Emerging Understanding of the $\Delta I = 1/2$ Rule from Lattice QCD

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There has been much speculation as to the origin of the $\Delta I = 1/2$ rule ($\mathrm{Re}A_0/\mathrm{Re}A_2 \simeq 22.5$). We find that the two dominant contributions to the $\Delta I = 3/2$, $K \to \pi\pi$ correlation functions have opposite signs, leading to a significant cancelation. This partial cancelation occurs in our computation of $\mathrm{Re}A_2$ with physical quark masses and kinematics (where we reproduce the experimental value of A_2) and also for heavier pions at threshold. For $\mathrm{Re}A_0$, although we do not have results at physical kinematics, we do have results for pions at zero momentum with $m_\pi \simeq 420$ MeV [$\mathrm{Re}A_0/\mathrm{Re}A_2 = 9.1(2.1)$] and $m_\pi \simeq 330$ MeV [$\mathrm{Re}A_0/\mathrm{Re}A_2 = 12.0(1.7)$]. The contributions which partially cancel in $\mathrm{Re}A_2$ are also the largest ones in $\mathrm{Re}A_0$, but now they have the same sign and so enhance this amplitude. The emerging explanation of the $\Delta I = 1/2$ rule is a combination of the perturbative running to scales of $O(2\text{ GeV})$, a relative suppression of $\mathrm{Re}A_2$ through the cancelation of the two dominant contributions, and the corresponding enhancement of $\mathrm{Re}A_0$. QCD and electroweak penguin operators make only very small contributions at such scales.

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Introduction.—The "$\Delta I = 1/2$ rule" remains one of the longest-standing puzzles in particle physics. It refers to the surprising feature that in $K \to \pi\pi$ decays the final state is about 450 times more likely to have total isospin $I = 0$ than $I = 2$. In terms of the (predominantly real) $K \to \pi\pi$ amplitudes A_0 and A_2, where the suffix denotes I, this corresponds to $\mathrm{Re}A_0/\mathrm{Re}A_2 \simeq 22.5$. Perturbative running from the electroweak scale to about 1.5–2 GeV contributes a factor of approximately 2 to this ratio [1,2]; the remaining factor of about 10 should come from nonperturbative QCD or, just possibly, from new physics. Lattice QCD provides the opportunity for the nonperturbative evaluation of A_0 and A_2, although it is only very recently that such direct $K \to \pi\pi$ calculations have become feasible. In this Letter, we summarize the emerging explanation of the $\Delta I = 1/2$ rule from computations of A_0 and A_2 by the RBC and UKQCD Collaborations.

The first results from direct simulations of a kaon decaying into two pions were presented in Refs. [3–5]. The determination of A_0, where the two pions have vacuum quantum numbers, is particularly challenging, and so far it has not been calculated with physical masses and momenta. We are striving to overcome technical issues such as the efficient evaluation of disconnected diagrams and the projection of the physical state through the use of G-parity boundary conditions [6–9] in order to evaluate A_0 at physical kinematics in the near future. In the meantime, we have evaluated A_0 and A_2 for pions with masses of approximately 420 [3] and 330 MeV [10] at threshold, i.e., with the pions at rest. For these unphysical masses, we do find a significant enhancement of the ratio $\mathrm{Re}A_0/\mathrm{Re}A_2$, albeit a smaller one than 22.5 (see the first two rows of Table I). While investigating the origin of this enhancement, we found a surprising cancelation in the evaluation of $\mathrm{Re}A_2$, which significantly increases the ratio $\mathrm{Re}A_0/\mathrm{Re}A_2$. This suppression of $\mathrm{Re}A_2$ is the main result presented here.

We have also evaluated A_2 with physical masses and momenta, obtaining a result for $\mathrm{Re}A_2$ which agrees with the physical value and determining $\mathrm{Im}A_2$ for the first time [4,5] (see the third row of Table I). In the evaluation of $\mathrm{Re}A_2$ at physical kinematics, there is a similar cancelation; indeed, it is even more pronounced than at the unphysical masses in the first two rows of Table I.

