(s_1, s_2, s_3) = -(1 + G_1(s_1 + s_2 + s_3)^{-1} \{3G_2(s_1, s_2 + s_3)U_1(s_1)U_2(s_2, s_3) + G_3(s_1, s_2, s_3)G_1(s_1) G_1(s_2) G_1(s_3)\}$$

etc. From these follow the transforms of

$$H_1(s) = 1 - U_1(s) = G_1(s) (1 + G_1(s))^{-1}$$

$$H_2(s_1, s_2) = - U_2(s_1, s_2)$$

$$H_3(s_1, s_2, s_3) = - U_3(s_1, s_2, s_3)$$

etc. In the most important case the forward-loop is an odd order system when there is found

$$U_1(s) = (1 + G_1(s))^{-1}$$

$$U_3(s_1, s_2, s_3) = -(1 + G_1(s_1 + s_2 + s_3)^{-1} G_3(s_1. s_2, s_3) G_1(s_1) G_1(s_2) G_1(s_3)$$

$$U_5(s_1, s_2, s_3, s_4, s_5) = -\{1 + G_1(s_1 + s_2 + s_3 + s_4 + s_5)\}^{-1} \{10 G_3(s_1. s_2, s_3 + s_4 + s_5) G_1(s_1) G_1(s_2) G_1(s_3) + G_5(s_1, \ s_5) H(s_1) H(s_2) H(s_3) H(s_4) H(s_5)\}$$

etc

Chapter 4

MULTILINEAR NOTATION

4.1 Multilinear representation of analytic functions relating vectors

When dealing with analytic relations between vectors, generality is obtained by the use of an abstract notation. In this notation formulae apply equally well to continuous- or discrete-time and finite- or infinite-dimensional systems.

Consider functions between two vector spaces X and Y:

$$y = f(x), \quad x \varepsilon X, y \varepsilon Y.$$

The basic property of an analytic expansion is the possibility of representing the function with terms of successive degrees 0, 1, 2, ...

$$f(x) = f_0 + f_1(x) + f_2(x) + \ldots$$

For any scalar multiplier λ the term of nth degree satisfies the *homogeneity property*

$$f_n(\lambda x) = \lambda^n f_n(x), \quad n = 1, 2, \ldots$$

This homogeneity property is not in itself sufficient to characterize the expansion of being of power series type, e.g. the function of a vector $[x_1, x_2, \ldots, x_n]$

$$f(x) = \sqrt{(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

is homogeneous of degree 1 yet it is not linear. The correct approach to the abstract representation of power series was found by Fréchet* and uses multilinear functions.

A *multilinear function* $f_n(x^{(1)}, .., x^{(n)})$ is defined as a function linear in each of its variables $x^{(1)}, .., x^{(n)}$ in a vector space X and with value in another vector space Y.

A function f(x) relating vector variables $x \varepsilon X$., $y \varepsilon Y$ will be said to be *analytic* if it has a representation in terms of the multilinear functions $f_n(.,..,.)$, n = 0, 1,2,.. as

$$y = f(x) = f_0 + \sum_{n=1}^{\infty} \frac{1}{n!} f_n(x, \ldots, x)$$

The terms $f_n(x, .., x)$, n = 1,2,.. occurring in the expansion of f(x) then have the homogeneity property for any scalar λ

$$f_n(\lambda x, \ldots, \lambda x) = \lambda^n f(x, \ldots, x)$$

---

* See e.g. Fréchet: 'Pages choisies de l'analyse générale', Paris 1953

Example 1: Let $X \varepsilon R^M$, $Y \varepsilon R^N$. Then the expansion of the function f(.) is obtained on using the multilinear functions having jth components

$$f_n(x^{(1)}, \ldots, x^{(n)})_j = \sum_{i_1=1}^{M} \ldots \sum_{i_n=1}^{M} f_{j;\, i_1,\ \, i_n}\, x^{(1)}{}_{i_1}, \ldots x^{(n)}{}_{i_n}, \quad j = 1, \ \ , M$$

Example 2: Let $X = R^{[t_0,\, t_f]}$ be the space of real-valued functions $x = \{x(t), t \,\varepsilon\, [t_0, t_f]\}$ and $Y = R$. The Volterra expansion is obtained by using the multilinear functions

$$f_n(x^{(1)}, \ \ , x^{(n)}) = \int_{t_0}^{t_f} \ldots \int_{t_0}^{t_f} f_n(\tau_1, \ldots, \tau_n)\, x^{(1)}(\tau_1) \ldots x^{(n)}(\tau_n)\, d\tau_1 \ldots d\tau_n$$

A restriction is placed on the class of functions under consideration to make the integrals well-defined.

Example 3: Let $X \varepsilon R^{M\mathrm{x}[t_0,\, t_f]}$, $Y \varepsilon R^{N\mathrm{x}[t_0,\, t_f]}$. The representations of the last two examples may be combined when the variables are vector functions of time.

$$f_n(x^{(1)}, \ \ , x^{(n)})_j = \sum_{i_1=1}^{M} \ldots \sum_{i_n=1}^{M} \int_{t_0}^{t_f} \ldots \int_{t_0}^{t_f} f_n(\tau_1, \ldots, \tau_n)_{j;\, i_1,\ \, i_n}\, x^{(1)}(\tau_1)_{i_1} \ldots x^{(n)}(\tau_n)_{i_n}\, d\tau_1 \ldots d\tau_n$$

Two multilinear function $f_n(x^{(1)}, \ldots, x^{(n)})$, $g_n(x^{(1)}, \ldots, x^{(n)})$ will be said to be *equivalent* if they are equal for all values of $x^{(1)}, \ldots, x^{(n)}$.

Symmetry: a multilinear function f_n: $X^N \rightarrow Y$ is said to be *symmetrical* if, for all permutations $k_1, \ldots, k_n$ of 1, ...n,

$$f_n(x^{(1)}, \ldots, x^{(n)}) = f_n(x^{(k_1)}, \ldots, x^{(kn)})$$

Any multilinear function f_n has a corresponding symmetrized form defined as

$$f_n(x^{(1)}, \ldots, x^{(n)})_{sym} = \frac{1}{n!} \sum_{\text{perms } k} f_n(x^{(k_1)}, \ldots, x^{(kn)})$$

the sum on the right-hand side being over all the n! permutations $k_1, \ldots, k_n$ of 1, ..., n.
On putting here $x^{(1)} = \ldots = x^{(n)} = x$ there follows

$$f_n(x, \ldots, x)_{sym} = f_n(x, \ldots, x), \quad n = 1, \ldots, n$$

so that f_n is equivalent to its symmetrized form and can be replaced by it in the power series representation of any function. Thus it is possible to make the *symmetry convention* that all multilinear functions occurring in the power series representation of a function are symmetrical.

4.2 Uniqueness of power series representation

It is convenient to use the notation

$$f_n[x] = f_n(x, .., x) \qquad n = 1, 2, ...$$

for the *homogeneous terms* in a power series. This formula expresses homogeneous terms by multilinear functions. Conversely it is possible to express symmetrized multilinear functions in terms of homogeneous ones. For it is simple to verify

$$f_2(x_1, x_2) = \frac{1}{2!}\{f_2[x_1 + x_2] - (f_2[x_1] + f_2[x_2])\}$$

$$f_3(x_1, x_2, x_3) = \frac{1}{3!}\{f_3[x_1 + x_3 + x_3] - (f_3[x_2 + x_3] + f_3[x_1 + x_2] + f_3[x_2 + x_3]) + (f_3[x_1] + f_3[x_2] + f_3[x_3])\}$$

etc. leading to the nth degree formula

$$f_n(x^{(1)}, .., x^{(n)}) = \frac{1}{n!}\{f_n[x^{(1)} + ... + x^{(n)}] - (f_n[x^{(2)} + ... + x^{(n)}] + ...) + \ldots \text{ etc} + (-1)^n (f_n[x^{(1)}] + ...)\}$$

As a deduction from this formula it follows that if $f_n[x] = 0$ for all $x \varepsilon X$ then $f_n(x^{(1)}, .., x^{(n)}) = 0$ for all $x^{(1)},.., x^{(n)} \varepsilon X$.

Theorem: If an analytic function f(x) is zero for all x, its multilinear terms vanish.

Proof: For any vector x and scalar λ,

$$0 = f(\lambda x) = f_0 + \sum_{n=1}^{\infty} \frac{1}{n!} f_n(\lambda x, .., \lambda x) = f_0 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} f_n(x, .., x)$$

Since this holds for all λ it follows that for all n

$$f_n(x, .., x) = f_n[x] = 0$$

so all multilinear operators $f_n(x^{(1)}, .., x^{(n)})$ also vanish.

Corollary: if $f_n(x^{(1)}, .., x^{(n)})$ and $g_n(x^{(1)}, .., x^{(n)})$ are multilinear functions of the same degree such that $f_n[x] = g_n[x]$ for all $x \varepsilon X$ then they are equal for all $x^{(1)}, .., x^{(n)}$. In particular, if they are both symmetrical then they coincide. It is only necessary to apply the lemma to the difference of the functions. This result justifies the procedure of 'equating coefficients' of two equal power series.

Equivalence of multilinear functions: two multilinear functions f_n, g_n are defined to be *equivalent* if $f_n = g_n$ or equally, $f_n[.] = g_n[\ ]$. This is an equivalence relation in the accepted mathematical sense. From the equality of two power series can be deduced equivalence of the multilinear functions for each degree. This is 'equating coefficients'. The use of the weaker equivalence instead of the stronger equality between the symmetrized forms of the multilinear functions can abbreviate formulae and is often useful for this purpose.

4.3 Analytic functions in a normed space

The generalizations here from the Volterra series case are completely straightforward

X, Y spaces with norms denoted by $\| \ \|$. Typical cases are:

(a) Finite n-dimensional vector space,
$\| x \| = \max | x_i |$, i=1,..,n.

I

(b) Real function on range $[t_0, t_f]$,
$\| x \| = \max | x(t) |$ $t \in [t_0, t_f]$.

(c) n-dimension vector function on range $[t_0, t_f]$,
$\| x \| = \max |x_i(t)|$, i=1,..,n, $t \in [t_0, t_f]$.

A multilinear function $a_n(x^{(1)}, \ldots, x^{(n)})$ in $x^{(i)} \in X$, i = 1, …,n with value in Y is called *bounded* if there is a constant A such that $\|a_n(x^{(1)}, \ldots, x^{(n)})\| \leq A \|x^{(1)}\| \ldots \|x^{(n)}\|$ for all values of the x's The least such A (i.e. infimum) is defined as $\| a_n \|$

A class of analytic functions X→Y is defined by a formal series

$$y = f(x) = a_0 + a_1(x) + \frac{a_2}{2!}(x, x) + \ldots$$

When the $a_n()$ n=0,1,2, .. are bounded multilinear functions $X^n \rightarrow Y$ a majorant series X > 0, Y > 0 may be formed

$$Y = F(X) = A_0 + A_1 X + \frac{A_2}{2!} X^2 + \ldots$$

where $A_n \geq \|a_n\|$ (equality giving *least majorant*) so that if $\|x\| \leq X$ each term of the series is less or equal in norm to the corresponding term of the majorant series.

$$\|a_n(x, x, \ldots x)\| \leq A_n X^n$$

It is easy to demonstrate the following analogue of the scalar case following the proof in Appendix 1. The normed spaces should be complete, i.e .Banach spaces

Theorem If the majorant series converges for $X \in S$ then the series for f(x)
(i) is absolutely convergent when $\|x\| \in S$
(ii) is uniformly convergent on $S_x = \{x| \ \|x\| \leq X \in S\}$
(iii) when $\|x\| \leq X \in S$, $\|f(x)\| \leq F(X)$

Reference: Lyusternik & Sobolev: Elements of Functional Analysis. (Russian) Moscow-Leningrad 1951; English translation New York 1961 (Ungar).

4.4 Series substitution

Use of the notation of multilinear functions effectively reduces calculations with power series to the scalar case. There are only a few differences.

Suppose that h: X |→ Y, k: Y |→ Z are two functions having power series representation

$$y = h(x) = \sum_{n=1}^{\infty} \frac{1}{n!} h^{(n)}(x, ..., x)$$

$$z = k(y) = \sum_{n=1}^{\infty} \frac{1}{n!} k^{(n)}(y, ..., y)$$

By substitution there is found a function **g = k o h** with power series representation say

$$z = g(x) = \sum_{n=1}^{\infty} \frac{1}{n!} g^{(n)}(x, ..., x)$$

where

$$g_1(x) = k_1(h_1(x))$$

$$g_2(x, x) = k_1(h_2(x, x)) + k_2(h_1(x), h_1(x))$$

$$g_3(x, x, x) = k_1(h_3(x, x, x)) + 3\, k_2(f_1(x), f_2(x, x)) + k_3(h_1(x), h_1(x), h_1(x))$$

Following a similar procedure as with scalar power series, the general term is found as

$$g_n(x, x, ...\, x) =$$

$$\sum_{1\le m\le n} \{ \sum_{r_1} \sum_{r_2} ... \sum_{r_m} P[n \mid r_1, r_2, ...r_m]\, k_m(\underbrace{h_1(x), ...h_1(x)}_{r_1}, \underbrace{h_2(x, x), ...h_2(x, x)}_{r_2}, ...\}$$

where the sum inside the bracket on the rhs is over all positive integers $r_1, r_2, ...$ such that

$$r_1 + 2r_2 + ... = n, \quad r_1 + r_2 + \quad = m$$

Omission of the variable x does not lead to ambiguity and results in the abbreviated forms:

$$g_1 = k_1(h_1)$$

$$g_2 = k_1(h_2) + k_2(h_1, h_1)$$

$$g_3 = k_1(h_3) + 3\, k_2(h_1, h_2) + k_3(h_1, h_1, h_1)$$

....

$$g_n = \sum_{1\le m\le n} \{ \sum_{r_1} \sum_{r_2} ... \sum_{r_m} P[n \mid r_1, r_2, ...r_m]\, k_m (\underbrace{h_1, ...h_1}_{r_1}, \underbrace{h_2, ...h_2}_{r_2}, ..., \underbrace{h_m, ...h_m}_{r_m})$$

Discussion of majorants follows the scalar case of Appendix 1

4.5 Inversion, reversion:

If two systems are inverse to one another the cascaded system is the identity. Then follow

$$I = k_1(h_1)$$

$$0 = k_1(h_2) + k_2(h_1, h_1)$$

$$0 = k_1(h_3) + 3\ k_2(h_1, h_2) + k_3(h_1, h_1, h_1)$$

.... etc

These equations can be solved either for the k in terms of h or vice versa. Solving for the h in terms of the k gives the same result as the recursion method. To do this write

$$k_1(h_1) = I$$

$$k_1(h_2) = -\ k_2(h_1, h_1)$$

$$k_1(h_3) = -\ \{3\ k_2(h_1, h_2) + k_3(h_1, h_1, h_1)\}$$

.... etc

From which

$$h_1 = k^{[-1]}(x)$$

$$h_2 = -\ k^{[-1]}(x)\ k_2(h_1, h_1)$$

$$h_3 = -\ k^{[-1]}(x)\{3\ k_2(h_1, h_2) + k_3(h_1, h_1, h_1)\}$$

.... etc

These determine the h recursively. The same result is found from reversion writing

$$y = k^{[-1]}(x) - k^{[-1]}\left(\sum_{r=2}^{\infty} \frac{1}{r!}\ k_r(y, y, \ldots y)\right)$$

and substituting with unknown h_n

$$y = \sum_{n=1}^{\infty} \frac{1}{n!}\ h_n(x, x, \ldots x)$$

resulting in the recursive equations for the kernels h:

Discussion of convergence by majorant series is similar to the scalar case of Appendix 1.

4.6 Solution of vector differential equations

Equations of the type

$$L(p)y(t)+g\{y(t)) = x(t)$$

Between vectors x(t) and y(t) can be solved by the same method as in the scalar case previously considered. L() will be a matrix differential operator which, when stable, can be inverted to give a matrix impulse response function h() and the equation converted to the nonlinear integral form on the infinite time interval as

$$y\,(t) \;=\; \int h(t-t')\, x(t')dt' - \int h(t-t')\; g\{y(t')\}dt'$$

The multilinear operator form of the procedure was given by Halme et al. For this it is convenient to redefine the variables x and y to represent the vectors x(t), y(t) over the appropriate time interval, e.g. $(-\infty, \infty)$. Then the equation can be written in the form

$$y = h(x) - h \circ g(y)$$

The solution developed by the reversion technique. The comparison equation is again

$$Y = H\,X + H\,G(X)$$

Here H will exceed the norm of the matrix h() i.e.

$$H \geq \max_{j} \int \sum_{i} |h_{ij}(t-t')|\; dt'$$

If (X^{+}, Y^{+}) is the turning values of the comparison equation the reverted form of the equation gives a contraction map for $||X|| \leq X^{+}$, (Halme et al 1971).

The state-space equation: With a little modification the reversion method also extends to the state-space equation written in the usual vector notation as

$$\frac{dx}{dt} = f(x, u)$$

where x is the state-vector and u the control vector and f is assumed analytic in these variables. In the neighbourhood of a stable point taken at x=0 the linear term in f(x,u) can be separated out and the equation written

$$\frac{dx}{dt} = Ax + g(x, u)$$

where A is a stable Hurwitz matrix and g(x, u) contains no linear terms in x.

Reference: Halme, Orava & Blomberg, Int. J. sys. Sci 1971

Then on the infinite interval the equation can be reverted to the integral form

$$x(t) = \int h(\tau)\, g(x(t-\tau), u(t-\tau))\, d\tau$$

where h() will be the exponential matrix

$$h(\tau) = \exp A\tau$$

An explicit solution can be given of the solution found by iteration if multilinear notation is used. To prove convergence within a certain range the comparison equation can be formed

$$X = H\, G(X, U)$$

where $H \geq \|h()\|$ and G(X, U) majorizes g(x, u). The determination of a range of convergence for values $\|x\| \leq X$, $\|y\| \leq Y$ is made possible by the use of Hille's theorem -for details see the references in the footnote below

The bilinear equation on the infinite time interval: The procedure is especially simple for the multidimensional bilinear equation

$$\frac{dx_i(t)}{dt} = \sum_{j=i}^{r} a_{ij}\, x_j(t) + \sum_{k=1}^{r} b_{i,k}\, u_k(t) + \sum_{j=1}^{s} \sum_{k=1}^{r} c_{i,j,k}\, x_j(t)\, u_k(t) \quad i = 1, 2. \ldots, n$$

On inverting the linear terms in x it becomes a nonlinear integral equation which in multilinear form is

$$x = hbu + hc(u, x))$$

where h is the linear integral operator, b is a linear matrix operation and c(., .) is a bilinear operator. The solution may be immediately be generated as an infinite series by iteration. The comparison equation is

$$X = HBU + HCXU$$

where A, B, C exceed the norms of operators a, b, c. Solved for X this is

$$X = HBU/(1 - HCU)$$

which for $U \leq 1/HC$ generates the convergent majorant series

$$X = HBU\{1 + HCU + (HCU)^2 + \ldots\}$$

So the solution of the bilinear equation on the infinite interval is valid if $\|u\| \leq 1/HC$

References: Hille's theorem, described in Appendix 1, was originally applied to the state-space problem by Kielkiewicz: Bull. Acad. Pol. Sci., 1969. For a detailed derivation of the results see the writer's papers in Comptes Rendus Acad. Sci. Paris`1980 and Proc IEE 1981. On the bilinear equation Bruni c., di Pillo , Koch: Ric. Autom. 1971

Chapter 5

TRANSIENT RESPONSE

5.1 Association of variables

One method for calculating transient response referred to in the literature is the method of association of variables due to George (1959). It is as follows. Suppose we wish to find the response of a Volterra system with kernels $h_n(.)$ to an input $x(t)$ having Laplace transform $X(s)$. The input will typically be a delta function, step function, ramp function etc. To evaluate the nth order term of the Volterra series a function $y_n(t_1', ..., t_n')$ of n time variables $t_1, ..., t_n$ is defined by

$$y_n(t_1, ..., t_n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} h_n(t_1 - t_1', ..., t_n - t_n')\, x(t_1') ... x(t_n')\, dt_1' ... dt_n'$$

Since $y_n(t_1, ..., t_n)$ is a convolution it has multidimensional transform

$$Y_n(s_1, ..., s_n) = H_n(s_1, ..., s_n)\, X(s_1), ..., X(s_n)$$

The term $y_n(t)$ is obtained from $y_n(t_1, ..., t_n)$ by putting $t_1 = t_2 = ... = t_n = t$.
In principle there are two methods by which $y_n(t)$ could be calculated.

