Threshold Effects in Multivariate Error Correction Models*

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Abstract

In this paper we propose a testing procedure for assessing the presence of threshold effects in nonstationary Vector autoregressive models with or without cointegration. Our approach involves first testing whether the long run impact matrix characterising the VECM type representation of the VAR switches according to the magnitude of some threshold variable and is valid regardless of whether the system is purely I(1), I(1) with cointegration or stationary. Once the potential presence of threshold effects is established we subsequently evaluate the cointegrating properties of the system in each regime through a model selection based approach whose asymptotic and finite sample properties are also established. This subsequently allows us to introduce a novel non-linear permanent and transitory decomposition of the vector process of interest.

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1 Introduction

A growing body of research in the recent time series literature has concentrated on incorporating nonlinear behaviour in conventional linear reduced form specifications such as autoregressive and moving average models. The motivation for moving away from the traditional linear model with constant parameters has typically come from the observation that many economic and financial time series are often characterised by regime specific behaviour and asymmetric responses to shocks. For such series the linearity and parameter constancy restrictions are typically inappropriate and may lead to misleading inferences about their dynamics.

Within this context, and a univariate setting, a general class of models that has been particularly popular from both a theoretical and applied perspective is the family of threshold models which are characterised by piecewise linear processes separated according to the magnitude of a threshold variable which triggers the changes in regime. When each linear regime follows an autoregressive process for instance we have the well known threshold autoregressive class of models, the statistical properties of which have been investigated in the early work of Tong and Lim (1980), Tong (1983, 1990), Tsay (1989), Chan (1990, 1993) and more recently reconsidered and extended in Hansen (1996, 1997, 1999a, 1999b, 2000), Caner and Hansen (2001), Gonzalez and Gonzalo (1997), Gonzalo and Montesinos (2000), Gonzalo and Pitarakis (2002) among others. The two key aspects on which this theoretical research has focused on were the development of a distributional theory for tests designed to detect the presence of threshold effects and the statistical properties of the resulting parameter estimators characterising such models.

Given their ability to capture a very rich set of dynamic behaviour including persistence and asymmetries, the use of this class of models has been advocated in numerous applications aiming to capture economically meaningful nonlinearities. Examples include the analysis of asymmetries in persistence in the US output growth (Beaudry and Koop (1993), Potter (1995)), asymmetries in the response of output prices to input price increases versus decreases (Borenstein, Cameron and Gilbert (1997), Peltzman (2000)), nonlinearities in unemployment rates (Hansen (1997), Koop and Potter (1999)), threshold effects in cross-country growth regressions (Durlauf and Johnson (1995)) and in international relative prices (Michael, Nobay and Peel (1997), Obstfeld and Taylor (1997), O'Connell and Wei (1997), Lo and Zivot (2001)) among numerous others.

Although the vast majority of the theoretical developments in the area of testing and estimation of univariate threshold models have been obtained under the assumption of stationarity and ergodicity, another important motivation for their popularity came from the observation that a better description of the dynamics of numerous economic variables can be achieved by interacting the pervasive nature of unit roots with that of threshold effects within the same specification. This was also motivated by the observation that there might be much weaker support for the unit root hypothesis when the alternative hypothesis under consideration allows for the presence of threshold type effects in the time series of interest. In Pippenger and Goering (1993) for instance the authors documented a substantial fall in the power of the Dickey Fuller test when the stationary alternative was allowed to include threshold effects. This also motivated the work of Enders and Granger (1998), who proposed a simple test of the null hypothesis of a unit root against asymmetric adjustment instead of a linear stationary alternative.

One important property of threshold models that contributed to this line of research is their ability to capture persistent behaviour while remaining globally stationary. This can be achieved for instance by allowing a time series to follow a unit root type process such as a random walk within one regime while being stationary in another. Numerous economic and financial variables such as unemployment rates or interest rates for instance must be stationary by the mere fact that they are bounded. However at the same time conventional unit roots tests are typically unable to reject the null hypothesis of a unit root in their autoregressive representation. This observation has prompted numerous researchers to explore the possibility that the dynamics of these series may be better described by threshold models that allow the nonstationary component to occur within a corridor regime. A well known example highlighting this point is the behaviour of real exchange rate series which are typically found to be unit root processes, implying lack of international arbitrage and violation of the PPP hypothesis. Once allowance is made for the presence of threshold effects capturing aspects such as transaction costs however it has been typically found that this nonstationarity only occurs locally (e.g. between transaction cost bounds) and that the process is in fact globally stationary (see Bec, Ben-Salem and Carrasco (2001) and references therein). Within a related context, Gonzalez and Gonzalo (1998) also introduced a globally stationary process referred to as a threshold unit root model that combines the presence of a unit root with threshold effects, and found strong support in favour of such a specification for modelling interest rate series.

Although all of the above mentioned research operated under a univariate setup the recent

time series literature has also witnessed a growing interest in the inclusion of threshold effects in multivariate settings such as vector error correction models. A key factor that triggered this line of research has been the observation that threshold effects may also have an intuitive appeal when it comes to modelling the adjustment process towards a long run equilibrium characterising two or more variables.

From the early work of Engle and Granger (1987), for instance, it is well known that two or more variables that behave like unit root processes individually may in fact be linked via a long run equilibrium relationship making particular linear combinations of these variables stationary or, as commonly known, cointegrated. When this happens, the variables in question admit an error correction model representation that allows for the joint modelling of both their long run and short run dynamics. In its linear form, such an error correction specification restricts the adjustment process to remain the same across time thereby ruling out the possibility of lumpy and discontinuous adjustment. An important paper, which proposed to relax this linearity assumption by introducing the possibility of threshold effects in the adjustment process towards the long run equilibrium and thereby capturing phenomena such as changing speeds of adjustment was, Balke and Fomby (1997) where the authors introduced the concept of threshold cointegration (see also Tsay (1998)).

The inclusion of such nonlinearities in error correction models has been found to have a very strong intuitive and economic appeal allowing for instance for the possibility that the adjustment process towards the long run equilibrium behaves differently depending on how far off the system is from the long run equilibrium itself (i.e depending on the magnitude of the equilibrium error). This naturally also allows for the possibility that the adjustment process shuts down over certain periods. Consider, for instance, the prices of the same asset in two different geographical regions. Although both prices will be equal in the long run equilibrium it could be that due to the presence of transaction costs arbitrage solely kicks in when the difference in price (i.e. the equilibrium error) is sufficiently large.

The concept of threshold cointegration as introduced in Balke and Fomby (1997) has attracted considerable attention from practitioners interested in uncovering nonlinear adjustment patterns in relative prices and other variables (see Wohar and Balke (1998), Baum, Barkoulas and Caglayan (2001), Enders and Falk (1998), Lo and Zivot (2001), O'Connell and Wei (1997)). From a method-

ological point of view, Balke and Fomby (1997) proposed to assess such occurences within a simple setup which consisted in adapting the approach developed in Hansen (1996) to an Engle-Granger type test performed on the cointegrating residuals. Their setup also implicitly assumed the existence of a known and single cointegrating vector linking the variables of interest. In a related study, Enders and Siklos (2001) extended Balke and Fomby's methodology by adapting the work of Enders and Granger (1998) to a cointegrating framework.

Despite the substantial interest generated by the introduction of the concept of threshold cointegration in Balke and Fomby (1997) a full statistical treatment within a formal multivariate error correction type of specification has only been available following the recent work of Hansen and Seo (2002). See also Tsay (1998) who introduced an arranged regression approach for testing for the presence of threshold effects in VARs. Although also dealing with a multivariable cointegration setup, the methodology proposed in Balke and Fomby (1998) or Enders and Siklos (2001) focused on the direct treatment of the cointegrating residuals akin to the familiar Engle-Granger test for cointegration. In Hansen and Seo (2002) however, the authors developed a maximum likelihood based estimation and testing theory starting directly from a vector error correction model representation of a cointegrated system with potential threshold effects in its adjustment process. More specifically, Hansen and Seo (2002) considered a VECM assumed to contain a single cointegrating vector and in which the threshold effects are driven by the error correction term. Their analysis also implicitly assumes that the researcher knows in advance the cointegration properties of the system (i.e the system is known to be cointegrated with a single cointegrating vector) and interest solely lies in detecting the presence of threshold effects in the adjustment process towards the equilibrium. This simplifying assumption avoids the need to test for cointegration in the presence of a potentially nonlinear adjustment process. In more recent research, Seo (2004) concentrated on this latter issue by developing a new distributional theory for directly testing the null of no cointegration against the alternative of threshold cointegration. In Seo's (2004) framework it is again the case that cointegration if present is solely characterised by a single cointegrating vector and as in Hansen and Seo (2002) the threshold variable of interest is taken to be the error correction term itself.

In the present research our goal is to contribute further to the analysis of threshold effects in possibly cointegrated multivariate systems of the vector error correction type. Our initial goal is to evaluate the properties of a Wald type test for testing the null of linearity against threshold

nonlinearity in the long run impact matrix of a VECM. Our analysis does not presume any specific cointegration properties of the system and is valid regardless of whether the system is cointegrated or not. One additional difference from previous work is our view about the threshold variable that induces the presence of threshold effects. Instead of taking the error correction term to be the variable whose magnitude triggers threreshold effects, we consider a general external threshold variable which could be any economic or financial variable that is stationary and ergodic such as the growth rate in the economy. Having established the existence of threshold effects in the VECM representation of our system, we subsequently evaluate the properties of least squares based estimators of the threshold parameter focusing on both its large and small sample properties followed by the analysis of the formal cointegration properties of the system when applicable. This then allows us to formally obtain a nonlinear permanent and transitory decomposition of the vector process of interest following the same methodology as in Gonzalo and Granger (1995).

The plan of the chapter is as follows. Section II develops the theory for testing for the presence of threshold effects in a Vector Error Correction type of model. Section III focuses on the theoretical properties of estimators of the threshold parameters. Section IV proposes a methodology for assessing the cointegration properties of the system, Section V introduces a nonlinear permanent and transitory decomposition based on a VECM with threshold effects and Section VI concludes. All proofs are relegated to the appendix.

2 Testing for Threshold Effects in a Multivariate Framework

2.1 The Model and Test Statistic

We let the p-dimensional time series $\{Y_t\}$ be generated by the following vector error correction type specification, which allows for the presence of threshold effects in its long run impact matrix:

$$\Delta Y_t = \boldsymbol{\mu} + \boldsymbol{\Pi}_1 Y_{t-1} I(q_{t-d} \leq \gamma) + \boldsymbol{\Pi}_2 Y_{t-1} I(q_{t-d} > \gamma) + \sum_{j=1}^{k} \boldsymbol{\Gamma}_j \Delta Y_{t-j} + u_t \qquad (1)$$

where $\boldsymbol{\Pi}_1$, $\boldsymbol{\Pi}_2$ and $\boldsymbol{\Gamma}_j$ are $p \times p$ constant parameter matrices, q_{t-d} a scalar threshold variable, $I(.)$ is the indicator function, γ the threshold parameter, k and d the known lag length and delay parameter and u_t is the p-dimensional random disturbance vector.

The model in (1) is a multivariate generalisation of an autoregressive model with threshold

effects whose dynamics are characterised by piecewise linear vector autoregressions. The regime switches are governed by the magnitude of the threshold variable q_t crossing an unknown threshold value γ. The specification in (1) is similar to the one considered in Seo (2004) with the difference that no assumptions are made about the rank structure of either $\mathbf{\Pi}_1$ or $\mathbf{\Pi}_2$, and the threshold variable is not necessarily given by the error correction term such as $q_t = \beta' Y_t$ with β denoting the single cointegrating vector for instance.

The initial question of interest in the context of the specification in (1) is whether the long run impact matrix is truly characterised by threshold effects driven by the threshold variable q_t. Under the absence of such effects we have a standard linear VECM with $\mathbf{\Pi}_1 = \mathbf{\Pi}_2$ and this restriction can be tested via a conventional Wald type test statistic against the alternative $H_1 : \mathbf{\Pi}_1 \neq \mathbf{\Pi}_2$.

