COVERS FOR S-ACTS AND CONDITION (A) FOR A MONOID S

ALEX BAILEY, VICTORIA GOULD, MIKLÓS HARTMANN, JAMES RENSHAW, AND LUBNA SHAHEEN

Abstract. A monoid S satisfies *Condition (A)* if every locally cyclic left S-act is cyclic. This condition first arose in Isbell's work on left perfect monoids, that is, monoids such that every left S-act has a projective cover. Isbell showed that S is left perfect if and only if every cyclic left S-act has a projective cover and Condition (A) holds. Fountain built on Isbell's work to show that S is left perfect if and only if it satisfies Condition (A) together with the descending chain condition on principal right ideals, M_R. We note that a ring is left perfect (with an analogous definition) if and only if it satisfies M_R. The appearance of Condition (A) in this context is therefore monoid specific.

Condition (A) has a number of alternative characterisations, in particular, it is equivalent to the ascending chain condition on cyclic subacts of any left S-act. In spite of this, it remains somewhat esoteric. The first aim of this article is to investigate the preservation of Condition (A) under basic semigroup-theoretic constructions.

Recently, Khosravi, Ershad and Sedaghatjoo have shown that every left S-act has a strongly flat or Condition (P) cover if and only if every cyclic left S-act has such a cover and Condition (A) holds. Here we find a range of classes of S-acts $\mathcal{C}$ such that every left S-act has a cover from $\mathcal{C}$ if and only if every cyclic left S-act does and Condition (A) holds. In doing so we find a further characterisation of Condition (A) purely in terms of the existence of covers of a certain kind.

Finally, we make some observations concerning left perfect monoids and investigate a class of monoids close to being left perfect, which we name *left $\mathcal{IP}a$-perfect.*

1. Introduction

Throughout this article, S denotes a monoid. Our aim is to add to the understanding of the so-called Condition (A) for S. Let A and B be left S-acts and let $\theta : A \to B$ be an onto S-morphism. We say that θ is *co-essential* if for any proper S-subact C of A, the restriction of θ to C is not onto. In this case we say A is a *cover* for B (more properly, (A, θ) is a cover for B). If $\mathcal{C}$ is a class of left S-acts then S is said to be *left $\mathcal{C}$-perfect* if every left S-act has a $\mathcal{C}$-cover, that is, a cover lying in $\mathcal{C}$ [4]. A *left perfect* monoid is one which is left $\mathcal{P}r$-perfect, where $\mathcal{P}r$ is the class of projectives. Left perfect monoids were shown by Fountain [2] and Isbell [5] to be exactly those satisfying Condition (A) and M_R

Date: January 26, 2014.

2000 *Mathematics Subject Classification.* 20 M 30, 20 M 50.

Key words and phrases. S-acts, projective, strongly flat, Condition (A), cover.

The authors acknowledge the support of EPSRC grant no. EP/I032312/1. Research also partially supported by the Hungarian Scientific Research Fund (OTKA) grant no. K83219.

(the descending chain condition on principal right ideals). We refer the reader to [7] for background details concerning acts over S.

After some preliminaries, we give known equivalent characterisations of Condition (A) in Section 2. Section 3 is devoted to the preservation of Condition (A) under some standard constructions. Next, for the convenience of the reader and for notational consistency, we have a short section defining classes of S-acts related to projectivity and flatness. These classes are used in Section 5 to find a new description of Condition (A) purely in terms of the existence of covers of a certain kind.

We stress that our techniques in Section 5 are essentially based on interpreting existing work. In Section 6 we then apply our results to investigate classes of left S-acts having a cover which is a disjoint union of cyclic left S-acts, or more particularly, of principal left ideals (thus, a cover from a rather larger class than $\mathcal{P}r$). In the commutative case we can generalise known results for left perfect monoids. Our final section contains a number of examples and counterexamples.

2. Condition (A)

A left S-act A is *cyclic* if $A = Sa$ for some $a \in A$ (equivalently, $A \cong S/\rho$ for a left congruence ρ on A) and *locally cyclic* if for any $a, b \in A$ there exists $c \in A$ such that $a, b \in Sc$.

Definition 2.1. A monoid S has Condition (A) if every locally cyclic left S-act is cyclic.

The following lemma gives a number of alternative characterisations of Condition (A), taken from [5, 2] and [4], with the exception of (v), which clearly follows from the equivalence of its predecessors.

Lemma 2.2. *The following conditions are equivalent for a monoid S:*

(i) S satisfies Condition (A);

(ii) every left S-act satisfies the ascending chain condition on cyclic subacts;

(iii) for every sequence $a_1, a_2, \ldots$ of elements of S, there exists $n \in \mathbb{N}$ such that for all $m \geq n$, there exists $k \geq 1$ such that $Sa_m a_{m+1} \ldots a_{m+k} = Sa_{m+1} \ldots a_{m+k}$;

(iv) for each left S-act A, there is a set $\{A_i : i \in I\}$ of locally cyclic left S-acts such that $A = \bigcup_{i \in I} A_i$ and for all $j \in I$, $A_j \not\subseteq \bigcup_{i \neq j} A_i$;

(v) for each left S-act A, there is a set $\{A_i : i \in I\}$ of cyclic left S-acts such that $A = \bigcup_{i \in I} A_i$ and for all $j \in I$, $A_j \not\subseteq \bigcup_{i \neq j} A_i$.

Further equivalent characterisations for Condition (A) may be found in [12, Lemma 3.1]. Clearly Condition (A) implies the ascending chain condition on principal left ideals of S but it is, in general, stronger [5].

Remark 2.3. In checking Condition (A) by part (iii) of Lemma 2.2, it is enough to consider sequences not containing the identity and not containing subproducts $a_i a_{i+1} \ldots a_j$ which are right zeros.

Proof. First note that if the sequence contains only finitely many non-identities then it is enough to choose n such that $a_i = 1$ for every $i > n$. On the other hand, if it contains infinitely many non-identities then it is easy to check that it satisfies the required condition if and only if the subsequence consisting of all non-identity elements does.

If for every m there exists i, j with $m < i \leq j$ such that $a_i a_{i+1} \dots a_j$ is a right zero, then the sequence $a_1, a_2, \dots$ clearly satisfies the condition. So we can suppose that there exists $n \in \mathbb{N}$ such that the sequence $a_n, a_{n+1}, \dots,$ does not have any right zero subproduct $a_i a_{i+1} \dots a_j$. It is straightforward that the sequence $a_1, a_2, \dots$ satisfies the required condition if and only if the sequence $a_n, a_{n+1}, \dots$ does, so the remark is proved. □

Corollary 2.4. *Let S be a monoid. Then S satisfies Condition (A) if and only if S^0 satisfies Condition (A).*

3. Constructions and Condition (A)

In this section we are going to investigate when submonoids, homomorphic images, direct and semidirect products satisfy Condition (A).

Lemma 3.1. *The class of monoids satisfying Condition (A) is closed under homomorphic images.*

Proof. Let S be a monoid that satisfies Condition (A) and $\phi : S \to T$ a homomorphism of monoids. Given any sequence $s_1\phi, s_2\phi, \dots$ of elements in the image of ϕ, there exists $n \in \mathbb{N}$ such that for all $m \geq n$ there exist $k \geq 1$, $s \in S$ such that $ss_m s_{m+1} \dots s_{m+k} = s_{m+1} \dots s_{m+k}$, so $(s\phi)(s_m\phi) \dots (s_{m+k}\phi) = (s_{m+1}\phi) \dots (s_{m+k}\phi)$ and the result follows. □

We now turn our attention to submonoids. Since a group clearly satisfies Condition (A), the next result shows that the class of monoids satisfying Condition (A) is not closed under submonoids.

Lemma 3.2. *A cancellative monoid satisfies Condition (A) if and only if it is a group.*

Proof. If S is cancellative and $a^n \,\mathcal{L}\, a^{n+1}$, then it follows that a is a unit. Thus if S has Condition (A), considering sequences of the form $a, a, \dots$ gives that S is a group. □

On the positive side we have the following lemma, the proof of which is clear.

Lemma 3.3. *Let T be a submonoid of S such that for any $a, b \in T$ we have*

$$a \,\mathcal{L}\, b \text{ in } T \Leftrightarrow a \,\mathcal{L}\, b \text{ in } S.$$

If S satisfies Condition (A), then so does T.

Submonoids T of S satisfying the condition of Lemma 3.3 include regular submonoids and retracts of S either as submonoids, or in the category of right T-acts. Examples of the latter are right self-injective submonoids, by which we mean that T is injective as a right T-act. (See Comments at the end of [7, Section IV.5] for some examples of self-injective monoids). Further, it is shown in [1, Proposition 5.14] that if T is a pure submonoid (for the definition, see [1]), then again T inherits Condition (A) from S.

As the following lemmas show, Condition (A) is preserved under finite direct products, but not under infinite direct products or free products.