(a) Firstly there is the direct calculation of inverse transform

$$y_n(t_1, ..., t_n) = \frac{1}{(2\pi i)^n} \int_C .... \int Y_n(s_1, ..., s_n) \exp(s_1 t_1 + ... + s_n t_n)\, ds_1 ... ds_n$$

Then there is put $t_1 = t_2 = ... = t_n = t$. This method is not normally possible as it involves inversion of an n-dimensional transformation.

(b) Secondly it is possible to proceed as follows

$$\begin{aligned} y_n(t) &= y_n(t, ..., t) \\ &= \frac{1}{(2\pi i)^n} \int_C ..... \int Y_n(s_1, ..., s_n) \exp(s_1 + ... + s_n)t\; ds_1 ... ds_n \\ &= \frac{1}{(2\pi i)} \int e^{st}\, Y_n(s)\, ds \end{aligned}$$

where $Y_n(s)$, called the associated transform, is

$$Y_n(s) = \frac{1}{(2\pi i)^{n-1}} \int_{s_1 + ... + s_n = s} \int Y_n(s_1, ..., s_n)\, ds_1 ... ds_n$$

Ref: George: Continuous nonlinear systems. D.Sc. thesis MIT 1959: www.dspace.mit.edu.

These two possible ways of calculating $y_n(t)$ from $Y_n(s_1, \ldots., s_n)$ are shown in the diagram below.

$$\begin{tikzcd}[column sep=huge, row sep=large]
Y_n(s_1, \ldots., s_n) \arrow[r, "\text{Inverse n-dimensional Laplace transform}"] \arrow[d, dash, "\begin{array}{l}\text{Association}\\ \text{of variables}\end{array}"] & y_n(t_1, \ldots., t_n) \arrow[d, dash, "t_1 = \ldots. = t_n = t"] \\
Y_n(s) \arrow[r, "\text{Inverse 1-dimensional Laplace transformation}"'] & y_n(t)
\end{tikzcd}$$

Fig. Illustrating the process of association of variables.

The process of going from $Y_n(s_1, \ldots, s_n)$ to $Y_n(s)$ is called *association of variables* and $Y_n(s)$ is called the *associated transform* of $Y_n(s_1, \ldots, s_n)$

The basic idea of the method of George can be explained for n = 2 as follows. Suppose $Y_2(s_1, s_2)$ has the form

$$Y_2(s_1, s_2) = M_2(s_1 + s_2)\, G(s_1)\, K(s_2)$$

Then the associated transform is

$$Y_2(s) = \frac{1}{2\pi i} \int\!\!\int_{s_1 + s_2 = s} Y_2(s_1, s_2)\, ds_1\, ds_2$$

$$= M_2(s) \frac{1}{2\pi i} \int\!\!\int_{s_1 + s_2 = s} G(s_1)\, K(s_2)\, ds_1\, ds_2$$

The integral represents a convolution of G() and K() which corresponds to multiplication in the time domain .Using this fact, the integral may be evaluated.

Suppose, in the simplest case, G(), K() may be put into partial fraction form

$$G(s) = \sum \frac{A}{s + \beta} \qquad K(s) = \sum \frac{B}{s + \gamma}.$$

Then

$$g(t) = \sum A\, e^{\beta t}, \qquad g(t) = \sum B\, e^{\gamma t}$$

and so

$$g(t)\, k(t) = \sum \sum AB\, e^{(\beta + \gamma)t}$$

Consequently,

$$\frac{1}{2\pi i} \int\!\!\int_{s_1 + s_2 = s} G(s_1)\, K(s_2)\, ds_1\, ds_2 = \sum \sum \frac{AB}{s + \beta + \gamma}.$$

To summarize, with $Y_2(s_1, s_2)$ expressed as

$$Y_2(s_1, s_2) = M(s_1 + s_2)\, G(s_1)\, K(s_2) = M(s_1 + s_2) \sum \sum \frac{A\,B}{(s_1 + \beta)\,(s_2 + \gamma)}.$$

it is possible to pass immediately to the associated transform which is

$$Y_2(s_1, s_2) = M(s) \sum \sum \frac{A\,B}{(s + \beta + \gamma)}.$$

More generally, with a partial fraction decomposition

$$G(s) = \sum \frac{A}{(s_1 + \beta)^m}, \qquad K(s) = \sum \frac{B}{(s_2 + \gamma)^n}.$$

there is found

$$Y_2(s_1, s_2) = M(s) \sum \sum \frac{(m + n - 2)!}{(m - 1)!\,(n - 1)!} \frac{A\,B}{(s + \beta + \gamma)^{m+n-1}}.$$

Lubbock & Bansal (1969) compiled an extensive table of multidimensional transforms to facilitate such calculations.

When there are more than two variables to be associated a method has been described by Chen & Chiu (1973). The same rules are used to associate the variables two at a time the others being held constant, e.g. with n variables $s_1, s_2, \ldots, s_n$ we first associate s_{n-1} with s_n the other values $s_1, s_2, \ldots, s_{n-2}$ being treated as constant. The new variable which replaces s_{n-1} and s_n may then be called s_{n-1}. The number of variables has now been reduced by one and the process should be repeated until only one variable remains.

Refs: Lubbock & Bansal: Proc. IEE, 1969, vol. 116, pp 2075-2081.
Chen & Chiu: J. Syst. Sci. Jul. 1973, vol 14, no 4, pp 647-664

5.2 Example: Step response of the phase-locked loop

The analysis of a phase-locked loop using multidimensional transforms was made by George (1959) and Van Trees (1964) and was one of the first applications of the method of association of variables..

As previously described, the forward loop operator consists of a cascade of an instantaneous nonlinearity and an integrator. It has transforms to order five of

$$G_1(s) = \frac{K}{s}$$

$$G_3(s_1, s_2, s_3) = -\frac{K}{6}\frac{1}{(s_1 + s_2 + s_3)}$$

$$G_5(s_1, ..., s_5) = \frac{K}{120}\frac{1}{(s_1 + ... + s_5)}$$

The corresponding closed-loop operator is found to have transforms to third order of

$$H_1(s) = \frac{K}{s + K}$$

$$H_3(s_1, s_2, s_3) = \frac{K}{6}\cdot\frac{1}{(K + s_1 + s_2 + s_3)(K + s_1)(K + s_2)(K + s_3)}$$

Step response: The input is A1(t) (A constant) with Laplace transform = A/s The output is found by association of variables from

$$Y_1(s) = \frac{K}{s + K}\cdot\frac{A}{s} = \frac{A}{s} - \frac{A}{s + K} \;\text{-->}\; A\,1(t)\,(1 - \exp(-Kt))$$

$$Y_3(s_1, s_2, s_3) = -\frac{KA^3}{6}\frac{1}{(K + s_1 + s_2 + s_3)}$$

$$\text{-->}\; -\frac{KA^3}{6}\cdot\frac{1}{(s + K)(s + 3K)}$$

$$\text{-->}\; -\frac{A^3}{12}(e^{-Kt} - e^{-3Kt}).1(t)$$

To this approximation the step response is consequently

$$1(t)\,\{A(1 - \exp(-Kt)) - \frac{A^3}{12}(e^{-Kt} - e^{-3Kt}).\}$$

5.3 Calculation of transients using a nonlinear Volterra integral equation

Consider the equation for transients

$$L(p)\, y(t) + g\,(y\,(t)) = 0$$

where L(p) is a linear polynomial operator and g(y) is an analytic function containing no constant or linear term

$$L(p) = p^n + a_1\, p_{n-1} + \ldots + a_n$$

$$g(y) = g_2 \frac{x^2}{2!} + g_3 \frac{x^3}{3!} + \ldots = \frac{1}{n!} \sum_{n=2}^{\infty} g_n\, y^n$$

The initial values of y and its first n-1 derivatives at t = 0 are assumed given.

The differential equation may be transformed to the equivalent integral equation using the same method as before and written

$$y(t) + \frac{1}{L(p)}\, g\,(y\,(t)) = \frac{1}{L(p)}\, 0$$

which is interpreted on the semi-infinite interval t > 0 as

$$y(t) + \int_0^t h(t-t')\, g(y(t'))\, dt' = z(t)$$

h(.) being the impulse response of operator L(p) and z(t) a solution for t > 0 of the differential equation

$$L(p)\, z(t) = 0$$

Now since

$$h(0) = 0,\ h'(0) = 0,\ \ldots,\ h^{(n-1)}(0) = 0$$

it follows on differentiating the integral equation, that

$$y(0) = z(0),\ y'(0) = z'(0),\ \ldots,\ y^{(n-1)}(0) = z^{(n-1)}(0),$$

Consequently it is seen that if z(.) satisfies the same initial conditions as y(.) then the integral equation is equivalent to the differential equation for y. The integral equation may now be written in the form

$$y(t) = z(t) - \int_0^t h(t-t')\, g(y(t'))\, dt'$$

and solved by successive approximation.

5.4 Operational solution of the nonlinear Volterra integral equation

Consider again the equation on the semi-infinite interval $t > 0$.

$$y(t) = z(t) - \int_0^t h(t - t')\, g(y(t'))\, dt'$$

where h(.) and g() are as before. The equation can be solved by Laplace transform using the following method due to Pipes (1965). Laplace transformation $\mathbf{L}_s$ gives

$$Y(s) = Z(s) - H(s) \sum_{n=2}^{\infty} \frac{1}{n!} g_n\, \mathbf{L}_s\{y(t)^n\}$$

where Y(s), Z(s), H(s) are the transforms of y(t), z(t), h(t) respectively. The method of successive approximation is applied to this equation using the recurrence equation

$$Y_1(s) = Z(s)$$

$$Y_{m+1}(s) = Z(s) - H(s) \sum_{n=2}^{\infty} \frac{1}{n!} g_n\, \mathbf{L}_s\{y_m(t)^n\} \qquad m = 1, 2, \ldots$$

The computation proceeds

$$Y_m(s) \xrightarrow{\mathbf{L}^{-1}} y_m(t) \longrightarrow y_m(t)^n \xrightarrow{\mathbf{L}} Y_{m+1}(s) \longrightarrow$$

For low order approximation use may be made of the following table quoted from Pipe's book. (A similar table is found also in the book of Rugh). For higher order approximation the method is in a form suitable for computer algebra.

Table for Pipe's Method

..F(s).	$L\{L^{-1}f(s)\}^2$	$L\{L^{-1}F(s)\}^3$
$\frac{A}{s+a}$	$\frac{A^2}{s+2a}$	$\frac{A^3}{s+3a}$
$\frac{A}{(s+a)^n}$	$\frac{(2n-2)!}{[n-1]^2}\frac{A^2}{(s+2a)^{2n-1}}$	$\frac{(3n-3)!}{[(n-1)!]^3}\frac{A^3}{(s+3a)^{3n-2}}$
$\frac{A}{s^2+a_1 s+a_2}$	$\frac{A^2}{\Delta}\left\{\frac{2s+a_1}{(s^2+2a_1 s+a_2)} - \frac{2}{(s+a_1)}\right\}$	$\frac{A^3}{\Delta}\left\{\frac{3}{s^3+3a_1 s-9a_2} - \frac{2}{s^2+3a_1 s+(2a_1^2+a_2)}\right\}$

$$\Delta = a_1^2 - 4a_2$$

Reference: Pipes L.A. Operational Methods in Nonlinear Mechanics.
Reprinted New York 1965 (Dover)

Chapter 6

SINUSOIDAL INPUT

6.1 Response of an instantaneous nonlinearity to a sinusoid

It is convenient to consider first the sinusoidal response of an instantaneous analytic system. So suppose the input-output relation is

$$y(t) = f(x(t)) = f^{(0)} + f^{(1)}\,x(t) + \frac{f^{(2)}}{2!}\,x(t)^2 + \frac{f^{(3)}}{3!}\,x(t)^3 + \ldots$$

A constant term $f^{(0)}$ has been added on the right hand side because the output may have constant (dc) contributions. Suppose the input is

$$x(t) = a\cos(\omega t + \varphi) = \tfrac{1}{2}\left(A\,e^{i\omega t} + A^*\,e^{-i\omega t}\right)$$

Here A is the complex amplitude $ae^{i\varphi}$ and asterisk denotes complex conjugate. Successive terms in the series for f may be arranged in a triangular array as follows:

Constant (dc) term:

$$f^{(0)}$$

Linear term:

$$(\tfrac{1}{2})\,f^{(1)}\left[e^{i\omega t}A + e^{-i\omega t}A^*\right]$$

Quadratic term:

$$1/2!\,(\tfrac{1}{2})^2 f^{(2)}\left[e^{2i\omega t}A^2 + 2AA^* + e^{-2i\omega t}A^{*2}\right]$$

Cubic terms:

$$1/3!\,(\tfrac{1}{2})^3 f^{(3)}\left[e^{3i\omega t}A^3 + 2\,e^{i\omega t}A^2A^* + 2\,e^{-i\omega t}AA^{*2} + e^{-3i\omega t}A^{*3}\right]$$

Here AA^* may be replaced by a^2 and then a little rearrangement gives

Constant (dc) term:

$$f^{(0)}$$

Linear terms:

$$f^{(1)}\frac{A}{2}e^{i\omega t} + f^{(1)}\frac{A^*}{2}e^{-i\omega t}$$

Quadratic terms:

$$\frac{f^{(2)}}{2!}\frac{A^2}{2^2}e^{2i\omega t} + f^{(2)}\frac{a^2}{2^2} + \frac{f^{(2)}}{2!}\frac{A^{*2}}{2^2}e^{-2i\omega t}$$

Cubic terms:

$$\frac{f^{(3)}A^3}{3!\,2^3}e^{3i\omega t} + \frac{f^{(3)}}{2!}\frac{A}{2}\frac{a^2}{2^2}e^{i\omega t} + \frac{f^{(3)}}{2!}\frac{A^*}{2}\frac{a^2}{2^2}e^{-i\omega t} + \frac{f^{(3)}A^{*3}}{3!\,2^3}e^{-3i\omega t}$$

etc. The constant terms are in the centre, the positive powers of the exponential are to the left and the negative powers, their conjugates, to the right.

If the process is continued, vertical summation gives the output in the form

$$\ldots + B_3^* e^{-3i\omega t} + B_2^* e^{2i\omega t} + B_1^* e^{i\omega t} + B_0 + B_1 e^{i\omega t} + B_2 e^{2i\omega t} + B_3 e^{3i\omega t} + \ldots$$

$$= \sum_{m=-\infty}^{\infty} B_m e^{mi\omega t}$$

B_{-m} here is defined as B_m^*. Alternatively the output can be written in real form as

$$B_0 + 2\mathrm{Re}\,[B_1 e^{i\omega t} + B_2 e^{2i\omega t} + \ldots + B_m e^{mi\omega t} + \ldots]$$

The complex coefficients B up to order 3 are found to be

$$B_0 = \left\{ f_0 + \frac{f_2}{1!} \frac{a^2}{2^2} + \frac{f_4}{2!} \frac{a^4}{2^4} + \right\}$$

$$B_1 = \frac{A}{2}\left\{ \frac{f_1}{1!} + \frac{f_3}{2!1!} \frac{a^2}{2^2} + \frac{f_5}{3!2!} \frac{a^4}{2^4} + \ldots \right\}$$

$$B_2 = \frac{A^2}{2^2}\left\{ \frac{f_2}{2!} + \frac{f_4}{3!1!} \frac{a^2}{2^2} + \frac{f_6}{4!2!} \frac{a^4}{2^4} + \ldots \right\}$$

$$B_3 = \frac{A^3}{2^3}\left\{ \frac{f_3}{3!} + \frac{f_5}{4!\,1!} \frac{a^2}{2^2} + \frac{f_7}{5\,2!} \frac{a^4}{2^4} + \ldots \right\}$$

The pattern of formation, as seen from these terms, leads to the general formula

$$B_m = \frac{A^m}{2^m} \sum_{k=0}^{\infty} f^{(m+k)} \frac{a^{2k}}{2^{2k}} \frac{1}{(m+k)!\,k!} = \frac{A^m}{2^m} \chi_m \qquad m = \ldots, -2, -1, 0, 1, 2, \ldots$$

where

$$\chi_m = \frac{A^m}{2^m}\left\{ f^{(m)} \frac{1}{m!0!} + \frac{a^2}{2^2} f^{(m+1)} \frac{1}{(m+1)!\,1!} + \frac{a^2}{2^4} f^{(m+2)} \frac{1}{(m+2)!\,2!} + \ldots \right\}$$

Here the factors χ_m are real so that $\chi_{-m} = \chi_m$. The response is

$$\chi_0 + 2\mathrm{Re}\,[\frac{A}{2} e^{i\omega t} \chi_1 + \frac{A^2}{2^2} e^{2i\omega t} \chi_2 + \ldots + \frac{A^m}{2^m} e^{mi\omega t} \chi_m + \ldots]$$

$$= \chi_0 + 2\mathrm{Re}\,[\frac{a}{2} e^{i(\omega t+\varphi)} \chi_1 + \frac{a^2}{2^2} e^{2i(\omega t+\varphi)} \chi_2 + \ldots + \frac{a^m}{2^m} e^{mi(\omega t+\varphi)} \chi_m + \ldots]$$

Since the χ_m are real the result can be written as.

$$\chi_0 + 2\,[\frac{a}{2} \chi_1 \cos(\omega t+\varphi) + \frac{a^2}{2^2} \chi_2 \cos 2(\omega t+\varphi) + \ldots + \frac{a^m}{2^m} \chi_m \cos m(\omega t+\varphi) + \ldots]$$

6.2 Response of a Volterra system to sinusoidal input

Let the input-output relation be

$$y(t) = h_0 + \int_{-\infty}^{\infty} h_1(t-t_1)\, x(t_1)\, dt + \frac{1}{2!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(t - t_1, t - t_2)\, x(t_1)\, x(t_2)\, dt_1 dt_2$$

$$+ \frac{1}{3!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_3(t - t_1, t - t_2, t - t_3)\, x(t_1)\, x(t_2)\, x(t_3)\, dt_1\, dt_2\, dt_3 \; + \ldots$$

A constant term h_0 has been added on the right hand side because the output may have constant dc contributions. Suppose the input is as before,

$$x(t) = a \cos(\omega t + \varphi) = ½\left(A\, e^{i\omega t} + A^*\, e^{-i\omega t}\right)$$

Here the complex amplitude $ae^{i\varphi}$ and asterisk denotes complex conjugate. Substituting for x(t) in the Volterra series and using the kernel multidimensional transforms $H^{(0)}$ ($= h^{(0)}$), $H^{(1)}(s)$, $H^{(2)}(s_1, s_2)$, ...etc, which are assumed symmetrical in their variables, successive terms are

Linear term:

$$(½)\, [e^{i\omega t} A \;\; H^{(1)}(i\omega) + \\ e^{-i\omega t} A^* \, H^{(1)}(-i\omega)]$$

Quadratic term:

$$1/2!\, (½)^2\, [\, e^{2i\omega t} A^2\, H^{(2)}(i\omega, i\omega) \quad + \; AA^*H^{(2)}(i\omega, -i\omega) \\ e^{-2i\omega t}\, A^{*2}\, H^{(2)}(-i\omega, -i\omega) + \; AA^*H^{(2)}(-i\omega, i\omega)]$$

Cubic terms:

$$1/3!\, (½)^3\, [e^{3i\omega t} A^3\, H^{(3)}(i\omega, i\omega, i\omega) \quad + e^{i\omega t}\, 3A^2A^*\, H^{(3)}(-i\omega, i\omega, i\omega) \\ e^{-3i\omega t}\, A^{*3}\, H^{(3)}(-i\omega, -i\omega, -i\omega) + e^{-i\omega t}\, 3AA^{*2}\, H^{(3)}(i\omega, -i\omega, -i\omega)]$$

Combining complex conjugates, these may be written in real form as

Linear term:

$$2\, \mathrm{Re}\, [e^{i\omega t} AH^{(1)}(i\omega)]$$

Quadratic term:

$$1/2!\, (½)^2\, 2\, \mathrm{Re}\, [e^{2i\omega t} A^2\, H^{(2)}(i\omega, i\omega) \; + \; a^2\, H^{(2)}(i\omega, -i\omega)]$$