At this stage it is important to note that the sole purpose of testing the above null hypothesis is to uncover the presence or absence of threshold effects in the long run impact matrix. More importantly we wish to conduct this set of inferences regardless of the stationarity properties of Y_t, in the sense that our null hypothesis may hold under a purely stationary set up or a unit root set up with or without cointegration. If the null hypothesis is not rejected we can then carry on with the process of exploring the stochastic properties of the data following for instance Johansen's methodology (see Johansen (1998) and references therein). Before proceeding further and to motivate our working model we consider two simple examples illustrating particular cases of our specification in (1).

EXAMPLE 1: Here we present a bivariate system of cointegrated I(1) variables with threshold effects in their adjustment process. Specifically, with $Y_t = (y_{1t}, y_{2t})'$ we write $y_{1t} = \beta y_{2t} + z_t$ where $\Delta y_{2t} = \epsilon_{2t}$ and $\Delta z_t = \rho_1 z_{t-1} I(q_{t-1} \leq \gamma) + \rho_2 z_{t-1} I(q_{t-1} > \gamma) + \epsilon_{1t}$ with $\rho_i < 0$ for $i = 1, 2$ and for simplicity we take q_t to be an iid random variable. In this example both y_{1t} and y_{2t} are I(1) and cointegrated with cointegrating vector $(1, -\beta)$ since z_t is a covariance stationary process following a threshold autoregressive scheme. It is now straightforward to reformulate the above model as in (1) writing,

$$\begin{pmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{pmatrix} = \begin{pmatrix} \rho_1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} \leq \gamma) + \begin{pmatrix} \rho_2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} > \gamma) + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \tag{2}$$

with $u_{1t} = \epsilon_{1t} + \beta\epsilon_{2t}$ and $u_{2t} = \epsilon_{2t}$.

EXAMPLE 2: Here we consider a purely stationary bivariate system with both variables following a threshold autoregressive process. Consider $\Delta y_{1t} = \rho_{11} y_{1t-1} I(q_{t-1} \leq \gamma) + \rho_{21} y_{1t-1} I(q_{t-1} > \gamma) + u_{1t}$ and $\Delta y_{2t} = \rho_{12} y_{2t-1} I(q_{t-1} \leq \gamma) + \rho_{22} y_{2t-1} I(q_{t-1} > \gamma) + u_{2t}$ with $\rho_{i1} < 0$ and $\rho_{i2} < 0$ for $i = 1, 2$. We can again reformulate this sytem as in (1) by writing

$$\begin{pmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{pmatrix} = \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{12} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} \leq \gamma) + \begin{pmatrix} \rho_{21} & 0 \\ 0 & \rho_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} > \gamma) + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}. \tag{3}$$

In order to explore the properties of the Wald type test for the above null hypothesis, it will be convenient to reformulate (1) in matrix form. In what follows, for the clarity and simplicity of the exposition, we focus on a restricted version of (1) setting the constant term as well as the coefficients on the lagged dependent variables equal to zero. Since our framework does not consider threshold effects in those parameters it would be straightforward to concentrate (1) with respect to $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ using an appropriate projection matrix. This is also with no loss of generality since our distributional results presented in Propositions 1 and 2 below would remain unaffected. We now write

$$\Delta Y = \mathbf{\Pi}_1 Z_1 + \mathbf{\Pi}_2 Z_2 + U \tag{4}$$

where ΔY, Z_1 and Z_2 are all $p \times T$ matrices stacking the vectors ΔY_t, $Y_{t-1} I(q_{t-d} \leq \gamma)$ and $Y_{t-1} I(q_{t-d} > \gamma)$, respectively. Within the formulation in (4) we have $\Delta Y = (\Delta y_1, \Delta y_2, \ldots, \Delta y_T)$ $Z_1 = (y_0 I(q_{0-d} \leq \gamma), \ldots, y_{T-1} I(q_{T-d} \leq \gamma))$ and $Z_2 = (y_0 I(q_{0-d} > \gamma), \ldots, y_{T-1} I(q_{T-d} > \gamma))$. Similarly U is a $p \times T$ matrix of random disturbances given by $U = (u_1, \ldots, u_T)$. We note that within our parameterisation the regressor matrices Z_1 and Z_2 are orthogonal due to the presence of the two indicator functions. Their dependence on γ is omitted for notational parsimony. For later use we also introduce the $p \times T$ matrix $Z = (y_0, \ldots, y_{T-1})$, which is such that $Z = Z_1 + Z_2$.

The unknown parameters of the model in (4) can be estimated via concentrated least squares proceeding conditional on a known γ. Indeed, since given γ the model is linear in its parameters the least squares estimators of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are given by $\widehat{\mathbf{\Pi}}_1(\gamma) = \Delta Y Z_1' (Z_1 Z_1')^{-1}$ and $\widehat{\mathbf{\Pi}}_2(\gamma) = \Delta Y Z_2' (Z_2 Z_2')^{-1}$. For later use we also introduce the vectorised versions of the parameter matrices,

writing $\hat{\boldsymbol{\pi}}_1 \equiv vec\ \hat{\boldsymbol{\Pi}}_1$ and $\hat{\boldsymbol{\pi}}_2 \equiv vec\ \hat{\boldsymbol{\Pi}}_2$, and the null hypothesis of interest can be equivalently expressed as $H_0 : \boldsymbol{\pi}_1 = \boldsymbol{\pi}_2$ or $H_0 : R\boldsymbol{\pi} = 0$ with $R = [I_{p^2}, -I_{p^2}]$ and $\boldsymbol{\pi} = (\boldsymbol{\pi}_1', \boldsymbol{\pi}_2')'$.

The Wald statistic for testing the above null hypothesis takes the following form

$$W_T(\gamma) \quad = \quad (R\hat{\boldsymbol{\pi}})' \left[R((DD')^{-1} \otimes \hat{\Omega}_u)R'\right]^{-1} (R\hat{\boldsymbol{\pi}}) \tag{5}$$

where $\otimes$ is the Kronecker product operator, $\hat{\boldsymbol{\pi}}_1 = [(Z_1Z_1')^{-1}Z_1 \otimes I_p]vec\ \Delta Y$, $\hat{\boldsymbol{\pi}}_2 = [(Z_2Z_2')^{-1}Z_2 \otimes I_p]vec\ \Delta Y$ and $D = [Z_1\ Z_2]$. The $p \times p$ matrix $\hat{\Omega}_u$ refers to the least squares estimator of the covariance matrix defined as $\hat{\Omega}_u = \hat{U}\hat{U}'/T$ with $\hat{U} = \Delta Y - \hat{\boldsymbol{\Pi}}_1(\gamma)Z_1 - \hat{\boldsymbol{\Pi}}_2(\gamma)Z_2$. Since Z_1 and Z_2 are orthogonal it also immediately follows that $DD' = diag(Z_1Z_1', Z_2Z_2')$ and $(DD')^{-1} \otimes \hat{\Omega}_u = diag[(Z_1Z_1')^{-1} \otimes \hat{\Omega}_u, (Z_2Z_2')^{-1} \otimes \hat{\Omega}_u]$. We can thus also reformulate the Wald statistic in (5) as

$$W_T(\gamma) \quad = \quad (\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2)' \left[(Z_2Z_2')(ZZ')^{-1}(Z_1Z_1') \otimes \hat{\Omega}_u^{-1}\right] (\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2) \tag{6}$$

where $ZZ' = Z_1Z_1' + Z_2Z_2'$.

At this stage it is also important to reiterate the fact that when implementing our test of the null hypothesis of linearity with say $\boldsymbol{\Pi}_1 = \boldsymbol{\Pi}_2 = \boldsymbol{\Pi}$, the corresponding characteristic polynomial $\Phi(z) = (1-z)I_p - \boldsymbol{\Pi}z$ will be assumed to have all its roots either outside or on the unit circle and the number of unit roots present in the system will be given by $p - r$ with $0 \leq r \leq p$. Our analysis rules out instances of explosive behaviour or processes that may be integrated of order two. This also allows us to have a direct correspondence between the stochastic properties of Y_t under the null hypothesis and the rank structure of the long run impact matrix $\boldsymbol{\Pi}$. In the particular case where all the roots of the characteristic polynomial are outside the unit circle, the series will be referred to as $I(0)$.

2.2 Assumptions and Limiting Distributions

Throughout this section we will be operating under the following set of assumptions

(A1) $u_t = (u_{1t}, \ldots, u_{pt})'$ is a zero mean iid sequence of p dimensional random vectors with a bounded density function, covariance matrix $E[u_tu_t'] = \Omega_u > 0$ and with $E|u_{it}|^{2\delta} < \infty$ for some $\delta > 2$ and $i = 1, \ldots, p$;

(A2) q_t is a strictly stationary and ergodic sequence that is independent of u_{is} $\forall t, s$, $i = 1, \ldots, p$ and has distribution function F that is continuous everywhere;

(A3) the threshold parameter γ is such that $\gamma \in \Gamma = [\gamma_L, \gamma_U]$ a closed and bounded subset of the sample space of the threshold variable.

Assumption (A1) above is required for our subsequent limiting distribution theory. It will ensure, for instance, that the functional central limit theorem can be applied to the sample moments used in the construction of Wald and related tests. Assumption (A2) restricts the behaviour of the scalar random variable that induces threshold effects in the model in (1). Although it allows q_t to follow a very rich class of processes, it requires it to be external in the sense of being independent of the u_t sequence and also rules out the possibility of q_t being I(1) itself for instance. Finally assumption (A3) is standard in this literature. The threshold variable sample space Γ is typically taken to be $[\gamma_L, \gamma_U]$, with γ_L and γ_U chosen such that $P(q_{t-d} \leq \gamma_L) = \theta_1 > 0$ and $P(q_{t-d} \leq \gamma_U) = 1 - \theta_1$. The choice of θ_1 is commonly taken to be 10% or 15%. Restricting the parameter space of the threshold in this fashion ensures that there are enough observations in each regime and also guarantees the existence of nondegenerate limits for the test statistics of interest.

In what follows, we will be interested in obtaining the limiting behaviour of $W_T(\gamma)$ defined in (6). In this context it will be important to explore the distinctive features of the limiting null distribution of the test statistic when the maintained model is either a pure multivariate unit root process with no cointegration (i.e. $\Delta Y_t = u_t$) or a VECM in the form $\Delta Y_t = \mathbf{\Pi} Y_{t-1} + u_t$ with $Rank(\mathbf{\Pi}) = r$ such that $0 < r \leq p$. The case where $r = p$ would correspond to a purely stationary specification. We note that under all these instances the null hypothesis of linearity holds. Before proceeding further it is also important to emphasise the fact that we are facing a nonstandard inference problem, since under the null hypothesis the threshold parameter γ is not unidentified. This is now a well known and documented problem in the literature on testing for the presence of various forms of nonlinearities in regression models and is commonly referred to as the Davies problem. Under a stationary setting where $Rank(\mathbf{\Pi}) = p$ and taking γ as fixed and given, we would expect $W_T(\gamma)$ to behave like a χ^2 random variable in large samples. Since we will not be assuming that γ is known, however we will follow Davies (1977, 1987) and test the null hypothesis of linearity using $SupW = \sup_{\gamma \in \Gamma} W_T(\gamma)$. In what follows we also make use of the equality $I(q_{t-d} \leq \gamma) = I(F(q_{t-d}) \leq F(\gamma))$, which allows us to use uniform random variables (see Caner and Hansen (2001), p. 1586). In this context we let $\lambda \equiv F(\gamma) \in \Lambda$ with $\Lambda = [\theta_1, 1 - \theta_1]$ and throughout this chapter we will be using λ and $F(\gamma)$ interchangeably.

In the following proposition we summarise the limiting behaviour of the Wald statistic for testing the null hypothesis of linearity when it is assumed that the system is purely stationary.