Lemma 3.4. *The class of monoids satisfying Condition (A) is closed under finite direct products.*

Proof. It is sufficient to show preservation for a direct product of two monoids. Let $S = S_1 \times S_2$ be the direct product of monoids S_1 and S_2 that both satisfy Condition (A). Now given any sequence $(a_1, b_1), (a_2, b_2), \ldots \in S$ there exist $n_1, n_2 \in \mathbb{N}$ such that for all $m \geq n_1, n_2$ there exist $k_1, k_2 \geq 1$ such that

$$\begin{aligned} S_1 a_m a_{m+1} \ldots a_{m+k_1} &= S_1 a_{m+1} \ldots a_{m+k_1} \\ S_2 b_m b_{m+1} \ldots b_{m+k_2} &= S_2 b_{m+1} \ldots b_{m+k_2}. \end{aligned}$$

Let $N = \max\{n_1, n_2\}$ and for all $M \geq N$, let $K = \max\{k_1, k_2\}$. Then

$$S(a_M, b_M)(a_{M+1}, b_{M+1}) \ldots (a_{M+K}, b_{M+K}) = S(a_{M+1}, b_{M+1}) \ldots (a_{M+K}, b_{M+K})$$

and so S satisfies Condition (A). □

The following examples show that the class of monoids satisfying Condition (A) is not closed under infinite direct products nor under free products.

Example 3.5. Let $S = \prod_{i \in \mathbb{N}} T_i$ where $T_i = T$ is a monoid containing an element t which has no left inverse. Considering the sequence

$$s_1 = (t, 1, 1, \ldots), s_2 = (1, t, 1, \ldots), s_3 = (1, 1, t, \ldots), \ldots$$

we see that S does not satisfy Condition (A). Note that S is residually finite if T is finite.

Example 3.6. Let S_1, S_2 be non-trivial monoids and let $S = S_1 * S_2$. Take any sequence $s_1, s_2, s_1, s_2, s_1 \ldots$ where $s_1 \in S_1$ and $s_2 \in S_2$ are non-identities. Then for any $m, k \geq 1$, $s_m s_{m+1} \ldots s_{m+k}$ is a word of length $k+1$ and $s_{m+1} \ldots s_{m+k}$ is a word of length k so the principal ideals generated by these words can never be equal.

We say that a monoid T acts on a monoid S by endomorphisms on the left, if for every $t \in T$ there exists a monoid endomorphism $\phi_t : S \to S$ such that $\phi_t(\phi_u(s)) = \phi_{tu}(s)$ (and $\phi_1 = id_S$) for all $t, u \in T$, $s \in S$. We denote $\phi_t(s)$ by ${}^t s$.

Given two monoids S and T, with T acting on S by endomorphisms on the left, the semidirect product $S \rtimes T$ is a monoid with underlying set $S \times T$, with binary operation

$$(s_1, t_1)(s_2, t_2) = (s_1 \, {}^{t_1} s_2, t_1 t_2).$$

and identity $(1, 1)$. It is then clear that $(s_1, t_1) \ldots (s_k, t_k) = (s_1 \, {}^{t_1} s_2 \, {}^{t_1 t_2} s_3 \ldots {}^{t_1 \ldots t_{k-1}} s_k, t_1 \ldots t_k)$.

The wreath product $S \wr T$ of a monoid S by a monoid T is the semidirect product $S^T \rtimes T$ where T acts on S^T by $t'({}^t \phi) = (tt')\phi$ for all $t' \in T$ where $\phi \colon T \to S$.

Lemma 3.7. *Let S be a monoid and T a monoid acting on S by endomorphisms on the left. If $S \rtimes T$ satisfies Condition (A) then S and T satisfy Condition (A).*

Proof. Note that $\phi \colon S \rtimes T \to T$, $(s, t) \mapsto t$ is a surjective homomorphism and so T satisfies Condition (A) by Lemma 3.1.

To show that S also does, let $s_1, s_2, \ldots$ be a sequence of elements of S. Let us consider the sequence $(s_1, 1), (s_2, 1), \ldots$ in $S \rtimes T$. By Condition (A) there exists $n \in \mathbb{N}$ such that for every $m \geq n$ there exist $k \geq 1$ and $(s, t) \in S \rtimes T$ satisfying

$$(s, t)(s_m, 1)(s_{m+1}, 1) \ldots (s_{m+k}, 1) = (s_{m+1}, 1) \ldots (s_{m+k}, 1).$$

As a consequence we have that $t = 1$, so $ss_m s_{m+1} \ldots s_{m+k} = s_{m+1}s_{m+2} \ldots s_{m+k}$, which proves that S satisfies Condition (A). □

Lemma 3.8. *Let G be a group and S a monoid satisfying Condition (A) acting on G by endomorphisms on the left. Then the semidirect product $G \rtimes S$ satisfies Condition (A).*

Proof. Given any sequence $(g_1, s_1), (g_2, s_2), \ldots \in G \rtimes S$, there exists $n \in \mathbb{N}$ such that for all $m \geq n$ there exist $k \geq 1$, $s \in S$ such that $ss_m s_{m+1} \ldots s_{m+k} = s_{m+1} \ldots s_{m+k}$. Now let

$$\begin{aligned} h &= {}^{s}g_m \, {}^{ss_m}g_{m+1} \, {}^{ss_m s_{m+1}}g_{m+2} \ldots {}^{ss_m \ldots s_{m+k-1}}g_{m+k} \\ \text{and} \qquad g &= g_{m+1} \, {}^{s_{m+1}}g_{m+2} \ldots {}^{s_{m+1} \ldots s_{m+k-1}}g_{m+k}. \end{aligned}$$

(Note that if $k = 1$ then $h = {}^{s}g_m \, {}^{ss_m}g_m$ and $g = g_{m+1}$.) Calculating,

$$\begin{aligned} (gh^{-1}, s)(g_m, s_m)(g_{m+1}, s_{m+1}) \ldots (g_{m+k}, s_{m+k}) &= (gh^{-1}h, ss_m s_{m+1} \ldots s_{m+k}) \\ &= (g, s_{m+1} \ldots s_{m+k}) \\ &= (g_{m+1}, s_{m+1}) \ldots (g_{m+k}, s_{m+k}), \end{aligned}$$

and so $G \rtimes S$ satisfies Condition (A). □

Corollary 3.9. *Let G be a group and S a monoid satisfying Condition (A), then a wreath product $G \wr S$ satisfies Condition (A).*

Proof. Recall that $G \wr S$ is a semidirect product of the form $G^S \rtimes S$ and cartesian products of groups are still groups. □

As the following theorem shows, the most frequently used semidirect products preserve Condition (A).

Lemma 3.10. *A monoid semidirect product $S \rtimes G$ where G is a group satisfies Condition (A) if and only if S satisfies Condition (A).*

Proof. If $S \rtimes G$ satisfies Condition (A) then by Lemma 3.7 so does S. Conversely, if S satisfies Condition (A) then let $(s_1, g_1), (s_2, g_2), \ldots$ be a sequence in $S \rtimes G$. Let us consider the sequence s_1, ${}^{g_1}s_2$, ${}^{g_1 g_2}s_3, \ldots$ in S. Since S satisfies Condition (A), there exists $n \in \mathbb{N}$ such that for every $m \geq n$ there exist $k \geq 1$ and $s \in S$ satisfying

$$s \, {}^{g_1 \cdots g_{m-1}}s_m \, {}^{g_1 \cdots g_m}s_{m+1} \ldots {}^{g_1 \cdots g_{m+k-1}}s_{m+k} = {}^{g_1 \cdots g_m}s_{m+1} \ldots {}^{g_1 \cdots g_{m+k-1}}s_{m+k}.$$

As a consequence

$${}^{g_m^{-1} g_{m-1}^{-1} \cdots g_1^{-1}}s \, {}^{g_m^{-1}}s_m s_{m+1} \, {}^{g_{m+1}}s_{m+2} \ldots {}^{g_{m+1} \cdots g_{m+k-1}}s_{m+k} = s_{m+1} \, {}^{g_{m+1}}s_{m+2} \ldots {}^{g_{m+1} \cdots g_{m+k-1}}s_{m+k},$$

thus

$$({}^{g_m^{-1} g_{m-1}^{-1} \cdots g_1^{-1}}s, g_m^{-1})(s_m, g_m)(s_{m+1}, g_{m+1}) \ldots (s_{m+k}, g_{m+k}) = (s_{m+1}, g_{m+1}) \ldots (s_{m+k}, g_{m+k}),$$

which shows that $S \rtimes G$ satisfies Condition (A). □

Corollary 3.11. *Let S be a monoid that satisfies Condition (A) and G a finite group, then a wreath product $S \wr G$ satisfies Condition (A).*

