From integration by parts to state and boundary variables of linear differential and partial differential systems

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Abstract. We elaborate on an idea originally expressed in [13]: the remainders resulting from repeated integration by parts of a set of linear higher-order ordinary differential equations define *state vectors*. Furthermore, these remainders and the corresponding state maps can be easily computed by factorization of a certain two-variable polynomial matrix, which is directly derived from the one-variable polynomial matrix describing the set of higher-order differential equations. Recently [7] we have extended this same idea to the construction of state maps for systems of linear partial differential equations involving, apart from the time variable, also spatial variables. In the current paper we take a next step by considering partial differential equations on a bounded spatial domain, and we show how integration by parts yields, next to the construction of state maps, also a recipe to define boundary variables in a natural manner.

It is a great pleasure for the first author to congratulate Uwe Helmke on his sixtieth birthday. Starting from my first close encounters with Uwe, probably at the famous Edzell meetings in Scotland, connecting the Systems & Control groups of Warwick, Bremen and Groningen in the early 1980s, it was a continuing joy to meet him and to discuss with him on topics of common interest.

1 Recall of state maps for finite-dimensional linear systems

In [13] we have shown how the notion of *'state'* for linear systems described by higher-order differential equations is intimately related to the procedure of *integration by parts*, and how the articulation of this relation yields an insightful and direct way of computing state maps.

In particular, consider a linear system

$$P\left(\frac{d}{dt}\right)y(t) = Q\left(\frac{d}{dt}\right)u(t), \quad y(t) \in \mathcal{Y} := \mathbb{R}^p,\ u(t) \in \mathcal{U} := \mathbb{R}^m, \tag{1}$$

or more generally, without distinguishing between inputs u and outputs y and letting $w := \begin{bmatrix} y \\ u \end{bmatrix}, q := p+m$, consider $R\left(\frac{d}{dt}\right)w(t) = 0, w(t) \in \mathcal{W} := \mathbb{R}^q$.

In all these equations, $P\left(\frac{d}{dt}\right), Q\left(\frac{d}{dt}\right)$, and $R\left(\frac{d}{dt}\right)$ describe linear (higher-order) differential operators, or, equivalently, $P(\xi), Q(\xi)$, and $R(\xi)$ are polynomial matrices of appropriate dimensions in the indeterminate ξ.

It is well-known [4] that for an observable *input-state-output system*

$$\begin{aligned} \tfrac{d}{dt}x &= Ax + Bu, \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \\ y &= Cx + Du, \quad y(t) \in \mathbb{R}^p \end{aligned} \tag{2}$$

the state x can be written as a linear combination of the outputs and inputs and their derivatives, i.e., $x = X_y\left(\frac{d}{dt}\right)y + X_u\left(\frac{d}{dt}\right)u$ for certain linear differential operators $X_y\left(\frac{d}{dt}\right), X_u\left(\frac{d}{dt}\right)$, or more compactly

$$x = X\left(\frac{d}{dt}\right)w, \tag{3}$$

for some $n \times q$ polynomial matrix $X(\xi)$. We will call (3) a *state map*.
Conversely, consider the system of linear higher-order differential equations

$$R\left(\frac{d}{dt}\right)w(t) = 0, \quad w(t) \in \mathcal{W} = \mathbb{R}^q, \tag{4}$$

where $R(\xi) = R_0 + R_1\xi^1 + \ldots + R_N\xi^N \in \mathbb{R}^{p \times q}[\xi]$. How do we construct state maps $x = X\left(\frac{d}{dt}\right)w$, which also allow to represent the system (4) into state space form ?
Before answering this question we need to formalize the space of solutions of (4), as well as the notion of state. An ordinary N-times differentiable solution of (4) will be called a *strong solution*. Denote the space of locally integrable trajectories from $\mathbb{R}$ to $\mathbb{R}^q$ by $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$. Recall that $w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ is a *weak solution* of (4) if

$$\int_{-\infty}^{\infty} w^T(t)R^T\left(-\frac{d}{dt}\right)\varphi(t)dt = 0 \tag{5}$$

for all $\mathcal{C}^\infty$ test functions $\varphi : \mathbb{R} \to \mathbb{R}^p$ with compact support. The set of all weak solutions of (4), called the *behavior* $\mathcal{B}$, is denoted as

$$\mathcal{B} := \{w : \mathbb{R} \to \mathbb{R}^q \mid w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \text{ and (4) is satisfied weakly}\} \tag{6}$$

Consider now two solutions $w_1, w_2 \in \mathcal{B}$, and define the *concatenation* of w_1 and w_2 at time 0 as the time-function

$$(w_1 \wedge_0 w_2)(t) := \begin{cases} w_1(t) & , \quad t < 0 \\ w_2(t) & , \quad t \geq 0 \end{cases}, \quad t \in \mathbb{R}. \tag{7}$$