Proposition 1 *Under assumptions A1-A3,* $H_0 : \mathbf{\Pi}_1 = \mathbf{\Pi}_2$ *and* Y_t *a p-dimensional I(0) vector we have*

$$SupW \quad \Rightarrow \quad Sup_{\lambda \in \Lambda} G(\lambda)' V(\lambda)^{-1} G(\lambda) \tag{7}$$

where $G(\lambda)$ *is a zero mean* p^2*-dimensional Gaussian random vector with covariance* $E[G(\lambda_1)G(\lambda_2)] = V(\lambda_1 \wedge \lambda_2)$ *and* $V(\lambda) = \lambda(1-\lambda)(Q \otimes \Omega_u)$ *with* $Q = E[ZZ']$.

REMARK 1: It is interesting to note that the above limiting distribution is equivalent to a normalised squared Brownian Bridge process identical to the one arising when testing for the presence of structural breaks as in Andrews (1993, Theorem 3, p. 838). The same distribution also arises in particular parameterisations of self-exciting threshold autoregressive models when only the constant terms are allowed to be different in each regime (see Chan (1990)). We also note that for known and given γ, the quantity $G(\lambda)'V(\lambda)^{-1}G(\lambda)$ reduces to a χ^2 random variable with p^2 degrees of freedom. Since $G(\lambda)$ is $(Q \otimes \Omega_u)^{\frac{1}{2}} N(0, \lambda(1-\lambda)I_{p^2}) \equiv (Q \otimes \Omega_u)^{\frac{1}{2}}[W(\lambda) - \lambda W(1)]$, with $W(.)$ denoting a p^2-dimensional standard Brownian Motion, the result follows from the above definition of $V(\lambda)$. We also note that the limiting process is free of nuisance parameters, solely depending on the number of parameters being tested under the null hypothesis and is tabulated in Andrews (1993, Table 1, p. 840). For a more extensive set of p-values of the corresponding limiting distributions see also Hansen (1997).

In the next proposition we summarise the limiting behaviour of the same Wald test statistic when the system is assumed to be a p-dimensional pure I(1) process as $\Delta Y_t = u_t$ or alternatively I(1) but cointegrated as in $\Delta Y_t = \alpha\beta' Y_{t-1} + u_t$, with α and β having reduced ranks. In what follows a standard Brownian Sheet $W(s,t)$ is defined as a zero mean two-parameter Gaussian process indexed by $[0,1]^2$ and having a covariance function given by $Cov[W(s_1,t_1), W(s_2,t_2)] = (s_1 \wedge t_1)(s_2 \wedge t_2)$ while a Kiefer process K on $[0,1]^2$ is given by $K(s,t) = W(s,t) - tW(s,1)$. The Kiefer process is also a two-parameter Gaussian process with zero mean and covariance function $Cov[K(s_1,t_1), K(s_2,t_2)] = (s_1 \wedge s_2)(t_1 \wedge t_2 - t_1 t_2)$.

Proposition 2 *Under assumptions A1-A3,* $H_0 : \mathbf{\Pi}_1 = \mathbf{\Pi}_2$ *and* Y_t *a p-dimensional I(1) vector cointegrated or not:*

$$SupW \Rightarrow Sup_{\lambda\in\Lambda}\frac{1}{\lambda(1-\lambda)}tr\left(\int_0^1 W(r)dK(r,\lambda)'\right)'\left(\int_0^1 W(r)W(r)'\right)^{-1}\left(\int_0^1 W(r)dK(r,\lambda)'\right) \quad (8)$$

where $K(r,\lambda)$ *is a Kiefer process given by* $K(r,\lambda) = W(r,\lambda) - \lambda W(r,1)$ *with* $W(.)$ *denoting a p-dimensional standard Brownian Motion and* $W(r,\lambda)$ *a p-dimensional standard Brownian Sheet.*

Looking at the expression of the limiting distribution in Proposition 2, we again observe that for given and known λ the limiting random variable is $\chi^2(p^2)$ exactly as occurred under the purely stationary setup of proposition 1. This follows from the observation that $W(r)$ and $K(r,\lambda)$ are independent. Note that we have $E[W(r)K(r,\lambda)] = E[W(r)W(r,\lambda)] - \lambda E[W(r)^2]$ and since $E[W(r)W(r,\lambda)] = r\lambda$ and $E[W(r)^2] = r$ the result follows. It also follows that the limiting random variables in (7) and (8) are equivalent in distribution.

2.3 Simulation Based Evidence

Having established the limiting behaviour of the Wald statistic for testing the null of no threshold effects within the VECM type representation we next explore the adequacy of the asymptotic approximations presented in Propositions 1-2 when dealing with finite samples. This will also allow us to explore the documented robustness of the above limiting distributions to the absence or presence of unit roots and cointegration, and to the stochastic properties of the threshold variable q_t when faced with limited sample sizes.

We initially consider a purely stationary bivariate DGP, as the model under the null hypothesis, parameterised as $Y_t = \Phi Y_{t-1} + u_t$ with $\Phi = diag(0.5, 0.8)$ and $u_t = NID(0, I_2)$. As a candidate threshold variable required in the construction of the Wald statistic we consider two options. One in which q_t is taken to be a normal iid random variable (independent of u_{it}, $i = 1, 2$) and one where q_t follows a stationary AR(1) process given by $q_t = \theta q_{t-1} + \epsilon_t$ with $\theta = 0.5$ and $\epsilon_t = NID(0, 1)$ with $Cov(\epsilon_t, u_{is}) = 0\ \forall t, s$ and $i = 1, 2$. Regarding the magnitude of the delay parameter we set $d = 1$ throughout all our experiments, all conducted using samples of size $T = 200, 400, 2000$ across $N = 5000$ replications and with a 10% trimming of the sample space of the threshold variable. Another important purpose of our experiments is to construct a range of critical values

for the distributions presented in (7)-(8) and compare them with the corresponding tabulations in Andrews (1993, Table 1, p. 840). Results for the purely stationary system are presented in Table 1 below.

Table 1: Critical Values under an I(0) system and $p^2 = 4$

	T	90%	95%	99%
		$q_t : NID(0,1)$		
$SupW$	200	14.946	16.909	21.246
$SupW$	400	14.606	16.686	21.239
$SupW$	2000	14.762	16.596	20.741
		$q_t : AR(1)$		
$SupW$	200	15.135	17.252	21.331
$SupW$	400	14.836	17.024	21.323
$SupW$	2000	14.829	16.737	20.854
$Andrews$	∞	14.940	16.980	21.040

The critical values tabulated in Table 1 suggest that the finite sample distributions of the Wald statistic track their asymptotic counterpart (as judged by a sample of size T=2000) very accurately. As discussed in Remark 1 above we can also observe that the critical values obtained in Andrews (1993) are virtually identical to the ones obtained using our DGPs and multivariate framework with thresholds (note that within our bivariate VAR we are testing for the presence of threshold effects across p^2 parameters).

In Tables 2 and 3 below we concentrated on the limiting and finite sample behaviour of the Wald statistic for testing the absence of threshold effects when the true DGP is a system of I(1) variables. Table 2 focuses on the case of a purely I(1) system with no cointegration, given by $\Delta Y_t = u_t$ while Table 3 focuses on a cointegrated system given $\Delta y_{1t} = u_{1t}$ and $y_{2t} = 0.8y_{2t-1} + u_{2t}$. In this latter case the bivariate system is characterised by the presence of one stationary relationship and the corresponding rank of the long run impact matrix is one. The dynamics of q_t were maintained as above in both sets of experiments.

Table 2: Critical Values under a pure I(1) system and $p^2 = 4$

	T	90%	95%	99%
		$q_t : NID(0,1)$		
$SupW$	200	14.970	17.023	22.098
$SupW$	400	14.858	18.578	21.205
$SupW$	2000	15.012	16.947	20.967
		$q_t : AR(1)$		
$SupW$	200	15.369	17.197	22.164
$SupW$	400	14.948	18.527	21.358
$SupW$	2000	14.904	16.840	21.212
$Andrews$	∞	14.940	16.980	21.040

Table 3: Critical Values under a cointegrated system and $p^2 = 4$

	T	90%	95%	99%
		$q_t : NID(0,1)$		
$SupW$	200	15.030	17.236	21.431
$SupW$	400	14.685	16.879	20.926
$SupW$	2000	14.723	16.739	20.911
		$q_t : AR(1)$		
$SupW$	200	15.068	16.903	21.074
$SupW$	400	15.150	17.040	21.153
$SupW$	2000	14.961	16.758	21.013
$Andrews$	∞	14.940	16.980	21.040

The empirical results presented in Tables 2-3 above clearly illustrate the robustness of the limiting distributions to various parameterisations of the threshold variable. Our tabulations also corroborate our earlier observation that the limiting distributions are unaffected by the presence or absence of I(1) components.

3 Estimation of the Threshold Parameter

Once inferences based on the Wald test reject the null hypothesis of a linear VECM our next objective is to obtain a consistent estimator of the threshold parameter. The model under which we operate is now given by $\Delta Y = \mathbf{\Pi}_1 Z_1 + \mathbf{\Pi}_2 Z_2 + U$. We propose to obtain an estimator of γ

based on the least squares principle. Letting $\hat{U}(\gamma) = \Delta Y - \hat{\mathbf{\Pi}}_1(\gamma)Z_1(\gamma) - \hat{\mathbf{\Pi}}_2(\gamma)Z_2(\gamma)$ we consider

$$\hat{\gamma} \quad = \quad \arg\min_{\gamma\in\Gamma} |\hat{U}(\gamma)\hat{U}(\gamma)'|. \tag{9}$$

Before establishing the large sample behaviour of $\hat{\gamma}$ introduced in (9) it is important to highlight the fact that a VECM type of representation *with* threshold effects as in (4) is compatible with either a purely stationary Y_t or a system of I(1) variables that is cointegrated in a conventional sense and with threshold effects present in its adjustment process. Examples of such processes are provided in (2) and (3) above while a formal discussion of the stationarity properties of Y_t generated from (4) is provided below.

The following proposition summarises the limiting behaviour of the threshold parameter estimator defined above with γ_0 referring to its true magnitude.

Proposition 3 *Under assumptions (A1)-(A3) with Y_t I(0) or I(1) but cointegrated and generated as in (4) we have $\hat{\gamma} \xrightarrow{p} \gamma_0$ as $T \to \infty$.*

From the above proposition it is clear that the consistency property of the threshold parameter estimator remains unaffected by the presence of I(1) components. In order to empirically illustrate the above proposition, and explore the behaviour of $\hat{\gamma}$ in smaller samples, we conducted a Monte-Carlo experiment covering a range of parameterisations including purely stationary and cointegrated systems. Our objective was to assess the finite sample performance of the least squares based estimator of γ_0 in moderate to large samples in terms of bias and variability.

For the purely stationary case we consider the specification introduced in (3), setting $(\rho_{11}, \rho_{21}) = (-0.8, -0.4)$ and $(\rho_{12}, \rho_{22}) = (-0.2, -0.6)$. Regarding the choice of threshold variable we consider the case of a purely Gaussian iid process as well as an AR(1) specification given by $q_t = 0.5q_{t-1} + u_t$ with $u_t = NID(0, 1)$. The true threshold parameter is set to $\gamma_0 = 0.25$ under the AR(1) dynamics and to $\gamma_0 = 0$ when q_t is iid. The delay parameter is fixed at $d = 1$. For the cointegrated case we consider a system given by $y_{1t} = 2y_{2t} + z_t$ with $\Delta y_{2t} = \epsilon_{2t}$ and $z_t = 0.2z_{t-1}I(q_{t-1} \leq \gamma_0) + 0.8z_{t-1}I(q_{t-1} > \gamma_0) + \nu_t$, while retaining the same dynamics for q_t and the same threshold parameter configurations as above. Both ϵ_{2t} and ν_t are chosen as NID(0,1) random variables.

Results for these two classes of DGPs are presented in Table 4 below, which displays the empirical mean and standard deviation of $\hat{\gamma}$ estimated as in (9) using samples of size $T = 200$ and $T = 400$ across $N = 5000$ replications.