Cubic terms:

$$1/3!\, (½)^3\, 2\, \mathrm{Re}\, [e^{3i\omega t}\, A^3\, H^{(3)}(i\omega, i\omega, i\omega) + 3\, e^{i\omega t}\, A\, a^2\, H^{(3)}(-i\omega, i\omega, i\omega)]$$

If terms up to 5^{th} order are taken into account, the result can be written as

$$B_0 + 2\,\mathrm{Re}\,[B_1 e^{i\omega t} + B_2 e^{2i\omega t} + \ldots + B_5 e^{5i\omega t}]$$

where

$$B_0 = \left\{ H^{(0)} + \frac{a^2}{2^2} H^{(2)}(i\omega, -i\omega) + \frac{a^4}{2^2} \frac{H^{(4)}(i\omega, i\omega, -i\omega, -i\omega)}{2!\,2!} \right\}$$

$$B_1 = \frac{A}{2} \left\{ \frac{H^{(1)}(i\omega)}{1!\,0!} + \frac{a^2}{2^2} \frac{H^{(3)}(i\omega, -i\omega, -i\omega)}{2!\,1} + \frac{a^4}{2^2} \frac{H^{(5)}(i\omega, i\omega, i\omega, -i\omega, -i\omega)}{3!\,2!} \right\}$$

$$B_2 = \frac{A^2}{2^2} \left\{ \frac{H^{(2)}(i\omega, i\omega)}{2!\,0!} + \frac{a^2}{2^2} \frac{H^{(4)}(i\omega, i\omega, i\omega, -i\omega)}{3!\,1!} \right\}$$

$$B_3 = \frac{A^3}{2^3} \left\{ \frac{H^{(3)}(i\omega, i\omega, i\omega)}{3!\,0!} + \frac{a^2}{2} \frac{H^{(5)}(i\omega, i\omega, i\omega, i\omega, -i\omega)}{4!\,1!} \right\}$$

$$B_4 = \frac{A^4}{2^4} \frac{H^{(4)}(i\omega, i\omega, i\omega, i\omega)}{4!\,0!}$$

$$B_5 = \frac{A^5}{2^5} \frac{H^{(5)}(i\omega, i\omega, i\omega, i\omega, i\omega)}{5!\,0!}$$

The pattern of formation can be seen from these terms. Using symmetrized transforms, the nth order contribution will contain terms of the type

$$\frac{1}{n!} \frac{A^{*k} A^{n-k}}{2^{2n}} e^{(n-2k)i\omega t} \frac{n!}{(n-k)!\,k!} H^{(n)}(\underbrace{i\omega, \ldots, i\omega}_{\ldots n-k \ldots}; \underbrace{-i\omega, \ldots, -i\omega}_{\ldots\ldots k \ldots\ldots})$$

where there are n-k plus and k minus signs for $\pm i\omega$, which can happen in $n!/(n-k)!\,k!$ ways. This will make a contribution to B_m where $m = n - 2k$. Consequently,

$$B_m = \frac{A^m}{2^m} \sum_{k=0}^{\infty} \frac{a^{2k}}{2^{2k}} \frac{1}{(m+k)!\,k!} . H^{(m+2k)}(\underbrace{i\omega, \ldots, i\omega}_{\ldots m+k \ldots}; \underbrace{-i\omega, \ldots, -i\omega}_{\ldots k \ldots}) = \frac{A_m}{2^m} \chi_m \text{ say}$$

where

$$\chi_m = \left\{ \frac{H^{(m)}(i\omega, i\omega, \ldots i\omega)}{m!\,0!} + \frac{a^2}{2^2} \frac{H^{(m+2)}(i\omega, i\omega, \ldots i\omega; i\omega, -i\omega)}{(m+1)!\,1!} + \frac{a^4}{2^4} \frac{H^{(m+4)}(i\omega, i\omega, \ldots; i\omega, i\omega, -i\omega, -i\omega)}{(m+2)!\,2!} + \ldots \right\}$$

The mth negative harmonic correspondingly has complex amplitude $B_{-m} = B_m^*$. Since B_0 is consequently real, the resulting response can be written

$$\sum_{m=-\infty}^{\infty} B_m e^{mi\omega t} = B_0 + 2\,\mathrm{Re}\,[B_1 e^{i\omega t} + B_2 e^{2i\omega t} + \ldots + B_m e^{mi\omega t} + \ldots]$$

$$= \chi_0 + 2\mathrm{Re}\,[\frac{a}{2} e^{i(\omega t + \varphi)} \chi_1 + \frac{a^2}{2^2} e^{2i(\omega t + \varphi)} \chi_2 + \ldots + \frac{a^m}{2^m} e^{mi(\omega t + \varphi)} \chi_m + \ldots]$$

Since now the factors χ_m are complex no further simplification is possible.

Reference: This derivation is a simplified form of that of Mircea & Sinnreich (1969)

6.3 Example

$$L(p)\,y(t) + \frac{a_2}{2!} y(t)^2 + \frac{a_3}{3!} y(t)^3 + \quad = x(t)$$

The left hand side of the differential equation is interpreted as a Volterra series with transforms

$$K^{(1)}(s) = L(s), \quad K^{(n)}(s_1,\ldots,s_n) = a_n \ \text{ for } n > 1$$

Application of the formula for finding pre-inverse give the inverse kernels as

$$\begin{aligned}
H_1(s) = L(s)^{-1} &= H(s) \text{ say} \\
H_2(s_1, s_2) &= - a_2 H(s_1 + s_2) H(s_1) H(s_2) \\
H_3(s_1, s_2, s_3) &= - H(s_1 + s_2 + s_3)H(s_1) H(s_2) H(s_3)\{3a_2^2 H(s_1 + s_2) - a_3\}
\end{aligned}$$

Here all the terms of $H_2(s_1, s_2)$ are symmetrical but for $H_3(s_1, s_2, s_3)$ they are not since the term $3H(s_1 + s_2)$ should be interpreted as an abbreviation for the symmetrical form

$$H(s_2 + s_3) + H(s_3 + s_1) + H(s_1 + s_2)$$

which must be used in the formula for the sinusoidal response. The rest of the terms are already symmetrical in s_1, s_2, s_3. The chi factors are found from the formulae

$$\begin{aligned}
\chi_0 &= \frac{a^2}{2^2} H^{(2)}(i\omega, -i\omega) \\
\chi_1 &= \frac{H^{(1)}(i\omega)}{1!\,0!} + \frac{a^2}{2^2} \frac{H^{(3)}(-i\omega, i\omega, i\omega)}{2!\,1} \\
\chi_2 &= \frac{H^{(2)}(i\omega, i\omega)}{2!\,0!} \\
\chi_3 &= \frac{H^{(3)}(i\omega, i\omega, i\omega)}{3!\,0!}
\end{aligned}$$

They are in this case,

$$\begin{aligned}
\chi_0 &= \frac{a^2}{2^2} a_2 H(0) H(i\omega) H(-i\omega) \\
\chi_1 &= H(i\omega) - \frac{a^2}{2^2} \frac{1}{2} H(i\omega)^3 H(-i\omega)\{3a_2^2\{H(2i\omega) + 2H(0)\} - a_3\} \\
\chi_2 &= -\frac{a_3}{2} H(2i\omega) H(i\omega)^2 \\
\chi_3 &= -\frac{1}{6} H(3i\omega) H(i\omega)^3\{3a_2^2 H(2i\omega) - a_3\}
\end{aligned}$$

6.4 Example: Odd-order system

Consider the same equation but with odd nonlinearity up to the 5th term.

$$L(p)\,y(t) + \frac{k_3}{3!}\,y(t)^3 + \frac{k_5}{5!}\,y(t)^5 + \quad = x(t)$$

The formulae to be used are:

$$\chi_1 = \frac{H^{(1)}(i\omega)}{1!\,0!} + \frac{a^2}{2^2}\,\frac{H^{(3)}(-i\omega, i\omega, i\omega)}{2!\,1} + \frac{a^4}{2^2}\,\frac{H^{(5)}(i\omega, i\omega, i\omega, -i\omega, -i\omega)}{3!\,2!}$$

$$\chi_3 = \frac{H^{(3)}(i\omega, i\omega, i\omega)}{3!\,0!} + \frac{a^2}{2^2}\,\frac{H^{(5)}(i\omega, i\omega, i\omega, i\omega, -i\omega)}{4!\,1!}$$

$$\chi_5 = \frac{H^{(5)}(i\omega, i\omega, i\omega, i\omega, i\omega)}{5!\,0!}$$

The values already found in the last example can be used for the terms not involving the 5th order kernel in the 2nd and 3rd equations. The kernels are:

$$H^{(1)}(s) = K_1(s)^{-1} = H(s) \text{ say}$$

$$H^{(3)}(s_1, s_2, s_3) = H(s_1 + s_2 + s_3)\; k_3\, H(s_1)\, H(s_3)\, H(s_3)$$

$$H^{(5)}(s_1, s_2, s_3, s_4, s_5) = H(s_1 + s_2 + s_3 + s_4 + s_5) H(s_1)\, H(s_2)\, H(s_3)\, H(s_4)\, H(s_5)\; \{10 k_3^2\, H(s_3+s_4+s_5) + k_5\}$$

Here the kernels of the first two equations are symmetrical but in the third there occurs an abbreviation 10 $H(s_3+s_4+s_5)$ which must be expanded into the different ways of choosing the three s terms from s_1, s_2, s_3, s_4, s_5.

The 10 possible partitions of s_1, s_2, s_3, s_4, s_5 occurring in the formula for $H_5()$ are

s_1	s_2	$s_3 + s_4 + s_5$	s_2	s_4	$s_1 + s_3 + s_5$
s_1	s_3	$s_2 + s_4 + s_5$	s_2	s_5	$s_1 + s_3 + s_4$
s_1	s_4	$s_2 + s_3 + s_5$	s_3	s_4	$s_1 + s_2 + s_5$
s_1	s_5	$s_2 + s_3 + s_4$	s_3	s_5	$s_1 + s_2 + s_4$
s_2	s_3	$s_1 + s_4 + s_5$	s_4	s_5	$s_1 + s_2 + s_3$

Putting 2 values (e.g. s_1, s_2) equal to $-i\omega$ and the other 3 equal to $i\omega$ there is found:

1 term of the type $\quad s_1 = -i\omega,\; s_2 = -i\omega,\; s_3 + s_4 + s_5 = 3i\omega$

6 terms of the type $\quad s_1 = -i\omega,\; s_3 = i\omega,\; s_2 + s_4 + s_5 = i\omega$

3 terms of the type $\quad s_3 = i\omega,\; s_4 = i\omega,\; s_1 + s_2 + s_5 = -i\omega$

so

$$H^{(5)}(i\omega, i\omega, i\omega, -i\omega, -i\omega) =$$

$$H(i\omega)\, H(i\omega)^3\, H(-i\omega)^2 \{k_3^2\, (H(3i\omega) + 6\, H(i\omega) + 3\, H(-i\omega)) + k_5\}$$

Putting 1 value $-i\omega$ and 4 values $i\omega$ there is found

4 terms of the type $s_1 = -i\omega,\ s_2 = i\omega,\ s_3 + s_4 + s_5 = 3i\omega$

6 terms of the type $s_2 = i\omega,\ s_3 = i\omega,\ s_1 + s_4 + s_5 = i\omega$

giving

$$H^{(5)}(i\omega, i\omega, i\omega, i\omega, -i\omega) =$$

$$H(3i\omega)\, H(i\omega)^4\, H(-i\omega)\{k_3^2\,(4H(3i\omega) + 6H(i\omega)) + k_5\}$$

Further, using symmetry,

$$H^{(5)}(i\omega, i\omega, i\omega, i\omega, i\omega) = H(5i\omega)\, H(i\omega)^5\, \{10k_3^2\, H(3i\omega) + k_5\}$$

The chi factors are then given by the formulae:

$$\chi_1 = \frac{H^{(1)}(i\omega)}{1!\,0!} + \frac{a^2}{2^2}\frac{H^{(3)}(-i\omega, i\omega, i\omega)}{2!\,1} + \frac{a^4}{2^2}\frac{H^{(5)}(i\omega, i\omega, i\omega, -i\omega, -i\omega)}{3!\,2!}$$

$$\chi_3 = \frac{H^{(3)}(i\omega, i\omega, i\omega)}{3!\,0!} + \frac{a^2}{2^2}\frac{H^{(5)}(i\omega, i\omega, i\omega, i\omega, -i\omega)}{4!\,1!}$$

$$\chi_5 = \frac{H^{(5)}(i\omega, i\omega, i\omega, i\omega, i\omega)}{5!\,0!}$$

6.5 Volterra system response to a sum of sinusoids (Mircea-Sinnreich formula)

A general formula was found by Mircea and Sinnreich (1969) for the response of a Volterra system to a sum of sinusoids. It is briefly given here in slightly modified form and notation. The previous formulae for single input are special cases.

Let the input be

$$x(t) = \sum_{p=1}^{N} a_p \cos(\omega_p t + \varphi_p) = \tfrac{1}{2} \sum_{p=1}^{N} \left(A_p e^{i\omega_p t} + A_p^* e^{-i\omega_p t}\right)$$

where ω_p, p = 1, 2, ... N are incommeasurable frequencies and

$$A_p = a_p \exp i\varphi_p$$

On writing $A_{-p} = A_p^*$, $\omega_{-p} = -\omega_p$, the input may be written

$$x(t) = \tfrac{1}{2} \sum_{p=-N}^{N} A_p e^{i\omega_p t}$$

On substitution of x(t) into the Volterra series the output can be written as

$$y(t) = B_0 + 2 \operatorname{Re} \left(\sum_{m_1,\ldots,m_N} \ldots \sum B_{m_1,\ldots,m_N} \exp i(m_1\omega_1 + m_2\omega_2 + \ldots + m_N\omega_N) \right)$$

summed over all positive zero and negative integers. The B are complex given by

$$B_{m_1,\ldots,m_N} = \frac{A_1^{m_1}}{2^{m_1}} \cdots \frac{A_N^{m_N}}{2^{m_N}} \chi_{m_1,\ldots,m_N}$$

$$\chi_{m_1,\ldots,m_N} = \sum_{r_1} \ldots \sum_{r_N} \frac{|A_1|^{2r_1} \ldots\ldots\ldots |A_N|^{2r_N}}{r_1!(m_1+r_1)! \ldots r_N!(m_N+r_N)!}$$

$$H_{m+2r}(\underbrace{i\omega_1, \ldots, i\omega_1}_{m_1+r_1}; \underbrace{-i\omega_1, \ldots, -i\omega_1}_{r_1}; \ldots\ldots \underbrace{i\omega_N, \ldots, i\omega_N}_{m_1+r_N}; \underbrace{-i\omega_N, \ldots, -i\omega_N}_{r_N})$$

The output consequently contains harmonics, combination and difference frequencies.

Mircea and Sinnreich also considered the limiting case of an infinite number of sinusoids which leads to the formula for spectrum resulting from an input Gaussian process (see later).

Reference: Mircea A & Sinnreich H: Distortion noise in frequency dependent networks. Proc.IRE 116 1969 1644-1648. See also Bedrosian & Rice (1969).

Chapter 7

GAUSSIAN INPUT AND HERMITE EXPANSIONS

7.1 Instantaneous nonlinear operation with Gaussian input

Calculation of higher order statistics of the output from a nonlinear system with Gaussian input is extremely complicated. What will be done here is to calculate the second order statistics which is not too difficult. These second order statistics are of interest as they give the output spectrum. The calculation is similar for both instantaneous and Volterra systems which will be emphasized by similar layout of the calculation.

The system: The input/output relation will be assumed analytic as

$$y = f(x) = f_0 + f_1 x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \ldots$$

Input $x(t)$ is assumed to be a stationary Gaussian process having zero mean, variance σ^2, autocorrelation function $R_{xx}(\tau)$ and spectral density $\Phi_{xx}(\omega)$.

Output mean value: The probability density function of x at any time is

$$p(x) = \frac{1}{\sqrt{(2\pi)}\,\sigma} \exp(-x^2/2\sigma^2)$$

For this probability density moments of odd order are zero and of even order are:

$$E\{x^{2n}\} = (2n-1)(2n-3)\ldots 3.1\,\sigma^{2n} = \frac{(2n)!}{2^n\, n!}\,\sigma^{2n}$$

So the mean value of output is found immediately as

$$E\{y\} = E\{f(x)\} = f_0 + \frac{f_2\,\sigma^2}{2!\,2} + \frac{f_4\,\sigma^4}{2!\,2^2} + \ldots$$

Output auto-correlation: The output correlation is

$$E\{y(t)\,y(t')\} = E\{f(x(t)f(x(t')\} = \sum_m \sum_n \frac{f_m\, f_n}{m!\, n!} E\{x^m(t)\, x^n(t')\}$$

Here odd-order moments (m+n odd) are zero so m + n can be assumed even. Then $E\{x^m(t)\, x^n(t')\}$ will be the sum of products of second order moments with say

(a) s cross-pairs between variables x(t) and x(t'),
(b) p pairings of variables x(t) among themselves,
(c) q pairings of variables x(t') among themselves.

In this case, $m = 2p + s$, $n = 2q + s$ and the number of terms can be shown to be

$$\frac{m!\, n!}{2p!\, 2q!\, s!}$$

On evaluating possibilities it is found that the right hand side can be written as the sum of contributions from $s = 0$ and $s > 0$ as

$$g_0^2 + \sum_{s=1}^{\infty} \frac{1}{s!} g_s^2 E\{x(t)\, x(t')\}^s$$

where g_0 is the output mean found above and g_s for $s > 0$ is

$$g_s = \sum_{p=0}^{\infty} \frac{f_{s+2p}}{p!} \frac{\sigma^{2p}}{2^p}$$

Putting $\tau = t - t'$ output autocorrelation function is consequently

$$R_{yy}(\tau) = E\{y(t)\, y(t')\} - E\{y(t)\}^2 = \sum_{s=1}^{\infty} \frac{1}{s!} g_s^2 R_{xx}^s(\tau)$$

The relation with Hermite polynomials: Hermite polynomials* of degrees 0, 1, 2, 3,.. etc associated with Gaussian probability density are

$$he_0(x) = 1,\ he_1(x) = x,\ he_2(x) = x^2 - \sigma^2,\ he_3(x) = x^3 - 3\sigma^2 x, \ldots \text{ etc}$$

Conversely, powers of x may be expressed in terms of the Hermite polynomials as

$$1 = he_0(x),\ x = he_1(x),\ x^2 = he_2(x) + \sigma^2 he_0(x),\ x^3 = he_3(x) + 3\sigma^2 he_1(x), \ldots \text{ etc}$$

If these relations are substituted into the power series for f(x) there is found the series

$$y = f(x) = g_0\, he_0(x) + g_1\, he_1(x) + \frac{g_2}{2!} he_2(x) + \frac{g_3}{3!} he_3(x) + \ldots$$

where the g's given by.

$$g_s = f_s + f_{s+2} \frac{\sigma^2}{2} + \frac{f_{s+4}}{2!} \frac{\sigma^4}{2^2} + \ldots \qquad s = 0, 1, 2, \ldots$$

which are the same as the coeffcients g found in the previous calculation. So the previous coefficients g are identified as coefficients in a Hermite polynomial expansion of the input/output function f().

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* Hermite polynomials and their properties are described in detail in appendix 4.