We say that $w_1, w_2 \in \mathcal{B}$ are *equivalent at time* 0, denoted as $w_1 \sim_0 w_2$, if for all $w \in \mathcal{B}$:

$$w_1 \wedge_0 w \in \mathcal{B} \Leftrightarrow w_2 \wedge_0 w \in \mathcal{B}. \tag{8}$$

Thus equivalent trajectories admit the same continuations starting from time $t = 0$.
Let $X(\xi) \in \mathbb{R}^{n \times q}[\xi]$. Then the differential operator

$$\begin{aligned} X\left(\frac{d}{dt}\right) : \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) &\to \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \\ w &\mapsto x := X\left(\frac{d}{dt}\right)w \end{aligned}$$

is said to be a *state map* [8] for the system (4), with set of solutions $\mathcal{B}$ defined in (6), if for all $w_1, w_2 \in \mathcal{B}$ and corresponding $x_i := X\left(\frac{d}{dt}\right)w_i$, $i = 1,2$, the following property (the *state property*) holds:

$$x_1(0) = x_2(0) \text{ and } x_1, x_2 \text{ continuous at } t = 0 \implies w_1 \sim_0 w_2 \,. \tag{9}$$

If (9) holds, then the vector x contains all the information necessary to conclude whether any two trajectories in $\mathcal{B}$ admit the same continuation at time $t = 0$. For this reason the vector $x(0) = X\left(\frac{d}{dt}\right)w(0)$ is called a *state* of the system at time 0 corresponding to the time-function w, and $\mathcal{X} = \mathbb{R}^n$ is called a *state space* for the system.

Remark 1. In the context of *linear* systems (as in this paper) equation (7) is equivalent to requiring that $w_1 \wedge_0 w$ and $w_2 \wedge_0 w \in \mathcal{B}$ for *some* $w \in \mathcal{B}$. Furthermore in this case, since $w_2 \wedge_0 w_2 \in \mathcal{B}$, it follows that $w_1 \sim_0 w_2$ if and only $w_1 \wedge_0 w_2 \in \mathcal{B}$. Because of the symmetry of this last condition, it also means that equivalence of $w_1, w_2 \in \mathcal{B}$ at $t = 0$ amounts to w_1 and w_2 having the same *precursors*. (Note that for *nonlinear* systems these equivalences in general do not hold; see [11] for some initial ideas about the construction of state maps in this case.)

The basic idea of [13] is to show how state maps can be obtained from the *integration by parts* formula. Take any N-times differentiable functions $w : \mathbb{R} \to \mathbb{R}^q$ and $\varphi : \mathbb{R} \to \mathbb{R}^p$, and denote $w^{(i)} := \frac{d^i}{dt^i}w$, $i \in \mathbb{N}$, and analogously for φ. For each pair of time instants $t_1 \le t_2$ repeated integration by parts yields

$$\int_{t_1}^{t_2} w^T(t)R^T\left(-\frac{d}{dt}\right)\varphi(t)dt = \int_{t_1}^{t_2} \varphi^T(t)R\left(\frac{d}{dt}\right)w(t)dt + B_\Pi(\varphi, w)\big|_{t_1}^{t_2} \,, \tag{10}$$

where we call the expression $B_\Pi(\varphi, w)(t)$ the *remainder*, which has the form

$$B_\Pi(\varphi, w)(t) = \begin{bmatrix} \varphi^T(t) & \varphi^{(1)T}(t) & \dots & \varphi^{(N-1)T}(t) \end{bmatrix} \tilde{\Pi} \begin{bmatrix} w(t) \\ w^{(1)}(t) \\ \vdots \\ w^{(N-1)}(t) \end{bmatrix} , \tag{11}$$

for some constant matrix $\tilde{\Pi}$ of dimension $Np \times Nq$.

The *differential version* of the integration by parts formula (10) (obtained by dividing (10) by $t_2 - t_1$ and letting t_1 tend to $t_2 = t$) is

$$w^T(t)R^T\left(-\frac{d}{dt}\right)\varphi(t) - \varphi^T(t)R\left(\frac{d}{dt}\right)w(t) = \frac{d}{dt}B_\Pi(\varphi, w)(t) \,, \tag{12}$$

Both sides of this equality define a bilinear differential operator form, or briefly a *bilinear differential form* (BDF), i.e., a bilinear functional of two trajectories and of a finite number of their derivatives. Formally, a bilinear differential form B_Φ as defined in [15] is a bilinear map $B_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ involving two vector-valued functions and a finite set of their time-derivatives, that is, at any time t

$$B_\Phi(\varphi, w)(t) = \sum_{k,l=0}^{M-1} \left[\frac{d^k}{dt^k}\varphi(t)\right]^T \Phi_{k,l} \frac{d^l}{dt^l}w(t) \tag{13}$$

for certain constant $p \times q$ matrices $\Phi_{k,l}, k,l = 0,\cdots,M-1$. The matrix $\tilde{\Phi}$ whose (k,l)-th block is the matrix $\Phi_{k,l}$ for $k,l = 0,\dots,M-1$, is called the *coefficient matrix* of the bilinear differential form B_Φ. It follows that the coefficient matrix of the bilinear differential form B_Π corresponding to the remainder is precisely the matrix $\tilde{\Pi}$ as defined in (11).