Table 4: Empirical Mean and Standard Deviation of $\hat{\gamma}$

	q_t $AR(1), \gamma_0 = 0.15$		q_t $iid, \gamma_0 = 0$	
	$E(\hat{\gamma})$	$Std(\hat{\gamma})$	$E(\hat{\gamma})$	$Std(\hat{\gamma})$
	Stationary System			
$T = 200$	0.142	0.278	−0.014	0.247
$T = 400$	0.145	0.108	−0.004	0.100
	Cointegrated System			
$T = 200$	0.140	0.266	−0.006	0.229
$T = 400$	0.144	0.101	−0.003	0.091

From both of the above experiments we note that $\hat{\gamma}$ as defined in (9) displays a reasonably small and negative finite sample bias of approximately 0.5% under both configurations of the dynamics of the threshold variable and system properties. At the same time, however, we note that $\hat{\gamma}$ is characterised by a substantial variability across all model configurations. Its empirical standard deviation is virtually twice the magnitude of γ_0 under T=200 and, although clearly declining with the sample size, remains substantial even under T=400. Similar features of threshold parameter estimators have also been documented in Gonzalo and Pitarakis (2002).

Taking the presence of threshold effects as given together with the availability of a consistent estimator of the unknown threshold parameter, our next concern is to explore further the stochastic properties of the p-dimensional vector Y_t.

4 Stochastic Properties of the System and Rank Configuration of the VECM with Threshold Effects

So far the test developed in the previous sections allows us to decide whether the inclusion of threshold effects into a VECM type specification is supported by the data. Given the simplicity of its implementation, and the fact that the limiting distribution of the test statistic is unaffected by the stationarity properties of the variables being modelled, the proposed Wald based inferences can be viewed as a useful pre-test before implementing a formal analysis of the integration and cointegration properties of the system. If the null hypothesis is not rejected for instance we can proceed with the specification of a linear VECM using for instance the methodology developed in Johansen (1995 and references therein).

Our next concern is to explore the implications of the rejection of the null hypothesis of linearity for the stability and, when applicable, cointegration properties of Y_t whose dynamics are now known to be described by the specification in (4). Although rejecting the hypothesis that $\mathbf{\Pi}_1 = \mathbf{\Pi}_2$ rules out the scenario of a purely I(1) system with no cointegration as traditionally defined, since having $\mathbf{\Pi}_1 \neq \mathbf{\Pi}_2$ is trivially incompatible with the specification $\Delta Y = U$, as shown below, it remains possible that the system is either purely covariance stationary or I(1) with cointegration in a sense to be made clear (see for instance the formulation in (2) under example 1).

4.1 Stability Properties of the System

In the context of our specification in (4), and maintaining the notation $\mathbf{\Phi}_1 = I_p + \mathbf{\Pi}_1$ and $\mathbf{\Phi}_2 = I_p + \mathbf{\Pi}_2$ so that the system can be formulated as $Y_t = \mathbf{\Phi}_t Y_{t-1} + u_t$ with $\mathbf{\Phi}_t = \mathbf{\Phi}_1 I(q_{t-d} \leq \gamma) + \mathbf{\Phi}_2 I(q_{t-d} > \gamma)$, the stability properties of the system are summarised in the following proposition where for a square matrix $\mathbf{M}$ the notation $\boldsymbol{\rho}(\mathbf{M})$ refers its spectral radius.

Proposition 4 *Under assumptions (A1)-(A3), Y_t generated from (4) is covariance stationary iff* $\boldsymbol{\rho}(F(\gamma)(\mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1) + (1 - F(\gamma))(\mathbf{\Phi}_2 \otimes \mathbf{\Phi}_2)) < 1$.

From the above proposition it is interesting to note that even if one of the two regimes has a root on the unit circle the model could still be covariance stationary. In fact the system could even be characterised by an explosive behaviour in one of its regimes while still being covariance stationary, if for instance the magnitudes of the transition probabilities are such that switching occurs very often. Note also that the condition ensuring the covariance stationarity of Y_t is also equivalent to requiring the eigenvalues of $E[\mathbf{\Phi}_t \otimes \mathbf{\Phi}_t]$ to have moduli less than one.

EXAMPLE 3: We can here consider the example of a bivariate process given by $Y_t = I_2 Y_{t-1} I(q_{t-d} \leq \gamma) + \mathbf{\Phi}_2 Y_{t-1} I(q_{t-d} > \gamma) + u_t$ and let $\Phi_2 = \phi I_2$ with $|\phi| < 1$ where I_2 denotes a two dimensional identity matrix. This system can be seen to be characterised by a random walk type of behaviour in one regime and is covariance stationary in the second regime. In matrix form we have

$$\begin{pmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} \leq \gamma) + \begin{pmatrix} \phi - 1 & 0 \\ 0 & \phi - 1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} > \gamma) + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}. \tag{10}$$

Letting $\mathbf{M} = F(\gamma)(\mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1) + (1 - F(\gamma))(\mathbf{\Phi}_2 \otimes \mathbf{\Phi}_2))$, it is straightforward to establish that in the case of (10) we have $\boldsymbol{\rho}(\mathbf{M}) = F(\gamma) + \phi^2(1 - F(\gamma)) < 1$, since $\phi^2 < 1$ and thus implying that $Y_t = (y_{1t}, y_{2t})'$ is covariance stationary.

EXAMPLE 4: Another example of a covariance stationary system is given by

$$\begin{pmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \phi - 1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} \leq \gamma) + \begin{pmatrix} \phi - 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} I(q_{t-1} > \gamma) + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \tag{11}$$

for which we have $\boldsymbol{\rho}(\mathbf{M}) = (1 - F(\gamma))(1 - \phi)^2 < 1$ if $F(\gamma) < 0.5$, and $\rho(M) = F(\gamma)(1 - \phi)^2 < 1$ if $F(\gamma) > 0.5$. On the other hand, if we concentrate on the specification given in (2), it is straighforward to establish that $\rho(M) = 1$ thus violating the requirement for Y_t to be covariance stationary.

For later use it is also important at this stage to observe the correspondence between the ranks of the long run impact matrices presented in the above examples and the covariance stationarity of each system. In example 3, for instance, we note that $r_1 \equiv Rank(\mathbf{\Pi}_1) = 0$ and $r_2 \equiv Rank(\mathbf{\Pi}_2) = 2$, while in model (11) we have $(r_1, r_2) = (1, 1)$. This highlights the fact that within a nonlinear specification, as in (4), the correspondence between the rank structure of the long run impact matrices and the stability/cointegration properties of the system will be less clearcut than within a simple linear VECM. Before exploring further this issue it will be important to clarify the type of threshold nonlinearities that are compatible with an I(1) system and its VECM representation in (4).

4.2 I(1)'ness and Cointegration within a nonlinear VECM

The recent literature on the inclusion of nonlinear features in models with I(1) variables and cointegration can typically be categorised into two strands. Single equation approaches, which aim to detect the presence of nonlinearities in regressions with I(1) processes known to be cointegrated (see Saikkonen and Choi (2004), Hong (2003), Arai (2004)). In Saikkonen and Choi (2004), for instance, the authors included a smooth transition type of function $g(.)$ within a postulated cointegrating regression model of the form $y_{1t} = \beta y_{2t} + \theta y_{2t}\ g(y_{2t}; \gamma) + u_t$ and proposed a methodology for testing the null hypothesis of no such effects given here by $H_0 : \theta = 0$. The presence of such

nonlinearities within a cointegrating relationship implies some form of switching equilibria in the sense that the cointegrating vector is allowed to be different depending on the magnitude of y_{2t}. In both Hong (2003) and Arai (2004), the authors focused on a similar setup without an explicit choice of functional form. This was achieved through the inclusion of additional polynomial terms in the y_2 variable in the right hand side of a cointegrating regression.

Another strand of the same literature focused on the treatment of nonlinearities within a multivariate error correction framework. The motivation underlying this research was again to detect the presence of nonlinear cointegration, but here defined as a nonlinear adjustment towards the long run equilibrium while maintaining the assumption that the cointegration relationship is itself linear. Another important maintained assumption in this line of research is the existence of a single cointegrating vector (see Balke and Fomby (1997), Seo and Hansen (2002), Seo (2004)). Regarding the theoretical properties of multivariate models with nonlinearities, Bec and Rahbek (2004) have explored the strict stationarity and ergodicity properties of multivariate error correction models with general cointegrating rank and nonlinearities in their adjustment process.

One aspect that seems not to have been emphasised in the literature is the fact that, when operating within a VECM type framework, an important aspect of restricting the presence of nonlinearity to occur solely in the adjustment process stems from representation concerns. More specifically it can be shown that two I(1) variables that are linearly cointegrated but with a nonlinear adjustment process continue to admit a "nonlinear" VECM representation similar to (4) above. If we also wish to explore the possibility of nonlinearities in the cointegrating relationship itself however it becomes difficult to justify the existence of a VECM representation à la Granger.

To highlight this point let us consider the following simple *nonlinear* cointegrating relationship which is characterised by the presence of a threshold type of nonlinearity

$$\begin{aligned} y_{1t} &= \beta y_{2t} + \theta y_{2t} I(q_{t-1} > \gamma) + z_t \\ \Delta y_{2t} &= \epsilon_{2t} \\ \Delta z_t &= \rho z_{t-1} + u_t. \end{aligned} \tag{12}$$

with $\rho < 0$ and z_t representing the stationary equilibrium error.

If we were in a linear setup with $\theta = 0$ it would be straightforward to reformulate the above specification as $\Delta y_{1t} = \rho z_{t-1} + \nu_t$, with $\nu_t = u_t + \beta\epsilon_{2t}$, and we would have a traditional VECM

representation with ρ playing the role of the adjustment coefficient to equilibrium and $z_{t-1} = (y_{1t-1} - \beta y_{2t-1})$ denoting the previous period's equilibrium error. At this stage it is important to note that a key aspect of the linear setup that allows us to move towards an ECM type representation is the fact that taking y_{2t} to be an I(1) variable, as in (12), directly implies that y_{1t} is also difference stationary since taking the first difference of both sides of the first equation gives $\Delta y_{1t} = \beta \Delta y_{2t} + \Delta z_t$ and both the left and right hand side are characterised by the same integration properties.

When we introduce nonlinearities in the relationship linking y_{1t} and y_{2t}, however, the stochastic properties of the system become less obvious. Specifically, taking y_{2t} to be I(1) or equivalently difference stationary no longer implies that y_{1t} is also difference stationary. Indeed, it becomes straightforward to show that although the I(1)'ness of y_{2t} makes y_{1t} nonstationary this nonstationarity of y_{1t} can no longer be removed by first differencing. Differently put, although the variance of y_{1t} behaves in a manner similar to the variance of a random walk, first differencing y_{1t} will no longer make it stationary. More formally, if we take the first difference of the first equation in (12) and using the notation $I_t \equiv I(q_t > \gamma)$ we have

$$\begin{aligned} \Delta y_{1t} &= \beta \Delta y_{2t} + \theta \Delta (y_{2t} I_{t-1}) + \Delta z_t \\ &= \rho z_{t-1} + \theta y_{2t-1} \Delta I_{t-1} + \nu_t \end{aligned} \tag{13}$$

where $\nu_t = \theta \epsilon_{2t} I_{t-1} + \beta \epsilon_{2t} + u_t$. Clearly the presence of the term $y_{2t-1}\Delta I_t$, in the right hand side of (13) precludes the possibility of a traditional ECM type representation à la Granger. If we take q_t to be an iid process for instance it is straightforward to establish that $V(y_{2t-1}\Delta I_t) = 2F(\gamma)(1 - F(\gamma))(t-1)$. Similarly, y_{1t} cannot really be viewed as a difference stationary process as would have been the case within a linear framework. As dicussed in Granger, Inoue and Morin (1997), where the authors introduced a specification similar to (13), the correct but not directly operational form of the error correction model could be formulated as

$$\Delta y_{1t} - \theta y_{2t-1} \Delta I_{t-1} = \rho z_{t-1} + \nu_t$$

where now both the left and right hand side components are stationary. Practical tools and their theoretical properties for handling models such as the above are developed in Gonzalo and Pitarakis (2005a).