7.2 Direct use of Hermite polynomials

The coefficients g in the expansion of a function into a Hermite series

$$f(x) = \sum_{n=0}^{\infty} \frac{g_n}{n!} he_n(x)$$

may be found directly by using the orthogonality property

$$\int_{-\infty}^{\infty} he_m(x)\, he_n(x)\, p(x)\, dx = 0 \qquad m \neq n$$
$$= n!\, \sigma^{2n}, \quad m = n$$

Statistically interpreted this means that polynomials of different degrees are uncorrelated. A coefficient g_m is found by multiplication by $he_m(x)\, p(x)$ and integration giving

$$\int_{-\infty}^{\infty} f(x)\, he_m(x)\, p(x)\, dx = \int_{-\infty}^{\infty} he_m(x) \sum_{n=0}^{\infty} \frac{g_n}{n!} he_n(x)\, p(x)\, dx$$
$$= \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_{-\infty}^{\infty} he_m(x)\, he_n(x)\, p(x)\, dx$$

$$= g_m\, \sigma^{2m}$$

So

$$g_m = \frac{1}{\sigma^{2m}} \int_{-\infty}^{\infty} f(x)\, he_m(x)\, p(x)\, dx$$

If f(x) has an mth derivative $f^{(m)}(x)$ this expression may be put into a very convenient form. Integration by parts shows g_m to be

$$g_m = \int_{-\infty}^{\infty} f^{(m)}(x)\, p(x)\, dx = E_x\{f^{(m)}(x)\}, \quad m = 0, 1, 2, \ldots$$

In particular, g_0 is the mean value of f(x) and g_1 is the average gradient:

$$g_0 = E\{y\} = E\{f(x)\}, \quad g_1 = E\{f\,'(x)\}$$

Output autocorrelation: The output autocorrelation function may be found by using the generalized orthogonality property

$$\int \int he_m(x_1)\, he_n(x_2)\, p(x_1, x_2\,;\tau)\, dx_1\, dx_2 = 0, \qquad m \neq n,$$
$$= n!\; E(x_1x_2)^2, \quad m=n$$

Here $p(x_1, x_2\,;\tau)$ is the joint probabilty density of two values x, x time τ apart. (see appendix 4 for derivation)

The covariance of the output is

$$E\{f(x(t))\, f(x(t+\tau))\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1)\, f(x_2)\, p(x_1, x_2\,;\tau)\, dx_1\, dx_2$$

On substituting the expansion for f(x) in terms of Hermite polynomial and using the generalized orthogonality property there follows the value

$$E\{f(x(t))\, f(x(t+\tau))\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{g_m\, g_n}{m!\; n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} he_m(x_1)\, he_n(x_2)\, p(x_1, x_2;\tau)\, dx_1\, dx_2$$
$$= \sum_{n=0}^{\infty} \frac{g_n^{\,2}}{n!} R_{xx}(\tau)^n$$

Consequently since the first term g_0 is the mean value E(y)

$$R_{yy}(\tau) = E\,(f(x(t))\, f(x(t+\tau)) - E(y)^2$$
$$= \sum_{n=1}^{\infty} \frac{g_n^{\,2}}{n!} R_{xx}(\tau)^n$$

Cross-correlation function: Easy extension of this calculation gives the cross-correlation function of input with output as

$$R_{xy}(\tau) = g_1\, R_{xx}(\tau)$$

This is known as Bussgang's result: output autocorrelation is proportional to input auto-correlation*. The above calculation identifies the constant of proportionality as the average gradient of f(x).

$$g_1 = \int_{-\infty}^{\infty} f(x)\, x\, p(x)\, dx = E_x\{f(x)\, he_n(x)\} = E_x\{f\,'(x)\}$$

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* Bussgang: Res. Lab. Electron. MIT Tech. Rep. 216, 1952

7.3 The output spectrum:

The Wiener-Khinchin theorem gives the relation between any autocorrelation function $R(\tau)$ and the corresponding spectral density $\Phi(\omega)$ as

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega) \exp i\omega\tau \, d\omega$$

There follows

$$R^2(\tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\omega_1) \Phi(\omega_2) \exp i(\omega_1 + \omega_2)\tau \, d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \Phi(\omega_1) \Phi(\omega - \omega_1) \, d\omega_1 \right\} \exp i\omega\tau \, d\omega$$

i.e. to multiplication of R corresponds convolution of Φ

In general,

$$R^n(\tau) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Phi(\omega_1) \Phi(\omega_2) \dots \Phi(\omega_n) \exp\{(\omega_1 + \omega_2 \dots + \omega_n) \tau\} d\omega_1 d\omega_2 \dots d\omega_n$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi^{(n)}(\omega) \exp i\omega\tau \, d\omega$$

where

$$\Phi^{(n)}(\omega) = \frac{1}{(2\pi)^{n-1}} \int \int \dots \int_{\omega_1 + \omega_2 + \dots + \omega_n = \omega} \Phi(\omega_1) \Phi(\omega_2) \dots \Phi(\omega_n) \, d\omega_1 d\omega_2 \dots d\omega_n$$

representing the n fold convolution of Φ with a constant multiplier, i.e.

$$R^n(\tau) \text{ has Fourier transform } \frac{1}{(2\pi)^{n-1}} \underbrace{\Phi * \Phi * \dots * \Phi}_{n \text{ times}} (\omega) = \Phi^{(n)}(\omega)$$

The series for output autocorrelation then transforms to a corresponding series in the spectra:

$$\Phi_{yy}(\omega) = \sum_{n=1}^{\infty} \frac{g_n^2}{n!} \Phi_{xx}^{(n)}(\omega)$$

7.4 Narrow-band input

The significance of the expression for output spectrum becomes clearer when the input is narrow-band, concentrated near a frequency ω_0 say. In this case the convoluted spectra resulting from Fourier transformation of the powers of the autocorrelation function are concentrated near to frequencies $0, \pm\,\omega_0, \pm\,2\omega_0, \pm\,3\omega_0$, .. etc similarly to a nonlinearly transformed sinusoid.

A narrow-band input can be represented by a sinusoid with slowly varying amplitude and phase in the form

$$x(t) = A(t)\cos(\omega_0 t + \varphi(t))$$

with autocorrelation function given by

$$R_{xx}(\tau) = R_{AA}(\tau)\cos\omega_0 t$$

(Rice 1944-45) On taking Fourier transforms follows

$$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} R_{AA}(\tau)\cos\omega_0 t\,\exp(-i\omega t)\,d\omega$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty} R_{AA}(\tau)\,\{\exp - i(\omega - \omega_0)\,t + \exp - i(\omega + \omega_0)\,t\}\,d\omega$$

$$= \tfrac{1}{2}\,\{\Phi_{AA}(\omega - \omega_0) + \Phi_{AA}(\omega + \omega_0)\}$$

Since A(t) is a slowly-varying function of t, $\Phi_{AA}(\omega)$ will have a peak concentrated near to $\omega = 0$. So there result peaks to $\Phi_{xx}(\omega)$ near to $\omega = \pm\,\omega_0$.

If $n \geq 1$ then similarly

$$R_{xx}(\tau)^n = R_{AA}(\tau)^n\cos^n\omega_0 t = (\tfrac{1}{2})^n R_{AA}(\tau)^n\,(\exp i\omega_0 t + \exp -i\omega_0 t)^n$$

$$= R_{AA}(\tau)^n\{\exp in\omega_0 t + n\exp i(n-2)\omega_0 t + \frac{n(n-1)}{1.2}\exp i(n-4)\omega_0 t + \ldots + \exp -in\omega_0 t\}$$

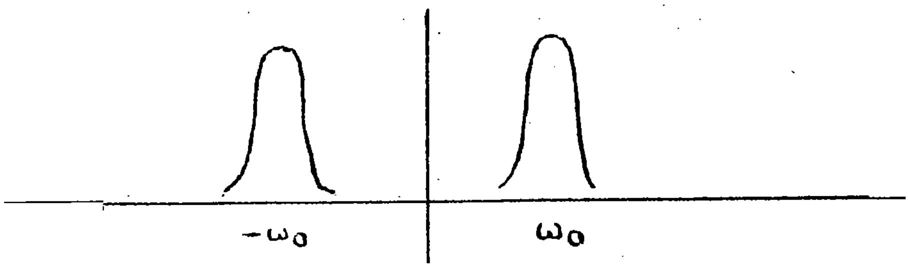
On transformation there results

$$\Phi_{xx}^{(n)}(\omega) = \int_{-\infty}^{\infty} R_{AA}(\tau)^{(n)}\exp(-i\omega t)\,d\omega$$

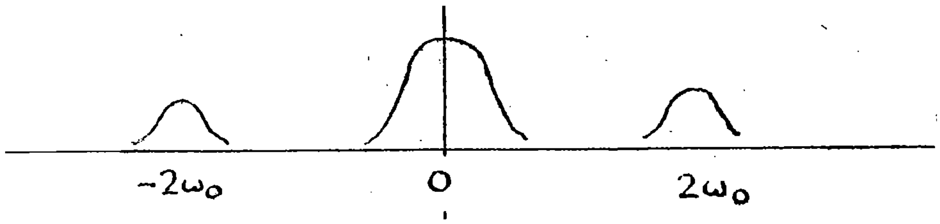
$$= \frac{1}{2^{n-1}}\{\Phi_{AA}^{(n)}(\omega - n\omega_0) + n\,\Phi_{AA}^{(n)}(\omega - (n-2)\omega_0) + \frac{n(n-1)}{1.2}\Phi_{AA}^{(n)}(\omega - (n-4)\omega_0) \ldots + \Phi_{AA}^{(n)}(\omega + n\omega_0)\}$$

Consequently $\Phi_{xx}^{(n)}(\omega)$ has peaks near to $\pm n\omega_0$, $\pm (n-2)\omega_0$, …etc and the ratio of the heights of these peaks is given by binomial coefficients. Convolution has a smoothing effect so that the peaks progressively become less peaked at higher frequencies and, since they arise from repeated convolution, tend towards a Gaussian form. The general effect is shown in the figures below.

$\Phi_{xx}^{(1)}(\omega) = \Phi_{xx}(\omega)$



$\Phi_{xx}^{(2)}(\omega)$



$\Phi_{xx}^{(3)}(\omega)$

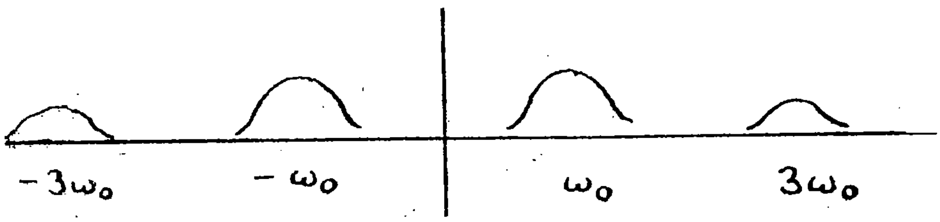


Fig. Convoluted Spectra

7.5 A Volterra system with Gaussian input

An analogous method to that for an instantaneous relation applies to the passage of Gaussian noise through a Volterra system. The Volterra system will be taken in time-invariant form with symmetric kernels which are essential in this calculation.

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(t - t_1, \dots, t - t_n)\, x(t_1) \dots x(t_n)\, dt_1 \dots dt_n$$

The input Gaussian process x(t), $-\infty < t < \infty$ will be assumed to be stationary with zero mean and autocorrelation function R(.).

Output mean value: Using known values of Gaussian moments there follows

$$E\{y(t)\} = \sum_{p=0}^{\infty} \frac{1}{(2p)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_{2p}(t- t_1, \dots, t -.t_{2p})\, R(t_{s+1} - t_{s+2}) \dots R(t_{s+2p-1} - t_{s+2p})\, dt_{s+1}, \dots, .dt_{s+2p}$$

Output correlation: On taking expectation of multiplied output values at different times there follows

$$E\{y(t)\, y(t')\} = \sum_{m}^{\infty} \sum_{n}^{\infty} \frac{1}{m!\, n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_m(t - t_1, \dots, t - t_m)\, h_n(t' - t'_1, \dots, t' -t'_n)\, E\{x(t_1)x(t_2) \dots x(t_m)\, x(t'_1)x(t'_2) \dots x(t'_n)\} dt_1 \dots dt_m\, dt'_1 \dots dt'_n$$

Odd-order cross-moments (m+n odd) are zero so m + n may be supposed even. Then

$$E\{x(t_1)x(t_2) \dots x(t_m)\, x(t'_1)x(t'_2) \dots x(t'_n)\}$$

will be the sum of all (m + n)! possible products of second order moments with say

(a) s cross-pairs between variables $x(t_1), x(t_2), \dots x(t_m)$ and $x(t'_1), x(t'_2), \dots, x(t'_n)$,
(b) p pairings of variables $x(t_1), x(t_2), \dots, x(t_m)$ among themselves,
(c) q pairings of variables $x(t'_1), x(t'_2), \dots, x(t'_n)$ among themselves.

For this it is necessary that

$$m = 2p + s, \quad n = 2q + s$$

The choice of the s values for cross-pairing from $x(t_1), x(t_2), \dots x(t_m)$ can be done in m!/(2p)!s! possible ways and similarly the choice of s values for cross-pairing from $x(t'_1), x(t'_2), \dots, x(t'_n)$ can be done in n!/(2q)!s! ways. When the two sets of s values are chosen they can be cross-paired in s! ways.

So the total number of possibilities is

$$s!\,\frac{(2p+s)!}{(2p)!\,s!}\,\frac{(2q+s)!}{(2q)!\,s!} = \frac{m!\,n!}{(2p)!\,(2q)!\,s!}.$$

By symmetry all these terms will all make the same contribution to the integral. So it is possible to concentrate on one of them when $x(t_1)$, $x(t_2)$,... $x(t_s)$ and $x(t'_1)$, $x(t'_2)$,.., $x(t'_s)$ are paired in that order so giving a term $E\{x(t_1)\,x(t'_1)\}E\{x(t_2)\,x(t'_2)\}\ldots E\{x(t_s)\,x(t'_s)\}$. This term is then multiplied by cross-moments resulting from the 2p remaining x(t) paired among themselves and the 2q remaining x(t') among themselves. These cross-moments will occur in all possible combinations and can be reconstituted as $E\{x(t_{s+1})\ldots x(t_{s+2p})\}$ and $E\{x(t'_{s+1})\ldots x(t'_{s+2q})\}$ respectively. So the complete symmetry of the kernels makes it possible to write the cross-moment $E\{y(t)\,y(t')\}$ as the sum of terms

$$\frac{1}{(2p)!\,(2q!)\,s!}\int\ldots\int dt_1 \ldots dt_m\, dt'_1 \ldots dt'_n$$

$$h_{2p+s}(t-t_1, \ldots, t-t_{2p+s})\, h_{2q+s}(t';\, t'-t'_1, \ldots, t'-t'_{2q+s})$$

$$E\{x(t_{s+1})\ldots x(t_{s+2p})\}E\{x(t'_{s+1})\ldots x(t'_{s+2q})\}$$

$$E\{x(t_1)\,x(t'_1)\}E\{x(t_2)\,x(t'_2)\}\ldots E\{x(t_s)\,x(t'_s)\}$$

This expression must be summed over all non-negative integral values of s, p and q. On summing first over p and q arise identical sums

(i) for $s = 0$ the value g_0^2 where $g_0 = E\{y\}$
(ii) for $s = 1, 2, 3, \ldots$ the value

$$g_s(t-t_1, \ldots, t-t_s) = \int\ldots\int dt_1 \ldots dt_s\, dt'_1 \ldots dt'_s$$

$$\sum_{p=0}^{\infty}\frac{1}{(2p)!}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} h_{s+2p}(t-t_1, \ldots, t-t_{s+2p})\; E\{x(t_{s+1})\ldots x(t_{s+2p})\}\, dt_{s+1}, \ldots, dt_{s+2p}$$

Output autocorrelation: On summation with respect to s there is finally found the autocorrelation function of the output

$$R_{yy}(t-t') = E[y(t)\,y(t')] - E[y(t)]^2 =$$

$$= \sum_{s=1}^{\infty}\frac{1}{s!}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} g_s(t-t_1, \ldots, t-t_s)\, g_s(t'-t'_1, \ldots, t'-t'_s)\; R(t_1-t'_1)\ldots R(t_s-t'_s)\, dt_1 \ldots dt_s\, dt'_1 \ldots dt'_s$$

Ref: Bedrosian & Rice Proc IEEE 1971.

7.6 Multidimensional Hermite polynomials and Hermite functional expansion

For an instantaneous nonlinearity it was seen that the passage of a Gaussian process is most conveniently analysed by using Hermite polynomials. The same is true for the passage through a Volterra system except that in this case a functional Hermite polynomials must be used. These are based on the multidimensional Hermite polynomials for N variables described in appendix 4

For a stationary Gaussian stochastic process with zero mean and auto-correlation function R(,) the Hermite polynomial functionals are

$$\begin{aligned}
he^{(0)}(x) &= 1 \\
he^{(1)}(x; t) &= x(t) \\
he^{(2)}(x; t_1, t_2) &= x(t_1)\, x(t_2) - R(t_2 - t_1) \\
he^{(3)}(x; t_1, t_2, t_3) &= x(t_1)\, x(t_2)\, x(t_3) - x(t_1)\, R(t_3 - t_2) - x(t_2)\, R(t_1 - t_3) - x(t_3)\, R(t_2 - t_1)
\end{aligned}$$

etc. The general term follows from the finite dimensional case (appendix 4).

The inverse formulae are similar but with positive signs:

$$\begin{aligned}
x(t) &= he^{(1)}(x; t) \\
x(t_1)\, x(t_2) &= he^{(2)}(x; t_1, t_2) + R(t_2 - t_1) \\
x(t_1)\, x(t_2)\, x(t_3) &= he^{(3)}(x; t_1, t_2, t_3) + \\
&\quad R(t_3 - t_2)\, he^{(1)}(x; t_1) + R(t_1 - t_3)\, he^{(1)}(x; t_2) + R(t_2 - t_1)\, he^{(1)}(x; t_3)
\end{aligned}$$

etc. When this transformation is made to the Volterra series with symmetrical kernels

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(t - t_1, \dots, t - t_n)\, x(t_1) \dots x(t_n)\, dt_1 \dots dt_n$$

there results the Hermite series with kernels g.

$$g_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g_n(t - t_1, \dots, t - t_n)\, he^{(n)}(x; t_1, \dots, t_n)\, dt_1 \dots dt_n$$

The kernels g are related to the kernels h by the same formulae as were found before, g_0 being the mean value of output.y(t) given above and for s >0,

$$g_s(t - t_1, \dots, t - t_s) = \sum_{p=0}^{\infty} \frac{1}{(2p)!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_{s+2p}(t - t_1, \dots, t - t_{s+2p})\, R(t_{s+1} - t_{s+2}) \dots R(t_{s+2p-1} - t_{s+2p})\, dt_{s+1}, \dots, dt_{s+2p}$$

The previous kernels g are consequently identified as kernels of a Hermite expansion of the input/output relation.

From the relation between products of the inputs x and the Hermite polynomials the inverse formulae are easily found, it only being necessary to change the sign of R in this expression

$$h_s(t - t_1, \ldots, t - t_s) = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g_{s+2p}(t - t_1, \ldots, t - t_{s+2p}) \, R(t_{s+1} - t_{s+2}) \ldots R(t_{s+2p-1} - t_{s+2p}) \, dt_{s+1}, \ldots, dt_{s+2p}$$

The output autocorrelation function: The previously derived formula for the output autocorrelation function now follows immediately from the orthogonality property of the Hermite functional polynomials

Two Hermite functional polynomials of different degrees m and n are orthogonal:

$$E\{he^{(m)}(x; t_1, \ldots t_m)\, he^{(n)}(x: t_1', \ldots t_n')\} = 0, \quad m \neq n$$

while for the same degree n:

$$E\{he^{(n)}(x; t_1, \ldots t_n)\, he^{(n)}(x: t_1', \ldots t_n')\} = \sum_{\text{perms } \{p_1, p_2, \ldots p_n\}} R(t'_{p_1} - t_1) \ldots R(t'_{p_n} - t_n)$$

Here the sum on the right is over n! permutations $\{p_1, p_2, \ldots p_n\}$ of the set $\{1, 2, \ldots, n\}$ Applying these formulae to the shifted product y(t) y(t') of two Hermite functional series there is found

$$E[y(t)\, y(t')] = g_0^2 + \sum_{s=1}^{\infty} \frac{1}{s!} \int \ldots \int g_s(t - t_1, \ldots, t - t_s)\, g_s(t' - t'_1, \ldots, t' - t'_s) \, R(t'_1 - t_1) \ldots R(t'_s - t_s)\, dt_1 \quad dt_s\, dt'_1 \quad dt'_s$$

from which the formula for the output autocorrelation function follows as before.

It should finally be remarked that, as in the scalar case previously considered, the Hermite functional expansion is valid for functionals of finite variance.

Reference: This type of Hermite functional polynomial expansion was introduced by the writer in the report 'Use of functionals ...' in 1955 and later. When the input is white Gaussian noise it includes as a special case the so-called Wiener functionals described in the following chapter.