In the next section, we summarize the simulations we have performed, highlighting features of immediate relevance for the $\Delta I = 1/2$ rule and referring to earlier publications for other details. We then explain the partial cancelation of the two contributions to $\mathrm{Re}A_2$, which contradicts naïve expectations from the factorization (vacuum insertion) hypothesis. We also show that these two contributions have the same sign in $\mathrm{Re}A_0$. We conclude by

TABLE I. Summary of simulation parameters and results obtained on three DWF ensembles. The errors with the Iwasaki action are statistical only; the second error for ReA_2 at physical kinematics from the IDSDR simulation is systematic and is dominated by an estimated 15% discretization uncertainty, as explained in Ref. [5].

	a^{-1}[GeV]	m_π[MeV]	m_K[MeV]	ReA_2[10^{-8} GeV]	ReA_0[10^{-8} GeV]	$\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2}$	Notes
16^3 Iwasaki	1.73(3)	422(7)	878(15)	4.911(31)	45(10)	9.1(2.1)	Threshold calculation
24^3 Iwasaki	1.73(3)	329(6)	662(11)	2.668(14)	32.1(4.6)	12.0(1.7)	Threshold calculation
32^3 IDSDR	1.36(1)	142.9(1.1)	511.3(3.9)	1.38(5)(26)	⋯	⋯	Physical kinematics
Experiment	⋯	135–140	494–498	1.479(4)	33.2(2)	22.45(6)	

explaining how these features combine to provide an emerging understanding of the $\Delta I = 1/2$ rule. Of course, a full quantitative explanation will require a calculation of ReA_0 at physical kinematics, which is underway.

Calculation of the decay amplitudes.—Our evidence is based on calculations from three domain wall fermion (DWF) ensembles with $2+1$ sea-quark flavors (see Table I). Papers [4,5] describe a complete calculation of A_2 on a 32^3 spacial lattice using the Iwasaki + dislocation suppressing determinant ratio (IDSDR) gauge action [11] for (almost) physical pion and kaon masses and realistic kinematics. The ensemble was generated at a single lattice spacing a ($a^{-1} \simeq 1.4$ GeV), chosen so that the volume is sufficiently large to accommodate the propagation of physical pions. In Ref. [3], a complete calculation of both A_0 and A_2 was carried out with the Iwasaki gauge action at $a^{-1} \simeq 1.7$ GeV for $m_\pi \simeq 422$ MeV, $m_K \simeq 737$, 878, and 1117 MeV (here, we present results for $m_K \simeq$ 878 MeV, which corresponds to almost energy-conserving decays). Although the calculation was performed at threshold, this was the first time a signal for ReA_0 had been obtained in the direct evaluation of the $K \to \pi\pi$ matrix elements. A similar threshold calculation was presented in Ref. [10] on a larger volume (24^3) with $m_\pi = 329$ MeV. The increased time extent of this lattice suppresses "around-the-world" effects in which one of the pions from the sink propagates in the forward time direction, crossing the periodic boundary and reaching the weak operator with the kaon. The calculation also used two-pion sources in which the single-pion wall sources are separated in time by a small number of time slices δ (the results presented here are for $\delta = 4$). We find that this suppresses the (unphysical) vacuum contributions in the $I = 0$ channel, significantly reducing the noise. In this way, ReA_0 was resolved using only 138 configurations, compared to 800 in Ref. [3]. With the actions used here, lattice artifacts scale parametrically as $O(a^2)$, although at present we are not in a position to take the continuum limit.

The amplitudes A_0 and A_2 can be expressed in terms of the "master formula"

$$A_I = F_I \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} \sum_{j=1}^{7} \{[z_i(\mu) + \tau y_i(\mu)] \times Z_{ij}^{\mathrm{lat}\to\overline{\mathrm{MS}}} M_j^{\Delta I,\mathrm{lat}}\} \quad (I = 0, 2). \tag{1}$$

$\tau = -V_{ts}^* V_{td}/V_{us}^* V_{ud}$ and the V_{ij} are elements of the Cabibbo-Kobayashi-Maskawa matrix. $M_i^{\Delta I,\mathrm{lat}} \equiv \langle(\pi\pi)_I|Q_i^{\mathrm{lat}}|K\rangle$ are the matrix elements calculated on the lattice. They are determined by fitting three-point correlation functions composed of a kaon source at $t = 0$, a two-pion sink at $t = \Delta$, and one of the operators Q_i^{lat} in the weak Hamiltonian inserted at all times $0 < t < \Delta$. We fit the correlation functions $C_{I,i}(\Delta, t)$

$$C_{I,i}(\Delta, t) \approx M_i^{\Delta I,\mathrm{lat}} N_{\pi\pi} N_K e^{-E_{(\pi\pi)_I}\Delta} e^{-(m_K - E_{(\pi\pi)_I})t} \tag{2}$$

for $0 \ll t \ll \Delta$, using a one parameter exponential fit to determine the matrix elements $M_i^{\Delta I,\mathrm{lat}}$. $E_{(\pi\pi)_I}$ is the energy of the two-pion channel with isospin I. All these correlation functions can be expressed in terms of the 48 contractions enumerated in Sec. 4 of Ref. [3] and labeled ① through ㊽. The contractions are functions of Δ and t, but we leave this dependence implicit, writing for example $C_{2,1}(\Delta, t) = i\sqrt{2/3}\{① + ②\}$.