Our specification in (13) has highlighted the difficulties of handling switching phenomena within the cointegrating relationship itself if we want to operate within the traditional VECM framework.

It is also worth emphasising that similar conceptual difficulties will arise in non-VECM based approaches to the treatment of nonlinearities in cointegrating relationships. Writing $y_{1t} = \beta_t y_{2t} + u_t$, with y_{2t} an I(1) variable and u_t an I(0) error term, defines a stationary relationship between y_{1t} and y_{2t} which is not invalid per se. However, it would be inaccurate to refer to it as a cointegrating relationship linking two I(1) variables since y_{1t} cannot be difference stationary due to the time varying nature of β_t.

In summary, a system such as (12) which has a switching cointegrating vector cannot admit a VECM representation as in (4) in which both the left and right hand sides are balanced in the sense of both being stationary. Equivalently, for an I(1) vector to admit a formal VECM representation as in (4) it must be the case that the threshold effects are solely present in the adjustment process.

4.3 Rank Configuration under Alternative Stochastic Properties of Y_t

Our objective here is to further explore the correspondence between the rank characteristics of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ and the stability properties of Y_t akin to the well known relationship between the rank of the long run impact matrix of a linear VECM specification and its cointegration properties. We are interested for instance in the rank configurations of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ that are consistent with covariance stationarity of Y_t. Similarly, we also wish to explore the correspondence between the presence of threshold effects in the adjustment process of a cointegrated I(1) system and the rank configurations of the two long run impact matrices that are compatible with such a system.

Within a linear VECM specification, whose corresponding lag polynomial has roots either on or outside the unit circle, it is well known that having matrix $\mathbf{\Pi}$ that has full rank also implies that the underlying process is I(0). Although, from our result in proposition 4 it is straightforward to see that if both *or either* of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ have full rank then Y_t generated from (4) is going to be covariance stationary as well, it is also true that the full rank condition is not necessary for covariance stationarity. Our examples in (2) and (11), for instance, have illustrated the fact that two identical rank configurations, say $(r_1, r_2) = (1, 1)$ may be compatible with either a purely I(1) system as in (2) or a covariance stationary system as in (11). Similarly, example 3 with $(r_1, r_2) = (0, 2)$ illustrated the possibility of having a covariance stationary DGP in which either $\mathbf{\Pi}_1$ or $\mathbf{\Pi}_2$ have zero rank. These observations highlight the difficulties that may arise when attempting to clearly define the meaning of "nonlinear cointegration" when operating within an Error Correction type

of model.

Drawing from our analysis in section 4.2, if we take the à priori view that Y_t is I(1) and (4) is the correct specification it must then be the case that the rejection of the null hypothesis of linearity $H_0 : \mathbf{\Pi}_1 = \mathbf{\Pi}_2$ *directly* implies that we have threshold cointegration, here undestood to mean that the adjustment process has a threshold type nonlinearity driven by the external variable q_t while the cointegrating relationship itself is stable over time. Differently put, we can formulate $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ as $\mathbf{\Pi}_1 = \boldsymbol{\alpha}_1\boldsymbol{\beta}'$ and $\mathbf{\Pi}_2 = \boldsymbol{\alpha}_2\boldsymbol{\beta}'$.

At this stage it is also important to note that even under the maintained assumption that the cointegrating relationship itself is linear, and is not characterised by threshold effects, this does necessarily imply that $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ must have identical ranks. This feature of the system can be illustrated by considering our earlier example in (2) in which we set $\rho_1 = 0$ and $\rho_2 < 0$. This specific parameterisation implies, for instance, that $r_1 \equiv Rank(\mathbf{\Pi}_1) = 0$ and $r_2 \equiv Rank(\mathbf{\Pi}_2) = 1$. Alternatively, we could also have set $\rho_2 = 0$ and $\rho_1 < 0$ implying the rank configuration $(r_1, r_2) = (1, 0)$ within the same example. Obviously our system could also be characterised by a parameterisation such as $\rho_1 < 0$ and $\rho_2 < 0$ with a corresponding rank configuration given by $(r_1, r_2) = (1, 1)$ as in example 1.

Using our result in proposition 4, and our discussion above, it is straightforward to observe that within a sytem whose characteristic roots may lie either on or outside the complex unit circle (excluding roots that induce explosive behaviour) I(1)'ness with cointegration characterised by threshold adjustment may only occur if the rank configuration of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ is such that $(r_1, r_2) \in \{(0, 1), (1, 0), (1, 1)\}$. Note, however, that the scenario whereby $(r_1, r_2) = (1, 1)$ may also be compatible with a purely stationary Y_t as for instance in example 2 above with $\rho_{11} = 0$ and $\rho_{12} = 0$ among other possible configurations. At this stage it is also important to recall that within our operating framework cases involving processes that are integrated with an order higher than one are ruled out. The above observations are summarised more formally in the following proposition.

Proposition 5 *Letting $r_j \equiv Rank(\mathbf{\Pi}_j)$ for $j = 1, 2$ and assuming that $p = 2$, we have that (i) Y_t is covariance stationary if either r_1 or r_2 is equal to 2, (ii) Y_t is I(1) with threshold cointegration if $(r_1, r_2) = (0, 1)$ or $(r_1, r_2) = (1, 0)$, (iii) Y_t is either covariance stationary or I(1) with threshold cointegration if $r_1 = r_2 = 1$.*

According to the above proposition, even if at most one of the two long run impact matrices characterising the model in (4) is found to have full rank it must be that Y_t itself is covariance stationary. On the other hand if we have a rank configuration such as $(r_1, r_2) = (0, 1)$ or $(r_1, r_2) = (1, 0)$ then this would imply that Y_t described by (4) is I(1) and the model is characterised by threshold effects in its adjustment process towards its long run equilibrium. Intuitively, such a rank configuration captures the idea of an adjustment process that shuts off when the threshold variable q_t crosses above or below a certain magnitude given by γ. Finally, the case whereby $(r_1, r_2) = (1, 1)$ is compatible with either a purely covariance stationary system or an I(1) system with an underlying adjustment process characterised by different speeds of adjustment depending on the magnitude of q_t.

4.4 Estimation of r_1 and r_2

Having established the correspondence between alternative rank configurations and the stochastic properties of Y_t, our next objective is to estimate each individual rank r_1 and r_2. In what follows we will take the view that Y_t is known to be I(1), so that the rejection of the null hypothesis of linearity directly implies threshold effects in the adjustment process towards equilibrium. Furthermore, for the simplicity of the exposition, we will be assuming that the system under consideration is bivariate, setting $p = 2$ in (4). Thus we wish to decide whether $(r_1, r_2) = (0, 1)$, $(r_1, r_2) = (1, 0)$ or $(r_1, r_2) = (1, 1)$ in the true specification. Note that any other configuration of (r_1, r_2) would imply that Y_t is covariance stationary and is therefore ruled out by our operating framework.

Before introducing our proposed methodology for estimating r_1 and r_2 we define the following sample quantities. We let $\widehat{\Delta Y_1} = \Delta Y * I(q \leq \hat{\gamma})$, $\widehat{\Delta Y_2} = \Delta Y * I(q > \hat{\gamma})$ and $\hat{Z}_1$ and $\hat{Z}_2$ are as in (4) with γ replaced with its estimated counterpart $\hat{\gamma}$. The residual vector is obtained as $\hat{U} = \Delta Y - \hat{\Pi}_1 \hat{Z}_1 - \hat{\Pi}_2 \hat{Z}_2$ and we also define $\hat{U}_1 = \widehat{\Delta Y_1} - \hat{\Pi}_1 \hat{Z}_1$ and $\hat{U}_2 = \widehat{\Delta Y_2} - \hat{\Pi}_2 \hat{Z}_2$, from which we note the equality $\hat{\Omega} = \hat{\Omega}_1 + \hat{\Omega}_2$ where $\hat{\Omega}_1 = \hat{U}_1 \hat{U}_1' / T$, $\hat{\Omega}_2 = \hat{U}_2 \hat{U}_2' / T$ and $\hat{\Omega} = \hat{U} \hat{U}' / T$. For later

use we also introduce the following moment matrices corresponding to each regime j

$$\begin{aligned} S_{11}^{j} &= \frac{\hat{Z}_j \hat{Z}_j'}{T}, \\ S_{00}^{j} &= \frac{\widehat{\Delta Y_j}\widehat{\Delta Y_j}'}{T}, \\ S_{01}^{j} &= \frac{\widehat{\Delta Y_j}\hat{Z}_j'}{T}, \\ S_{10}^{j} &= (S_{01}^{j})' \end{aligned} \tag{14}$$

with $j = 1, 2$. Using (14) we can now reformulate the estimated covariance matrices as $\hat{\Omega}_j = S_{00}^{j} - S_{01}^{j}(S_{11}^{j})^{-1}S_{10}^{j}$ $j = 1, 2$, and for later use it will also be useful to note that the eigenvalues of $(S_{00}^{j})^{-1}S_{01}^{j}(S_{11}^{j})^{-1}S_{10}^{j}$ are the same as those of $I - (S_{00}^{j})^{-1}\hat{\Omega}_j$ for $j = 1, 2$.

We now propose to estimate the unknown ranks of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ using a model selection approach as introduced and investigated in Gonzalo and Pitarakis (1998, 1999, 2002). We view the problem of the estimation of r_1 and r_2 from a model selection perspective in which our main task is to select the optimal model among a portfolio of nested specifications. The selection is made via the optimisation of a penalised objective function. The latter has one component which decreases as the number of estimated parameters increases (e.g. as r_j increases) and another component that increases to penalise overfitting. The use of a model selection based approach for inferences similar to the above has been advocated in numerous related areas of the econometric literature. In Gonzalo and Pitarakis (2002), for instance, the authors explore the properties of a model selection based approach for estimating the number of regimes of a stationary time series characterised by threshold effects. In Cragg and Donald (1997), the authors used AIC and BIC type criteria for estimating the rank of a normally distributed matrix. Similarly, in Phillips and Chao (1999) the authors developed a new information theoretic criterion used to determine the rank and short run dynamics of an error correction models.

Formally, letting $\hat{\Omega}_j(r_j)$ denote the sample covariance matrices obtained from each regime characterising (4) under the restriction that $rank(\mathbf{\Pi}_j) = r_j$, our estimator of r_j is defined as

$$\hat{r}_j = \arg\min_{r_j} IC_j(r_j) \tag{15}$$

where

$$IC(r_j) = \ln|\hat{\Omega}_j(r_j)| + \frac{c_T}{T}m(r_j) \tag{16}$$

with $m(r_j)$ denoting the number of estimated parameters (here $m(r_j) = 2pr_j - r_j^2$) and c_T a deterministic penalty term. Next, using the fact that

$$\ln|\hat{\Omega}_j(r_j)| \quad = \quad \ln|S_{00}^j| + \sum_{i=1}^{r_j}(1-\hat{\lambda}_i^j) \tag{17}$$

and noting that S_{00}^j is independent of the magnitude of r_j, we can instead focus on the optimisation of the following modified criterion

$$\overline{IC}(r_j) = \sum_{i=1}^{r_j}\ln(1-\hat{\lambda}_i^j) + \frac{c_T}{T}(2pr_j - r_j^2). \tag{18}$$

A clear advantage of using (18) stems from the simplicity of its empirical implementation, requiring solely the availability of the eigenvalues of $I-(S_{00}^j)^{-1}\hat{\Omega}_j$ for $j=1,2$. It is also interesting to observe the close similarity between conducting inferences using (18) and, for instance, a formal likelihood ratio based testing procedure. Focusing on the estimation of r_1 for instance our model selection based approach involves selecting $\hat{r}_1 = 0$ as the optimal choice if $\overline{IC}(r_1=0) < \overline{IC}(r_1=1)$ and $\hat{r}_1 = 1$ if $\overline{IC}(r_1=1) < \overline{IC}(r_1=0)$. Equivalently, the model selection based approach points to $\hat{r}_1 = 1$ if $-T\ln(1-\hat{\lambda}_1^1) > 3c_T$ and to $\hat{r}_1 = 0$ otherwise under a bivariate setting. This is equivalent to the formulation of a likelihood ratio statistic for testing the null $H_0 : r_1 = 0$ against $H_1 : r_1 = 1$, except that here the decision rule is dictated by the magnitude of the penalty term and the number of estimated parameters. A formal distribution theory for an LR test based approach for the determination of r_1 and r_2 à la Johansen can be found in Gonzalo and Pitarakis (2005a). We next summarise the asymptotic properties of the model selection approach in the following proposition.

