UNIVERSITY OF SOUTHAMPTON

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Southampton High Energy Physics Group

Holographic Descriptions of Large N gauge Dynamics

by

Raul Alvares

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ABSTRACT

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DOCTOR OF PHILOSOPHY

Holographic descriptions of large N gauge dynamics

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We use the AdS/CFT correspondence to study different aspects of large N dynamics of gauge theories in the strongly coupled regime. We present three models designed to capture some of the physics present in QCD at low energies or QCD-type theories such as walking tecnhicolor. We use the D3/D7 system to study chiral symmetry breaking in two different contexts: In the first model we break chiral symmetry with an arbitrary running coupling which has a pion in its spectrum. We derive integral equations for the quark condensate and pion decay constant by matching our model to a low energy chiral lagrangian and discuss the implications for technicolor theories. In the second model we study the on set of chiral symmetry breaking at the edge of the conformal window in QCD in the Veneziano limit and show a BKT-type transition. Finally, in the context of AdS/QCD we extend the "hard wall" model to include the tensor operator $\bar{\phi}\sigma^{\mu\nu}\psi$ dual to a two form satisfying a self duality condition and study numerically the spectrum of vector mesons.

Contents

List of Figures

List of Tables

Declaration of Authorship

I, Raul Alvares, declare that this thesis, entitled "Holographic descriptions of large N gauge dynamics" and the work presented in it are my own. I confirm that

- This work was done wholly or mainly while in candidature for a research degree at this university.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.
- Work contained in this thesis has previously been published in references...

Signed:

Date:

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Finally, I would like to thank my closest family with an emphasis on my strong mother, late grandfather and my late late uncle for feeding my interest in physics.

—Raul

God is a theoretical physicist.

One day he was model building when suddenly

he stopped in his tracks...

It was clear, couldn't be avoided,

cascading down his equations, he had just cal-

culated Man.

Too complex, it seemed

'I don't know where this is going', he said

'The idea is so sexy the math will never add

up!'

—The author

Chapter 1

Introduction

Since the AdS/CFT correspondence was first established [1, 2] towards the end of the last century it has been regarded as a promising framework to study strongly coupled gauge theories. Ironically, string theory itself was first proposed as a model to describe mesons and hadrons in the sixties by considering the different particles as different modes of vibration of a string. This model had its virtues describing qualitatively the lightest modes of the hadron spectrum but with the development of QCD and the discovery of quarks it attracted less attention. QCD seemed like the natural candidate to explain the strong force since it was a natural extension to a non-abelian gauge theory of the field theory techniques that successefully had given us a description of the electroweak force. This programme was carried out with the promise of understanding the distinctive fixtures of the strong force when compared to its abelian partner QED, namely, assymptotic freedom, confinement and chiral symmetry breaking. At high energies, for example as the result of a high energy collision in a particle accelerator or at the beggining of the universe, the theory is decoupled, the quarks and gluons are free and form a plasma that can be described using thermodynamics. As this plasma expands and cools down, and the quarks and gluons try to move away from each other, the coupling grows quickly and we find that the theory confines and no assymptotic states of quarks and gluons are observed. Instead it is experimentally observed that at low energies the theory's natural degrees of freedom become mesons and baryons. As successful as it is, we can only use QCD perturbation theory at high energies and low energy physics such as confinement is hard to unveil, in fact up to this day it has not yet been derived analytically from the QCD lagrangian. Somewhere between the quark-gluon plasma picture and the low energy assymptotic states it is hard to keep track of what happens and make any predictions. Considering this state of affairs the challenge for particle physicists was

laid: How to go beyond perturbative physics?
With considerable overlap, attempts to understand QCD in its strongly coupled regime either focus on QCD itself trying to improve our ability to compute or extract physics from strongly coupled problems, e.g. lattice QCD [3], Dyson-Swinger equations [4], chiral perturbation theory (see [5] and chapter 3). Or focus on expanding our understanding of gauge theories in general in hopes that one may find tractable limits to QCD-like scenarios from which something can be learned about QCD itself. These include Large N QCD, QCD in the Veneziano limit (see chapter 5), and AdS/CFT correspondence.

1.1 Aspects of QCD

QCD is the gauge theory of the strong interactions with SU(3) gauge group. Quarks are the spin-1/2 matter fields and gluons the gauge bosons. In addition there are 6 *flavours* of quarks: up, down, charm, strange, top and bottom. The lagrangian is:

$$\mathcal{L}_{QCD} = \sum_{flavour} \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a \tag{1.1}$$

The quarks come in different masses but the group $m_u, m_d, m_c \ll 1GeV$ is considerable lighter then the remaining three and it is useful to consider these three only when studying QCD, and the fact that $m_p \gg 2m_u + m_d$ suggests that the *chiral limit* where the masses of the three lightest quarks are taken to zero is a good approximation to the theory. This fact is better stated by saying that QCD has an approximate $SU(3)_R \times SU(3)_L \times U(1)_B \times U(1)_A$ global symmetry where $SU(N)_{R/L}$ rotates right/left quark fields, $U(1)_B$ is the baryon number and it is an exact symmetry whereas $U(1)_A$ is anomalous in the quantum thery. The conserved currents are:

$$L^{\mu,a} = \bar{q}_L\gamma^\mu\frac{\lambda^a}{2}q_L \tag{1.2}$$

$$R^{\mu,a} = \bar{q}_R\gamma^\mu\frac{\lambda^a}{2}q_R \tag{1.3}$$

Where λ^a are the Gell-Mann matrices. Alternatively, taking the combinations $V^{\mu,a} = R^{\mu,a} + L^{\mu,a}$ and $A^{\mu,a} = R^{\mu,a} - L^\mu$ we can write these currents as parity eigenstates transforming as vector and axial vector current densities, together with the vector singlet $V^\mu = \bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d + \bar{c}\gamma^\mu c$. In this case the global symmetry group can be

written as $SU(3)_V \times SU(3)_A \times U(1)_V$.
For non-zero quark masses but $m_u = m_d = m_s$ the $SU(3)_A$ is explicitly broken and for non-zero and *all different* quark masses all but the baryon number symmetry are broken. Adding to this picture even in the massless quark limit chiral symmetry is spontaneously broken by the QCD vaccum. Evidence for this comes from the experimental observation that the masses of the octet of pseudoscalar mesons (π, K, η) are small compared to the mass of vector mesons which suggests that these states may be the goldstone bosons (otherwise massless) of the spontaneously broken $SU(3)_A$ approximate symmetry. It turns out that a non-vanishing scalar quark condensate $\langle \bar{q}q \rangle \neq 0$ is a sufficient condition for spontaneously symmetry breaking, in particular we have:

$$\langle 0 | A^a_\mu(0) | \pi^b(p) \rangle = i p_\mu f_\pi \delta^{ab} \tag{1.4}$$

Where $\pi(p)$ is a massless one pion state and f_π the the pion's decay constant. A non-zero value for f_π is a necessary and a sufficient condition for spontaneous chiral symmetry breaking.

1.1.1 beta function

The two loop QCD beta function is:

$$\mu \frac{d\lambda}{d\mu} = -b_0 \lambda^2 + b_1 \lambda^3 \tag{1.5}$$

where

$$b_0 = \frac{2}{3} \frac{(11 - 2x)}{(4\pi)^2}, \;\; b_1 = -\frac{2}{3} \frac{34 - 13x}{(4\pi)^4} \tag{1.6}$$

and $x = N_f/N_c$. By inspection we see that for $N_f = 6$ we have $\beta(g_s) < 0$ so the coupling will grow as we move to low energies which is consistent with confinement and assymptotic freedom. However, taking the limit $N_c \to \infty$ with fixed x one can treat x as a continuous variable and study how the QCD phase diagram depends on the number of flavours (see chapter 5). The one loop solution for $N_c = 3$ and $N_f = 6$ looks like:

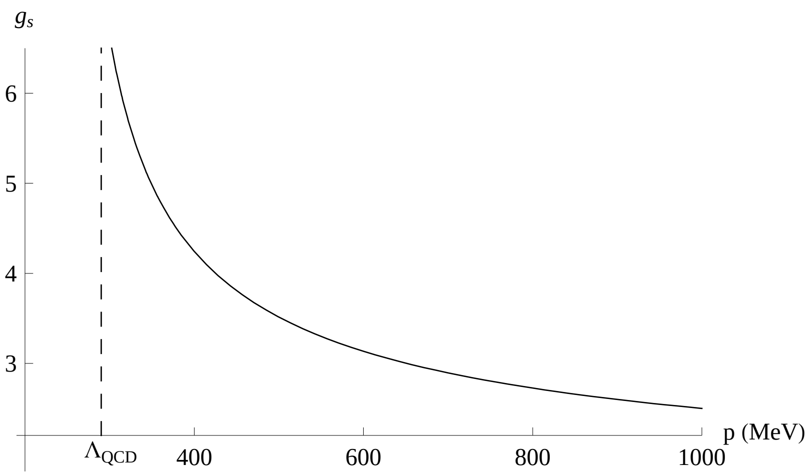


Figure 1.1: *Plot of the solution to the first loop beta function for QCD*

We see from the figure above that even for massless QCD there is a dynamically generated scale Λ_{QCD}, defined by the scale at which the coupling goes to infinity.

1.1.2 A relationship between field theory and string theory

Since perturbative field theory does not work at low energies, there have been attempts at alternative descriptions of confining quarks. It was noticed that when one tries to pull two quarks apart they form string-like objects called flux tubes and since this picture seemed to have some of phenomenological fixtures, there were attempts to write a string theory discribing the strong interactions with flux tubes as the basic objects. So maybe the original intuition that led to the development of string theory was not that far off the money. This intuition became more refined with the study of the large N limit of gauge theories [6].

It turns out that when one studies the $N_c \to \infty$ limit of gauge theories they get considerably simplified. In the *t'Hooft limit* , when one allows $N_c \to \infty$ with $\lambda = g_{YM} N$ fixed, where g_{YM} is the Yang-Mills coupling, the leading terms in the beta function remain the same and the theory has a perturbative expansion in $\frac{1}{N}$. The crucial thing is to realize what happens to the relative weight of the different feynman diagrams in this perturbative expansion. The different vertices, propagators and loops can be identififed with the vertices, edges and faces of a simplicial decomposition defining a surface. A bit of counting shows that the field theory may be written as a double expansion of the form:

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{1.7}$$

Where f_g is some polynomial in λ and g is the genus of the surface defined by each feynman diagram. We see that in the large N limit diagrams of minimal genus dominate which are surfaces topologically equivalent to a sphere or a plane. Now, such an expansion is exactly what we find in closed string theory if we identify $1/N$ with the string coupling constant g_s. Perturbative string theory naturally includes a sum of diagrams discribing interacting strings splitting and joining defining world sheets with different topologies that come with factors of g_s^{2g-2}.

This argument is very general and although it suggested a relationship between fields theories and string theory it did not specify which field theory was related to each string theory.

1.2 Gauge-gravity duality

Meanwhile string theory followed its own development as a promising theory of quantum gravity and much more since any type of closed string theory included the graviton. Consistency required supersymmetry, extra dimensions, and a complete picture includes spatially extended objects called D-branes. These objects came into play in two different ways and were later proven to be the same. First, from closed string theory as sources of the supergravity fields that appear in the low energy effective theory and secondly as dynamical hypersurfaces where open strings can end. It was recognized that the open string description of branes naturally realized Yang-Mills type gauge theories. The fact that the 'gravity' branes and the 'open strings' branes were shown to be the same object suggested a duality between open strings and closed strings which was in fact, in the low energy limit a duality between SYM field theories and supergravity. In [1, 2] these ideas were realized explicitly in the limit of Large N and large $\lambda = g_{YM}N$ and the AdS/CFT was established as an holographic duality: the physics of a classical gravitational system in anti de Sitter spacetime in $d+1$ dimensions was related to a conformal SYM field theory in d dimensions defined at the boundary of this spacetime.

Peculiar to this construction, when one side of the duality is strongly coupled the other is weakly coupled, when gravity is semiclassical, the gauge theory which is equivalent to is strongly coupled. This fact spurred lots of interest and open a new field of research since the duality suggested that there were hopes to probe the physics of strongly coupled gauge theories by studying the physics of its weakly coupled grav-

ity dual.
QCD became one of the natural targets of this framework but as promising as it is it is not without limitations. The fact that string theory is only perturbative for large N restricts the class of QCD-like theories we may try to study to large N. Nonetheless even in such extreme limit when one introduces flavour [7, 8] these theories seem do display several qualitatively features that one expects to find in QCD at low energies such as chiral symmetry breaking and mesonic degrees of freedom. Furthermore, other types of strongly coupled scenarios can be studied such as walking technicolor or condensed matter systems.
Two main approaches have been followed in attempts to use gauge/gravity duality to understand QCD, top-down and bottom-up. In the former, one tries do deform the supergravity construction in a consistent way to construct a QCD dual. In the later, one starts with QCD observables, then uses the AdS/CFT dictionary to write down effective gravity actions that in the dual description match these observables.

1.3 Overview

In this thesis, within the framework of gauge/gravity duality, we present three different models of large N gauge dynamics. In chapter 2 we introduce the AdS/CFT correspondence and its main ingredients. In chapter 3 we study chiral symmetry breaking in the holographic $D3/D7$ system in a simple model with an arbitrary coupling and discuss the implications for technicolour theories. In chapter 4 in a simple holographic model we study chiral symmetry breaking at the edge of the conformal window in QCD in the Veneziano limit. In chapter 5 in the context of a bottom-up approach, we complete analyse the sector of dimension-three vector meson operators in the "hard wall" model of holographic QCD.

Chapter 2

AdS/CFT correspondence

The goal of this chapter is to justify the AdS/CFT correspondence. We will not go into all the details, there are good reviews available [9], but we will comment on all the ingredients that make this framework a plausible one to study aspects of strongly coupled gauge theories. We will focus on the particular example of $U(N)$ $\mathcal{N} = 4$ SYM and its conjectured gravity dual, type IIB supergravity in $AdS_5 \times S^5$. The relation between this two different theories comes from the combination of two different insights. First, the realization that in the large N limit a field theory looks like a free string theory, and second, the two-fold understanding [10] of branes as both solitonic solutions to type IIB supergravity, p-branes, and hypersurfaces were open strings can end, D-branes.

2.1 Basics of string theory

One can start learning about string theory by studying the Nambu-Goto action which discribes the area that a relativistic bosonic string world sheet sweeps out in time. By introducing a worldsheet metric h^{ab} it can be written as:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}. \tag{2.1}$$

α' is the open string Regge slope parameter which is related to the string length: $\alpha' = l_s^2$. The functions $X^\mu(\sigma, \tau)$ describe the embedding of the worldsheet in spacetime. $G_{\mu\nu}$ is the spacetime background. Solving the equations of motion we find that with open string boundary conditions, the right moving modes combine with the left moving modes to form standing waves, where in the closed string they move independently. In the case of the classical bosonic string the mass formula of the open and closed strings is:

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}.\alpha_n \tag{2.2}$$

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n}.\alpha_n + \tilde{\alpha}_{-n}.\tilde{\alpha}_n) \tag{2.3}$$

Where it is explicit the two independent modes in the close string. $N = \sum_{n=1}^{\infty} \alpha_{-n}.\alpha_n$ is the number operator. These formulas get slightly motived in the quantum theory. In order to avoid a tachyon in the spectrum it is necessary to add supersymmetry. One finds that in order to have a theory with spacetime supersymmetry free of tachyons, we need $D = 10$, where D is the spacetime dimension, and to perform a GSO *projection* to remove states. This operator projects the spacetime fermions in the close string onto states of definite chirality. Depending on the relative chirality between right moving modes and left moving modes we will have different string theories. In type IIB string theory the chiral projections are opposite, in type IIA the same chirality is projected in each case. The massless closed string spectrum is quite rich since one is taking tensor products of left and right moving fermionic states which can also satisfy periodic and antiperiodic boundary conditions. NS *Neveu-Schwarz* boundary conditions, antiperiodic and R *Ramond* boundary conditions, periodic. The NS-NS sector is the same for type IIA and IIB and it includes the metric, G^{MN}, the dilaton, Φ, and an antisymmetric two-index tensor, B^{MN}. The R-R sector will be different for the two theories. For type IIB string theory it includes a zero-form field, a gauge field, A^M, a two-form gauge field, C_2, a four-form gauge field, C_4, and a self-dual field strength F_5. The R-NS and NS-R sectors include all the fermionic partners.

2.1.1 Supergravity and p-branes solutions

In the section above we only described the massless spectrum of superstrings since when α' is taken to be small, all but the lightest modes are unimportant and since $l_s^2 = \alpha'$ we effectively zoom out and the string looks point-like. On the other hand, when one considers strings moving in spacetime, in principle we should consider summing over all possible topologies for the string world sheet, however one finds that these come with factors of $e^{-\phi\chi}$ where $\chi = 2 - 2g$ is the *Euler characteristic* of the particular topology defined by the world sheet. ϕ is the expectation value of the dilaton. The string coupling is defined as $g_s = e^{\phi}$ and one concludes that if g_s

is small enough higher genus diagrams become less important. In low energy string theory, α' is small but it can not be of the order of the Planck scale or quantum gravity effects become important. We have:

$$l_P = g_s^{1/3} l_s \tag{2.4}$$

i.e. when g_s is small the Planck length is some orders of magnitude shorter then the string length and we can ignore quantum gravity effects.
In this limit, closed string becomes supergravity. For completeness, the type IIB supergravity action is:

$$\begin{aligned} S_{IIB} = \quad & \frac{1}{4k_B^2}\int dx^{10}\sqrt{G}e^{-2\Phi}(2R_G + 8\partial_\mu\Phi\partial^\mu\Phi - \mid H_3 \mid^2) \quad (2.5) \\ & -\frac{1}{4k_B^2}\int dx^{10}[\sqrt{G}(\mid F_1 \mid^2 + \mid \tilde{F}_3 \mid^2 + \frac{1}{2}\mid \tilde{F}_5 \mid^2) + C_4 \wedge H_3 \wedge F_3] \quad (2.6) \end{aligned}$$

With the definitions:

$$F_1 = dC,\ H_3 = dB,\ F_3 = dC_2,\ F_5 = dC_4 \tag{2.7}$$

$$\tilde{F}_3 = F_3 - CH_3,\ \tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3 \tag{2.8}$$

And the additional self duality condition: $\star\tilde{F}_5 = \tilde{F}_5$.
Where k_B is the gravitational coupling constant and is related to g_s and l_s: $2k_B^2 = \frac{1}{2\pi}(2\pi l_s)^2 g_s^2$. This action picks an infinite series of α' corrections and an infinite series of g_s corrections. It turns out that strings are not the only objects that appear in string theory, the theory naturally includes higher dimensional objects that are charged under the several form-fields that appear in the superstring spectrum. We have seen the massless spectrum of the theory includes n-forms gauge fields which can be interpreted as n-form generalizations of the ordinary one-form Maxwell field, $A_\mu dx^\mu$. Furthermore, in the Maxwell theory, we have a two-form field strength tensor, $F^{\mu\nu}$, describing the electric and magnetic fields, as well as electrical and magnetic sources. Although magnetic charges have not been observed, it is believed that they may exist at higher energies. We can define the electrical and magnetic charges as:

$$e = \int_{S^2} \star F \tag{2.9}$$

$$g = \int_{S^2} F \tag{2.10}$$

Where S^2 is a two dimensional surface surrounding the point-charges.
Extending this idea, we say that the same way a one-form field is sourced by a point-like charge, that we will call a 0-brane, a $(p+1)$-form gauge field is sourced by a p-dimensional charge, a p-brane, in D dimensions. To make it precise we need to generalize the Gauss law written above and instead of integrating over a S^2 sphere surrounding a point charge, if a p-brane is electrically charged we will have an integral $\mu_p = \int \star F_{p+2}$ over a S^{D-p-2} sphere where μ_p is the p-brane's charge. Likewise, we compute the magnetic flux of the magnetic dual brane, $\int F_{p+2}$ by surrounding it with a S^{p+2} sphere. If such objects exist in string theory it remains to be seen if they are stable. Looking into the type IIB spectrum, we have $0, 2, 4$-form fields which should couple to Dp-branes with odd values of p. Doing the same with type IIA we find that the theory could have p-branes with even values of p. Can we find such solutions?
Yes. The action (2.6) has a variety of solutions that correspond to extended black holes [11] whose description falls outside the scope of this text, but as an example that will be important to establish the AdS/CFT correspondence we will mention $D3$ brane solutions. Essentially, according to what was said above, a $D3$ brane sources the four form tensor field, C_4 so we look for a black hole solution carrying electric charge with respect to this form. Such solution does exist with a metric:

$$dS^2 = (1 + \frac{R^4}{y^4})^{-1/2} \eta_{ij} dx^i dx^j + (1 + \frac{R^4}{y^4})^{1/2} (dy^2 + y^2 d\Omega^5) \tag{2.11}$$

Where $R^4 = 4\pi g_s N \alpha'^2$. x^i are coordinates in the $D3$-brane world volume, $\vec{y}$ are the 6 spatial coordinates transverse to the branes, η_{ij} is the $3+1$ Minkoswki metric and N is the number of $D3$-branes. We have:

$$\int_{S^5} F_5 = N \tag{2.12}$$

Where $F_5 = dC_4$. The statement that this is a low energy solution means we are taking the limit $\frac{R^4}{\alpha'^2} = 4\pi g_s N \gg 1$, i.e. the radius of this spacetime is much larger then the string length scale. This example will be important later on.

2.1.2 T-duality and D-branes

We will introduce T-duality for the bosonic string, but the argument generalizes to superstrings. Let $X^M(\sigma, \tau)$ be a solution to a closed bosonic string in $D = 26$, where σ and τ are the worldsheet coordinates. Let $M = 25$ be an S^1 of radius R, i.e. a compact direction. To describe the string along the compact direction we need to impose the boundary condition:

$$X^{25}(\sigma,\tau) = X^{25}(\sigma,\tau) + 2\pi RW, \;\; W \in \mathbb{Z} \tag{2.13}$$

Where W is called the winding number and counts the number of times the string winds around the circle. On the other hand, the momentum p^{25} along this direction is necessarily quantized:

$$p^{25} = \frac{K}{R}, \;\; K \in \mathbb{Z} \tag{2.14}$$

Where K is the Kaluza-Klein excitation number. With this in mind, formula (2.3) gets modified to:

$$\alpha' M^2 = \alpha'[(\frac{K}{R})^2 + (\frac{WR}{\alpha'})^2] + 2N_L + 2N_R - 4 \tag{2.15}$$

Where N_L and N_R are the number operators for the left and right moving modes along the non compact directions and satisfy $N_R - N_L = WK$. Essentially the formula above includes the contributions to the mass of the string from the momentum and winding number along the compact direction. Interestingly, the equation above is invariant under the interchange of W and K if we take $R \to \tilde{R} = \alpha'/R$. In other words, the theory of a string compactified on a circle of radius R is equivalent to a theory where a string is compactified on a circle of radius $\tilde{R} = \frac{\alpha'}{R}$. This symmetry is called T-duality. One finds that under T-duality transformation the right moving mode of string flips sign, $X_R \longrightarrow -X_R$ and $X_L \longrightarrow X_L$, so if $\tilde{X}(\sigma,\tau)$ is the string solution along the T-dualized direction we have:

$$\tilde{X}(\sigma,\tau) = X_L(\sigma+\tau) - X_R(\tau-\sigma) = \tilde{x} + 2\alpha'\frac{K}{R}\sigma + 2RW\tau + ... \tag{2.16}$$

Where $\tilde{x}$ is the coordinate parametrizing the dual circle.
In the case of open strings T-duality leads to a suprising conclusion. Note that the difference between closed strings and open strings is in the boundary conditions. When one varies the action for an open string, in order to set $\delta S = 0$ we need to impose boundary conditions on the end points of the string. In order to retain Poincaré invariance the only consistent choice is Neumann boundary conditions:

$$\frac{\partial}{\partial\sigma}X^M(\sigma,\tau) = 0, \;\; for\ \sigma = 0, \pi. \tag{2.17}$$

Along all spacetime components. The mode expansion of a solution that satisfies this boundary condition looks like:

$$X(\sigma,\tau) = x + p\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}cos(n\sigma) \tag{2.18}$$

Where α' is set to 1/2. Unlike the close strings, open strings do not have a winding number since the open string is topologically equivalent to a point, however, away from the ends the open string looks locally like a closed string so if we compactify along a circle of radius R and perform a T-duality transformation, $X_R \longrightarrow -X_R$ and $X_L \longrightarrow X_L$, we get for the T-dualized direction:

$$\tilde{X}(\tau,\sigma) = X_L - X_R = \tilde{x} + 2\tilde{R}K\sigma + \sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}sin(n\sigma) \tag{2.19}$$

So what happened? Comparing with (2.18) there is no momentum in the 25th direction since there is no winding mode and the string has only oscilatory motion. On the other hand, the momentum along the original 25^{th} direction becomes a winding number along the dualized direction: $\tilde{X}(\tau,\pi) - \tilde{X}(\tau,0) = 2\pi\tilde{R}K$, i.e. the end points of the string are fixed. T-duality maps Neumann boundary conditions into Dirichlet boundary conditions. The endpoints of a dual open string are fixed on the hyperplane $\tilde{X} = \tilde{x}$ which is called D-brane, short for Dirichlet-brane. These hypersurfaces can be understood as physical objects in themselves. The dynamics of D-branes describe the dynamics of strings ending on them and vice-versa. Specifically, if we compactify n directions, in the dual theory the string endpoints will be restricted to a $(25-n)$-brane. The original massless gauge boson living in 26 dimensions will be split between the spacetime vector vibrating along the brane's world volume $\alpha_{-1}^{M}|0,k\rangle$ with $\mu = 0, ..., 25-n$ and n massless scalars which describe excitations along the directions transverse to the brane. If we allow these to depend on the worldvolume coordinates $\phi(x_1, ..., x_n)$ we are indeed describing fluctuations of the brane itself in spacetime. These fluctuations are a consequence of breaking spacetime translational symmetry in the first place and can be interpreted as goldstone bosons.

Chan-Paton charges and $U(N)$ gauge theories

If we stack N D-branes together there is no way to distinguish the particular brane in which the string ends. In fact since in type IIB superstring theory we distinguish between the two end points of the string there is a N^2 multiplicity. The Chan-Paton factors associate these N degrees of freedom to each of the end points of the string which describe a $U(N)$ gauge group. In this case an arbitrary string state will fall in a representation of the $U(N)$ algebra and will look like:

$$|k,\lambda\rangle = \sum_{i,j=1}^{N} |k,ij\rangle\, \lambda_{ij}. \tag{2.20}$$

Where λ_{ij} are matrices transforming in the adjoint of $U(N)$.
It turns out that in order for D-branes to be stable they need conserved charges otherwise they will decay. The form fields in the superstring spectrum couple naturally to this objects that in turn will act as sources to these fields.

DBI action

We now procede to write an action for generic D-branes. The idea is that the open string modes that end on the D-brane can be described by fields in its world volume. In energies which are low compared to the string scale we may only take the massless modes. On the other hand, in an analogy with the action for strings, the action for a D-brane should describe the volume swept by the brane in spacetime. The scalar excitations of the open string along the transverse directions will play the role of embedding functions $X(\xi)$ whereas excitations in the world volume are accounted by the gauge field $A_a(\xi)$. Taking $\xi^a, a = 0, ..., p+1$ to parametrize the brane's world volume we have the Dirac-Born-Infeld action (DBI) [10]:

$$S = -\mu_p \int d_\xi^{p+1} e^{-\phi} \sqrt{-det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} + S_{WZ} + fermions \tag{2.21}$$

Where G_{ab} and B_{ab} are the pull-back of the spacetime fields to the brane, $\mu_p = \frac{1}{(2\pi)^p}\alpha'^{-(p+1)/2}$ is the brane's tension, S_{WZ} are the Chern-Simons terms that include the coupling of the brane to the background form fields and ϕ is the dilaton.