Remark 2. For a scalar polynomial or a square polynomial matrix $R(\xi)$ the formula's (10) and (12) are classically referred to as *Green's*, respectively *Lagrange's, identity*, while the matrix $\tilde{\Pi}$ for a scalar $R(\xi)$ is called the *bilinear concomitant*, see [3].

There is a useful one-to-one correspondence between the bilinear differential form B_Φ in (13) and the two-variable polynomial matrix $\Phi(\zeta,\eta)$ defined as

$$\Phi(\zeta,\eta) := \sum_{k,l=0}^{M-1} \Phi_{k,l}\zeta^k\eta^l \,. \tag{14}$$

The crucial observation, see [1, 15], is that for any bilinear differential form B_Φ the bilinear differential form corresponding to its *time-derivative*, defined as

$$\begin{aligned} B_\Psi(\varphi,w)(t) &:= \tfrac{d}{dt}\left(B_\Phi(\varphi,w)\right)(t) \\ &= \textstyle\sum_{k,l=0}^{M-1}\left[\frac{d^{k+1}}{dt^{k+1}}\varphi(t)\right]^T \Phi_{k,l}\frac{d^l}{dt^l}w(t) + \left[\frac{d^k}{dt^k}\varphi(t)\right]^T \Phi_{k,l}\frac{d^{l+1}}{dt^{l+1}}w(t), \end{aligned} \tag{15}$$

corresponds, by the product rule of differentiation, to the two-variable polynomial matrix

$$\Psi(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)\,. \tag{16}$$

As a consequence, the differential version of the integration by parts formula (12) has associated to it the *two-variable polynomial matrix equality*

$$R(-\zeta) - R(\eta) = (\zeta+\eta)\Pi(\zeta,\eta) \tag{17}$$

From this formula it follows how the two-variable polynomial matrix $\Pi(\zeta,\eta)$ and its coefficient matrix $\tilde{\Pi}$ (corresponding to the remainder) can be easily computed: since the two-variable polynomial matrix $R(-\zeta)-R(\eta)$ is zero for $\zeta+\eta=0$, it directly follows that $R(-\zeta)-R(\eta)$ contains a factor $\zeta+\eta$, and thus we can *define* the two-variable polynomial matrix $\Pi(\zeta,\eta)$ as

$$\Pi(\zeta,\eta) := \frac{R(-\zeta)-R(\eta)}{\zeta+\eta}\,. \tag{18}$$

It now turns out that state maps for a system $R(\frac{d}{dt})w = 0$ can be computed from a *factorization* of the two-variable polynomial matrix $\Pi(\zeta,\eta)$ into a product of single-variable polynomial matrices. Indeed, any factorization $\Pi(\zeta,\eta) = Y^T(\zeta)X(\eta)$ of the two-variable polynomial matrix $\Pi(\zeta,\eta)$ leads from (17) to the matrix polynomial equality

$$R(-\zeta) - R(\eta) = (\zeta+\eta)Y^T(\zeta)X(\eta)\,, \tag{19}$$

and to the corresponding bilinear differential form equality, expanding (12)

$$\begin{aligned} w^T(t)R^T\left(-\tfrac{d}{dt}\right)\varphi(t) \quad - \quad & \varphi^T(t)R\left(\tfrac{d}{dt}\right)w(t) = \\ = \quad & \tfrac{d}{dt}\left[\left(Y\left(\tfrac{d}{dt}\right)\varphi(t)\right)^T X\left(\tfrac{d}{dt}\right)w(t)\right], \end{aligned} \tag{20}$$

which immediately yields (see [6, 13] for further developments)

Theorem 3. *For any factorization* $\Pi(\zeta,\eta) = Y^T(\zeta)X(\eta)$ *the map*

$$w \mapsto x := X\left(\frac{d}{dt}\right)w$$

is a state map.

Remark 4. Furthermore [13], $Y(\xi)$ can be seen to define a state map for the *adjoint system* of (4).

2 State maps for linear systems of partial differential equations on an unbounded spatial domain

In [7] the approach of the previous section has been extended to the case of systems described by linear *partial differential* equations, involving a time variable t, and spatial variables $z_1,\cdots,z_k$. In particular for $k = 1$ (single spatial variable) we consider systems described by linear PDEs

$$R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial z}\right)w = 0\,, \tag{21}$$

where $R(\xi,\delta) = \sum_{i,j=0}^{L} R_{ij}\xi^i\delta^j$ with ξ and δ the indeterminates, $R_{ij} \in \mathbb{R}^{p\times q}$ for $i,j = 0,\ldots,N$. An N-times differentiable (both in t and in z) solution of (21) will be called a *strong solution*. Furthermore, denote by $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^q)$ the space of locally integrable functions from $\mathbb{R}^2$ to $\mathbb{R}^q$. Then $w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^q)$ is a *weak solution* of (21) if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^\top(t,z)\left[R\left(-\frac{\partial}{\partial t}, -\frac{\partial}{\partial z}\right)^\top \varphi(t,z)\right] dt\, dz = 0 \tag{22}$$

for all infinitely-differentiable test functions $\varphi : \mathbb{R}^2 \to \mathbb{R}^p$ with compact support. The *behavior* $\mathcal{B}$ is defined as the set of weak solutions of (21) , i.e.,

$$\mathcal{B} := \{w : \mathbb{R}^2 \to \mathbb{R}^q \mid w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^q) \text{ and (21) is satisfied weakly}\} \tag{23}$$

