Controller architectures for optimum performance in practical active acoustic metamaterials

M. Reynolds[a], S. Daley[a], Y. Gao[a], V. Humphrey[a] and S. A. Pope[b]

[a]ISVR, University of Southampton, SO17 1BJ Southampton, UK
[b]University of Sheffield, Department of Automatic Control and Systems Engineering, Mappin Street, S1 3JD Sheffield, UK
mjr304@soton.ac.uk

Over the last decade there has been significant interest in the design and production of acoustic metamaterials with physical qualities not seen in naturally occurring media. Progress in this area has been stimulated by the desire to create materials that exhibit novel behaviour when subject to acoustic waves, such as negative refraction or the appearance of band gaps in the frequency response of the material. Proposed designs range from locally resonant phononic crystals to arrays of Helmholtz resonators within ducts and past research has investigated both passive and active materials. Much of the research into active acoustic metamaterials remains theoretical, therefore to determine whether such materials are physically realisable and of potentially practical use it is important to understand the physical constraints that may arise in a produced active metamaterial. In this paper a 1-dimensional active acoustic metamaterial derived from a passive, Helmholtz resonator based design is considered where the applied control forces produce controllable double negative behaviour. The physical dimensions and active forces required to achieve the desired novel behaviour are explored for different architectures and any trade-offs that might have to be considered when producing a practically useful active metamaterial are identified

1   Introduction

The concept of 'left-handed' electromagnetic materials was first proposed by Vesalago [16], where it was proposed that materials that displayed negative electromagnetic permeability and permittivity could create interesting effects on waves incident to the medium. Later, metamaterials were proposed as a means of realising such a material [14]. The analogy of acoustic and electromagnetic waves has led to a great deal of research into acoustic metamaterials [7, 10, 11, 17], where the analogues of permeability and permittivity are density and bulk modulus respectively. Like their electromagnetic counterparts when an acoustic metamaterial displays either negative effective bulk modulus or effective mass density the refractive index becomes complex and the medium partially blocks the transmission of the wave. When both become negative simultaneously (where the material is said to be in its double negative (DNG) region), the refractive index has a negative sign so negative refraction occurs. In this region Snell's law still applies, however the path of the reflected wave lies to the opposite side of the incident normal than one would expect in a regular transmission medium. Band gaps appear in the dispersion characteristics of metamaterials, at high frequencies due to Bragg scattering effects [10] related to the periodic properties of the material and, in materials where low frequency resonances occur, at frequencies around two orders of magnitude lower [11]. This leads to high levels of attenuation in the transmission characteristics of the material at these frequencies. These novel properties mean that metamaterials are of particular interest and have been proposed as a potential solution to achieve acoustic cloaking [3, 5, 9], transmission blocking [4, 8, 17] and subwavelength acoustic lenses [11, 18].

One of the major limitations of many proposed materials is the fixed, narrow frequency bands in which the attenuation and negative behaviour occur. This is due to the materials being constructed from passive components with fixed material properties. One potential solution to this problem is to introduce an active control architecture into the material, allowing the material parameters to be altered and the region of novel behaviour associated with low frequencies to be adapted [1, 2]

In this paper a viscoelastic active metamaterial is proposed based on previous work carried out by the authors [12]. By giving the material arbitrary dimensions and a realistic excitation force the dimensions, forces and size of a practical metamaterial are investigated. By coupling the feedback forces to not only the motion of resonant mass but to adjacent transmission masses, the active architecture exhibits be-

haviour not possible with passive materials. The advantages of this active architecture when considering the magnitude of material's physical parameters are demonstrated. Finally, by considering the the resonators as independent mechanical filters the low frequency resonant band gap can be widened, and potential future work is identified using established control techniques to optimise the filter design.