2.2 The gauge gravity duality

In the section above we have shown how branes have two alternative descriptions in string theory. In this section we will show how the low energy limit of both these descriptions gives us two systems that are related to each other. In particular type IIB string theory compactified on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super Yang-Mills.

2.2.1 The correspondence

D3 branes and $\mathcal{N}=4$ SYM

If we stack N $D3$-branes on top of each other in $9+1$ spacetime the resulting theory will have close string excitations propagating in the bulk and open strings ending on the $D3$ branes. The action of this theory has three pieces:

$$S = S_{bulk} + S_{brane} + S_{int} \tag{2.22}$$

Where S_{bulk} describes the closed strings propagating in the 9+1 bulk, S_{brane} describes the open strings in the $3+1$ world volume of the $D3$ brane and S_{int} describes interactions between them. If we imagine separating one of the branes to a distance r from the pack we can take the low energy limit:

$$\alpha' \to 0, \;\; U \equiv \frac{r}{\alpha'} = fixed \tag{2.23}$$

Where the second condition is a rescaling to make sure we keep the mass of the strings fixed. In this limit, the interaction between the closed strings in the bulk and the open strings on the brane has a small cross section. We are left with an effective theory of open strings living on the world volume of the $D3$ branes decoupled from supergravity in flat space which in its massless limit is $\mathcal{N}=4$ SYM. This field theory will have a $U(N)$ gauge group and due to supersymetry all the fields transform in the adjoint of this gauge group. The theory has a gauge field A_μ invariant under $SU(4)$ R-symmetry group, describing excitations along the branes' world volume, 4 Weyl fermions in the 4 of $SU(4)$ and 6 scalars in the 6 of $SU(4)$ describing excititions along the tranverse directions to the brane. The theory has a vanishing β-function and a $SO(4,2)$ conformal group.

D3 branes and supergravity

We have seen in the section above that $D3$ branes can alternatively be described as massive charged objects that source the supergravity fields. The metric (2.11) describes a spacetime with a black hole with the $D3$ branes at the origin of the radial direction y. In this system there are two kinds of low energy excitations. First, any excitation near the horizon gets red shifted as it travels up the potential well. Second, all the large wavelength excitations that travel in the bulk. It can be shown that these two excitations decouple and the near horizon region decouples from the asymptotic region. In order to take this limit while keeping the energies on the throat constant, analogously to the argument in the section above we need to take $\alpha' \to 0$ with $\frac{y}{\alpha'}$ fixed. With this limit in mind we can zoom in the near horizon geometry and (2.11) becomes:

$$dS^2 = \frac{y^2}{R^2}\eta_{ij}dx^i dx^j + \frac{R^2}{y^2}(dy^2 + y^2 d\Omega_5^2) \tag{2.24}$$

Which is $AdS_5 \times S^5$ spacetime. $R = \sqrt{4\pi g_s N \alpha'^2}$ is the Anti-de Sitter radius and has a boundary at $y = \infty$, y is the radial direction. Null geodesics can reach this boundary in finite time. From the physics of D-branes the YM coupling g_{YM} can be related to the string coupling g_s and $\lambda = g_s N = g_{YM}^2 N$ is the t'Hooft coupling. Alternatively if we do $z = \frac{R^2}{y}$ we have:

$$dS^2 = R^2(\frac{\eta_{ij}}{z^2}dx^i dx^j + \frac{dz^2}{z^2} + d\Omega_5^2) \tag{2.25}$$

With a boundary at $z = 0$. The isometry group of AdS_5 spacetime is $SO(4,2)$. The isometry of the S^5 sphere is $SO(6)$ which are the same as the conformal and R-symmetry groups of $\mathcal{N} = 4$ SYM. So both descriptions of $D3$-branes share the same symmetries and have two decoupled low energy theories. The AdS/CFT correspondence states that these two descriptions are dual to each other. $\mathcal{N} = 4$ SYM in $3+1$ dimensions is dual to supergravity in $AdS_5 \times S^5$ with appropriate boundary conditions.
What is the validity of this conjecture? On the field theory side we can use perturbation theory when the t'Hooft coupling is small, i.e.:

$$g_{YM}^2 N \sim \frac{R^4}{l_s^4} \ll 1 \tag{2.26}$$

On the other hand the supergravity description is reliable when the radius of the spacetime is large compared to the string length:

$$\frac{R^4}{l_s^4} \gg 1 \tag{2.27}$$

Essentially when one side of the duality is strongly coupled the other one is weakly coupled which makes AdS/CFT correspondence non trivial and hard to test but also useful. The argument was based on perturbative string theory and it is not clear if it is valid for any value of $\lambda = g_s N$. In its weakest form, and the one we will assume, the AdS/CFT correspondence is valid only for large N and $g_s \to 0$ but with $\lambda \to \infty$ where the gravity description is semiclassical. A stronger statement is that the correspondence may be valid in the *t'Hooft limit* when λ is finite but $N \to \infty$, i.e., α' corrections agree with field theory but g_s corrections may not. Finally, in its stronger form the full IIB string theory is dual to the field theory and only needs to be assymptotically $AdS_5 \times S^5$.

2.2.2 The dictionary

The gravity side of the duality has an extra non-compact dimension y, the radial direction of AdS spacetime which corresponds to the field theory energy scale, the correspondence is holographic [12]. We can see this by noting that under gauge theory dilations $x \to e^{\alpha} x$, an inverse energy dimension, in order to keep AdS metric invariant we need y to scale like energy $y \to e^{-\alpha} y$.
How to match the physics from both theories? We need a correspondence between supergravity fields and field theory observables. This is done in [1, 2] where a *field-operator map* is proposed that gives a precise recipe to compute correlation functions through the dependence of the supergravity action on its asymptotic behavior at the boundary. Specifically we have:

$$\langle e^{\int d^d x \, \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \rangle_{\text{CFT}} = \mathcal{Z}_{\text{Sugra}} \Big|_{\phi(0,\vec{x}) = \phi_0(\vec{x})}, \tag{2.28}$$

The left hand side is the generating functional of correlation functions in the field theory where ϕ_0 is the source of a gauge invariant operator $\mathcal{O}$ and the right hand side is the partition function of string theory with the boundary condition that the field $\phi(z, \vec{x})$ at the boundary $z = 0$ matches the source of the operator. There is a one-to-one correspondence between supergravity fields and field theory operators. The operators and the fields will fall in the same representation of $SU(4)$ and the scaling dimension Δ of the operator will be related to the mass of the field ϕ, to see this lets consider the simplest example of a scalar field in AdS with an action:

$$S = \int dx^4 dz \sqrt{-g} (g^{ab} \partial_a \phi \partial_b \phi - m^2 \phi^2) \tag{2.29}$$

Expanding the solution near the bounday $z = 0$ we have:

$$\phi(z) \sim c_1 z^{4-\Delta} + c_2 z^{\Delta} \tag{2.30}$$

Where

$$m^2 = \Delta(\Delta - 4) \tag{2.31}$$

For ϕ to be invariant under dilatations, since z transforms as a length, we need the mass dimensions of c_1 and c_2 to be $4-\Delta$ and Δ respectively. If we recall the structure of the left hand side of (2.28) we see that we should identify $c_1 \to \phi_0$, the source of the operator $\mathcal{O}$, and $c_2 \to \langle \mathcal{O} \rangle$, its vaccum expectation value. The formula (2.31) can be generalized for fields in different representations of the Lorentz group. For a generic p-form in AdS_d we have:

$$(\Delta + p)(\Delta + p - d) = m^2 \tag{2.32}$$

Some examples of gravity fields and its field theory duals are the graviton $\leftrightarrow$ stress energy tensor, gauge fields $\leftrightarrow$ conserved currents.

2.3 $D3/D7$ system

2.3.1 $D3/D7$ interception

In order to use the framework of AdS/CFT correspondence to study problems of strongly coupled gauge theories in general and QCD in particular, we need to add flavour in the fundamental representation together with the appropriate symmetries. As the construction stands, the open strings endpoints move in the $D3$ branes world volume which describe fields in the adjoint representation of $SU(N_c)$ (one endpoint acts as a charge in the fundamental representation, the other one as point charge in the anti-fundamental representation). The endpoints of the strings can be arbitraly close to each other hence describing massless excitations. If we add N_f D-branes as in [7] we can have new strings stretching from the $D3$ brane to the D-branes, which generate matter in the fundamental representation, and strings with both points ending on the flavour D-branes will describe matter in the adjoint of $U(N_f)$, i.e. 'mesonic-like' degrees of freedom. If we separate the two branes along some direction transverse to both branes we will have a string of minimum length L stretching between both branes which amounts to a finite energy in this string responsible for the quark's mass ($m_q = L/2\pi\alpha'$). In principle we can consider any D-branes of odd numbers: $D5$, $D7$ and $D9$ but the $D7$ brane is the best candidate for our purposes. $D9$ fills the whole space, hence it cannot be separated from the stack of $D3$ branes, $D5$ and $D3$ branes lead to defect field theories which we will not consider.

The diagram above shows how we extend the $D7$ branes in space-time. They overlap the $D3$ branes in the Minkowski 0123 directions, then extend over the 4567 directions which in the gravity description of $AdS_5 \times S^5$ corresponds to the filling of

	0	1	2	3	4	5	6	7	8	9
D3	X	X	X	X						
D7	X	X	X	X	X	X	X	X		

Table 2.1: *The D3/D7-brane intersection in* $9+1$ *dimensional flat space.*

the AdS radial direction and 3 directions of the S^5 sphere. In principle we could pick different directions to extend the $D7$ brane, but these choices break different amounts of supersymmetry. In order to have as much control of the system as possible we would like to preserve as much supersymmetry as we can. The choice above retains 1/4 of the total supersymmetry in $\mathcal{N}=4$, 8 supercharges, which is the maximum we can preserve. One can ask the question, if adding N_f $D7$ branes, by construction, gives us a theory with a $U(N_f)$ gauge group in the world volume of the $D7$ branes, how do we find a global flavour group? One can answer this question by looking into the field theory of the $D3/D7$ brane intersection. As we described earlier, massless strings with both ends on the $D3$ brane will be responsible for a $\mathcal{N}=4$ multiplet, whereas strings stretching form a $D7$ brane to the $D3$ brane will generate a $\mathcal{N}=2$ hypermultiplet. Now, if we look at the theory in the $D7$ world volume, the eight-dimensional t' Hooft coupling, λ' is proportional to $\lambda l_s^4 \frac{N_f}{N}$ which at low energies, where the string length is made arbitrarily small ($l_s \to 0$), ensures that the $7-7$ strings effectively decouple from the $3-3$ and $3-7$ strings. In this limit, the $U(N_f)$ gauge group plays the role of a global flavour group in the four-dimensional theory.

2.3.2 The new field theory

As we have described above, the field theory corresponding to this brane set-up is a $\mathcal{N}=4$ SYM coupled to a $\mathcal{N}=2$ $U(N)$ gauge theory containing N_f hypermultiplets in the fundamental representation. Without writing the full Lagrangian of this theory, (it can be found explicitly in [13]), we want to understand the key differences from the original $\mathcal{N}=4$ SYM. We conveniently described the $\mathcal{N}=4$ theory using three chiral superfield (Φ_1, Φ_2, Φ_3) and one vector multiplet (W_α), which, along with its fermionic partners, gave us six scalars, ϕ_i, (transforming under the 6 of $SU(4)$) and a gauge field, that we could interpret as fluctuations of the $D3$ brane along its transverse directions and its world volume respectively. Essentially, the $D7$ brane gives us a new fundamental hypermultiplet that we will denote as Q_r,$\tilde{Q}_r$, $(r=1,...,N_f)$, which will couple to one of the chiral multiplets of the $\mathcal{N}=4$ theory. This amounts to a term in the superpotential of the type $\tilde{Q}_r(m_q+\Phi_3)Q^r$. This term will break the original $SO(6)_R$ to $SO(4)_R \times SO(2)_R$. $SO(4)_R$ in turn is isomorphic to $SU(2)_\Phi \times SU(2)_{\tilde{R}}$

where $SU(2)_\Phi$ mixes the scalars in the adjoint multiplet (ϕ_1,ϕ_2,ϕ_3,ϕ_4) and $SU(2)_R$ is an $\mathcal{N} = 2$ R-symmetry. These are better understood if we look into the gravity description of this field theory: by introducing a $D7$ brane as in the figure above, we explicitly break the symmetry in the directions transverse to the original $D3$ branes. Since the R-symmetries became isometries of space-time, $SO(4)_R$ can be understood as rotations in the 4567 directions and $SO(2)_R$ as rotations in the 89 plane.
A few more comments are in order. Since $U(N_f)$ is a global symmetry, the diagonal $U(1)_B$ acts as a baryon number where the fundamental hypermultiplet has a charge of $+1$ under this symmetry and the remaing fields are neutral.
The $U(1)_R$ acts as a chiral symmetry, the two fermions in the fundamental hypermultiplet (left and right-handed Weyl fermions) carry opposite charges under this symmetry. It will be important to us because it is analogous to the axial symmetry in QCD and will be responsible for the holographic dual of chiral symmetry breaking. It is anomalous if N_c is finite but we will not worry about this scenario even though it can be studied using AdS/CFT. Note that adding a quark mass m_q explicitly breaks this symmetry as in $\tilde{Q}_r(m_q + \Phi_3)Q^r$. In the gravity theory this means we are separating the $D7$ brane from the $D3$ stack by a minimum length L proportional do m_q, which forces the brane to chose a direction in the 89 plane that effectively breaks this symmetry.

2.3.3 The probe limit

In order to study this system we will take an important approximation. We have not mentioned what happens to the conformal symmetries when we introduce the $D7$ branes. If $m_q \neq 0$ conformal symmetry is broken explicitly, however, even when $m_q = 0$, if N_f is of the same order as N_c, quantum effects will be important and the flavour fields will dynamically generate a scale. This effect, from the field theory point of view, comes from adding the contribution of quark loops which in the gravity picture means to consider the backreaction of the $D7$ brane on the geometry. In this case, the coupling will start running, getting stronger in the UV which is unlike QCD. However, since we will be taking the large N_c limit and small N_f it is justifiable to ignore this effects and the conformal symmetry is restored in the case of $m_q = 0$. On the other hand, in the probe limit the $U(1)_R$ symmetry is not anomalous. As we will see later, taking this approximation gives us the freedom to break conformal symmetry in a controlled way, by switching on new fields in the background, which will allow us to engineer a running of the coupling that we can use to explore QCD-like scenarios.

2.3.4 Embedding equation

In the sections above we gave a field theoretical description of the $D3/D7$ system. On the gravity side we need to determine how the $D7$ brane sits in spacetime by solving its embedding equation. In general it will depend on the fields we turn on in background or any particular deformation of AdS spacetime which will reflect the change of physics on the field theory related to this deformation. As an example we will explore the simplest case. Lets write the $AdS_5 \times S^5$ metric as:

$$dS^2 = \frac{y^2}{R^2}\eta_{ij}dx^i dx^j + \frac{R^2}{y^2}(d\rho^2 + \rho^2 d\Omega_5^2 + d\omega_5^2 + \omega_6^2) \tag{2.33}$$

Where $\rho^2 = \omega_1^2 + ... + \omega_4^2$ and $y^2 = \rho^2 + \omega_5^2 + \omega_6^2$. Note that $\omega_{5,6}$ are the coordinates transverse to the $D7$ brane. We chose this coordinate system since its the most convenient considering how the inclusion of the $D7$ brane breaks the $SO(6)$ isometry. The action (2.21) describing the $D7$ brane can be written in this coordinate system as:

$$S_{D7} = -\mu_7 \int d^8\xi \rho^3 \sqrt{1 + \dot{\omega_5} + \dot{\omega_6}} \tag{2.34}$$

Where $\dot{\omega} = \partial_\rho \omega$ and F_{ab} was set to zero. Clearly the embedding is defined by the functions $\omega_{5,6}(\rho)$, with the equation of motion:

$$\frac{d}{d\rho}[\frac{\rho^3}{\sqrt{1 + \dot{\omega}_5^2 + \dot{\omega}_6^2}}\frac{d\omega_{5,6}}{d\rho}] = 0 \tag{2.35}$$

The solution to this equations is a constant L, giving the position of the brane in the ω_5, ω_6 plane. Strings now stretching from the $D3$ to the $D7$ have a minimum length of L which is proportional to the quark mass which in the field theory side correponds to a term of the type $m_q \tilde{Q}_r Q^r$ mentioned above. ρ is a radial direction of AdS so it has dimensions of energy and is interpreted as the renormalization group scale of the field theory. The fact that the embeddings are flat reflect the fact that the quark mass is not renormalizable if supersymmetry is present. In a more general scenario the solutions will look like:

$$\omega = L + \frac{c}{\rho^2} \tag{2.36}$$

Where $L = m_q 2\pi\alpha'$ and $c = \langle\bar{q}q\rangle(2\pi\alpha')^3$. For the constant solution, the induced metric on the $D7$ brane is [8]:

$$dS^2 = \frac{\rho^2 + L^2}{R^2}\eta_{ij}dx^i dx^j + \frac{R^2}{\rho^2 + L^2}d\rho^2 + \frac{R^2\rho^2}{\rho^2 + L^2}d\Omega_3^2 \tag{2.37}$$

If $L = 0$ this is the metric for $AdS_5 \times S^3$ and the theory is conformal. As expected if $L \neq 0$ there is a finite quark mass that breaks conformality. Note that in the UV of the theory when $\rho \gg L$ conformality is restored asymptotically.

2.3.5 Meson spectra

The $D3/D7$ system also includes bound states of quarks which are strings with both their endpoints in the $D7$ brane [8, 14]. We can calculate their masses by studying fluctuations of the $D7$ brane's world volume fields. As an example we will show the steps for computing the mesons masses from the scalar fields, in which case we can set $F_{ab} = 0$ in the DBI action and ignore the Wess-Zumino term. Taking the ansatz:

$$\omega_5 = 0 + \delta\omega_5, \;\; \omega_5 = L + \delta\omega_6 \tag{2.38}$$

we plug it in the action (2.34) and expand it to quadratic order to get:

$$\mathcal{L} \neq -\mu_7 \sqrt{-det g_{ab}}(1 + \frac{R^2}{r^2} g^{cd} \partial_c \phi_{5,6} \partial_d \phi_{5,6}) \tag{2.39}$$

From which we derive the e.o.m.:

$$\frac{R^4}{(\rho^2 + L^2)^2} \partial^\mu \partial_\mu \phi_{5,6} + \frac{1}{\rho} \partial_\rho (\rho^3 \partial_\rho \phi_{5,6}) + \frac{1}{\rho^2} \bigtriangledown^i \bigtriangledown_i \phi_{5,6} = 0 \tag{2.40}$$

where $\bigtriangledown_i$ is the covariant derivative on the S^3 sphere. We identify mesons as the normalizable solutions to these equations, which makes the spectrum discrete with a mass scale set by L. Making the separation of variables:

$$\phi = \phi(\rho) e^{ikx} \mathcal{Y}^l(S^3) \tag{2.41}$$

Where $\mathcal{Y}^l(S^3)$ are the scalar harmonics on the S^3 and satisfy:

$$\bigtriangledown^i \bigtriangledown_i \mathcal{Y}^l = -l(l+1)\mathcal{Y}^l \tag{2.42}$$

The meson masses are defined as $M^2 = -k^2$.and will label solutions of the equation for $\phi_{5,6}(\rho)$:

$$\partial_\varrho^2 \phi + \frac{3}{\varrho} \partial_\varrho \phi + (\frac{\tilde{M}^2}{(1+\rho^2)^2} - \frac{l(l+2)}{\varrho^2})\phi = 0 \tag{2.43}$$

After the redefinitions: $\varrho = \frac{\rho}{L}$ and $\tilde{M}^2 = -\frac{k^2 R^4}{L^2}$. The solution can be cast in terms of an hypergeometric function and we find for the mass of scalar mesons:

$$M(n,l) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2)} \tag{2.44}$$

Where l is the angular momentum.

Chapter 3

Holographic Integral equations and Walking Technicolour

The D3-D7 system [7, 15, 16] in AdS-like spaces has allowed the study, through gauge/gravity duality or holography [2,9,17], of many aspects of strongly interacting gauge theories with quarks [13]. The system has been used to study quark confinement [1, 18], mesons [8], transport properties at finite temperature [19–21], and chiral symmetry breaking in the presence of a running coupling [22–24] or a magnetic field [25].

In this chapter we present a very simple model of chiral symmetry breaking and the associated Goldstone boson (essentially pion) in this system. The simple model consists of embedding the D7s in pure $AdS_5 \times S^5$ but with an arbitrary dilaton profile to represent the running coupling of the dual gauge theory. This basic model, although the metric is not back reacted to the dilaton's presence, provides a simple encapsulation of the chiral symmetry breaking mechanism in the D3-D7 system. In particular it will allow us to elucidate in the holographic equations of motion why there is a Goldstone boson present for the symmetry breaking. Further it will allow us to write integral equations for the parameters of the low energy chiral Lagrangian involving just the form of the running coupling and the quark self energy function (the D7 brane embedding function). These equations are very similar in spirit to the Pagels-Stokar formula [26] for the pion decay constant, f_π, and constituent quark model [27] estimates of the chiral condensate and so forth.

The formulae we will present for these low energy parameters allow one to develop intuition about how the low energy theory depends on the underlying gauge dynamics. We explore this and as a particular example look at walking [28, 29] Technicolour [30, 31] theories to see if the holographic model matches the folk lore from constituent quark models. Our results support the expectation that a walking regime will enhance the quark condensate relative to the pion decay constant.

In the final section of this chapter we will perform a similar study for the non-supersymmetric D3/D5 system with a four dimensional overlap. We interpret this system as a walking gauge theory where the quark condensate has a dimension of $2+\sqrt{3}$ in the far UV. This theory is not of any obvious phenomenological use but the walking paradigm does seem to explain the physics of the system.

3.1 A simple $D3/D7$ chiral symmetry breaking model

We will consider a gauge theory with a holographic dual described by the Einstein frame geometry $AdS_5 \times S^5$

$$ds^2 = \frac{1}{g_{uv}}\left[\frac{r^2}{R^2}dx_4^2 + \frac{R^2}{r^2}\left(d\rho^2 + \rho^2 d\Omega_3^2 + dw_5^2 + dw_6^2\right)\right] \tag{3.1}$$

where we have split the coordinates into the x_{3+1} of the gauge theory, the ρ and Ω_3 which will be on the D7 brane world-volume and two directions transverse to the D7, w_5, w_6. The radial coordinate, $r^2 = \rho^2 + w_5^2 + w_6^2$, corresponds to the energy scale of the gauge theory. The radius of curvature is given by $R^4 = 4\pi g_{uv}^2 N \alpha'^2$ with N the number of colours. g_{uv}^2 is the $r \to \infty$ value of the dilaton. In addition we will allow an arbitrary running as $r \to 0$ to represent the gauge theory coupling

$$e^{\phi} \quad = \quad g_{YM}^2(r^2) \quad = \quad g_{uv}^2\ \beta(\rho^2 + w_5^2 + w_6^2) \tag{3.2}$$

where the function $\beta \to 1$ as $r \to \infty$. The $r \to \infty$ limit of this theory is dual to the $\mathcal{N} = 4$ super Yang-Mills theory and g_{uv}^2 is the constant large r asymptotic value of the gauge coupling.

We will introduce a single D7 brane probe [7] into the geometry to include quarks - by treating the D7 as a probe we are working in a quenched approximation although we can reintroduce some aspects of quark loops through the running coupling's form if we wish. Although this system only has a U(1) axial symmetry on the quarks corresponding to rotations in the $w_5 - w_6$ plane (formally this symmetry is an R-symmetry of the model but it is broken by a quark mass or condensate) we believe it is a good setting for studying the dynamics of the quark condensation.

That process is driven by the strong dynamics rather than the global symmetries so the absence of a non-abelian axial symmetry should not be important[1].

We must find the D7 embedding function eg $w_5(\rho), w_6(\rho) = 0$. The Dirac Born Infeld action in Einstein frame is given by (2.21)

$$\begin{aligned} S_{D7} &= -T_7 \int d^8\xi e^{\phi} \sqrt{-\det P[G]_{ab}} \\ &= -\overline{T_7} \int d^4x \; d\rho \; \rho^3 \beta \sqrt{1 + (\partial_\rho w_5)^2} \end{aligned} \tag{3.3}$$

where $T_7 = 1/(2\pi)^7 \alpha'^4$ and $\overline{T_7} = 2\pi^2 T_7 / g_{uv}^2$ when we have integrated over the 3-sphere on the D7. The equation of motion for the embedding function is therefore

$$\partial_\rho \left[\frac{\beta \rho^3 \partial_\rho w_5}{\sqrt{1 + (\partial_\rho w_5)^2}} \right] - 2 w_5 \rho^3 \sqrt{1 + (\partial_\rho w_5)^2} \frac{\partial \beta}{\partial r^2} = 0 \tag{3.4}$$

The UV asymptotic of this equation, provided the dilaton returns to a constant so the UV dual is the $\mathcal{N} = 4$ super Yang-Mills theory, has solutions of the form

$$w_5 = m + \frac{c}{\rho^2} + ... \tag{3.5}$$

where we can interpret m as the quark mass ($m_q = m/2\pi\alpha'$) and c is proportional to the quark condensate as we'll see below.

The embedding equation (4.3) clearly has regular solutions $w_5 = m$ when g_{YM}^2 is independent of r - the flat embeddings of the $\mathcal{N} = 2$ Karch-Katz theory [7]. Equally clearly if $\partial\beta/\partial r^2$ is none trivial in w_5 then the second term in (4.3) will not vanish for a flat embedding. We conclude that for any non-trivial gauge coupling the asymptotic solutions must contain the parameter c, a quark condensate. Whether $c \to 0$ or not as $m \to 0$ depends on the precise form of the running coupling chosen (note that $w_5 = 0$ is always a solution of (4.3)). However, if the coupling grows towards $r = 0$ as one would expect in a confining theory then there is clearly a growing penalty in the action for the D7 to approach the origin and we expect c to be non-zero.

As an example one can consider a gauge coupling running with a step of the form

$$\beta = a + 1 - a \tanh\left[\Gamma(r - \lambda)\right] \tag{3.6}$$

[1]The Sakai Sugimoto model [32] is an example of a gravity dual with a non-abelian chiral symmetry but it is fundamentally five dimensional and a clear prescription for including a quark mass is lacking - the result is that we would not know how to do this analysis in that model since we can not identify the quark self energy nor the quark condensate.

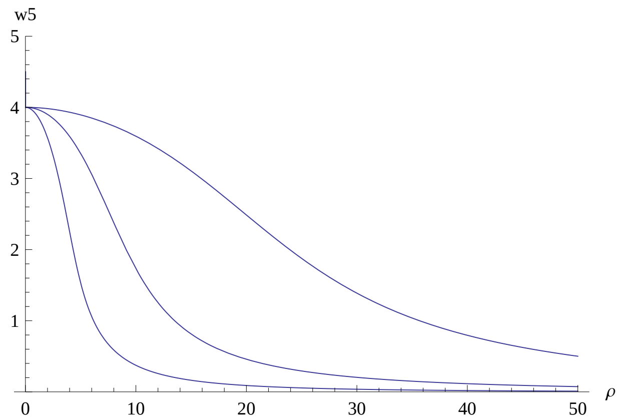


Figure 3.1: *The D7 brane embeddings/quark self energy plots for the coupling ansatz in (3.6) - in each case the parameter* $a = 3$ *and from left to right:* $\lambda = 3.19, \Gamma = 1$*;* $\lambda = 4.55, \Gamma = 0.3$*;* $\lambda = 10.4, \Gamma = 0.1$.

