A view on parametric uncertainties of structural and vibro-acoustic systems

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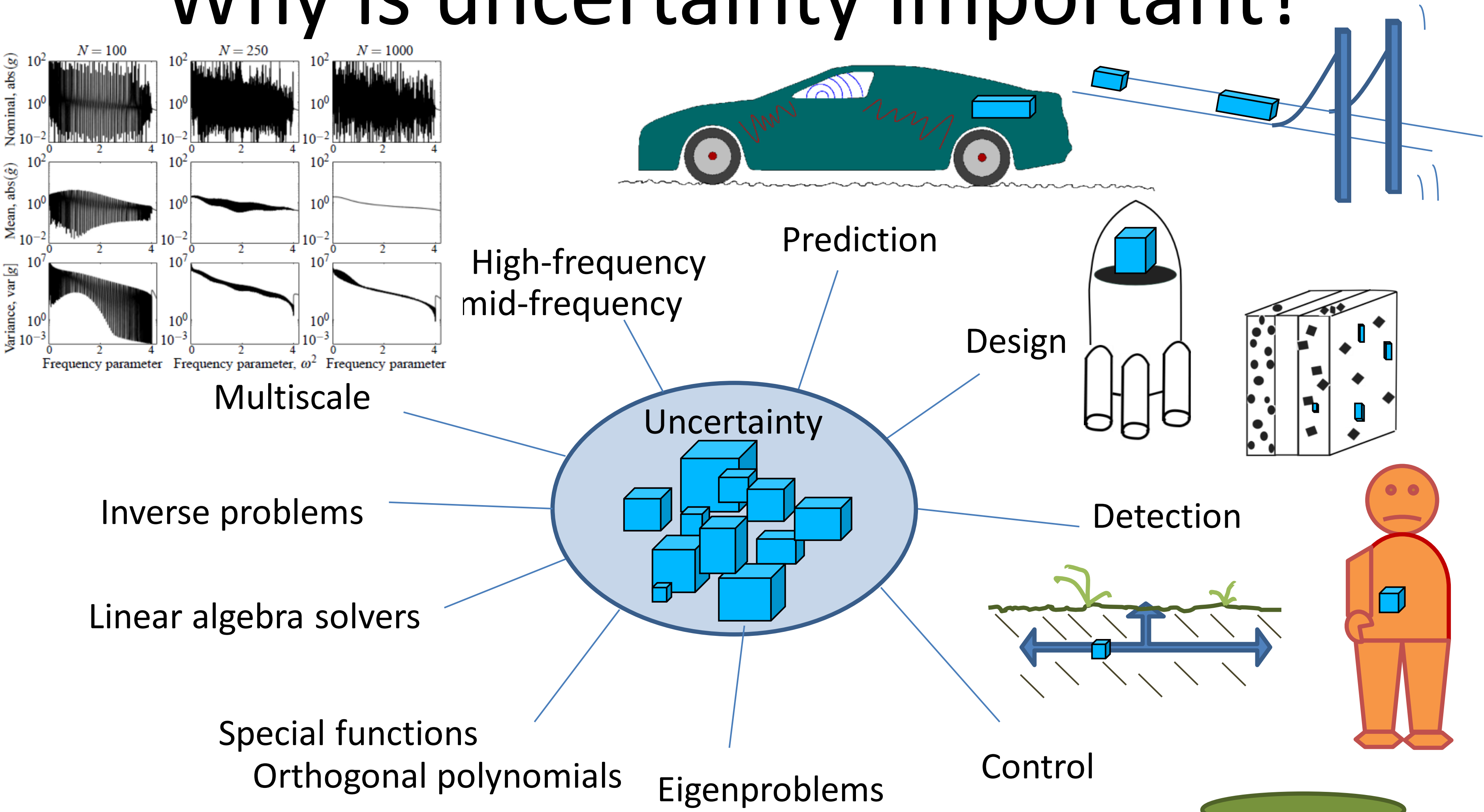
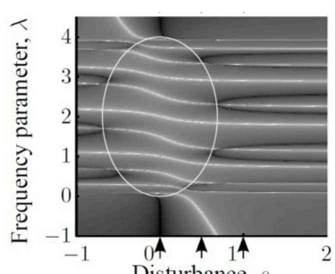
Why is uncertainty important?





Plan of talk

- Definition of stochastic systems
- Propagation of uncertainties
- Bayesian identification of uncertain parameters
- Matrix point of view

Zooming in on linear algebra

Consider a structural or vibro-acoustic system

- Nominally,

$$\mathbf{A}(\omega)\boldsymbol{x}(\omega, 0) = \mathbf{f}$$

$\omega = 2\pi f$ is a frequency parameter

- Dynamic stiffness matrix, $\mathbf{A}$
- Force vector, $\boldsymbol{f}$
- Response vector, $\boldsymbol{x}$

  – For example,
    - $\left(\mathbf{K} + \mathrm{i}\,\omega\,\mathbf{C} - \omega^2\,\mathbf{M}\right)\boldsymbol{x}(\omega, 0) = \mathbf{f}$
      Quadratic function of the parameter ω
    - $(\mathbf{A_0} + \omega\mathbf{A_1} + \omega^2\mathbf{A_2} + \cdots)\boldsymbol{x}(\omega) = \boldsymbol{f}(\omega)$
      Polynomial system
    - $\left(\mathrm{i}\omega\mathbf{I} - \mathbf{B}_0 - \sum_{i=1}^{m}\mathbf{B}_i\,\mathrm{e}^{-\mathrm{i}\omega\tau_i}\right)\boldsymbol{x}(\omega) = \boldsymbol{f}$
      Non-polynomial (time-delay) system