7.7 Relation between transforms of Volterra and Hermite kernels

On introducing the Fourier transforms of the kernels $h_n(.)$, $g_n(.)$, $n = 1, 2, ...$

$$H_n(i\omega_1,.., i\omega_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1,..\tau_n) \exp i(\omega_1\tau_1+..+ \omega_n\tau_n)\, d\tau_1....d\tau_n$$

$$G_n(i\omega_1,.., i\omega_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g_n(\tau_1,..\tau_n) \exp i(\omega_1\tau_1+..+ \omega_n\tau_n)\, d\tau_1....d\tau_n$$

the relations between the two sets of kernels become (changing p to k)

(a) Volterra to Hermite:

$$G_0 = H_0 + \sum_{k=0}^{\infty} \frac{1}{2^k k!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} H_{2k}(i\omega_1', - i\omega_1',..., i\omega_k', - i\omega_k') \frac{\Phi(\omega_1')}{2\pi} .. \frac{\Phi(\omega_k')}{2\pi} d\omega_1' \dots d\omega_k'$$

$$G_n(\omega i_1,...,\omega i_n) = H_n(i\omega_1, ..., i\omega_n) + \sum_{k=0}^{\infty} \frac{1}{2^k k!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} H_{n+2k}(\omega_1,...,\omega_n, i\omega_1', -i\omega_1',..., i\omega_k', -i\omega_k') \frac{\Phi(\omega_1')}{2\pi} \dots \frac{\Phi(\omega_k')}{2\pi} d\omega_1' \dots d\omega_k'$$

(b) Hermite to Volterra:

$$H_0 = G_0 + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} G_{2k}(i\omega_1', - i\omega_1',..., i\omega_k', - i\omega_k') \frac{\Phi(\omega_1')}{2\pi} .. \frac{\Phi(\omega_k')}{2\pi} d\omega_1' \dots d\omega_k'$$

$$H_n(\omega i_1,...,\omega i_n) = G_n(i\omega_1, ..., i\omega_n) + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} G_{n+2k}(\omega_1,...,\omega_n, i\omega_1', -i\omega_1',..., i\omega_k', -i\omega_k') \frac{\Phi(\omega_1')}{2\pi} \dots \frac{\Phi(\omega_k')}{2\pi} d\omega_1' \dots d\omega_k'$$

7.8 The output spectrum

The previous equations relating input and output autocorrelations may be transformed into the frequency domain giving corresponding relations between input and output spectra. Then on introducing the inverse transform:

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega) \exp i\omega\tau \, d\omega$$

into terms on the right-hand side of the equations there is found, on setting $\tau = t - t'$

(a) for the term with n = 1

$$\int_{-\infty}^{\infty} |G_n(i\omega)|^2 \, \Phi_{xx}(\omega) \exp i\omega\tau \, d\omega$$

(b) for the terms with n = 2, 3, ...

$$\frac{1}{n!} \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} |G_n(i\omega_1, .., i\omega_n)|^2 \, \Phi_{xx}(\omega_1) \ldots \Phi_{xx}(\omega_n) \exp i(\omega_1\tau_1 + .. + \omega_n\tau_n) \, d\omega_1, ., d\omega_n$$

These last terms may be written in the form

$$\frac{1}{n!} \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \{...\} \exp i\omega\tau \, d\omega$$

where the expression in the bracket is

$$\frac{1}{n!} \frac{1}{(2\pi)^n} \int_{\omega_1, + .. + \omega_n = \omega} \ldots\ldots \int |G^{(n)}(i\omega_1, .., i\omega_n)|^2 \, \Phi_{xx}(\omega_1) \ldots \Phi_{xx}(\omega_n) \, d\omega_1, ., d\omega_n$$

On substituting terms (a), (b) into the expression for output autocorrelation it becomes

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \Phi_{yy}(\omega) \exp i\omega\tau \, d\omega$$

which identifies the output spectral density as

$$\Phi_{yy}(\omega) = |G_1(i\omega)|^2 \, \Phi_{xx}(\omega) +$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(2\pi)^{n-1}} \int_{\omega_1 + .. + \omega_n = \omega} \ldots \int |G_n(i\omega_1, .., i\omega_n)|^2 \, \Phi_{xx}(\omega_1) \quad \Phi_{xx}(\omega_n) \, d\omega_1, ..., d\omega_n$$

Chapter 8

WHITE NOISE INPUT

8.1 The use of a Hermite expansion for white noise input

We now consider the response of Volterra systems to white noise which was the first application of Volterra functional series in engineering with the initial paper by Wiener in 1942.

The white noise input process denoted by w(t), $-\infty < t < \infty$, will be assumed to have zero mean value and power N If this white noise is input to a Volterra system there result terms of the type:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(t-t_1, t-t_2, \dots, t-t_n)\, w(t_1)\, w(t_2) \dots w(t_n)\, dt_1\, dt_2 \dots dt_n$$

Here arises immediately the question as to what is the correct interpretation of this expression in view of the fact that there occur powers of white noise whose meaning is unclear when certain of the $t_1, t_2, \dots, t_n$ are equal i.e. down the 'diagonals' of the n-dimensional region of integration. This question is clarified by use of the Hermite form which is now considered

Hermite polynomial functionals for white Gaussian noise: For a stationary Gaussian white noise process x(t), $-\infty < t < \infty$ having zero mean and noise power N, the autocorrelation function is

$$R(\tau) = N\, \delta(\tau)$$

The corresponding Hermite functionals follow from the values given in the last chapter as

$$he^{(0)}(x) = 1$$

$$he^{(1)}(x; t) = x(t)$$

$$he^{(2)}(x; t_1, t_2) = x(t_1)\, x(t_2) - N\, \delta(t_2 - t_1)$$

$$he^{(3)}(x; t_1, t_2, t_3) = x(t_1)\, x(t_2)\, x(t_3) + x(t_1)\, N\, \delta(t_3 - t_2) - x(t_2)\, N\, \delta(t_1 - t_3) - x(t_3)\, N\, \delta(t_2 - t_1)$$

.... etc

The formulae inverse to these have a similar form and are:

$$x(t) = he^{(1)}(x; t)$$

$$x(t_1)\, x(t_2) = he^{(2)}(x; t_1, t_2) + N\, \delta(t_2 - t_1)$$

$$x(t_1)\, x(t_2)\, x(t_3) = he^{(3)}(x; t_1, t_2, t_3) + N\, \delta(t_3 - t_2)\, he^{(1)}(x; t_1) + N\, \delta(t_1 - t_3)\, he^{(1)}(x; t_2) + N\, \delta(t_2 - t_1)\, he^{(1)}(x; t_3)$$

.... etc.

When this transformation is made to the Volterra series considered above i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int \int \ldots \int h_n(t - t_1, \ldots, t - t_n)\, x(t_1) \ldots x(t_n)\, dt_1 \ldots dt_n$$

there results the Hermite series with kernels g() i.e..

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int \int \ldots \int g_n(t - t_1, \ldots, t - t_n)\, he^{(n)}(x; t_1, \ldots.\, t_n)\, dt_1 \ldots dt_n$$

Here and below, the range of integration is understood as infinite. On first sight the Hermite form of the expansion appears to involve delta functions which occur in the expressions for the $he^{(n)}()$ but this is not really so since they are absorbed into the kernels. e.g. for n = 2

$$\int \int g_2(t - t_1, t - t_2)\, he^{(2)}(x; t_1, t_2)\, dt_1\, dt_2$$

$$= \int \int g_2(t - t_1, t - t_2)\{x(t_1)\, x(t_2) - N\, \delta(t_2 - t_1)\} dt_1\, dt_2$$

$$= \int \int g_2(t - t_1, t - t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2$$

$$- N \int \int g_2(t - t_1, t - t_2) \delta(t_2 - t_1)\} dt_1\, dt_2$$

$$= \int \int g_2(t - t_1, t - t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2 - N \int g_2(t - t_1, t - t_1)\, dt_1$$

$$= \int \int g_2(t - t_1, t - t_2)\, x(t_1)\, x(t_2)\, dt_1\, dt_2 - N \int g_2(t', t')\, dt'$$

Similarly there is found

$$\int \int \int g_3(t - t_1, t - t_2, t - t_3)\, he^{(3)}(x; t_1, t_2, t_3)\, dt_1\, dt_2\, dt_3$$

$$= \int \int \int g_3(t - t_1, t - t_2, t - t_3)\, x(t_1)\, x(t_2)\, x(t_3)\, dt_1\, dt_2\, dt_3$$

$$- 3N \int \int g_3(t - t_1, t', t')\, x(t_1)\, dt_1\, dt'$$

The last step assumes symmetric kernels. In the literature such formulae are ascribed to Wiener, the kernels being described as Wiener kernels*.

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* The connection with Hermite polynomial functionals as defined here did not occur in Wiener's 1958 book and so he did not explicitly define the Hermite kernels although these were mentioned in subsequent literature.

8.2 Illustration: a LNL system with noise input.

With white noise input the LNL (linear-nonlinear-linear) system has the form below.

$$\xrightarrow{w} \boxed{H(p)} \xrightarrow{x} \boxed{f(.)} \xrightarrow{y} \boxed{K(p)} \xrightarrow{z}$$

Fig: A LNL system with noise input.

The initial linear filter with transfer function H(s) is a shaping filter acting on white noise w giving Gaussian input to an instantaneous nonlinear element having input-output characteristic

$$y = f(x)$$

This output then passes through another linear filter with transfer function K(s).
The Gaussian input x to the nonlinear element has variance

$$\sigma^2 = N \int_{-\infty}^{\infty} h(t')^2 \, dt'$$

The function f(.) relative to this input is then of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} b_n he_n(x)$$

The Hermite polynomials on the right are the polynomials $he_n(x, \sigma)$ (appendix 4) They are now expressible by Hermite functionals, e.g. $he_2(x)$ is

$$x(t)^2 - \sigma^2 = \left(\int_{-\infty}^{\infty} h(t-t') \, w(t') \, dt'\right)^2 - N \int_{-\infty}^{\infty} h(t')^2 \, dt'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-t_1) \, h(t-t_2) \{w(t_1) w(t_2) - N \, \delta(t_2 - t_1)\} dt_1 \, dt_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-t_1) \, h(t-t_2) \, he^{(2)}(x; t_1, t_2) \, d\,t_1 \, dt_2$$

In general there is found

$$he_n(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h(t-t_1) \, h(t-t_2) \ldots h(t-t_n) \, he^{(n)}(x; t_1, t_2, \ldots, t_n) \, dt_1 \, dt_2 \ldots dt_n$$

On convoluting this system with the final linear filter it is seen that the input-output relation of the system is a white noise Hermite functional series with nth order kernel

$$b_n \int_{-\infty}^{\infty} k(t-t') \, h(t'-t_1) \, h(t'-t_2) \ldots h(t'-t_n) \, dt'$$

8.3 Definitions using integration with respect to the Wiener process

Wiener's approach did not use the concept of white noise and so was more rigorous mathematically. Instead integrating with respect to white noise, he used Stieltjes integrals with respect to a one-dimensional Brownian motion or 'Wiener process'. The Wiener process W(t) is integrated white noise so white noise can be thought of intuitively as its (strictly non-exisiting) derivative There is defined from some initial instant t_0:

$$W(t) = \int_{t_0}^{t} w(t')\, dt'$$

Initially Wiener considered linear functionals of the process W(t), $t \varepsilon [0,T]$ of the type

$$\int_0^T k(t)\, dW(t)$$

This linear functional will be a Gaussian variable of zero mean and mean square value

$$N^2 \int_0^T k(t)^2\, dt$$

For this to be finite the function k(.) must be taken to be square integrable in the Lebesgue sense. For such functions however, the above Stieltjes integral is not defined according to normal theory which would require k(.) to be of bounded variation since W(.) is continous. So another way is used to define the integral, sometimes known as the Paley-Wiener-Zygmund (PWZ) integral. First the integral is defined for a step function approximation to k(t) based on regular subdivision of the interval of integration as indicated in the figure below. The value taken in each interval is the mean value of k(t) in that interval

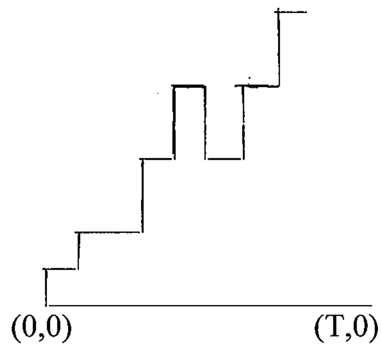

Fig: Stepwise kernel approximation

Note: Actually Wiener in all his writings used the Wiener process in the form W(t, α) where α is a parameter lying in the range $0 \leq \alpha \leq 1$. He had shown in 1930 that the Brownian motion paths can be uniquely specified by such a parameter. Lebesgue integration with respect to this parameter gives expected (mean) values. Nowadays it is the custom to omit the parameter as will be done here.

If the number of subintervals is M, the mean value of k() in the nth subinterval I_n is

$$\bar{k}_n = \frac{1}{\Delta t} \int_{I_n} k(t')\, dt' \quad n = 1,\ldots, M$$

The value of the integral for this stepwise function is taken as

$$S_M = \sum_{n=1}^{M} \bar{k}_n \, \Delta W$$

the sum being over M subintervals. With increasing M it can be shown the sequence S_M converges in mean square as $M \to \infty$ to a limit so defining the integral*.

The multidimensional Wiener integral: In his 1939 paper on turbulence 'The homogeneous chaos' Wiener used multidimensional integrals of the form

$$\int \int \ldots \int k(t_1, t_2, \ldots, t_n)\, dW(t_1)\, dW(t_2) \ldots dW(t_n)$$

taken over a rectangular region. The implicit assumption was that such integrals could be defined in a similar way to the one-dimensional linear case. However a significant difference occurs with the multidimensional form** which Wiener appears to have been unaware of at the time as he did not discuss it either then or later.

The difference may be explained in the two-dimensional case by considering the integral

$$\int_0^T \int_0^T k(t_1, t_2)\, dW(t_1)\, dW(t_2)$$

extended over the square of side T as in the diagram below where, for the approximating integral, both axes are divided into M subintervals of equal size.

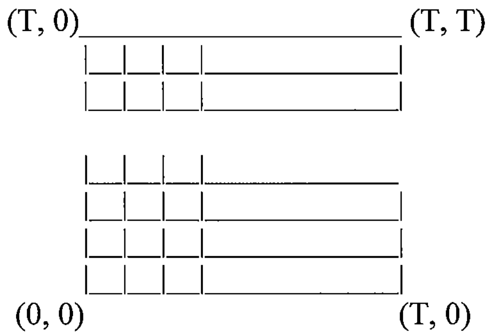

Fig. Uniform partition for the definition of a 2 dimensional Wiener integral.

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* The Cauchy general principle of convergence for mean square convergence is satisfied.

** As observed by V Mandreker, Wiener's multidimensional integral was of a type later called the Stratonovich integral. See: Notices Amer.Math.Soc.vol.42(6), 1995, p.668

By the same method of definition as for the linear case, the integral would be defined as a mean square limiting value of approximating sums taken over all cells.

$$\sum_{m} \sum_{n} \bar{k}_{m,n}\, \Delta W_m\, \Delta W_n$$

This sum may be split up over the upper triangle, the diagonal, and the lower triangle.

$$\sum\sum_{m<n} \bar{k}_{m,n}\, \Delta W_m\, \Delta W_n + \sum\sum_{m=n} \bar{k}_{m,n}\, \Delta W_m\, \Delta W_n + \sum\sum_{m>n} \bar{k}_{m,n}\, \Delta W_m\, \Delta W_n$$

Of special interest is the sum over the diagonal cells (m = n). In the normal definition of an integral the contribution of these would tend to zero with decreasing mesh size since their total area tends to zero and in the limit they approach the diagonal line. However for the Wiener integral this is not so; the diagonal terms summing to give a finite limiting value arising from their autocorrelation as expressed by the equation

$$E\,(\Delta W)^2 = N\,\Delta t$$

where ΔW is the increment of the Wiener process in a time interval Δt. As a result the sum of the diagonal elements tends in the limit to the value

$$\int_0^T k(t, t)\,(N\,dt) = N \int_0^T k(t, t)\,dt$$

It shows that the Wiener integral has quite different properties to ordinary integrals. The value of the original integral is in consequence made up of 3 contributions from the upper triangle the diagonal and the lower triangle.

$$\Big\{ \iint_{0 \le t_1 < t_2 \le T} + \iint_{0 \le t_1 = t_2 \le T} + \iint_{0 \le t_2 < t_1 \le T} \Big\}\, k(t_1, t_2)\, dW(t_1)\, dW(t_2)$$

Similar behaviour is observed in Wiener integrals of higher dimension and the region of integration has to be divided up in order to evaluate them.

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* Apparently first pointed out by P Lévy (see Doob's book: Stochastic Processes 1953).

8.4 The Itô integral

Based on such considerations the Wiener integral was redefined by Itô in 1951 as

$$\int\int \ldots \int_{0 \le t_1 < \ldots < t_n \le T} k(t_1, t_2, \ldots, t_n)\, dW(t_1)\, dW(t_2) \ldots dW(t_n)$$

The integral is taken over a open triangular or tetrahedral region. The kernel function k() being assumed square integrable these integrals may be defined analogously to the one dimensional case. Such integrals have the property not shared by the Wiener's form in that integrals of different orders are orthogonal i.e. uncorrelated. Such a property is of central importance

By breaking up the region $0 \le t_1, t_2, \ldots, t_n \le T$ of integration into tetragonal subregions in the previous Wiener integral it can be shown that Itô and Hermite representations are equivalent; in fact they differ only by a numerical factor:

$$n! \int\int \ldots \int_{0 \le t_1 < \ldots < t_n \le T} k(t_1, t_2, \ldots, t_n)\, dW(t_1)\, dW(t_2) \ldots dW(t_n)$$

$$= \int_0^T \int_0^T \ldots \int_0^T k(t_1, t_2, \ldots, t_n)\, he^{(n)}(w; t_1, t_2, \ldots, t_n)\, dt_1\, dt_2 \ldots dt_n$$

The Itô expansion corresponding to the Hermite expansion previously considered will consequently not have the factorial multipliers

The verification in the 2 dimensional case is as follows. From above and using the symmetry of the kernel,

$$\int_0^T \int_0^T k(t_1, t_2)\, dW(t_1)\, dW(t_2) = 2 \int\int_{0 \le t_1 < t_2 \le T} k(t_1, t_2)\, dW(t_1)\, dW(t_2) + N \int_0^T k(t, t)\, dt$$

Rearranging and putting $dW(t) = w(t)\, dt$ there follows

$$\begin{aligned} 2 \int\int_{0 \le t_1 < t_2 \le T} k(t_1, t_2)\, dW(t_1)\, dW(t_2) &= \int_0^T \int_0^T k(t_1, t_2)\, dW(t_1)\, dW(t_2) - N \int_0^T k(t, t)\, dt \\ &= \int_0^T \int_0^T k(t_1, t_2)\, \{w(t_1)\, w(t_2) - N\, \delta(t_2 - t_1)\} dt_1 dt_2 \\ &= \int_0^T \int_0^T k(t_1, t_2)\, he^{(2)}(w; t_1, t_2)\, dt_1\, dt_2 \end{aligned}$$

Showing equivalence of the Hermite and Itô forms. Similar calculations exist for higher dimensional integrals.

Reference: K Itô: J. Math. Soc. Japan 1951

Consider now a 3rd order Wiener integral

$$\int_0^T \int_0^T \int_0^T k(t_1, t_2, t_3)\, dW(t_1)\, dW(t_2)\, dW(t_3)$$

The region $0 \leq t_1, t_2, t_2 \leq T$ must be subdivided into 6 (= 3!) tetrahedral regions of the type $0 \leq t_1 < t_2 < t_3 \leq T$ and 3 regions of the type $0 \leq t_1 < t_2 = t_3 \leq T$ Followoing the same method as before,

$$\int_0^T \int_0^T \int_0^T k(t_1, t_2, t_3)\, dW(t_1)\, dW(t_2)\, dW(t_3) =$$

$$6 \int\int\int_{0 \leq t_1 < t_2 < t_3 \leq T} k(t_1, t_2, t_3)\, dW(t_1)\, dW(t_2)\, dW(t_3) \;+\; 3 \int\int\int_{0 \leq t_1 < t_2 = t_3 \leq T} k(t_1, t_2, t_3)\, dW(t_1)\, dW(t_2)\, dW(t_3)$$

The last integral is

$$\int\int\int_{0 \leq t_1 < t_2 = t_3 \leq T} k(t_1, t, t)\, dW(t_1)\, (N\, dt)$$

Using white noise w there results in a similar way to case n = 2,

$$6 \int\int\int_{0 \leq t_1 < t_2 < t_3 \leq T} k(t_1, t_2, t_3)\, dW(t_1)\, dW(t_2)\, dW(t_3) =$$

$$= \int_0^T \int_0^T \int_0^T k(t_1, t_2, t_3)\{w(t_1)\, w(t_2)\, w(t_3) - 3\, N\, w(t_1)\, \delta(t_2 - t_1)\} dt_1\, dt_2\, dt_3$$

$$= \int_0^T \int_0^T \int_0^T k(t_1, t_2, t_3)\, he^{(3)}(w;\, t_1, t_2, t_3)\, dt_1\, dt_2\, dt_3$$

8.5 Wiener's G-functionals

In 1958 Wiener published a book 'Nonlinear Problems of Random Theory' giving his version of the theory. In doing so he corrected his earlier oversight regarding orthogonality. Calling the method expansion by G-functionals he gave no description of the prior forms of the Itô and Hermite expansion*. His form of the expansion was not new but his reputation in this field soon led to this type of orthogonal expansion becoming known as the Wiener expansion..