This form introduces conformal symmetry breaking at the scale $\Lambda = \lambda/2\pi\alpha'$ which triggers chiral symmetry breaking. The parameter a determines the increase in the coupling across the step but the solutions have only a small dependence on the value chosen because the area of increasing coupling is avoided by the D7 brane. An extreme choice of the profile is to let the coupling actually diverge at the barrier to represent the one loop blow up in the running of the QCD coupling - the solutions show the same behaviour as for a finite step provided the transition is not infinitely sharp. The parameter Γ spreads the increase in the coupling over a region in r of order Γ^{-1} in size - the effect of widening the step is to enhance the tail of the self energy function for the quark. We show the symmetry breaking embeddings in figure 3.1.

We will interpret the D7 embedding function as the dynamical self energy of the quark, similar to that emerging from a gap equation. The separation of the D7 from the ρ axis is the mass at some particular energy scale given by ρ - in the $\mathcal{N} = 2$ theory where the embedding is flat the mass is not renormalized, whilst with the running coupling an IR mass forms - we have picked parameters in Figure 3.1 that generate the same dynamical quark mass at $\rho = 0$. We call the embedding function Σ_0 below.

3.2 Goldstone mode

The embedding above lies at $w_6 = 0$ but there is clearly a set of equivalent solutions given by rotating that solution in the $w_5 - w_6$ plane. That degeneracy of the solutions is the vacuum manifold. We therefore expect a Goldstone mode associated with a fluctuation of the vacuum in the angular direction. For small fluctuations about the embedding above we may look at fluctuations in w_6. The quadratic order expanded action for such a small fluctuation is

$$S_7 = -\overline{T_7} \int d\rho dx^4 \rho^3 \beta \sqrt{1 + (\partial_\rho \Sigma_0)^2} \left(1 + \frac{\partial_{r^2}\beta}{\beta} w_6^2 \right. \tag{3.7}$$

$$\left. + \frac{1}{2} \frac{(\partial_\rho w_6)^2}{1 + (\partial_\rho \Sigma_0)^2} + \frac{1}{2} \frac{R^4}{r^4} (\partial_\mu w_6)^2 + ... \right) \tag{3.8}$$

note r, β and $\partial_{r^2}\beta$ are evaluated on the solution Σ_0 here and henceforth.

As usual we will seek solutions of the form $w_6(\rho, x) = f_n(\rho) e^{ik.x}$, $k^2 = -M_n^2$. Here n takes integer values - the solutions are associated with the Goldstone boson and its tower of radially excited states. The f_n satisfy the equation

$$\partial_\rho \left(\frac{\beta \rho^3 \partial_\rho f_n}{\sqrt{1 + (\partial_\rho \Sigma_0)^2}} \right) - 2\rho^3 \sqrt{1 + (\partial_\rho \Sigma_0)^2} (\partial_{r^2}\beta) f_n + \frac{1}{r^4} \rho^3 \beta \sqrt{1 + (\partial_\rho \Sigma_0)^2} R^4 M_n^2 f_n = 0 \tag{3.9}$$

The presence of a Goldstone boson is now immediately apparent - there is a solution with $M_n^2 = 0$ and $f_0 = \Sigma_0$. With these substitutions the equation is exactly the embedding equation (4.3), a result of the symmetry between between w_5 and w_6. This is the pion like bound state of this theory - although there is only a broken U(1) axial symmetry, the absence of anomaly effects at large N make it closer in nature to the pions than the η' of QCD.

Naively the argument just given makes it appear there is a massless Goldstone for any w_5 solution including those where there is an explicit quark mass in the asymptotic fall off in (3.5). This is not the case though because to interpret the solution as a Goldstone requires f_0 to fall off at large ρ as $1/\rho^2$ - it must be a fluctuation in the condensate not the explicit mass. The naive massless solution is related to the fact that the theory has a spurious symmetry where $\bar{\psi}_L \psi_R \to e^{i\alpha} \bar{\psi}_L \psi_R$ and simultaneously $m \to e^{-i\alpha} m$. This spurious symmetry must be present in the string construction.

3.2.1 The low energy Chiral Lagrangian

The Goldstone field's low energy Lagrangian must take the form of a chiral Lagrangian, non-linear realization of the broken symmetry [33]. We can substitute the form $w_6 = f_0(\rho)\pi^a(x) = \Sigma_0\pi^a(x)$ into (3.7) and integrate over ρ to obtain this Lagrangian

$$\begin{aligned}\mathcal{L} &= -\overline{T_7}\int \mathrm{d}\rho\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\left(1+\frac{1}{2}\frac{R^4}{r^4}\Sigma_0^2(\partial_\mu\Pi)^2\right. \\ &\quad \left. +\frac{1}{4}\frac{R^4}{r^4}\left(\frac{2}{\beta}\frac{d\beta}{dr^2}\Sigma_0^4+\frac{\Sigma_0^2(\partial_\rho\Sigma_0)^2}{1+(\partial_\rho\Sigma_0)^2}\right)Tr([\partial_\mu\Pi,\Pi]^2)+...\right) \quad (3.10)\end{aligned}$$

where we've used the equation of motion (3.9) to eliminate the second and third terms in (3.7) in the massless limit. We have also included the $[\partial_\mu\Pi,\Pi]^2$ term from the fourth order expansion from which we will determine f_π, where $\Pi = \pi^a(x)\tau^a$ and τ^a are the generators of $U(N_f)$.

This should be compared to the standard chiral Lagrangian form where $U = \exp(2i\pi/f_\pi)$

$$\begin{aligned}\mathcal{L} &= V_0 + \frac{f_\pi^2}{4}Tr(\partial_\mu U^\dagger\partial^\mu U) + \mathcal{O}(p^4) \\ &= V_0 + \frac{1}{2}(\partial_\mu\pi(x)^a)^2 + \frac{1}{48f_\pi^2}Tr([\partial_\mu\Pi,\Pi]^2) + \mathcal{O}(\pi(x)^6) + \mathcal{O}(p^4)\end{aligned} \quad (3.11)$$

where V_0 is the vacuum energy and f_π is the pion decay constant.

We must rescale $\pi(x)$ in (3.10) to the canonical normalization in (3.11) and then we can read off an integral expression for the pion decay constant. To ensure all factors of α' are absent from physical answers, as they must be, we must express our answer as the ratio of two physical scales. Here we will use the scale Λ in the gauge coupling running (3.6) that encodes the scale of the chiral symmetry breaking as our reference - we have

$$\frac{f_\pi^2}{\Lambda^2} = \frac{-N}{48\pi^2\lambda^2}\left.\frac{\left[\int d\rho\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2}\right]^2}{\left[\int d\rho\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\frac{1}{4(\rho^2+\Sigma_0^2)^2}\left(\frac{2}{\beta}\frac{d\beta}{dr^2}\Sigma_0^4+\frac{\Sigma_0^2(\partial_\rho\Sigma_0)^2}{1+(\partial_\rho\Sigma_0)^2}\right)\right]}\right|_{r^2=\rho^2+\Sigma_0^2} \quad (3.12)$$

Note that $\partial_{r^2}\beta$ is typically negative for the embeddings we have explored above so that f_π^2 is positive. Employing the embedding equation (4.3) the denominator may be simplified leaving

$$\frac{f_\pi^2}{\Lambda^2} = \frac{-N}{12\pi^2\lambda^2} \frac{\left[\int d\rho \rho^3 \beta \sqrt{1+(\partial_\rho \Sigma_0)^2} \frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2}\right]^2}{\left[\int d\rho \frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2} \partial_\rho \left(\frac{\beta \rho^3 \Sigma_0 (\partial_\rho \Sigma_0)}{\sqrt{1+(\partial_\rho \Sigma_0)^2}}\right)\right]} \tag{3.13}$$

We can also extract an integral equation for the quark condensate (evaluated in the UV where there is no running) from our analysis. We use the fact that the expectation value of $\bar{q}_L q_R$ is given by $\frac{1}{Z}\frac{\partial Z}{\partial m_q}|_{m_q \to 0}$. For an infinitessimal value of m in the boundary embedding (3.5) we expect the full embedding, to leading order, to simply take the form $w_5 = 2\pi\alpha' m_q + \Sigma_0$. We insert this form into the vacuum energy and expand to leading order in m_q - the coefficient is just the quark condensate

$$\frac{\langle \bar{q}_L q_R \rangle}{\Lambda^3} = \frac{-N}{4\pi\lambda^3 g_{uv}^2 N} \int d\rho \ \rho^3 \Sigma_0 \sqrt{1+(\partial_\rho \Sigma_0)^2} \partial_{r^2} \beta \Big|_{r^2=\rho^2+\Sigma_0^2} \tag{3.14}$$

One may use the embedding equation (4.3) to turn this into a surface term that is then, given β becomes unity asymptotically, proportional to $\rho^3 \partial_\rho \Sigma_0|_{\rho\to\infty}$ which is just proportional to the constant c in (3.5) confirming the interpretation of c as the condensate. The integral form of the equation though allows intuition for the value of the condensate from the shape of the embedding as we will see. Note that if the 'tHooft coupling $g_{uv}^2 N$ is kept fixed both f_π and the condensate grow as N as expected.

The integral equations (3.13) and (3.14) that link low energy parameters to the underlying UV physics are our main results. They are very reminiscent of constituent quark model [27] results which input the quark self energy, $\Sigma(q)$, (for example determined from a gap equation [29]) to determine the same quantities. In particular those models give for the condensate

$$\langle \bar{q} q \rangle = \frac{N}{2} \int q^3 dq \frac{\Sigma}{q^2 + \Sigma^2} \tag{3.15}$$

and the Pagels Stokar formula [26] for the pion decay constant

$$f_\pi^2 = \frac{N}{8\pi^2} \int q^3 dq \frac{\Sigma^2 - \frac{1}{2} q^2 \Sigma \Sigma'}{(q^2+\Sigma^2)^2} \tag{3.16}$$

where a prime indicates a derivatives with respect to q^2. Although our formulae are more complex and include the underlying gauge coupling's running there are nevertheless a number of common features. We will compare them for the case of walking Technicolour below.

It must be stressed that we have derived our expressions (3.13) and (3.14) in a toy holographic model of chiral symmetry breaking. Of course one can not just impose any random running of the gauge coupling and assume one is in a real gauge theory. We have also not included any back reaction of the space's metric to the presence of a non-trivial dilaton. The analysis is very similar in spirit to the chiral quark model assumption of an arbitrary choice of $\Sigma(q^2)$. Despite these flaws, we hope the simplicity of the expressions allows one to analytically understand the typical response of the holographic descriptions to different types of running coupling.

3.3 Walking Technicolour

The constituent quark model expressions (3.15) and (3.16) have underpinned much of the folk lore for walking Technicolour theories [28, 29]. In brief, in walking Technicolour the gauge coupling is assumed to transition from perturbative to non-perturbative behaviour at one scale, Λ_1 but then the running slows, only crossing some critical value for inducing chiral symmetry breaking at a scale, Λ_2, several orders of magnitude below Λ_1. In the period between Λ_1 and Λ_2 we imagine that the anomalous dimension ϵ of the quark condensate is negative (so $\bar{q}q$ has dimension less than three) - the condensate evaluated in the UV is then enhanced taking the rough value $\Lambda_2^{3-\epsilon}\Lambda_1^{\epsilon}$.

Gap equation analysis [29] provides an alternative but equivalent explanation for the enhancement of the quark condensate. There walking, which has a larger coupling value further into the UV, enhances the large q tail of the quark self energy $\Sigma(q)$. Looking at the constituent quark model expressions for low energy parameters one can see that f_π is dominated at small q (there is a q^4 in the denominator) and so f_π is broadly unchanged by walking. In a Technicolour model f_π sets the W and Z masses and hence the weak scale. On the other hand the condensate in (3.15) is given by a simple integral over $\Sigma(q)$ and hence grows if the tail of $\Sigma(q)$ is raised. The condensate is enlarged in walking theories relative to the weak scale. In extended Technicolour models [34] the condensate determines the standard model fermion masses - increasing it drives up the extended Technicolour scale, potentially suppressing flavour physics below current experimental bounds.

Do our holographic expressions agree with this story? The challenge is to simulate walking in a holographic setting. The problem is that we are always at strong coupling (large N) if we have a weakly coupled gravity dual. As we have seen, the introduction of any conformal symmetry breaking through the running coupling causes chiral symmetry breaking[2]. We can not therefore reproduce directly the physics at the scale Λ_1 discussed above where the theory moved to strong coupling but without causing chiral symmetry breaking.

As a first attempt to address this point we can be led by the solutions in Figure 3.1 as a result of the coupling ansatz in (3.6). If we increase the parameter Γ we effectively smear the scale at which the chiral symmetry breaking is induced over a range of $r \sim \Gamma^{-1}$. Could we use this smeared range to represent the separation between Λ_1 and Λ_2 above? The effect of the smearing is to enhance the tail of the self energy just as expected in walking theories.

If we now turn to the holographic expressions (3.13) & (3.14) we see that they naively share the same response to enhancing the tail of Σ_0 as the constituent quark model expressions (3.15) & (3.16) did to raising the tail of $\Sigma(q)$. In particular again f_π has a $1/\rho^4$ factor in the denominator of each integral involved, making it, one would expect, insensitive to changes in the tail of Σ_0. The expression for the condensate though is sensitive to the tail and should grow as walking is introduced. In fact though this analysis neglects the dependence of these functions on the derivatives of the gauge coupling and the self energy function Σ_0 - this additional understanding of dynamics coming from the gauge coupling running lies beyond the constituent quark model pictures. Both (3.13) and (3.14) are dominated around the points of maximum change in the coupling and Σ_0. Note though that the derivative of the coupling, $\partial_{r^2}\beta$, is evaluated on the brane, which in the cases above has precisely embedded itself so as to avoid large derivatives in β. By smoothing these functions through increasing Γ we include extra functional behaviour. In fact these changes in the derivatives are more numerically important than the rise in the tail of Σ_0 for the plots in Figure 3.1. This means that the more "walking" looking self energies in fact give a slightly lower condensate for a fixed value of f_π. The simple coupling ansatz in (3.6) does not therefore accommodate a behaviour we can interpret in the usual walking picture. The model does suggest that there could be considerable variation in the ratio of the condensate to f_π in gauge theories with rather different speeds of IR running though. A recent lattice analysis suggest this ratio could vary as the number of quark flavours is changed in QCD [37].

To take advantage of the similarities between (3.13) & (3.14) and (3.15) & (3.16) one would need to keep the derivatives of the coupling and Σ_0 roughly fixed as the scale at which that change occurred was moved out to larger ρ. Our equations would in such a scenario provide the enhancement of the condensate that one looks

[2] Attempts to find backreacted holographic models of gauge theories with a walking profile such as those in [35,36] could fall foul of this problem were they used to generate chiral symmetry breaking.

for in a walking theory. Essentially one would want a self energy that rose sharply at large ρ but then flattened to meet the w_5 axis at the same value as the curves in Figure 3.1. This in fact matches the crucial signal of walking that one would expect $\Sigma_0(\rho = 0) \ll \Lambda$ with Λ the scale at which the high scale running occurs. Within holographic models this should be the crucial signal of walking.

This scenario suggests we are mimicking a slightly different walking dynamics in the gauge theory than that discussed above - imagine a theory in which the coupling ran to strong coupling (call this scale Λ_1 again) and then entered a conformal regime with coupling value slightly above the critical value needed to form a condensate. If the coupling was tuned from above sufficiently close to the critical value in its conformal window then a self energy would form but with a size considerably below Λ_1.

Realizing this sort of walking behaviour can be done in a straightforward, if adhoc, fashion. We need to break the symmetry between ρ and w_5, w_6 in the coupling ansatz β. A simple ansatz is just to shift our previous ansatz out to larger ρ:

$$\begin{aligned} \beta &= a + 1 - a \tanh\left[\Gamma(\sqrt{(\rho-\lambda_1)^2 + w_5^2 + w_6^2} - \lambda)\right] \quad & \rho \geq \lambda_1 \\ \beta &= 1 & \rho < \lambda_1 \end{aligned} \tag{3.17}$$

This ansatz, which we sketch in Figure 3.2, leaves the derivative of β unchanged but shifted by λ_1 in ρ - this will ensure the condensate, which is given by (3.14) and dominated around λ_1 where the derivative of β is non-zero, will grow as λ_1^3. The embedding will still plateau around the same value of w_5 since above the step (which is quite sharp) the space is AdS and the embeddings must be flat. Below λ_1 the embedding becomes flat since the geometry is AdS (the first derivative of Σ_0 at $\rho = \lambda_1$ is smooth). Obviously this choice of β below λ_1 looks peculiar - one could though imagine that in that region there is a sharp step function to large coupling at small w_5, w_6 - the embeddings would remain the same.

With the embeddings from this walking β ansatz we can analytically see how the expressions for f_π and $\langle \bar{q}q \rangle$ change with λ_1. In (3.13) the numerator will become independent of λ_1 as it grows whilst the denominator, which is proportional to the derivatives of Σ_0 and β will fall as $1/\lambda_1$. f_π will therefore scale as $\lambda_1^{1/2}$. The condensate expression (3.14) is dominated around λ_1 where the derivative of β is non-zero - it will grow as λ_1^3. Therefore if we raise λ_1 at fixed f_π the condensate will grow as $\lambda_1^{3/2}$. The rise is consistent with the usual claims that a walking theory will enhance the condensate.

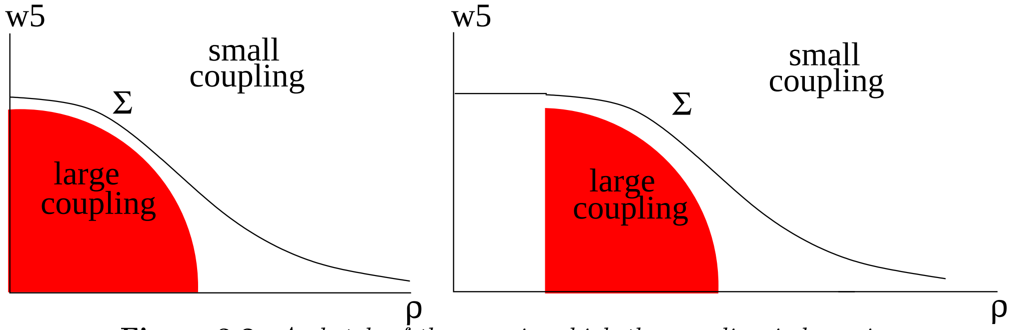


Figure 3.2: *A sketch of the area in which the coupling is large in our ansatz in (3.17) and the resulting form of the embeddings* Σ_0 *- on the left for* $\lambda_1 = 0$ *and on the right for a non-zero* λ_1

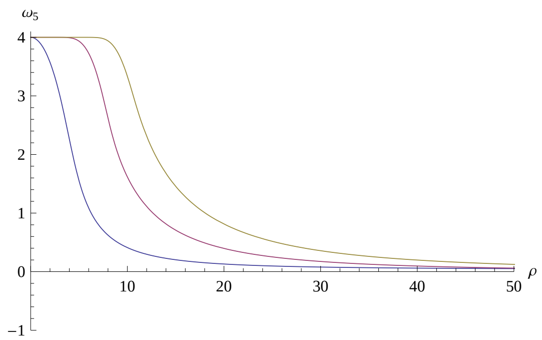


Figure 3.3: *Numerically determined embeddings for the coupling ansatz in (3.17). These curves all have* $a = 3$ *and* $\lambda = 3.19$ *in addition the curves from from left to right correspond to the parameter choices* $\lambda_1 = 0, \Gamma = 1,\ \lambda_1 = 5, \Gamma = 3.51, \lambda_1 = 8, \Gamma = 3.63$.

It is also possible to numerically confirm this behaviour at least for small λ_1. In Figure 3.3 we show numerical embeddings, displaying the behaviour shown in Figure 3.2, as λ_1 is increased from 0 to 8. To keep the plateau value exactly equal we have tuned Γ in the coupling ansatz (it changes from 1 to 3.6 across these plots). The condensate grows by an order of magnitude across these plots and in the large λ_1 limit will presumably match the analytic behaviour discussed although more and more tuning of Γ would be needed. Note that breaking the symmetry between ρ and w_5, w_6 in the β ansatz is still consistent with the symmetries of the D3-D7 system. In fact interestingly a distinction between the ρ and w_5, w_6 directions is precisely what one would expect in a geometry backreacted to the D7 branes [15, 38]. It is therefore plausible that one could fine tune the number of quark flavours in some D3-D7 system to obtain these forms of ansatz for the dilaton.

3.4 The D3-D5 System

We now turn to an alternative attempt to describe aspects of walking dynamics with holography. On first meeting the D3/(probe)D7 system it seems as if that system should fundamentally be a walking gauge theory - the $\mathcal{N} = 4$ gauge dynamics is conformal and strongly coupled in the UV. When we introduce running in the IR that triggers chiral symmetry breaking, should the physics not be closer in spirit to that of a walking theory rather than QCD? Why did we have to work so hard above to make that system walk? The reason it is not a walking theory is that the UV of the D3/D7 system possesses $\mathcal{N} = 2$ supersymmetry which both forbids a quark condensate and protects the dimension of the $\bar{q}q$ condensate at three. That the self energy profiles Σ_0 fall off as $1/\rho^2$ in the analysis above is driven by that UV supersymmetry and mimics the behaviour of asymptotically free QCD.

It is natural then to look for a way to introduce quarks into $\mathcal{N} = 4$ super Yang Mills which breaks supersymmetry even in the far UV. Using a D5 probe to introduce quarks seems the simplest example to explore. Here we consider the system with a four dimensional overlap of the D3 and the D5 not a three dimensional overlap as studied in [39]. Note that the strings between the D3 and D5 remain bi-fundamental fields of the gauge symmetry and global symmetry. The lowest energy modes of those strings are still at heart the gauge field, that would be present if the strings were free to move in the whole space, which become scalar fields, and the gaugino partners that become the fermionic quarks. In a non-supersymmetric theory the scalars will most likely become massive leaving fermionic quark multiplets in the $\mathcal{N} = 4$ theory.

The metric of $AdS_5 \times S^5$ can be written in coordinates appropriate to the D5 embedding as:

$$ds^2 = \frac{1}{g_{uv}} \left[\frac{r^2}{R^2} \eta_{ij} dx^i dx^j + \frac{R^2}{r^2} \left(d\rho^2 + \rho^2 d\Omega_1^2 + d\omega_3^2 + d\omega_4^2 + d\omega_5^2 + d\omega_6^2 \right) \right] \tag{3.18}$$

with $r^2 = \rho^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2$ and $\rho^2 = \omega_1^2 + \omega_2^2$. R is the radius of AdS $R^4 = 4\pi g_{uv}^2 N \alpha'^2$ The D3 brane is extended in the x_i dimensions. The D5-brane will be extended in the ρ and Ω_1 directions. The $\omega_3, \omega_4, \omega_5$ and ω_6 are perpendicular to the D5-brane. g_{uv}^2 is the value of the dilaton for $r \to \infty$.

Let us first analyze the system with a constant dilaton

$$e^{\phi} = g_{uv}^2 \tag{3.19}$$

The action for a probe D5 brane assuming the embedding $\omega_5(\rho), \omega_3 = \omega_4 = \omega_6 = 0$

is:

$$\begin{aligned} S_{D5} &= -T_5 \int d^8\xi e^{\phi} \sqrt{-\text{det} P \left[G\right]_{ab}} \\ &= -\overline{T_5} \int d^4 x d\rho r^2 \rho \sqrt{1 + (\partial_\rho w_5)^2}, \end{aligned} \tag{3.20}$$

where $T_5 = 1/(2\pi)^5 \alpha'^3$ and $\overline{T_5} = T_5 2\pi / R^2 g_{uv}$. The embedding equation is

$$\partial_\rho \left[r^2 \rho \frac{(\partial_\rho \omega_5)}{\sqrt{1 + (\partial_\rho \omega_5)^2}} \right] - 2\omega_5 \rho \sqrt{1 + (\partial_\rho \omega_5)^2} = 0 \tag{3.21}$$

The large ρ behaviour of these solutions is

$$\omega_5 \sim m\rho^{\sqrt{3}-1} + c/\rho^{1+\sqrt{3}} \tag{3.22}$$

The full embeddings are shown on the left hand of Figure 3.4. Note that as $m \to 0$ in the UV asymptotics the full solutions lie along the ρ axis indicating that the condensate $c = 0$ and there is no spontaneous chiral symmetry breaking - this is a simple result of the absence of a scale in the conformal field theory.

We continue to interpret the parameter m in the D5 brane embedding as the quark mass. Then from equation (3.22) we can see that there is an effective anomalous dimension present for that mass - its dimension is $2 - \sqrt{3}$. The parameter c is then the quark condensate and has dimension $2 + \sqrt{3}$. the change in the dimension of these operators in the UV conformal regime is exactly the physics that underlies the walking idea. Amusingly though here the anomalous dimension of the quark condensate is positive rather than negative as usually envisaged in walking theories. The D3/D5 system will not apparently be much use for constructing a phenomenological technicolour model. On the other hand here we are simply interested in testing the intuition for walking theories so we will continue to investigate for more formal reasons.

3.4.1 D3-D5 Embedding with a Non-Trivial Dilaton

Let us now include a non-trivial dilaton (gauge coupling) profile as we did above in the D3-D7 system

$$e^{\phi} = g^2_{YM}(r^2) = g^2_{uv}\beta(r^2). \tag{3.23}$$

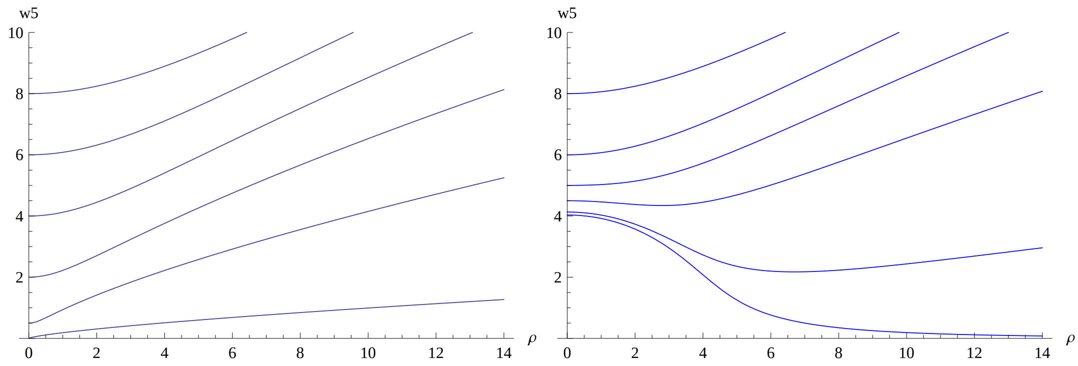


Figure 3.4: *The regular embeddings of a D5 brane in pure AdS with $\beta = 1$ on the left. On the right the chiral symmetry breaking embeddings for the ansatz for β in (3.6) with $\Gamma = 1, \lambda = 3, a = 5$.*

For $r \to \infty$ $\beta \to 1$. The action is now

$$S_{D5} = -\overline{T_5} \int d^4x d\rho r^2 \beta \rho \sqrt{1 + (\partial_\rho w_5)^2}. \tag{3.24}$$

The embedding equation is

$$\partial_\rho \left[r^2 \beta \rho \frac{(\partial_\rho \omega_5)}{\sqrt{1 + (\partial_\rho \omega_5)^2}} \right] - 2\omega_5 \rho \sqrt{1 + (\partial_\rho \omega_5)^2} \left[\beta + r^2 (\partial_{r^2} \beta) \right] = 0 \tag{3.25}$$

The embeddings can be seen on the right in Figure 3.4 for the ansatz for β in (3.6). There is again chiral symmetry breaking with a non-zero $w_5(\rho = 0)$ as $m \to 0$ in the UV. The self energy curves fall off faster at large ρ which matches expectations from gap equations in a theory where the quark condensates dimension grows in the walking regime.