- Stiffness matrix, $\mathbf{K}$
- Damping matrix, $\mathbf{C}$
- Mass matrix, $\mathbf{M}$
- Taylor matrices, $\mathbf{A}_j$
- Delay matrices, $\mathbf{B}_j$

Stochastic system

- If a disturbance affects the matrix

$$[\mathbf{A}(\omega) - \mathbf{D}(\omega)]\boldsymbol{x}\big(\omega, \mathbf{D}(.)\big) = \boldsymbol{f}$$

and if D(w) can be expressed in terms of parameters s_1, s_2,...

$$\left(\mathbf{A}(\omega) - \sum_j s_j\, \mathbf{D}_j(\omega)\right) \boldsymbol{x}(\omega, s_1, s_2, \dots) = \boldsymbol{f}$$

There are then several parameters, $\omega, s_1, s_2, \dots$

- The system is called stochastic if $s_1, s_2, \dots$ are random variables with known pdf, $p(s_1), p(s_2), \dots$

Stochastic systems (cont'd)

There are several questions, including:

- What is the effect of the random variables on
  - The response (pdf, mean, variance, etc.)
  - The resonances and zeroes of the system
  - The extreme values
- Can one identify the random parameters
  - What are they?
  - Where are they?
  - What are their statistical properties?
- How does one solve such a multi-parameter system?
  - Sampling all parameters?
  - Reducing the system?
    - Modal condensation? What are the "modes"?
    - Something else?
    - Rational interpolation?
- Are there simplifications, properties, invariants, bounds, etc …?

Propagation

Bayesian identification

Inherent properties

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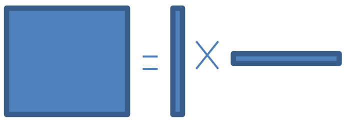
Propagation of uncertainty

- Start by looking at the "simplest" case…

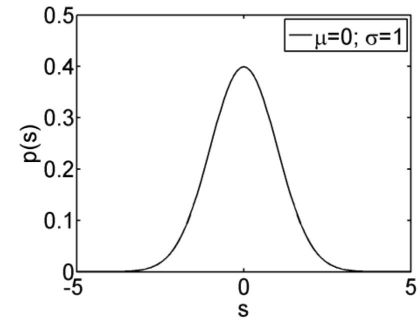
a) Single random component $\left(\mathbf{A}(\omega) - s\mathbf{D}(\omega)\right)\,\boldsymbol{x}(\omega, s) = \boldsymbol{f}$

b) Rank-one disturbance matrix, $\mathbf{D}(\omega) = \boldsymbol{d}_l(\omega)\boldsymbol{d}_r^T(\omega)$

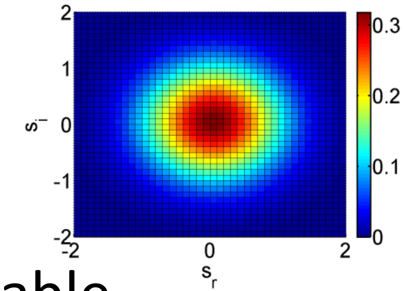


c) Normal random variable, s

✓ Real normal: $p(s) = \sqrt{\dfrac{1}{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{(s-\mu)^2}{2\sigma^2}}$



✓ Complex normal: $p(s) = \dfrac{1}{\pi\sigma^2}\,\mathrm{e}^{-\frac{\overline{(s-\mu)}\,(s-\mu)}{\sigma^2}}$



- It appears that it is not that simple

There *was* no entirely analytical solution available

Propagation – Low rank update

- One can easily show that the rank-one update for any transfer function $g(\omega, s) = \boldsymbol{c}^T \boldsymbol{x}(\omega, s)$ is

$$g(\omega, s) = g(\omega, 0) + s_1(\omega) t(\omega) \left[\frac{s_1(\omega)}{s_1(\omega) - s} - 1 \right]$$

1

Where $s_1(\omega) = [\boldsymbol{d}_r^T(\omega) \mathbf{A}(\omega)^{-1} \boldsymbol{d}_l(\omega)]^{-1}$ 2

$t(\omega) = [\boldsymbol{c}^T \mathbf{A}(\omega)^{-1} \boldsymbol{d}_l(\omega)][\boldsymbol{d}_r^T(\omega) \mathbf{A}(\omega)^{-1} \boldsymbol{f}]$

3 4

- Several advantages including isolation of s in ratio
- It only requires four transfer functions
- It allows the exact derivation of the statistics (pdf, mean, variance and covariance) of *any* transfer function

Propagation – Mean and variance

- The mean, $\hat{g}(\omega)=\int_{\mathcal{D}(s)} g(\omega, s)p(s)\mathrm{d}s$ requires a "simple" integral

$$\hat{g}(\omega) = g(\omega, 0) + s_1(\omega)t(\omega)[e_1(\omega) - 1]$$

$$e_1(\omega) = \int_{\mathcal{D}(s)} \frac{s_1(\omega)}{s_1(\omega) - s} p(s)\mathrm{d}s$$

- Covariance (possibly at different frequencies and for different input and output vectors), var $(g_A(\omega_A), g_B(\omega_B))$=

$$\int_{\mathcal{D}(s)} [g_A(\omega_A, s) - \hat{g}_A(\omega_A)]\left[\overline{g_B(\omega_B, s) - \hat{g}_B(\omega_B)}\right] p(s)\mathrm{d}s$$

requires another integral

$$v_{11}(\omega_A, \omega_B) = \int_{\mathcal{D}(s)} \frac{s_1(\omega_A)}{(s_1(\omega_A) - s)} \frac{\bar{s}_1(\omega_B)}{(\bar{s}_1(\omega_B) - \bar{s})} p(s)\mathrm{d}s$$