The renormalization factors $Z_{ij}^{\mathrm{lat}\to\overline{\mathrm{MS}}}$ provide the connection between the bare lattice operators and those renormalized in the modified minimal subtraction with naïve dimensional reduction ($\overline{\mathrm{MS}}$-NDR) scheme at the scale μ

$$Q_i^{\overline{\mathrm{MS}}}(\mu) = Z_{ij}^{\mathrm{lat}\to\overline{\mathrm{MS}}}(\mu, a) Q_j^{\prime\mathrm{lat}}(a). \tag{3}$$

The operators Q_i on the left of (3) correspond to the conventional ten-operator "physical" basis, which is overcomplete (see, e.g., Ref. [12]). When calculating the renormalization factors, it is convenient to work in an equivalent "chiral" basis of seven linearly independent operators Q'_j with definite $SU(3)_L \times SU(3)_R$ transformation properties [see Eqs. (172)–(175) in Ref. [12]]. $z_i(\mu) + \tau y_i(\mu)$ are Wilson coefficient functions. F_I is the Lellouch-Lüscher factor relating the finite-volume Euclidean-space matrix element to the physical decay amplitude [13].

Evaluation of ReA_2.—A_2 receives contributions from the electroweak penguin (EWP) operators Q_7 and Q_8 as well as a single operator $Q_{(27,1)}^{3/2}$

$$Q_{(27,1)}^{3/2} = (\bar{s}^i d^i)_L\{(\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L\} + (\bar{s}^i u^i)_L(\bar{u}^j d^j)_L, \tag{4}$$

where the superscript 3/2 denotes ΔI and the subscript (27, 1) denotes how the operator transforms under $SU(3)_L \times SU(3)_R$ chiral symmetry. i and j are color labels,

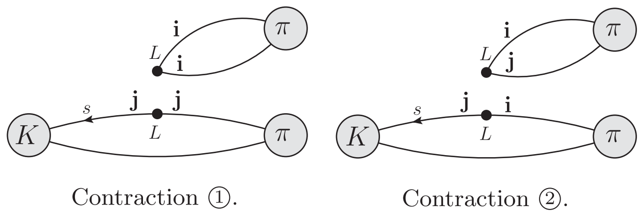

FIG. 1. The two contractions contributing to ReA_2. They are distinguished by the color summation (**i** and **j** denote color). s denotes the strange quark and L that the currents are left handed.

and the spinor indices are contracted within each pair of parentheses. The subscript L denotes *left*, so that, e.g., $(\bar{s}^i d^i)_L(\bar{u}^j u^j)_L = [\bar{s}^i\gamma^\mu(1-\gamma^5)d^i][\bar{u}^j\gamma_\mu(1-\gamma^5)u^j]$. The $\Delta I = 3/2$ components of the operators Q_1, Q_2, Q_9, and Q_{10} are all proportional to $Q^{3/2}_{(27,1)}$. From all our simulations, we confirm that the contribution from the EWP operators to ReA_2 is about 1%; e.g., for physical kinematics, we find Re$A_2 = (1.381 \pm 0.046 \pm 0.258) \times 10^{-8}$ GeV to which the EWP operators contribute -0.0171×10^{-8} GeV [4,5] (the physical value is Re$A_2 = 1.479(4) \times 10^{-8}$ GeV). We therefore neglect the EWP operators in the following discussion. Chiral symmetry implies that $Q^{3/2}_{(27,1)}$ does not mix with the EWP operators, so that ReA_2 is proportional to its lattice matrix element; the constant of proportionality is the product of the Wilson coefficient, the renormalization constant, finite-volume effects, and kinematical factors (see Ref. [5] for a detailed discussion, including an explicit demonstration that the mixing is indeed negligible in the DWF simulation).