Proposition 6 *Letting r_j^0 denote the true rank of Π_j for $j=1,2$ and $\hat{r}_j$ defined as in (15), with c_T such that (i) $c_T \to \infty$ and (ii) $c_T/T \to 0$ as $T \to \infty$, we have $\hat{r}_j \xrightarrow{p} r_j^0$.*

The above proposition establishes the weak consistency of the rank estimators obtained through the model selection based approach. A possible candidate for the choice of the penalty term satisfying both (i) and (ii) is $c_t = \ln T$ corresponding to the well known BIC type criterion. It is clear, however, that other functionals of the sample size may be equally valid (e.g. $c_T = 2\ln\ln T$) making it difficult to argue in favour of a universally optimal criterion.

Having established the limiting properties of our rank estimators we next concentrate on their finite and large sample performance across a wide range of possible model configurations. Following

Gonzalo and Pitarakis (2002) we implement our experiments using $c_T = \ln T$ as the penalty term in (18).

We initially consider the DGP given in (2) under example 1. We have a bivariate system that is I(1) with a single cointegrating vector $(1, -\beta)$. We set $\beta = 2$ and consider $(\rho_1, \rho_2) = (0, -0.4)$ so that the system is characterised by a true rank configuration given by $(r_1, r_2) = (0, 1)$. In a second set of experiments we set $(\rho_1, \rho_2) = (-0.2, -0.6)$ so that this second system has $(r_1, r_2) = (1, 1)$. Our results are summarised in Table 5 below, which presents the decision frequencies for each possible magnitude of r_j. Throughout all our experiments q_t is assumed to follow the AR(1) process given by $q_t = 0.5q_{t-1} + \epsilon_t$ with $\epsilon_t = iid(0, 1)$ and the true threshold parameter is set at $\gamma_0 = 0$. As in our earlier experiments the delay parameter is set at $d = 1$ throughout.

Table 5: Decision Frequencies in an I(1) System

	$\hat{r}_1 = 0$	$\hat{r}_1 = 1$	$\hat{r}_1 = 2$	$\hat{r}_2 = 0$	$\hat{r}_2 = 1$	$\hat{r}_2 = 2$
	$(r_1^0 = 0, r_2^0 = 1), \beta = 2, (\rho_1, \rho_2) = (0.0, -0.4)$					
$T = 200$	85.26	14.74	0.00	0.00	100.00	0.00
$T = 400$	93.42	6.58	0.00	0.00	100.00	0.00
$T = 1000$	100.00	0.00	0.00	0.00	100.00	0.00
	$(r_1^0 = 1, r_2^0 = 1), \beta = 2, (\rho_1, \rho_2) = (-0.2, -0.6)$					
$T = 200$	34.76	65.24	0.00	0.02	99.98	0.00
$T = 400$	10.16	89.84	0.00	0.00	100.00	0.00
$T = 1000$	0.00	100.00	0.00	0.00	100.00	0.00
	$(r_1^0 = 1, r_2^0 = 0), \beta = 2, (\rho_1, \rho_2) = (-0.4, 0.0)$					
$T = 200$	0.02	99.98	0.00	84.76	15.24	0.00
$T = 400$	0.00	100.00	0.00	93.50	0.00	0.00
$T = 1000$	0.00	100.00	0.00	100.00	0.00	0.00

From the decision frequencies presented in Table 5 above it is clear that the proposed model selection procedure performs remarkably well across the three alternative specifications. As expected from our result in Proposition 6 it is pointing to the true magnitude of each rank 100% of the times under T=1000, while maintaining very high correct decision frequencies even under T=200. Under the specification in (2), for instance, with $(r_1^0, r_2^0) = (0, 1)$, the procedure picked $r_1 = 0$ about 85% of the times and $r_2 = 1$ 100% of the times under T=200, with the correct decision frequency increasing to about $(93\%, 100\%)$ under T=400.

To provide further empirical support for our proposed approach we next consider a set of threshold DGPs that restrict Y_t to be covariance stationary. For this purpose we have focused on the specification given in (3) under example 2 and considered two alternative rank configurations. First, imposing $(\rho_{11}, \rho_{12}) = (0, 0)$ and $(\rho_{21}, \rho_{22}) = (-0.2, -0.4)$ we have a covariance stationary system with $(r_1, r_2) = (0, 2)$. Second, setting $(\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) = (-0.4, 0.0, 0.0, -0.2)$ we have another covariance stationary system this time with $(r_1, r_2) = (1, 1)$. All simulation results are presented in Table 6 below.

Table 6: Decision Frequencies in a Stationary System

	$\hat{r}_1 = 0$	$\hat{r}_1 = 1$	$\hat{r}_1 = 2$	$\hat{r}_2 = 0$	$\hat{r}_2 = 1$	$\hat{r}_2 = 2$
	$(r_1^0 = 0, r_2^0 = 2), (\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) = (0.0, 0.0, -0.2, -0.4)$					
$T = 200$	88.36	10.24	1.40	0.00	0.00	100.00
$T = 400$	94.16	5.32	0.52	0.00	0.00	100.00
$T = 1000$	100.00	0.00	0.00	0.00	0.00	100.00
	$(r_1^0 = 1, r_2^0 = 1), (\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) = (-0.4, 0.0, 0.0, -0.2)$					
$T = 200$	0.00	86.90	13.10	0.56	86.94	12.50
$T = 400$	0.00	90.38	0.00	0.00	91.00	9.00
$T = 1000$	0.00	92.64	7.36	0.00	92.96	7.04

From the empirical decision frequencies presented above it is again the case that the various estimators of r_1 and r_2 point to their true counterparts as T is allowed to increase. Although the accuracy of the estimators is somehow determined by the DGP specific parameters it is also clear that under both experiments the frequency of pointing to the true rank is high, reaching levels ranging between 90 and 100% accuracy.

5 A Nonlinear Permanent and Transitory Decomposition

Having established the threshold cointegration properties of Y_t, we next investigate how this vector process of interest can be decomposed into a permanent and transitory component following the methodology developed in Gonzalo and Granger (1995).

Recall that in the linear case with Y_t following a VECM of the form $\Delta Y_t = \alpha\beta' Y_t + u_t$ we are

interested in decomposing the p-dimensional vector Y_t into two sets of components as

$$Y_t = A_1 f_t + \tilde{Y}_t \tag{19}$$

where A_1 is the $p \times (p-r)$ loading matrix, f_t the $(p-r) \times 1$ common I(1) factors and $\tilde{Y}_t$ is the I(0) component. The above decomposition of Y_t is such that the factors f_t are linear combinations of Y_t and $A_1 f_t$ and $\tilde{Y}_t$ form a Permanent-Transitory decomposition (see Gonzalo and Granger (1995) for the detailed definitions of each component).

As shown in Gonzalo and Granger (1995), the above two conditions are sufficient to identify the permanent and transitory components. Formally we can write

$$Y_t = A_1 f_t + A_2 z_t \tag{20}$$

with $f_t = \alpha_\perp Y_t$, $z_t = \beta' Y_t$ and $A_1 = \beta_\perp(\alpha'_\perp \beta_\perp)^{-1}$, $A_2 = \alpha(\beta'\alpha)^{-1}$. Note that $\alpha'_\perp \alpha = \beta'_\perp \beta = 0$.

Now, let us consider the following VECM with threshold effects

$$\Delta Y_t = \alpha_1 \beta' Y_{t-1} I(q_{t-d} \leq \gamma) + \alpha_2 \beta' Y_{t-1} I(q_{t-d} > \gamma) + u_t.$$

Following the same reasoning as in Gonzalo and Granger (1995) it is now straightforward to establish the following Threshold Permanent-Transitory decomposition for Y_t

$$Y_t = A_1 f_{1t} I(q_{t-d} \leq \gamma) + A_2 f_{2t} I(q_{t-d} > \gamma) + (A_3 I(q_{t-d} \leq \gamma) + A_4 I(q_{t-d} > \gamma) z_t \tag{21}$$

where $f_{1t} = \alpha'_{1\perp} Y_t$, $f_{2t} = \alpha_{2\perp} Y_t$ and $z_t = \beta' Y_t$. The corresponding loading matrices are then given by $A_1 = \beta_\perp(\alpha'_{1\perp}\beta_\perp)^{-1}$, $A_2 = \beta_\perp(\alpha'_{2\perp}\beta_\perp)^{-1}$ and similarly $A_3 = \alpha_1(\beta'\alpha_1)^{-1}$ and $A_4 = \alpha_2(\beta'\alpha_2)^{-1}$. Given our estimator of the threshold parameter γ defined in (9) together with the corresponding sample moment matrices introduced in (14), the practical implementation of the above Threshold Permanent and Transitory decomposition becomes straightforward (see Gonzalo and Pitarakis (2005b)) and is obtained following the same approach as in Gonzalo and Granger (1995).

Despite the representational complications that would arise if we were to also allow the cointegrating vector β to be characterised by the presence of threshold effects as say $\beta_t = \beta_1 I(q_{t-d} \leq \gamma) + \beta_2 I(q_{t-d} > \gamma)$ (see our discussion in section 4.2) the above threshold based decomposition would translate naturally to such a framework by reformulating it as $Y_t = A_1 f_{1t} I(q_{t-d} \leq$

$\gamma) + A_2 f_{2t} I(q_{t-d} > \gamma) + A_3 z_{1t} I(q_{t-d} \leq \gamma) + A_4 z_{2t} I(q_{t-d} > \gamma)$, with $z_{1t} = \beta_1' Y_t$, $z_{2t} = \beta_2' Y_t$. The corresponding loading matrices would then be given by $A_1 = \beta_{1\perp}(\alpha_{1\perp}'\beta_{1\perp})^{-1}$, $A_2 = \beta_{2\perp}(\alpha_{2\perp}'\beta_{2\perp})^{-1}$, $A_3 = \alpha_1(\beta_1'\alpha_1)^{-1}$ and $A_4 = \alpha_2(\beta_2'\alpha_2)^{-1}$.

6 Conclusions

This chapter has focused on the issue of introducing and testing for threshold type nonlinear behaviour into the conventional multivariate error correction model. The threshold nonlinearities we considered were driven by a stationary and external random variable triggering the regime switches. Within this context we obtained the limiting properties of a Wald type test statistic for testing for the presence of such threshold effects characterising the long run impact matrix of the VECM. An interesting property of the proposed test is its robustness to the presence or absence of unit roots in the system, displaying the same limiting null distribution under a wide range of stochastic properties of the system.

We subsequently proceeded with the interpretation and further analysis of the system following a rejection of the null hypothesis of linearity. We showed that cointegration as traditionally defined was compatible with such an error correction type specification only if the nonlinearities are present in the adjustment process rather than the long run equilibrium itself. We then introduced a model selection based approach designed to gain further insight into the stochastic properties of the system through the determination of the rank structure of the long run impact matrices characterising each regime. This then allowed us to extend the permanent and transitory decomposition of Gonzalo and Granger (1995) into a nonlinear permanent and transitory decomposition.

Much remains to be done in the area of nonlinear multivariate specifications such as the VAR/VECMs considered here. In this chapter for instance we restricted our analysis to models with no deterministic trends. Similarly our results also ignored the possibility of having such components together with the lagged dependent variables and cointegrating vectors display threshold switching behaviour. Extensions along these lines together with a formal representation theory for such models are topics currently being investigated by the authors.