Propagation – stochastic integrals

	Real normal	Complex normal
Mean	$e_1^{(g)}(\omega) = \frac{s_1(\omega)}{\sqrt{2\pi\sigma^2}} I_{-1}\left(\frac{s_1(\omega)-\mu}{\sqrt{2}\sigma}\right)$ $I_{-1}(b) = -\mathrm{i}\pi w(b)$ if $\mathrm{Im}(b) > 0$ $I_{-1}(b) = \mathrm{i}\pi \overline{w}\left(\overline{b}\right)$ if $\mathrm{Im}(b) < 0$ $I_{-1}(b) = 2\sqrt{\pi}F(b)$ in a p.v. sense if $\mathrm{Im}(b) = 0$ Dawson's or Faddeeva function **Evaluable**	**CLOSED-FORM** $e_1^{(g_C)}(\omega) = \frac{s_1(\omega)}{s_1(\omega)-\mu}\left[1 - \mathrm{e}^{-\left(\frac{\lvert s_1(\omega)-\mu\rvert^2}{\sigma^2}\right)}\right]$ Real $\in [0,1]$ if $\mu = 0$
Variance Covariance	If $s_1(\omega_A) \neq \overline{s}_1(\omega_B)$ $v_{11}(\omega_A, \omega_B) = \frac{e_1(\omega_A)\overline{s}_1(\omega_B) - s_1(\omega_A)\overline{e}_1(\omega_B)}{\overline{s}_1(\omega_B) - s_1(\omega_A)}$ $v_{11}(\omega_A, \omega_B) = q_{11}(\omega_A) = E\left[\left(\frac{s_1(\omega_A)}{s_1(\omega_A)-s}\right)^2\right]$ $q_{11}(\omega) = -s_1(\omega)^2 \int_{-\infty}^{\infty} \frac{1}{s_1(\omega)-s}\frac{\partial p(s)}{\partial s}\mathrm{d}s$ $= \frac{1}{\sigma^2}\left[s_1(\omega)^2\left(e_1^{(g)}(\omega) - 1\right) - \mu s_1(\omega) e_1^{(g)}(\omega)\right]$	$v_{11}^{(g_C)}(\omega_A, \omega_B) = \frac{s_1(\omega_A)\overline{s}_1(\omega_B)}{\pi\sigma^2} J_{-2}(b_A, b_B)$ $J_{-2}(b_A, b_B) = -\pi \mathrm{e}^{-\left(b_A\overline{b}_B\right)}\left[\mathrm{Ei}\left(b_A\overline{b}_B - \lvert b_A\rvert^2\right) + \mathrm{Ei}\left(b_A\overline{b}_B - \lvert b_B\rvert^2\right) - \mathrm{Ei}\left(b_A\overline{b}_B\right)\right]$ **"New" function…** One singularity (no variance) One zero

Propagation - pdf

- The probability density function (pdf) for *any transfer function* is also available from the rank-one update expression

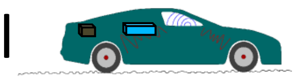
$$p(\mathbf{g}(\omega,\mathbf{s})=\mathbf{g}) = \left(\Pi_{i=1,\ldots,m} J(\omega,g_i)\right)\left(\Pi_i p\left(s(\omega,g_i)\right)\right)$$

$$J(\omega,g) = \frac{|t(\omega)|}{\left|t(\omega)+[g-g(\omega,0)]\,\mathbf{d}_r(\omega)^T\mathbf{A}^{-1}(\omega)\mathbf{d}_l(\omega)\right|^2}$$ Jacobian

$$s(\omega,g) = \frac{g-g(\omega,0)}{t(\omega)+[g-g(\omega,0)]\,\mathbf{d}_r(\omega)^T\mathbf{A}^{-1}(\omega)\mathbf{d}_l(\omega)}$$ Inverse of update function

- The expression is valid for any kind of random variable, s, with known pdf, $p(s)$
- If s is real, the values of both s and $g(\omega, s)$ are along a line in the complex plane

Propagation - Application - Half-car

- Application on half-car model 
- Transfer function from road height to occupant's (heads) acceleration
- Change in mass of passenger
  - Real normal variable

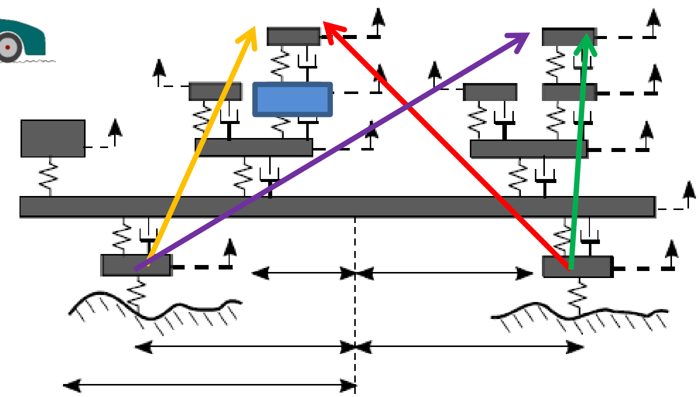
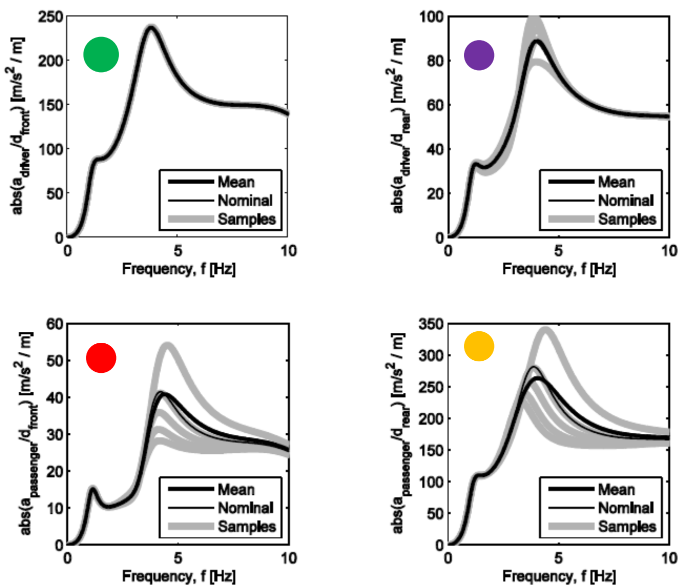