2   Definition of the Metamaterial

A commonly cited example of a 1-dimensional acoustic metamaterial is a duct fitted with an array of Helmholtz resonators to create a periodic material with locally resonant elements [7]. Figure 1(a) is an example of such a material. By creating lumped parameter model of this material, as suggested in [12], one can envisage a solid 1D metamaterial, shown in Figure 1(b) that could used as a vibration isolator, for example. The material would consist of solid layers separated by a viscoelastic layer, represented in lumped parameter form by the transmission masses, m_t, and the transmission stiffness and viscous damping terms k_t and c_t, respectively. Each individual transmission layer would have connected a separate resonator mass, m_r (equivalent to the mass of the air in the neck of the Helmholtz resonators in the ducted array), via a spring and viscous damper, k_r and c_r. The resonator mass would be connected to an immobile 'earth' connection via an additional spring and viscous damper, k_h and c_h, as shown in Figure 1(c).

The classical equation of motion of the lumped parameter model in Figure 1(b) can be constructed using D'Alemberts principle, and it can be shown [12] that the material dynamics are governed by

$$-M_e x_n \omega^2 = (C_e i\omega + K_e)(x_{n-1} + x_{n+1} + 2x_n) + f_n \quad (1)$$

Where f_n is a harmonic disturbance applied to the first transmission mass and M_e, K_e and C_e are the effective mass, stiffness and damping coefficients of the Kelvin-Voigt material respectively. By comparing the classical equation of motion of the material with Eq. 1 the effective mass of the material becomes dependent on the excitation frequency, ω, and will become negative at certain frequencies. Presuming that all the layers of the material have the same parameters ($m_{t,1} = m_{t,2}... = m_{t,N}$ etc) the effective mass becomes Eq. 2. Here $K_e = k_t$ and $C_e = c_t$ so both remain positive and independent of frequency.
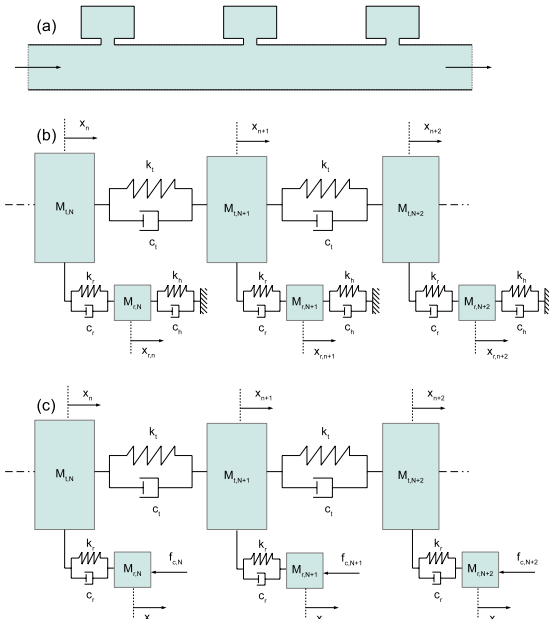
Figure 1: (a) A duct fitted with an array of Helmholtz resonators to create a 1D acoustic metamaterial. (b) A lumped parameter equivalent model of the metamaterial (c) An active metamaterial where the static 'earth' connection and it's associated springs and dampers are replaced by an actively fed back force applied to the resonator masses in a 'skyhook' configuration.

$$M_e = m_t - \frac{(-m_r\omega^2 + c_h i\omega + k_h)(c_r i\omega + k_r)}{\omega^2(-m_r\omega^2 + (c_h + c_r)i\omega + k_h + k_r)} \quad (2)$$

M_e is complex, where the real part describes the actual effective mass of the dynamic system and the imaginary part represents an additional damping component related to the resonators. At the natural frequency of the resonators the effective mass becomes negative [7, 11, 17], and there will be a deep, asymmetric band-gap in this region caused by the wave vector becoming complex. This band gap occurs at a frequency associated with the resonators, and hence can be created at frequencies much lower than those due to Bragg scattering [11]. Therefore the resonant band gap is of great interest when considering disturbances in the audible frequency range where the wavelengths are such that the periodicity of material would have to be of the order of several metres to induce Bragg scattering.