In order to define state maps, we consider partitions $(\mathcal{S}_-,\mathcal{S}_c,\mathcal{S}_+)$ of $\mathbb{R}^2$ induced by vertical lines $t = c$, with $c \in \mathbb{R}$, as depicted in Figure 1 on the next page;

$$\begin{aligned} \mathcal{S}_- &:= \{(t,z) \in \mathbb{R}^2 \mid t < c\}, \\ \mathcal{S}_c &:= \{(t,z) \in \mathbb{R}^2 \mid t = c\}, \\ \mathcal{S}_+ &:= \{(t,z) \in \mathbb{R}^2 \mid t > c\}. \end{aligned}$$

Since the behavior described by (21) is invariant with regard to shifts in t a special

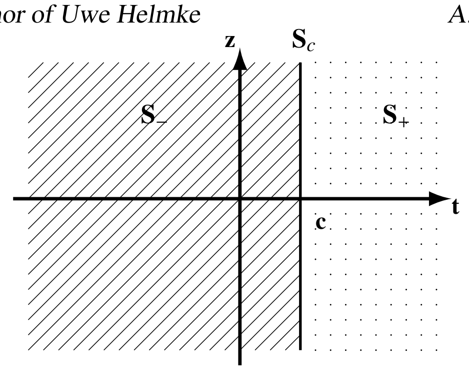

Figure 1: A partition of the plane induced by a vertical line.

role will be played by the partition $(\mathcal{S}_-, \mathcal{S}_0, \mathcal{S}_+)$ of $\mathbb{R}^2$ induced by the vertical line $t = 0$.

Let $w_1, w_2 \in \mathcal{B}$; we define the *concatenation of* w_1 *and* w_2 *along* $\mathcal{S}_0$ *as*

$$\left(w_1 \wedge_{\mathcal{S}_0} w_2\right)(t,z) := \begin{cases} w_1(t,z) & , \quad (t,z) \in \mathcal{S}_- \\ w_2(t,z) & , \quad (t,z) \in \mathcal{S}_0 \cup \mathcal{S}_+ \end{cases} .$$

We may again define an equivalence on the space of solutions.

Definition 5. $w_1, w_2 \in \mathcal{B}$ are *equivalent along* $\mathcal{S}_0$, denoted by $w_1 \sim_{\mathcal{S}_0} w_2$, if for all $w \in \mathcal{B}$:

$$\left[w_1 \wedge_{\mathcal{S}_0} w \in \mathcal{B}\right] \Leftrightarrow \left[w_2 \wedge_{\mathcal{S}_0} w \in \mathcal{B}\right] .$$

If we interpret the partition $(\mathcal{S}_-, \mathcal{S}_0, \mathcal{S}_+)$ as imposing a distinction between "past" $\mathcal{S}_-$, "present" $\mathcal{S}_0$ and "future" $\mathcal{S}_+$, the equivalence of trajectories corresponds to w_1 and w_2 admitting the same future continuations. In the $1D$ case $\mathcal{S}_- = (-\infty, 0)$, $\mathcal{S}_0 = \{0\}$ and $\mathcal{S}_+ = (0, +\infty)$, and consequently equivalence of trajectories corresponds to w_1 and w_2 bringing the system to the same state at time $t = 0$ (see [8, 13]). For the $2D$ case, a similar property to our definition of equivalence is the notion of Markovianity, see [9, 10]. Note however that in our case there is a clear distinction between the time variable t and the spatial variable z.

Under which conditions are two weak solutions w_1 and w_2 equivalent along $\mathcal{S}_0$? We will first consider this question for *strong* solutions w_1 and w_2; the general answer will then follow from the fact that the strong solutions are dense in the set of weak solutions, cf. [5] and a similar argument in [13]. Write

$$R(\xi, \delta) = \sum_{i=0}^{L} R_i(\delta)\xi^i ,$$

where $R_i \in \mathbb{R}^{p \times q}[\delta]$, and $R_L(\delta) \neq 0$. Observe that $w_i \wedge_{\mathcal{S}_0} w \in \mathcal{B}$, $i = 1, 2$, if and only if