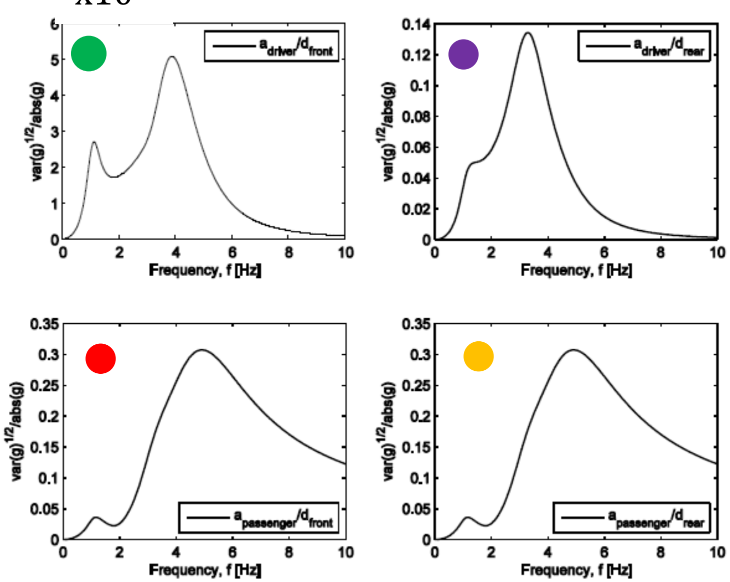
Fig. 1 Schematic view of the nominal half-car model

Mean

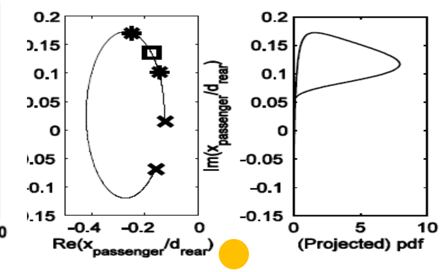


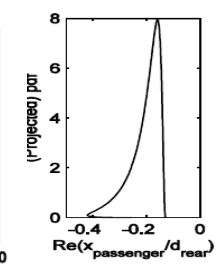
Relative variance

$x10^{-3}$



Pdf at 5 Hz





Bounded

Propagation - Application - Half-car

- Application on half-car model
- Transfer function from road height to occupant's (heads) acceleration
- Change in stiffness AND damping suspension (Complex normal variable)

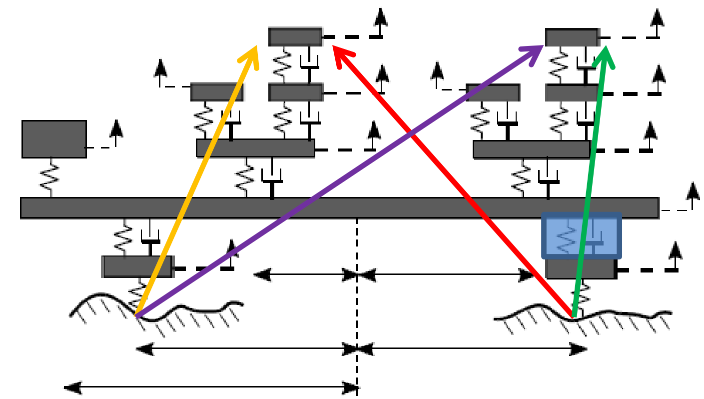
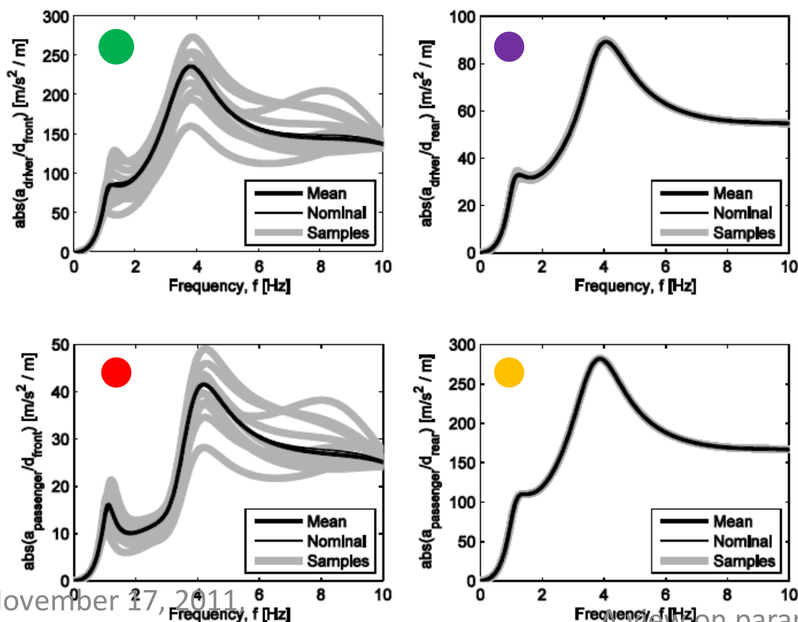
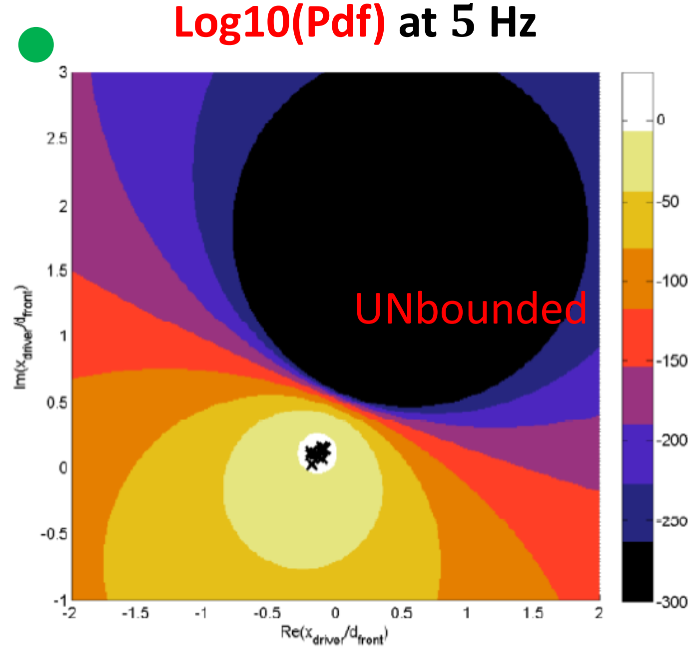