An alternative approach to the passive metamaterial described in 1(b) is to replace k_h and c_h with an active control architecture providing a feedback force as a function of the displacement and velocity of the resonator mass. Such an active control system is known as 'skyhook' control, and is shown in Figure 1(c). This model gives the same dynamic performance as the passive example but the resonator parameters k_h and c_h are replaced with a force $f_{c,n} = k_c x_n + c_c \dot{x}$. When $k_c = k_h$ and $c_c = c_h$ the dynamic response of the system is identical to that of its passive equivalent, however by changing these control parameters the region of negative effective mass and resonant behaviour can be adapted.

The above system provides negative mass, however does not have the frequency dependent stiffness term necessary to

achieve negative effective stiffness and hence double negativity. Following the approach used in [12] the control system can be extended by making the control forces a function of the displacement and velocity not only of the resonator itself, but of elements of neighbouring elements, in this case the $(n-1)th$ and $(n+1)th$ transmission masses. This modified control force, $f_{c,n} = k_c(x_{t,n-1} + x_{t,n+1} + 2x_{r,n}) + c_c(\dot{x}_{t,n-1} + \dot{x}_{t,n+1} - 2\dot{x}_{r,n})$ results in an expression for the effective mass and a combined expression for the effective damping-stiffness of the system, given by Eq. (3) and Eq. (4) respectively.

$$M_e = m + \frac{m_r(c_r i\omega + k_r)}{-m_r\omega^2 + (2c_c + c_r)i\omega + 2k_c + k_r} \quad (3)$$

$$C_e + K_e = ci\omega + k + \frac{(c_c i\omega + k_c)(c_r i\omega + k_r)}{-m_r\omega^2 + (2c_c + c_r)i\omega + 2k_c + k_r} \quad (4)$$

Eq. (4) is a complex expression describing the combined effective damping and stiffness. As such the real part represents the pure stiffness elements within the system, and the imaginary part the dissipative damping elements within the system. So calculating the real part of Eq. (4) yields the effective stiffness of the system.

$$K_e = k +$$
$$\frac{m_r c_c c_r \omega^4 + (-m_r k_c k_r + 2c_c^2 k_r + c_r^2 k_c)\omega^2 + k_c k_r(2k_c + k_r)}{m_r^2\omega^4 + (-2m_r(2k_c + k_r) + (2c_c + c_r)^2)\omega^2 + (2k_c + k_r)^2} (5)$$

Now using this extended, 'parallel coupled', control force the material may have a region not only of negative effective mass but of negative effective stiffness, becoming 'double negative' and allowing negative refraction to occur.

3 Physical properties of a metamaterial

If the metamaterials described in Section 2 are to be viable both the physical dimensions and the material parameters must be practical. Insight into these properties, for a given scenario, can be gained by interrogating the equations of motion. The dimensions of a producible material are dependent on the displacements of the transmission and resonant masses for a particular excitation force and the number of layers required to produce useful transmission blocking, for example. The active feedback forces that would be required are then a function of the displacements and velocities of the relevant masses. Since the low frequency band gap and onset of negative material parameters occurs at frequencies associated with the resonant parts of the material, it is the behaviour at these frequencies that is critical. The dynamic behaviour of a metamaterial can be expressed using D'alembert's principle to construct the equations of motion, presuming harmonic excitation.

$$[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]\mathbf{q} = \mathbf{f} + \mathbf{F}_c \quad (6)$$

Where $\mathbf{M}, \mathbf{C}, \mathbf{K}$ and $\mathbf{F}_c$ are the mass, damping, stiffness and feedback force matrices respectively, $\mathbf{f}$ is the excitation force vector, and ω is angular frequency in rad/s. The examples presented here have material coefficients described by Table 1.