Fierz transformations allow the $K \to \pi\pi$ correlation function of $Q^{3/2}_{(27,1)}$ to be reduced to the sum of the two contractions illustrated in Fig. 1, labeled by ① and ②. The two contractions are identical except for the way that the color indices are summed. A_2 is proportional to the matrix element extracted from the sum ① + ②. The main message of this Letter is our observation from all three simulations that ① and ② have opposite signs and are comparable in size. This is illustrated in Fig. 2 for the results at physical kinematics from Refs. [4,5], where we plot ①, −②, and ① + ② as functions of t. We extract A_2 by fitting ① + ② in the interval $t \in [5, 19]$, where there is a significant cancelation between the two terms. A similar, although not quite so pronounced cancelation occurs at threshold for physical masses and for the heavier masses studied in Refs. [3,10]; see Fig. 3, for example.

We stress that it is only the correlation function ① + ② which has a time behavior corresponding to $E_{(\pi\pi)_2}$. Because the calculation is performed in a finite volume, $E_{(\pi\pi)_2} \neq E_{(\pi\pi)_0}$ and ① and ② individually have an isospin 0 component. If $E_{(\pi\pi)_2} = m_K$, then ① + ② is independent of t away from the kaon and two-pion sources, and this is what we observe, particularly in Fig. 2, where the energies are most precisely matched.

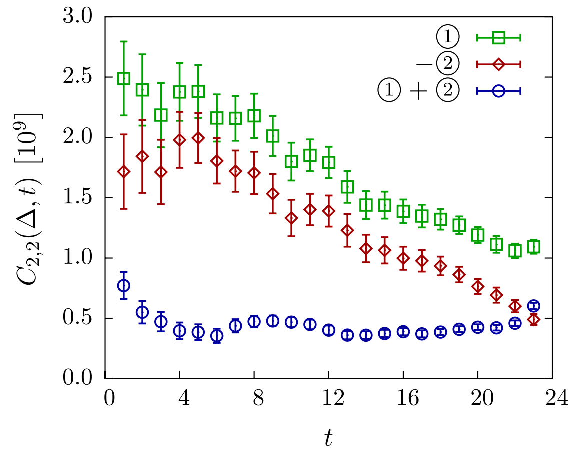

FIG. 2 (color online). Contractions ①, −②, and ① + ② as functions of t from the simulation at physical kinematics and with $\Delta = 24$.

It has been argued that the factorization hypothesis [14] works reasonably well in reproducing the experimental value of A_2 (see, e.g., Sec. VIII-4 in Ref. [15]). In this approach, the gluonic interactions between the quarks combining into different pions are neglected and A_2 is related to the decay constant f_π and the $K_{\ell 3}$ form factor close to zero-momentum transfer. On the basis of color counting, one might therefore expect that ② ≃ 1/3①, whereas, for physical kinematics, we find ② ≃ −0.7① and that nevertheless ① + ② leads to the correct result for A_2. Thus, the expectation based on the factorization hypothesis proves to be unreliable here.

Following the discovery that ① and ② have opposite signs, we examined separately the two contributions to the matrix element $\langle \bar{K}^0 | (\bar{s}d)_L(\bar{s}d)_L | K^0 \rangle$, which contains the nonperturbative QCD effects in neutral kaon mixing [11]. The two contributions correspond to Wick contractions in which the two quark fields in the K^0 interpolating

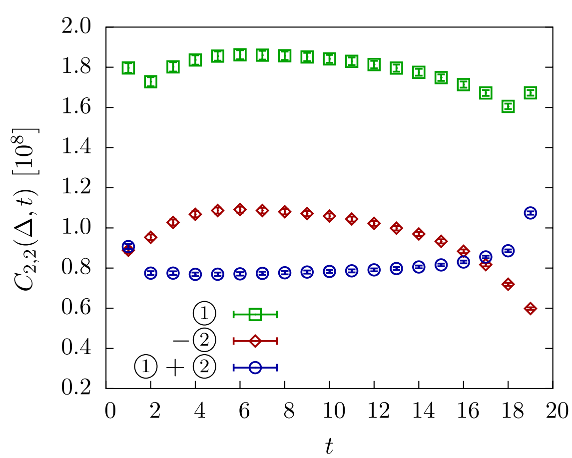

FIG. 3 (color online). Contractions ①, −②, and ① + ② as functions of t from the simulation at threshold with $m_\pi \simeq$ 330 MeV and $\Delta = 20$.

operator are contracted (i) with fields from the same current in $(\bar{s}d)_L(\bar{s}d)_L$ and (ii) with one field from each of the two currents. Color counting and the vacuum insertion hypothesis suggest that the two contributions come in the ratio 1:1/3, whereas we find that in QCD they have the opposite sign. This had been noticed earlier; see e.g., [16] and references therein.