He obtained his G-functionals by successive orthogonalization of the sequence

$$k,\ \int_0^T k(t)\,dW(t),\ \int_0^T\!\!\int_0^T k(t,t')\,dW(t)\,dW(t'),\ \int_0^T\!\!\int_0^T\!\!\int_0^T k(t,t',t'')\,dW(t)\,dW(t')\,dW(t''),\ \ldots\text{etc}$$

The first four G-functionals are easy to find as:

$$G_0(k_0, x(t)) = k_0$$

$$G_1(k_1, x(t)) = \int_0^T k_1(t)\,dW(t)$$

$$G_2(k_2, x(t)) = \int_0^T\!\!\int_0^T k_2(t,t')\,dW(t)\,dW(t') - N\int_0^T k_2(t,t)\,dt$$

$$G_3(k_3, x(t)) = \int_0^T\!\!\int_0^T\!\!\int_0^T k_3(t,t',t'')\,dW(t)\,dW(t')\,dW(t'') - 3N\int_0^T\!\!\int_0^T k_3(t,t,t')\,dt\,dW(t')$$

These are of course exactly the same as those which would be found by either the Hermite or Itô theory. Wiener's definition does not easily lead to an expression for the nth order G-functional. A systems analysis restatement using white noise was made by Lee in 1964** who wrote the equations in the form

$$G_1(k_1, x(t)) = \int_{-\infty}^{\infty} k_1(t-\tau_1)\,x(\tau_1)\,d\tau_1$$

$$G_2(k_2, x(t)) = \int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty} k_2(\tau_1,\tau_2)\,x(t-\tau_1)\,x(t-\tau_2)\,d\tau_1\,d\tau_2 - A\int_{-\infty}^{\infty} k_2(\tau_1,\tau_1)\,d\tau_1$$

$$G_3(k_3, x(t)) = \int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty} k_3(\tau_1,\tau_2,\tau_3)\,x(t-\tau_1)\,x(t-\tau_2)\,x(t-\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3 - 3A\int_{-\infty}^{\infty} k_3(\tau_1,\tau_2,\tau_2)\,x(t-\tau_1)\,d\tau_1\,d\tau_2$$

This notation then became adopted in its sampled data form by Marmarelis and Rugh*** which led on to rediscovering the Hermite connection.

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* Apparently Wiener was not aware of Itô's work but on p.38 of this book he referenced the writer's 1957 report describing Hermite functionals which was known to his students.
** Y.W. Lee: pp.17-33 in 'Selected Papers of Norbert Wiener' SIAM & MIT Press 1964
*** See the books of P.Z. Marmarelis & V.S. Marmarelis 1978; W.J. Rugh 1981

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Appendix 1

POWER SERIES

1 Remarks on notation

An analytic function is characterised by a power series in x

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

The Taylor-MacLaurin form gives coefficients in terms of derivatives at x zero

$$a_n = \frac{f^{(n)}(0)}{n!} \quad n = 0, 1, 2, \dots$$

Correspondingly it is convenient to use the representation

$$f(x) = f_0 + f_1 x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \dots$$

where

$$f_n = f^{(n)}(0) \quad n = 0,1,2,..$$

the coefficients f_n then being called *Taylor-MacLaurin coefficients*. This notational convention can extend to functions of several variables

The most important case is when the constant term f_0 is zero so that $x = 0$ corresponds to $f(x) = 0$ This can be arranged by change of scale. Attention will be restricted to this case in what follows.

2 Bell polynomials

It is frequently necessary to form powers of quantities expressed by power series. Suppose that

$$y = f_1 x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + f_4 \frac{x^4}{4!} + \dots$$

Then it is found that

$$\frac{y^2}{2!} = (f_1^2) \frac{x^2}{2!} + (3f_1f_2) \frac{x^3}{3!} + (3f_2^2 + 4f_1f_3) \frac{x^4}{4!} + ..$$

$$\frac{y^3}{3!} = (f_1^3) \frac{x^3}{3!} + (6f_1^2f_2) \frac{x^4}{4!} + (10f_1^2f_3 + 15f_1f_2^2) \frac{x^5}{5!} + ..$$

$$\frac{y^4}{4!} = (f_1^4) \frac{x^4}{4!} + (10f_1^3f_2) \frac{x^5}{5!} + (20f_1^3f_3 + 45\, f_1^2f_2^2) \frac{x^6}{6!} + ..$$

The equation for the nth power has the form

$$\frac{y^n}{n!} = B_{n,n}(f)\frac{x^n}{n!} + B_{n+1,n}(f)\frac{x^{n+1}}{(n+1)!} + B_{n+2,n}(f)\frac{x^{n+2}}{(n+2)!} + .. = \sum_{m \geq n} B_{m,n}(f)\frac{x^m}{m!}$$

The quantities $B_{m,n}(f)$ defined by this equation are polynomials in the Taylor-MacLaurin coefficients f called *Bell Polynomials* after the American mathematician E.T. Bell.
The polynomial $B_{m,n}(f)$ depends on $f_1, ..., f_{m-n+1}$ and is also written $B_{m,n}(f_1, ..., f_{m-n+1})$
The first few polynomials may be set out to correspond with the above equations as:

$B_{1,1}(f) = f_1$ $B_{2,1}(f) = f_2$, $B_{3,1}(f) = f_3$, $B_{4,1}(f) = f_4$, $B_{5,1}(f) = f_5$,

$B_{2,2}(f) = f_1^2$ $B_{3,2}(f) = 3f_1f_2$, $B_{4,2}(f) = 4f_1f_3 + 3f_2^2$, $B_{5,2}(f) = 5\, f_1f_4 + 10f_2f_3$,

$B_{3,3}(f) = f_1^3$ $B_{4,3}(f) = 6f_1^2f_2$, $B_{5,3}(f) = 10f_1^2f_3 + 15f_1f_2^2$,

$B_{4,4}(f) = f_1^4$ $B_{5,4}(f) = 10f_1^3f_2$,

$B_{5,5}(f) = f_1^5$

3 Series substitution

Suppose given two power series relating y to x and z to x:

$$y = h_1 x + h_2 \frac{x^2}{2!} + h_3 \frac{x^3}{3!} + \ldots$$

$$z = k_1 y + k_2 \frac{y^2}{2!} + k_3 \frac{y^3}{3!} + \ldots$$

Substitution for y from the first into the second series results in a series relating z to x

$$z = g_1 x + g_2 \frac{x^2}{2!} + g_3 \frac{x^3}{3!} + \ldots$$

The first few coefficients are:

$$g_1 = k_1 h_1$$
$$g_2 = k_1 h_2 + k_2 h_1^2$$
$$g_3 = k_1 h_3 + k_2 (3h_1h_2) + k_3 h_1^3$$
$$g_4 = k_1 h_4 + k_2 (4\, h_1 h_3 + 3\, h_2^2) + k_3 (6h_1^2 h_2) + k_4 h_1^4$$
$$g_5 = k_1 h_5 + k_2 (5\, h_1h_4 + 10\, h_2 h_3) + k_3 (10\, h_1^2 h_3 + 15\, h_1 h_2^2) + k_4 (10\, h_1^3 h_2) + k_5 h_1^5$$

The general term may be written down using Bell polynomials as

$$g_n = \sum_{m \leq n} k_m B_{n,m}(h)$$

---

*Reference: Riordan: 'Combinatorial Analysis', Wiley 1958.

4 Series inversion and the reversion method

Two power series of the form

$$y = h_1 x + h_2 \frac{x^2}{2!} + h_3 \frac{x^3}{3!} + \dots$$
$$x = k_1 y + k_2 \frac{y^2}{2!} + k_3 \frac{y^3}{3!} + \dots$$

are called *inverse* the relation being mutual. Such a relation implies that h_1 and k_1 are both nonzero. It is frequently necessary to know the relation between the two sets of coefficients. To keep the notation conforming with other calculations in the book it will be convenient to solve for the second series from the first.

The reversion method: A direct method of inversion is to rearrange the equation to be inverted as

$$y = k_1^{-1} x - k_1^{-1}\{k_2 \frac{y^2}{2!} + k_3 \frac{y^3}{3!} + \dots\}$$

and iterate using successive polynomial approximations for y on the right side starting with the linear term. This quickly gives the first few terms. This method is easily generalized to higher dimensions and it leads directly to the proof of convergence of the inverse series. The successive terms may be found alternatively and more systematically by substituting the power series for y into the right hand side giving the recursive equations for the undetermined coefficients h.

$$\begin{aligned} h_1 &= k_1^{-1} \\ h_2 &= - k_1^{-1}\{k_2 h_1^2\} \\ h_3 &= - k_1^{-1}\{k_2 (3h_1h_2) + k_3 h_1^3\} \\ h_4 &= - k_1^{-1}\{k_2 (4 h_1 h_3 + 3 h_2^2) + k_3 (6h_1^2 h_2) + k_4 h_1^4\} \\ &\dots\dots\dots\dots\dots\dots \quad \text{etc} \end{aligned}$$

The same equations result from using the previous substitution formula and equating to unity the series found by substituting the first series into the second i.e.

$$\begin{aligned} 1 &= k_1 h_1 \\ 0 &= k_1 h_2 + k_2 h_1^2 \\ 0 &= k_1 h_3 + k_2 (3h_1h_2) + k_3 h_1^3 \\ 0 &= k_1 h_4 + k_2 (4 h_1 h_3 + 3 h_2^2) + k_3 (6h_1^2 h_2) + k_4 h_1^4 \\ &\dots\dots\dots\dots\dots\dots \quad \text{etc.} \end{aligned}$$

The explicit solution for the h's is found as

$$\begin{aligned} h_1 &= k_1^{-1} \\ h_2 &= - k_1^{-3} k_2 \\ h_3 &= - k_1^{-4} k_3 + 3 k_1^{-5} k_2^2 \\ h_4 &= - k_1^{-5} k_4 + 10 k_1^{-6} k_2 k_3 - 15 k_1^{-7} k_2^3 \\ &\dots\dots\dots\dots\dots\dots \quad \text{etc} \end{aligned}$$

5 Analytic implicit functions

An implicit function relation has the form

$$y = f(x,y) = f_{10}\, x \;+\; f_{01}\, y \;+\; \sum_{m+n>1} \sum \frac{1}{m!\, n!} f_{mn}\, x^m y^n$$

It will be assumed that $f_{01} \neq 1$ otherwise the y term cancels. Then the term $f_{01}y$ may be transferred to the left hand side and division by $1 - f_{01}$ again comes to a similar implicit relation but without a linear term in y on the right hand side, i.e.

$$y = f(x,y) = f_{10}\, x \;+\; \sum_{m+n>1} \sum \frac{1}{m!\, n!} f_{mn}\, x^m y^n$$

This may be called the *normal form* of an implicit function relation. A series solution of the equation is now found by substituting a series with unknown coefficients

$$y = h(x) = h_1\, x \;+\; \frac{1}{2!} h_2\, x^2 \;+\; \frac{1}{3!} h_3\, x^3 \;+\; \ldots.$$

Then equating coeffients of powers of x which gives recursive equations for these coefficients:

$$\begin{aligned} h_1 &= f_{10} \\ h_2 &= f_{20} + 2\, f_{11}\, h_1 + f_{02}\, h_1^2 \\ h_3 &= f_{30} + 3\, f_{21}\, h_1 + 3\,(f_{11}\, h_2 + f_{12}\, h_1^2) + 3\, f_{02}\, h_1 h_2 + f_{03}\, h_1^3 \end{aligned}$$

....

$$h_n = \sum_{q=0}^{n} \frac{n!}{(n-q)!\, q!} \sum_{r=1}^{q} f_{n-q,\, r}\, \mathbf{B}_{q\, r}(h_1, \ldots\, h_{q-r+1})$$

This expression is derived by noting that on expanding f(x, y), terms in x^n result from

$$\frac{1}{(n-q)!\, r!} f_{n-q\, r}\, x^{n-q}\, y^r\,, \qquad q = 0, 1, \ldots n,\;\; r \leq q$$

where $y^r/r!$ contributes a term $\mathbf{B}_{q,\, r}\, x^q/q!$ for r = 1, 2, ..., q.

The recursive equations give explicitly

$$\begin{aligned} h_1 &= f_{10} \\ h_2 &= f_{20} + 2\, f_{11}\, f_{10} + f_{02}\, f_{10}^2 \\ h_3 &= f_{30} + 3\, f_{21}\, f_{10} + 3\,(f_{11}\,(f_{20} + 2\, f_{11}\, f_{10} + f_{02}\, f_{10}^2) + f_{12}\, f_{10}^2) \\ &\quad + 3\, f_{02}\, f_{10}\,(f_{20} + 2\, f_{11}\, f_{10} + f_{02}\, f_{10}^2) + f_{03}\, f_{10}^3 \end{aligned}$$

etc

6 Convergence - the method of majorant series

The method of majorant series is a convenient and flexible technique for dealing with convergence of power series and it extends easily to vector and functional cases.

Suppose the power series whose convergence is to be discussed is

$$f(x) = f_0 + f_1 x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} f_n x^n$$

Definition: A scalar power series

$$F(X) = F_0 + F_1 X + F_2 \frac{X^2}{2!} + F_3 \frac{X^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} F_n X^n$$

is a *majorant* (or *majorizing*) *series* to the preceeding series if its coefficients satisfy

$$F_n \geq |f_n|, \quad n = 0, 1, 2, \dots$$

The *least majorant series* is one for which

$$F_n = |f_n|, \quad n = 0, 1, 2,$$

Consequently any majorant series has non-negative coefficients. The variable X is also considered to be positive, $X > 0$. When $|x| \leq X$, every term of the series for f(x) is less or equal in absolute value to the corresponding term of the majorant series for F(X). There follows:

Theorem: If F(X) majorizes f(x) and has radius of convergence X* finite or infinite, then
(a) The series for f(x) is absolutely convergent when $|x| < X^*$,
(b) If $X < X^*$ the series for f(x) is uniformly convergent when $|x| \leq X$,
(c) The inequality $|f(x)| \leq F(X)$ holds when $|x| \leq X < X^*$.

Proof: (a) and (b) follow from the inequality

$$\left| \sum \frac{1}{n!} f_n x^n \right| \leq \sum \frac{1}{n!} F_n X^n$$

The sum here can be over any selection of terms and so can transfer the Cauchy general principle for convergence from the series for F(X) to the series for f(x). (c) is immediate.

All its terms being non-negative, any majorant series converges monotonically either to a finite positive limit or to $+\infty$. Assuming a nonzero radius of convergence there are two possibilities:

Case (a): The majorant series converges for all $X > 0$, i.e. $X^* = +\infty$ the majorized series also converges for all x (an *entire function* e.g. exp X)

Case (b): The majorant series has a finite radius of convergence X^*. The majorized series converges for $|x| < X^*$ and possibly also when $|x| = X$

Operations on power series: The relationship of majorant is conserved under many operations with power series.

(1) *Addition*: If f(x), g(x) are majorized by F(X), G(X) respectively then f(x) + g(x) is majorized by F(X) + G(X).

(2) *Multiplication by a positive constant*: If f(x) is majorized by F(X) and $C > 0$ then C f(x) is majorized by C F(X).

(3) *Multiplication of series*: If f(x), g(x) are majorized by F(X), G(X) respectively then f(x) g(x) is majorized by F(X) G(X).

(4) *Substitution of series*: If y = h(x) and z = k(y) are majorized by Y = H(X) and Z = K(Y) respectively then z = g(x) = k(h(x)) is majorized by Z = G(X) = K(H(X)).

The proofs are straightforward for (1) to (3). For (4) the formulae for the coefficients of substituted series give the following inequalities from which the result follows.

$$
\begin{array}{lll}
|g_1| = |k_1|\,|h_1| & = K_1 H_1 & = G_1 \\
|g_2| \le |k_1|\,|h_2| + |k_2|\,|h_1|^2 & \le K_1 H_2 + K_2 H_1^2 & = G_2 \\
|g_3| \le |k_1|\,|h_3| + |k_2|\,(3|h_1||h_2|) + |k_3|\,|h_1|^3 & \le K_1 H_3 + K_2 (3H_1 H_2) + K_3 H_1^3 & = G_3
\end{array}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$
|g_n| \le \sum_{m \le n} |k_m|\, B_{n,m}(|h_1|, \ldots, |h_{n-m+1}|) \quad \le \sum_{m \le n} K_m\, B_{n,m}(H_1, \ldots, H_{n-m+1}) \quad = G_n
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \text{etc.}$$

5) *Series reversion.* The determination of the majorant in series reversion requires a slightly different technique. Consider the inverse relations

$$x = k(y) = k_1 y + k_2 \frac{y^2}{2!} + k_3 \frac{y^3}{3!} + \dots \quad (k_1 \neq 0)$$
$$y = h(x) = h_1 x + h_2 \frac{x^2}{2!} + h_3 \frac{x^3}{3!} + \dots \quad (h_1 \neq 0)$$

The reverted form of the first equation for finding the second from the first is

$$y = k_1^{-1} x - k_1^{-1} \sum_{n=2}^{\infty} \frac{1}{n!} k_n y^n$$

From this a comparison equation is set up:

$$Y = H X + H \sum_{n=2}^{\infty} \frac{1}{n!} K_n Y^n$$

$$H \geq |k_1|^{-1}, \quad K_n \geq |k_n|, \quad n = 2, 3, \dots$$

This will have a series solution with positive terms:

$$Y = H(X) = H_1 X + H_2 \frac{X^2}{2!} + H_3 \frac{X^3}{3!} + \dots$$

From the recursive equations for the coefficients h_n, H_n, $n = 1, 2, \dots$, are found the inequalities.

$$|h_1| = |k_1^{-1}| \leq H = H_1$$
$$|h_2| = |k_1^{-1}|\{|k_2|\,|h_1|^2\} \leq H_1\{K_2 H_1^2\} \leq H_2$$
$$|h_3| \leq |k_1^{-1}|\{|k_2|\,\{3|h_1||h_2|) + |k_3|\,|h_1|^3\} \leq H_1\{K_2\,\{3H_1H_2) + K_3 H_1^3\} \leq H_3$$

....

$$|h_n| \leq |k_1^{-1}| \sum_{2 \leq m \leq n} |k_m|\, B_{m,n}(|h_1|, \dots, |h_{n-m+1}|) \leq K_1^{-1} \sum_{2 \leq m \leq n} K_m\, B_{m,n}(H_1, \dots, H_{n-m+1}) \leq H_n$$

i.e. the series solution of the comparison equation provides a majorant for the reverted series.

6) *Solution by series of implicit function equations.* A similar technique applies to the general implicit function equation in the normal form

$$y = f_{10}\,x + \sum_{m+n>1}\sum \frac{1}{m!\,n!} f_{mn}\,x^m y^n$$

having series solution

$$y = h_1\,x + \frac{1}{2!} h_2\,x^2 + \frac{1}{3!} h_3\,x^3 + \ldots.$$

Set up the comparison equation

$$Y = F_{10}\,X + \sum_{m+n>1}\sum \frac{1}{m!\,n!} F_{mn}\,X^m Y^n$$

$$F_{mn} \geq |f_{mn}|$$

The series solution defines a function H(X)

$$Y = H_1\,X + \frac{1}{2!} H_2\,X^2 + \frac{1}{3!} H_3\,X^3 + \ldots = H(X)$$

From these recursive equations for coefficients follow the inequalities

$$|h_1| = |f_{10}| \leq F_{10} \qquad = H_1$$

$$|h_2| \leq |f_{20}| + 2\,|f_{11}|\,|h_1| + |f_{02}|\,|h_1|^2$$

$$\leq |F_{20}| + 2\,|F_{11}|\,|h_1| + |F_{02}|\,|h_1|^2 \qquad = H_2$$

....

$$|h_n| = \sum_{q=0}^{n} \frac{n!}{(n-q)!\,q!} \sum_{r=1}^{q} |f_{n-q,\,r}|\,\mathbf{B}_{q\,r}(|h_1|, \ldots |h_{q-r+1}|)$$

$$\leq \sum_{q=0}^{n} \frac{n!}{(n-q)!\,q!} \sum_{r=1}^{q} F_{n-q,\,r}\,\mathbf{B}_{q\,r}(H_1, \ldots H_{q-r+1}) = H_n$$

i.e. the series solution H(x) of the comparison equation provides a majorant to the series solution of the given implicit function relation.