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(w_i \wedge_{\mathcal{S}_0} w\right)^\top (t,z) \left[R\left(-\frac{\partial}{\partial t}, -\frac{\partial}{\partial z}\right)^\top \varphi(t,z)\right] dt\, dz = 0 , \qquad (24)$$

for all test functions φ. Now integrate (24) by parts with respect to t and z repeatedly till all derivatives of the function φ have disappeared. Recalling that φ is of compact support (and thus equal to zero for t and z equal to $-\infty$ and ∞), and that $R(\frac{\partial}{\partial t},\frac{\partial}{\partial z})w_i = 0$ in $(-\infty,0]\times\mathbb{R}$, $i=1,2$, it follows that $w_1 \sim_{\mathcal{S}_0} w_2$ if and only if for all compact support infinitely-differentiable test functions φ and for $i=1,2$ it holds that

$$\int_{-\infty}^{+\infty}\begin{bmatrix}\varphi(0,z)\\ \frac{\partial\varphi}{\partial t}(0,z)\\ \vdots\\ \frac{\partial^{L-1}\varphi}{\partial t^{L-1}}(0,z)\end{bmatrix}^{\top}\begin{bmatrix}\Pi_{00}(\frac{\partial}{\partial z}) & \dots & \Pi_{0,L-1}(\frac{\partial}{\partial z})\\ \Pi_{10}(\frac{\partial}{\partial z}) & \dots & \Pi_{1,L-1}(\frac{\partial}{\partial z})\\ \vdots & \dots & \vdots\\ \Pi_{L-1,0}(\frac{\partial}{\partial z}) & \dots & \Pi_{L-1,L-1}(\frac{\partial}{\partial z})\end{bmatrix}\begin{bmatrix}w_1(0,z)\\ \frac{\partial w_1}{\partial t}(0,z)\\ \vdots\\ \frac{\partial^{L-1}w_1}{\partial t^{L-1}}(0,z)\end{bmatrix}dz$$
$$=\int_{-\infty}^{+\infty}\begin{bmatrix}\varphi(0,z)\\ \frac{\partial\varphi}{\partial t}(0,z)\\ \vdots\\ \frac{\partial^{L-1}\varphi}{\partial t^{L-1}}(0,z)\end{bmatrix}^{\top}\begin{bmatrix}\Pi_{00}(\frac{\partial}{\partial z}) & \dots & \Pi_{0,L-1}(\frac{\partial}{\partial z})\\ \Pi_{10}(\frac{\partial}{\partial z}) & \dots & \Pi_{1,L-1}(\frac{\partial}{\partial z})\\ \vdots & \dots & \vdots\\ \Pi_{L-1,0}(\frac{\partial}{\partial z}) & \dots & \Pi_{L-1,L-1}(\frac{\partial}{\partial z})\end{bmatrix}\begin{bmatrix}w_2(0,z)\\ \frac{\partial w_2}{\partial t}(0,z)\\ \vdots\\ \frac{\partial^{L-1}w_2}{\partial t^{L-1}}(0,z)\end{bmatrix}dz\,, \tag{25}$$

where $\Pi_{i,j}(\frac{\partial}{\partial z})\in\mathbb{R}^{p\times q}[\frac{\partial}{\partial z}]$ for $i,j=0,\dots,L$, are certain matrix differential operators (in the spatial variable z) summarizing the remainders at $t=0$ in the repeated integration by parts procedure. (Note that since w_1,w_2 are strong solutions the remainders arising from repeated integration by parts with respect to the spatial variable z are at $-\infty$ and at ∞, and are thus equal to zero.)

Furthermore, the polynomial matrices $\Pi_{i,j}\in\mathbb{R}^{p\times q}[\delta]$ can be easily obtained from a 2*D bilinear differential form* (see [14]) obtained from $R(\xi,\delta)$. In fact, since the three-variable polynomial matrix $R(-\zeta,\delta)-R(\eta,\delta)$ is zero whenever $\zeta+\eta=0$, we can factorize

$$R(-\zeta,\delta)-R(\eta,\delta)=(\zeta+\eta)\Pi(\zeta,\eta,\delta)\,, \tag{26}$$

for some three-variable polynomial matrix $\Pi(\zeta,\eta,\delta)$. It turns out that

$$\Pi(\zeta,\eta,\delta)=\begin{bmatrix}I_p & \dots & I_p\zeta^{L-1}\end{bmatrix}\begin{bmatrix}\Pi_{00}(\delta) & \dots & \Pi_{0,L-1}(\delta)\\ \vdots & \dots & \vdots\\ \Pi_{L-1,0}(\delta) & \dots & \Pi_{L-1,L-1}(\delta)\end{bmatrix}\begin{bmatrix}I_q\\ \vdots\\ I_q\eta^{L-1}\end{bmatrix}, \tag{27}$$

where $\Pi_{ij}(\frac{\partial}{\partial z})\in\mathbb{R}^{p\times q}[\frac{\partial}{\partial z}]$ are the matrix differential operators as obtained in the integration by parts procedure.

Remark 6. Thus the remainder at $t=0$ given by the polynomial matrices $\Pi_{i,j}\in\mathbb{R}^{p\times q}[\delta]$ is obtained from $R(\xi,\delta)$ by performing the same procedure as in the previous section (for ordinary differential equations) *only with respect to* the indeterminate ξ corresponding to the time variable t.