The embedding breaks the SO(4) symmetry in the $\omega_3 - \omega_6$ directions so we expect there to be Goldstone modes present. For example, there should be an equivalent solution when rotating the embedding in e.g. the $\omega_5 - \omega_6$ plane. Let's look at small fluctuations around the embedding Σ_0 in the ω_6 direction to find a Goldstone boson. The action for such fluctuations in quadratic order is

$$\begin{aligned} S_5 = &- \overline{T_5} \int d^4x d\rho r^2 \beta \rho \sqrt{1 + (\partial_\rho \Sigma_0)^2} \left[1 + (\partial_{r^2} \beta) w_6^2 \right. \\ &\left. + \frac{1}{2} \frac{(\partial_\rho \omega_6)^2}{1 + (\partial_\rho \Sigma_0)^2} + \frac{1}{2} \frac{R^4}{r^4} (\partial_\mu \omega_6)^2 + \dots \right] \end{aligned} \tag{3.26}$$

where again r^2, β and $\partial_{r^2} \beta$ are all evaluated on the the D7 brane world volume

Σ_0. We seek fluctuations of the form $\omega_6(x,\rho) = f_n(\rho)e^{ik\cdot x}$ with $k^2 = -M_n^2$. The equation of motion for the fluctuations give the following equations for f_n

$$\partial_\rho \left[r^2\beta\rho \frac{(\partial_\rho f_n)}{\sqrt{1+(\partial_\rho\Sigma_0)^2}} \right] + \frac{R^4}{r^2}\beta\rho\sqrt{1+(\partial_\rho\Sigma_0)^2}M_n^2 f_n - 2\rho\sqrt{1+(\partial_\rho\Sigma_0)^2}\left[\beta + r^2\partial_{r^2}\beta\right] f_n = 0 \tag{3.27}$$

The equation with $M^2 = 0$ and $f_0 = \Sigma_0$ is the embedding equation (4.3) revealing the presence of the Goldstone mode.

The Lagrangian for the Goldstone field is found by writing $\omega_6^a = f_0(\rho)\pi^a(x) = \Sigma_0\pi^a(x)$ in (3.26) and integrating over ρ. We can expand $r^2\beta$ with $r^2 = \rho^2 + \Sigma_0^2 + \Sigma_0^2(\pi^4(x))^2$ as $r^2\beta(r^2) = r^2\beta(r^2)|_{r^2=\rho^2+\Sigma_0^2} + \Sigma_0^2(\pi^a(x))^2\partial_{r^2}(r^2\beta)|_{r^2=\rho^2+\Sigma_0^2}$ and then use the equation of motion (3.27) to eliminate the second and third terms in (3.26) for $M_n = 0$. This procedure gives the Lagrangian to quartic order

$$\mathcal{L} = -\overline{T_5}\int d\rho r^2\beta\rho\sqrt{1+(\partial_\rho\Sigma_0)^2} \left[1 + \frac{1}{2}\frac{R^4}{r^4}\Sigma_0^2(\partial_\mu\Pi)^2 + \frac{1}{4}\frac{R^4}{r^4}\left(\frac{(\partial_\rho\Sigma_0)^2\Sigma_0^2}{1+(\partial_\rho\Sigma_0)^2} + 2\Sigma_0^4\frac{\partial_{r^2}\left(\beta r^2\right)}{\beta r^2}\right) Tr\left(\partial_\mu\Pi,\Pi]^2\right) + \ldots\right]. \tag{3.28}$$

We can now rescale $\pi(x)$ in (3.28) and get an expression for f_π. We find

$$\frac{f_\pi^2}{\Lambda^2} = \frac{-2N^{1/2}}{24\pi^{3/2}\lambda^2}\frac{\left[\int d\rho\beta\rho\sqrt{1+(\partial_\rho\Sigma_0)^2}\frac{\Sigma_0^2}{\rho^2+\Sigma_0^2}\right]^2}{\left[\int d\rho\frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2}\partial_\rho\left(\frac{(\rho^2+\Sigma_0^2)\beta\rho\Sigma_0(\partial_\rho\Sigma_0)}{\sqrt{1+(\partial_\rho\Sigma_0)^2}}\right)\right]} \tag{3.29}$$

We also want to find out the value of the quark condensate. We expand $r^2\beta$ in (3.28) with $r = \rho^2 + (\Sigma_0 + m\rho^{\sqrt{3}-1})^2$ as $r^2\beta = r^2\beta|_{r^2=\rho^2+\Sigma_0^2} + \partial_{r^2}(r^2\beta)|_{r^2=\rho^2+\Sigma_0^2}(2m\rho^{\sqrt{3}-1}\Sigma_0 + \mathcal{O}(m^2))$. Then we can compare the vacuum energy, V_0, in (3.28) with the vacuum energy of the chiral Lagrangian to find the quark condensate

$$\frac{\langle\overline{q}q\rangle}{\Lambda^{2+\sqrt{3}}} = \frac{-N^{1/2}}{2g_{uv}^2 N\pi^{1/2}\lambda^{2+\sqrt{3}}}\int d\rho\rho^{\sqrt{3}}\sqrt{1+(\partial_\rho\Sigma_0)^2}\Sigma_0\partial_{r^2}(r^2\beta)\Big|_{r^2=\rho^2+\Sigma_0^2}. \tag{3.30}$$

These expressions for f_π and $\langle\bar{q}q\rangle$ are in some ways similar to those in the D3/D7 system. f_π is again dominated at low ρ whilst the condensate is more sensitive to the tail of Σ_0. In the D5 setting Σ_0 falls off more quickly in the UV and will suppress the condensate. This matches the chiral quark model results. On the other hand the factor of $N^{1/2}$ before each expression suggests some radical redistribution of the degrees of freedom in the UV conformal regime which we can offer no explanation for.

It is important to also note that one can not directly compare the condensates in the D5 and D7 cases since they have different intrinsic dimension even in the far UV. In fact to convert the D3/D5 theory to the usual walking set up would require the inclusion of extra UV physics (equivalent to that at the scale Λ_1 in the walking discussion above) where the condensate's dimension changes to three. The condensate above that scale would be suppressed by a further factor of roughly $\Lambda_1^{\sqrt{3}-1}$.

Whilst the D3/D5 system may not form the basis of any helpful phenomenological model we do believe that the walking paradigm is the correct way to interpret the system and the anomalous dimensions present in the UV.

3.5 Conclusions

We have presented a general description of chiral symmetry breaking in the D3/D7 system that describes a strongly coupled gauge theory with quarks. The model allows one to compute the dependence of the parameters of the low energy chiral Lagrangian on the running coupling or dilaton form. Our integral formulae for f_π and the quark condensate allow analytic understanding of how these quantities depend on the coupling and the dynamical mass of the quark in a similar way to the results of chiral quark models and the Pagels-Stokar formula. Our model is not complete since we do not back react the geometry to the dilaton. However, we view this as a necessary evil to construct intuition in this type of set up to the response to different dilaton profiles. This toy environment should provide good guidance for those wishing to construct fully backreacted solutions that show specific phenomena.

We have used our results to understand how walking like gauge dynamics could be included in a holographic framework. The crucial signal of walking should be that the quark self energy at zero momentum should be much less than the scale at which conformal symmetry breaking is introduced. We displayed in figure 3.2 the form a dilaton profile must take to achieve walking. Our integral equations support the usual hypothesis that walking in a gauge theory would tend to boost the value of the quark condensate relative to the value of f_π.

Finally we studied the non-supersymmetric D3/D5 system with a four dimensional overlap and proposed that the conformal UV of the theory should be considered as a walking phase of a gauge theory. The anomalous dimensions of the quark mass and condensate were computed - in this theory the dimension of the quark condensate is $2 + \sqrt{3}$ which is greater than the canonical dimension 3. Normally walking is constructed to lower this dimension but this theory hopefully nevertheless adds to our knowledge of walking behaviour.

Chapter 4

Holography of the Conformal Window

4.1 Introduction

There has been much interest in how the phase of QCD depends on the number of quark flavours for many years now. In the Veneziano limit, where the number of colours $N_c \to \infty$ with fixed $x = N_f/N_c$, we may treat x as a continuous variable. At $x = 11/2$ the one loop beta function vanishes. Just below that value of x the theory is known to be asymptotically free and to have a Banks-Zak fixed point [40, 41] at which the one and two loop beta functions balance to give a non-trivial, perturbative, conformal, IR fixed point. As x falls the fixed point value rises until the perturbative regime is lost. Based on the observation of chiral symmetry breaking in $N_c = 3, N_f = 3$ QCD it is presumed that at some critical x_c the IR conformal theory is replaced by one with a chiral condensate and a mass gap.

A number of methods have been used to estimate x_c. Truncated Schwinger-Dyson equations suggest $3.5 < x_c < 4$ [42, 43]. In these models chiral symmetry breaking is triggered when the anomalous dimension of the quark anti-quark operator hits of order one[1] [44]. The precise value for x_c then depends on the truncation scheme, the choice made for the running coupling profile with energy scale, μ, and

[1]In the Schwinger Dyson analysis the criticality condition for chiral symmetry breaking is when $\gamma_m(2-\gamma_m) = 1$ and hence $\gamma_m = 1$. To predict the critical value x_c one needs to make an assumption for the dependence of γ_m on x at strong coupling. In [43] the one loop form $\gamma_m^{(1)}$ is used and the criticality condition expanded at small $\gamma_m^{(1)}$ giving $\gamma_m^{(1)} = 1/2$ as the condition if extended to strong coupling. Using the two loop form for the running coupling in $\gamma_m^{(1)}$ gives the prediction $x_c = 4$.

the anomalous dimension relation for the quark mass term, γ_m. Other attempts to estimate the critical value have been made in [45]- [46] and typically give a similar estimate. Recently there has been much interest in simulating such theories on the lattice too [47]- [48] the simulations are still in an early phase but already support the general picture from all these analyses.

Jarvinen and Kiritsis [49] have recently proposed a holographic model of the strongly coupled near conformal regime around x_c (the work in [50–55] is also very relevant). Their model consists of 5D supergravity with a dilaton field dual to the running coupling and a "tachyon" field dual to the chiral condensate. They impose potentials for all these fields that generate the known two loop running for the coupling and the perturbative relation for the anomalous dimension in the UV of their description. They predict the range $3.7 < x_c < 4.2$ and that the transition at x_c is a BKT type transition in which the condensate grows exponentially from the transition (as expected - see [56, 57]). In a sense all this physics is imposed by the choice of potentials in the model but those choices are reasonable and it is encouraging that the results match other estimates.

In this chapter we wish to attempt a similar construction using an alternative holographic model of chiral symmetry breaking. The D3-D7 system [7] provides the simplest and best understood holographic description of a strongly coupled gauge theory with quark fields. At heart it consists of $SU(N_c)$ $\mathcal{N} = 4$ super Yang-Mills theory with N_f $\mathcal{N} = 2$ quark hypermultiplets. In the quenched approximation the theory is conformal and on the gravity side is described by probe D7 branes in $AdS_5 \times S^5$. The theory is 3+1 dimensional at all energy scales and has a conformal UV in which the identification of the operator matching between the field theory and the gravity description is clean. The simplest description of chiral symmetry breaking is found by imposing a background magnetic field on this theory [58] - the description is regular throughout and the interpretation again clear cut. Interestingly the DBI action for the probe with the magnetic field present is equivalent to the same theory with a particular choice of running gauge coupling. This effective dilaton is not backreacted on the geometry. It therefore seems natural to move to phenomenological models where one simply imposes some running coupling on the theory by hand - the underlying reaction of the holographic description seems likely to correctly capture the resulting physics of chiral symmetry breaking. Indeed in a recent paper it was studied the phase structure of just such a model with a running coupling with a step change between two conformal regimes [59]. The imposition of the two loop QCD running is a very similar analysis which we explore here, concentrating though on the transition to chiral symmetry breaking. Placing the probe brane in the presence of the dilaton matching the two loop gauge running essentially looks at the dynamics of one of the N_f quark flavours in a background backreacted to the full dynamics of the N_f quarks. Our model is more direct than that of Jarvinen and Kiritsis [49] in that we simply impose the running of the gauge coupling, and also in a later analysis the QCD anomalous dimension relation, rather than imposing a potential and then solving for these quantities. If one had the correct gravity dual of the gauge theory then the more involved process would capture more of the dynamics but if we are simply modeling the gauge theory then our approach may be sufficient.

First we will very naively impose the two loop QCD beta-function on the D3/D7 dynamics. We find that chiral symmetry breaking is induced for $x < 2.9$ and that the transition to the chiral symmetry breaking phase is second order in nature. This value of x_c is low relative to other estimates and the transition type is at odds with that argued for in QCD in [56, 57]. To understand this we recast the DBI action for small fluctuations about the chirally symmetric phase as a slipping mode in AdS$_5$ (we study this analysis in more detail in the Appendix). One can then plot its mass squared as a function of the radial coordinate and seek a violation of the Breitenlohner Freedman (BF) bound [60] which would lead to an instability. The model only shows an anomalous dimension for the chiral condensate in the regimes where the dilaton is running and the size of the anomalous dimension is proportional to the strength of that running. Our critical value of $x_c = 2.9$ corresponds to the theory which first has sufficiently strong running present. In terms of the slipping mode mass squared the BF bound must be violated over a sufficient interval in the radial direction of the gravity description - within that interval the BF bound is substantially violated at the transition leading to the second order behaviour.

This analysis highlights a failure of the D3/D7 system as analyzed so far - it has too much supersymmetry present. In the IR conformal regime the background gauge dynamics returns to that of $\mathcal{N} = 4$ Super Yang-Mills. It has too much symmetry and does not induce an anomalous dimension for the quark mass/condensate no matter how large the gauge coupling. This is in direct contradiction to QCD where the anomalous dimension γ_m is directly proportional to the magnitude of the coupling, at least in the perturbative regime [61]. Simply put we need to introduce more supersymmetry breaking into the description. We show how by a choice of background dilaton the QCD one-loop anomalous dimension relation can be imposed on the model by hand. We next impose on top the two loop QCD running profile within the anomalous dimension relation. In this model the slipping mode's mass squared, m^2, asymptotes to -3 in the UV and to some lower IR fixed point value. As it passes through the BF bound of -4 chiral symmetry breaking is triggered. The two loop running's IR fixed point implies that at the transition the IR mass squared lies at exactly -4 and this is the condition needed for a BKT transition (see [62, 63] for the first examples of holographic BKT transitions), which we indeed observe. In this model $x_c = 4$.

It's worth stressing that the BF bound is violated in these holographic models precisely when $m^2 = -4$ and, using the usual conformal AdS mass-operator dimension relation, $\gamma_m = 1$. This seems a robust holographic prediction, particularly since we envisage a conformal IR regime where the AdS mass-dimension relation is expected to hold. Note that x_c and the BKT transition behaviour are completely determined by the IR fixed point behaviour of the coupling and the precise non-perturbative running is not crucial. There are more vagueries in the precise prediction of x_c since we must assume a non-perturbative relation between the anomalous dimension and the value of the IR coupling. We have used the leading perturbative relation between m^2 and the one loop anomalous dimension and extended it to the non-perturbative regime, giving criticality when $\gamma_m^{(1)} = 1/2$ and $x_c = 4$. Given the full QCD dynamics this value may be different though.

4.2 The D3/D7 System

Our starting point is the holographic D3/D7 system [7]. Strings tied to the surface of the N_c D3 branes generate the adjoint representation fields of the $\mathcal{N} = 4$ gauge theory. Strings stretched from the D3 to the D7 are the quark fields in the fundamental representation of the $SU(N)$ group.

In the strong coupling limit the D3 branes are replaced by the geometry that they induce. We will consider a gauge theory with a holographic dual described by the Einstein frame geometry $\mathrm{AdS}_5 \times S^5$

$$\mathrm{d}s^2 = \frac{r^2}{R^2}\mathrm{d}x_4^2 + \frac{R^2}{r^2}\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_3^2 + \mathrm{d}w_5^2 + \mathrm{d}w_6^2\right) , \tag{4.1}$$

where we have split the coordinates into the x_{3+1} of the gauge theory, the ρ and Ω_3 which will be on the D7 brane world-volume and two directions transverse to the D7, w_5, w_6. The radial coordinate, $r^2 = \rho^2 + w_5^2 + w_6^2$, corresponds to the energy scale of the gauge theory. The radius of curvature is given by $R^4 = 4\pi g_s N \alpha'^2$ with N the number of colours. The $r \to \infty$ limit of this theory is dual to the $\mathcal{N} = 4$ super Yang-Mills theory where $g_s = g_{\mathrm{UV}}^2$ is the constant large r asymptotic value of the gauge coupling.

In addition we will allow ourselves to choose the profile of the dilaton as $r \to 0$. Simplistically this represents the running of the gauge theory coupling, $e^\phi \equiv \beta$, where the function $\beta \to 1$ as $r \to \infty$. For the coupling profiles we will consider later the UV form of β will have weak logarithmic running present - we will impose a UV cut off when $\beta = 1$ corresponding roughly to the scale where the holographic dual should be matched to perturbative QCD. Above that cut off we simply set $\beta = 1$. The physics we study is all in the IR and not affected by the precise form of this cut off though. In the final section we will use the dilaton function as an input to the DBI action for the quark physics to enforce the QCD anomalous dimension relation. At this point the relation between the dilaton, the gauge coupling and phenomenological corrections to the DBI action become less clear but our philosophy is simply to phenomenologically enforce the correct quark physics in the DBI action.

We will introduce a single D7 brane probe into the geometry to represent the dynamics of one quark in the theory - by treating the D7 as a probe we are working in a quenched approximation although we can reintroduce some aspects of the N_f quark loops through the running coupling's form. This system has a $U(1)$ axial symmetry on the quarks, corresponding to rotations in the w_5-w_6 plane, which will be broken by the formation of a quark condensate. The hope is that the dynamics of chiral symmetry breaking for the quark described by the probe is generic across many gauge theories and the results will be applicable to QCD.

We find the D7 embedding function e.g. $w_5(\varrho), w_6 = 0$. The Dirac Born Infeld (DBI) action in Einstein frame is given by

$$\begin{aligned} S_{D7} &= -T_7 \int \mathrm{d}^8\xi e^\phi \sqrt{-\det P[G]_{ab}} \\ &= -\overline{T}_7 \int \mathrm{d}^4x \; \mathrm{d}\rho \; \rho^3 \beta \sqrt{1 + (\partial_\rho L)^2} \, , \end{aligned} \tag{4.2}$$

where $w_5 \equiv L$, $T_7 = (2\pi)^{-7}\alpha'^{-4} g_{\mathrm{UV}}^{-2}$ and $\overline{T}_7 = 2\pi^2 T_7$ when we have integrated over the 3-sphere on the D7. The equation of motion for the embedding function is therefore

$$\partial_\rho \left[\frac{\beta \rho^3 \partial_\rho L}{\sqrt{1 + (\partial_\rho L)^2}} \right] - 2L\rho^3 \sqrt{1 + (\partial_\rho L)^2} \frac{\partial \beta}{\partial r^2} = 0 \, . \tag{4.3}$$

The UV asymptotics of this equation, provided the dilaton returns to a constant so the UV dual is the $\mathcal{N} = 4$ super Yang-Mills theory, has solutions of the form $w_5 = d + c/\rho^2 + \cdots$, where we can interpret d as the quark mass ($m_q = d/2\pi\alpha'$) and c is proportional to the quark condensate.

The embedding equation (4.3) clearly has regular solutions $w_5 = m$ when β is independent of r - the flat embeddings of the $\mathcal{N} = 2$ Karch-Katz theory. Equally clearly if $\partial\beta/\partial r^2$ is none trivial in w_5 then the second term in (4.3) will not vanish for a flat embedding.

There is always a solution $w_5 = 0$ which corresponds to a massless quark with zero quark condensate ($c = 0$). In the pure $\mathcal{N} = 2$ gauge theory with $\beta = 1$ this is the true vacuum. In the symmetry breaking geometries [22, 59] this configuration is a local maximum of the potential.

If the coupling is larger near the origin then the D7 brane will be repelled from the origin ending at $\rho = 0$ with $L'(0) = 0$. The symmetry breaking of these solutions is visible directly [22]. The $U(1)$ symmetry corresponds to rotations of the solution in the w_5-w_6 plane. An embedding along the ρ axis corresponds to a massless quark with the symmetry unbroken. The symmetry breaking configurations though map onto the flat case at large ρ (the UV of the theory) but bend off axis breaking the symmetry in the IR. $L(0)$, the IR quark mass, is a good order parameter for studying the chiral symmetry breaking that also reflects the bound state masses of the theory.

4.3 Imposing the 2-loop QCD Running

Our first analysis is straightforward. We impose the two loop running of the QCD gauge coupling on the dilaton profile of the D3/D7 system. That running is determined by

$$\mu \frac{d\lambda}{d\mu} = -b_0 \lambda^2 + b_1 \lambda^3 \,, \tag{4.4}$$

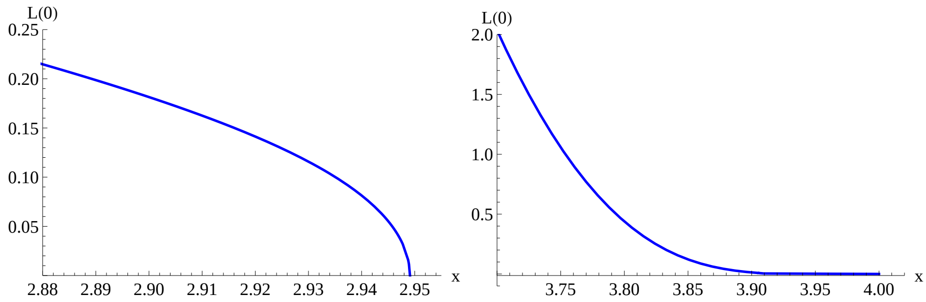
where

$$b_0 = \frac{2}{3} \frac{(11 - 2x)}{(4\pi)^2}, \qquad b_1 = -\frac{2}{3} \frac{(34 - 13x)}{(4\pi)^4} \,. \tag{4.5}$$

In the b_1, we omitted a subleasing term $\mathcal{O}(N_c^{-2})$ at large N_c. We simply identify the radial direction r with the RG scale of the field theory and set $\lambda = \beta$. As is well known these equations have logarithmic running in the UV and an IR fixed point that grows from zero as x is reduced from $x = 5.5$[2].

Here the UV is not strictly conformal although it approaches it at weak coupling asymptotically. Nevertheless it is easy to look for chiral symmetry breaking. We continue to associate massless quarks with D7 embeddings that approach the ρ axis at large ρ and seek solutions that bend off axis with that UV boundary condition. In fact the simplest identifier of chiral symmetry breaking is to look for solutions that

[2]Note for reference that in the usual gap equation analysis [42, 43] the critical coupling is given by $\lambda_c = 8\pi^2/3$ which is first achieved in the IR for $x_c = 4$



(a) The model of section III where we naively impose the QCD running coupling and shows a second order transition at $x = 2.949$.

(b) The model of section IVa where we impose the IR QCD anomalous dimension relation and shows a BKT transition at $x_c = 4$.

Figure 4.1: *Plots of the IR mass L(0) against $x = N_f/N_c$ for our two models.*

begin with $L'(0) = 0$ and shoot out to lie below $L = 0$ in the UV. We use the value of $L(0)$, the IR quark mass, as the order parameter for chiral symmetry breaking.

In Fig 4.1a we show a plot of $L(0)$ vs x. The transition is clearly second order and by fitting we determine it to be mean field with critical exponent 1/2. This second order nature is of course at odds with expectations that the transition with x should be of the BKT type [56, 57].

To understand this behaviour let us perform a linearized analysis on our DBI action to see why the flat embedding $L = 0$ becomes unstable.

We have an action, which is proportional to (4.2),

$$S = \int \mathrm{d}\rho \lambda(r) \rho^3 \sqrt{1 + L'^2}\,, \tag{4.6}$$

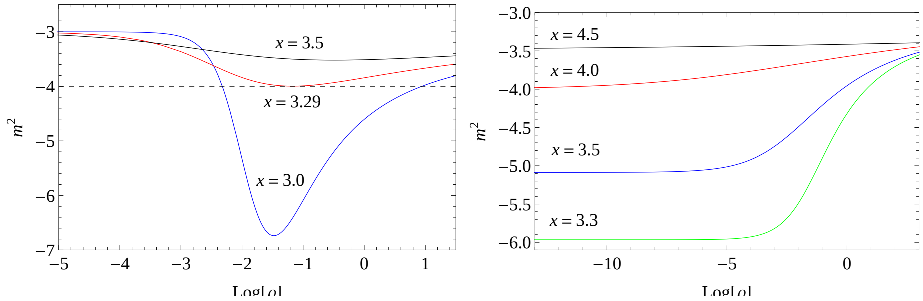
where $r^2 = L^2 + \rho^2$. We expand for small L

$$S = \int \mathrm{d}\rho \left(\frac{1}{2} \lambda(r) \Big|_{L=0} \rho^3 L'^2 + \rho^3 \left. \frac{d\lambda}{dL^2} \right|_{L=0} L^2 \right)\,, \tag{4.7}$$

where $L' \equiv dL(\rho)/d\rho$. To make the kinetic term canonical, we can now make a coordinate transformation[3] on ρ

$$\lambda(\rho)\rho^3 \frac{d}{d\rho} = \tilde{\rho}^3 \frac{d}{d\tilde{\rho}}\,, \qquad \tilde{\rho} = \sqrt{\frac{1}{2} \frac{1}{\int_\rho^\infty \frac{d\rho}{\lambda \rho^3}}}\,, \tag{4.8}$$

[3]See Appendix for more detailed and general discussion on the coordinate transformation.



(a) The model with the QCD running imposed in section III ($x = 3.5, 3.29, 3.0$).

(b) The model of section IVa where the QCD anomalous dimension is imposed in the IR ($x = 4.5, 4, 3.5, 3.3$).

Figure 4.2: *Plots of the AdS_5 slipping mode m^2 against r in our two models.*

Now the first term in our action can be recast by setting $L = \tilde{\rho}\phi$

$$S = \int \mathrm{d}\tilde{\rho}\frac{1}{2}\tilde{\rho}^3 L'^2 = \int \mathrm{d}\tilde{\rho}\frac{1}{2}\left(\tilde{\rho}^5\phi'^2 - 3\tilde{\rho}^3\phi^2\right) , \qquad (4.9)$$

where $L' \equiv dL(\widetilde{\rho})/d\widetilde{\rho}$. This is the action of a canonical $m^2 = -3$ scalar in AdS_5. The remaining term in the action becomes

$$S = \int \mathrm{d}\tilde{\rho}\frac{1}{2}\lambda\frac{\rho^5}{\tilde{\rho}}\frac{d\lambda}{d\rho}\phi^2 . \qquad (4.10)$$

So we have a AdS_5 scalar with ρ dependent mass squared

$$m^2 = -3 - \delta m^2 , \qquad \delta m^2 \equiv -\lambda\frac{\rho^5}{\tilde{\rho}^4}\frac{d\lambda}{d\rho} . \qquad (4.11)$$

We plot this mass against ρ in Fig 4.2a for a variety of choices of x. We first note that the mass squared of the slipping mode approaches -3 in both the UV and IR, which we will return to shortly. The instability that is causing our phase transition with x is in the intermediate period where the mass squared is falling below the BF bound of -4. Note that $x = 3.29$ is the first case where the BF bound is met at one point in ρ but that this value of x is significantly above the critical value $x_c = 2.95$ found above. Apparently the BF bound must be violated in a region of ρ for the instability to trigger a transition. At the point of transition the BF bound is violated in a range of ρ and the effective mass squared lies considerably below the BF bound in the mid-region. Such a scenario has caused a second order transition. See (A.15) and footnote 1 for more examples.