We postpone a discussion of the implications of these results to the $\Delta I = 1/2$ rule until the next section, but we believe that the partial cancelation observed in the evaluation of A_2 is a significant component.

Evaluation of ReA_0.—The evaluation of A_0 at physical kinematics has not yet been completed. The results presented here are obtained at threshold, with the two pions in their zero-momentum ground state with each pion at rest up to finite-volume effects. Even at threshold, we have had to overcome many theoretical and technical problems, including the evaluation of the 48 contractions contributing to the correlation functions, the renormalization of the operators in the effective Hamiltonian, the subtraction of power divergences, and the evaluation of the finite-volume corrections. The threshold calculations do not require, however, the isolation of an excited state. The pions in a physical decay each have a nonzero momentum in the center-of-mass frame, which corresponds to an excited state in lattice calculations. Given the poor statistical signals after the subtraction of power divergences and the evaluation of disconnected diagrams, the evaluation of A_0 at physical kinematics is currently impracticable with standard techniques and is the main motivation for our development of G-parity boundary conditions [6–9].

With the two pions at threshold, we find [3,10]

$$\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} = \begin{cases} 9.1(2.1) & \text{for } m_K = 878\ \text{MeV},\quad m_\pi = 422\ \text{MeV} \\ 12.0(1.7) & \text{for } m_K = 662\ \text{MeV},\quad m_\pi = 329\ \text{MeV}. \end{cases} \tag{5}$$

While these results differ significantly from the observed value of 22.5, because the calculations are not performed at physical kinematics, there is nevertheless already a significant enhancement in the ratio and it is interesting to understand its origin. In Table II, we present the contributions to ReA_0 from each of the lattice operators in the 24^3 simulation with $a^{-1} = 1.73(3)$ GeV and from each $\overline{\mathrm{MS}}$-NDR operator at a renormalization scale 2.15 GeV. In both cases, the dominant contribution comes from the current-current operators Q_2.

Since, in a finite volume, $E_{(\pi\pi)_2} \neq E_{(\pi\pi)_0}$, one cannot satisfy the condition $m_K = E_{\pi\pi}$ for both isospin channels simultaneously with the same quark masses. Here, we quote results using the fixed meson masses quoted in Eq. (5), which is sufficient for our current discussion. For these masses $E_{(\pi\pi)_0} = 766(29)$ MeV [629(15) MeV] and $E_{(\pi\pi)_2} = 876(15)$ MeV [668(11) MeV] for the 16^3 (24^3) lattice. A study that interpolates in the kaon mass to make both decays energy conserving may be found in Ref. [3].

TABLE II. Contributions from each operator to ReA_0 for $m_K = 662$ MeV and $m_\pi = 329$ MeV. The second column contains the contributions from the seven linearly independent lattice operators with $1/a = 1.73(3)$ GeV and the third column those in the ten-operator basis in the $\overline{\mathrm{MS}}$-NDR scheme at $\mu = 2.15$ GeV. The numbers in parentheses represent the statistical errors.

i	Q_i^{lat} [GeV]	$Q_i^{\overline{\mathrm{MS}}\text{-NDR}}$ [GeV]
1	$8.1(4.6)\times10^{-8}$	$6.6(3.1)\times10^{-8}$
2	$2.5(0.6)\times10^{-7}$	$2.6(0.5)\times10^{-7}$
3	$-0.6(1.0)\times10^{-8}$	$5.4(6.7)\times10^{-10}$
4	$\cdots$	$2.3(2.1)\times10^{-9}$
5	$-1.2(0.5)\times10^{-9}$	$4.0(2.6)\times10^{-10}$
6	$4.7(1.7)\times10^{-9}$	$-7.0(2.4)\times10^{-9}$
7	$1.5(0.1)\times10^{-10}$	$6.3(0.5)\times10^{-11}$
8	$-4.7(0.2)\times10^{-10}$	$-3.9(0.1)\times10^{-10}$
9	$\cdots$	$2.0(0.6)\times10^{-14}$
10	$\cdots$	$1.6(0.5)\times10^{-11}$
ReA_0	$3.2(0.5)\times10^{-7}$	$3.2(0.5)\times10^{-7}$