Due to the arbitrariness of the test function φ, the following result follows [7].

Proposition 7. *Let $R \in \mathbb{R}^{p\times q}[\xi,\delta]$, and define $\mathcal{B}$ as in (23). Let $w_1, w_2 \in \mathcal{B}$; then $w_1 \sim_{\mathcal{S}_0} w_2$ if and only if*

$$\begin{aligned}&\begin{bmatrix}\Pi_{00}(\frac{\partial}{\partial z}) & \dots & \Pi_{0,L-1}(\frac{\partial}{\partial z})\\ \Pi_{10}(\frac{\partial}{\partial z}) & \dots & \Pi_{1,L-1}(\frac{\partial}{\partial z})\\ \vdots & \dots & \vdots\\ \Pi_{L-1,0}(\frac{\partial}{\partial z}) & \dots & \Pi_{L-1,L-1}(\frac{\partial}{\partial z})\end{bmatrix}\begin{bmatrix}w_1(0,z)\\ \frac{\partial w_1}{\partial t}(0,z)\\ \vdots\\ \frac{\partial^{L-1}w_1}{\partial t^{L-1}}(0,z)\end{bmatrix}\\ &=\begin{bmatrix}\Pi_{00}(\frac{\partial}{\partial z}) & \dots & \Pi_{0,L-1}(\frac{\partial}{\partial z})\\ \Pi_{10}(\frac{\partial}{\partial z}) & \dots & \Pi_{1,L-1}(\frac{\partial}{\partial z})\\ \vdots & \dots & \vdots\\ \Pi_{L-1,0}(\frac{\partial}{\partial z}) & \dots & \Pi_{L-1,L-1}(\frac{\partial}{\partial z})\end{bmatrix}\begin{bmatrix}w_2(0,z)\\ \frac{\partial w_2}{\partial t}(0,z)\\ \vdots\\ \frac{\partial^{L-1}w_2}{\partial t^{L-1}}(0,z)\end{bmatrix},\end{aligned} \tag{28}$$

where $\Pi_{ij} \in \mathbb{R}^{p\times q}[\delta]$, $i,j = 0,\dots,L$, are defined from (26)–(27).

Furthermore, the condition stated in Proposition 7 amounts to first-order representation with respect to only the time variable. Recall the definition of $\Pi(\zeta,\eta,\delta)$, and define

$$x := \begin{bmatrix}\Pi_{00}(\frac{\partial}{\partial z}) & \dots & \Pi_{0,L-1}(\frac{\partial}{\partial z})\\ \vdots & \dots & \vdots\\ \Pi_{L-1,0}(\frac{\partial}{\partial z}) & \dots & \Pi_{L-1,L-1}(\frac{\partial}{\partial z})\end{bmatrix}\begin{bmatrix}I_q\\ \vdots\\ \frac{\partial^{L-1}}{\partial t^{L-1}}I_q\end{bmatrix} w\,. \tag{29}$$

Proposition 8. *Let $R \in \mathbb{R}^{p\times q}[\xi,\delta]$, and define $\mathcal{B}$ as in (23). Let $w_1, w_2 \in \mathcal{B}$, and define correspondingly x_1, x_2 as in (29). Then*

$$w_1 \sim_{\mathcal{S}_0} w_2 \iff x_1(0,z) = x_2(0,z) \text{ for all } z \in \mathbb{R}\,.$$

Thus, x contains all information necessary to determine whether two trajectories in $\mathcal{B}$ admit the same continuation; for this reason we call x a *state* for $\mathcal{B}$, and we call the polynomial differential operator acting on w on the right of (29) a *state map* for the system of linear PDEs.

Finally, the state x defined in (29) corresponds to a description of $\mathcal{B}$ involving first-order (in time) equations in x, and zeroth-order (in time) equations in w. Observe that from (26), for every $w \in \mathcal{B}$ and corresponding x defined by (29), and every test function φ it holds that

$$\begin{bmatrix}\varphi & \dots & \frac{\partial^{L-1}\varphi}{\partial t^{L-1}}\end{bmatrix}\frac{\partial}{\partial t}x + \begin{bmatrix}\frac{\partial}{\partial t}\varphi & \dots & \frac{\partial^{L}\varphi}{\partial t^{L}}\end{bmatrix}x = \begin{bmatrix}\varphi & \dots & \frac{\partial^{L}\varphi}{\partial t^{L}}\end{bmatrix}\begin{bmatrix}R_0(\frac{\partial}{\partial z})\\ \vdots\\ (-1)^L R_L(\frac{\partial}{\partial z})\end{bmatrix} w\,.$$