APPENDIX

Lemma A1: *Under assumptions A1-A3 and Y_t a p-dimensional vector of I(0) variables we have as $T \to \infty$*

(a) $\frac{ZZ'}{T} \xrightarrow{p} Q \equiv E[ZZ']$,

(b) $\frac{Z_1 Z_1'}{T} \xrightarrow{p} F(\gamma)Q$, $\frac{Z_2 Z_2'}{T} \xrightarrow{p} (1-F(\gamma))Q$,

(d) $\frac{UZ'}{T} \xrightarrow{p} 0$, $\frac{UZ_j'}{T} \xrightarrow{p} 0$ *for* $j = 1, 2$,

(e) $\hat{\Omega}_u \xrightarrow{p} \Omega_u$.

where Q denotes a positive definite $p \times p$ matrix.

Proof: Under the stated assumptions parts (a) and (d) follow directly from the ergodic theorem. Parts (b) and (d) follow from Lemma 1 in Hansen (1996) and part (e) is obvious.

Lemma A2: *Letting $H_T(\gamma) \equiv \frac{1}{\sqrt{T}}(Z_1 \otimes I) vec\, U$, under assumptions A1-A3 and Y_t a p-dimensional vector of I(0) variables we have $H_T(\gamma) \Rightarrow H(\gamma)$ as $T \to \infty$, where $H(\gamma)$ is a zero mean gaussian process with covariance kernel $F(\gamma_1 \wedge \gamma_2)(Q \otimes \Omega_u)$.*

Proof: The use of the central limit therem for martingale differences applied to the sequence $\{Y_{t-1} u_t I(q_{t-d} \leq \gamma)\}$ leads to the required gaussianity for each $\gamma \in \Gamma$. This combined with the componentwise tightness of $H_T(\gamma)$ which follows from Hansen (1996, Theorem 1) leads to the desired result.

Proof of Proposition 1: From Lemma A1 it directly follows that

$$(Z_2 Z_2'/T)(ZZ'/T)^{-1}(Z_1 Z_1'/T) \otimes \hat{\Omega}_u^{-1} \quad \xrightarrow{p} \quad F(\gamma)(1-F(\gamma))Q \otimes \Omega_u^{-1} \tag{22}$$

and the Wald statistic in (6) can be formulated as

$$W_T(\gamma) \quad = \quad F(\gamma)(1-F(\gamma))\sqrt{T}(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2)'(Q \otimes \Omega_u^{-1})\sqrt{T}(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2) + o_p(1). \tag{23}$$

Standard least squares algebra together with Lemma A1 also imply

$$\begin{aligned}\sqrt{T}(\hat{\boldsymbol{\pi}}_1 - \boldsymbol{\pi}) &= \sqrt{T}[(Z_1 Z_1')^{-1} Z_1 \otimes I_p] vec\ U \\ &= \left[\left(\frac{Z_1 Z_1'}{T}\right)^{-1} \otimes I_p\right] \frac{1}{\sqrt{T}} (Z_1 \otimes I_p) vec\ U \\ &= \frac{1}{F(\gamma)} (Q^{-1} \otimes I_p) \frac{1}{\sqrt{T}} (Z_1 \otimes I_p) vec\ U + o_p(1) \qquad (24)\end{aligned}$$

and

$$\begin{aligned}\sqrt{T}(\hat{\boldsymbol{\pi}}_2 - \boldsymbol{\pi}) &= \left[\left(\frac{Z_2 Z_2'}{T}\right)^{-1} \otimes I_p\right] \frac{1}{\sqrt{T}} (Z_2 \otimes I_p) vec\ U \\ &= \frac{1}{(1 - F(\gamma))} (Q^{-1} \otimes I_p) \frac{1}{\sqrt{T}} (Z_2 \otimes I_p) vec\ U + o_p(1). \qquad (25)\end{aligned}$$

Combining (24) and (25) above and using the fact that $Z_2 = Z - Z_1$ we have

$$\sqrt{T}(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2) = \frac{(Q^{-1} \otimes I)}{F(\gamma)(1 - F(\gamma))} \left[\frac{1}{\sqrt{T}} (Z_1 \otimes I) vec\ U - F(\gamma) \frac{1}{\sqrt{T}} (Z \otimes I) vec\ U\right] + o_p(1). \qquad (26)$$

We can now write the Wald statistic as

$$\begin{aligned}W_T(\gamma) &= \left[\frac{1}{\sqrt{T}} (Z_1 \otimes I) vecU - F(\gamma) \frac{1}{\sqrt{T}} (Z \otimes I) vecU\right]' V(\gamma)^{-1} \\ &\quad \left[\frac{1}{\sqrt{T}} (Z_1 \otimes I) vecU - F(\gamma) \frac{1}{\sqrt{T}} (Z \otimes I) vecU\right] + o_p(1). \qquad (27)\end{aligned}$$

where $V(\gamma) = F(\gamma)(1 - F(\gamma))(Q \otimes \Omega_u)$. Next letting $G_T(\gamma) \equiv [(Z_1 \otimes I) vecU - F(\gamma)(Z \otimes I) vecU] / \sqrt{T}$, Lemmas A1-A2 together with the fact that $\frac{1}{\sqrt{T}} (Z \otimes I) vec\ U \xrightarrow{d} N(0, Q \otimes \Omega_u)$ which follows directly from the CLT imply $G_T(\gamma) \Rightarrow G(\gamma)$, where $G(\gamma)$ is a zero mean gaussian random vector with covariance $E[G(\gamma_1) G(\gamma_2)] = V(\gamma_1 \wedge \gamma_2) \equiv F(\gamma_1 \wedge \gamma_2)(1 - F(\gamma_1 \wedge \gamma_2))(Q \otimes \Omega_u)$. It now follows that the limiting distribution of the Wald statistic $W_T(\gamma)$ is given by $W_T(\gamma) \Rightarrow G(\gamma)' V(\gamma)^{-1} G(\gamma)$ and the final result follows from the continuous mapping theorem.

Lemma A3: *Under assumptions A1-A3 and Y_t a p-dimensional vector of I(1) variables with $\Delta Y = U$ we have as $T \to \infty$*

(a) $\dfrac{ZZ'}{T^2} \Rightarrow \displaystyle\int_0^1 W(r) W(r)' dr,$

(b) $\dfrac{Z_1 Z_1'}{T^2} \Rightarrow F(\gamma) \displaystyle\int_0^1 W(r) W(r)' dr,$

(c) $\dfrac{Z_2 Z_2'}{T^2} \Rightarrow (1 - F(\gamma)) \displaystyle\int_0^1 W(r) W(r)' dr$

where $W(r)' = (W_1(r), \ldots, W_p(r))$ *is a p-dimensional standard Brownian Motion.*

Proof: Part (a) follows directly from Phillips and Durlauf (1986). For part (b) we first write

$$\frac{Z_1 Z_1'}{T^2} = F(\gamma)\frac{ZZ'}{T^2} + \frac{W_1 W_1'}{T^2} \tag{28}$$

where $W_1 W_1'$ stacks the elements of the form $Y_{t-1}Y_{t-1}'(I(q_{t-d} \leq \gamma) - F(\gamma))$. It now suffices to show that $\frac{W_1 W_1'}{T^2} = o_p(1)$. We let $S_t = \sum_{i=1}^{t}(I(q_{t-1} \leq \gamma) - F(\gamma))$ and with no loss of generality set $d = 1$ and take zero initial conditions. Using summation by parts we can write $\sum_{t=1}^{T}(I(q_{t-1} \leq \gamma) - F(\gamma))Y_{t-1}Y_{t-1}' = S_{T-1}Y_T Y_T' - \sum_{t=1}^{T-1} S_t(Y_{t+1}Y_{t+1}' - Y_t Y_t')$. Next using the fact that $Y_{t+1}Y_{t+1}' = Y_t Y_t' + Y_t u_{t+1}' + u_{t+1}Y_t' + u_{t+1}u_{t+1}'$ we also have

$$\begin{aligned}\frac{1}{T^2}W_1 W_1' &= \frac{S_{T-1}}{T}\frac{Y_T Y_T'}{T} - \frac{1}{T^2}\sum_{t=1}^{T-1} Y_t u_{t+1}' S_t - \frac{1}{T^2}\sum_{t=1}^{T-1} u_{t+1}Y_t' S_t - \\ &\quad \frac{1}{T^2}\sum_{t=1}^{T-1}(u_{t+1}u_{t+1}' - \Omega_u)S_t - \frac{1}{T^2}\Omega_u \sum_{t=1}^{T-1} S_t. \end{aligned} \tag{29}$$

Under the maintained assumptions the ergodic theorem ensures that $S_{T-1}/T \xrightarrow{p} 0$. Since $Y_T Y_T'/T$ is stochastically bounded it thus follows that the first term in the right hand side of (29) is $o_p(1)$. Next, we consider the components $y_{it}u_{jt+1}S_t$. We have $E[y_{it}u_{jt+1}S_t] = 0$ and it is also straightforward to establish that

$$\lim_{T\to\infty} E\left[\frac{1}{T^2}\sum_{t=1}^{T-1} y_{it-1}u_{jt}S_t\right]^2 = 0$$

and both the second and third terms in the right hand side of (29) are also $o_p(1)$. Proceeding similarly, the third and fourth components can also be seen to be $o_p(1)$ and the final result follows from (a). Part (c) can be shown to hold in exactly the same manner as part (b).

Lemma A4: *Under assumptions A1-A3 and* Y_t *a p-dimensional vector of I(1) variables with* $\Delta Y = U$ *we have as* $T \to \infty$

(a) $\frac{1}{T}(Z \otimes I_p)vec\ U \Rightarrow vec\left[\int_0^1 dW(r)W(r)'\right]$,

(b) $\frac{1}{T}(Z_1 \otimes I_p)vec\ U \Rightarrow vec\left[\int_0^1 dW(r, F(\gamma))W(r)'\right]$

Proof: Part (a) follows directly from Phillips and Durlauf (1986). For part (b), the result follows from $L_T(\gamma) \equiv \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} u_t I(q_{t-1} \leq \gamma) \Rightarrow W(r, F(\gamma))$ where $W(r, F(\gamma))$ denotes a standard

Brownian Sheet (see Theorem 1 in Diebolt, Laib and Wandji (1997)) and Theorem 2 in Caner and Hansen (2001).

Proof of Proposition 2 We assume that the underlying null model is a pure unit root process as $\Delta Y = U$. Within the present I(1) framework we consider the following normalisation of the Wald statistic

$$T(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2)' \left[\left(\frac{Z_2 Z_2'}{T^2} \right) \left(\frac{ZZ'}{T^2} \right)^{-1} \left(\frac{Z_1 Z_1'}{T^2} \right) \otimes \hat{\Omega}_u^{-1} \right] T(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2).$$

and with no loss of generality in what follows we will impose $\Omega_u = I_p$. Next, from Lemma A3 it follows that

$$\left[\left(\frac{Z_2 Z_2'}{T^2} \right) \left(\frac{ZZ'}{T^2} \right)^{-1} \left(\frac{Z_1 Z_1'}{T^2} \right) \otimes \hat{\Omega}_u^{-1} \right] \quad \Rightarrow \quad F(\gamma)(1 - F(\gamma)) \int_0^1 W(r) W(r)' dr \otimes I_p. \tag{30}$$

and we formulate the test statistic of interest as

$$W_T(\gamma) = F(\gamma)(1 - F(\gamma)) T(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2)' \left[\int_0^1 W(r) W(r)' dr \otimes I_p \right] T(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2) + o_p(1).$$

We next focus on the large sample behaviour of $T(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2)$ when the true DGP is given by $\Delta Y = U$. We have

$$\begin{aligned} T\hat{\boldsymbol{\pi}}_1 &= \left[\left(\frac{Z_1 Z_1'}{T^2} \right)^{-1} \otimes I_p \right] \frac{1}{T} (Z_1 \otimes I_p) vec\, U \\ &= \frac{1}{F(\gamma)} \left[\left(\int_0^1 W(r) W(r)' dr \right)^{-1} \otimes I_p \right] \frac{1}{T} (Z_1 \otimes I_p) vec\, U + o_p(1). \end{aligned} \tag{31}$$

Proceeding similarly for $\hat{\boldsymbol{\pi}}_2$ and rearranging as above we have

$$T(\hat{\boldsymbol{\pi}}_1 - \hat{\boldsymbol{\pi}}_2) = \frac{1}{F(\gamma)(1 - F(\gamma))} \left[\left(\int_0^1 WW' \right)^{-1} \otimes I_p \right] \left[\frac{1}{T} (Z_1 \otimes I_p) vec\, U - F(\gamma) \frac{1}{T} (Z \otimes I_p) vec\, U \right]$$

Next, using Lemma A4 it follows that

$$\begin{aligned} \frac{1}{T}(Z_1 \otimes I) vec U - F(\gamma) \frac{1}{T} (Z \otimes I) vec U \quad &\Rightarrow \quad vec \left[\int_0^1 dW(r, F(\gamma)) W(r)' \right] - F(\gamma) vec \left[\int_0^1 dW(r, 1) W(r)' \right] \\ &= \quad vec \left[\int_0^1 [dW(r, F(\gamma)) - F(\gamma) dW(r, 1)] W(r)' \right] \\ &= \quad vec \left[\int_0^1 dK(r, F(\gamma)) W(r)' \right] \end{aligned} \tag{32}$$

where we let $K(r, F(\gamma)) = W(r, F(\gamma)) - F(\gamma)W(r, 1)$. Using the above in the expression of the Wald test statistic and rearranging we obtain the required result. The case for a cointegrated system follows along the same lines.

Proof of Proposition 3 From $\hat{U}(\gamma) = \Delta Y - \hat{\mathbf{\Pi}}_1 Z_1 - \hat{\mathbf{\Pi}}_2 Z_2 + U$ we can write

$$\begin{aligned}\hat{U}(\gamma)\hat{U}(\gamma)' &= (\Delta Y - \hat{\mathbf{\Pi}}_1 Z_1 - \hat{\mathbf{\Pi}}_2 Z_2)(\Delta Y' - Z_1'\hat{\mathbf{\Pi}}_1' - Z_2'\hat{\mathbf{\Pi}}_2') \\ &= \Delta Y \Delta Y' - \Delta Y Z_1'(Z_1 Z_1')^{-1} Z_1 \Delta Y' - \Delta Y Z_2'(Z_2 Z_2')^{-1} Z_2 \Delta Y' \qquad (33)\end{aligned}$$

where we made use of the fact that $Z_i Z_j' = 0 \ \forall i \neq j$ and $i, j = 1, 2$. Next, letting γ_0 denote the true threshold parameter we write the model evaluated at γ_0 as $\Delta Y = \mathbf{\Pi}_1 Z_1^0 + \mathbf{\Pi}_2 Z_2^0 + U$ where $Z_1^0 = (y_0 I(q_{0-d} \leq \gamma_0), \ldots, y_{T-1} I(q_{T-d} \leq \gamma_0))$ and $Z_2^0 = Z - Z_1^0$ with $Z_1^0 Z_2^{0\prime} = 0$. Inserting into (33) and rearranging gives

$$\begin{aligned}\hat{U}(\gamma)\hat{U}(\gamma)' = \mathbf{\Pi}_1 Z_1^0 Z_1^{0\prime}\mathbf{\Pi}_1' + \mathbf{\Pi}_2 Z_2^0 Z_2^{0\prime}\mathbf{\Pi}_2' + 2\mathbf{\Pi}_1 Z_1^0 U' + 2\mathbf{\Pi}_2 Z_2^0 U' + UU' - \mathbf{\Pi}_1 Z_1^0 M_1 Z_1^{0\prime}\mathbf{\Pi}_1' - \\ \mathbf{\Pi}_2 Z_2^0 M_1 Z_2^{0\prime}\mathbf{\Pi}_1' - 2\mathbf{\Pi}_1 Z_1^0 M_1 Z_2^{0\prime}\mathbf{\Pi}_2' - 2\mathbf{\Pi}_1 Z_1^0 M_1 U' - 2\mathbf{\Pi}_2 Z_2^0 M_1 U' - U M_1 U' - \mathbf{\Pi}_1 Z_1^0 M_2 Z_1^{0\prime}\mathbf{\Pi}_1' - \\ \mathbf{\Pi}_2 Z_2^0 M_2 Z_2^{0\prime}\mathbf{\Pi}_2' - 2\mathbf{\Pi}_1 Z_1^0 M_2 Z_2^{0\prime}\mathbf{\Pi}_2' - 2\mathbf{\Pi}_1 Z_1^0 M_2 U' - 2\mathbf{\Pi}_2 Z_2^0 M_2 U' - U M_2 U'\end{aligned}$$

where $M_1 = Z_1'(Z_1 Z_1')^{-1} Z_1$ and $M_2 = Z_2'(Z_2 Z_2')^{-1} Z_2$. We next evaluate the limiting behaviour of the above quantity for $\gamma < \gamma_0$, $\gamma = \gamma_0$ and $\gamma > \gamma_0$. Applying appropriate normalisations we obtain the following uniform convergence in probability result over $\gamma \in \Gamma$ for the case $\gamma < \gamma_0$

$$\begin{aligned}\frac{\hat{U}(\gamma)\hat{U}(\gamma)'}{T} &\xrightarrow{p} (\mathbf{\Pi}_1 - \mathbf{\Pi}_2)[(G(\gamma_0) - G(\gamma))(G - G(\gamma))^{-1}(G - G(\gamma_0))](\mathbf{\Pi}_1 - \mathbf{\Pi}_2)' + \Omega_u \\ &\equiv \frac{(F(\gamma_0) - F(\gamma))(1 - F(\gamma_0))}{1 - F(\gamma)}(\mathbf{\Pi}_1 - \mathbf{\Pi}_2) Q (\mathbf{\Pi}_1 - \mathbf{\Pi}_2)' + \Omega_u. \qquad (34)\end{aligned}$$

Proceeding similarly for the case $\gamma > \gamma_0$ we have

$$\begin{aligned}\frac{\hat{U}(\gamma)\hat{U}(\gamma)'}{T} &\xrightarrow{p} (\mathbf{\Pi}_1 - \mathbf{\Pi}_2)[G(\gamma_0)G(\gamma)^{-1}(G(\gamma) - G(\gamma_0))](\mathbf{\Pi}_1 - \mathbf{\Pi}_2)' + \Omega_u \\ &\equiv \frac{F(\gamma_0)(F(\gamma) - F(\gamma_0))}{F(\gamma)}(\mathbf{\Pi}_1 - \mathbf{\Pi}_2) Q (\mathbf{\Pi}_1 - \mathbf{\Pi}_2)' + \Omega_u.\end{aligned}$$

Finally with

$$\frac{\hat{U}(\gamma_0)\hat{U}(\gamma_0)'}{T} \xrightarrow{p} \Omega_u$$

we have that the objective function converges uniformly in probability to a nonstochastic limit that is uniquely minimised at $\gamma = \gamma_0$ and the required result follows from Theorem 2.1 in Newey and McFadden (1994).

Proof of Proposition 4: We are interested in the covariance stationarity of the stochastic recurrence given by $Y_t = \mathbf{\Phi}_1 Y_{t-1} I_{1t-d} + \mathbf{\Phi}_2 Y_{t-1} I_{2t-d} + u_t$ where we use the notation $I_{1t-d} \equiv I(q_{t-d} \leq \gamma)$ and $I_{2t-d} \equiv I(q_{t-d} > \gamma)$. Note first that given assumption A2 we have $E[Y_{t-1} I_{1t-d}] = E[I_{1t-d}]E[Y_{t-1}] = F(\gamma)E[Y_{t-1}]$ and $E[Y_{t-1} I_{2t-1}] = (1 - F(\gamma))E[Y_{t-1}]$. With Y_t denoting a solution to the stochastic recurrence we have $\forall t$ $E[Y_t] = 0$ and

$$\begin{aligned} E[Y_t Y_t'] &= E[\mathbf{\Phi}_1 Y_{t-1} Y_{t-1}' \mathbf{\Phi}_1 I_{1t-1}] + E[\mathbf{\Phi}_2 Y_{t-1} Y_{t-1}' \mathbf{\Phi}_2 I_{2t-1}] + E[u_t u_t'] \\ &= F(\gamma)\mathbf{\Phi}_1 E[Y_{t-1} Y_{t-1}']\mathbf{\Phi}_1 + (1 - F(\gamma))\mathbf{\Phi}_2 E[Y_{t-1} Y_{t-1}']\mathbf{\Phi}_2 + \Omega_u. \end{aligned}$$

Letting $V_t = E[Y_t Y_t']$ the above stochastic difference equation can be written more compactly as $V_t = F(\gamma)\mathbf{\Phi}_1 V_{t-1}\mathbf{\Phi}_1' + (1 - F(\gamma))\mathbf{\Phi}_2 V_{t-1}\mathbf{\Phi}_2' + \Omega_u$. Next, vectorising both sides and letting $v_t \equiv vec(V_t)$ and $\omega \equiv vec(\Omega_u)$ we have $v_t = [F(\gamma)(\mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1) + (1 - F(\gamma))(\mathbf{\Phi}_2 \otimes \mathbf{\Phi}_2)]\, v_{t-1} + \omega$. For Y_t to be covariance stationary it is thus necessary that V_t converges and this is ensured by the requirement that $\boldsymbol{\rho}(F(\gamma)(\mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1) + (1 - F(\gamma))(\mathbf{\Phi}_2 \otimes \mathbf{\Phi}_2)) < 1$. Following the same line of proof as in Brandt (1986) and Karlsen (1990) it is also straightforward to establish that if $\boldsymbol{\rho}(F(\gamma)(\mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1) + (1 - F(\gamma))(\mathbf{\Phi}_2 \otimes \mathbf{\Phi}_2)) < 1$ then the above stochastic recurrence admits a unique covariance stationary solution. We can thus conclude that the above threshold VAR admits a unique covariance stationary solution *if and only if* $\boldsymbol{\rho}(F(\gamma)(\mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1) + (1 - F(\gamma))(\mathbf{\Phi}_2 \otimes \mathbf{\Phi}_2)) < 1$.

Proof of Proposition 6: We first consider the case $r_j > r^0$ and establish that under the stated conditions $P[\overline{IC}(r_j) < \overline{IC}(r^0)] \to 0$ as $T \to \infty$. From the definition of $\overline{IC}(.)$ in (18) we have $P[\overline{IC}(r_j) < \overline{IC}(r^0)] = P[-T \sum_{i=r^0+1}^{r_j} \ln(1 - \hat{\lambda}_i^j) > c_T(2pr_j - 2pr^0 - r_j^2 + (r^0)^2)]$. Since $-T \sum_{i=r^0+1}^{r_j} \ln(1 - \hat{\lambda}_i^j)$ is $O_p(1)$ and the right hand side diverges towards infinity we have that $\lim_{T\to\infty} P[\overline{IC}(r_j) < \overline{IC}(r^0)] = 0$ and thus the procedure does not overrank asymptotically. For the case $r_j < r^0$ we have $P[\overline{IC}(r_j) < \overline{IC}(r^0)] = P[\sum_{i=r_j+1}^{r^0} \ln(1 - \hat{\lambda}_i^j) < \frac{c_T}{T}(2pr^0 - (r^0)^2 + r_j^2 - 2pr_j)]$. Since $-\sum_{i=r_j+1}^{r^0} \ln(1 - \hat{\lambda}_i^j) \xrightarrow{p} \theta > 0$ and $\frac{c_T}{T} \to 0$ it follows that for $r_j < r^0$, $\lim_{T\to\infty} P[\overline{IC}(r_j) < \overline{IC}(r^0)] = 0$ as required.

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