UNIVERSITY OF Southampton
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Simplicity and intricacy of the propagation of uncertainty in linear structural dynamic systems

ISVR Engineering Research Seminar Series

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Linear systems

- Most results presented here are valid for any linear system,

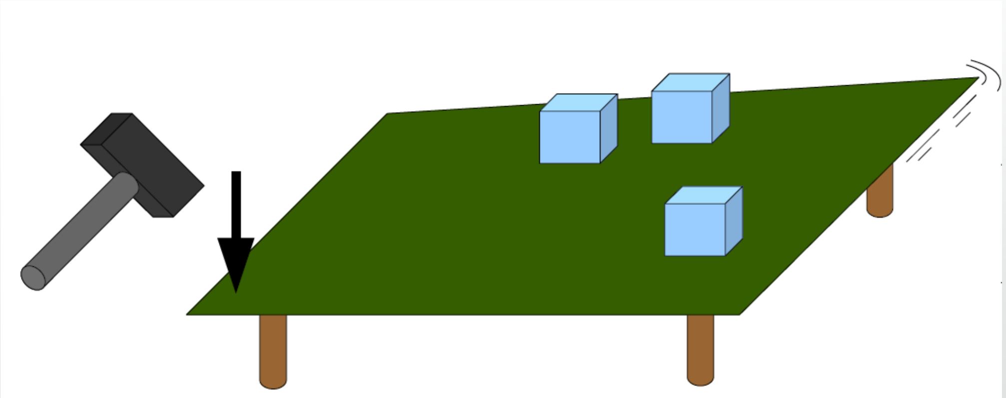
$$Ax=f$$

  - Linearity between input or force vector, f, output or response vector, x, is represented by stiffness matrix, A

- **Main question:** How does the response x change if the system matrix A changes?

$$[A-sD]x(s)=f$$

  - D deterministic,
  - s random, p(s) known



Linear structural dynamics systems

- The matrices and vectors can be parameterised. We use the frequency parameter $\omega=2\pi f$.

- We consider responses, $x(\omega)$, and transfer functions, $g(\omega)$:

$$[A(\omega)-sD(\omega)]\, x(\omega,s) = f$$

$$g(\omega,s)=c^T x(\omega, s)$$

- Can be ANY linear structural dynamic system, e.g.

$$A(\omega)=K(1+i\zeta)-\omega^2 M$$

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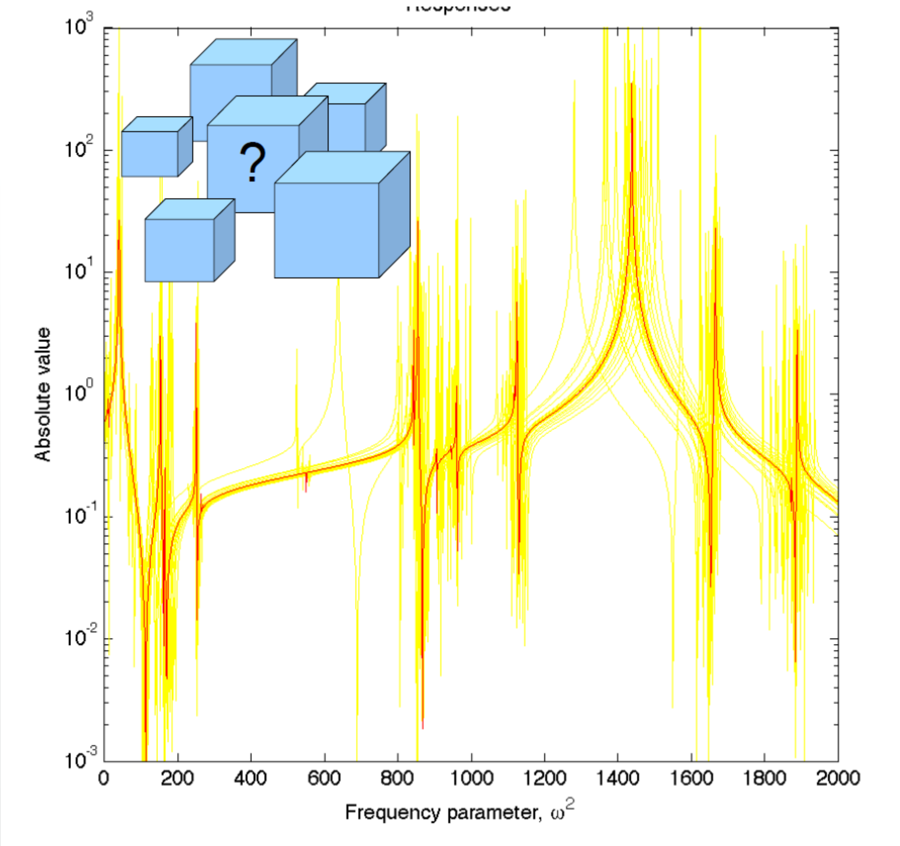
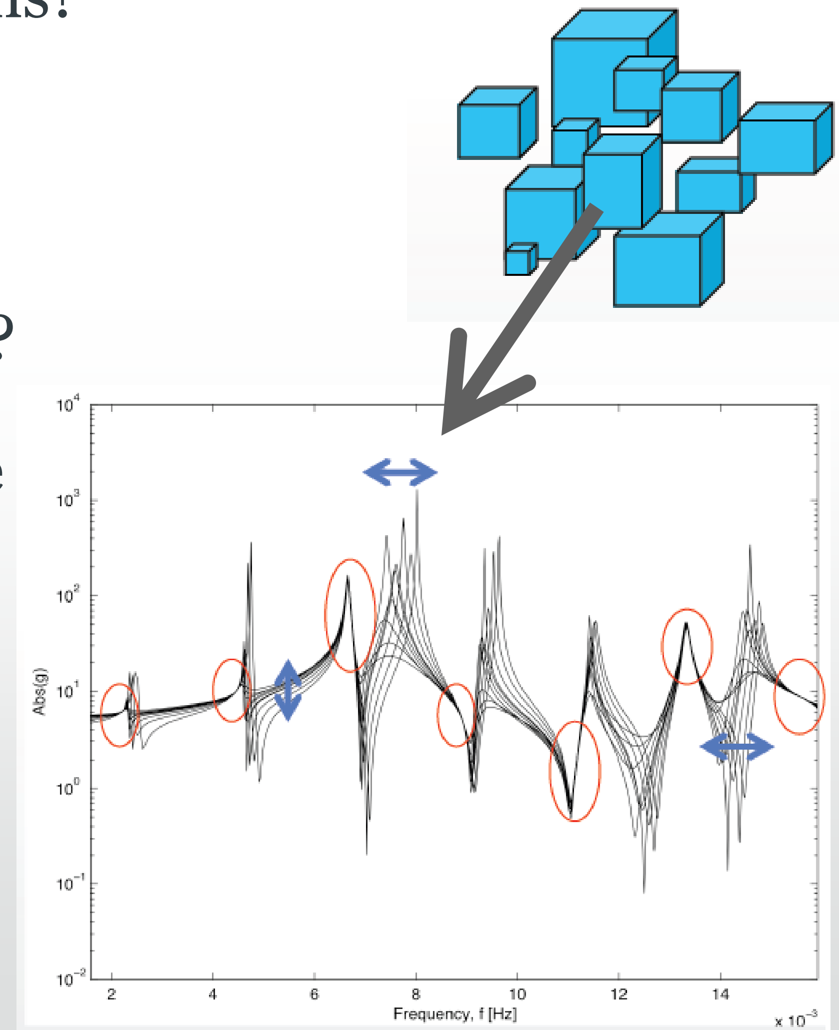