Denoting with n the number of variables of the state x the above equation can be rewritten as

$$\begin{bmatrix}\frac{\partial}{\partial t}\varphi & \dots & \frac{\partial^{L}\varphi}{\partial t^{L}}\end{bmatrix}\left(\begin{bmatrix}I_n\\ 0_{p\times n}\end{bmatrix}\frac{\partial}{\partial t}x + \begin{bmatrix}0_{p\times n}\\ I_n\end{bmatrix}x + \begin{bmatrix}R_0(\frac{\partial}{\partial z})\\ \vdots\\ -(-1)^L R_L(\frac{\partial}{\partial z})\end{bmatrix}w\right) = 0\,.$$

From the arbitrariness of φ we thus conclude that

$$\begin{bmatrix} I_n \\ 0_{p\times n} \end{bmatrix} \frac{\partial}{\partial t} x + \begin{bmatrix} 0_{p\times n} \\ I_n \end{bmatrix} x + \begin{bmatrix} -R_0(\frac{\partial}{\partial z}) \\ \vdots \\ -(-1)^L R_L(\frac{\partial}{\partial z}) \end{bmatrix} w = 0 \,. \tag{30}$$

It is a matter of straightforward verification to check that by *eliminating* x in (30), the set of w-trajectories for which there exists x such that (30) holds is precisely equal to the solutions of the PDEs $R(\frac{\partial}{\partial t}, \frac{\partial}{\partial z})w = 0$; consequently we call (30) a *state representation* of $\mathcal{B}$.

3 From integration by parts to the definition of boundary variables

In the preceding section we considered linear partial differential equations on an unbounded spatial domain $z \in (-\infty, \infty)$. In many cases of interest the spatial domain is *bounded*, and there is an essential role to be played by *boundary variables*. These boundary variables are either *prescribed*, giving rise to partial differential equations with *boundary conditions*, or are the variables through which the system interacts with its environment, leading to *boundary control systems*. Note that in fact the first case (boundary conditions) can be seen to be a special case of the second case (interaction with the environment) in the sense that boundary conditions may be interpreted as corresponding to interaction with a *static* environment (e.g., an ideal constraint or a source system). In this section we will show how integration by parts leads to a natural definition of boundary variables for systems of linear partial differential equations.

Consider a set of linear partial differential equations as before, but now on a *bounded* spatial domain $[a,b]$, that is

$$R(\frac{\partial}{\partial t}, \frac{\partial}{\partial z})w = 0, \quad z \in [a,b] \tag{31}$$

We now perform the same integration by parts procedure as in the previous section, however interchanging the t and z variable, and replacing the line $t = 0$ by the two lines $z = a$ and $z = b$. Dually to the situation considered in the previous section this will correspond to the factorization

$$R(\xi, -\gamma) - R(\xi, \varepsilon) = (\gamma + \varepsilon)\Sigma(\xi, \gamma, \varepsilon) \,, \tag{32}$$

for some three-variable polynomial matrix $\Sigma(\xi, \gamma, \varepsilon)$ (i.e., we do the factorization with respect to the indeterminate δ corresponding to the spatial variable z). As before in the case of factorization with respect to ξ we thus obtain

$$\Sigma(\xi, \gamma, \varepsilon) = \begin{bmatrix} I_p & \dots & I_p \gamma^{N-1} \end{bmatrix} \begin{bmatrix} \Sigma_{00}(\xi) & \dots & \Sigma_{0,N-1}(\xi) \\ \vdots & \dots & \vdots \\ \Sigma_{N-1,0}(\xi) & \dots & \kappa_{N-1,N-1}(\xi) \end{bmatrix} \begin{bmatrix} I_q \\ \vdots \\ I_q \varepsilon^{N-1} \end{bmatrix}, \tag{33}$$

where $\Sigma_{ij}(\frac{\partial}{\partial t}) \in \mathbb{R}^{p\times q}[\frac{\partial}{\partial t}]$ equal the matrix differential operators obtained in integration by parts with respect to the spatial variable z.

Then define the vectors $w_\partial(a)(t), w_\partial(b)(t)$ (functions of time t) as

$$\begin{aligned} w_\partial(a)(t) &:= \begin{bmatrix} \Sigma_{00}(\frac{\partial}{\partial t}) & \dots & \Sigma_{0,N-1}(\frac{\partial}{\partial t}) \\ \vdots & \dots & \vdots \\ \Sigma_{N-1,0}(\frac{\partial}{\partial t}) & \dots & \Sigma_{N-1,N-1}(\frac{\partial}{\partial t}) \end{bmatrix} \begin{bmatrix} I_q \\ \vdots \\ \frac{\partial^{L-1}}{\partial z^{N-1}} I_q \end{bmatrix} w(t,a) \\ w_\partial(b)(t) &:= \begin{bmatrix} \Sigma_{00}(\frac{\partial}{\partial t}) & \dots & \Sigma_{0,N-1}(\frac{\partial}{\partial t}) \\ \vdots & \dots & \vdots \\ \Sigma_{N-1,0}(\frac{\partial}{\partial t}) & \dots & \Sigma_{N-1,N-1}(\frac{\partial}{\partial t}) \end{bmatrix} \begin{bmatrix} I_q \\ \vdots \\ \frac{\partial^{L-1}}{\partial z^{N-1}} I_q \end{bmatrix} w(t,b) \end{aligned} \tag{34}$$