A circular cross-section metamaterial is considered with a diameter of 100mm, incident with an acoustic plane wave

	Parameter	Value
Transmission Media	m_t	$0.01g$
	k_t	$3000Nm^{-1}$
	c_t	$0.01Nsm^{-1}$
Resonators	m_r	$0.1g$
	k_h	$1000Nm^{-1}$
	c_h	$0.1Nsm^{-1}$
	k_r	$1000Nm^{-1}$
	c_r	$0.002Nsm^{-1}$
Equivalent feedback parameters	k_c	$1000Nm^{-1}$
	c_c	$0.01Nsm^{-1}$

Table 1: The material parameters used to simulate a the metamaterials described in Section 2

Figure 3: The maximum elemental displacement within the region of the resonant band gap as a function of the number of metamaterial layers. Passive lumped parameter (-x-), skyhook (-o-) and parallel coupled (-△-)

with an SPL of 90dB (ref $20\mu Pa$). The displacement of elements within metamaterials peak at modal frequencies. Since it is in the resonant band gap region that the desired novel behaviour occurs it is the displacement and active forces occurring within the band gap that are presented here. Figure 2 shows the maximum mass element displacement within a 4 layer metamaterial as a function of frequency.
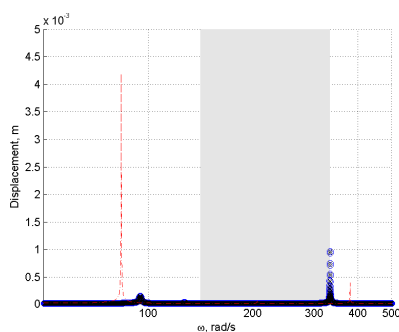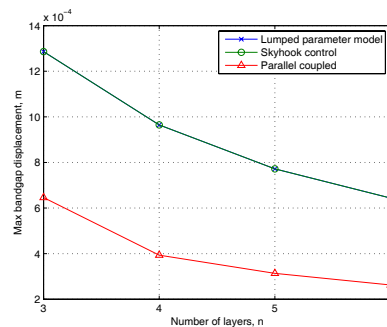

Figure 2: The maximum elemental displacement of a lumped parameter equivalent of the Helmholtz resonator metamaterial (-x-), an active metamaterial with 'skyhook' active control (-o-) and an active metamaterial where the active feedback force is parallel coupled to the motion of the adjacent transmission masses (dotted line). The shaded area represents the region where the passive and skyhook metamaterials have negative effective mass, which coincides with their resonant band gap.

As Figure 2 shows there is a large peak in displacement, of an order of 1mm, associated with the resonant band gap. The passive and skyhook materials have equivalent performance, as expected. The results suggest that the parallel coupling active architecture can be used to reduce the maximum displacement within this region. However, as the results show, this may be associated with increased displacement at other frequencies; an inevitable consequence of the 'waterbed effect' [13]. Figure 3 demonstrates how a parallel coupled active metamaterial can achieve a smaller maximum displacement per layer in the resonant band gap region. Compared to a passive construction, therefore, the parallel coupling material achieves greater performance efficiency with respect to material thickness.

Results were calculated for metamaterials consisting of 3-6 layers. The passive lumped parameter material and active skyhook material always experience maximum displacement in the first transmission mass. This can be understood physically as each layer acting independently and consisting of the transmission medium attached to a mechanical filter. As the energy passes through each layer it is attenuated by the action of the filter so the energy transferred to the next layer is lower, and elemental displacements are smaller. The parallel coupled model cannot be considered in such simple terms since the active forces acting on the resonators are coupled to the adjacent transmission masses so the cells no longer act independently. Here the maximum displacement alternates between the first and last transmission masses depending on whether there are an odd or even number of layers. This may be physically explained by examining how the mode shape changes for a material when the number of layers is changed from odd to even, and requires further investigation.

Used in an active metamaterials physical size of the actuators will be a function of the magnitude of the force required, which here is a function of the displacements and velocities of the mass elements. This data can therefore again be extracted for any given case from the equations of motion. Figure 4 shows the maximum required feedback forces for a skyhook active metamaterial (a), a parallel coupling active metamaterial (b) and the required force as a function of the numbers of layers (c).