Proof. Recall that $S \wr G$ is a semidirect product of the form $S^G \rtimes G$ and so the result follows by Lemmas 3.4 and 3.10. $\square$

Lemma 3.12. *Let T be a semigroup satisfying $uv \mathrel{\mathcal{L}} v$ for every $u, v \in T$ and let $S = \mathcal{M}^0[T; I; \Lambda; P]$ be a Rees matrix semigroup with zero over T such that the matrix P is regular (that is, every row and column of P contains a nonzero element). Then S^1 satisfies Condition (A).*

Proof. To show that S^1 satisfies Condition (A), by Remark 2.3, we need only consider sequences $s_1 = (i_1, t_1, \lambda_1), s_2 = (i_2, t_2, \lambda_2), \ldots$ of elements in S. By the same remark, we can also assume there are no pairs in the sequence whose product is zero, so that $p_{\lambda_r i_{r+1}} \in T$ for all $r \geq 1$. Let $m \in \mathbb{N}$, so that $s_m s_{m+1} = (i_m, t_m p_{\lambda_m i_{m+1}} t_{m+1}, \lambda_{m+1})$. Since P is regular, there exists some $\mu \in \Lambda$ (depending on m), such that $p_{\mu i_m} \in T$. Since $p_{\mu i_m} t_m p_{\lambda_m i_{m+1}} t_{m+1} \mathrel{\mathcal{L}} t_{m+1}$, there must exist some $t \in T$ such that $t p_{\mu i_m} t_m p_{\lambda_m i_{m+1}} t_{m+1} = t_{m+1}$. Let $s = (i_{m+1}, t, \mu)$ and note that $s s_m s_{m+1} = s_{m+1}$ and so S satisfies Condition (A). $\square$

Corollary 3.13. *Every completely* 0*-simple and completely simple semigroup with a* 1 *adjoined is left perfect.*

Proof. By the Rees Theorem, every completely 0-simple semigroup is a Rees matrix semigroup with zero over a group, so the resulting monoid satisfies Condition (A) by Lemma 3.12. For the completely simple case, note that by Corollary 2.4, a monoid M satisfies Condition (A) if and only if M^0 does.

By definition, completely (0-)simple semigroups have M_R, whence certainly so do the corresponding monoids. $\square$

Lemma 3.14. *Let L be a semigroup satisfying $uv \mathrel{\mathcal{L}} v$ for every $u, v \in L$ and let*

$$S = \{(n, x, m) : n, m \in \mathbb{N}, n \geq m, x \in L\} \cup \{0\}$$

with multiplication given by

$$(n, x, m)(m, y, l) = (n, xy, l),$$

all other products being 0 *(that is, S is a subsemigroup of the 'Brandt' semigroup $\mathcal{B}^0(L, \mathbb{N})$), and let $M = S^1$.*

Then the monoid M satisfies Condition (A) and does not have M_L.

Proof. We first show that M satisfies Condition (A). Let $a_1, a_2, \ldots$ be a sequence of elements in M. As in the proof of Lemma 3.12, by Remark 2.3 we can suppose that $a_i \neq 1$ and that $a_i a_{i+1} \neq 0$ for every $1 \leq i$. Putting $a_i = (n_i, x_i, m_i)$, we have $m_i = n_{i+1}$ for all $1 \leq i$. Hence $n_1 \geq m_1 = n_2 \geq m_2 = \ldots$; clearly the descending sequence stabilises with w such that $n_w = m_w = n_{w+1} = \ldots$.

For every $u \geq w$ we have that $x_u x_{u+1} \,\mathcal{L}\, x_{u+1}$ so there exists $l \in L$ such that $x_{u+1} = l x_u x_{u+1}$ which implies that

$$(n_w, l, n_w)a_u a_{u+1} = (n_w, l, n_w)(n_w, x_u, n_w)(n_w, x_{u+1} n_w) = (n_w, x_{u+1}, n_w) = a_{u+1},$$

so that Condition (A) is satisfied.

To see that M does not satisfy M_L, fix $x \in L$ and note that

$$M(1, x, 1) \supset M(2, x, 1) \supset M(3, x, 1) \supset \dots$$

is an infinite strictly descending chain of principal left ideals of M.

□

4. The classes

We now describe the classes of left S-acts which will form the main object of our concern in later sections of this article. Further details may be found, for example, in [7].

A left S-act A is *decomposable* if there exist left S-acts B and C such that $A = B \cup C$ with $B \cap C = \emptyset$. A left S-act which is not decomposable is called *indecomposable*. Every left S-act A can be uniquely written as a disjoint union of indecomposable left S-acts and these indecomposable components are the classes of $\sim$, where $\sim$ is the transitive closure of $\{(sa, ta) : s, t \in S, a \in A\}$.

It is clear that every locally cyclic left S-act is indecomposable, but the converse is only true in case S is a group [9, 12].

Definition 4.1. Let X be a property of left S-acts. Then $\mathcal{IX}$ is the class of left S-acts, the indecomposable components of which have property X.

Notice that classes of the form $\mathcal{IX}$ are precisely those that are closed with respect to taking coproduct (disjoint union) and indecomposable components.

Let LC, C, Pa, Pe and S denote the properties of left S-acts of being locally cyclic, cyclic, isomorphic to a principal left ideal, isomorphic to an idempotent generated principal left ideal and isomorphic to S (regarded as a left S-act), respectively. Then $\mathcal{IS}$ and $\mathcal{IP}e$ are the classes $\mathcal{F}$ and $\mathcal{P}r$ of free and projective left S-acts, and we have the class inclusions

$$\mathcal{F} = \mathcal{IS} \subseteq \mathcal{P}r = \mathcal{IP}e \subseteq \mathcal{IP}a \subseteq \mathcal{IC} \subseteq \mathcal{ILC}.$$

Of course if S is regular or even left abundant (that is, every principal left ideal Sa is S-isomorphic to one generated by an idempotent e, where the isomorphism takes a to e), then Pa is just Pe and $\mathcal{P}r = \mathcal{IP}e = \mathcal{IP}a$.

Lemma 4.2. *A left S-act A lies in $\mathcal{ILC}$ if and only if for all $a, b \in A$, if $Sa \cap Sb \neq \emptyset$ then $Sa \cup Sb \subseteq Sc$ for some $c \in A$.*

Proof. Let $A \in \mathcal{ILC}$ and $a, b \in A$ and suppose that $Sa \cap Sb \neq \emptyset$. Then a and b lie in the same indecomposable component say C, since this is locally cyclic, $a, b \in Sc$ for some $c \in C$, i.e. $Sa \cup Sb \subseteq Sc$.

Conversely, let U be an indecomposable component of A. Let $u, v \in U$ so that as $u \sim v$ there exists a sequence

$$u = s_1 a_1,\ t_1 a_1 = s_2 a_2, \cdots, t_n a_n = v$$

where $n \in \mathbb{N}, a_i \in A, s_i, t_i \in S$ for $1 \leq i \leq n$. If $n = 1$, then $u, v \in Sa_1$ where clearly $a_1 \in U$.

Suppose inductively that $1 \leq k < n$ and $Sa_1 \cup \cdots \cup Sa_k \subseteq Sw_k$ for some $w_k \in U$. With $a_k = rw_k$ we have $t_k r w_k = s_{k+1} a_{k+1}$ and again we call upon our assumption to obtain $Sw_k \cup Sa_{k+1} \subseteq Sw_{k+1}$ for some $w_{k+1} \in U$. Hence $Sa_1 \cup \cdots \cup Sa_{k+1} \subseteq Sw_{k+1}$ and finite induction gives the result. $\square$

Finally in this section we consider three further classes of left S-acts, namely $\mathcal{SF}$, $\mathcal{P}$, and $\mathcal{WPF}$, consisting of the *strongly flat, Condition (P)* and *weakly pullback flat* left S-acts respectively. We recall that a left S-act A is strongly flat if it is a direct limit of finitely generated free left S-acts, and this is equivalent to satisfying Conditions (P) and (E):

(P) for all $s, t \in S$ and $a, b \in A$, if $sa = tb$ then $su = tv, a = uc$ and $b = vc$ for some $u, v \in S$ and $c \in A$;

(E) for all $s, t \in S$ and $a \in A$, if $sa = ta$ then $su = tu$ and $a = uc$ for some $u \in S$ and $c \in A$.

Condition (E)$'$ is defined as follows:

(E)$'$ for all $s, t, z \in S$ and $a \in A$, if $sa = ta$ and $zs = zt$, then $a = uc$ and $su = tu$ for some $u \in S$ and $c \in A$.

A left S-act A is *weakly pullback flat* if it satisfies (P) and (E)$'$. It is known that $\mathcal{P}r \subseteq \mathcal{SF}$ and, using Lemma 4.2, it is clear that

$$\mathcal{SF} = \mathcal{ISF} \subseteq \mathcal{WPF} = \mathcal{IWPF} \subseteq \mathcal{P} = \mathcal{IP} \subseteq \mathcal{ILC}.$$

5. The general result for covers

We now consider the question of existence of covers. The proof of our result is easy, since the hard steps all follow from Lemma 2.2.