This plot of the mass squared of the slipping mode against ρ reveals a number of failings of our most naive model. In particular the mass squared returns to -3 in the IR. The reason is that the gauge coupling becomes constant in the IR conformal regime and the brane construction returns to that of the $\mathcal{N}=4$ gauge theory with quarks. The mass squared is -3 because the model returns to a highly supersymmetric configuration in the IR where the anomalous dimension of $\bar{q}q$ is protected to be 3. This is quite unlike in non-supersymmetric QCD where the anomalous dimension, γ_m, of the quark mass m_q (the dimension of $\bar{q}q$ is $3-\gamma_m$), at one loop, is given by

$$\gamma_m^{(1)} = \mu \frac{d \ln m_q}{d\mu} = \frac{3\lambda}{(4\pi)^2} \,. \tag{4.12}$$

At the IR fixed point one expects a non-zero γ_m. However, in our holographic model, using the naive scalar mass operator dimension relation, $m^2 = \Delta(\Delta-4)$, we find

$$\gamma_m = 3 - \Delta = 1 - \sqrt{1-\delta m^2} \tag{4.13}$$

$$\delta m^2 = 1 - (1-\gamma_m)^2 \,, \tag{4.14}$$

where δm^2 is defined in (4.11). Strictly the relation $m^2 = \Delta(\Delta-4)$ is valid only in conformal regimes where the scalar mass is constant but we allow ourselves to use it slightly more liberally here. Therefore, the model we present only conjures an anomalous dimension in the regime in which the coupling is running ($\delta m^2 \neq 0$), breaking both conformal invariance and supersymmetry. For a model that is precociously asymptotically free such as $x=1$ QCD this deviation of our model from QCD is probably not so important for the phenomenology - in both cases the coupling grows rapidly and a quark condensate is triggered. If we wish to model the transition to chiral symmetry breaking though with changing x, where we leave an IR fixed point theory, it is more crucial. Our estimate of $x_c = 2.95$ is most likely an under estimate because we have not included the contribution to γ_m from the absolute value of the coupling λ.

4.4 Imposing the QCD anomalous dimension

Our naive model above of the x dependence of QCD suffered from an excess of supersymmetry in the IR regime, left over from our underlying construction. The model included the running gauge coupling but not the QCD anomalous dimension relation. We will now enforce the perturbative QCD form of that relation (with the two loop QCD form for λ) on the model as an alternative way to include the QCD physics. That is we use the two loop relation to fix the coupling at the IR fixed point and then use the one loop anomalous dimension relation to predict x_c from the point where the slipping mode mass becomes -4.

4.4.1 IR Physics

Let us first consider the IR conformal regime where we want a constant non-zero value of γ_{IR}. Our model predicts the slipping mode mass (4.11)

$$m_{IR}^2 = -3 + \beta \frac{\rho^5}{\tilde{\rho}^4} \frac{d\beta}{d\rho}, \tag{4.15}$$

where we substitute β for λ since we have given up the identification of β in the DBI action with the gauge coupling. Here we are concentrating on making γ_m match QCD instead so the quark physics is correct. The choice of β that gives such a constant m_{IR}^2 is $\beta \sim \frac{1}{\rho^q}$ $(0 \leq q < 2)$ for which we find

$$m_{IR}^2 = -3 - \delta m^2, \qquad \delta m^2 = \frac{4q}{(2-q)^2}. \tag{4.16}$$

By (4.13), δm^2 is related to γ_m

$$\gamma_m = 1 - \sqrt{1 - \frac{4q}{(2-q)^2}}. \tag{4.17}$$

Here the use of (4.13) is more valid than in the previous section; the scalar mass in the action given by (4.9) and (4.10) is constant and by ansatz $\tilde{\rho}$ matches the RG scale of QCD. It is possible that back-reaction between the geometry and the scalar might disturb this relation but it seems fairly sound. Note the conditions that $m_{IR}^2 = -4$ and $\gamma_m = 1$ are the same,where $q = 0.536$.

This model displays a BKT transition as q is changed continuously through $\gamma_m = 1$. As usual the BKT transition occurs due to the presence of an infinite number of unstable, Efimov modes at the transition point [57]. We can see them here explicitly by considering the action for static, linearized, mesonic solutions around the $L = 0$ embedding. The action is

$$S = \int \mathrm{d}\rho \frac{\rho^3}{2} \left(\beta L'^2 - \frac{\beta}{\rho^4} \dot{L}^2 + \frac{\partial \beta}{\partial \rho^2} L^2 \right), \tag{4.18}$$

where a prime is a ρ derivative and dot a time derivative. If we move to the inverse

z-coordinate ($z = 1/\rho$) we have

$$S = \int \mathrm{d}z \frac{z^{-5}}{2} \left(\beta z^4 L'^2 - \beta z^4 \dot{L}^2 - z^4 \frac{\partial \beta}{\partial z^2} L^2 \right) , \tag{4.19}$$

where a prime is now a z derivative. We can now write the equation of motion for a solution of the form $L = e^{-i\omega t} z^{(1-q)/2} \psi$ and $\beta = z^q$

$$-\psi'' + \frac{(3 - 8q + q^2)}{4} \frac{1}{z^2} \psi = \omega^2 \psi \, , \tag{4.20}$$

which is a 1D Schrodinger equation form with a $1/z^2$ potential. This problem is known and becomes unstable when the coefficient of the $1/z^2$ term is equal to $-1/4$. This condition is equivalent to $\gamma_m = 1$ in (4.17). At that point an infinite number of unstable negative energy modes emerge from $E = 0$. At the critical value of $-1/4$ all of those modes play a role in the transition generating the BKT transition.

This discussion so far has been restricted to the IR and a more complete model would require that $\beta \to 1$ in the UV. A simple fix is to set $\beta = 1 + c/\rho^q$. In this case,

$$\delta m^2(\rho; q, c) = c \, q \, \rho^{-2q} (c + \rho^q)_2 F_1^2 \left[1, \frac{2}{q}, \frac{2+q}{q}, -c\rho^{-q} \right] \tag{4.21}$$

Its IR asymptotic behavior is

$$\delta m^2 \sim \frac{4q}{(2-q)^2} \left(1 - \frac{1}{c(1-q)} \rho^q + \cdots \right) \tag{4.22}$$

which is the same as (15) with a ρ-dependent correction. The IR behaviour matches our discussion above.

At this point we can make a simple model to extract the critical value of N_f in QCD. The two loop QCD beta function has a fixed point at

$$\lambda_* = \frac{11 - 2x}{13x - 34} (4\pi)^2 \, . \tag{4.23}$$

In the Banks-Zak regime where perturbation theory applies, $\gamma_{m*} = \frac{3\lambda_*}{(4\pi)^2}$. the order λ relation between the δm_*^2 and γ_{m*} is given by

$$\delta m_*^2 \sim 2\gamma_{m*}^{(1)} = \frac{6\lambda_*}{(4\pi)^2} \tag{4.24}$$

where we used (4.14) and (4.12) and $\gamma_{m*}^{(1)}$ denote the order λ relation.

Of course we have no true idea how to continue this relation into the non-perturbative regime but following the spirit of [43] we will simply assume (4.24) applies at all values of the coupling. The holographic model tells us that the transition will occur when $m^2 = -4$ ($\delta m_*^2 = 1$) so we find, using the one loop QCD anomalous dimension result

$$1 = \frac{6\lambda_*}{(4\pi)^2} = 6\frac{11 - 2x}{13x - 34}\,. \tag{4.25}$$

This gives $x_c = 4$. Note that this amounts to $\gamma_{m*}^{(1)} = 1/2$, which coincides to the one-loop perturbative field theory computation [43].

Finally we can numerically check the BKT nature of the transition as well. We can simply set $\lambda = 1/\rho^q$ with q and x related, through the IR relations (4.16) and (4.24),

$$\frac{4q}{(2-q)^2} = 6\frac{11 - 2x}{13x - 34}\,. \tag{4.26}$$

We then numerically solve for the D7 embedding, L as a function of ρ. $L(0)$, the IR quark mass, is a useful order parameter - we show the result for $L(0)$ vs x in Fig 4.1b - the BKT type transition is apparent with $x_c = 4$. Close to x_c this simple model and the case $\beta = 1 + 1/\rho^q$ coincide since the dynamics is dominated in the far IR.

4.4.2 All RG scales

To construct a full model of the RG flow in the conformal window, one should enforce the QCD anomalous dimension formula (4.24) at all energy scales or ρ. In particular we want

$$\beta\frac{\rho^5}{\tilde{\rho}^4}\frac{d\beta}{d\rho} = -\frac{6\lambda(\rho)}{(4\pi)^2}\,. \tag{4.27}$$

To find the associated β one can re-arrange for $\tilde{\rho}$,

$$\tilde{\rho} = \sqrt{\frac{1}{2}\frac{1}{\int_\rho^\infty \frac{d\rho}{\beta\rho^3}}}\,, \tag{4.28}$$

differentiate, and find the differential equation

$$\frac{2}{\beta\rho^3} + \partial_\rho\left[\frac{-6\lambda}{(4\pi)^2\rho^5\beta\beta'}\right]^{1/2} = 0\,. \tag{4.29}$$

We can solve for β numerically by shooting from some initial value of ρ and trialling various values of the initial condition β'. Typically the true solution lies on the crossover between solutions that are real at all ρ and those that go complex so the correct initial condition can be tuned to. Once found the numerical solution can be tested that it is a good solution of (4.29) and that it has the IR fixed point behaviour $1/\rho^q$ where q and x are related by (4.26).

We can then use these solutions to solve for L as a function of ρ - close to x_c the results are again those in Fig 4.1b since the dynamics is entirely determined by the IR fixed point.

4.5 Summary

In this chapter we have presented two simple holographic models of $x = N_f/N_c$ behaviour of QCD at large N_c. In our first model we imposed the QCD two loop running directly on the D3/D7 system through a non-backreacted dilaton profile. We found chiral symmetry breaking sets in at $x_c = 2.95$ at a second order transition. The transition is expected to be at a larger value of x and to be of BKT type [56, 57] and we highlighted that this discrepancy is due to the IR supersymmetry of the model forcing $\gamma_m = 0$. In a second model we imposed the perturbative QCD γ_m relation and found a BKT transition at $x_c = 4$.

Whilst these models are much less sophisticated than the very nice model of Jarvinen and Kiritsis [49], in which the AdS-space backreacts to the running coupling and the quark condensate, we believe they highlight the key ingredients. One must input into the model, either directly as we do, or indirectly through supergravity potentials as in [49], the form of the running coupling and the impact that has on the quark anomalous dimension. Since we do not have the true QCD dual all of this is the model builder's choice. The clear prediction from AdS is that the chiral transition will occur when the AdS slipping mode associated to the quark condensate hits a mass squared at the IR fixed point of -4, the BF bound. This corresponds to $\gamma_m = 1$. The Miransky scaling or BKT nature of the transition is then also very clear in the holographic description through the presence of Effimov modes.

Chapter 5

An improved model of vector mesons in holographic QCD

5.1 Introduction

Popular phenomenological models of QCD, such as the "hard wall model" [64, 65] as well as "the soft wall model" [66], rely on the assumption that one can map QCD to an effective theory in the bulk of the holographic fifth dimension. This is a very strong assumption, which is not fully justified. Such an effective description implies a large hierarchy of scales between meson masses and flux tube tension, which is not present in QCD. This is the main reason why these are at best phenomenological models. Nevertheless, they are a useful resource as they provide quick and easy ways to estimate many quantitative properties, provided one is willing to live with errors which, in most instances where these models can be compared to data, turn out to be in the 10–30% range. These simple tools are nice to have for QCD. They are even more valuable when studying QCD-like theories in particle physics, most notably as a theory of technicolor, or more generally as a potential "hidden sector" which may leave an imprint on LHC data. As far as, for example, meson spectra in QCD are concerned, holographic models will never be competitive with lattice gauge theories. However, when exploring theories of technicolor or hidden sectors one does not know, a priori, what the correct Lagrangian is. So one needs to explore many different models, each of which would require years of extensive computer simulations on the lattice, but only days in a holographic model.

While using a 5D effective theory for QCD is not necessarily justified, it should at least be done self-consistently. In the 5D effective theory, not only are higher derivative interactions and high dimension operators suppressed — one is also suppressing an infinite number of additional fields. In holography, we know that every boundary operator should correspond to a field in the bulk. Fields kept in the simplest holographic models correspond to boundary operators of UV dimension[1] 3. One can say that bulk fields dual to operators of higher dimension, which are more massive and hence can be integrated out, are being neglected. In a top-down holographic theory, such as the $\mathrm{AdS}_5 \times S^5$ dual of large $\mathcal{N} = 4$ SYM with a large number of colors N_c and at strong 't Hooft coupling λ [1, 2, 17], a similar reduction to a small subset of boundary operators and hence bulk fields is entirely justified: there is only a finite number of ten-dimensional fields dual to BPS operators, which retain their free field dimension. All other operators acquire anomalous dimensions of order $\lambda^{1/4}$, so their dual fields in the bulk acquire masses of the same order in λ and can safely be integrated out. In the phenomenological approach one includes the fields dual to dimension 3 operators, but hopes to be able to neglect operators of dimensions 4, 5, 6 and so on.

There are, however, two additional dimension 3 operators, $\bar{\psi}\sigma^{\mu\nu}\psi$ and $\bar{\psi}\sigma^{\mu\nu}\gamma_5\psi$ ($\sigma^{\mu\nu} = i/2[\gamma^\mu, \gamma^\nu]$ being the antisymmetrized product of two gamma matrices), whose dual field is not included in the simplest bulk model. The corresponding field, a complex bifundamental anti-symmetric rank-two tensor field $B_{\mu\nu}$, should be included for self-consistency of the model. Real and imaginary part of this field correspond to the tensor operators with and without γ_5 insertion respectively. An immediate benefit resulting from the inclusion of this extra field is that one will obtain masses for isospin triplet vector mesons with $J^{PC} = 1^{+-}$, starting with the b_1 at a mass of 1235 MeV. They clearly should be part of the setup which, as it stands, can otherwise only incorporate 1^{--} and 1^{++} vector mesons like the ρ and a_1.

Including a new field in the Lagrangian comes with new interaction terms and coupling constants. The original work on the hard wall model in ref. [64] has proposed a rigorous procedure to fix those, which so far has been very successful phenomenologically: first of all, we assume that the bulk is described by an effective field theory so we only write down interaction terms of bulk field theory dimension 5 or less (not to be confused with the dimension of boundary operators, which after all maps to the mass of the bulk field). The corresponding coupling constants as well as the normalization of the kinetic terms and the masses are obtained by demanding that at large momentum correlation functions in the bulk agree with the corresponding

[1] In an asymptotically free gauge theory such as QCD the dimensions of operators evolve with scale together with the coupling constant. In the UV the theory becomes free and all operators take on the free field dimensions.

field theory values at weak coupling. This is basically the statement that one trusts the effective field theory picture in the bulk all the way into the UV. While this is a very strong assumption, it at least is a hypothesis that can be tested as it allows one to make predictions for particle masses and decay constants based on very few inputs. In the original hard wall model this procedure was used to fix the mass of the vectors and the scalar in the bulk, the five-dimensional gauge coupling and, as was pointed out later in ref. [67], it was also needed to fix the map between the asymptotic form of the scalar field and the quark mass, as well as the quark condensate $\langle\bar{\psi}\psi\rangle$ in the field theory. Out of these, only the mass of the vectors could have been justified[2] without the assumption of a valid effective theory in the UV. But already for the determination of the gauge coupling one had to rely on the two-point functions of the currents, whose overall normalization is not protected.

Preliminary advances in including the $B_{\mu\nu}$ field in the bulk has recently appeared in the pioneering work [68], but only in a form which does not account for chiral symmetry breaking. In that work only the real part of $B_{\mu\nu}$ is considered; this field propagates the new 1^{+-} tensor mesons, but also an additional copy of 1^{--} vector mesons. More importantly, only quadratic terms were included in the action for the new B-field. Its mass is fixed to 1 in AdS units by requiring that the dual operator has dimension 3; the normalization of the kinetic term is fixed, as in the original hard wall model, by requiring the large momentum limit of two-point functions to agree with asymptotically free QCD. However there is one more bulk operator of dimension 5 or less that needs to be included in the action to communicate the effects of chiral symmetry breaking to the $B_{\mu\nu}$ field and ensure that there is only a single set of vector mesons and no double counting, that would happen since gauge fields and the $B_{\mu\nu}$ fields have the same degenerate spectra in the absence of this term. This extra term has the form $\mathrm{tr}\,(X^\dagger F_L B + B F_R X^\dagger + h.c.)$, where X is the bifundamental scalar responsible for chiral symmetry breaking and $F_{L/R}$ are the field strengths for the bulk gauge fields dual to the chiral symmetry currents. The coupling constant in front of this term may be fixed by demanding the correct OPE structure of the correlator in the UV.

More progress in this direction was made in ref. [69], where it was proposed that the action of the complex $B_{\mu\nu}$ field should be first order, in such a way that the four-dimensional components satisfy a complex self-duality condition. The reason behind this choice is that in four dimensions the tensor operators $\bar{\psi}\sigma^{\mu\nu}\psi$ and $\bar{\psi}\sigma^{\mu\nu}\gamma_5\psi$ are

[2] A gauge field always has to be massless; the dual field theory current is conserved and so has dimension 3 protected by a Ward identity

not independent, but given the definition of $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, they are related by

$$\bar{\psi}\sigma^{\mu\nu}\gamma_5\psi = \frac{i}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}\bar{\psi}\sigma^{\alpha\beta}\psi. \tag{5.1}$$

Hence a similar condition must be imposed on the two-form field. Following a similar procedure, we choose the action for the $B_{\mu\nu}$ to be a Chern-Simons action with a mass term,[3] schematically

$$S_B = 2g_B \int d^5x\sqrt{g}\mathrm{tr}\left[\frac{i}{3}\varepsilon^{MNLPQ}(B_{MN}H^{\dagger}_{LPQ} - B^{\dagger}_{MN}H_{LPQ}) - m_B B^{\dagger}_{MN}B^{MN}\right]. \tag{5.2}$$

The duality condition follows from the equations of motion. We differ from ref. [69] in several ways. Just like in ref. [68], the authors do not include the effects of the dimension 5 bulk operator that communicates chiral symmetry breaking to the tensor sector, even though they correctly point out that its effect should be included. Secondly, in order to fix the degeneracies resulting from this truncation to a free field theory, they let the mass of the bulk $B_{\mu\nu}$ field take different values, so that the field is dual to operators of dimension Δ different than 3. While it is true, as we pointed out above, that using matching to free field theory is not a well justified procedure, it is at least a testable assumption and so far has met with surprisingly large phenomenological success. If one abandons this, one should not just treat the mass of the $B_{\mu\nu}$ field as a new free parameter, but also the five-dimensional gauge coupling, the mass and normalization of the bulk scalar field as well as the normalization of the $B_{\mu\nu}$ kinetic term, all of which affect the correlation functions in the boundary theory. In this case the model loses virtually all predictive power. Given the surprising accuracy with which the hard wall model so far has predicted particle masses, we believe it is premature to abandon the procedure of matching to UV correlators at this stage. We fix our parameters to reproduce the boundary expansion of a field dual to an operator of dimension $\Delta = 3$.

In this chapter we will explicitly carry out the calculation for the short distance behavior of the bulk correlation functions of $B_{\mu\nu}$. Comparing to the operator product expansion (OPE) of weakly coupled QCD we will indeed be able to completely fix all new coupling constants in the bulk. In fact, the set of conditions we obtain for the couplings is overdetermined and the fact that we can find values that allow us to reproduce all QCD correlation functions to leading order is a nice consistency check. The upshot is that this improved model has no new undetermined parameters. For now this serves as a proof of principle that this matching can be done. One has

[3] We could also add a kinetic term of the form $(dB)^2$ to the action, we will comment more on this in section 5.4.

an improved hard wall model with no new free input parameters but several new predictions (masses and decay constants for the tensor and axial tensor mesons). Whether the phenomenological success of the model survives these additions will be a good test to what extent the underlying assumption of an effective description in five dimensions gives an accurate picture of real QCD. We analyze the meson spectrum, we demonstrate that the cubic coupling indeed removes all the unwanted degeneracies of masses that were present in the case of a free $B_{\mu\nu}$, but unfortunately the meson spectrum that we observe does not match with what has been measured for QCD.

The organization of this chapter is as follows: in section 5.2 we will review the short distance structure of the correlation functions involving the dimension 3 vector operators in QCD, as this is what we want to reproduce. In section 5.3 we present the improved holographic model. We derive equations of motion and the renormalized action in sections 5.4 and 5.5 respectively. In section 5.5 we calculate the short distance correlation functions in the theory with massive quarks and extract the bulk coupling constants from comparing to QCD. As some of the correlation functions have the leading short distance terms proportional to the mass, in the chiral limit several correlators are dominated by the subleading term involving the chiral condensate. As our bulk Lagrangian at this stage is entirely determined, reproducing these correlators is a non-trivial check of our construction. We demonstrate that this indeed works out in section 5.6. In section 5.7 we analyze the meson spectrum and the phenomenological issues that appear in the new model. In section 5.8 we summarize our results.

5.2 Correlation functions in QCD

In two-flavor QCD, the relevant two-point functions in the vector sector are:

$$\Pi^{\mu\nu,\,ab}_{VV}(q^2) = i\int d^4x\, e^{iqx}\, \langle\Omega|\, T\{V^{\mu\, a}(x)V^{\nu\, b\,\dagger}(0)\}\,|\Omega\rangle\,, \tag{5.3}$$

$$\Pi^{\mu;\nu\rho,\,ab}_{VT}(q^2) = i\int d^4x\, e^{iqx}\, \langle\Omega|\, T\{T^{\nu\rho\, a}(x)V^{\mu\dagger\, b}(0)\}\,|\Omega\rangle\,, \tag{5.4}$$

$$\Pi^{\mu\nu;\alpha\beta,\,ab}_{TT}(q^2) = i\int d^4x\, e^{iqx}\, \langle\Omega|\, T\{T^{\mu\nu\, a}(x)T^{\alpha\beta\dagger\, b}(0)\}\,|\Omega\rangle\,, \tag{5.5}$$

where $V^{\mu\, a}(x) = \bar\psi(x)\gamma^\mu\tau^a\,\psi(x)$ and $T^{\mu\nu\, a}(x) = \bar\psi(x)\sigma^{\mu\nu}\tau^a\,\psi(x)$ are the vector and tensor isospin triplet currents respectively, and $|\Omega\rangle$ is the non-perturbative vacuum. We choose a normalization for the isospin generators such that $\mathrm{tr}\,(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}$. The two-point functions above have the following kinematic structure:

$$\Pi_{VV}^{\mu\nu,\,ab}(q^2) = \delta^{ab}(q^\mu q^\nu - q^2\eta^{\mu\nu})\Pi_{VV}(q^2), \tag{5.6}$$

$$\Pi_{TT}^{\mu\nu;\alpha\beta,\,ab}(q^2) = \delta^{ab}\Pi_{TT}^{+}(q^2)F_{+}^{\mu\nu;\alpha\beta} + \delta^{ab}\Pi_{TT}^{-}(q^2)F_{-}^{\mu\nu;\alpha\beta}, \tag{5.7}$$

$$\Pi_{VT}^{\mu;\nu\rho,\,ab}(q^2) = i\delta^{ab}(\eta^{\mu\nu}q^\rho - \eta^{\mu\rho}q^\nu)\Pi_{VT}(q^2), \tag{5.8}$$

where, defining the projector $q^2 P_{\mu\nu} = q^2\eta_{\mu\nu} - q_\mu q_\nu$,

$$P^\alpha_{[\mu}P^\beta_{\nu]} = \frac{1}{q^2}F^{\alpha\beta}_{+\mu\nu}; \tag{5.9}$$

projects onto positive parity. Its counterpart is

$$F_{-}^{\mu\nu;\alpha\beta} = F_{+}^{\mu\nu;\alpha\beta} - q^2(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}), \tag{5.10}$$

so $F_{-} \sim F_{+} - 1$ is the negative parity projector. Notice that the sign of F_{-} is chosen so it is actually minus the projector

$$F_{-}^{\mu\nu;\alpha\beta} = -(q^\nu q^\beta\eta^{\mu\alpha} + q^\mu q^\alpha\eta^{\nu\beta} - q^\nu q^\alpha\eta^{\mu\beta} - q^\mu q^\beta\eta^{\nu\alpha}). \tag{5.11}$$

We also have:

$$(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu) - P_{[\mu}{}^{\alpha}P_{\nu]}{}^{\beta} = -\frac{1}{q^2}F_{-\mu\nu}{}^{\alpha\beta}, \tag{5.12}$$

with

$$F_{\pm}{}^{\mu\nu}{}_{\alpha\beta}F_{\pm}{}^{\alpha\beta}{}_{\sigma\rho} = \pm 2q^2 F_{\pm}{}^{\mu\nu}{}_{\sigma\rho}. \tag{5.13}$$

In the large-N_c limit the two-point functions above are saturated by single-particle exchange of an infinite number of stable mesons, in this approximation to real QCD we can write, up to subtractions, the two-point functions above as:

$$\Pi_{VV}(q^2) = \sum_n \frac{f^2_{\rho,n}}{M^2_{\rho,n} - q^2}; \quad \Pi^{-}_{TT}(q^2) = \sum_n \frac{(f^T_{\rho,n})^2}{M^2_{\rho,n} - q^2} \tag{5.14}$$

$$\Pi^{+}_{TT}(q^2) = \sum_n \frac{f^2_{b,n}}{M^2_{b,n} - q^2}; \quad \Pi_{VT}(q^2) = \sum_n \frac{f_{\rho,n}f^T_{\rho,n}}{M^2_{\rho,n} - q^2}$$

with the decay constants defined as:

$$\langle\Omega| V^a_\mu \left|\rho^b_n(p,\lambda)\right\rangle = M_{\rho,n}\delta^{ab} f_{\rho,n}\epsilon_\mu(p,\lambda), \tag{5.15}$$

$$\langle\Omega| T^a_{\mu\nu} \left|\rho^b_n(p,\lambda)\right\rangle = i\delta^{ab} f^T_{\rho,n}[p_\mu\epsilon_\nu(p,\lambda) - p_\nu\epsilon_\mu(p,\lambda)], \tag{5.16}$$

$$\langle\Omega| T^a_{\mu\nu} \left|b^b_n(p,\lambda)\right\rangle = i\delta^{ab} f_{b,n}\varepsilon_{\mu\nu\alpha\beta}p^\alpha\epsilon^\beta(p,\lambda). \tag{5.17}$$

As it is made explicit by the notation above, the current V^μ produces vector mesons ($J^{PC} = 1^{--}$) like the ρ, while the tensor operator $T^{\mu\nu}$ produces both vector mesons and their even-parity partners ($J^{PC} = 1^{+-}$), like the b_1 meson. For large Euclidean momentum $Q^2 = -q^2 \to \infty$ contributions to these correlators can be organized according to the operator product expansion (OPE), with a leading perturbative contribution plus an expansion on the several vacuum condensates, $\langle\bar\psi\psi\rangle$, $\langle\alpha_s G^2\rangle$, etc., that capture the non-perturbative effects. This was originally done for three colors in refs. [70–72]. Expressions for a general number of colors can also be found in refs. [73–75]. To leading order we have:

$$\lim_{Q^2\to\infty} \Pi_{VV}(Q^2) = -\frac{N_c}{24\pi^2}\log\frac{Q^2}{\mu^2} + \mathcal{O}\left(\frac{\alpha_s}{Q^4}\right), \tag{5.18}$$

$$\lim_{Q^2\to\infty} \Pi^\pm_{TT}(Q^2) = -\frac{N_c}{48\pi^2}\log\frac{Q^2}{\mu^2} \mp \frac{N_c}{8\pi^2}\frac{m^2}{Q^2}\log\frac{Q^2}{\mu^2} + \mathcal{O}\left(\frac{\alpha_s}{Q^4}\right), \tag{5.19}$$

$$\lim_{Q^2\to\infty} \Pi_{VT}(Q^2) = \frac{N_c}{16\pi^2} m\log\frac{Q^2}{\mu^2} - \frac{\langle\bar\psi\psi\rangle}{4Q^2} + \mathcal{O}\left(\frac{\alpha_s}{Q^4}\right), \tag{5.20}$$

where m is the quark mass. This is the large momentum behaviour of the correlators that we will use to fix the free parameters of the five-dimensional action.