The amount of feedback force required to realise the parallel coupled metamaterial is over an order of magnitude higher than that required for the skyhook example. Being a function of displacement, a peak in the required force occurs in the resonant band gap. However it was observed in the parallel coupled material that at frequencies approaching 2 octaves lower the maximum feedback force peaked at almost twice that experienced within the band gap so, if the metamaterial is intended to be used in regions other than the band gap, larger actuators will be required. Increasing the number of layers can greatly reduce the magnitude of the feedback force required to realise the parallel coupled material, as Figure 4(c) shows, however this tends towards a finite value.

The force and displacements of the metamaterial are linearly proportional to the magnitude of the excitation force. Following the approach detailed above will provide vital information when designing an active metamaterial, providing a quick and easy tool to assess the viability of a material of a given specification and to gain insight into the trade offs involved when specifying material performance where there is a limit on the numbers of layers, total material thickness and the available transducers. These preliminary results in-
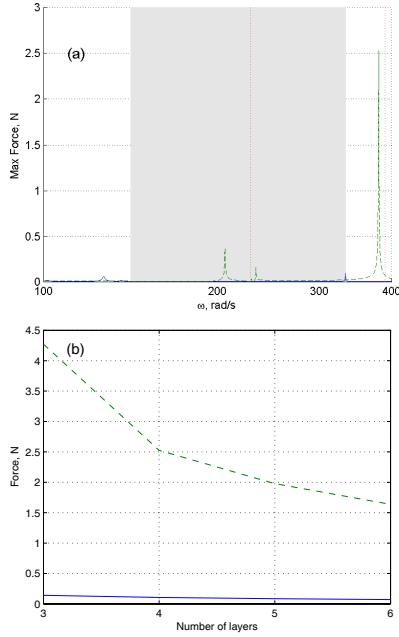
Figure 4: (a)The maximum applied feedback control force within a 4 layer active metamaterial with skyhook architecture (solid line) and parallel coupled architecture (dashed). Also shown is the region of negative effective mass for the skyhook material (shaded), and parallel coupled material (between the dotted lines). (b) The maximum required force in the band gap as a function of the number of layers.

dicate that for a DNG material there is an optimum trade-off between displacement and force, and both dictate physical size.

4 Isolation Performance of Active Metamaterials

One of the major drawbacks of the metamaterials described in Section 2, and limiting their potential use, is that the bandwidth in which the desired behaviour occurs is very narrow. This is particularly true of the region of attenuation associated with the resonator elements which has a deep but narrow, asymmetric attenuation profile. As shown in Section 3, the layers of passive metamaterial can be considered to be practically independent of each other with each resonator acting like a mechanical filter. For transmission blocking the resonators are performing the function of a traditional vibration absorber, but with an additional stiffness and damping term related to the static earth connection. By adapting vibration absorber theory [15], the approximate frequency at which the resonator acts (neglecting damping) is

$$\omega_r = \sqrt{\frac{k_r + k_h}{m_r}} \qquad (7)$$

By considering the Kelvin-Voigt model of an equivalent viscoelastic material shown in Figure 5(a), the behaviour of each layer of a material can be described in terms of the motion of the previous layer using a transfer matrix, $\mathbf{T}_n$.

$$\left\{ \begin{array}{c} f_{n+1} \\ x_{n+1} \end{array} \right\} = \left[\begin{array}{cc} T_{n,1} & T_{n,2} \\ T_{n,3} & T_{n,4} \end{array} \right] \left\{ \begin{array}{c} f_n \\ x_n \end{array} \right\} \qquad (8)$$

Where x_n and f_n are, respectively, the displacement of the equivalent mass M_{e_n} and the force acting on that mass due to the viscoelastic connections to its adjacent layer. Assuming all layers are identical, calculating the dynamic response of the material the transfer matrix, $\mathbf{T}$, is

$$\mathbf{T} = \left[\begin{array}{cc} 1 & -\omega^2 M_e \\ \frac{1}{K_e + i\omega C_e} & 1 - \frac{\omega^2 M_e}{K_e + i\omega C_e} \end{array} \right] = \frac{a}{b} \left[\begin{array}{cc} \frac{b}{a} & \frac{-\omega^2 M_e b}{a} \\ \frac{1}{a} & 1 \end{array} \right] \qquad (9)$$