Fig. 1 Schematic view of the nominal half-car model

Mean

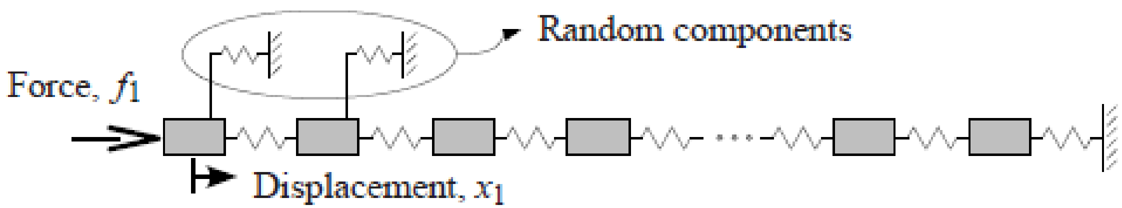


Log10(Pdf) at 5 Hz

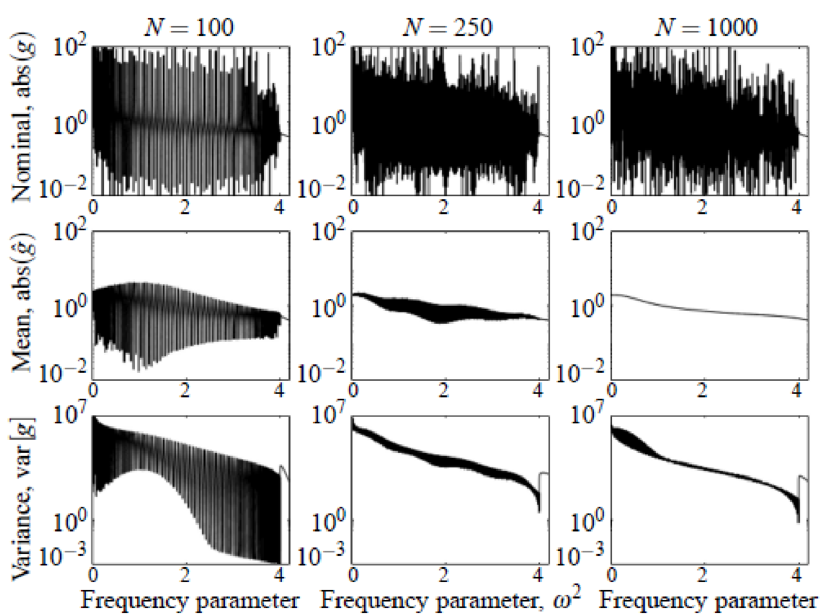


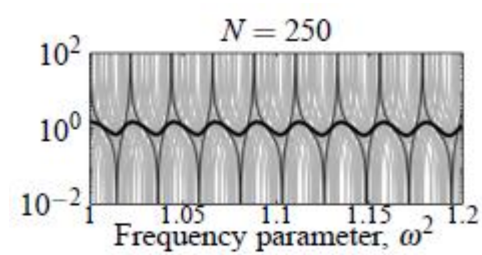
Propagation of uncertainty

- Application at mid- and high-frequency
- Collinear spring-mass system with added stiffness



- Smooth mean at higher modal density (from overlap)

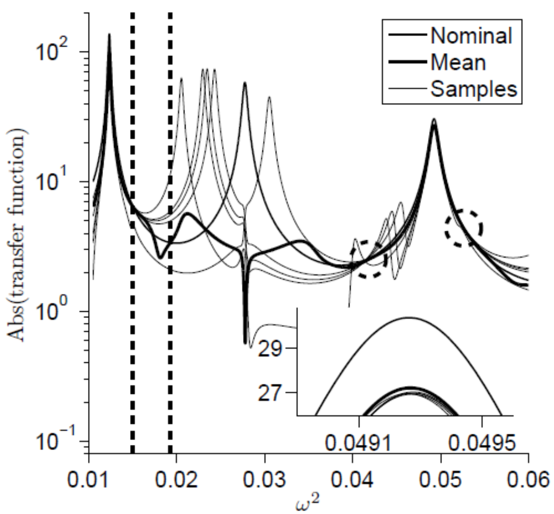


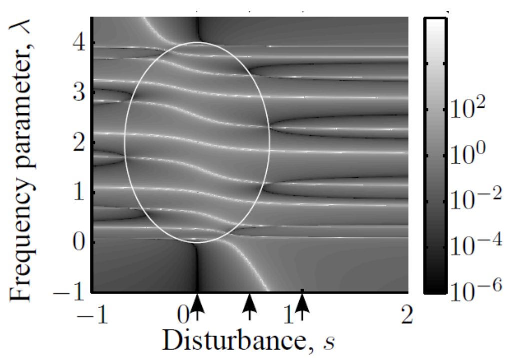


Christophe Lecomte. Vibration analysis of an ensemble of structures using an exact theory of stochastic linear systems. In Robin S. Belyaev, Alexander K. Langley, editor, *Proceedings of the IUTAM Symposium on the Vibration Analysis of Structures with Uncertainties*, pages 301–315. Springer-Verlag, 2011.