Theorem 5.1. *Let X be a property of left S-acts such that $\mathcal{IX} \subseteq \mathcal{ILC}$, that is, X is stronger than being locally cyclic. Then S is left $\mathcal{IX}$-perfect if and only if every cyclic left S-act has an $\mathcal{IX}$-cover and Condition (A) holds.*

Proof. Suppose that every left S-act has an $\mathcal{IX}$-cover and let A be a left S-act. It follows from (ii) [4, Theorem 2.2] (and is easy to see from the definition of cover), that

$$A = \bigcup_{i \in I} A_i,\ A_j \not\subseteq \bigcup_{i \neq j} A_i \text{ for all } j \in I,$$

where each A_j is the image of an indecomposable S-act B_j with property X. Hence B_j is locally cyclic and consequently, A_j is locally cyclic. From Lemma 2.2, S satisfies Condition (A).

Conversely, suppose that every cyclic left S-act has an $\mathcal{IX}$-cover and Condition (A) holds. By Lemma 2.2, every locally cyclic left S-act is cyclic, hence has an $\mathcal{IX}$-cover.

Let A be a left S-act. Since S satisfies Condition (A), Lemma 2.2 gives

$$A = \bigcup_{i \in I} A_i, \;\; A_j \not\subseteq \bigcup_{i \neq j} A_i \text{ for all } j \in I,$$

where each $A_i = Sa_i$ is a cyclic S-subact of A. Now each A_i has an $\mathcal{IX}$-cover B_i (which must actually be cyclic), so there is a co-essential S-morphism $\theta_i : B_i \to A_i$. Let B be the disjoint union $\bigcup_{i \in I} B_i$, so that $B \in \mathcal{IX}$, and let $\theta : B \to A$ restrict to θ_i on each B_i. If θ is not co-essential, there is some $j \in I$ and (possibly empty) proper S-subact C_j of B_j such that $\theta : \bigcup_{i \neq j} B_i \cup C_j \to A$ is onto. Hence either $a_j = c\theta_j$ for some $c \in C_j$ (contradicting θ_j being co-essential) or $a_j = b\theta_i$ for some $b \in B_i$ with $i \neq j$, contradicting $A_j \not\subseteq \bigcup_{i \neq j} A_i$. Hence θ is co-essential. ☐

We immediately have our promised characterisation of Condition (A) by covers.

Theorem 5.2. *The following conditions are equivalent for a monoid S:*
(i) S has Condition (A);
(ii) S is left $\mathcal{ILC}$-perfect;
(iii) S is left $\mathcal{IC}$-perfect.

Proof. Every cyclic left S-act is its own $\mathcal{ILC}$-cover and $\mathcal{IC}$-cover. ☐

We now proceed to deduce some known results.

Corollary 5.3. [4, Corollary 2.3] *A monoid is left $\mathcal{F}r$-perfect if and only if it is a group.*

Proof. As pointed out in [4], it is clear that the trivial left S-act Θ has a free cover if and only if S is a group, and groups satisfy Condition (A). Moreover, if S is a group then S is a free cover of S/ρ via the natural S-morphism, for any left congruence ρ. ☐

Corollary 5.4. [5] *A monoid S is left perfect if and only if every cyclic left S-act has a projective cover and Condition (A) holds.*

A submonoid T of S is *right unitary* if for any $s, t \in S$, if $st, t \in T$, then $s \in T$. From [5, 1.3], a submonoid is right unitary (referred to as a *block* in that article) if and only if it is the class of the identity for some left congruence on S. It is well known and easy to see that if $\rho_T = \langle T \times T \rangle$, that is, ρ_T is the left congruence generated by $T \times T$, then $T = [1]$.

We observe that if a cyclic left S-act S/ρ has a projective cover, this must necessarily be cyclic, hence of the form Se for some idempotent $e \in S$. Let $\theta : Se \to S/\rho$ be co-essential. We cannot immediately deduce that $e \in [1]$. However, if $(pe)\theta = [1]$, then it follows from the co-essentiality of θ that $qpe = e$ for some $q \in S$. It is easy to check that $peq \in E(S) \cap [1]$ and $peq \,\mathcal{D}\, e$. Isbell goes on to show:

Proposition 5.5. [5] *Every cyclic left S-act has a projective cover if and only if S satisfies Condition (D):*
(D) every right unitary submonoid has a minimal idempotent generated left ideal.

Thus Corollary 5.4 and Proposition 5.5 completely describe left perfect monoids. Fountain [2] shows that the conjunction of Conditions (A) and (D) is equivalent to S satisfying

Condition (A) and M_R, thus providing an alternative description of left perfect monoids. Further, he showed that a monoid is left perfect if and only if $\mathcal{SF} = \mathcal{P}r$.

Choosing X to be strongly flat or Condition (P) immediately yields:

Corollary 5.6. [4] *A monoid S is left $\mathcal{SF}$-perfect (left $\mathcal{P}$-perfect) if and only if every cyclic left S-act has a strongly flat cover (Condition (P) cover) and Condition (A) holds.*

A monoid S is said to be *left reversible* if for all $s, t \in S$ there exists $p, q \in S$ with $sp = tq$ and *right collapsible* if for all $s, t \in S$ there exists $r \in S$ with $sr = tr$. Further, S is said to be *weakly right collapsible* if for any $p, q, r \in S$ with $rp = rq$ there exists $u \in S$ such that $pu = qu$.

The next lemma follows from the definition of ρ_T, [10, Lemma 1.4] and [8, Lemma 7]. Note that if T is left reversible, then ρ_T takes on the simpler form that $a \,\rho_T\, b$ if and only if $au = bv$ for some $u, v \in T$.

Lemma 5.7. *(Cf.* [6, 8, 10]*) Let T be a right unitary right collapsible (left reversible and weakly right collapsible, left reversible) submonoid of S. Then S/ρ_T is strongly flat (weakly pullback flat, Condition (P)) and $[1] = T$.*

For comparison with what follows we recall the next result from [10, 11]:

Proposition 5.8. [10, Theorems 3.2, 4.2], [11, Theorem 4.3] *Every cyclic left S-act has a $\mathcal{SF}$-cover ($\mathcal{WPF}$-cover, $\mathcal{P}$-cover) if and only if every right unitary submonoid T of S contains a right collapsible (left reversible and weakly right collapsible, left reversible) submonoid R such that for all $u \in T$ we have $Su \cap R \neq \emptyset$.*

There appears to exist no natural chain conditions binding those of Proposition 5.8 with Condition (A), as in Fountain's result for left perfect monoids.

6. Left $\mathcal{IX}$-perfect monoids

The aim of this section is to give new and non-trivial applications of Theorem 5.1.

We begin with some notation. For an element a of a left S-act A we denote by $L(a)$ the set $\{t \in S : ta = a\}$, the (right unitary) submonoid of left identities of a, and by $\ell(a)$ the set $\{(u, v) \in S \times S : ua = va\}$, the *left annihilator congruence* σ_a of a. It is clear that Sa is isomorphic to S/σ_a under the S-isomorphism $sa \mapsto [s]$. The next lemma slightly reformulates results in Section 2 of [10].

Lemma 6.1. *(Cf.* [10, Section 2]*) Let $\mathcal{D}$ be a class of left S-acts. Then the following conditions are equivalent:*

(i) every cyclic left S-act has a $\mathcal{D}$-cover;

(ii) for every right unitary submonoid T of S there is a cyclic left S-act $Sa \in \mathcal{D}$ such that $\ell(a) \subseteq \rho_T$ and for all $u \in T$ we have $Su \cap L(a) \neq \emptyset$;

(iii) for every right unitary submonoid T of S there is a left congruence σ on S such that $S/\sigma \in \mathcal{D}$, $\sigma \subseteq \rho_T$ and for each $u \in T$ there is a $v \in S$ with $vu \,\sigma\, 1$.

Proof. $(i) \Rightarrow (ii)$ Let T be a right unitary submonoid of S. Then $T = [1]$ where $[1]$ is the ρ_T-class of the identity. By assumption, S/ρ_T has a $\mathcal{D}$-cover, which must be cyclic as S/ρ_T is. There is therefore a cyclic S-act $Sa \in \mathcal{D}$ and a co-essential S-morphism $\theta : Sa \to S/\rho_T$. By co-essentiality we may assume that $a\theta = [1]$. Since θ is well defined we have $\ell(a) \subseteq \rho_T$. If $u \in T$ then

$$a\theta = [1] = [u] = u[1] = u(a\theta) = (ua)\theta,$$

so that $\theta|_{Sua} : Sua \to S/\rho_T$ is onto. By co-essentiality we have $Sua = Sa$ and so $a = vua$ for some $v \in S$. Hence $Su \cap L(a) \neq \emptyset$.