Where $a = -M_e\omega^2 + i\omega C_e + K_e$ and $b = i\omega C_e + K_e$. If $\mathbf{\Gamma}$ is used to denote the square matrix on the right hand side of Eq. 9, then by applying the boundary condition $f_N = 0$ the transfer function across the material becomes

$$H(s) = \frac{x_N}{x_0} = \left(\frac{-\mathbf{\Gamma}_{1,2}^N \mathbf{\Gamma}_{2,1}^N}{\mathbf{\Gamma}_{1,1}^N} + \mathbf{\Gamma}_{2,2}^N \right) \times \left(\frac{a}{b} \right)^N \qquad (10)$$

Where the subscript (p, q) denotes the pth row and qth column of $\mathbf{\Gamma}^N$. From Eqs. 8 and 9 it is clear that for the metamaterials described in Section 2, the material parameters are constant for all n and the transfer matrices associated with the layers are identical. With reference to the common factor, $\frac{a}{b}$, the locations of the poles and zeros associated with this term will remain identical, layer to layer. With multiple layers this term is merely raised to the power n and therefore the attenuation associated with $(\frac{a}{b})^n$ will increase in magnitude as more zeros occupy the same location, but the frequency at which the attenuation occurs will remain the same. A more desirable objective may be to widen rather than deepen the band gap . This can be achieved by altering $\mathbf{T}$ with each layer by changing the value of $M_{e,n}$ so the zeros associated with $\frac{a}{b}$ move as a function of n.

In a multiple layer material this can be achieved by staggering the frequencies at which the resonators act [6]. Now the region of attenuation can be widened at the expense of notch depth, as shown in Figure 5(b). In addition, this pushes the band gap profile towards a more symmetric shape; potentially a desirable characteristic. Whilst it would be possible to achieve this with a passive material by varying $\frac{kr+kh}{mr}$ from layer to layer, when manufacturing such a material it would be more practical to produce identical layers that include active components. Now the value of Eq. 7 can be adapted by replacing kh with an active feedback control force. This would simplify the production of such materials whilst also creating a metamaterial where the region of the low frequency band gap and DNG behaviour is both widened and adaptable.

Whilst the appearance of a wider band gap may be desirable, designing such a material poses a problem. There are trade offs to be made when designing the shape and width of the resultant band gap. For a given target frequency and number of layers it is not obvious what the frequency gap between adjacent resonators should be to achieve an optimum solution. Whilst each resonator will absorb maximum energy at its tuned frequency, either side of this frequency the response of the element will exhibit a resonant peak. If the filter frequencies are too far apart this peak will not be filtered out by adjacent resonators and will compromise the performance of the band gap. Alternatively, if the filters are too close in frequency the full potential bandwidth of the gap will not be exploited. Additionally, due to the asymmetric nature of each resonators band gap response, shaping the overall band gap in a controllable way is non trivial.
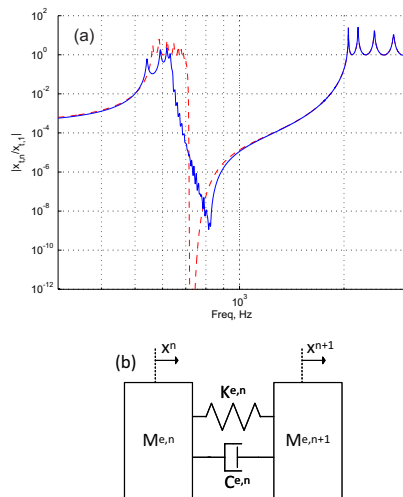
Figure 5: (a) The low frequency resonant band gap when the resonators are all tuned to the same frequency (dotted line) and when they are staggered about a target frequency (solid line). (b) The equivalent Kelvin-Voigt viscoelastic representation of a metamaterial.

By approaching the above problem with an active control mindset, a solution can be sought using existing control design methods. The active system as a whole can be modeled so the feedback forces of the system are represented by gain coefficients within the feedback loop. Now, a band gap shape can be specified and a solution be found using established design methods such as H_∞ optimisation. The viability of this approach is currently being investigated by the authors.

5 Conclusion

A one dimensional, lumped-parameter active acoustic metamaterial was derived from a passive design based on a duct fitted with an array of Helmholtz resonators. The control architecture was extended to include forces on the resonators as a function of adjacent transmission masses, and the advantages this architecture provides the physical properties of the material were demonstrated. The difficulty of designing a metamaterial to operate as a useful isolator was highlighted and it was shown that the potential of applying control optimisation techniques, the physical advantages active control systems over their passive counterparts and the adaptability of their working range could lead to metamaterials that not only have compact, practical dimensions, but enhanced performance over a useful, wideband frequency range.

6 Acknowledgments

References

[1] L. Airoldi and M. Ruzzene, "Design of tunable acoustic metamaterials through periodic arrays of resonant shunted piezos", *New Journal of Physics*, **13**, (2011)

[2] W. Akl and A. Baz, "Multi-cell Active Acoustic Metamaterial with Programmable Bulk Modulus", *Journal of Intelligent Material Systems and Structures*, **21**(5), (2010)

[3] H. Y. Chen and C. T. Chan, "Acoustic cloaking and transformation acoustics", *Journal of Physics D-Applied Physics*', **43**(11), (2010)

[4] Y. Cheng, J. Y. Xu and J. X. Liu, "Broad forbidden bands in parallel-coupled locally resonant ultrasonic metamaterials", *Applied Physics Letters*, **92**(5), (2008)

[5] S. A. Cummer and D Schurig, "One path to acoustic cloaking", *New Journal of Physics*, **9**, (2007)

[6] C. L. Ding and X. P. Zhao, "Multi-band and broadband acoustic metamaterial with resonant structures", *Journal of Physics D: Applied Physics*, **44**, (2011)

[7] N. Fang et al., "Ultrasonic metamaterials with negative modulus", *Nature Materials*, **5**(6), (2006)

[8] X. C. Huang et al., "Design and optimization of periodic structure mechanical filter in suppression of foundation resonances", *Journal of Sound and Vibration*, **330**(20), (2011)

[9] X. J Liu et al., "A multilayer structured acoustic cloak with homogeneous isotropic materials", *Applied Physics Letters*, **92**(15), (2008)

[10] Z. Y. Liu et al., 'Elastic wave scattering by periodic structures of spherical objects: Theory and Experiment", *Physical Review B*, **62**(4), (2000)

[11] Z. Y. Liu et al., "Locally resonant sonic materials", *Science*, **289**(5485), (2000)

[12] S. A. Pope and S. Daley, "Viscoelastic locally resonant double negative metamaterials with controllable effective density and elasticity", *Physics Letters A*, **374**, (2010)

[13] S. Skogestad and I. Postlethwaite, "Multivariable Feedback Control: Analysis and Design", Chichester: Wiley, (1996)

[14] D. R. Smith et al, "Composite medium with simultaneously negative permeability and permittivity", *Physical Review Letters*, **84**(18), (2000)

[15] W. Thomson, "Theory of Vibration with Applications", 4th edition, London: Chapman & Hall, (1993)

[16] V. Veselago, The electrodynamics of substances with simultaneously negative values of permittivity and permeability, *Soviet Physics Uspekhi-Ussr*, **10**(4), (1968)

[17] Y. Xiao et al, "Formation and coupling of band gaps in a locally resonant elastic system comprising a string with attached resonators", *Physics Letters A*, **375**(12), (2011)

[18] S. X. Yang et al., "Focusing of sound in a 3D phononic crystal, *Physical Review Letters*, **93**(2), (2004)