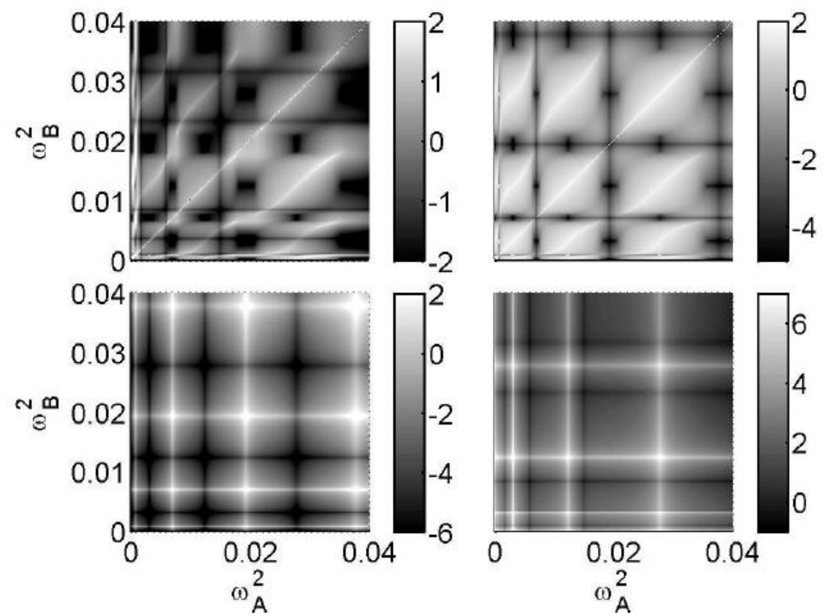
Propagation - more

- There is much more to that… such as…
  - Exact (analytical) location of eigenvalues and zeroes
  - Statistical pseudo-spectrum and robustness
  - Stable and unbiased points
  - Covariance





Covariance (log10(abs(.)))

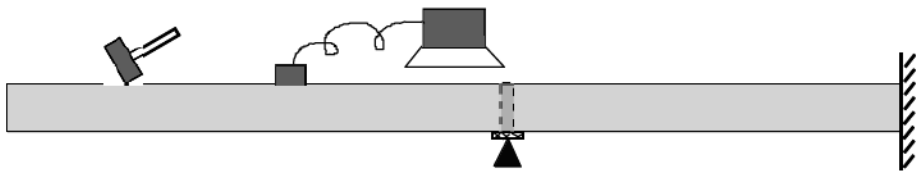


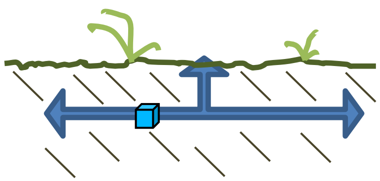
Plan of talk

- Definition of stochastic systems
- Propagation of uncertainties
- Bayesian identification of uncertain parameters
- Matrix point of view

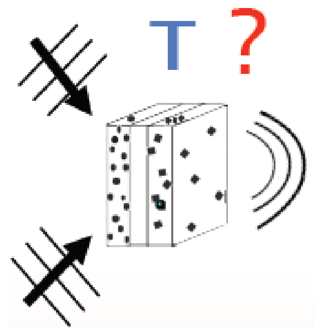
Bayesian identification

- This is the reverse identification of parameters, $\boldsymbol{\theta}$, of an uncertain system from measurements, $\boldsymbol{Y}$

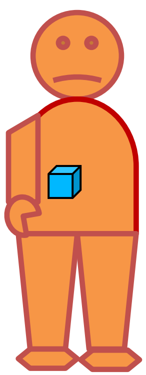




$\boldsymbol{\theta}$: Location, layers, properties
$\mathbf{Y}$: Ground vibration



$\boldsymbol{\theta}$: Layers, each layer properties
$\mathbf{Y}$: Transmissibility



$\boldsymbol{\theta}$: Location, stiffness, body model
$\mathbf{Y}$: Ultrasound, MRI, etc.

Bayesian – Bayes theorem

- Bayes theorem (trivial to prove) provides the probability of the parameters posterior to measurements