Convergence: The classical result of Cauchy is that the method of series substitution gives a solution of an analytic implicit function equation convergent for sufficiently small values of the independent variable. The idea of the proof is to construct a majorant which can easily be proved convergent. A clear description is given in Goursat's Course of Mathematical Analysis*.

---

* *References*: Cauchy A-L: Comptes Rendus Acad Sci Paris 1839, 587-588; Goursat E: Cours d'analyse mathématique, Paris 1915; Eng. Transl. by Dunkel and Hedrik, 1916 (Ginn); rpr Dover (1950). See vol.I, p.400-

Cauchy's result was considerably strengthened by Hille (1959). By using methods of the theory of functions of a complex variable, Hille showed that the radius of convergence X^+ of the series solution of the implicit function equation may be determined as the unique solution of the simultaneous equations for the majorant function F(X, Y)

$$Y^+ = F(X^+, Y^+), \quad 1 = \frac{\partial F}{\partial X}(X^+, Y^+)$$

These equations have the simple geometrical interpretation on the graph of the comparison equation Because of the positivity of terms in the expansion of F(X, Y), the graph is increasing from the origin and curves upward having a vertical tangent and turning back at the point (X^+, Y^+) satisfying the above equations called the *turning value*.
The series solution of the implicit function equation for the majorant is consequently convergent for values X on the range $0 \leq X \leq X^+$ i.e up to the turning value. The series solution on this range defines an analytic function H(X) which represents the arc OP over its entire length, P being the turning value with coordinates (X^+, Y^+). This function H(X) may be termed the *principal solution* of the implicit function since there is a second solution for the upper branch of the graph. Since H(X) majorizes h(x) the series solution of the original implicit function equation y = f(x, y) will also be convergent over the same interval. So Hille's theorem may be stated as:

Statement of Hille's Theorem: The series solution of an analytic implicit function equation in normal form y = f(x, y) is convergent when $|x| < X^+$ where X^+ is the turning value of X on the graph of the comparison equation Y = F(X, Y).

Proof: Hille's proof using complex analysis is given in his book. It can also be proved by real analysis. (see footnote below)

Series reversion: This is the special case considered in the text with majorant equation

$$Y = H X + H K(Y)$$

The values (X^+, Y^+) are here determined by the equations

$$Y^+ = H X^+ + H K(Y^+), \quad 1 = H \frac{\partial K}{\partial Y}(Y^+)$$

The second equation does not involve X^+ and may be solved immediately for Y^+ from which X^+ may then be found from the first equation.

Hille E: Analytic Function Theory, Boston 1959 (Ginn). A proof using only real analysis was given by the writer in the report: 'A generalization of a result of Hille', Report 79-Wsk-03 1979, Dept Math Tech, Highschool, Eindhoven, NL

Appendix 2

LINEAR SYSTEMS

1 Linear continuous-time systems

These satisfy the *superposition principle* and so may be characterized by *impulse response function* h(t, t') defined as response at time t due to unit impulse input at time $t' < t$

Physical realizability condition: Due to causality this function must satisfy

$$h(t, t') = 0 \quad t' > t$$

So linear superposition results in the input/output relation:

$$y(t) = \int_{-\infty}^{t} h(t, t')\, x(t')\, dt'$$

This can also be written over the doubly infinite range as

$$y(t) = \int_{-\infty}^{\infty} h(t, t')\, x(t')\, dt'$$

Time-invariant condition: if the system input/output relation is not time-varying then $h(t+\tau, t'+\tau) = h(t, t')$ for all values of τ. On putting $\tau = -t'$ there follows identically $h(t, t') = h(t - t', 0)$ and so h(t, t') depends only on the time-difference $t - t'$. $h(t - t', 0)$ is abbreviated as $h(t - t')$
The realizability condition then becomes

$$h(t - t') = 0 \quad \text{unless} \quad t \geq t'$$

The input-output equation becomes

$$y(t) = \int_{-\infty}^{t} h(t - t')\, x(t')\, dt' = \int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt'$$

Or in terms of $\tau = t - t'$

$$h(\tau) = 0 \quad \text{unless} \quad \tau \geq 0$$

$$y(t) = \int_{0}^{\infty} h(\tau)\, x(t - \tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau$$

Systems are equally defined by their *transfer function* H(s), the Laplace transform of the impulse-transform function

$$H(s) = \int_0^{\infty} h(t) \exp(-st)dt$$

The integral here may also be written over the doubly infinite range (bilateral Laplace transform) which allows easy transition to the Fourier transform

$$H(i\omega) = \int_{-\infty}^{\infty} h(t) \exp(-i\omega t)\, dt$$

Stability condition: Stability follows from the boundedness condition

$$\int_0^{\infty} |h(\tau)|\, d\tau < \infty$$

a condition which appears to have originated in the book of James, Nichols, Phillips (1946) From it is deduced that when $\sigma > 0$

$$|H(\sigma + i\omega)| = \left| \int_{-\infty}^{\infty} h(t) \exp -(\sigma+i\omega)t\, dt \right| \leq \int_{-\infty}^{\infty} |h(t) \exp -\sigma t\,|\, dt \leq \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

So H(s) is bounded in the right hand s plane and any infinities must lie in the region $\sigma < 0$. Further there follows the BIBO condition

$$||y|| = \max |y(t)| \leq \left| \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)\, x(t - \tau)|\, d\tau \leq ||h||\; ||x||$$

The stability of a general time-varying linear system may be defined by the condition that the response to any bounded input is bounded (Zadeh 1952). This will be the case if

$$\int_{-\infty}^{t} |h(t, t')|\, dt' < \infty$$

For time invariant systems this is the same as the BIBO condition but for general time-varying systems it is not and a slightly stronger condition is needed:

$$||\, h\, || = \max_{-\infty<t<\infty} \int_{-\infty}^{t} |\, h(t, t')|\, dt' < \infty$$

From this follows

$$||\, y\, || \leq ||\, h\, ||\; ||\, x\, ||$$

2 The use of delta functions.

The need to use delta functions for time-domain representation of systems is already evident in linear systems analysis. Consider the system with input-output relation

$$y(t) = H(p)\, x(t)$$

where H(.) is a rational function:

$$H(s) = P(s)/Q(s)$$

P(s), Q(s) being polynomials in s without common factor and Q(s) in addition satisfying the stability condition. The time-domain representation will be of the form

$$y(t) = \int_{-\infty}^{\infty} h(t-t')\, x(t')\, dt'$$

where h(.) the impulse response function is the inverse transform of H(s). If H(s) is a proper rational fraction in s, so that the degree of Q(s) exceeds the degree of P(s) then h(.) is an absolutely integrable function:

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

However if H(s) is not a proper rational fraction then h(.) is no longer a regular function and will involve delta function and derivatives of delta functions. In this case the linear operator can also be written in terms of instantaneous values and derivatives in the form

$$y(t) = \sum_{r=0}^{n} c_r\, x^{(r)}(t) + \int_{-\infty}^{\infty} h(t-t')\, x(t')\, dt'$$

Consequently it is necessary to define the impulse response function as

$$g(t) = \sum_{r=0}^{n} c_r\, \delta^{(r)}(t) + h(t)$$

Then the input-output relation can be written in the usual form

$$y(t) = \int_{-\infty}^{\infty} g(t-t')\, x(t')\, dt'$$

3 Inversion of time-invariant linear relations

Inverse time-invariant linear relations are characterized by simultaneous satisfaction on the infinite time interval $(-\infty, \infty)$ of the equations

$$y(t) = \int_{-\infty}^{\infty} h(t - t')\, x(t')\, dt'$$

$$x(t) = \int_{-\infty}^{\infty} k(t - t')\, y(t')\, dt'$$

By substitution we find that

$$\delta(t - t'') = \int_{-\infty}^{\infty} h(t - t')\, k(t' - t'')\, dt'$$

$$\delta(t - t'') = \int_{-\infty}^{\infty} k(t - t')\, h(t' - t'')\, dt'$$

Both these relations are equivalent to the Laplace transform relation

$$1 = H(s)\, K(s)$$

This relation implies that the poles and zeros of the two transforms are interchanged so that from the stability condition of both it follows that each must have poles and zeros in the left hand half plane, i.e. both operators must be minimum phase. In the most important case where the transforms are rational so that

$$H(s) = P(s)/Q(s)$$

$$K(s) = Q(s)/P(s)$$

Here $P(s)$, $Q(s)$ are stable polynomials. There are three cases:

I degree $P(s) <$ degree $Q(s)$

II degree $P(s) =$ degree $Q(s)$

III degree $P(s) >$ degree $Q(s)$

In cases I and III one of the operators represents a differential operator and the other an integral operator. In case II both operators will be of the type occurring in the Volterra integral equation.

Appendix 3

PROBABILITY THEORY

1 Probability distributions.

A convenient way to characterize probability distributions is by the expectation operator. For a distribution having a probability density p(x), the expectation operator E is defined by

$$E\{f(x)\} = \int f(x)\, p(x)\, dx$$

The integral is extended over the whole range of the distribution. E has the properties

1) E is linear
2) E is positive: if $f(x) \geq 0$ then $E\{f(x)\} \geq 0$
3) $E\{1\} = 1$
4) E is continuous relative to monotone convergence:

In condition (4) monotone convergence means that if $f_n(x)$, n = 1, 2, 3,... is a monotone increasing sequence of positive functions with f(x) as limit, then $E\{f_n(x)\} \to E\{f(x)\}$ as $n \to \infty$

According to a theorem of Riesz an operator with these properties defines a probability distribution

One dimensional distributions: The *moments* of the distribution may be defined by:

$$\mu_n = E\{x^n\} \qquad n = 0,1,2, \ldots$$

The *characteristic function* is

$$c(iu) = E\{\exp iux\}$$

It defines the moments through the series expansion

$$c(iu) = 1 + \mu_1 (iu) + \frac{1}{2!} \mu_2 (iu)^2 + \frac{1}{3!} \mu_3 (iu)^3 + \ldots$$

If a probability density function p(x) exists the characteristic function is its Fourier transform:

$$c(iu) = \int_{-\infty}^{\infty} \exp iux \,.p(x)\, dx$$

It is known that under conditions normally satisfied the characteristic function uniquely characterizes the probability distribution (Lévy's theorem). The moments will also normally define the characteristic function and thereby the probability distribution

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Reference: Cramér H: Mathematical Methods of Statistics, Princeton, 1948

The Gaussian (normal) distribution: This, the most well known distribution, has the probability density for a scalar real valued random variable x,

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\{-\tfrac{1}{2}(x-m)^2/\sigma^2\}$$

where m is the mean and σ^2 is the variance. Most frequently, by suitable choice of scale, the mean is taken to be zero, the characteristic function then being

$$c(iu) = \exp\{-\tfrac{1}{2}(u\sigma)^2\}$$

Expansion now shows that even ordered moments are zero while even order moments are:

$$\mu_{2n} = (2n-1)(2n-3)\ldots3.1\,\sigma^{2n} = (2n)!\,\sigma^{2n} / 2^n\,n!$$

Probability distributions in N dimensions: The axioms and Riesz's theorem carry over to finite dimensional vectors $(x_1, x_2 \ldots x_N)$. The moments are defined by

$$\mu_{m_1 \ldots, m_N} = E[x_1^{m_1}\,x_2^{m_2} \ldots x_N^{m_N}]$$

The characteristic function has an expansion

$$c(iu) = E\{\sum_{m=0}^{\infty} \frac{1}{m!}(iu.x)^m\}$$

$$= E\{\sum_{m_1=0}^{\infty} \ldots \sum_{m_N=0}^{\infty} \frac{1}{m_1!\,m_2! \ldots m_N!}.\,(iu_1x_1)^{m_1}(iux_2)^{m_2}\ldots(iux_N)^{m_N}\}$$

$$= \sum_{m_1=0}^{\infty} \ldots \sum_{m_N=0}^{\infty} \frac{1}{m_1!\,m_2! \ldots m_N!}.\,(iu_1)^{m_1}(iu)^{m_2}\ldots(iu)^{m_N}\,\mu_{m_1\ldots m_N}$$

The N-dimensional Gaussian distribution: The probability density is:

$$p(x) = \frac{1}{\sqrt{\det A}} \exp -\tfrac{1}{2}(x-m)^T A^{-1}(x-m)$$

m is the vector mean and A is the covariance matrix, here assumed non-singular. If A is singular then a similar formula exists with A inverse replaced by generalized inverse. The mean value m is usually made to be zero. For zero mean the characteristic function is

$$c(iu) = \exp\{-\tfrac{1}{2}u^T A\,u\} = \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r\,r!}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\,u_i\,u_j\right)^r$$

From expansion of this result it is found that the odd order moments are zero while those of even order 2n are formed by summing products of pairs of covariances. e.g the moment of order 4 from the product of random Gaussian variables x_1, x_2, x_3, x_4 is

$$E[x_1\,x_2\,x_3\,x_4] = a_{12}\,a_{34} + a_{13}\,a_{24} + a_{14}\,a_{23}$$

The general formula is

$$E[x_{i_1} x_{i_2} \dots x_{i_{2n}}] = \sum_{\text{pairs}} a_{j_1 j_2} a_{j_3 j_4} \dots a_{j_{2n-1} j_{2n}}$$

the sum being over $(2n)!/ 2^n n!$ selections of n pairs $(j_1, j_2)(j_3, j_4)\dots(j_{2n-1}, j_{2n})$ from 2n suffixes $i_1 \dots i_{2n}$

Two-dimensional (bivariate) Gaussian distribution: An important special case of the N-dimensional distribution. Two correlated Gaussian random variables x_1 and x_2 having zero means, equal variances σ^2 and normalized cross-correlation ρ have joint probability density

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2\sqrt{(1 - \rho^2)}} \exp - \frac{1}{2} \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\sigma^2(1 - \rho^2)}$$

This corresponds through Fourier transformation to the characteristic function

$$c(iu_1, iu_2) = \exp - \frac{\sigma^2}{2} (u_1^2 + u_2^2 - 2\rho u_1 u_2)$$

Mehler's Formula: This useful formula relates two dimensional Gaussian to one dimensional Gaussian distributions. Expanding the two dimensional characteristic function in powers of ρ there results the series representation

$$c(iu_1, iu_2) = \exp - \frac{\sigma^2}{2} (u_1^2 + u_2^2)\{1 + \sum_{n=1}^{\infty} \frac{1}{n!} (\sigma^2\rho)^n (u_1u_2)^n\}$$

Now use is made of the fact that there are Fourier pairs

$$c(iu) \longleftrightarrow p(x), \qquad (iu)^n c(iu) \longleftrightarrow (\sigma^2\, d/dx)^n p(x)$$

Then it is seen that the inverse Fourier transformed series expansion is:

$$p(x_1, x_2) = p(x_1)\, p(x_2) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \sigma^{2n} \frac{d^n p(x_1)}{dx_1^n} \sigma^{2n} \frac{d^n p(x_2)}{dx_2^n}$$

This is Mehler's formula which may be interpreted in terms of the homogeneous Hermite polynomials associated with the one dimensional distributions of x_1, x_2 (see appendix 4)

$$p(x_1, x_2) = p(x_1)\, p(x_2) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} he_n(x_1, \sigma)\, he_n(x_2, \sigma)\, p(x_1)\; p(x_2)$$

A generalized orthogonality property: using Mehler's formula inside the integral gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} he_m(x_1, \sigma)\, he_n(x_2, \sigma)\, p(x_1,x_2)\; dx_1dx_2 = \begin{cases} 0 & m\neq n \\ n!\, 2^n\, E\,(x_1\, x_2)^n & m=n \end{cases}$$

showing orthogonality of Hermite polynomials extends to correlated Gaussian random variables.

2 Random (stochastic) processes

From multidimensional distributions it is possible to go to stochastic processes by defining the probability distribution of each collection of values $x_1 = x(t_1)$, $x_2 = x(t_2)$, ..., $x_n = x(t_n)$. Statistical parameters for the process become functions of time, e.g

(1) mean value $m(t) = E\{x(t)\}$
(2) autocovariance function $a(t_1, t_2) = E\{(x(t_1) - m(t_1))(x(t_2) - m(t_2))\}$

A Gaussian stochastic process has a multidimensional Gaussian distribution, for each finite set of values $x_1 = x(t_1)$, $x_2 = x(t_2)$, ..., $x_n = x(t_n)$ for all times $t_1, t_2, ..., t_n$ The process is completely determined by its mean value $m(t)$ and autocovariance $a(t_1, t_2)$.

Stationary stochastic process: The most important case when all probability distributions are independent of time. Then the first two moments $m(t)$ and $a(t_1,t_2)$ are independent of time origin which implies

$$m(t) = m = \text{const}, \qquad a(t_1+t', t_2+t') = a(t_1, t_2) \text{ for all } t'$$

The last of these implies that the covariance a(..) has the form

$$a(t_1, t_2) = R(t_2 - t_1)$$

where R() is the *autocorrelation function.*

Since R() is an autocovariance it has positive definite property and this implies according to a theorem of Bochner that it can be represented as

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp i\omega\tau \, dF(\omega)$$

where $F(\omega)$ is an increasing function of ω (*integrated spectrum*). By decomposition of $F(\omega)$ into two components, one representing step discontinuities and the other continuous increase the process may be separated into a deterministic and purely random components. Normally attention is restricted to the continuous purely random part characterized by the *power spectrum* (spectrum for short) $\Phi(\omega)$, the derivative of $F(\omega)$. Then follows the *Wiener-Khinchin theorem*

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega) \exp i\omega\tau \, d\omega$$

with inverse relation

$$\Phi(\omega) = \int_{-\infty}^{\infty} R(\tau) \exp i\omega\tau \, d\omega$$

In his derivation of this relation Wiener had considered the harmonic analysis of a long sample of the process which assumes that such a long sample gives approximations to the statistical averages (*ergodic hypothesis*). Certain important processes may be proved to have this property, such as a stationary Gaussian process with continuous spectrum. {Itô: Proc.Imp.Acad.Japan 1944). It follows from the work of Wiener, that processes derived from Brownian motion are ergodic. Similar results hold for the Poisson process and processes derived from it (see Doob: 'Stochastic Processes').

White noise: White noise does not fit into the normal definitions of stochastic process theory and Wiener approached the problem by considering its integrated form, now referred to as the Wiener process. Thus white noise becomes the (strictly non-existent) derivative of the Wiener process. It has infinite flat spectrum and delta function autocorrelation.

$$\Phi(\omega) = N, \text{ const} \qquad R(\tau) = N\,\delta(\tau)$$

where N is the *power* of the noise

The Wiener process is an additive process i.e. the increments in successive time intervals are statistically independent. It is not the only the only process with this property the other important example being the Poisson process which has step increases occurring purely randomly in time. The formal derivative of this gives another form of white noise (*shot noise*). By a theorem of Kolmogorov a general additive process may be regarded as the sum of a Wiener process and a range of Poisson processes for different parameters*. Correspondingly there is the representation of general white noise in terms of Gaussian white noise and a range of shot noise processes.

A simple derivation was given by the writer in J. Sound & Vibr. 1964, vol.1.

Appendix 4

HERMITE POLYNOMIALS

1 Homogeneous one-dimensional Hermite polynomials

In the literature one dimensional Hermite polynomials are found with two definitions, one for weight function exp - x^2 and the other for weight function exp - ½ x^2. The former are used in physics. For statistical problems the second form is appropriate and it is most convenient to modify the normal definition by choosing weight function as a Gaussian distribution of zero mean and variance σ so having probability density function

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp - \frac{x^2}{2\sigma^2}$$

For this weight function homogeneous Hermite polynomials can be defined as follows.