We claim that the variables $w_\partial(a), w_\partial(b)$ qualify as a natural set of *boundary variables*. Indeed, they provide just enough information to *extend* a solution on the spatial domain $[a,b]$ to a *weak solution* of the same set of partial differential equations on a *larger* spatial domain $[c,d]$, with $c \le a, b \le d$. Indeed, as in the previous section for the case of the computation of the state at $t = 0$, the vector $w_\partial(a)$ provides just enough information to extend a solution $w(t,z)$ to a weak solution for values of the spatial variable z to the left of a; while the same holds for $w_\partial(b)$ with regard to extension of the solution to a weak solution for values of z to the right of b.

Example 9. Consider a system of linear conservation laws

$$\begin{aligned} \frac{\partial w_1}{\partial t}(t,z) &= -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial w_2}(w_1(t,z), w_2(t,z)) \\ \frac{\partial w_2}{\partial t}(t,z) &= -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial w_1}(w_1(t,z), w_2(t,z)) \end{aligned}$$

for a quadratic Hamiltonian density

$$\mathcal{H}(w_1,w_2) = \frac{1}{2}\begin{bmatrix} w_1 & w_2 \end{bmatrix} Q \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

with Q a symmetric 2×2 matrix, on a spatial domain $z \in [a,b]$. Note that many physical systems, including the telegrapher's equations of the dynamics of an ideal (lossless) transmission line and the equations of a linear vibrating string, are of this form, with $\int_a^b \mathcal{H}(w_1,w_2)dz$ denoting the total energy stored in the system, see [12]. Computing the boundary vectors $w_\partial(a), w_\partial(b)$ amounts to

$$w_\partial(a)(t) = \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial w_2}(t,a) \\ \frac{\partial \mathcal{H}}{\partial w_1}(t,a) \end{bmatrix} \quad w_\partial(b)(t) = \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial w_2}(t,b) \\ \frac{\partial \mathcal{H}}{\partial w_1}(t,b) \end{bmatrix}$$

These are exactly the boundary variables as defined in [12] based on physical considerations. For example, in the case of the telegrapher's equations, the variables w_1, w_2 will be the charge, respectively, flux density, while the boundary vectors

$w_\partial(a), w_\partial(b)$ will be the vector of current and voltage at $z = a$, respectively $z = b$. Clearly, these are the natural boundary variables.

Similarly, in the case of a vibrating string the vector of boundary variables at $z = a, b$ will consist of the velocity and force at these boundary points.

4 Conclusions and outlook

Although we have restricted ourselves in this paper to PDEs involving a single spatial variable z the construction of state maps given immediately extends to systems of partial differential equations involving multiple spatial variables, of the general form

$$R(\frac{\partial}{\partial t}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_k})w = 0\,. \tag{35}$$

Indeed, by factorizing

$$R(-\zeta, \delta_1, \cdots, \delta_k) - R(\eta, \delta_1, \cdots, \delta_k) = (\zeta + \eta)\Pi(\zeta, \eta, \delta_1, \cdots, \delta_k)\,, \tag{36}$$

the polynomial matrix $\Pi(\zeta, \eta, \delta_1, \cdots, \delta_k)$, written out as

$$\Pi(\zeta, \eta, \delta_1, \cdots, \delta_k) = \\ \begin{bmatrix} I_p & \ldots & I_p\zeta^{L-1} \end{bmatrix} \begin{bmatrix} \Pi_{00}(\delta_1, \cdots, \delta_k) & \ldots & \Pi_{0,L-1}(\delta_1, \cdots, \delta_k) \\ \vdots & \ldots & \vdots \\ \Pi_{L-1,0}(\delta_1, \cdots, \delta_k) & \ldots & \Pi_{L-1,L-1}(\delta_1, \cdots, \delta_k) \end{bmatrix} \begin{bmatrix} I_q \\ \vdots \\ I_q\eta^{L-1} \end{bmatrix}, \tag{37}$$

defines the state map

$$x := \begin{bmatrix} \Pi_{00}(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_k}) & \ldots & \Pi_{0,L-1}(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_k}) \\ \vdots & \ldots & \vdots \\ \Pi_{L-1,0}(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_k}) & \ldots & \Pi_{L-1,L-1}(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_k}) \end{bmatrix} \begin{bmatrix} I_q \\ \vdots \\ \frac{\partial^{L-1}}{\partial t^{L-1}} I_q \end{bmatrix} w\,. \tag{38}$$

In a similar fashion the construction of boundary variables can be extended to higher-dimensional spatial domains.

A very much challenging avenue for further research concerns the extension of the ideas put forward in this paper to *nonlinear* higher-order ordinary or partial differential equations. Some initial ideas for doing this, based on considering the variational (i.e., linearized) systems, have been proposed in [11], also drawing inspiration from some results in [2].

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