$(ii) \Rightarrow (i)$ Let ρ be a left congruence and let $T = [1]$, the ρ-class of the identity, so that T is a right unitary submonoid. Notice that $T \times T \subseteq \rho$ and so $\rho_T \subseteq \rho$. Pick Sa satisfying the given conditions. Since $\ell(a) \subseteq \rho_T \subseteq \rho$ we have $\theta : Sa \to S/\rho$ given by $(ta)\theta = [t]$ is a well defined onto S-morphism. If $\theta|_{Sya} : Sya \to S/\rho$ is onto, then we must have that $(xya)\theta = [xy] = [1]$ for some $x \in S$. From $u = xy \in [1] = T$ we obtain $v \in S$ with $vu \in L(a)$ and so $vxya = a$. This gives that $Sa = Sya$ and θ is co-essential as required.

$(ii) \Leftrightarrow (iii)$ This follows from the remarks preceding the lemma. □

As an immediate consequence of Theorem 5.1 and Lemma 6.1 we have our first description of left $\mathcal{IX}$-perfect monoids for suitable X, and, in particular, of left $\mathcal{IP}a$-perfect monoids.

Corollary 6.2. (i) *Let X be a property of left S-acts such that $\mathcal{IX} \subseteq \mathcal{ILC}$. Then S is left $\mathcal{IX}$-perfect if and only if S satisfies Condition (A) and for every right unitary submonoid T of S there is a cyclic left S-act $Sa \in \mathcal{IX}$ such that $\ell(a) \subseteq \rho_T$ and for all $u \in T$ we have $Su \cap L(a) \neq \emptyset$.*

(ii) *Every cyclic left S-act has an $\mathcal{IP}a$-cover if and only if for every right unitary submonoid T of S there is an element $s \in S$ such that $\ell(s) \subseteq \rho_T$ and for all $u \in T$ we have $Su \cap L(s) \neq \emptyset$.*

(iii) *A monoid S is left $\mathcal{IP}a$-perfect if and only if it satisfies Condition (A) and the condition in (ii) above.*

We note that under the conditions in (ii) of Corollary 6.2 above, we immediately have that $L(s) \subseteq T$. Unfortunately, from this latter condition we cannot deduce that $\ell(s) \subseteq \rho_T$. This is essentially because $L(s) \times L(s)$ need not generate $\ell(s)$ nor a suitable $\ell(t)$ (compare with the analogous situations in [10]).

Example 6.3. Let X be a set and let S be the null semigroup on X with an identity adjoined (so that $S = X \cup \{1, 0\}$). Let ρ be the Rees congruence associated with the ideal $S \setminus \{1\}$. Pick any $x \in X$. The map $\theta : Sx \to S/\rho$ given by $(ux)\theta = [u]$ is an onto S-morphism which is clearly co-essential. However, $L(x) = \{1\}$ and the congruence generated by $L(x) \times L(x)$ is not $\ell(x)$ (nor indeed $\ell(u)$ for any $u \in S$ such that there is a co-essential S-morphism from Su onto S/ρ).

We now present a construction which will allow us to improve upon the description of left $\mathcal{IX}$-perfect monoids in Corollary 6.2, in particular in the case where S is commutative.

Let T be a submonoid of S. Let F be the free left S-act on $\{x_a : a \in T\}$, let ρ be the congruence on F generated by $H = \{(x_a, bx_{ab}) : a, b \in T\}$ and put $F(T) = F/\rho$.

Lemma 6.4. *Let T be a right unitary submonoid of S and let $F(T)$ be constructed as above. Then*

(i) *for any $s,t \in S$ and $a,b \in T$ we have $[sx_a] = [tx_b]$ if and only if $s = t, a = b$ or there exist $u_1,\dots,u_n \in S$ and $v_1,\dots,v_n,c_1,d_1,\dots,c_n,d_n \in T$ such that*

$$s = u_1c_1,\ u_1d_1 = u_2c_2,\ \dots,\ u_nd_n = t$$

and

$$a = v_1c_1,\ v_1d_1 = v_2c_2,\ \dots,\ v_nd_n = b;$$

(ii) *if T is left reversible, then $[sx_a] = [tx_b]$ if and only if $sh = tk$ and $ah = bk$ for some $h,k \in T$;*

(iii) *if T is right collapsible, then $[sx_a] = [tx_b]$ if and only if $sw = tw$ for some $w \in T$;*

(iv) *if T has a right zero z, then $[sx_a] = [tx_b]$ if and only if $sz = tz$.*

Proof. (i) If $s = t$ and $a = b$ then clearly $[sx_a] = [tx_b]$. On the other hand, if $u_1,\dots,u_n \in S$ and $v_1,\dots,v_n,c_1,d_1,\dots,c_n,d_n \in T$ exist connecting s to t and a to b as given, then

$$[sx_a] = [u_1c_1x_{v_1c_1}] = [u_1x_{v_1}] = [u_1d_1x_{v_1d_1}] = [u_2c_2x_{v_2c_2}] = \dots = [u_nd_nx_{v_nd_n}] = [tx_b].$$

Conversely, if $[sx_a] = [tx_b]$, then either $sx_a = tx_b$, so that $s = t$ and $a = b$, or there exists a ρ-sequence connecting $[sx_a]$ to $[tx_b]$. If the length of this sequence is 1, then

$$sx_a = wy,\ wz = tx_b$$

for some $w \in S$ and $(y,z) \in H \cup H^{-1}$. Without loss of generality suppose $(y,z) = (x_u, vx_{uv})$ for some $u,v \in T$. Then $s = w1$, $wv = t$ and $a = u1, uv = b$ so the result is true with $n = 1, w = u_1, u = v_1, c_1 = 1$ and $d_1 = v$. Suppose for induction that sx_a is connected to tx_b by a ρ-sequence of length n, and the result is true for all shorter sequences; moreover, we assume that a ρ-sequence of length $m < n$ is replaced by a pair of sequences of length m. Then

$$sx_a = wy,\ wz = rx_c$$

for some $w \in S$ and $(y,z) \in H \cup H^{-1}$ where rx_c is connected to tx_b via a ρ-sequence of length $n-1$. From the above and our inductive hypothesis we have $u_2,\dots,u_n \in S$ and $v_2,\dots,v_n,c_2,d_2,\dots,c_n,d_n \in T$ such that

$$r = u_2c_2,\ u_2d_2 = u_3c_3 \dots,\ u_nd_n = t$$

and

$$c = v_2c_2,\ v_2d_2 = v_3c_3,\ \dots,\ v_nd_n = b$$

and $u_1 \in S$, $v_1,c_1,d_1 \in T$ with $s = u_1c_1, u_1d_1 = r$, $a = v_1c_1, v_1d_1 = c$. The result is just a matter of glueing the two pairs of sequences together.

(ii) If $sh = tk$ for some $h,k \in T$ with $ah = bk$ then

$$s = s1,\ sh = tk,\ t1 = t \text{ and } a = a1,\ ah = bk,\ b1 = b,$$

so that $[sx_a] = [tx_b]$ by (i).

Suppose now that T is left reversible and $[sx_a] = [tx_b]$. If $s = t$ and $a = b$, take $h = k = 1$. Otherwise there exist $u_1, \ldots, u_n \in S$ and $v_1, \ldots, v_n, c_1, d_1, \ldots, c_n, d_n \in T$ such that

$$s = u_1c_1,\ u_1d_1 = u_2c_2,\ \ldots,\ u_nd_n = t$$

and

$$a = v_1c_1,\ v_1d_1 = v_2c_2,\ \ldots,\ v_nd_n = b.$$

If $n = 1$ then use left reversibility of T to choose $h, k \in T$ with $c_1h = d_1k$, so that $sh = u_1c_1h = u_1d_1k = tk$ and similarly $ah = bk$. If $n > 1$ we use induction to obtain $h, k, p, q \in T$ with $sh = u_2c_2k, ah = v_2c_2k$, $u_1d_1p = tq, v_1d_1p = bq$. Now pick $z, z' \in T$ such that $kz = pz'$ and we see that $shz = tqz', ahz = bqz'$ as required.

(iii) If T is right collapsible then it is certainly left reversible. If $sw = tw$, where $w \in T$, then choose $z' \in T$ with $az' = bz'$ and then $z'' \in T$ with $wz'' = z'z'' = z$ say, to obtain $sz = tz$ and $az = bz$ so that $[sx_a] = [tx_b]$ by (ii).

Conversely, if $[sx_a] = [tx_b]$ then we have in particular that $sh = tk$ for some $h, k \in T$ and so take $z \in T$ with $hz = kz$ to obtain $sw = tw$ where now $w = hz$.