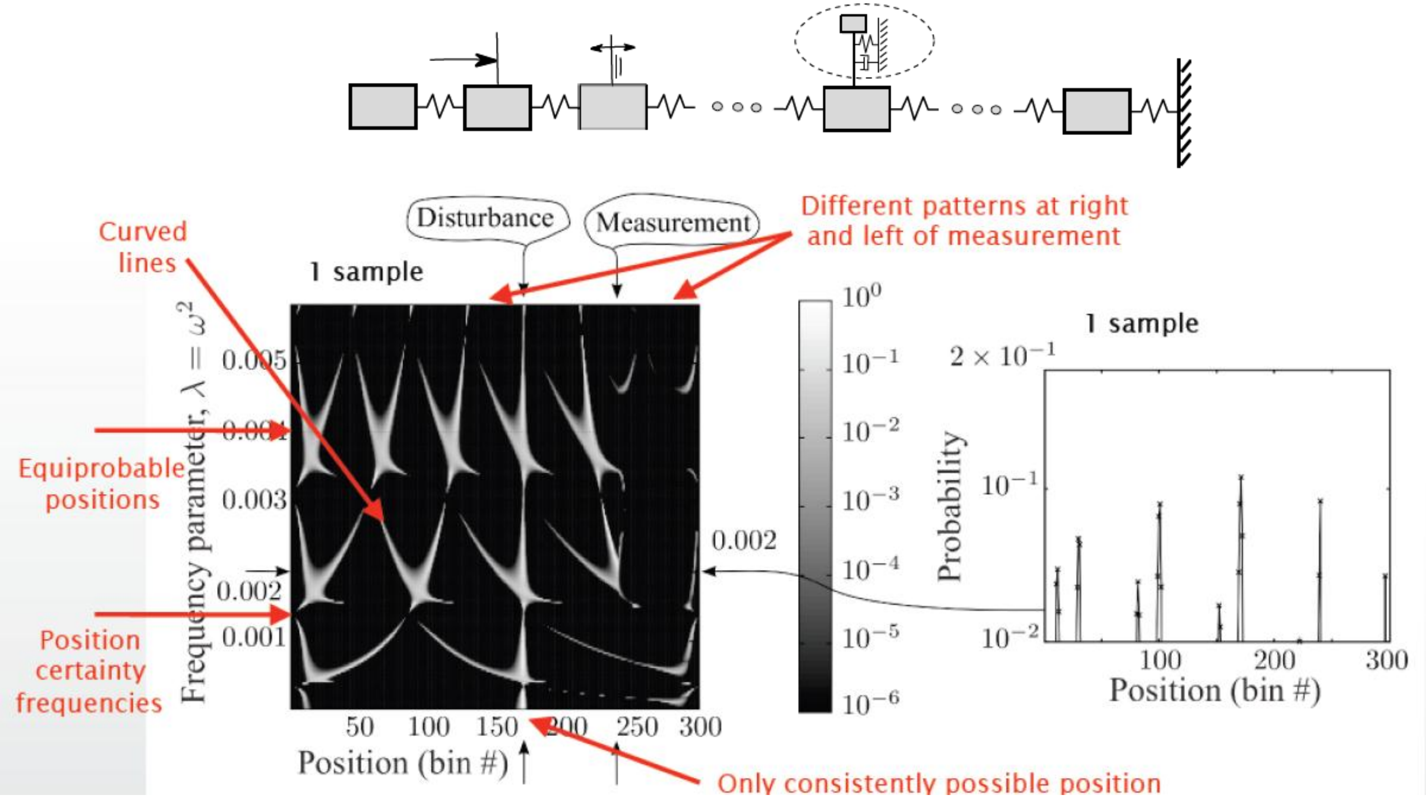
$$p(\boldsymbol{\theta}|\mathbf{Y}) = \frac{p(\mathbf{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{Y})}$$

  - Prior knowledge (or assumption) from the engineer (based on past or present knowledge) is included
  - The likelihood describes the propagation of uncertainty in the system
  - The denominator is a scaling factor (dependent on the numerator)

- Despite a very simple proof, the application of Bayes theorem can lead to very intricate theory and properties

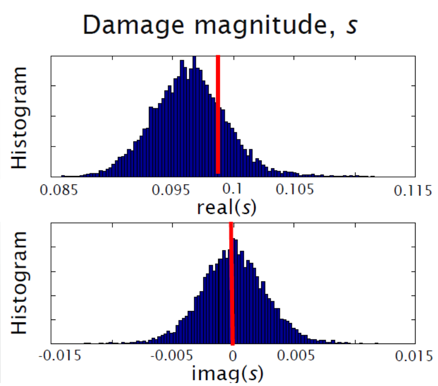
Bayesian – Analytical

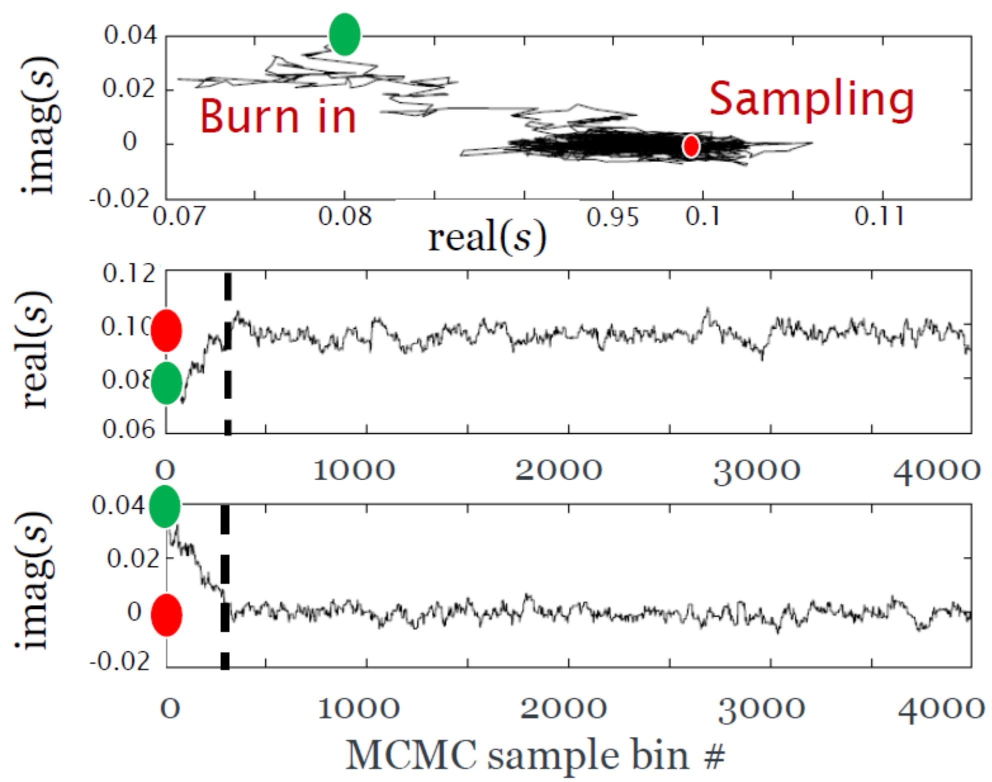
- For given combinations of the prior and likelihood there exist analytical expressions of the posterior
- The (statistics, i.e. pdf of) the location of a damage (random component) of known statistics can be evaluated analytically.