5.3 Improved model of holographic QCD

The model we consider is an extension of the hard wall model of ref. [64], but it can be generalized to other holographic QCD models like the soft wall model of ref. [66]. We use a five-dimensional geometry to describe the dynamics of four-dimensional QCD with a large number of colors $N_c \to \infty$. The metric is that of AdS_5 with a radius ℓ, we choose a mostly minus signature and work with the coordinate system

$$ds^2 = g_{MN}dx^M dx^N = \frac{\ell^2}{z^2}\left(-dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right). \tag{5.21}$$

In these coordinates the boundary is at $z = 0$. In order to recover some of the physics of confinement, we introduce a cutoff in the radial coordinate z_m. Since the radial coordinate maps to a renormalization group scale in the dual theory, with $z = 0$ corresponding to the UV, the cutoff $1/z_m$ can be interpreted as an IR scale where the theory becomes confining.

We introduce a set of fields $\phi(x, z)$ in the five-dimensional theory that are dual to mesonic operators $\mathcal{O}(x)$ in the field theory with conformal dimensions $\Delta \leq 3$ and spin $J \leq 1$. In previous works this was done considering scalar and vector fields. This included both scalar and pseudoscalar mesons, as well as vector mesons 1^{--} and axial vector mesons 1^{++}, but the full set of vector mesons include also 1^{+-} states, that were missing in the original formulation. These can be included by considering a complex two-index antisymmetric field $B_{\mu\nu}$, or two-form for short. The five-dimensional theory has a $U(2)_L \times U(2)_R$ gauge symmetry, that maps to the global flavor symmetry of two-flavor QCD. The fields $A_{L\,\mu}$ and $A_{R\,\mu}$ will be the associated gauge bosons, while the complex fields X and $B_{\mu\nu}$ are in a bifundamental representation

$$X \longrightarrow U_L X U_R^\dagger, \quad B_{\mu\nu} \longrightarrow U_L B_{\mu\nu} U_R^\dagger. \tag{5.22}$$

The map between operators and fields can be summarized as:

$4D : \mathcal{O}(x)$	$5D : \phi(x, z)$	Δ	$m_\phi^2 \ell^2$
$\sqrt{2}\overline{\psi}_L \gamma_\mu \tau^a \psi_L$	$A^a_{L\,\mu}$	3	0
$\sqrt{2}\overline{\psi}_R \gamma_\mu \tau^a \psi_R$	$A^a_{R\,\mu}$	3	0
$\overline{\psi}^\alpha_L \psi^\beta_R$	$X^{\alpha\beta}$	3	-3
$\overline{\psi}^\alpha_L \sigma_{\mu\nu} \psi^\beta_R$	$B^{\alpha\beta}_{\mu\nu}$	3	1

Where $m_\phi^2 \ell^2$ is the mass of the field. We have chosen masses such that the conformal dimension Δ of the dual operator matches with its free value. Although quantum corrections will change the conformal dimension of operators in the IR, QCD is a free theory in the UV and is in this regime where we will do the matching with our model, hence our choice of masses both for the scalar and the two-form field. We can also form the real combinations

$4D : \mathcal{O}(x)$	$5D : \phi(x,z)$	Δ	$m_\phi^2\ell^2$
$\overline{\psi}\gamma_\mu\tau^a\psi$	$V_\mu^a = (A_{R\,\mu}^a + A_{L\,\mu}^a)/\sqrt{2}$	3	0
$\overline{\psi}\gamma_\mu\gamma_5\tau^a\psi$	$A_\mu^a = (A_{R\,\mu}^a - A_{L\,\mu}^a)/\sqrt{2}$	3	0
$\overline{\psi}^\alpha\psi^\beta$	$X_+^{\alpha\beta} = X^{\alpha\beta} + X^{\dagger\,\alpha\beta}$	3	-3
$i\,\overline{\psi}^\alpha\gamma_5\psi^\beta$	$X_-^{\alpha\beta} = i\left(X^{\alpha\beta} - X^{\dagger\,\alpha\beta}\right)$	3	-3
$\frac{1}{\sqrt{2}}\overline{\psi}^\alpha\sigma_{\mu\nu}\psi^\beta$	$B_{+\,\mu\nu}^{\alpha\beta} = (B_{\mu\nu}^{\alpha\beta} + B_{\mu\nu}^{\dagger\,\alpha\beta})/\sqrt{2}$	3	1
$\frac{i}{\sqrt{2}}\overline{\psi}^\alpha\sigma_{\mu\nu}\gamma_5\psi^\beta$	$B_{-\,\mu\nu}^{\alpha\beta} = i\left(B_{\mu\nu}^{\alpha\beta} - B_{\mu\nu}^{\dagger\,\alpha\beta}\right)/\sqrt{2}$	3	1

Although the flavor representation is correct, in four dimensions a complex two-form has too many degrees of freedom, the reason is that the tensor operators are not all independent, but satisfy the duality condition (5.1). This relation implies that the complex two-form has to be imaginary anti self-dual.

Defining F_L and F_R as the field strengths of A_L and A_R, $H_{ABC} = \partial_{[A}B_{BC]} - iA_{L,\,[A}B_{BC]} + iB_{[BC}A_{R,\,A]}$ as the three-form field strength of B_{BC}, and $D_MX = \partial_MX - iA_{L,\,M}X + iXA_{R,\,M}$ as the covariant derivative of X, the action takes the form

$$S = \int d^5x\sqrt{g}\mathrm{tr}\Big[-\frac{1}{2g_5^2}(F_L^2 + F_R^2) + g_X^2\big(\,|DX|^2 + 3\,|X|^2\,\big) - \\ + 2g_B\left(\frac{i}{3}\varepsilon^{MNLPQ}\left(B_{MN}H_{LPQ}^\dagger - B_{MN}^\dagger H_{LPQ}\right) - m_B|B|^2\right) + \lambda(X^\dagger F_L B + BF_R X^\dagger + h.c.)\Big]\,. \tag{5.23}$$

The trace is taken over the gauge indices. The factors of the AdS radius ℓ have been absorbed in the coupling constants or the masses.

Let us comment on the different terms. The first term is the kinetic action of the gauge fields, its coefficient was fixed in the original hard wall model comparing the expansion of the holographic vector-vector correlation function at large Euclidean momentum with the OPE of QCD [64]. The result was

$$\frac{1}{g_5^2} = \frac{N_c}{12\pi^2}\,. \tag{5.24}$$

The second term is the scalar action, that is usually canonically normalized, which can be achieved by rescaling $g_XX \to X$ in (5.23). The asymptotic value of the scalar field close to the boundary $z = 0$ determines the quark mass m and the condensate

$\langle\overline{\psi}\psi\rangle$ in the dual theory. With canonical normalization, g_X now appears in this relation:

$$X = \frac{1}{2}\left(g_X m\, z + \frac{\langle\overline{\psi}\psi\rangle}{g_X} z^3\right)\mathbf{1}_{2\times 2} \equiv \frac{g_X}{2} v(z)\mathbf{1}_{2\times 2}\,. \tag{5.25}$$

The reason that there are not two independent normalization constants in this relation is that the expectation value of the mass operator $\langle\overline{\psi}\psi\rangle$ can be obtained from varying the on-shell action with respect to m. The value of g_X was determined in ref. [67] comparing the holographic scalar-scalar correlation function at large Euclidean momentum with the OPE of QCD,

$$g_X^2 = \frac{N_c}{4\pi^2}\,. \tag{5.26}$$

The third term is the action of the two-form field. The kinetic term in the action has been replaced by a Chern-Simons term, and the mass m_B is a free parameter. We will see later that the right self-duality condition for the two-form field can be derived from the equations of motion of this action by fixing the mass. Then, following the usual procedure, we will compute the holographic tensor-tensor correlator, expand it at large Euclidean momentum, and match with the OPE of QCD. The last term is the most general gauge-invariant term of dimension five or less that couple isospin triplet vector mesons and preserve parity and charge conjugation in the dual theory [69].

We can rewrite the interaction term in (5.23) using real fields

$$\mathrm{tr}\left(X^\dagger F_L B + B F_R X^\dagger + h.c.\right) = \frac{1}{2}\mathrm{tr}\left(X_+\big(\{F_V,B_+\}+i[F_A,B_-]\big)+X_-\big(\{F_V,B_-\}-i[F_A,B_+]\big)\right). \tag{5.27}$$

The term (5.27) determines how chiral symmetry breaking affects to the isospin triplet 1^{--} vector mesons. Without this coupling the spectrum will be determined by the equations of motion of the V field, but once we introduce it, the B_+ field and the V field are coupled.

Using the expression (5.25) for the background scalar field and taking the trace in the action (5.23), we get in the vector sector

$$S_V = \int d^5x\sqrt{g}\Big[-\frac{1}{4g_5^2}\sum_{i=V,A} F_{i\,MN}F^{i\,MN} + \frac{g_B}{3}\varepsilon^{MNLPQ}\big(B_{-MN}H_{+LPQ} - B_{+MN}H_{-LPQ}\big) - \\ - g_B m_B \sum_{\alpha=+,-} B_{\alpha MN}B_\alpha^{MN} + \frac{\lambda}{2}v(z)F_{V\,MN}B_+^{MN}\Big]\,, \tag{5.28}$$

where we have suppressed gauge indices. In total we have introduced three new parameters, g_B, m_B and λ. We will now fix m_B imposing the self-duality condition and g_B and λ using the matching with the OPE of QCD. The only free parameters left in the model are the mass m, the condensate $\sigma = \langle \overline{\psi}\psi \rangle / g_X^2$ and the IR scale $1/z_m$. Although by introducing the B field we have added a new sector of vector mesons and therefore of masses and decay constants we can compare the model with, we have not increased the number of free parameters. Notice that the axial sector, involving the fields X and A is untouched.

5.4 Equations of motion

Our next step is to calculate the equations of motion for the fields $B_{\pm}^{\mu z}$,$B_{\pm}^{\mu\nu}$, and V^{μ} from the action (5.28). We will write explicitly all the factors involving the radial coordinate and raise and lower indices with the flat metric, using $g_{MN} = \frac{1}{z^2}\eta_{MN}$. Greek letters for the indices will refer to the flat Minkowski directions and capitalized italic letters will include the radial direction z. Let us consider first the case with no interaction, $\lambda = 0$. From (5.28) we get the equations of motion for the components of the two-form

$$\pm\frac{1}{3}\varepsilon^{MNLPQ}H_{\mp MNL} + m_B B_{\pm}^{PQ} = 0. \tag{5.29}$$

The epsilon tensor density is $\varepsilon^{MNLPQ} = z^5\epsilon^{MNLPQ}$ with the definition $\epsilon_{z\alpha\beta\mu\nu} \equiv \epsilon_{\alpha\beta\mu\nu} \Longrightarrow \epsilon^{z\alpha\beta\mu\nu} = -\epsilon^{\alpha\beta\mu\nu}$. Notice also that $B^{PQ} \sim z^4$, since indices are raised with the inverse metric $g^{MN} = z^2\eta^{MN}$. Then, writing explicitly powers of z we have:

$$\pm\epsilon^{z\alpha\beta\mu\nu}H_{\mp z\alpha\beta} + \frac{m_B}{z}B_{\pm}^{\mu\nu} = 0, \tag{5.30}$$
$$\pm\epsilon^{z\alpha\beta\gamma\mu}H_{\mp\alpha\beta\gamma} + \frac{3m_B}{z}B_{\pm}^{\mu z} = 0, \tag{5.31}$$

Where the first equation corresponds to $PQ = \mu\nu$ and the second to $PQ = \mu z$.

We can solve directly for $B_{\pm}^{\mu\nu}$ and $B_{\pm}^{\mu z}$ in the equations above. We then contract free indices with an epsilon tensor and use the relations

$$\begin{aligned} \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\gamma} &= -2\delta^{\mu}_{[\rho}\delta^{\nu}_{\gamma]}, \\ \epsilon^{\beta\alpha\mu\nu}\epsilon_{\beta\rho\sigma\tau} &= -\delta^{\alpha}_{[\rho}\delta^{\mu}_{\sigma}\delta^{\nu}_{\tau]}, \end{aligned} \tag{5.32}$$

where the antisymmetrization is made with unit weight. This gives expressions for $H_{\pm}^{z\mu\nu}$ and $H_{\pm}^{\alpha\mu\nu}$ that can be plugged back in the original equations, so the plus and minus components are decoupled. The equations of motion one finds are

$$z\partial_z\left(zH_\pm^{z\mu\nu}\right) + z^2\partial_\alpha H_\pm^{\alpha\mu\nu} + \frac{m_B^2}{4}B_\pm^{\mu\nu} = 0, \tag{5.33}$$

$$\partial_\alpha H_\pm^{\alpha\mu z} + \frac{m_B^2}{4z^2}B_\pm^{\mu z} = 0. \tag{5.34}$$

Note that taking the derivative ∂_Q of (5.29) (multiplied by a factor $\sqrt{g}$) will give a constraint $\partial_Q\sqrt{g}B^{PQ} = 0$ by virtue of the Bianchi identity. Equivalently, we contract ∂_μ with both equations (5.33) and (5.34) and use the fact that H^{MNL} is an antisymmetric tensor

$$\partial_\mu B_\pm^{\mu z} = 0, \tag{5.35}$$

$$\partial_\mu B_\pm^{\mu\nu} + z\partial_z\frac{1}{z}B_\pm^{z\nu} = 0. \tag{5.36}$$

Using (5.36) in (5.33) and (5.34) we can eliminate $\partial_z B_\pm^{\nu z}$ from the former and $B_\pm^{\mu\nu}$ from the latter. Expanding solutions in Fourier modes of four-momentum q^μ we get:

$$z^2\partial_z^2 B_\pm^{\mu\nu} + z\partial_z B_\pm^{\mu\nu} + \left(z^2q^2 - \frac{m_B^2}{4}\right)B_\pm^{\mu\nu} = 2izq^{[\mu}B_\pm^{\nu]z}, \tag{5.37}$$

$$\partial_z^2 B_\pm^{\mu z} - \frac{1}{z}\partial_z B_\pm^{\mu z} + \left(q^2 + \frac{4-m_B^2}{4z^2}\right)B_\pm^{\mu z} = 0. \tag{5.38}$$

Let us use the following decomposition

$$B_\pm^{\mu\nu} = iq^{[\mu}T_\pm^{\nu]} + i\epsilon^{\mu\nu\sigma\rho}q_\sigma\overline{T}_{\pm\,\rho}. \tag{5.39}$$

If the two-form field is dual to an operator of conformal dimensions $\Delta = 3$, then its expansion at small z should be

$$\begin{aligned} T_\pm^\mu &= \frac{1}{z}T^{(0)}{}_\pm^\mu + z\log zT^{(1)}{}_\pm^\mu + zT^{(2)}{}_\pm^\mu + \cdots \\ \overline{T}_\pm^\mu &= \frac{1}{z}\overline{T}^{(0)}{}_\pm^\mu + z\log z\overline{T}^{(1)}{}_\pm^\mu + z\overline{T}^{(2)}{}_\pm^\mu + \cdots \end{aligned} \tag{5.40}$$

This is possible if we set $m_B^2 = 4$. Setting $m_B = 2$ and using the original equations (5.30) and (5.31), from the leading $\sim 1/z^2$ term we get the conditions

$$\overline{T}^{(0)}{}_\mp^\mu = \pm T^{(0)}{}_\pm^\mu\,. \tag{5.41}$$

The conditions (5.41) above imply that

$$B^{(0)}{}_{+\,\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}B^{(0)}{}_{-}^{\alpha\beta} = 0. \qquad (5.42)$$

Let us define $b_{\mu\nu} = B^{(0)}{}_{+\,\mu\nu} - iB^{(0)}{}_{-\,\mu\nu}$. In terms of b, the condition above means it is imaginary anti-self-dual

$$b_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}b^{\alpha\beta} = 0. \qquad (5.43)$$

The conjugate $b^{\dagger}$ is imaginary self-dual.

Let us now study the effect of adding to the action a kinetic term for the two-form field, with a relative coefficient $C > 0$.

$$\Delta S = g_B C\int d^5x\sqrt{g}\Big[\sum_{\alpha=+,-} H_{\alpha MNL}H_{\alpha}^{MNL}\Big], \qquad (5.44)$$

Given arbitrary mass m_B, we find the following equations

$$C\nabla_L H_{\pm}{}^{LMN} \pm \frac{1}{3}\varepsilon^{MNABC}H_{\mp\,ABC} + m_B B_{\pm}{}^{MN} = 0\,. \qquad (5.45)$$

Where $\nabla_L \equiv \frac{1}{\sqrt{g}}\partial_L\sqrt{g}$. Assuming that the leading term in the small-z expansion of the two-form field is $B_{\pm}^{\mu\nu} \simeq z^{\eta}B^{(0)}{}_{\pm}^{\mu\nu}$, one finds that $B^{(0)}{}_{\pm}^{\mu\nu}$ satisfies an imaginary (anti) self-duality condition if

$$C\eta^2 \mp 2\eta - m_B = 0. \qquad (5.46)$$

The upper sign corresponds to the condition (5.42). The derivation is valid if $m_B - C\eta^2 \neq 0$, otherwise the leading term is a logarithm and the analysis is different. The equations for plus and minus components can be decoupled, giving a fourth order equation

$$(4{+}2m_BC)\nabla_K H_{\pm}{}^{KPQ}{+}m_B^2 B_{\pm}{}^{PQ}{+}\frac{C^2}{2}\nabla_L\left[\left(g^{KP}g^{CQ}g^{DL} + (PQL)\right)\partial_K\left(g_{CU}g_{DV}\nabla_S H_{\pm}{}^{SUV}\right)\right] = 0. \qquad (5.47)$$

Where by (PQL) we denote permutations with a minus sign if they are odd with respect to the first term. To leading order in z, the equation for the spacetime components $B_{\pm}^{\mu\nu}$ imposes a constraint on η in the form of a quartic equation

$$C^2\eta^4 - (4 + 2m_BC)\eta + m_B^2 = 0. \qquad (5.48)$$

We can rewrite this equation as

$$(C\eta^2 + 2\eta - m_B)(C\eta^2 - 2\eta - m_B) = 0. \tag{5.49}$$

Therefore, when the solution is (anti) self-dual the quartic equation is automatically satisfied. This shows that we can always impose the right self-duality condition, although we can also have solutions with opposite self-duality conditions if we change η, we have to set those to zero by hand. Instead, we will drop the kinetic term, we will show that it does not affect to properties of the model like the meson spectrum, although it can affect to correlation functions because it contributes to the boundary action.

If we set $C = 0$, then the equations (5.47) become

$$\nabla_K H_\pm{}^{KPQ} + \frac{m_B^2}{4} B_\pm{}^{PQ} = 0. \tag{5.50}$$

Let us now do the following trick, we can rewrite (5.45) as

$$C\left[\nabla_L H_\pm{}^{LMN} + \frac{\tilde{m}_B^2}{4} B_\pm{}^{MN}\right] \pm \frac{1}{3}\varepsilon^{MNABC} H_{\mp ABC} + \left(m_B - \frac{\tilde{m}_B^2 C}{4}\right) B_\pm{}^{MN} = 0\,. \tag{5.51}$$

Then, if

$$m_B = \tilde{m}_B + \frac{\tilde{m}_B^2 C}{4}, \tag{5.52}$$

the solutions to the $C = 0$ decoupled equation (5.50) with mass $\tilde{m}_B$ would make the term that multiplies C in the first bracket vanish, so one would recover the $C = 0$ equations again. Now let us fix the asymptotic expansion (5.40) ($\eta = -1$) and impose the self-duality condition (5.42). By fixing the mass to the value $m_B = 2 + C$ we can solve the system with $C \neq 0$ using the solutions for $C = 0$.[4] If one examines the equations involving the interaction term below one sees that the same is true if the coupling is rescaled appropriately. An exception to this rule may be the case $m_B = 0$, where there is an additional gauge invariance associated to the two-form field $\delta B_{MN} = \partial_{[M}\Lambda_{N]}$ and the separation in two parts of the equations of motion involves introducing gauge non-invariant terms. It also coincides with the case $\eta = 0$, where the leading solution is logarithmic. We will neglect this case and set $C = 0$ from now on. We now consider the interaction term, it does not affect to the leading asymptotic behavior, so the value of $m_B = 2$ is not changed. The equations of motion

[4]We can consider $C < 0$ by changing the overall sign of the action.

of the B_+ two-form are

$$z\partial_\alpha H_+^{\alpha\mu\nu} + \partial_z z H_+^{z\mu\nu} + \frac{B_+^{\mu\nu}}{z} = \frac{\lambda}{8g_B} v(z) \frac{F_V^{\mu\nu}}{z}, \tag{5.53}$$

$$\partial_\alpha z H_+^{\alpha\nu z} + \frac{B_+^{\nu z}}{z} = \frac{\lambda}{8g_B} \frac{v(z)}{z} F_V^{\nu z}, \tag{5.54}$$

$$z\partial_\alpha H_-^{\alpha\mu\nu} + \partial_z z H_-^{z\mu\nu} + \frac{B_-^{\mu\nu}}{z} = -\frac{\lambda}{8g_B} v'(z) F_{V\alpha\beta}\epsilon^{\alpha\beta\mu\nu}, \tag{5.55}$$

$$\partial_\alpha z H_-^{\alpha\nu z} + \frac{B_-^{\nu z}}{z} = 0. \tag{5.56}$$

The equations of motion of the vector fields and the constraints for the two-form fields are

$$\partial_z \frac{1}{z} B_+^{\nu z} - \frac{1}{z}\partial_\mu B_+^{\mu\nu} = \frac{\lambda}{8g_B}\left[\partial_z \frac{v(z)}{z} F_V^{\nu z} - \frac{v(z)}{z}\partial_\mu F_V^{\mu\nu}\right], \tag{5.57}$$

$$\partial_z \frac{1}{z} F_V^{\nu z} - \frac{1}{z}\partial_\mu F_V^{\mu\nu} = \lambda g_5^2 \left[\partial_z \frac{v(z)}{z} B_+^{\nu z} - \frac{v(z)}{z}\partial_\mu B_+^{\mu\nu}\right], \tag{5.58}$$

$$\partial_z \frac{1}{z} B_-^{\nu z} - \frac{1}{z}\partial_\mu B_-^{\mu\nu} = 0, \tag{5.59}$$

$$\partial_z \frac{1}{z} F_A^{\nu z} - \frac{1}{z}\partial_\mu F_A^{\mu\nu} = g_5^2 g_X^2 \frac{v^2(z)}{z^3} A^\nu. \tag{5.60}$$

We can simplify the two first equations above by eliminating $\partial_\mu F_V^{\mu\nu}$ and $\partial_\mu B_+^{\mu\nu}$ from the first and second equations respectively. Expanding in Fourier modes we have:

$$\partial_z B_+^{\nu z} - \frac{f(z)}{g(z)} B_+^{\nu z} - i q_\mu B_+^{\mu\nu} = \frac{\lambda}{8g_B}\frac{v'(z)}{g(z)}\partial_z V^\nu \tag{5.61}$$

$$\partial_z^2 V^\nu - \frac{f(z)}{g(z)}\partial_z V^\nu + q^2 V^\nu = \frac{\lambda g_5^2 v'(z)}{g(z)} B_+^{\nu z} \tag{5.62}$$

$$\partial_z B_-^{\nu z} - \frac{1}{z} B_-^{\nu z} - i q_\mu B_-^{\mu\nu} = 0 \tag{5.63}$$

$$\partial_z^2 A^\nu - \frac{1}{z}\partial_z A^\nu + q^2 A^\nu = g_5^2 g_X^2 \frac{v^2(z)}{z^2} A^\nu \tag{5.64}$$

Where $f(z) = \frac{1}{z} + \chi\, z v(z)(\frac{v(z)}{z})'$, $g(z) = 1 - \chi\, v(z)^2$ and $\chi = \lambda^2 g_5^2/(8g_B)$. Note that from the structure of the equations we can understand how the fields mix among themselves by considering parity conservation: the vector mode V^μ (negative parity) couples to its vector partner $B_+^{\nu z}$ (5.62) and to the tensor component of negative parity (5.53). Using the constraint (5.61) we can also simplify the equations of motion (5.57) and (5.58) for the two-form fields and we get:

$$\left[\partial_z^2 - \frac{f(z)}{g(z)}\partial_z - (C_1(z) - q^2)\right] B_+^{\nu z} = -\frac{\lambda}{8g_B}\left[C_2(z)V'^{\nu} + \frac{v'(z)q^2}{g(z)}V^{\nu}\right], \quad (5.65)$$

$$\left[\partial_z^2 - \frac{1}{z}\partial_z + q^2\right] B_-^{\nu z} = 0, \quad (5.66)$$

where

$$C_1(z) = \partial_z \frac{f(z)}{g(z)} + \frac{1}{z^2} + \frac{\chi v'^2}{g(z)^2}, \quad (5.67)$$

$$C_2(z) = \frac{v(z)}{z^2} - \partial_z \frac{v'(z)}{g(z)} - v'(z)\frac{f(z)}{g^2(z)}. \quad (5.68)$$

The equations for the tensor components of the two-form field are

$$\left[z\partial_z z\partial_z - 1 + z^2 q^2\right] B_+^{\mu\nu} = -i\frac{\lambda}{8g_B}\left[v(z)q^{[\mu}V^{\nu]} - \frac{z^2 v'(z)}{g(z)}\partial_z q^{[\mu}V^{\nu]}\right] + \left(z^2\frac{f(z)}{g(z)} + z\right) iq^{[\mu}B_+^{\nu]z}, \quad (5.69)$$

$$\left[z\partial_z z\partial_z - 1 + z^2 q^2\right] B_-^{\mu\nu} = -i\frac{\lambda}{8g_B} z v'(z) q_{[\alpha}V_{\beta]}\epsilon^{\alpha\beta\mu\nu} + 2ziq^{[\mu}B_-^{\nu]z}. \quad (5.70)$$

Our final equations of motion can then be divided in the vector (5.62), axial vector (5.64), and two-form components (5.65), (5.66), (5.69) and (5.70). A more convenient grouping is in four decoupled sets: $\{A^\mu\}$, $\{V^\mu, B_+^{\mu z}, (\delta_\alpha^\mu - P_\alpha^\mu)B_+^{\alpha\nu}, P_\alpha^\mu P_\beta^\nu B_-^{\alpha\beta}\}$, $\{P_\alpha^\mu P_\beta^\nu B_+^{\alpha\beta}\}$ and $\{B_-^{\mu z}, (\delta_\alpha^\mu - P_\alpha^\mu)B_-^{\alpha\nu}\}$. Normalizable solutions of the first two sets correspond to 1^{++} and 1^{--} mesons in the dual theory, respectively. The last two are not independent since they are coupled in the original system of first order equations (5.30), (5.31), and normalizable solutions correspond to 1^{+-} mesons. Notice that $b_+^{\mu\nu} \equiv zP_\alpha^\mu \mathcal{P}_\beta^\nu B_+^{\alpha\beta}$ satisfies the same equation as $B_-^{\mu z}$, and that this one is the same as the equation for vector mesons (5.62) in the absence of the interaction term $\lambda = 0$. Therefore, the interaction lifts the degeneracy between 1^{--} and 1^{+-} mesons.