(iv) Follows directly from (iii). □

The above result can be used to obtain a description of left $\mathcal{IX}$-perfect monoids (for suitable X), a little tighter than Corollary 6.2. For ease of notation, if s, t, a and b are connected as in (i) of Lemma 6.4, then we write $(s, a) \equiv_T (t, b)$.

Theorem 6.5. *Let X be a property of left S-acts such that $\mathcal{IX} \subseteq \mathcal{ILC}$. A monoid S is left $\mathcal{IX}$-perfect if and only if it satisfies Condition (A) and for every right unitary submonoid T of S, there is a cyclic left S-act Sa with property X and $c \in T$ such that for any $(p, q) \in \ell(a)$ we have $(p, c) \equiv_T (q, c)$ and for all $u \in T$ we have $Su \cap L(a) \neq \emptyset$.*

Proof. If S is left $\mathcal{IX}$-perfect then it satisfies Condition (A) by Theorem 5.1. Let T be a right unitary submonoid of S and consider $F(T)$. By assumption, and the fact that locally cyclic left S-acts are cyclic, it has an $\mathcal{IX}$-cover $G = \dot{\bigcup}_{i\in I} Sa_i$ where each Sa_i has property X. Let $\theta : G \to F(T)$ be co-essential and choose any $i \in I$. Then $a_i\theta = [ux_c]$ say, where $[x_c] = (wa_j)\theta$ for some $w \in S$ and $j \in I$. By co-essentiality $i = j$ and $Swa_i = Sa_i$; we may therefore assume that $a_i\theta = [x_c]$.

Write $a = a_i$. If $(p, q) \in \ell(a)$, then $p[x_c] = q[x_c]$ and so $(p, c) \equiv_T (q, c)$ by Lemma 6.4. Let $u \in T$. From (A) we have that $Su^n = Su^{n+1}$ for some $n \in \mathbb{N}$ and with $u^n = zu^{n+1}$ we see that

$$[x_c] = u^n[x_{cu^n}] = zu^{n+1}[x_{cu^n}] = zu[x_c],$$

so that by co-essentiality of θ we have that $Sa = Szua$, giving that $a = vzua$ and $vzu \in L(a)$, for some $v, z \in S$.

The converse is clear from Corollary 6.2, since if $(p, c) \equiv_T (q, c)$, then certainly $p \,\rho_T\, q$. □

We now specialise to the case of commutative S, where we can make more satisfactory progress. Where S is commutative, we drop the adjectives 'left, right', where appropriate. The reader could compare our next result with [4, Proposition 1.7].

Lemma 6.6. *Let S be commutative and let X be a property of left S-acts such that $\mathcal{IX} \subseteq \mathcal{IC}$. Then every strongly flat S-act which has a cover in $\mathcal{IX}$ lies in $\mathcal{IX}$.*

Proof. Suppose first that S/ρ is strongly flat and cyclic. Let Sa be a cover with property X and let $\theta : Sa \to S/\rho$ be a co-essential S-morphism with $a\theta = [1]$. If $(ua)\theta = (va)\theta$ then $[u] = [v]$ so that $u[1] = v[1]$ and by Condition (E) we have $uh = vh$ for some $h \in S$ with $[1] = h[1]$. Hence $\theta|_{Sha} : Sha \to S/\rho$ is onto, giving us that $kha = a$ for some $k \in S$. From $uhka = vhka$ we obtain $ua = va$, so that θ is an S-isomorphism as required.

Now consider a strongly flat S-act A having an $\mathcal{IX}$ cover. Write A as a disjoint union of indecomposable strongly flat S-acts and let B be one of these indecomposable components. By [12, Theorem 3.7], B is locally cyclic. Clearly B has an $\mathcal{IX}$ cover $\bigcup_{i\in I} Sa_i$ where each Sa_i has property X and the union is disjoint. By [4, Lemma 1.4], $|I| = 1$, so that B is cyclic. The result follows. $\square$

Example 7.1 demonstrates that Lemma 6.6 need not be true if S is not commutative.

We recall from [2] that S is left perfect if and only if every strongly flat left act is projective. Our next result is analogous.

Theorem 6.7. *Let S be commutative and let X be a property of S-acts such that $\mathcal{IX} \subseteq \mathcal{IC}$. Then the following are equivalent:*

(i) S is $\mathcal{IX}$-perfect;

(ii) every strongly flat S-act is in $\mathcal{IX}$;

(iii) S satisfies Condition (A) and for any unitary submonoid T of S, there exists a cyclic S-act Sa with property X such that for any $p, q \in S$, if $pa = qa$ then $pt = qt$ for some $t \in T$, and for any $u \in T$ we have $Su \cap L(a) \neq \emptyset$.

Proof. *(i)* $\Rightarrow$ *(ii)* This follows from Lemma 6.6.

(ii) $\Rightarrow$ *(iii)* Let $\underline{a} = (a_1, a_2, \ldots)$ be a sequence of elements of S and define $F(\underline{a})$ to be $F_{\mathbb{N}}/\sigma$, where $F_{\mathbb{N}}$ is the free S-act on $\{x_i : i \in \mathbb{N}\}$ and σ is generated by $\{(x_i, a_i x_{i+1}) : i \in \mathbb{N}\}$. From [2, Lemma 1] we have that $F(\underline{a})$ is strongly flat, hence in $\mathcal{IX}$ by assumption. It follows that $F(\underline{a})$ is cyclic. From [5, Lemma 1.2], as explicated in the ordered case in [3, Lemma 3.4], we deduce that S has Condition (A).

Suppose now that T is a unitary submonoid of S and let $F(T) = F/\rho$ be constructed as above. We first claim that for any $s, t \in S$ and $b, b' \in T$, we have $[sx_b] = [tx_{b'}]$ if and only if $sb'c = tbc$ for some $c \in T$. Indeed, if the latter condition holds, then with $h = b'c$ and $k = bc$ we have $sh = tk$ and $bh = b'k$, so that $[sx_b] = [tx'_b]$ holds by Lemma 6.4 (ii). Conversely, if $[sx_b] = [tx_{b'}]$ then by the same result, $su = tv$ and $bu = b'v$ for some $u, v \in T$. Then $sb'u = tvb' = tbu$ as required.

We now show that F/ρ is strongly flat. If $s, t \in S$ and $[ux_b], [vx_{b'}] \in F/\rho$ with $s[ux_b] = t[vx_{b'}]$, then from our proven claim we deduce $sub'd = tvbd$ for some $d \in T$. Now $[ux_b] = [ub'dx_{bb'd}] = ub'd[x_{bb'd}]$ and $[vx_{b'}] = [vbdx_{bb'd}] = vbd[x_{bb'd}]$, so that Condition (P) holds and similarly, so does (E). Moreover, F/ρ is locally cyclic, for given $[sx_b], [tx_{b'}] \in F/\rho$, we have $[sx_b] = sb'[x_{bb'}]$ and $[tx_{b'}] = tb[x_{bb'}]$. Since Condition (A) holds, we must have $F/\rho = S[x_c]$ for some $c \in T$.

Our assumption now gives that F/ρ is isomorphic to some cyclic S-act Sa with property X via an S-isomorphism $\theta : Sa \to F/\rho$ with $a\theta = [x_c]$.

Let $u \in T$. Then $[x_{cu}] = w[x_c]$ for some $w \in S$, and $a\theta = [x_c] = u[x_{cu}] = uw[x_c] = (uwa)\theta$ so that as θ is an S-isomorphism, we have $a = wua$ so that $Su \cap L(a) \neq \emptyset$.

Finally, if $pa = qa$ where $p, q \in S$, then $[px_c] = [qx_c]$ so our claim gives that $pcd = qcd$ for some $d \in T$, giving our result.

$(iii) \Rightarrow (i)$ By Theorem 5.1 we need only show that every cyclic S-act has a (cyclic) cover with property X.

Let ρ be a left congruence and let $T = [1]$, so that T is right unitary. Let Sa be the S-act guaranteed by our hypothesis and define $\theta : Sa \to S/\rho$ by $(ua)\theta = [u]$. If $ua = va$, then by assumption, $ut = vt$ for some $t \in T$ and so

$$u = u1 \,\rho\, ut = vt \,\rho\, v1 = v,$$

giving that θ is well defined. Clearly θ is an onto S-morphism. If $k \in S$ and $\theta|_{Ska} : Ska \to S/\rho$ is onto, then we must have $(hka)\theta = [1]$ for some $h \in S$, that is, $hk \in T$. By assumption, $whka = a$ for some $w \in S$ and it follows that $Ska = Sa$. Hence θ is co-essential as required. □

Remark 6.8. Theorem 6.7 can of course be applied to $\mathcal{F}r$ and to $\mathcal{P}r$, and then refined to produce existing results. The new applications of Theorem 6.7 are to $\mathcal{IP}a$ and to $\mathcal{IC}$. In the former case, the element a may of course be taken to be an element of S.