The dominant contribution from the lattice operator Q_2 to the $\Delta I = 1/2$ correlation function is proportional to the contractions 2①− ② and corresponds to *type 1* diagrams in the language of Ref. [3] (see Fig. 3 in Ref. [3]). In Fig. 4, we show the total contribution of Q_2 to the correlation function, as well as the total connected contribution and that of type 1 diagrams given by $\frac{i}{\sqrt{3}}\{2①-②\}$. The errors on the total contribution are dominated by the disconnected diagrams. The observation that ① and ② have opposite signs leads to an enhancement between the two terms rather than the suppression in the factorization approximation $② = \frac{1}{3}①$. Similarly, in the case of Q_1, the type 1 combination $\frac{i}{\sqrt{3}}\{2②-①\}$ is dominant. In this case, both the correlation function and the Wilson coefficient

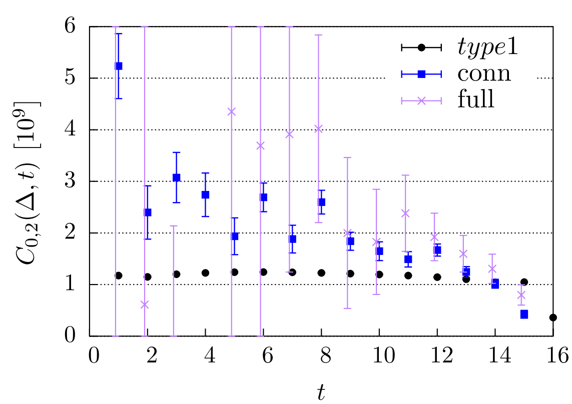

FIG. 4 (color online). Contributions of Q_2^{lat} to ReA_0 (purple crosses). The blue squares and black circles denote the connected and type 1 contractions, respectively.

$z_1(\mu) + \tau y_1(\mu)$ are negative, so that the overall contribution adds to that from the correlation function of Q_2.

Finally, we note that in our data ReA_2 shows a much stronger mass dependence than ReA_0, which was also expected in $SU(2)$ chiral perturbation theory [17]. We attribute this to the partial cancelation between ① and ② in ReA_2. Our results for ReA_2 and ReA_0 are given in Table I.

Conclusions.—From our recent computations of $K \to \pi\pi$ decay amplitudes, a likely explanation of the $\Delta I = 1/2$ rule is emerging. In particular, we find that, in the evaluation of ReA_2, which is proportional to the sum of two contractions ①+ ②, there is a significant cancelation between the two terms. The naïve expectation based on the factorization hypothesis suggests that ② $\approx \frac{1}{3}$①, whereas in QCD we find that they have the opposite sign. (The two terms contributing to B_K similarly have opposite signs, contradicting expectations from the vacuum insertion approximation.)

The evaluation of A_0 at physical kinematics has not yet been performed. Our simulations at threshold with $m_\pi =$ 329 and 422 MeV show that the dominant contributions to A_0 come from the current-current operators, with only small corrections from the penguin operators. This is true whether we express the results in terms of the bare lattice operators at $a^{-1} = 1.73$ GeV or the $\overline{\text{MS}}$-NDR renormalized operators at $\mu = 2.15$ GeV (see Table II). Although 48 contractions contribute to the $I = 0$ correlation function, in our simulations, the largest contributions again come from contractions ① and ② with relative signs which enhance ReA_0.

References to estimates of the amplitudes using analytic or model approximations are presented in the reviews [18,19]. We note that a suppression of ReA_2 and an enhancement of ReA_0 was found in Ref. [20] using the $1/N$ expansion with a particular ansatz for matching the short- and long-distance factors at scales 0.6–0.8 GeV.

The results presented above indicate that ReA_2 is very sensitive to the choice of quark masses and momenta, a sensitivity we attribute to the partial cancelation of the two contributing contractions. On the other hand, there is no such cancelation in ReA_0, and indeed the results depend much less on the masses, and the values we find are already close to the experimental result. Of course, before we can claim to understand the $\Delta I = 1/2$ rule quantitatively, we need to reproduce ReA_0/Re$A_2 = 22.5$ at physical quark masses and kinematics and in the continuum limit, and we are currently undertaking this challenge. Nevertheless, from the results and discussion of this Letter, it appears that, in addition to the well known perturbative enhancement of ReA_0/ReA_2, the explanation is a combination of a significant relative suppression of ReA_2 as well as some enhancement of ReA_0, with penguin operators contributing very little.

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