5.4.1 Boundary expansion

We now proceed to do a Frobenius expansion of solutions close to the boundary at $z = 0$. This will be useful for both the calculation of renormalized two-point functions and the calculation of the meson spectrum. Using (5.39) in (5.30), we find the conditions

$$\begin{aligned}
&\mp(\partial_z T_{\mp\mu} + B_{\mp\mu z}) + \frac{1}{z}\overline{T}_{\pm\mu} = 0, \\
&\partial_z \overline{T}^{\mu}_{-} + \frac{1}{z}T^{\mu}_{+} = \frac{\lambda}{8g_B}\frac{v}{z}V^{\mu}, \\
&-\partial_z \overline{T}^{\mu}_{+} + \frac{1}{z}T^{\mu}_{-} = 0.
\end{aligned} \tag{5.71}$$

And from (5.31) we have

$$\begin{aligned}
&-q^2\overline{T}^{\mu}_{-} + \frac{1}{z}B^{\mu z}_{+} = \frac{\lambda}{8g_B}\frac{v}{z}\partial_z V^{\mu}, \\
&q^2\overline{T}^{\mu}_{+} + \frac{1}{z}B^{\mu z}_{-} = 0.
\end{aligned} \tag{5.72}$$

The expansion of the vector components at small z is given by (5.40) and

$$\begin{aligned}
A^{\mu}_{\pm} &= A^{(0)}{}^{\mu}_{\pm} + z^2 \log z A^{(1)}{}^{\mu}_{\pm} + z^2 A^{(2)}{}^{\mu}_{\pm} + \cdots, \\
B^{\mu z}_{\pm} &= B^{(0)}{}^{\mu}_{\pm} + z^2 \log z B^{(1)}{}^{\mu}_{\pm} + z^2 B^{(1)}{}^{\mu}_{\pm} + \cdots,
\end{aligned} \tag{5.73}$$

where we have defined $A^{\mu}_{+} = V^{\mu}$ and $A^{\mu}_{-} = A^{\mu}$. Expanding (5.71), (5.72), (5.58) and (5.60) for small z we find a set of conditions that allows us to solve for the coefficients of the logarithmic terms and give us a relation between the leading terms, dual to sources in the field theory. In particular we recover the imaginary self-duality condition (5.41) for the components of the two-form field. Defining $\tilde{\lambda}_{+} = \lambda/(8g_B)$, $\tilde{\lambda}_{-} = 0$, $q^2_{+} = q^2$ and $q^2_{-} = q^2 - g_X^2 g_5^2 m^2$, we can write them in a compact form

$$\begin{aligned}
&B^{(0)}{}^{\mu}_{\pm} = \pm q^2 T^{(0)}{}^{\mu}_{\pm}, && \overline{T}^{(0)}{}^{\mu}_{\mp} = \pm T^{(0)}{}^{\mu}_{\pm}, \\
&T^{(1)}{}^{\mu}_{\pm} = \frac{q^2}{2}T^{(0)}{}^{\mu}_{\pm} - \frac{\tilde{\lambda}_{\pm}}{2}mA^{(0)}{}^{\mu}_{\pm}, && \overline{T}^{(1)}{}_{\mp} = \mp\frac{q^2}{2}T^{(0)}{}^{\mu}_{\pm} \pm \frac{\tilde{\lambda}_{\pm}}{2}mA^{(0)}{}^{\mu}_{\pm}, \\
&\overline{T}^{(2)}{}^{\mu}_{\mp} = \mp T^{(2)}{}^{\mu}_{\pm} \pm \frac{1}{2}(q^2 T^{(0)}{}^{\mu}_{\pm} + \tilde{\lambda}_{\pm} m A^{(0)}{}^{\mu}_{\pm}) && A^{(1)}{}^{\mu}_{\pm} = -\frac{1}{2}(q^2_{\pm}A^{(0)}{}^{\mu}_{\pm} - g_5^2\lambda_{\pm} m q^2 T^{(0)}{}^{\mu}_{\pm}).
\end{aligned} \tag{5.74}$$

5.5 Holographic renormalization

We will follow the usual holographic procedure to compute correlation functions, deriving the on-shell action with respect to the sources of dual operators. The action usually diverges, so we will introduce a cutoff at a small value of the radial coordinate $z = \varepsilon$ to regularize it. We will introduce counterterms following the usual prescription

[76] to make the action finite before removing the cutoff $\varepsilon \to 0$.

The on-shell regularized action is

$$S_{o.s.} = \int d^4x\sqrt{g}\left[\frac{1}{2g_5^2}g^{zz}g^{\mu\nu}\sum_{a=\pm}F_{\pm\,z\mu}A_{\pm\,\nu} - \lambda v g^{zz}g^{\mu\nu}B_{+,z\mu}A_{+,\nu}\right]_{z=\varepsilon} + S_{CS}. \quad (5.75)$$

Where we have introduced the cutoff ε and the overall sign corresponds to taking the lower limit in the z integral. S_{CS} is the contribution of the two-form Chern-Simons action. The action has the form

$$\begin{aligned} S_{CS} &= \frac{2ig_B}{3}\int d^5x\sqrt{g}\varepsilon^{MNLPQ}\text{tr}\left(B_{MN}H^\dagger_{LPQ} - B^\dagger_{MN}H_{LPQ}\right) \\ &= \frac{g_B}{3}\int d^5x\sqrt{g}\varepsilon^{MNLPQ}\left(B_{-MN}H_{+LPQ} - B_{+MN}H_{-LPQ}\right). \end{aligned} \quad (5.76)$$

A variation gives, to leading order,

$$\delta S_{CS} = g_B\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\left(B_{-\mu\nu}\delta B_{+\alpha\beta} - B_{+\mu\nu}\delta B_{-\alpha\beta}\right). \quad (5.77)$$

Where the fields are evaluated at the cutoff $z=\varepsilon$. The condition (5.42) implies that we cannot vary B_+ and B_- independently, but since $B_{+\,\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}B_-^{\alpha\beta} = 0$ at the boundary, we should treat $B_{+\,\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}B_-^{\alpha\beta}$ as the variable boundary value. In order to have a consistent variational principle we need to add a boundary term to the action, of the form

$$S_0 = 4g_B\int d^4x\,\sqrt{-\gamma}\text{tr}\left(\gamma^{\mu\alpha}\gamma^{\nu\beta}B^\dagger_{\mu\nu}B_{\alpha\beta}\right), \quad (5.78)$$

where the indices are raised with the induced boundary metric $\gamma_{\mu\nu} = \varepsilon^{-2}\eta_{\mu\nu}$. The variation of this term, with explicit ε factors, is

$$\delta S_0 = 2g_B\int d^4x\left(B_+^{\ \mu\nu}\delta B_{+\mu\nu} + B_-^{\ \mu\nu}\delta B_{-\mu\nu}\right). \quad (5.79)$$

The sum of the two variations gives

$$\delta(S_{CS}+S_0) = 2g_B\int d^4x\left(B_+^{\ \mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}B_{-\,\alpha\beta}\right)\delta\left(B_{+\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu}^{\ \ \sigma\rho}B_{-\,\sigma\rho}\right). \quad (5.80)$$

This gives a variational principle where the combination $B_{+\,\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}B_-^{\alpha\beta}$ is varied, in accord with the condition (5.42).

The total on-shell regularized action is

$$S_{o.s.} = \int d^4x \left[-\frac{1}{2g_5^2}\frac{1}{z}\sum_{a=\pm}\partial_z A_a^\mu A_{a\,\mu} + \lambda\frac{v}{z}B_{+\,z}^{\ \mu}V_\mu + g_B\sum_{a=\pm}B_{a\,\mu\nu}B_a^{\mu\nu}\right]_{z=\varepsilon} . \quad (5.81)$$

The bulk contribution vanishes on-shell.

Expanding for small ε we find that, for Fourier modes of momentum q_μ,

$$\frac{1}{z}\partial_z A_\pm^\mu A_{\pm\,\mu} \sim (2A^{(2)}{}_{\pm\,\mu} + A^{(1)}{}_{\pm\,\mu})A^{(0)}{}_\pm^\mu + \\ +2\log(Q\varepsilon)A^{(1)}{}_{\pm\,\mu}A^{(0)}{}_\pm^\mu, \quad (5.82)$$

$$\frac{v}{z}B_{\pm\,z}^{\ \mu}A_{\pm\,\mu} \sim -mB^{(0)}{}_{\pm\,\mu}A^{(0)}{}_\pm^\mu, \quad (5.83)$$

$$\sum_{a=\pm}B_{a\,\mu\nu}B_a^{\mu\nu} \sim 4q^2\log(Q\varepsilon)\left(q^2T^{(0)}{}_+^\mu T^{(0)}{}_{+\,\mu} + q^2\overline{T}^{(0)}{}_+^\mu\overline{T}^{(0)}{}_{+\,\mu} - \frac{\lambda}{8g_B}mT^{(0)}{}_+^\mu A^{(0)}{}_{+\,\mu}\right) + \\ +8q^2(T^{(0)}{}_+^\mu T^{(2)}{}_{+\,\mu} - \overline{T}^{(0)}{}_+^\mu\overline{T}^{(2)}{}_{+\,\mu}) + \cdots \quad (5.84)$$

Where the dots refer to local terms in the sources, they will not be relevant because we can remove them with finite counterterms.

To this action we have to add some boundary counterterms to remove the divergences that appear as $\varepsilon \to 0$. As expected, the leading divergence $1/\varepsilon^2$ does not appear in the action of the two-form field. There are however additional logarithmic divergences. In order to completely cancel them we need more counterterms, of the form H^2, F^2, $(dX)^2$ and XFB. More explicitly, we have that the finite regularized action is

$$S_{reg} = S_{o.s} + S_1 + S_2 + S_{3+} + S_{3-} + S_4 + S_{\rm finite} \quad (5.85)$$

where

$$S_1 = c_1\int d^4x\log(\mu\varepsilon)\sqrt{-\gamma}\sum_{a=\pm}H_{a\,\mu\nu\sigma}H_a^{\mu\nu\sigma}, \quad (5.86)$$

$$S_2 = c_2\int d^4x\log(\mu\varepsilon)\sqrt{-\gamma}X_+F_{V\,\mu\nu}B_+^{\mu\nu}, \quad (5.87)$$

$$S_{3,\pm} = c_{3,\pm}\int d^4x\log(\mu\varepsilon)\sqrt{-\gamma}F_{\pm\,\mu\nu}F_\pm^{\mu\nu}. \quad (5.88)$$

$$S_4 = c_4\int d^4x\log(\mu\varepsilon)\sqrt{-\gamma}(D_\mu X)^\dagger(D^\mu X). \quad (5.89)$$

Notice that we can also have finite counterterms, corresponding to S_1, S_2, $S_{3\pm}$ and S_4 with no log factors. We will introduce the finite counterterms in $S_{\rm finite}$ and use them later on.

In the $\varepsilon \to 0$ limit, the counterterms become

$$S_1 \sim 6(q^2)^2\Big(T^{(0)}{}_{+\,\mu}T^{(0)\mu}_{\;\;+} + \overline{T}^{(0)}{}_{+\,\mu}\overline{T}^{(0)\mu}_{\;\;+}\Big), \tag{5.90}$$

$$S_2 \sim 2mq^2A^{(0)}{}_{+\,\mu}T^{(0)\mu}_{\;\;+}, \tag{5.91}$$

$$S_{3\pm} \sim 2q^2A^{(0)}{}_{\pm\,\mu}A^{(0)\mu}_{\;\;\pm}, \tag{5.92}$$

$$S_4 \sim m^2A^{(0)\mu}_{\;\;-}A^{(0)}{}_{-\,\mu}. \tag{5.93}$$

One can cancel the quadratic and logarithmic divergences if

$$c_1 = -\frac{2g_B}{3}, \quad c_2 = \frac{\lambda}{2}, \quad c_{3\pm} = -\frac{1}{4g_5^2}, \quad c_4 = \frac{g_X^2}{2}. \tag{5.94}$$

Now one can take the $\varepsilon \to 0$ limit, and use finite counterterms to remove the pieces that are local in the sources

$$\begin{aligned} S_{ren} = \int \frac{d^4q}{(2\pi)^4}\Bigg[&-\frac{1}{g_5^2}\sum_{a=\pm}A^{(0)}{}_{a\,\mu}A^{(2)\mu}_{\;\;a} + 8g_Bq^2(T^{(0)\mu}_{\;\;+}T^{(2)}{}_{+\,\mu} - \overline{T}^{(0)\mu}_{\;\;+}\overline{T}^{(2)}{}_{+\,\mu}) + \\ &+q^2\log\frac{Q^2}{\mu^2}\Bigg(\frac{1}{4g_5^2}\sum_{a=\pm}\frac{q_a^2}{q^2}A^{(0)}{}_{a\,\mu}A^{(0)\mu}_{\;\;a} - \lambda mA^{(0)\mu}_{\;\;+}T^{(0)}{}_{+\,\mu} + \\ &+2g_Bq^2(T^{(0)\mu}_{\;\;+}T^{(0)}{}_{+\,\mu} + \overline{T}^{(0)\mu}_{\;\;+}\overline{T}^{(0)}{}_{+\,\mu})\Bigg)\Bigg]. \end{aligned} \tag{5.95}$$

5.5.1 Two-point functions and matching to QCD

We now proceed to write the renormalized action in terms of general sources v_μ and $t_{\mu\nu}$. The transverse vector field is

$$A^{(0)}{}_\mu = P_\mu{}^\alpha a_\alpha = \left(\delta_\mu^\alpha - \frac{q_\mu q^\alpha}{q^2}\right)a_\alpha. \tag{5.96}$$

The transverse tensor is

$$t^T_{\mu\nu} = \frac{1}{2}P_{[\mu}{}^\alpha P_{\nu]}{}^\beta t_{\alpha\beta} = \frac{1}{2q^2}F_{+\mu\nu}{}^{\alpha\beta}t_{\alpha\beta}. \tag{5.97}$$

The longitudinal part of the tensor is

$$t^L_{\mu\nu} = \frac{1}{2}\left[(\delta_\mu^\alpha\delta_\nu^\beta - \delta_\mu^\beta\delta_\nu^\alpha) - P_{[\mu}{}^\alpha P_{\nu]}{}^\beta\right]t_{\alpha\beta} = -\frac{1}{2q^2}F_{-\mu\nu}{}^{\alpha\beta}t_{\alpha\beta} \tag{5.98}$$

Using the expansion

$$t_{\mu\nu} = i\epsilon_{\mu\nu\sigma\rho}q^\sigma\overline{T}^{(0)\rho} + iq_{[\mu}T^{(0)}{}_{\nu]}, \tag{5.99}$$

we indeed find

$$t^T_{\mu\nu} = i\epsilon_{\mu\nu\sigma\rho} q^\sigma \overline{T}^{(0)\rho}, \quad t^L_{\mu\nu} = iq_{[\mu} T^{(0)}{}_{\nu]}. \tag{5.100}$$

Solutions to the equations of motion close to the boundary depend on two coefficients, the sources and a subleading term related to the expectation value of the dual operators. Imposing suitable boundary conditions (Dirichlet or Neumann) at the cutoff will fix subleading terms in the boundary expansion in terms of the sources. In order to compute two-point functions it is enough to consider a linear dependence:

$$\begin{aligned} A^{(2)}{}_{+\mu} &= G_{VV}(q^2) A^{(0)}{}_{+\mu} + G_{VT}(q^2) T^{(0)}{}_{+\mu}, \\ T^{(2)}{}_{+\mu} &= G^-_{TT}(q^2) T^{(0)}{}_{+\mu} + G_{TV}(q^2) A^{(0)}{}_{+\mu}, \\ \overline{T}^{(2)}{}_{+\mu} &= G^+_{TT}(q^2) \overline{T}^{(0)}{}_{+\mu}. \end{aligned} \tag{5.101}$$

Using (5.13)

$$\begin{aligned} A^{(0)}{}_\mu A^{(0)\mu} &= a_\mu P^\mu{}_\alpha P^{\alpha\nu} a_\nu = a_\mu P^{\mu\nu} a_\nu \\ T^{(0)}{}_\mu T^{(0)\mu} &= \frac{1}{(q^2)^2} q^\alpha t^L_{\alpha\mu} q^\beta t^L{}_\beta{}^\mu = -\frac{1}{4(q^2)^2} t_{\mu\nu} F^{\mu\nu;\alpha\beta}_- t_{\alpha\beta} \\ A^{(0)}{}_\mu T^{(0)\mu} &= -\frac{i}{2q^2} q_{[\mu} P^\sigma_{\nu]} a_\sigma t^{L\mu\nu} = \frac{i}{2q^2} a_\mu (\eta^{\mu\alpha} q^\beta - \eta^{\mu\beta} q^\alpha) t_{\alpha\beta} \\ \overline{T}^{(0)}{}_\mu \overline{T}^{(0)\mu} &= \frac{1}{4(q^2)^2} \epsilon^{\mu\nu\sigma\rho} \epsilon^{\alpha\beta\gamma\delta} g_{\mu\alpha} q_\nu q_\beta t^T{}_{\sigma\rho} t^T{}_{\gamma\delta} = -\frac{1}{4(q^2)^2} t_{\mu\nu} F^{\mu\nu;\alpha\beta}_+ t_{\alpha\beta}. \end{aligned}$$

Introducing these expressions in the renormalized action (5.95) and deriving twice with respect to the sources, we find the following correlation functions

$$\Pi^{\mu\nu,\,ab}_{VV}(q^2) = \delta^{ab}(q^\mu q^\nu - q^2\eta^{\mu\nu}) \Pi_{VV}(q^2), \tag{5.102}$$

$$\Pi^{\mu;\nu\rho,\,ab}_{VT}(q^2) = i\delta^{ab}(\eta^{\mu\nu} q^\rho - \eta^{\mu\rho} q^\nu) \Pi_{VT}(q^2), \tag{5.103}$$

$$\Pi^{\mu\nu;\alpha\beta,\,ab}_{TT}(q^2) = \delta^{ab} \Pi^+_{TT}(q^2) F^{\mu\nu;\alpha\beta}_+ + \delta^{ab} \Pi^-_{TT}(q^2) F^{\mu\nu;\alpha\beta}_-. \tag{5.104}$$

Where

$$\Pi_{VV}(q^2) = -\frac{1}{2g_5^2} \log \frac{Q^2}{\mu^2} + \frac{2}{g_5^2 q^2} G_{VV}(q^2), \tag{5.105}$$

$$\Pi_{VT}(q^2) = -\frac{\lambda}{2} m \log \frac{Q^2}{\mu^2} - \frac{1}{2g_5^2 q^2} G_{VT}(q^2) + 4g_B G_{TV}(q^2), \tag{5.106}$$

$$\Pi^+_{TT}(q^2) = -g_B \log \frac{Q^2}{\mu^2} + \frac{4g_B}{q^2} G^+_{TT}(q^2), \tag{5.107}$$

$$\Pi^{-}_{TT}(q^2) = -g_B \log\frac{Q^2}{\mu^2} - \frac{4g_B}{q^2} G^{-}_{TT}(q^2). \tag{5.108}$$

Comparing with the expressions (5.18), (5.19) and (5.20) we get

$$\frac{1}{g_5^2} = \frac{N_c}{12\pi^2}, \quad g_B = \frac{N_c}{48\pi^2}, \quad \lambda = -\frac{N_c}{8\pi^2}. \tag{5.109}$$

Together with (5.26), this fixes all the coupling constants of the bulk action.

5.6 Matching to the massless theory

We have used the coefficients of the logarithmic divergences of the correlation functions to fix the parameters of the model. Notice that in the massless limit $m \to 0$ the logarithmic contribution to the vector-tensor correlator (5.106) vanish. In QCD, perturbative contributions vanish to all orders, so the only contributions left come from non-perturbative physics, this is clear in (5.20), where the leading term when the mass is zero is proportional to the condensate. Therefore, for massless QCD this is the term we have to match to fix the value of the parameter λ. Since the coefficient of this term is independent of the mass we should get the same value for λ, we will see that this is indeed the case, so the model passes this non-trivial consistency check.

In order to find the non-perturbative contributions to the OPE, we need to compute the functions '$G(q^2)$' that appear in (5.101) and plug them in the expressions for the correlators that we have found in the previous section. The overall strategy will be to solve the relevant equations of motion and match the near boundary expansion of the solutions to the coefficients of the series defined in (5.40) and (5.73).

As we mentioned before, there are four coupled equations describing the negative parity mesons. Plugging (5.39) in equations (5.61), (5.62), (5.65), (5.69) and (5.70) and keeping only the negative parity modes $(V, T^{\nu}_{+}, \overline{T}^{\nu}_{-}, B^{z\nu}_{+})$, we have the equations:

$$\left[\partial_z^2 - \frac{f(z)}{g(z)}\partial_z - (C_1(z) - q^2)\right] B_+^{\nu z} = -\frac{\lambda}{8g_B}\left[C_2(z)V'^{\nu} + \frac{v'(z)q^2}{g(z)}V^{\nu}\right] \tag{5.110}$$

$$\partial_z^2 V^{\nu} - \frac{f(z)}{g(z)}\partial_z V^{\nu} + q^2 V^{\nu} = \frac{\lambda g_5^2 v'(z)}{g(z)} B_+^{\nu z}, \tag{5.111}$$

$$\left[z\partial_z z\partial_z - 1 + z^2 q^2\right] T_+^{\nu} = -\frac{\lambda}{8g_B}\left[v(z)V^{\nu} - \frac{z^2 v'(z)}{g(z)}\partial_z V^{\nu}\right] + \left(z^2\frac{f(z)}{g(z)} + z\right) B_+^{\nu z}, \tag{5.112}$$

$$\left[z\partial_z z\partial_z - 1 + z^2 q^2\right] \overline{T}_-^{\nu} = -\frac{\lambda}{8g_B} z v'(z) V^{\nu} \tag{5.113}$$

With the additional constraint:

$$\partial_z B_+^{\nu z} - \frac{f(z)}{g(z)} B_+^{\nu z} + iq^2 T_+^{\nu} = \frac{\lambda}{8g_B}\frac{v'(z)}{g(z)}\partial_z V^{\nu}. \tag{5.114}$$

The mixing term proportional to $\lambda v(z)$ in the equations of motion is a small perturbation for small values of z, as $v(z)$ falls off towards the boundary. As the large Q behavior of spatial correlators with $Q^2 = -q^2$ is dominated by the small z behavior of the solution, we can determine the short distance behavior of correlation functions analytically by treating λ as a small parameter and solving the the equations of motion perturbatively in λ. However, we don't need to solve all the four equations. First, the constraint above implies that $B^{\nu z}$ is not independent, and the relations (5.74) imply that close to the boundary, T_+^{ν} is not independent of $\overline{T}_-^{\nu}$. For convenience, we will focus on equations (5.112) and (5.113). Dropping all terms of order λ^2 in the equations above, and taking the ansatz:

$$V^{\nu} = V_0(x)v^{\nu} + V_{\lambda}(x)b^{\nu} \tag{5.115}$$

$$\overline{T}_-^{\nu} = T_0(x)\overline{t}_-^{\nu} + T_{\lambda}(x)v^{\nu} \tag{5.116}$$

Our problem is reduced to solving the equations below

$$\left(\partial_x^2 - \frac{1}{x}\partial_x + q^2\right) V_0(x) = 0, \tag{5.117}$$

$$\left(\partial_x^2 - \frac{1}{x}\partial_x + 1\right) V_{\lambda}(x) = \lambda(\alpha_1 + \alpha_2 x^2) B_0(x), \tag{5.118}$$

$$\left[x^2\partial_x^2 + \partial_x - 1 - x^2\right] \overline{T}_0(x) = 0, \tag{5.119}$$

$$\left[x\partial_x^2 + \partial_x - \frac{1+x^2}{x}\right] \overline{T}_{\lambda}(x) = -\lambda(\Gamma_1 x + \Gamma_2 x^3) V_0(x). \tag{5.120}$$

Where $\alpha_1 = \frac{g_5^2 m}{Q^2}$, $\alpha_2 = \frac{3g_5^2\sigma}{Q^4}$, $\Gamma_1 = \frac{m}{8g_B Q}$ and $\Gamma_2 = \frac{3\sigma}{8g_B Q^3}$. $V_0, \overline{T}_0$ are the homogeneous solutions and V_λ, $\overline{T}_\lambda$ the perturbative corrections of order λ. B_0 satisfies the same equation as V_0. Focusing first on the vector mode, V_0 and B_0 have well known solutions in terms of Bessel functions:

$$V_0(x) = x(aI_1(x) + K_1(x)), \tag{5.121}$$

$$B_0(x) = x(bI_1(x) + K_1(x)), \tag{5.122}$$

Where a and b are constants fixed after imposing the IR boundary condition. For $V_0(x)$, following previous work we are going to choose a Neumann boundary condition at x_m, $\partial_x V_0(x)\big|_{x_m} = 0$ which allows us to set $a = \frac{K_0(x_m)}{I_0(x_m)}$. To choose the appropriate boundary condition for B_0 we note that a Dirichlet boundary condition imposed on $B_0(x)$, which sets $b = -\frac{K_1(x_m)}{I_1(x_m)}$, implies, by use of the constrain (5.114) to leading order, a Neumann boundary condition for the leading tensor mode, but a choice of Neumann boundary condition for the former is not consistent. As B_0 and V_0 describe completely independent modes, this already captures the most general solution.

To compute V_λ we will use a Green's function method. Note that we can write a solution for equation (5.118) of the form:

$$V_\lambda(x) = \lambda \int_{x_\varepsilon}^{x_m} dx'[\alpha_1 + \alpha_2 x'^2] B_0(x') \frac{G_V(x,x')}{x'}. \tag{5.123}$$

Provided G_V satisfies the equation:

$$\left(\partial_x^2 - \frac{1}{x}\partial_x - 1\right) G_V(x,x') = x\delta(x-x'). \tag{5.124}$$

With boundary conditions $G_V(x_\varepsilon, x') = G'_V(x, x_m) = 0$. We solve the equation above in the two regions $x > x'$ and $x < x'$ and match the two solutions at $x = x'$. It is not hard to show that $G_V(x,' x)$ can be written as:

$$G_V(x,x') = \frac{xx'}{AD - BC}[AI_1(x_>) + BK_1(x_>)][CI_1(x_<) + DK_1(x_<)]. \tag{5.125}$$

Where $x_{<,>} = \{min, max\}(x,x')$ is book keeping notation to specify the two branches of the Green's function. The coefficients are $A = -K_0(x_m)$; $B = I_0(x_m)$;$C = K_1(x_\varepsilon)$;$D = -I_1(x_\varepsilon)$. Taking the limit $x_\varepsilon \to 0$ we can set above $D = 0$ and $C = 1$.