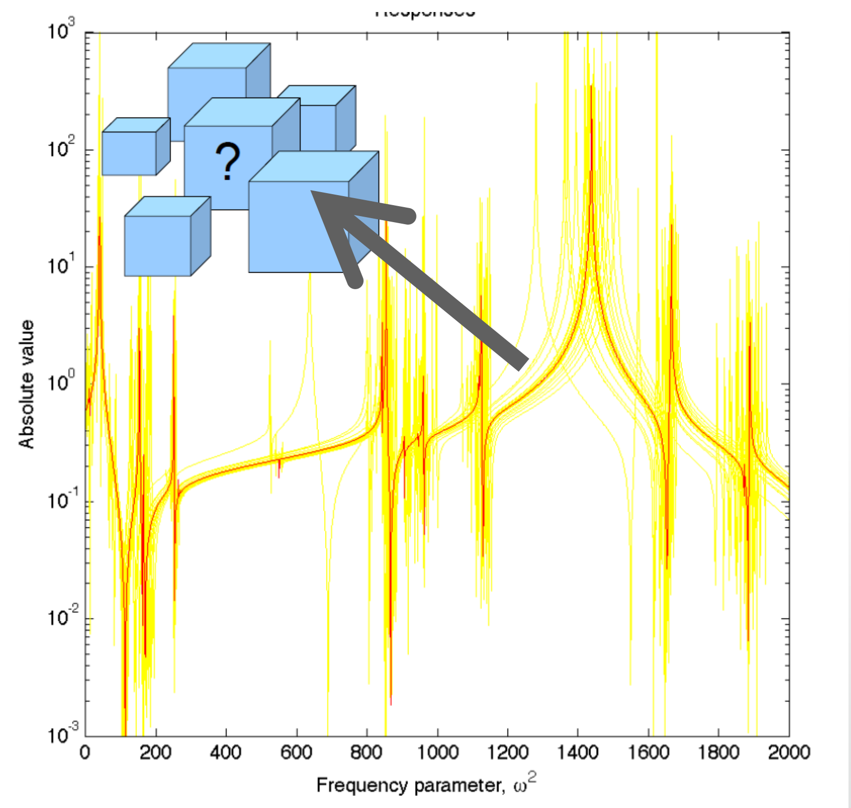
Uncertainty propagation (forward/backward)

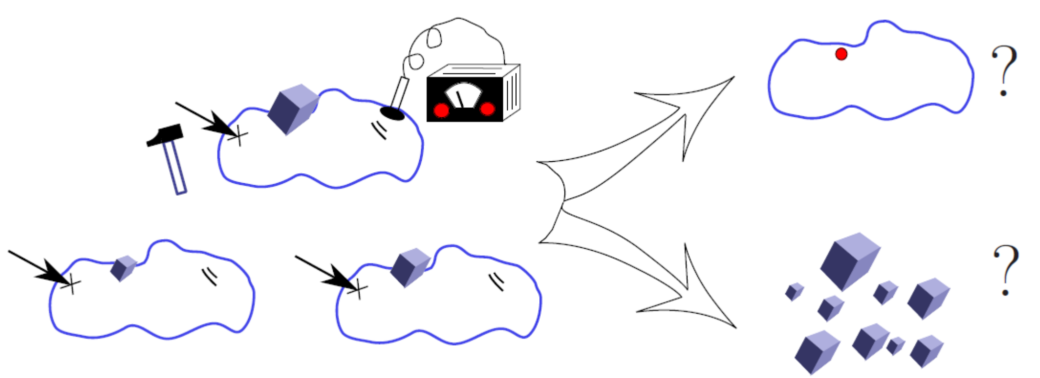
- Where do the poles go? Where do the zeros go? What is the effect on the transfer functions?
- Are there invariants?
- Average, variance, covariance of transfer function, poles, zeros?
- Is the variance the right measure of variability? Extreme values? Probability density functions?
- Effect of kind of randomness? (different distribution of s)
- Effect of frequency range?



Uncertainty propagation (forward/backward)

- What can be inferred of the system from `*measurements*'?
- Can one identify the randomness?
  Can one identify the nominal system despite the randomness?
- Effect of error in measurements?
- Any particular frequencies?
- Combination of frequencies,





Some `common' options for the forward analysis

- **Monte-Carlo simulations**
  - For each sampled value, $s = s_j$, exact expression

$$x(\omega, s_j) = [A(\omega) - s_j D(\omega)]^{-1} f$$

  - **Positive:** Statistics can be evaluated (approximated)
    Any pdf p(s) can be considered
  - **Negative:** Cost may be large for each s_j (Inversion/solution with large matrix)
    Might need millions (billions) of simulations
    No real assurance of `convergence'
    Harder to understand what is happening

- **Perturbation approach**

Some `common' options (continued)

- **Perturbation approach** (Recall [A(ω)-sD(ω)] x(ω,s) = f)

$$[\mathbf{A}(\omega) + s\mathbf{D}(\omega)]\left[\mathbf{x}(\omega,0) + s\Delta^{(1)}\mathbf{x}(\omega,0) + s^2\Delta^{(2)}\mathbf{x}(\omega,0) + \ldots\right] = f$$

$$s^0: \qquad \mathbf{A}(\omega)\mathbf{x}(\omega,0) = f$$

$$s^1: \qquad \mathbf{A}(\omega)\Delta^{(1)}\mathbf{x}(\omega,0) + \mathbf{D}(\omega)\mathbf{x}(\omega,0) = 0$$

$$\Delta^{(1)}\mathbf{x}(\omega,0) = -\left[\mathbf{A}(\omega)^{-1}\mathbf{D}(\omega)\right]\mathbf{x}(\omega,0)$$

$$s^2: \qquad \mathbf{A}(\omega)\Delta^{(2)}\mathbf{x}(\omega,0) + \mathbf{D}(\omega)\Delta^{(1)}\mathbf{x}(\omega,0) = 0$$

$$\Delta^{(2)}\mathbf{x}(\omega,0) = -\left[\mathbf{A}(\omega)^{-1}\mathbf{D}(\omega)\right]\Delta^{(1)}\mathbf{x}(\omega,0)$$

$$\mathbf{x}(\omega,s) \approx \left\{\mathbf{I} - s\left[\mathbf{A}(\omega)^{-1}\mathbf{D}(\omega)\right] + s^2\left[\mathbf{A}(\omega)^{-1}\mathbf{D}(\omega)\right]^2\right\}\mathbf{x}(\omega,0)$$

- **Positive:** Only solutions are x(ω,0) = A(ω) $^{-1}$ f and A(ω) $^{-1}$D(ω)

  Can then be used for <u>any</u> value of s

  Statistics easily approximated

$$\hat{\mathbf{x}}(\omega,s) \approx \left\{\mathbf{I} - \hat{s}\left[\mathbf{A}(\omega)^{-1}\mathbf{D}(\omega)\right] + E[s^2]\left[\mathbf{A}(\omega)^{-1}\mathbf{D}(\omega)\right]^2\right\}\mathbf{x}(\omega,0)$$

Some `common' options (continued)

- **Eigen sensitivity** (Disturbed eig.: [A(ω_j(s))-sD(ω_j(s))] ω_j(s) = 0)

  1. Expand the eigenvalues and eigenvectors

$$\omega_j(s) = \omega_j(0) + s\Delta^{(1)}\omega_j(0) + s^2\Delta^{(2)}\omega_j(0) + \dots$$
$$\phi_j(s) = \phi_j(0) + s\Delta^{(1)}\phi_j(0) + s^2\Delta^{(2)}\phi_j(0) + \dots$$

  2. Substitute in [A(ω_j(s))-sD(ω_j(s))] ω_j(s) = 0
  3. Solve for orders 0, 1, 2, etc.

- **Positive:** Only solutions are x(ω,0) = A(ω) $^{-1}$ f and A(ω) $^{-1}$D(ω)
  Can then be used for <u>any</u> value of s
  Statistics easily approximated

- **Negative:** What about large disturbance?

Low rank approach

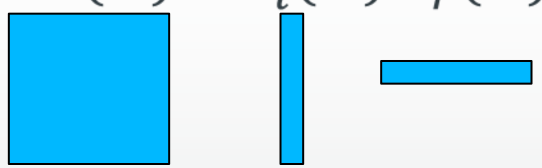
- Based on the facts that

  1. **<u>If</u>** the **<u>nominal</u>** (independent on s) solutions of $x(\omega,0) = A(\omega)^{-1} f$ and $A(\omega)^{-1} D(\omega)$ are available,

     <u>then</u> there is an **<u>exact</u>** expression of the response (and therefore transfer function) using these solutions for **<u>any</u>** value of s.

  2. **<u>If</u>** the disturbance matrix is **<u>low rank</u>**,

     $$D(\omega) = D_l(\omega) D_r(\omega)^T$$

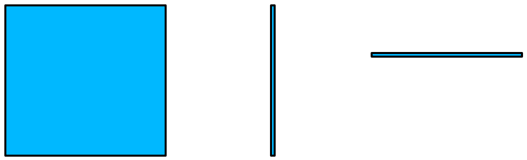
     

     the **<u>exact</u>** expression **<u>further simplifies</u>**:

Low rank approach (continued)

If the disturbance matrix is **rank-one**, i.e. if $D_l(\omega)$ and $D_r(\omega)$ are vectors $d_l(\omega)$ and $d_r(\omega)$,

$$D(\omega) = d_l(\omega) d_r(\omega)^T$$



$$s_1(\omega) = \left[\boldsymbol{d}_r(\omega)^T \boldsymbol{A}(\omega)^{-1} \boldsymbol{d}_l(\omega) \right]^{-1}$$

then the exact update expression is

$$\boldsymbol{x}(\omega, s) = \boldsymbol{x}(\omega, 0) + \boxed{s_1(\omega) \boldsymbol{A}(\omega)^{-1} \boldsymbol{d}_l(\omega) \boldsymbol{d}_r(\omega)^T \boldsymbol{A}(\omega)^{-1} \boldsymbol{f} \left[\frac{s_1(\omega)}{s_1(\omega) - s} - 1 \right]}$$

$\boldsymbol{A}(\omega)^{-1} \boldsymbol{f}$

$\boldsymbol{A}(\omega)^{-1} \boldsymbol{d}_l(\omega)$

Only occurrence

Low rank statistics

- We can evaluate exactly all average and variance or covariance of the response and transfer function
- Average: $\hat{\boldsymbol{x}}(\omega)=\int_{-\infty}^{\infty} \boldsymbol{x}(\omega,s)\,p(s)\,ds$ (by definition)

$$\hat{\boldsymbol{x}}(\omega)=\boldsymbol{x}(\omega,0)+\boxed{s_1(\omega)\,\boldsymbol{A}(\omega)^{-1}\boldsymbol{d}_l(\omega)\,\boldsymbol{d}_r(\omega)^T\boldsymbol{A}(\omega)^{-1}\boldsymbol{f}\left[e_1(\omega)-1\right]}$$

It only requires *first* stochastic coefficient

$$\boxed{e_1(\omega)=\int_{-\infty}^{\infty}\frac{s_1(\omega)}{s_1(\omega)-s}\,p(s)\,ds}$$

- Variance/covariance:

$$var\left[\boldsymbol{x}_A(\omega_A,s),\boldsymbol{x}_B(\omega_B,s)\right]=\boldsymbol{B}_1(\omega_A)\,\boldsymbol{f}_A\boldsymbol{f}_B^H\,\boldsymbol{B}_1(\omega_B)^H\left[v_{11}(\omega_A,\omega_B)-e_1(\omega_A)\bar{e}_1(\omega_B)\right]$$

Notation: $\boldsymbol{B}(\omega)=s_1(\omega)\,\boldsymbol{A}(\omega)^{-1}\boldsymbol{d}_l(\omega)\,\boldsymbol{d}_r(\omega)^T\boldsymbol{A}(\omega)^{-1}$

It only requires *second stochastic coefficient*
(Not always necessary...)

$$\boxed{q_{11}(\omega)=-s_1(\omega)^2\int_{-\infty}^{\infty}\left[\frac{1}{s_1(\omega)-s}\right]\frac{\partial p(s)}{\partial s}\,ds}$$

- Once we have evaluated these two simple integrals, we have EXACT expressions of all averages and covariances... `*Simple integrals*'...?

Stochastic coefficients

- More of these integrals are required in the multi-rank case

$$e_j(\omega) = \int_{-\infty}^{\infty} \frac{s_j(\omega)}{s_j(\omega) - s} p(s)\, ds \qquad j = 1, \dots, n$$

$$q_{jj}(\omega) = -s_j(\omega)^2 \int_{-\infty}^{\infty} \left[\frac{1}{s_j(\omega) - s} \right] \frac{\partial p(s)}{\partial s} ds$$

- They depend on the probability density function $p(s)$
- Note that we assumed s real (integrals from $-\infty$ to ∞ and change of variable), which is not a necessity
- They are actually difficult integrals… (If one wants to evaluate them correctly)

Stochastic coefficients (normal real variable)

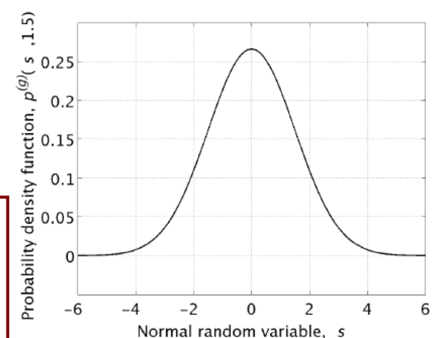
- We have an exact expression of the two coefficients!
- It can be evaluated cheaply and accurately (using last century work)!
- Second coefficient directly from first coefficient (simple).

- If s is a normal variable, with zero mean and standard deviation, σ, its probability density function is the Gaussian function,

$$p^{(g)}(s,\sigma)=\sqrt{\frac{1}{2\pi\sigma^2}}\,e^{-\frac{s^2}{2\sigma^2}}$$

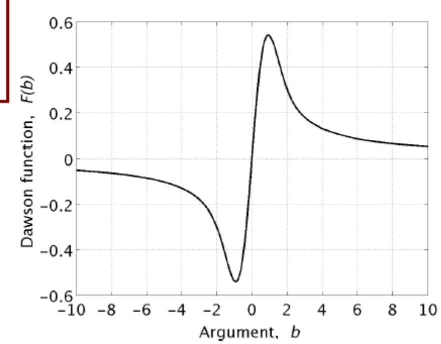
- The stochastic coefficients are then (exactly)

$$\boxed{\begin{aligned}
&e_j^{(g)}(\omega,\sigma)=-i\sqrt{\frac{\pi}{2}}\frac{s_j(\omega)}{\sigma}\,w\!\left(\frac{s_j(\omega)}{\sqrt{2}\,\sigma}\right) && \text{if}\quad \Im\left(s_j(\omega)\right)>0\\
&e_j^{(g)}(\omega,\sigma)\overset{\text{(P.V.)}}{=}\sqrt{2}\,\frac{s_j(\omega)}{\sigma}\,F\!\left(\frac{s_j(\omega)}{\sqrt{2}\,\sigma}\right) && \text{if}\quad \Im\left(s_j(\omega)\right)=0\\
&q_{jj}^{(g)}(\omega,\sigma)=\frac{s_j(\omega)^2}{\sigma^2}\left(e_j^{(g)}(\omega,\sigma)-1\right)
\end{aligned}}$$



- The special Faddeeva and Dawson's functions are

$$w(b)=e^{-b^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^b e^{t^2}\,dt\right)\qquad F(b)=e^{-b^2}\int_0^b e^{t^2}\,dt$$



Error function family (See *Abramowitz and Stegun*)

Other questions…

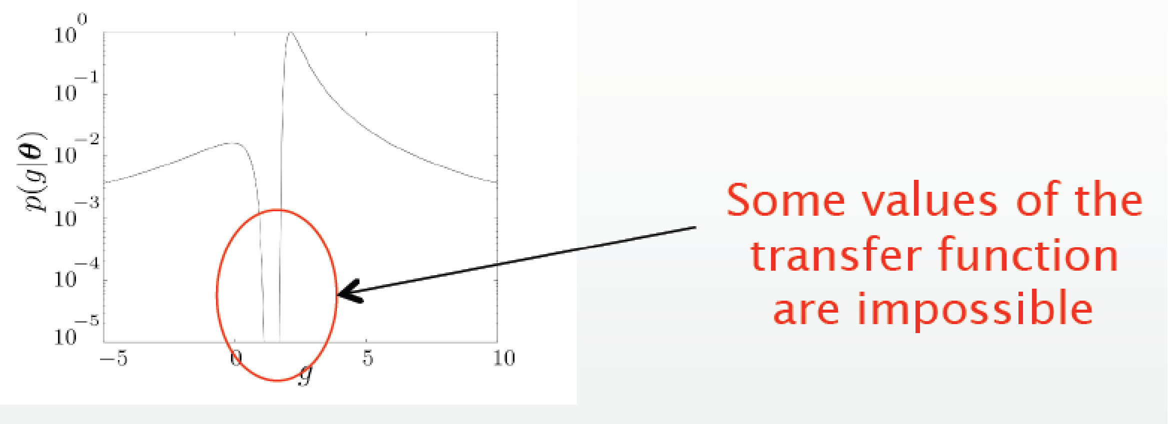
- What about a complex random variable, $s = s_r + \mathrm{i}\, s_i$?
  - Is the evaluation of the stochastic coefficients simple or difficult?
  - In the case of a normal complex variable, for example, are the real and complex stochastic coefficients related?
- What about other kinds of random variables?
- Are the average and variance of the responses and transfer functions the right information?
- Would the evaluation of the whole probability density functions of the transfer functions be simple of difficult?
- What about the statistics of the location of poles and zeros?

Some answers…

- About a complex random variable, $s = s_r + \mathrm{i}\, s_i$, with normal distribution…
  - If the normal variable is 'symmetric':
    - the real and complex stochastic coefficients are not (directly) related
    - the stochastic coefficients have very simple exact expression (but they are not necessarily easy to find… It took me some time.). Much simpler to evaluate than in the real case.
  - If the normal variable is not 'symmetric' (added spring with small damping):
    - The stochastic coefficients appear to be relatively complicated (though interesting) functions (I am not sure what they are. There is *some* link with continued fractions).

Some answers… (continued)