Bayesian - MCMC

- In general analytical evaluation is not possible (because unknown scaling factor that is too complicated to evaluate) but one can simple implicitly from the posterior.
- Markov Chain Monte Carlo methods allows to estimate the pdf of ALL the parameters (they could include the choice of model)

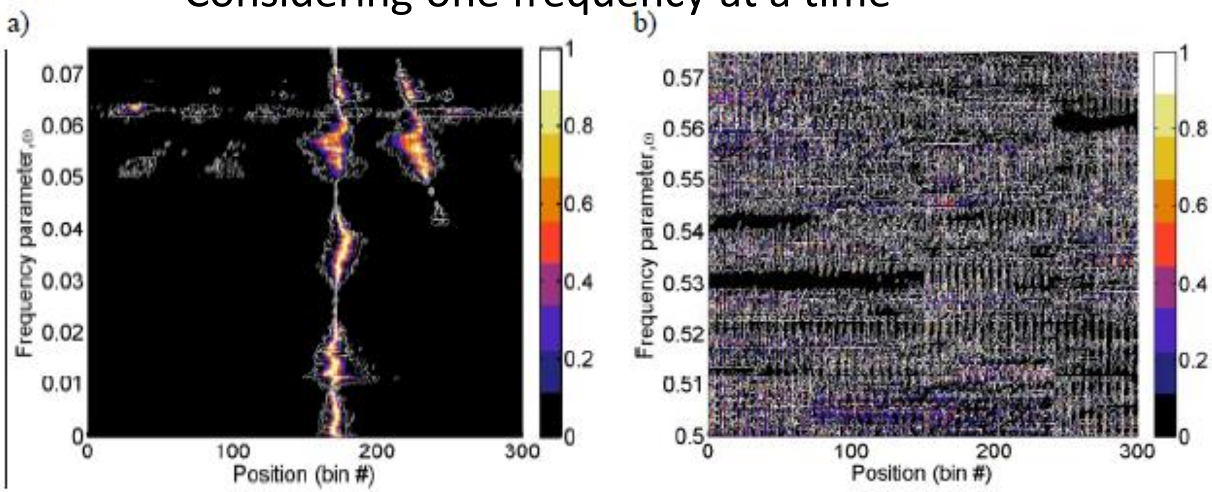




Bayesian - MCMC

- Damage identification again now with random damage magnitude and location, and random measurement error.
- Both damage magnitude and measurement error dependent on frequency (Gaussian process)

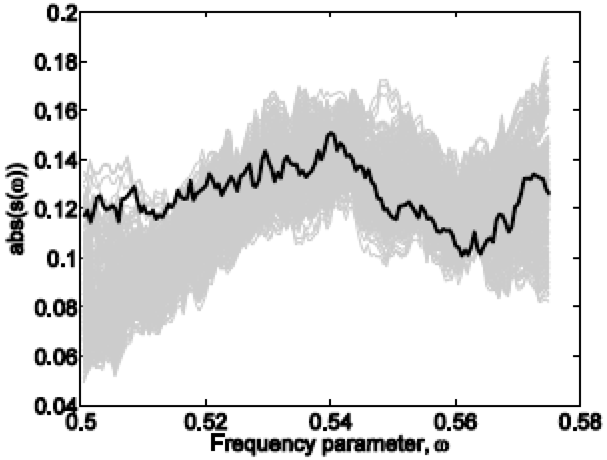
Considering one frequency at a time

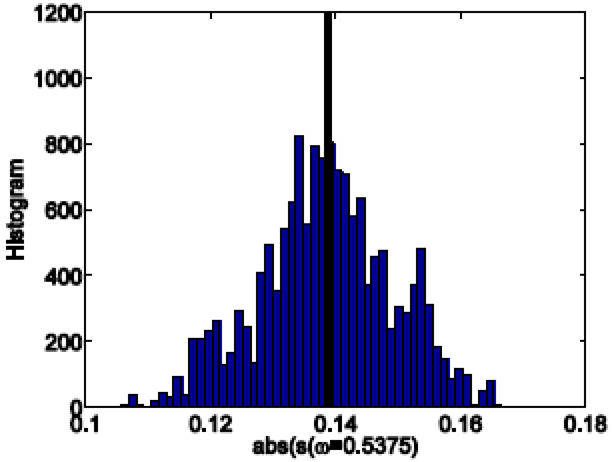


- Considering all frequencies together allows identification of damage location both at low and high frequency

Bayesian - MCMC

- Statistics on damage magnitude





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Matrix point of view

- This can be attacked from at least three points of view
  - Evaluate/approximate the means and variance directly
  - Use low-rank update in the inverse problem and use model reduction
  - Consider the parametric system directly
- All three very hot topics in linear algebra, inverse problems
  - Collaborations with KUL, BU
  - Workshops and minisymposia planned

Thank you !

Some slides based on Bayesian work pursued with Jon Forster, Brian Mace, and Neil Ferguson and presented elsewhere

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References available on request

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