Replacing back in (5.123) we have:

$$\begin{aligned} V_\lambda(x) = \quad & -\lambda \int_0^x dx'[\alpha_1 x' + \alpha_2 x'^3] K_1(x') I_1(x') - \\ & \lambda \frac{x^2}{2} \int_x^{x_m} dx'[\alpha_1 x' + \alpha_2 x'^3] K_1^2(x'). \end{aligned} \tag{5.126}$$

Where we used:

$$\frac{x(CI_1(x) + DK_1(x))}{AD - BC} = -xI_1(x) \simeq -\frac{x^2}{2}, \tag{5.127}$$

$$\frac{x(AI_1(x) + BK_1(x))}{AD - BC} \simeq -1, \tag{5.128}$$

and

$$B_0 \simeq xK_1(x). \tag{5.129}$$

We also have ignored all the terms proportional to $\frac{A}{B} \sim e^{-2x_m}$ since in the limit of large momentum these terms vanish quickly. Physically this means that as the momentum increases what happens in the IR region becomes less important, as expected. In fact, these solutions near the boundary and for large momentum become oblivious of the IR boundary conditions, since both Neumann and Dirichlet boundary conditions will enforce factors that fall off exponentially. Moreover, we will ignore the contribution of the first integral, that is negligible since these contributions vanish too quickly near the boundary.
We then have:

$$\begin{aligned} V_\lambda(x) \quad = \quad & -\lambda \frac{x^2}{2} \int_x^{x_m} dx'(\alpha_1 x' + \alpha_2 x'^3) K_1(x') K_1(x') \\ & \simeq \lambda \alpha_1 \frac{x^2}{2} \left(\frac{1}{2} + \log x \right) - \lambda \frac{x^2}{3} \alpha_2. \end{aligned} \tag{5.130}$$

The near boundary solution for V^ν is:

$$V^\nu(x) = V_0(x) v^\nu + V_\lambda b^\nu = \left(1 - \frac{x^2}{4} + \frac{x^2}{2} \log x \right) v^\nu + \left(\lambda \alpha_1 \frac{x^2}{2} \left(\frac{1}{2} + \log x \right) - \lambda \frac{x^2}{3} \alpha_2 \right) b^\nu. \tag{5.131}$$

Matching the solution above to the expansion defined in (5.73), and using (5.74), we find:

$$A^{(2)}_{+\mu} \equiv V^{(2)}_\mu = -\frac{Q^2}{4} V^{(0)} - \lambda Q^4 \left(\frac{\alpha_1}{4} - \frac{\alpha_2}{3} \right) T^{(0)}_+. \tag{5.132}$$

Therefore:

$$G^+_{VV}(Q^2) = -\frac{Q^2}{4}, \quad G^+_{VT}(Q^2) = -\lambda Q^4 \left(\frac{\alpha_1}{4} - \frac{\alpha_2}{3} \right). \tag{5.133}$$

Following similar steps, we can now compute near boundary solutions for equations (5.119) and (5.120). The homogeneous equation has a well known solution of the form:

$$\overline{T}_{-}(x) = cI_1(x) + K_1(x) \simeq K_1(x). \tag{5.134}$$

For the second equation, again, we can write a solution with a Green's function:

$$\overline{T}_{\lambda}(x) = -\lambda \int_{x_\epsilon}^{x_m} dx'(\Gamma_1 x' + \Gamma_2 x'^3) K_1(x') G_T(x, x'), \tag{5.135}$$

where analogously to the previous calculation, $G_T(x, x')$ satisfies:

$$\left[x\partial_x^2 + \partial_x - \frac{1 + x^2}{x} \right] G_T(x, x') = \delta(x - x'). \tag{5.136}$$

It can be shown that:

$$G_T(x, x') \simeq -K_1(x_>)I_1(x_<), \tag{5.137}$$

in the limit where $x_\epsilon \to 0$ and $x_m \to \infty$. Finally, we find that the solution for the tensor field is:

$$\overline{T}_{\lambda}(x) = \lambda \frac{x}{2} \int_x^\infty dx'[(\Gamma_1 x' + \Gamma_2 x'^3) K_1(x')] K_1(x') = \lambda \left(\frac{\Gamma_2}{3} - \frac{\Gamma_1}{4} \right) x, \tag{5.138}$$

therefore,

$$\overline{T}_{-}^{(2)\mu} = -\frac{Q^2}{4} \overline{T}_{-}^{(0)\mu} + \lambda Q \left(\frac{\Gamma_2}{3} - \frac{\Gamma_1}{4} \right) V^{(0)\mu}. \tag{5.139}$$

However, we are really after $T^{(2)\mu} = G_{TT}(Q^2)T^{(0)\mu} + G_{TV}^{+}(Q^2)V^{(0)\mu}$. To compute the latter, we use the relations we have found previously:

$$\overline{T}_{-}^{(2)\mu} = -T_{+}^{(2)\mu} + \frac{1}{2} \left(q^2 T_{+}^{(0)\mu} + \frac{\lambda}{8g_B} m V^{(0)\mu} \right) \tag{5.140}$$

$$\overline{T}_{-}^{(0)} = T_{+}^{(0)\mu}. \tag{5.141}$$

Solving for $T_{+}^{(2)\mu}$ we get:

$$T_{+}^{(2)\mu} = -\overline{T}_{-}^{(2)\mu} + \frac{1}{2} \left(q^2 T_{+}^{(0)\mu} + \frac{\lambda}{8g_B} m V^{(0)\mu} \right) = -\frac{Q^2}{4} T_{+}^{(0)\mu} - \left[\frac{3\lambda m}{32} + \frac{\sigma\lambda}{8g_B Q^2} \right] V^{(0)\mu}. \tag{5.142}$$

So the result is,

$$G_{TT}^{+}(Q^2) = -\frac{Q^2}{4}, \quad G_{TV}^{+}(Q^2) = -\frac{3\lambda m}{32} - \frac{\sigma\lambda}{8g_B Q^2}. \tag{5.143}$$

We can now compare with the OPE of vector and tensor correlators (5.105-5.108). Setting the mass to zero, the only nonzero contributions are

$$G_{VV} = -\frac{Q^2}{4}, \qquad G_{VT} = \lambda g_5^2 \sigma, \tag{5.144}$$

$$G_{TT}^{\pm} = -\frac{Q^2}{4}, \qquad G_{TV} = -\tfrac{\lambda\sigma}{8 g_B Q^2}. \tag{5.145}$$

The contributions G_{VV} and G_{TT} give contact terms that can be removed using counterterms in the regularized action. The only non-perturbative contributions to this order are

$$\Pi_{VT}(q^2) = \left(\frac{1}{2} - \frac{1}{2}\right)\frac{\lambda\sigma}{Q^2} = 0 \times \frac{\lambda}{g_X^2}\frac{\langle\overline{\psi}\psi\rangle}{Q^2} = 0. \tag{5.146}$$

Surprisingly the total contribution vanishes when we add the vector-tensor and tensor-vector contributions. At this moment we do not have a good understanding of why this is so. The results we have obtained are insensitive to the details of the IR, so they should be valid for any models that are asymptotically AdS space. However, the value of the condensate itself and other quantities like the meson spectrum will be sensitive to IR physics. In the next section we will study how the inclusion of the new terms in the action affect to some of these quantities.

5.7 Meson spectrum

So far we have discussed the UV physics of our model, focusing in the matching with the OPE of correlators in QCD. We will now comment on some of the IR physics, in particular the meson spectrum. In our analysis we have seen that the two-form field splits in a transverse part and a longitudinal part, that mixes with the vector fields. We can summarize the correspondence between the fields and meson states in the following table:

$B_{\mu\nu}$	mixes	J^{PC} mesons
transverse	–	1^{+-}
longitudinal	V_μ	1^{--}

The lightest isospin triplet states that can be found in the Particle Data Group (PDG) review [77], are

meson	J^{PC}	mass (MeV)
$b_1(1235)$	1^{+-}	$\sim 1229.5 \pm 3.2$
$\rho(770)$	1^{--}	$\sim 775.49 \pm 0.34$
$a_1(1260)$	1^{++}	$\sim 1230 \pm 40$

Notice that according to the PDG estimate, the b_1 and a_1 mesons are almost degenerate, although the error in the estimate of the a_1 mass is very large. Other estimates give a mass to the $a_1 \sim 1255$ MeV, with somewhat smaller errors [78].

In the holographic model the b_1 state is obtained from the transverse components of the B field, that are decoupled from the rest of the fields. The degeneracy between the ρ and the b_1 is broken in our model, thanks to the interaction term proportional to λ in ref. (5.23). Had we not considered this term, the spectrum would be degenerate, as has been observed in ref. [68]. So we should include this cubic interaction term both from the perspective of the large momentum OPE and from the properties of the meson spectrum.

We follow a similar procedure as in ref. [64] to compute numerically the lowest masses of the vector meson spectrum. We must specify suitable boundary conditions for the fields at the IR radial cutoff $z = z_m$ (Neumann or Dirichlet) and at the boundary $z = 0$ (normalizability). Solutions do not exist for any value of the four-momentum q^2, but only for a discrete set of values, which correspond to the masses of mesons in the holographic dual $m_n^2 = q^2$. We have checked that our results for the meson spectrum and the pion decay constant f_π coincide with those of ref. [64] when we set the coupling $\lambda = 0$.

We start with the spectrum of 1^{+-} mesons, dual to the field components $\{P^\mu_\alpha P^\nu_\beta B_+^{\alpha\beta}\}$ and $\{B_-^{\mu z}, (\delta^\mu_\alpha - P^\mu_\alpha) B_-^{\alpha\nu}\}$. Notice that we can solve first for $B_-^{\mu z}$ in (5.66) and then use (5.70) to solve for $(\delta^\mu_\alpha - P^\mu_\alpha) B_-^{\alpha\nu}$. As we have explained (5.69) is equivalent to (5.66), so for the purpose of finding the masses it is enough to focus on (5.66). Close to the boundary, a normalizable solution has the asymptotic expansion (5.73) with ${B^{(0)}}_-^\mu = 0$. At the cutoff we impose Neumann boundary conditions, since for Dirichlet boundary conditions there is a normalizable solution at $q^2 = 0$, which would be dual to a massless vector meson. Normalizable solutions are Bessel functions $B_-^{\mu z} = b^\mu z J_1(|q| z)$ and the Neumann boundary condition is satisfied for values of the momentum such that $J_0(|q|z_m) = 0$. Then, the mass of the lowest mode is

$$m_{b_1} z_m \simeq 2.405. \tag{5.147}$$

Notice that this value is independent of the quark mass and condensate. The remaining modes do depend on them and we have to solve numerically the equations.
We will first solve for modes dual to pseudoscalar mesons, whose lowest mode corresponds to the pion. We need to solve the set of equations [64]

$$\varphi'' - \frac{1}{z}\varphi' + g_5^2 g_X^2 \frac{v}{z^3}(\pi - \varphi) = 0, \tag{5.148}$$

$$-q^2\varphi' + \frac{g_5^2 g_X^2 v^2}{z^2}\pi' = 0. \tag{5.149}$$

For this, we first derive a single second order equation by solving algebraically for π in the first equation, plugging the result in the second equation and defining $\phi = \varphi'$. Then, using $g_X^2 g_5^2 = 3$ and defining $h(z) = 3v(z)^2/z^3$, we obtain

$$\phi''(z) + \frac{h'}{h}\phi(z) - \left(\left(\frac{h'}{h}\right)^2 - \frac{h''}{h} + zh - q^2\right)\phi(z) = 0. \tag{5.150}$$

Normalizable solutions at the boundary behave as $\phi(z) \sim z$ and we impose a Dirichlet boundary condition at the cutoff for the field ϕ. Then, for given values of mz_m and σz_m^3 we find the lowest value of $q_1^2 z_m^2 = m_\pi^2 z_m^2$ such that a solution satisfying the boundary conditions exists. We can then use the physical value of the pion mass $m_\pi = 139.6$ MeV to fix the scale z_m.
The spectrum of axial vector mesons 1^{++} can be found by solving equation (5.64). From (5.73) a normalizable solution $A^{(0)\mu}_{-} = 0$ vanishes at the boundary. At the cutoff, we impose Neumann boundary conditions. Finally, the spectrum of vector mesons can be computed from the system of coupled equations (5.62) and (5.65), with conditions $A^{(0)\mu}_{+} = 0$, $B^{(0)\mu}_{+} = 0$ in the expansions at the boundary (5.73). Regarding the boundary conditions at the cutoff, we must be careful since equations (5.65) and (5.69) have an additional singular point at z_* such that $g(z_*) = 1$. We are then constrained to values of the quark mass and the condensate such that $z_* > 1$ or to impose suitable boundary conditions at the singular point. A quick analysis shows that the two possible behaviors of solutions close to the singular point are $\sim (z_* - z)^{1/2}$ and ~ 1 for V^μ and $\sim (z - z_*)^{(1\pm\sqrt{13})/4}$ for $B_+^{\mu z}$. We can then make the solution regular by imposing a Neumann boundary condition for V^μ and a Dirichlet boundary condition for $B_+^{\mu z}$.
For the values of the mass and the condensate we have explored $0.0001 \leq mz_m \leq 0.1$,

$0.0125 \le \sigma z_m^3 \le 0.5$ we do not find a realistic spectrum of mesons, the lightest vector meson 1^{--} is always heavier than both parity even mesons 1^{++}, 1^{+-}. For larger values of the mass we can understand this as a consequence of the singularity at $z = z_*$. The curvature of AdS makes the classical problem of finding normalizable modes effectively as the quantum mechanical problem of finding the energy spectrum of a particle in a box, with one of the walls at the cutoff. For the 1^{--} modes we are forced to impose boundary conditions at the singularity $z_* < z_m$, so the "box" is smaller and the spectrum is lifted to higher values. This could be a problem of how infrared effects are implemented in this particular model, maybe different constructions like the soft wall could avoid this issue.

There is a way to find a more realistic meson spectrum, with the parity odd vector meson below the other modes. Instead of introducing the cutoff, we can impose boundary conditions for the vector mesons at the singularity even when it sits at a radial position beyond the cutoff $z_* > z_m$. For large enough values, the vector mesons become lighter and the spectrum can be tuned to realistic values, for instance for $mz_m = 0.0005$, $\sigma z_m^3 = 0.05375$ we find that $m_\rho \simeq 753.95$ MeV, $m_{a_1} \simeq 1238.24$ MeV and $m_{b_1} \simeq 1237.87$ MeV. Although this would fix the meson spectrum, there are other quantities that are important to determine whether the model is phenomenologically viable. One such quantity is the pion decay constant, f_π, that in QCD is approximately $f_\pi \simeq 91.92$ MeV. In the holographic model it is given by the formula [64]

$$f_\pi^2 = -\frac{1}{g_5^2} \frac{\partial_z A(z)}{z}\bigg|_{z=\varepsilon}, \tag{5.151}$$

where $A(z)$ is a solution to (5.64) satisfying $A(\varepsilon) = 1$, $A'(z_m) = 0$. With the parameters that give a realistic meson spectrum, the value of the pion decay constant is quite low $f_\pi \simeq 4.07$ MeV.

5.8 Conclusions

We have carried out, for the first time, a complete treatment of the hard-wall model including all fields dual to operators of free field theory dimension 3. We followed the standard procedure of fixing bulk parameters by matching the short distance behavior of correlation functions to perturbative QCD. Reassuringly, the structure of the correlators we obtained from our holographic model precisely matched the expressions in perturbative QCD, so this program can be carried out consistently. With this matching in hand, we calculated physical properties of mesons which, unfortunately, no longer match QCD. While this result casts into doubt whether the simple hard wall model can serve as a good stand-in for QCD, one may hope that an improved IR model could potentially lead to a better spectrum. As our analysis of the short-distance behavior of correlation functions only relies on the UV asymptotics of the geometry, the action we derived (including the numerical values of the coupling constants) should serve as the starting point for any such exploration of complete (in the sense of including all dimension 3 field theory operators) holographic bottom-up models with alternative IR boundary conditions. As we discussed in section 5.4, it is possible to modify the bulk action of the two-form field by adding a kinetic term, giving a one-parameter family of theories with the desired self-duality condition and asymptotic behavior. Since this will modify the boundary action, in principle the value of the bulk couplings will be shifted when the matching to QCD is done. It is possible then, that by changing this parameter, a more realistic spectrum can be found.

Let us point out some differences between our approach and what one expects in a top-down models like Sakai-Sugimoto [32, 79], based on a string theory construction. The matter content of the model is such that it coincides with large-N_c QCD at low energies in some region of parameter space where the UV theory is weakly coupled. In particular, 1^{+-} mesons should be part of the spectrum. However, in the holographic description where supergravity is valid such modes are missing. This should not come as a surprise: since the tensor operator is not a BPS protected operator, its conformal dimension can receive large corrections of order $\sim \lambda^{1/4}$, where λ is the 't Hooft coupling. In the holographic description this means that the tensor operator is dual to a field with a mass of order of the string scale, and therefore beyond the supergravity approximation. Since corrections to non-BPS operators are very large, it is even possible that the lowest 1^{+-} meson is not described by a field dual to the tensor operator we have considered in our model, but to a different operator with the same quantum numbers but larger conformal dimension in the free theory. This indeed seems to be the case in the Sakai-Sugimoto model, where the 1^{+-} mode is described by some components of a symmetric field in the bulk [80]. Clearly, in this case we do not expect that the OPE of the model will match with that of QCD, so in some sense the approach of refs. [68, 69] is closer to the top-down model. However, if the dimension of the tensor operator is chosen to be larger than 3, it is more difficult to argue that the effective theory description in the bulk stays valid anymore.

We have studied the extension of the model that takes into account 1^{+-} mesons, like b and ω. In principle the model can be further extended to include other modes in the QCD spectrum that have been observed experimentally. A mode that is somewhat heavier, but not that much, than vector and axial vector modes is the $\pi_1(1400)$ meson, with $J^{PC} = 1^{-+}$ and a mass $m_{\pi_1} \sim 1354 \pm 25$ MeV [77]. A peculiarity of this mode is that it cannot be predicted within the valence quark model, or in other words a simple quark bilinear operator would not create this kind of mode. An operator with the right quantum numbers would involve also a gluon field $\overline{\psi}F_{ij}\gamma_5\psi$. Then, in order to include mesons with the quantum numbers of π_1, we would have to introduce a field dual to the dimension-five operators $\overline{\psi}F_{\mu\nu}\gamma_5\psi$, and $\overline{\psi}F_{\mu\nu}\psi$. The obvious candidate is again a complex two-form field, with bulk mass $m^2\ell^2 = 9$ and no Chern-Simons action, since there is no self-duality constraint for these operators.

Appendix A

Effective scalar mass and the BF mass violation

In this appendix, we show how to identify the effective mass of the slipping mode of the probe brane in an effective AdS space in a more general context and in more detail.

In general the action of the embedding $L(\rho)$, which is a function of only ρ, a holographic direction, can be written as

$$S = \int \rho\, \beta(r(\rho), \rho)\, \rho^{d-1} \sqrt{1 + L'(\rho)^2}\,, \tag{A.1}$$

where $r(\rho) = \sqrt{L^2 + \rho^2}$ and d is an integer related to the dimension of the background and worldvolume spacetime. For example, for the D7(D5) probe brane in $\mathrm{AdS}_5 \times S^5$, $d = 4(3)$. We assume that

$$\beta(r(\rho), \rho) = \begin{cases} 1 & \rho \to \infty \quad \text{(UV)} \\ \frac{c}{\rho^q}\,, \quad (q \leq d-1) & \rho \to 0 \quad \text{(IR)}\,, \end{cases} \tag{A.2}$$

where c is constant. The first condition comes from the fact that the slipping mode ($\phi = L/\rho$) is a scalar in AdS_{d+1} in UV. The second condition restricts us to an effective IR AdS space. When $q = d - 1$, IR space is effectively AdS_2.

The linearized action in terms of the slipping mode reads

$$\begin{aligned} S &\sim \int \rho \frac{1}{2}\beta_0 \rho^{d-1}\left(\rho^2\phi'^2 + m^2\phi^2\right) \\ m^2 &= (1-d) - \frac{d\log\beta_0}{d\log\rho} + 2\frac{\rho^2}{\beta_0}\frac{\partial\beta}{\partial L^2}\bigg|_{L=0}\,, \end{aligned} \tag{A.3}$$

where $\beta_0 = \beta(r(\rho),\rho)|_{L=0}$.

A.1 Effective geometry changed

In the UV, $\beta_0 = 1$ and the action (A.3) corresponds to the scalar action in AdS_{d+1} space with the UV mass

$$m_{UV}^2 = 1 - d \geq -d^2/4\,, \quad \rho \to \infty\,. \tag{A.4}$$

For all d, the BF bound is satisfied (in AdS_{d+1}).

In the IR, the action is written as

$$S \sim \int \rho \frac{1}{2}\rho^{d-q-1}\left(\rho^2\phi'^2 + m^2\phi^2\right) \tag{A.5}$$

$$m^2 = (1-d+q) + 2\frac{\rho^2}{\beta_0}\frac{\partial\beta}{\partial L^2}\bigg|_{L=0}\,. \tag{A.6}$$

The scalar effectively lives in AdS_{d-q}, where $q \leq d-1$. To go further we consider two cases: $\beta = \beta(r(\rho))$ and $\beta = \beta(r(\rho),\rho)$.
case 1: For $\beta = \beta(r(\rho))$,

$$\frac{\partial\beta}{\partial L^2}\bigg|_{L=0} = \frac{1}{2\rho}\frac{d\beta_0}{d\rho}\,, \tag{A.7}$$

$$m_{IR}^2 = (1-d+q) + \frac{d\log\beta_0}{d\log\rho} = 1-d\,. \tag{A.8}$$

Note that the m^2 is the same in UV and IR. However, the stability criteria, the BF bound $-\frac{(d-q)^2}{4}$, is now changing and a function of q. Therefore, if q is a continuous parameter (for this purpose, let us continue q to real values), then the BKT transition occurs at $q = d - \sqrt{4(d-1)}$. For $d=4$, $q \sim 0.536$, which is the same value we obtained in section 4.4.
case 2: For $\beta = \beta(r(\rho),\rho)$, we have to study case by case, since (A.7) is not valid. As an example, let us consider D3/D7(D5) at finite B and density, **d** [62, 63, 81, 82].

D3/D5

$$S = \int \rho\, \beta(r(\rho), \rho)\, \rho^2 \sqrt{1 + L'(\rho)^2}\,, \tag{A.9}$$

where

$$\beta(r(\rho), \rho) = \sqrt{1 + \frac{\mathbf{d^2}}{\rho^4} + \frac{B^2}{r^4}} \tag{A.10}$$

By (A.4), the UV mass is -2 in AdS_4, while, by (A.6), the IR mass is $(d = 3, q = 2)$

$$m_{IR}^2 = -\frac{2B^2}{\mathbf{d}^2 + B^2} \tag{A.11}$$

in AdS_2. The BF bound is violated at $\mathbf{d} = \sqrt{7}B$ and this violation by the continuous parameter, $\mathbf{d}$ or B, implies the BKT transition. If $\mathbf{d} = 0$ then $m_{IR}^2 = -2$, which is consistent with (A.8).

D3/D7

$$S = \int \rho\, \beta(r(\rho), \rho)\, \rho^3 \sqrt{1 + L'(\rho)^2}\,, \tag{A.12}$$

where

$$\beta(r(\rho), \rho) = \sqrt{1 + \frac{\mathbf{d}^2}{\rho^6} + \frac{B^2}{r^4}}\,. \tag{A.13}$$

By (A.4), the UV mass is -3 in AdS_5, while, by (A.6), the IR mass is $(d = 4, q = 3)$
,

$$m_{IR}^2 = -\frac{B^2\rho^2}{\mathbf{d}^2} \to 0\,, \tag{A.14}$$

in AdS_2. It satisfies the BF bound for all B and $\mathbf{d}$. However, the instability is in the intermediate regime. We can see this by expanding the action in the regime $\mathbf{d}/B \ll \rho \ll \sqrt{B}$, of which linearized equation of motion is

$$L'' + \frac{1}{\rho}L' + 2\frac{1}{\rho^2}L = 0\,. \tag{A.15}$$

The slipping mode is effectively the scalar of $m^2 = -3$ in AdS$_3$, which violates the BF bound. It also can be seen more directly from (A.3), where the second term and third term cancel out, leaving the first term, $(1 - d) = -3$. Note that this instability happens only for a large enough B (or small enough $\mathbf{d}$) to satisfy the condition $\mathbf{d}/B \ll \rho \ll \sqrt{B}$. Note also that the BF mass violation is *finite* as we dial B for a fixed $\mathbf{d}$, and the phase transition turns out to be of mean-field type[1]. (The

[1] A non-mean field (but non-BKT) type transition also can be understood in the same way. In the model studied in [83], it can be shown that an instability can arise in the range $(\{(B/O)^{1/(2-\Delta)}, (\mathbf{d}/O)^{1/(2-\Delta)}\} \ll \rho \ll O^{1/\Delta})$, where O is a phenomenological operator with dimension Δ. This range is essentially where the operator O dominates over $B, \mathbf{d}$.

infinitesimal violation of the BF bound as in (A.11) is a characteristic of the BKT transition.)

A.2 Effective geometry fixed

There is alternative way, in which we keep the UV AdS space for all ρ. For this, we need to redefine the coordinate system by

$$\frac{\rho}{\tilde{\rho}} = \frac{\beta \rho^{d-1}}{\tilde{\rho}^{d-1}}\,, \tag{A.16}$$

so

$$\tilde{\rho} = \left(\frac{1}{(d-2)\int_{\rho}^{\infty} \frac{\rho}{\beta \rho^{d-1}}} \right)^{\frac{1}{d-2}} \tag{A.17}$$

which is defined only for $0 \le q < d-2$. Otherwise, the integral diverges and $\tilde{\rho}$ is not defined. (So, the previous examples of the D7(D3) probe in $\mathrm{AdS}_5 \times S^5$ cannot be analyzed in this way; q is too big.) In terms of a new coordinate $\tilde{\rho}$, the action reads

$$S \sim \int d\tilde{\rho} \frac{1}{2} \tilde{\rho}^{d-1} \left(\tilde{\rho}^2 \phi'^2 + m^2 \phi^2 \right) \tag{A.18}$$

$$m^2 = (1-d) + \beta_0 \beta_0' \frac{\rho^{2d-3}}{\tilde{\rho}^{2d-4}}\,, \tag{A.19}$$

where we consider only the case $\beta = \beta(r(\rho))$, so that we can use (A.7). Note that for a function $\beta_0 = \rho^{-q}$ (the normalization of β_0 does not matter, since any normalization factor is canceled in (A.18)), m^2 is constant:

$$m^2 = (1-d) - q \left(\frac{d-2}{d-2-q} \right)^2 . \tag{A.20}$$

One might wonder if this analysis is consistent with the previous one (Appendix A.1). For example, for $d = 4, q = 1$, both analyses are applicable. They must be consistent since the BF bound analysis is an effective tool and how to interpret the action should not change the physics. i.e. for $d = 4, q = 1$, we can interpret the action of either (1) a scalar in AdS_5 with $m^2 = -7$ or (2) a scalar in AdS_4 with $m^2 = -3$. However, both cases tell us the scalar mass violate their own BF bound, so they are consistent. To see this more clearly, let us check the BF bound conditions, which are

$$1 - d = -\frac{(d-q)^2}{4} \tag{A.21}$$

in Appendix A.1, and

$$1 - d = q\left(\frac{d-2}{d-2-q}\right)^2 - \frac{d^2}{4}\,. \tag{A.22}$$

in this subsection. These seemingly different conditions indeed give the same results: the BF bound is violated at the value of $q = q_c$,

$$q_c = d - 2\sqrt{d-1}\,. \tag{A.23}$$

Therefore we may interpret our analysis as either (1) m^2 does not change but the effective background is changing (2) m^2 is changing but the geometry does not change.

Of course, we can do a mixture: partial change of geometry and partial change of m^2. How this works in general can be seen by the following simple example. The equation of the scalar field at the boundary of AdS_{D+1} space ($z \to 0$) reads

$$\Phi'' + \frac{(1-D)}{z}\Phi' - m^2\frac{\Phi}{z^2} = 0\,. \tag{A.24}$$

By the definition $\Phi = z^{\frac{D-d}{2}}\phi$, it can be transformed to

$$\phi'' + \frac{(1-d)}{z}\phi' - \left(m^2 - \frac{d^2-D^2}{4}\right)\frac{\phi}{z^2} = 0\,. \tag{A.25}$$

It is formally interpreted as the scalar in AdS_{d+1} space with the modified mass $m^2 - (d^2+D^2)/4$. In both cases the BF bound is the same, $m^2 = -D^2/4$, so the physics does not change. Especially, the equation for $D = 2$ (or $d = 2$) corresponding to the effective AdS_2 is the Schrodinger equation with the $1/z^2$ potential term and the $-1/4z^2$ potential plays a role for the BKT transition as discussed in [57] and section 4.4.

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