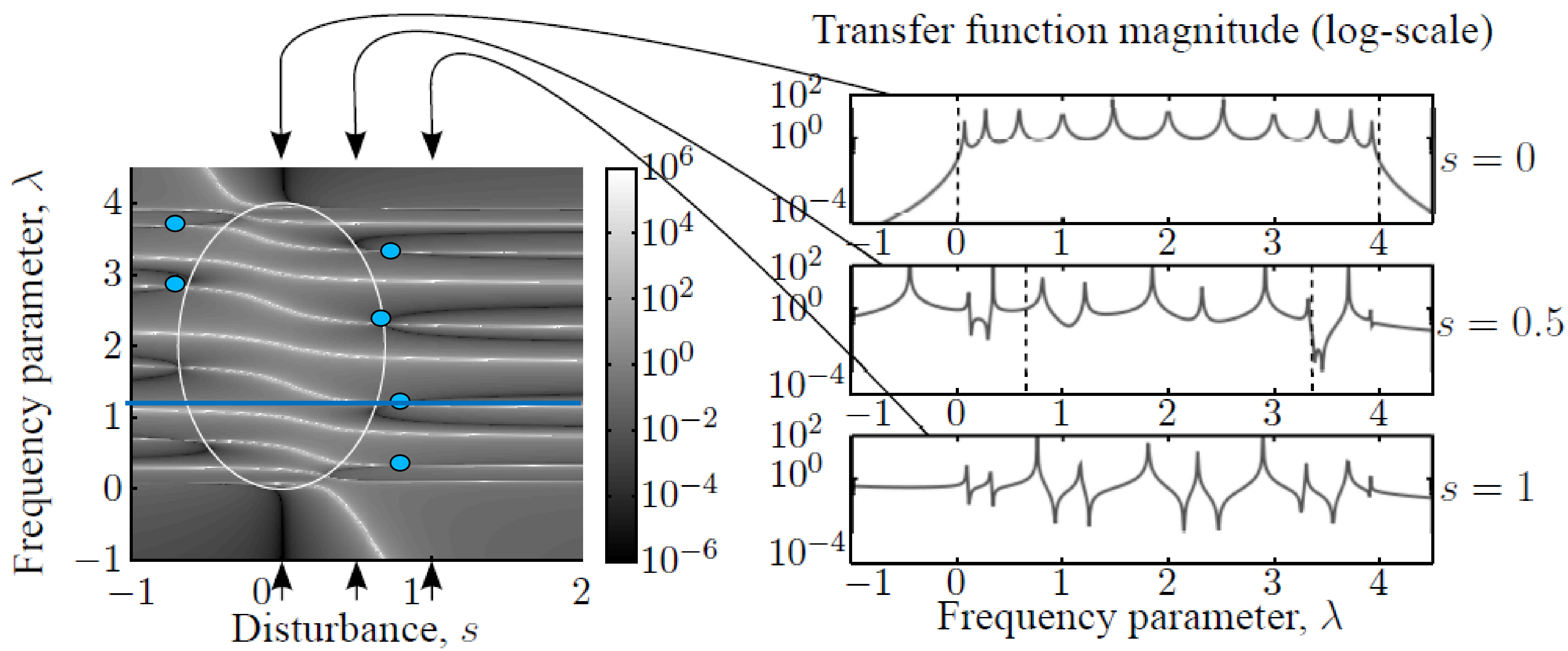
- About probability density functions of transfer functions:
  - Exact expression is available (for any random variable)
  - It has interesting properties



- About the statistics of the location of poles and zeros?
  - It appears (I am not sure) that their statistics (average and variance) cannot be evaluated as easily as those of the responses and transfer functions
  - A statistical estimation of the variance of the poles (i.e. of the robustness of the system) is possible. Exactly? I do not know yet.

Some answers… (continued)

- About (some of the) interesting patterns and invariants of the transfer functions…



- Invariants correspond to intersection of poles (whiter lines) and zeros (darker lines) in the frequency parameter – disturbance magnitude plane (need check of proof…).
- Note the ellipse in which there is no zero… (It is a property of the particular system)

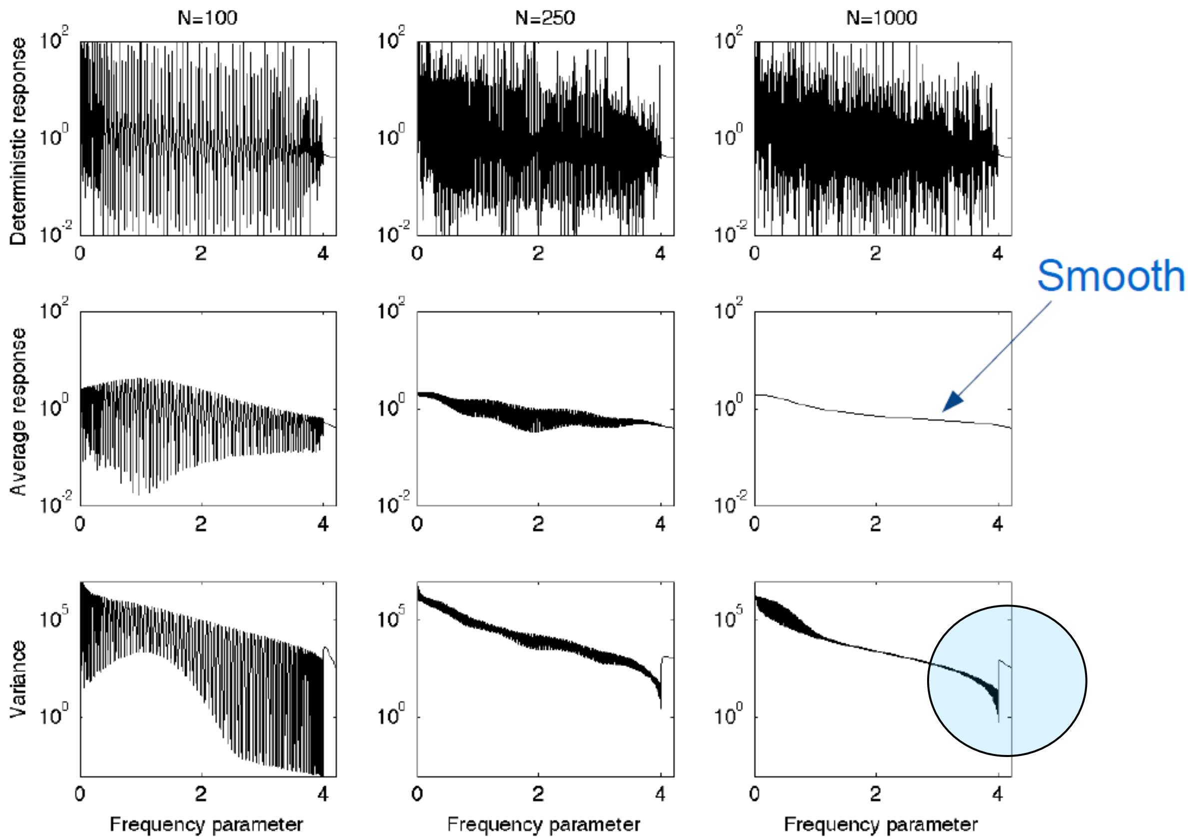
Some answers… (continued)

- About applicability and robustness: All appears fine. Monte-Carlo simulations tend to the evaluated exact expressions (*billions* of simulations if low damping…).
- Example: Application from low- to mid- and high-frequency range.

Nominal transfer functions

Exact average
(for given multi-rank
disturbance)
Evaluated using nominal
Transfer functions

Exact variance

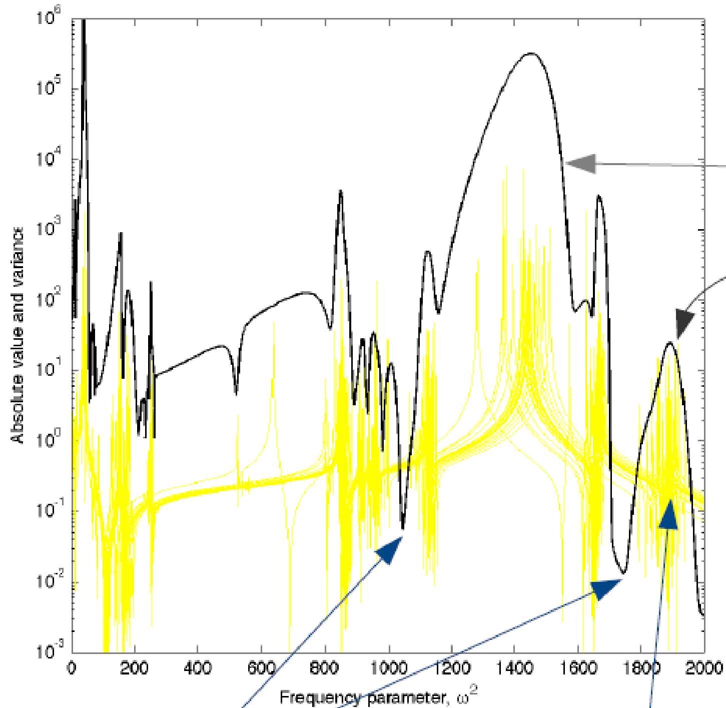


Some answers... (continued)

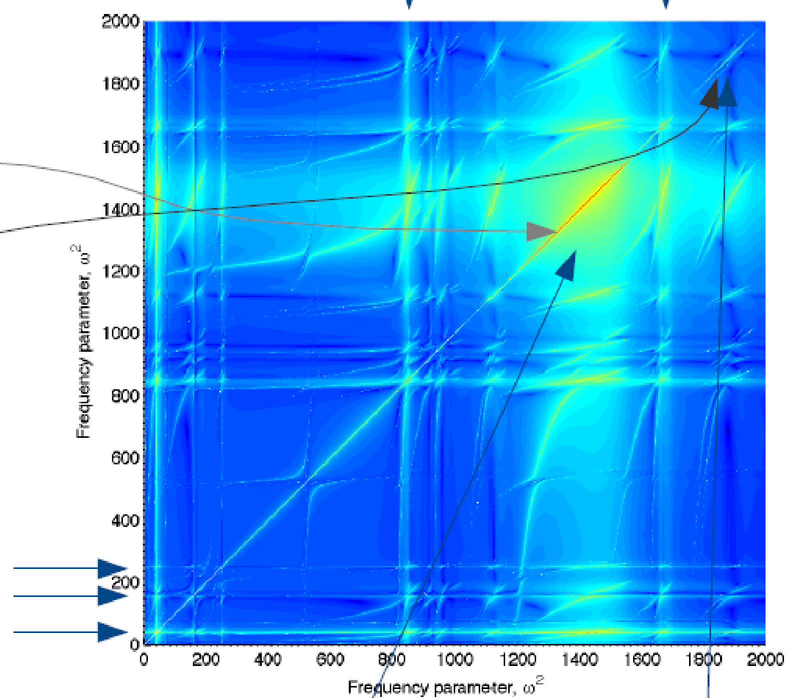
- Variance and covariance (between different frequencies)

Less affected by variability

- Variance of BCSST11 benchmark:



- Covariance:



Small variance

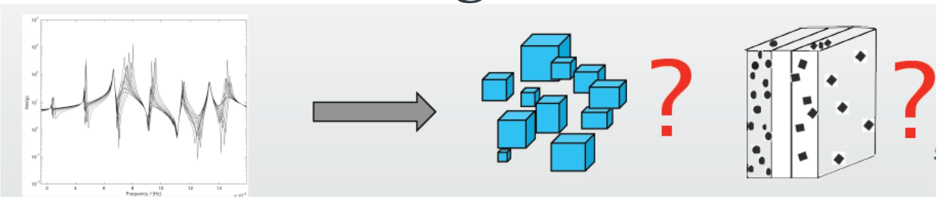
Pocket of variability

Strong resonance

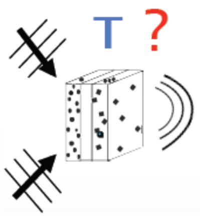
Weak resonance

Inversion of uncertainty propagation

- The objective is to characterise the nominal system and the disturbance from knowledge of some measurements.



- Motivation for inversion:



1. Absence of complete model and/or system parameters: characterise the randomness via simplified parameters. E.g. Sound transmissibility (Average? Variance?).
2. Assembled system more complicated than the sum of the individual components. E.g. Vulcanisation of tyres (Stiffness variability? Geometry variability?).
3. Validation of model, nominal system, presence and characteristics of disturbance. E.g. Checking presence of damage (Where? How much?); Update of nominal parameter values.
4. **Understanding the inverse problem's structure.** E.g. Gain insight into mid-frequency/mid-modal density problems.

- Use Bayes theorem.

Work in collaboration with Jon Forster, Brian Mace, Neil Ferguson

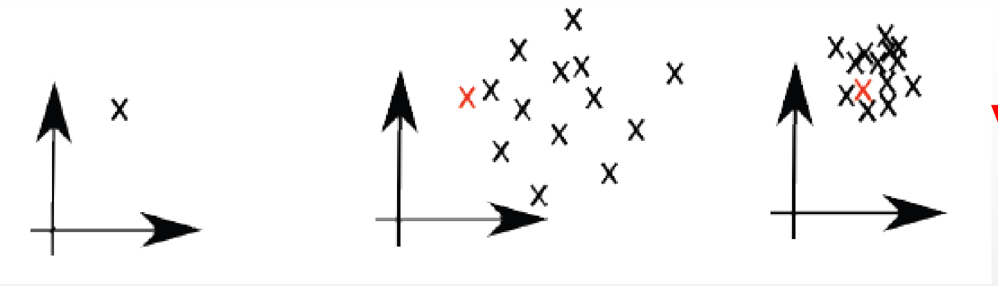
Bayesian inversion (intuitively and theorem)

- We know the structure of the (random) system but some of its parameters, θ, are unknown. For example, we know that the system behaves as a complex normal variable but we do not know the mean and/or variance.

- We can sample measurements, y, from the model…

- One sample does not give very probable information on the parameters, more samples do:

$$s = \mathcal{N}_c(\underbrace{\mu, \sigma}_{\theta})$$



y

- Mathematical expression of the probability of values of the parameters

Bayes theorem: $$p(\boldsymbol{\theta}|\mathbf{y}) = \frac{p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{y})}$$

Very simple proof: $$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Bayesian inversion components

- *Posterior* probability $p(\theta/y)$ is (only) function of θ (once y known)

$$p(\boldsymbol{\theta}|\mathbf{y}) = \frac{p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{y})}$$

- It is a function (product) of two functions:
    - *Likelihood* $p(y/\theta)$ describes the randomness propagation of the system (which needs to be known)
    - The *prior* probability $p(\theta)$ is a known or assumed probability (e.g. based on past or best knowledge)
- The probability $p(y)$ is a scaling factor (dependent on the same two functions…)

Bayesian inversion application

- Even though the proof of Bayes theorem is extremely simple, it leads to very intricate properties and theory.

  a) For given combinations of prior and likelihood, there exists analytical expression of the posterior.

  b) For more 'complicated' systems (combinations of likelihood and prior), one can implicitly sample from the posterior without knowing its expression (MCMC – Markov Chain Monte Carlo sampling)

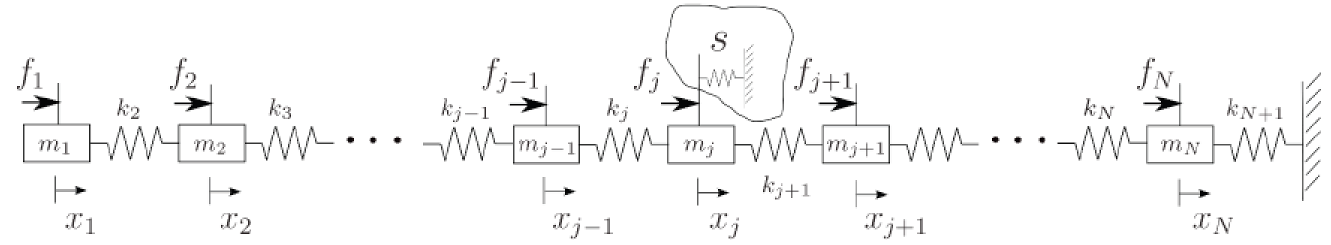
- It is a very general theorem that can

  - be applied with multiple measurements and parameters
  - allow to choose between different models for a given system
  - is not limited to linear systems or working in the frequency domain

- A big advantage is that other probabilities can be derived from the posterior probability $p(\theta|y)$

Bayesian inversion application (Analytical cases)

- First consider cases with analytical expression of the posterior.
    - A collinear chain of springs and masses with small proportional damping



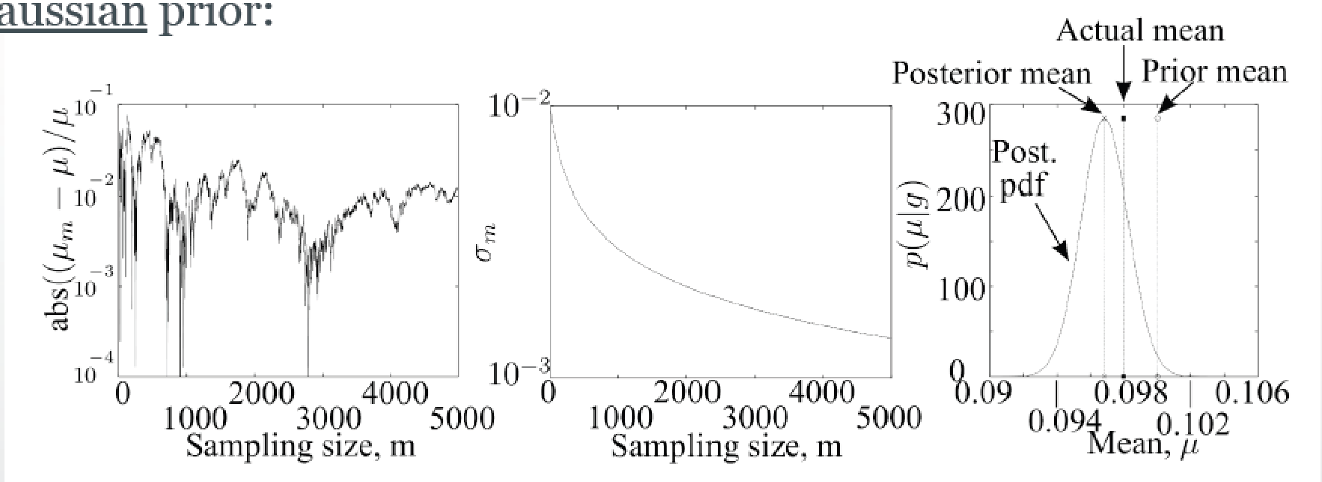
    - A single disturbance (in all cases) at a single location or connecting two masses. Disturbance is a spring (frequency independent).
    - Measurements are taken at a single position
    - There is no error in the measurements

- Three questions:

1. Identify the parameters of the disturbance (knowing its location)?
2. Identify the location of the disturbance (knowing its properties)?
3. Identify correction of the nominal system (knowing the disturbance)?

Case 1: Identification of disturbance

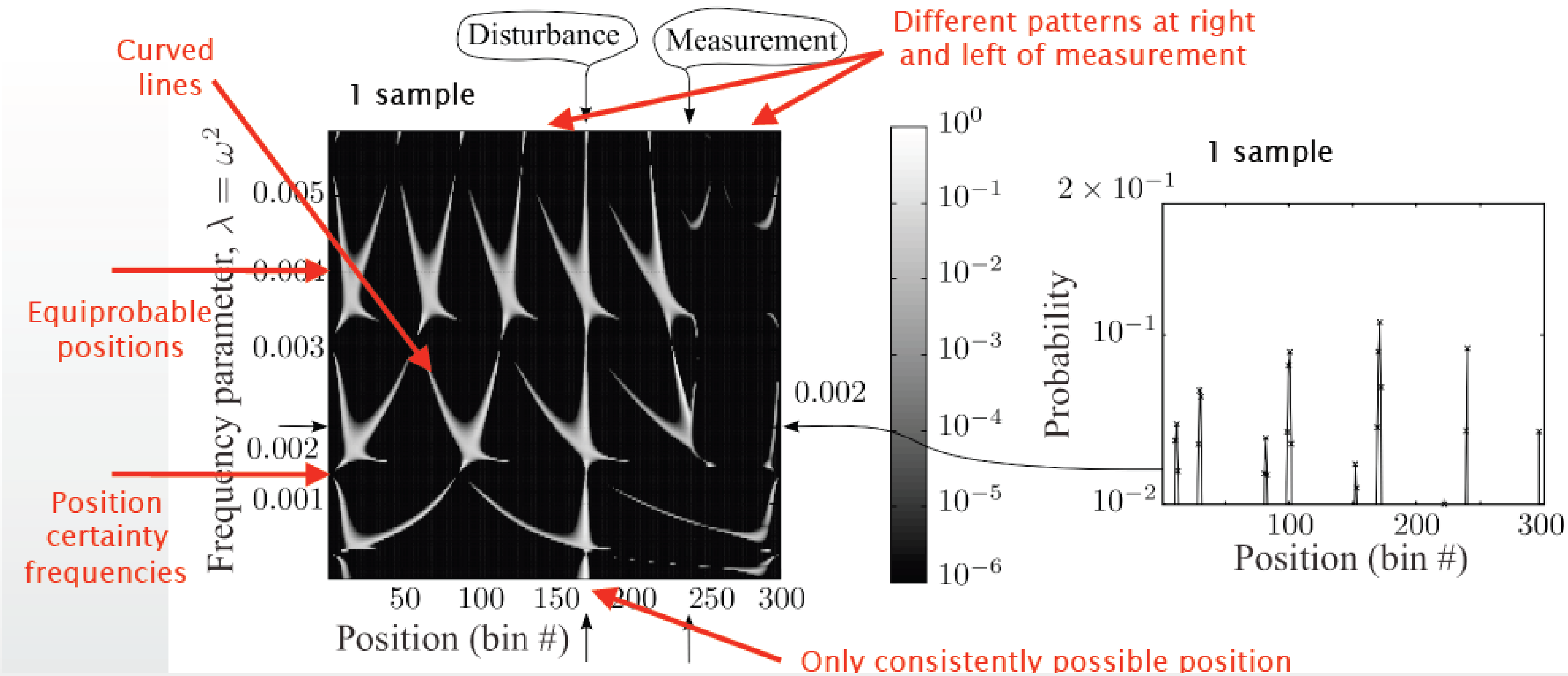
- Disturbance is a normal variable with known standard deviation
- The `Bayesian parameter' is the mean of the disturbance
- Posterior mean and standard deviation are available analytically from a Gaussian prior:



- Relatively trivial (simple)… for this particular choice of prior and disturbance (both normal).
- Posterior independent of frequency considered…

Case 2: Identification of location of damage

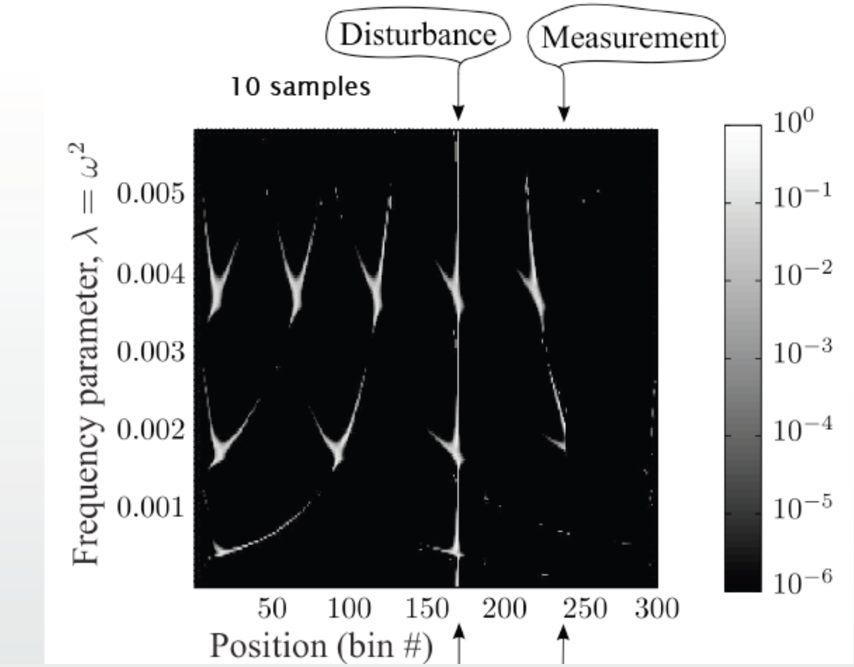
- Considering one frequency at a time (each row of the map independently)



- Very interesting pattern
  - If only one frequency is considered, there are better frequencies than other to identify the location
  - Considering all frequency together with give the location (1 sample!)

Case 2: Identification of location of damage (continued)

- Considering one frequency at a time (each row of the map independently) – Measurements on several samples…



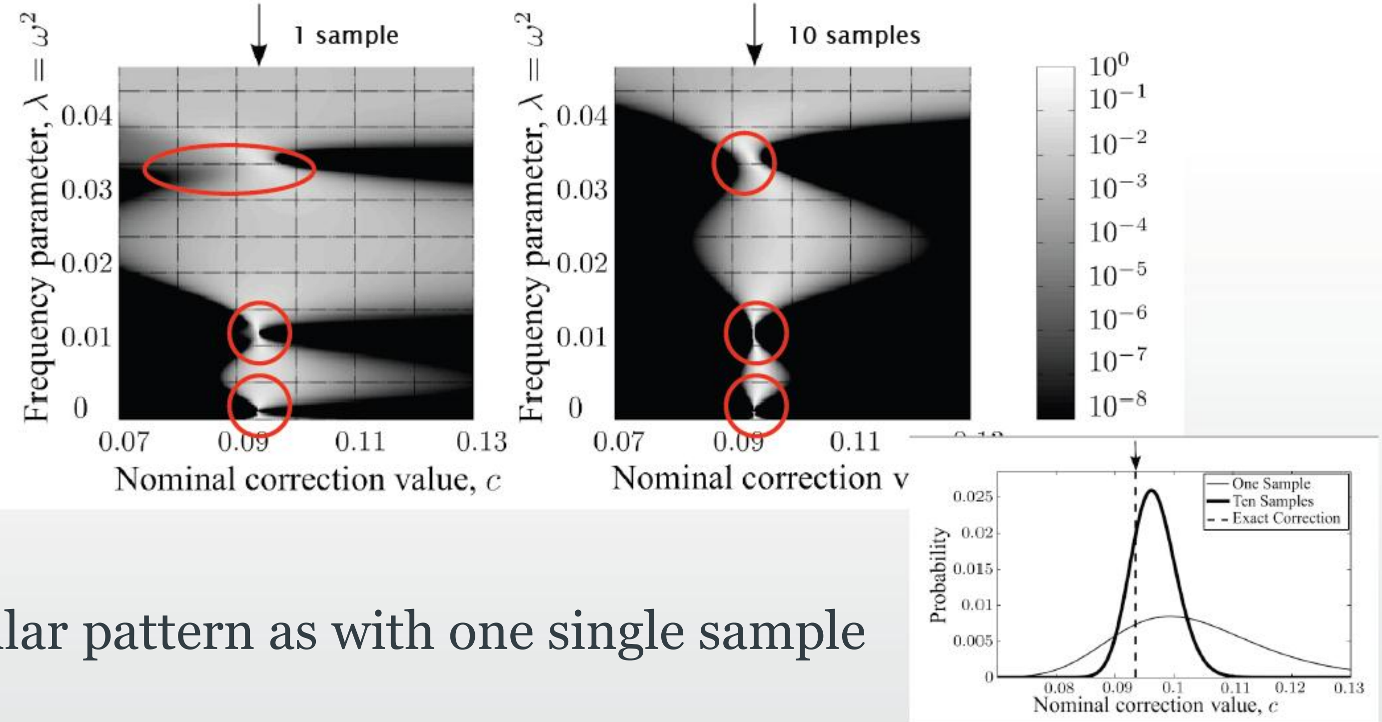
- Similar pattern as with one single sample

Case 3: Correction to nominal system

All members of the stochastic ensemble share a wiggly component with known statistical properties and location. The nominal system however has an unknown component (shared by all members).

Can one identify this deterministic component ***despite of the presence of the random component****?*

- Uniform prior for c in the range [0.07,0.13]



- Similar pattern as with one single sample

Conclusion

The analysis of forward and backward propagation of uncertainties in a structural dynamic systems low rank random disturbance is very interesting.

Some questions have answers simpler than expected while others questions have no obvious answers or show very intriguing patterns or generate other very interesting questions.

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