7. Examples

In this section we give a number of examples and counterexamples which we hope will be of interest in their own right. We focus on $\mathcal{IP}a$-covers and left $\mathcal{IP}a$-perfect monoids, and their relation to left perfect monoids. The first example is superceded by Example 7.13, but contains a useful construction.

Example 7.1. Let $\Sigma = \{x_0, x_1, x_2, \ldots, \}$ be an alphabet and let $\Omega = \Sigma \cup \{a\}$. If $u \in \Sigma^+$, let $i(u)$ be the highest x-index appearing in u, and define $i(\epsilon) = 0$ where ϵ is the empty word. Let $\phi \colon \Sigma^* \to \mathbb{N}$ and $\psi \colon \Sigma^* \times \Sigma^* \to \mathbb{N}$ be two injective maps having disjoint image such that $\phi(u) > i(u)$ and $\psi(u, v) > i(u), i(v)$ for every $u, v \in \Sigma^*$. Let τ be the congruence on Ω^* generated by the set

$$H = \{(x_{\phi(u)}ua, a), (ux_{\psi(u,v)}, vx_{\psi(u,v)}) : u, v \in \Sigma^*\} \subseteq \Omega^* \times \Omega^*.$$

Denote the monoid Ω^*/τ by U. Then there is a strongly flat cyclic left U-act that has an $\mathcal{IP}a$-cover, but which does not lie in $\mathcal{IP}a$.

We justify the above via a series of lemmas.

Lemma 7.2. *Let*

$$u = w_n a w_{n-1} a \ldots w_1 a w_0,\ v = w'_m a w'_{m-1} a \ldots w'_1 a w'_0 \in \Omega^*$$

where $w_0, \ldots, w_n, w'_0, \ldots, w'_m \in \Sigma^*$. *Then* $(u, v) \in \tau$ *if and only if* $n = m$, $(w_0, w'_0) \in \tau$ *and* $(w_i a, w'_i a) \in \tau$ *for all* $1 \le i \le n$.

As a consequence, if $u, v \in \Omega^*$ such that $(ua, va) \in \tau$, then

$$u\tau = w\tau u'\tau \text{ and } v\tau = w\tau v'\tau$$

where $u', v' \in \Sigma^*$.

Proof. Note that the letters a appearing in any word partition it into subwords such that elements of H can only be applied to the subwords. As a consequence we have that every element of U can be uniquely written in the form

$$(w_n a)\tau \cdot (w_{n-1}a)\tau \cdot \ldots \cdot (w_1 a)\tau \cdot w_0\tau$$

where $w_0, \ldots, w_n \in \Sigma^*$. $\square$

Definition 7.3. We say that a word $u \in \Sigma^*$ has *Property* (p) if for every factorisation $u = vx_iw$ where i is not contained in the image of ϕ, we have that v contains a letter x_j such that $j > i$ and j is contained in the image of ϕ.

Lemma 7.4. *Let $u, v \in \Sigma^*$ such that u has Property (p). Then if $(u, v) \in \tau$ or $(ua, va) \in \tau$, then v also has Property (p).*

Proof. Observe that taking prefixes of a word, adding a as a suffix, or applying relations from H, preserves (p). $\square$

Let M be the submonoid of U generated by $\{x_0\tau, x_1\tau, \ldots, \}$, denote by ρ the left congruence generated by $M \times M$, and let $[w\tau]$ be the ρ-class of $w\tau$, where $w \in \Omega^*$. It is clear from the definition of τ that M is a (right) unitary submonoid of U, which implies that $[\epsilon\tau] = M$.

Lemma 7.5. *The cyclic left U-act U/ρ is strongly flat.*

Proof. Since ρ is generated by $M \times M$, Lemma 5.7 tells us that it is enough to check that M is right collapsible. For this let $u\tau, v\tau \in M$ where $u, v \in \Sigma^*$. We have that $x_{\psi(u,v)}\tau \in M$, $(ux_{\psi(u,v)}, vx_{\psi(u,v)}) \in H$ and consequently, $u\tau \cdot x_{\psi(u,v)}\tau = v\tau \cdot x_{\psi(u,v)}\tau$, as required. $\square$

Lemma 7.6. *The cyclic left U-act U/ρ is not in $\mathcal{IP}a$.*

Proof. We have that $U/\rho \in \mathcal{IP}a$ if and only if there exists $s \in U$ such that $\rho = \ell(s)$. Suppose that such an element s exists. Then $s = (w_n a w_{n-1} a \ldots w_1 a w_0)\tau$ for some $n \geq 0$ and $w_n, \ldots, w_0 \in \Sigma^*$. If $n \geq 1$, then since the word $x_{\phi(w_n)} w_n a$ has (p), but $x_0 w_n a$ does not, we have by Lemma 7.4 that $(x_{\phi(w_n)} w_n a, x_0 w_n a) \notin \tau$. However, by Lemma 7.2 this implies that

$$(x_{\phi(w_n)} w_n a w_{n-1} a \ldots w_1 a w_0, x_0 w_n a w_{n-1} a \ldots w_1 a w_0) \notin \tau,$$

that is,

$$(x_{\phi(w_n)}\tau, x_0\tau) \notin \ell(s).$$

A similar argument holds if $n = 0$. Since $x_{\phi(w_n)}\tau, x_0\tau \in M$ we have $(x_{\phi(w_n)}\tau, x_0\tau) \in \rho$, so $\rho \neq \ell(s)$ for any $s \in U$. $\square$

Lemma 7.7. *The cyclic left U-act U/ρ has an $\mathcal{IP}a$-cover.*

Proof. We claim that $U(a\tau)$ is an $\mathcal{IP}a$-cover of U/ρ. For this we have to check that $\ell(a\tau) \subseteq \rho$ and that $U(u\tau) \cap L(a\tau) \neq \emptyset$ for every $u\tau \in [\epsilon\tau]$.

For the inclusion, let $(u\tau, v\tau) \in \ell(a\tau)$ where $u, v \in \Omega^*$. Thus $(ua, va) \in \tau$ which by Lemma 7.2 implies that

$$u\tau = w\tau \cdot u'\tau \text{ and } v\tau = w\tau \cdot v'\tau$$

where $u', v' \in \Sigma^*$. That is, $(u'\tau, v'\tau) \in \rho$, so that

$$(u\tau, v\tau) = (w\tau u'\tau, w\tau v'\tau) \in \rho,$$

which proves that $\ell(a\tau) \subseteq \rho$.

Note that $(x_{\phi(u)}u)\tau \in U(u\tau) \cap L(a\tau)$ for every $u \in \Sigma^*$, because $(x_{\phi(u)}ua, a) \in H$. Since $[\epsilon\tau] = M = \{u\tau : u \in \Sigma^*\}$, this proves the lemma and ends the justification of Example 7.1. □

If every cyclic left S-act has an $\mathcal{IP}a$-cover, then by considering the trivial act, it is clear that S has a minimal left ideal. To the converse, we can show the following.

Lemma 7.8. *If S has the descending chain condition M_L on principal left ideals, then every cyclic left S-act has a principal left ideal cover.*

Proof. Let S/ρ be a cyclic left S-act. The natural S-morphism $\nu_\rho : S = Sa_1 \to S/\rho$ is onto, where $a_1 = 1$. Suppose we have constructed a sequence of elements $a_1, a_2, \ldots, a_n$ of S such that $Sa_1 \supset Sa_2 \supset \ldots \supset Sa_n$ and $\nu_\rho|_{Sa_n} : Sa_n \to S/\rho$ is onto. If Sa_n is a cover we are done, but if not, there exists $a_{n+1} \in S$ with $Sa_{n+1} \subset Sa_n$ and $\nu_\rho|_{Sa_{n+1}} : Sa_{n+1} \to S/\rho$ onto. Since S has M_L this process must stop after a finite number of steps, producing a cover. □

Example 7.9. Let L be a left Baer-Levi semigroup, that is, L is left simple and left cancellative with no idempotents, and let $S = L^1$. Then S is left $\mathcal{IP}a$-perfect.

Proof. Clearly S has Condition (A) and M_L, so by Lemma 7.8 and Theorem 5.1 it is left $\mathcal{IP}a$-perfect. □

Isbell gives an example [5, page 106] of a left perfect monoid which does not have M_L. In fact, his is one of the kind given below.

Example 7.10. Let S be a monoid with zero such that for any $a_1, a_2, \ldots \in S \setminus \{1\}$, there is an $n \in \mathbb{N}$ such that $a_1 a_2 \ldots a_n = 0$. Then S is left perfect.

Proof. It is clear from Lemma 2.2 (iii) that Condition (A) holds. If T is a right unitary submonoid, then either $T = \{1\}$, or there is an $a \neq 1 \in T$. By assumption, $a^n = 0$ for some $n \in \mathbb{N}$, so that $0 \in T$. Hence T satisfies Condition (D). □

We now present an example of a left $\mathcal{IP}a$-perfect monoid that is not left perfect, and does not have M_L.