Definition: The homogeneous Hermite polynomials $he_n(x, \sigma)$, are defined by the equation

$$he_n(x, \sigma)\, p(x) = (- \sigma^2 d/dx\,)^n\, p(x) \qquad n = 0, 1, 2,...$$

These polynomials up to degree 5 are:

$$\begin{aligned}
he_0\,(x, \sigma) &= 1 \\
he_1\,(x, \sigma) &= x \\
he_2\,(x, \sigma) &= x^2 - \sigma^2 \\
he_3\,(x, \sigma) &= x^3 - 3\,x\,\sigma^2 \\
he_4\,(x, \sigma) &= x^4 - 6\,x^2\,\sigma^2 + 3\,\sigma^4 \\
he_5\,(x, \sigma) &= x^5 - 10\,x^3\,\sigma^2 + 5\,x\,\sigma^4
\end{aligned}$$

When σ = 1 these reduce to the usual polynomials based on weight function exp -½ x^2 and their properties are similar. In what follows $he_n\,(x, \sigma)$ will be written $he_n\,(x)$ for brevity, dependence on σ being understood.

It can be shown by induction that

$$\begin{aligned}
he_n(x) &= x^n - \frac{n(n-1)}{1!}\frac{\sigma^2}{2} x^{n-2} + \frac{n\,(n-1)(n-2)(n-3)}{2!}\frac{\sigma^4}{2^2} x^{n-4} + \\
&= \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!\,(n-2r)!\,2^r}\, \sigma^{2r}\, x^{n-2r}
\end{aligned}$$

This formula may be expressed in the convenient operator form

$$he_n(x) = \exp\left(- ½\, \sigma^2\, d^2/dx^2 \right) x^n$$

There follows the inverse operator equation

$$x^n = \exp(\tfrac{1}{2}\sigma^2 d^2/dx^2)\, he_n(x)$$

On expansion results the inverse relation with all signs positive

$$x^n = he_n(x) + \frac{n(n-1)}{1!}\frac{\sigma^2}{2} he_{n-2}(x) + \frac{n(n-1)(n-2)(n-3)}{2!}\frac{\sigma^4}{2^2} he_{n-4}(x) +$$

$$= \sum_{r=0}^{[n/2]} \frac{n!}{r!(n-2r)!}\frac{\sigma^{2r}}{2^r} he_{n-2r}(x)$$

The first few values are

$$\begin{aligned}
x &= he_1(x) \\
x^2 &= he_2(x) + \sigma^2 \\
x^3 &= he_3(x) + 3\, he_1(x)\, \sigma^2 \\
x^4 &= he_4(x) + 6\, he_2(x)\, \sigma^2 + 3\,\sigma^4 \\
x^5 &= he_5(x) + 10\, he_3(x)\, \sigma^2 + 5\, he_1(x)\, \sigma^4
\end{aligned}$$

These have the same form as the reverse equations with positive coefficients.

Orthogonality: The orthogonality property is

$$\int_{-\infty}^{\infty} he_m(x)\, he_n(x)\, p(x)\, dx = \begin{cases} 0 & (m \neq n) \\ n!\,\sigma^{2n} & (m = n) \end{cases}$$

This may be proved from the definition in terms of derivatives and integration by parts.

Expansion formula: Using the orthogonality property, and the known completeness property of the polynomials*, a function f(x) of finite mean square satisfying the condition

$$\int_{-\infty}^{\infty} f(x)^2\, p(x)\, dx < \infty$$

may be expanded into a mean square convergent series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\, b_n\, he_n(x)$$

where

$$b_n = \frac{1}{\sigma^{2n}} \int_{-\infty}^{\infty} f(x)\, he_n(x)\, p(x)\, dx$$

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* Completeness is proved in, e.g.
(a) Wiener *The Fourier integral* Cambridge Univ. Press 1933 p.64
(b) Courant & Hilbert: *Methods of Mathematical Physics.*

Chebyshev's Formula: If f(x) is n times differentiable then the coefficients b_n may be expressed in another way. From

$$b_n = \frac{1}{\sigma^{2n}} \int_{-\infty}^{\infty} f(x) \, (-\sigma^2 d/dx)^n \, p(x) \, dx$$

On integrating by parts n times and making the reasonable assumption that the end terms vanish at infinity it is found that

$$b_n = \int_{-\infty}^{\infty} (d/dx)^n f(x) \, p(x) \, dx$$

This formula generalizes that due to Chebyshev 1859 for the case $\sigma = 1$, n=1. Interpreted statistically, it takes the simple form

$$b_n = \mathrm{E}\,\{d^n f(x)/dx^n\}$$

where **E** is the expectation operator.

Hermite series for an analytic function: Suppose that

$$f(x) = a_0 + a_1 x + \frac{1}{2!} a_2 x^2 + \ldots$$

then

$$f^{(n)}(x) = a_n + a_{n+1} x + \frac{1}{2!} a_{n+2} x^2 + \ldots$$

so that Chebyshev's formula gives

$$c_n = a_n + a_{n+1} \mathrm{E}\{x\} + \frac{1}{2!} a_{n+2} \mathrm{E}\{x^2\} + \ldots$$

All odd moments are zero while even moments are

$$\mathrm{E}\{x^{2p}\} = (2p-1)(2p-3) \ldots 3.1 \, \sigma^{2p} = (2p)!/2^p \, p! \; \sigma^{2p}$$

Hence the Hermite coefficients are

$$c_n = \sum_{p=0}^{\infty} \frac{a_{n+2p}}{(n+2p)!} \mathrm{E}\{x^{2r}\}$$

$$= \sum_{p=0}^{\infty} \frac{a_{n+2p}}{(n+2p)!} \frac{(2p)!}{2^p \, p!} \sigma^{2p}$$

$$= a_n + \frac{1}{2.1!} a_{n+2} \sigma^2 + \frac{1}{2^2 \, 2!} a_{n+4} \sigma^4 + \ldots$$

2 Expansion of the delta function and functions derived from it

Several useful Hermite expansions result from use of the delta function. The delta function $\delta(x)$ has coefficients

$$b_n = \frac{1}{\sigma^{2n}} \int_{-\infty}^{\infty} \delta(x)\, he_n(x)\, p(x)\, dx = he_n(0)\, p(0)$$

All odd coefficients are zero since the odd Hermite polynomials vanish at x = 0.
The even ones for n = 2m are

$$b_{2m} = \frac{(-1)^m}{\sqrt{2\pi}\,\sigma} \frac{(2m)!}{m!\, 2^m\, \sigma^{2m}}.$$

giving

$$\delta(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, 2^m} \frac{he_{2m}(x)}{\sigma^{2m}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \{1 - \frac{1}{2} \frac{he_2(x)}{\sigma^2} + \frac{1}{2!} \frac{1}{2^2} \frac{he_4(x)}{\sigma^4} - \}$$

By integration and identification of the constant term is found the expansion for the Heaviside step function.

$$1(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, 2^m} \frac{he_{2m+1}(x)}{\sigma^{2m}}$$

The sign function is related to this by

$$\mathrm{sgn}(x) = 2\,\{1(x) - ½\}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}\,\sigma} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, 2^m} \frac{he_{2m+1}(x)}{\sigma^{2m}}$$

By a similar method is found the integrated step function, i.e. the unit ramp

$$\begin{aligned} f(x) &= x, \quad x > 0; \\ &= 0, \quad x < 0 \end{aligned}$$

Its expansion is

$$f(x) = \frac{\sigma}{\sqrt{2\pi}} + \frac{x}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, 2^m} \frac{he_{2m+2}(x)}{\sigma^{2m}}$$

3 The Mehler formula and generalized orthogonality of Hermite polynomials

If x_1 and x_2 are two correlated Gaussian random variables having zero means, equal variances σ^2, and normalized cross-correlation ρ then their bivariate probability density is

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho^2)}} \exp - \frac{1}{2} \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\sigma^2(1-\rho^2)}$$

This corresponds through Fourier transformation to the bivariate characteristic function

$$c(iu_1, iu_2) = \exp - \frac{\sigma^2}{2} (u_1^2 + u_2^2 - 2\rho u_1 u_2)$$

This has series representation

$$c(iu_1, iu_2) = \exp - \frac{\sigma^2}{2} (u_1^2 + u_2^2)\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (\sigma^2\rho)^n (u_1u_2)^n \}$$

Now make use of the fact that there are Fourier pairs

$$c(iu) \longleftrightarrow p(x) \qquad\qquad (iu)^n c(iu) \longleftrightarrow (\sigma^2 d/dx)^n p(x)$$

and also that the cross-correlation.is $\sigma^2\rho = E(x_1, x_2)$. Then the transformed series expansion will be:

$$p(x_1, x_2) = p(x_1) p(x_2)\{1 + \sum_{n=1}^{\infty} \frac{E(x_1x_2)^n}{n!} he_n(x_1, \sigma) he(x_2, \sigma)\}$$

This is Mehler's formula giving the bivariate probability distribution in terms of the one-dimensional distributions and the Hermite polynomials.

The generalized orthogonality property: Using Mehler's formula for $p(x_1, x_2)$ follows

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} he_m(x_1, \sigma) he_n(x_2, \sigma) p(x_1, x_2) dx_1dx_2 = \begin{cases} 0 & m \neq n \\ n! \, E(x_1x_2)^n & m = n \end{cases}$$

showing orthogonality of Hermite polynomials in correlated Gaussian random variables.

Ref: Mehler: J. f. Math.(Crelle) vol.66, 1866

4 General n dimensional Hermite polynomials

These are defined for a vector

$$x = [x_1,\ldots, x_n]^T$$

having an N dimensional Gaussian distribution with probability density

$$p(x) = \frac{1}{\sqrt{\det A}} \exp - \tfrac{1}{2} x^T B x$$

Here B is the inverse of A the covariance matrix $[a_{ij}]$, assumed non-singular.

Let Δ denote the operator $(\Delta_1, \ldots, \Delta_N)$ defined by

$$\Delta_i = \sum_{j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} \qquad i = 1,\ldots, N$$

The Hermite polynomials of order n form an array $he^{(n)}{}_{i_1 i_2 \ldots i_n}(x)$ of N^n terms defined by

$$he^{(n)}{}_{i_1 i_2 \ldots i_n}(x)\, p(x) = (-\Delta_{i_1})(-\Delta_{i_2}) \ldots (-\Delta_{i_n})\, p(x) \qquad i_1, \ldots, i_n = 1, \ldots, N$$

Up to order 3 they are

$$\begin{array}{llr} he^{(0)}(x) & = 1 & \\ he^{(1)}{}_{i_1}(x) & = x_{i_1} & i_1 = 1, \ldots, N \\ he^{(2)}{}_{i_1 i_2}(x) & = x_{i_1} x_{i_2} - a_{i_1 i_2} & i_1, i_2 = 1, \ldots, N \\ he^{(3)}{}_{i_1 i_2 i_3}(x) & = x_{i_1} x_{i_2} x_{i_3} - x_{i_1} a_{i_2 i_3} - x_{i_2} a_{i_3 i_1} - x_{i_3} a_{i_1 i_2} & i_1, i_1, i_3 = 1, \ldots, N \end{array}$$

The nth order term takes the form

$$he^{(n)}{}_{i1,\ldots,in}(x) = \sum_{r=0}^{[n/2]} (-1)^r \sum_{r \text{ pairings}} \left\{ \prod_{r \text{ pairs } p,q} a_{p,q} \prod_{(n-2r) \text{ terms}} x_{is} \right\} \qquad i_1, \ldots, i_n = 1, \ldots, N$$

Orthogonality: Using the definition of the polynomials it can be shown for $m \neq n$,

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} he^{(m)}{}_{i_1 i_2 \ldots i_n}(x)\, he^{(n)}{}_{j_1 j_2 \ldots j_n}(x)\, p(x)\, dx = 0$$

while if m = n the value is

$$\sum_{\text{perms}} a_{1 p_1} a_{2 p_{22}} \ldots a_{n p_n}$$

Here the sum is over the n! permutations $p_1, p_2, \ldots, p_n$ of 1, 2, ..., n.

Expansion formula: a function of finite mean square has a mean-square convergent expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 i_2 \ldots i_n} c^{(n)}{}_{i_1 i_2 \ldots i_n}(x)\, he^{(n)}{}_{i_1 i_2 \ldots i_n}(x)$$

The inner summation is over all permutations of 1, 2, ..., N. In the case when f() is sufficiently differentiable the coefficients may be written as an extension of Chebyshev's formula

$$c^{(n)}{}_{i_1 i_2 \ldots i_n}(x) = \sum_{\text{perms}} b_{i_1 p_1} b_{i_1 p_{22}} \ldots b_{i_n p_n} E_x \{f(x)\, he^{(n)}{}_{p1\, p2 \ldots pn}(x)\}$$

the sum being over the permutations of $i_1\, i_2 \ldots i_n$. Using the definition of the Hermite polynomials the equation reduces to

$$c^{(n)}{}_{i_1 i_2 \ldots i_n}(x) = E_x \{\partial^n f(x)/\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_n}\}$$

For a more detailed derivation of the properties of these polynomials see the appendix to the writer's 1980 paper referenced below.

Reference: IEE PROC Vol 127 Part D No 6 Nov 1980

5 Homogeneous Grad Hermite polynomials

A homogeneous form of Grad's (1948) N-dimensional polynomials can be defined for a vector

$$x = [x_1, \ldots, x_N]^T$$

having as weight function the probability distribution

$$p(x) = \frac{1}{(2\pi)^{N/2}\,\sigma^{N}} \exp - (x_1^2 + \ldots + x_N^2)/2\sigma^2$$

These are an important special case of the general polynomials which have additional properties. Grad in his original 1948 paper dealt with the special case when $\sigma = 1$.

The nth order polynomials form an array $he^{(n)}{}_{i_1 i_2 \ldots i_n}(x)$ defined by the general formula

$$he^{(n)}{}_{i_1 i_2 \ldots i_n}(x)\, p(x) = (-\sigma^2\, \partial/\partial x_{i_1})(-\sigma^2\, \partial/\partial x_{i_2}) \ldots (-\sigma^2\, \partial/\partial x_{i_n})\, p(x)$$

The suffixes i range from 1 to N. The first few polynomial arrays are

$$\begin{array}{lll}
he^{(0)}(x) & = 1 & \\
he^{(1)}{}_{i_1}(x) & = x_{i_1} & i_1 = 1, \ldots, N \\
he^{(2)}{}_{i_1 i_2}(x) & = x_{i_1} x_{i_2} - \sigma^2 \delta_{i_1 i_2} & i_1, i_1 = 1, \ldots, N \\
he^{(3)}{}_{i_1 i_2 i_3}(x) & = x_{i_1} x_{i_2} x_{i_3} - \sigma^2 x_{i_1} \delta_{i_2 i_3} - \sigma^2 x_{i_2} \delta_{i_3 i_1} - \sigma^2 x_{i_3} \delta_{i_1 i_2} & i_1, i_1, i_3 = 1, \ldots, N
\end{array}$$

From their definition in terms of derivatives these may be seen to be products of homogeneous one-dimensional Hermite polynomials. For suppose the sequence $i_1\, i_2 \ldots i_n$ contains integers 1, 2, ..., N repeated $r_1, r_2, \ldots, r_N$ times respectively. Then

$$\begin{aligned}
he^{(n)}{}_{i_1 i_2 \ldots i_n}(x)\, p(x) & \\
&= (-\sigma^2\, \partial/\partial x_1)^{r_1}(-\sigma^2\, \partial/\partial x_2)^{r_2} \ldots (-\sigma^2\, \partial/\partial x_N)^{r_N}\, p(x_1)\, p(x_2) \ldots p(x_N) \\
&= he_{r_1}(x_1)\, he_{r_2}(x_2) \ldots he_{r_N}(x)\, p(x)
\end{aligned}$$

This property does not hold for the general Hermite polynomials defined previously.

Orthogonality: From the last equation it follows immediately that two Hermite polynomials

$$he^{(m)}{}_{i_1 i_2 \ldots i_n}(x) \text{ and } he^{(n)}{}_{j_1 j_2 \ldots j_n}(x)$$

are orthogonal unless the sequences $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ are permutations of each other. This implies (i) they have the same degree ($m = n$) and (ii) both have the same decomposition of the degree as 1, 2, ..., N repeated $r_1, r_2, \ldots, r_N$ times. Then from the orthogonality of the one-dimensional polynomials it follows that

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} he^{(n)}{}_{i_1 i_2 \ldots i_n}(x)\, he^{(n)}{}_{j_1 j_2 \ldots j_n}(x)\; p(x)\, dx = r_1!, r_2!, \ldots, r_N!\; \sigma^{2n}$$

Reference: Grad H: A note on N-dimensional Hermite polynomials. Comm. Pure Appl. Math. 1949

The right hand side may also be written as

$$\sigma^{2n} \sum_{\text{perms}} \delta_{i_1 j_{p1}} \delta_{i_2 j_{p2}} \dots \delta_{i_n j_{pn}}$$

the sum being over the n! permutations $p_1, p_2, \dots, p_n$. of $1, 2, \dots, n$.

Expansion formula: a function of finite mean square

$$f(x) = f(x_1, x_2, \dots, x_N)$$

has a mean-square convergent expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum \sum \dots \sum_{i_1 + i_2 + \dots + i_n = n} f^{(n)}{}_{i_1 i_2 \dots i_n} \, he^{(n)}{}_{i_1 i_2 \dots i_n}(x)$$

where

$$f^{(n)}{}_{i_1 i_2 \dots i_n} = \frac{1}{\sigma^{2n}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x) \, he^{(n)}{}_{i_1 i_2 \dots i_n}(x) \, p(x) \, dx_1 \, dx_2 \dots dx_N$$

Proof: It is known that from the completeness of the Hermite polynomials follows the completeness of their products* so that

$$f(x) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \dots \sum_{r_N=0}^{\infty} \frac{1}{r_1! \, r_2! \dots r_N!} f^{(n)}{}_{r_1, r_2, \dots r_N} \, he_{r_1}(x) \, he_{r_2}(x) \dots he_{r_N}(x)$$

where

$$f^{(n)}{}_{r_1, r_2, \dots r_N} = \frac{1}{\sigma^{2N}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x) \, he_{r_1}(x_1) \, he_{r_2}(x_2) \dots he_{r_N}(x_N) \, p(x) \, dx_1 \, dx_2 \dots dx_N$$

Arranging terms by degree gives

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum \sum \dots \sum_{r_1 + r_2 + \dots + r_N = n} \frac{n!}{r_1! \, r_2! \dots r_N!} f^{(n)}{}_{r_1, r_2, \dots r_N} \, he_{r_1}(x) \, he_{r_2}(x) \dots he_{r_N}(x)$$

With a change of notation, this is seen to be the required expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum \sum \dots \sum_{i_1 + i_2 + \dots + i_n = n} f^{(n)}{}_{i_1 i_2 \dots i_n} \, he^{(n)}{}_{i_1 i_2 \dots i_n}(x)$$

Here the coefficients may be written in terms of derivatives by a straightforward extension of Chebyshev's formula

$$f^{(n)}{}_{i_1 i_2 \dots i_n}(x) = E_x \{ \partial^n f(x) / \partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n} \}$$

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* See e.g. Hilbert-Courant: Mathematical Physics

6 Historical remarks on Hermite polynomials

History: What are nowadays called Hermite polynomials were apparently first used in the 1812 classic of Laplace on probability although there they are hardly recognizable as such and there is little description of the familiar properties of these polynomials. The first treatment of these properties in remarkably modern form was given by Chebyshev in 1860 who also described several other types of orthogonal polynomials now in common use, e.g. those usually called Laguerre polynomials. Hermite's 1864 paper described, not the familiar one-dimensional polynomials, but their multidimensional form and only in this sense was his development new.

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C.R. Acad. Sci. Paris 58 1864 93-100; 266-273; Oeuvres Paris 1908; II 293-308.

Notation: There are many notations in the literature using the two weight functions exp $(-x^2)$ and exp $(-\frac{1}{2}x^2)$. The weight function exp $(-\frac{1}{2}x^2)$ apparently originates from the book of Appel & Kampé de Feriet (1926). Our notation follows Rosenhead & Fletcher's Mathematical Tables which recommends using $He_n(x)$ for weight function $\exp(-x^2)$ and $he_n(x)$ for weight function $\exp(-\frac{1}{2}x^2)$.

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