Example 7.11. Let L be a left Baer-Levi semigroup, and define the monoid M as in Lemma 3.14. Then the monoid M is left $\mathcal{IP}a$-perfect, is not left perfect, and does not have M_L.

Proof. Since L is left simple, by Lemma 3.14 M satisfies Condition (A) and does not have M_L. Let ρ be a left congruence on M: we show that M/ρ has an $\mathcal{IP}a$-cover. We need to find an element $s \in M$ such that $\ell(s) \subseteq \rho$ such that $Mu \cap L(s) \neq \emptyset$ for all $u \in [1]$. Note that if $[1] = \{1\}$ then $s = 1$ satisfies these properties, and similarly, if $\rho = M \times M$ (which happens if and only if $0 \in [1]$) then $s = 0$ does. So we can suppose that $[1] \neq \{1\}$ and $\rho \neq M \times M$. Let $a = (i, x, j), b = (k, y, l) \in [1]$. Then $a^2, ab \in [1]$ also, because $[1]$ is a submonoid of M, and it follows that $i = j = k = l$. As a consequence we have that there exists $i \in \mathbb{N}$ such that all non-identity elements of $[1]$ are of the form (i, x, i). We fix one such $s = (i, x, i) \in [1]$. If $(u, v) \in \ell(s)$, then $1 \;\rho\; s$ implies that $u \;\rho\; us = vs \;\rho\; v$, thus $(u, v) \in \rho$, and we have that $\ell(s) \subseteq \rho$. Now let $w \in [1]$; if $w = 1$ then clearly $1w \in L(s)$. Otherwise, $w = (i, y, i)$ for some $y \in L$, and as L is left simple, $x = zyx$ for some $z \in L$. Hence $(i, z, i)(i, y, i)(i, x, i) = (i, x, i)$ so with $r = (i, z, i)$ we have that $rw \in L(s)$ as required.

Finally we wish to show that M is not left perfect. First, we note that for any $x \in L$ we have

$$xL^1 \supset x^2L^1 \supset x^3L^1 \supset \dots$$

for if the sequence were to terminate, we would have $n \in \mathbb{N}$ such that $x^n \,\mathcal{R}\, x^{2n}$ and as certainly $x^n \,\mathcal{L}\, x^{2n}$ (L being left simple), we would have that $x^n \,\mathcal{H}\, x^{2n}$ so that by Green's theorem, $x^n \,\mathcal{H}\, e$ for some $e = e^2$, contradicting L being idempotent free. It is then easy to see that

$$(1, x, 1)M \supset (1, x^2, 1)M \supset (1, x^3, 1)M \supset \dots,$$

so that M does not have M_R and hence is not left perfect. $\square$

Remark 7.12. If we replace the Baer-Levi semigroup L in Example 7.11 by a group, then from Lemma 3.14 the resulting monoid has Condition (A) but does not have M_L. An easy calculation shows that it has M_R, so is left perfect.

We now show that the implication (i) $\Rightarrow$ (ii) in Theorem 6.7 need not hold if S is not commutative.

Example 7.13. Let

$$L = \{\phi \colon \mathbb{N} \to \mathbb{N} : \phi \text{ is one-one and } \overline{\operatorname{im} \phi} \text{ is infinite}\}$$

with composition of maps from *right to left*. Recall that L is an example of a Baer-Levi semigroup, and hence left simple, left cancellative and without any idempotents. Let $S = L^1 = L \cup \{I_{\mathbb{N}}\}$. By Example 7.9, S is left $\mathcal{IP}a$-perfect. However, S has a strongly flat cyclic left S-act that is not in $\mathcal{IP}a$.

Proof. Let $\mathbb{N} = \bigcup_{i=0}^{\infty} B_i$ be a partition of $\mathbb{N}$ into infinite subsets, and for every $1 \leq i$, let $A_i = \bigcup_{j=i}^{\infty} B_j$,

$$M_i = \{\alpha \in S : \alpha \text{ fixes every element of } A_i\},$$

and let $M = \bigcup_{i=1}^{\infty} M_i$. Note that $M_1, M_2, \dots, M$ are submonoids of S.

We first show that the submonoid M is right unitary and right collapsible.

To see that M is right unitary, let $\alpha, \beta \in S$ such that $\beta, \alpha\beta \in M$. Then there exists an i such that $\beta, \alpha\beta \in M_i$, that is, both β and $\alpha\beta$ fix every element of A_i. If $x \in A_i$, we have $\alpha(x) = \alpha(\beta(x)) = (\alpha\beta)(x) = x$, so that $\alpha \in M_i$. Thus M is indeed right unitary.

To see that M is right collapsible, let $\alpha, \beta \in M$. Then there exists an i such that $\alpha, \beta \in M_i$. Let $\gamma \colon \mathbb{N} \to \mathbb{N}$ be such that γ fixes all elements of A_{i+1} (so that $\gamma \in M_{i+1}$), and maps $\mathbb{N} \setminus A_{i+1}$ into $B_i = A_i \setminus A_{i+1}$ injectively. Notice that the image of γ is contained in A_i, so that $\gamma \in S$ and $\alpha\gamma = \beta\gamma = \gamma$, which proves that M is right collapsible.

Denote by ρ the left congruence generated by $M \times M$. Lemma 5.7 and the above argument imply that the cyclic left S-act S/ρ is strongly flat. In order to counter Theorem 6.7 we have to show that it is not contained in $\mathcal{IP}a$, that is, that ρ is not the left annihilator congruence of any element of S.

Let $\gamma \in S$ and let $x \in \mathbb{N}$. Then there exists $0 \leq i$ such that $\gamma(x) \in B_i$. Define a map α such that it fixes all elements of A_{i+2} and maps $\mathbb{N} \setminus A_{i+2}$ into B_{i+1} injectively. Then $B_i \cap \text{ im } \alpha = \emptyset$, so that $\alpha \in L$, and hence $\alpha \in M_{i+2} \subseteq M$. Since ρ is generated by $M \times M$ we conclude that $(\alpha, I_{\mathbb{N}}) \in \rho$. However, $\gamma(x) \in B_i$, so $\gamma(x) \notin A_{i+2}$, and it follows that $\alpha(\gamma(x)) \in B_{i+1}$ and hence that $\alpha(\gamma(x)) \neq \gamma(x)$. As a consequence $\alpha\gamma \neq \gamma$, that is, $(\alpha, I_{\mathbb{N}}) \notin \ell(\gamma)$, which shows that $\rho \neq \ell(\gamma)$ and hence S/ρ cannot lie in $\mathcal{IP}a$. $\square$

References

[1] A. Bailey and J. Renshaw, 'Covers of acts over monoids and pure epimorphisms', to appear in *Proc. Edinburgh Math. Soc.*

[2] J. B. Fountain, 'Perfect semigroups', *Proc. Edinburgh Math. Soc.* **20** (1976), 87-93.

[3] V. Gould and L. Shaheen, 'Perfection for pomonoids', *Semigroup Forum* **81** (2010), 102–127.

[4] R. Khosravi, M. Ershad and M. Sedaghatjoo, 'Strongly flat and condition (P) covers of acts over monoids', *Communications in Algebra* **38** (2010), 4520–4530.

[5] J. R. Isbell, 'Perfect monoids', *Semigroup Forum* **2** (1971), 95–118.

[6] M. Kilp, 'On monoids over which all strongly flat cyclic right acts are projective', *Semigroup Forum* **52** (1996), 241–245.

[7] M. Kilp, U. Knauer, A. V. Mikhalev, *Monoids, Acts, and Categories*, de Gruyter, Berlin (2000).

[8] M. Kilp, V. Laan, 'On flatness properties of cyclic acts', *Communications in Algebra* **28** (2000), 2919-2926.

[9] U. Knauer, 'Projectivity of acts and Morita equivalence of monoids', *Semigroup Forum* **3** (1972), 359–370.

[10] M. Mahmoudi and J. Renshaw, 'On covers of cyclic acts over monoids', *Semigroup Forum* **77** (2008), 325–338.

[11] H. Qiao and L. Wang, 'On flatness covers of cyclic acts over monoids', *Glasgow Math. J.* **54** (2012), 163–167.

[12] J. Renshaw, 'Monoids for which Condition (P) acts are projective', *Semigroup Forum* **61** (2000), 46–56.

E-mail address: `alex.bailey@soton.ac.uk`

School of Mathematics, University of Southampton, Highfield, Southampton SO17 1BJ, UK

E-mail address: `victoria.gould@york.ac.uk`

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

E-mail address: miklos.hartmann@york.ac.uk

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

E-mail address: J.H.Renshaw@soton.ac.uk

School of Mathematics, University of Southampton, Highfield, Southampton SO17 1BJ, UK

E-mail address: lls502@york.ac.uk

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK