Looking at Computations from a Different Angle

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Abstract. We present a novel framework to reason about programs based on encodings of computations as graphs. The main insight here is to rearrange the programs such that given a bound k, each computation can be explored according to any tree decomposition of width k of the corresponding behaviour graph. This produces under-approximations parameterized on k, which result in a complete method when we restrict to classes of behaviour graphs of bounded tree-width. As an additional feature, the transformation of the input program can be targeted to existing tools for the analysis, and thus, off-the-shelf tools based on fixed-point, or capable of analyzing sequential programs with scalar variables and nondeterminism, can be used.
To illustrate our approach, we develop this framework for sequential programs and discuss how to extend it to handle concurrency. For the case of sequential programs, we develop a compositional approach to generate on-the-fly tree decompositions of nested words, which is based on graph-summaries. To illustrate our technique, we also implement our algorithms for C programs.

1 Introduction

Program computations are typically described as runs of flat transitions systems with possibly infinite states. The basic information stored in a state is the current control location and the valuation of the statically allocated variables. Depending on the class of programs, a state can also store heap structures if the program uses dynamic memory allocation, and the call stack in presence of recursive calls, and in general, additional *data structures* to handle concurrency (multiple call stacks, and/or FIFO channels, etc.).

Computations can be represented also as graphs (*behaviour graphs*) where the nodes capture the basic (finite) information and different types of edges are used to capture the transitions and the relations deriving from the use of the additional data structures (see [17]). For example, a stack can be captured by linking the pair of states corresponding to a push and a matching pop, a queue by linking an enqueue to a matching dequeue, and so on.

Depending on the aspects and the granularity of the information we wish to capture, several classes of behaviour graphs can be defined. Nested words naturally capture the control-flow structure of sequential programs [1], multiply nested words that of shared-memory multi-threaded programs and stack-queue graphs that of both distributed programs with recursive calls and sequential programs with queues and stacks [17], and more definitions are possible.

A very general result on the decidability of problems on classes of behaviour graphs is the decidability of MSO for all MSO-definable classes of graphs of bounded tree-width [17], which generalizes to classes of graphs Courcelle/Seese's theorem [6, 20]. For interesting classes of programs, most of the decidability results of relevant decision problems in verification, such as reachability, model-checking and decidability of logics, are indeed subsumed by this general result.

A class of graphs has tree-width k if for each graph there is a tree decomposition of *width* at most $k+1$, that is, the graph can be rearranged on a tree by assigning to each node a set of at most $k+1$ graph vertices (*bag*) such that all vertices and edges are covered and vertices replicated in two nodes also belong to all the bags on the path connecting them. Essentially, for a behaviour graph G, a tree decomposition T of width k ensures that we can execute the computation described by G by checking the consistency of the information at each vertex (i.e., its program counter and variable valuation) locally to a node and its neighbors: in fact, to check the consistency across the edges of G, it is sufficient to consider one bag at a time, and to ensure that each vertex has the same information in any bag, it is sufficient to check this among the bags of a node and its children.

This way of looking at the tree decompositions is the crucial intuition of the approach we present in this paper. We design a general framework for analyzing programs where given a parameter k, we transform an input program P such that the resulting encoding P' interprets the behaviours of P as described above, according to all the tree decompositions of width k of P behaviour graphs, and then P' is analyzed in a separate tool.

Our approach gives a novel and natural way of representing and analyzing systems and has several other features. First, each tree decomposition rearranges the transitions of a computation and gets a way to explore them in a totally independent order, and thus our approach is likely to be less sensitive to "pathological patterns". Second, the width of a tree decomposition gives a natural parameter for bounding the additional storage needed to explore the program computations, and thus we get under-approximation methods of adjustable precision for arbitrary classes of systems. Third, and probably more importantly, we get a general way to encode unbounded heap structures captured in configurations into a *thick* tree where we need to associate a fixed amount of data stored at each node. The tree encodings of computation graphs will allow us to encode complex features of programs (recursion, concurrency, heap, etc.) in a uniform way where we can exploit off-the-shelf solutions for computing the fixed point of finite relations, or sequential programs that handle only recursion, to analyze these otherwise complex systems.

We develop our framework for sequential programs with recursive procedure calls and use as behaviour graphs the nested words augmented with the program counters and the variable valuations (*program nested words*). A crucial aspect is to find an efficient way to generate tree decompositions for this class of graphs. For this, we introduce the concept of *shape of a program nested word* (pnw-shape) to summarize portions of program nested words. Namely, a pnw-shape is either a fragment of a nested word (*ground pnw-shape*), or a *merge* of two "compatible" pnw-shapes, or a *contraction* of a pnw-shape on a set of vertices. By compatible we essentially mean that the pnw-shapes can share nodes but edges do not overlap, and the contraction has the effect of keeping only the vertices in the contraction set and projecting on them the edges of the starting pnw-shape. Essentially, in the construction of a tree decomposition, we use these pnw-shapes to summarize the information of the portion of the nested word covered by the nodes of a subtree. In particular, we start from the leaves that are labeled each with a ground pnw-shape. At each internal node v, we add a ground pnw-shape which is compatible with the pnw-shape of the children of v, and compute the pnw-shape of v by first merging these three pnw-shapes and then contracting the resulting pnw-shape on the vertices of the ground one (which form the bag of v). We give a test for the root to ensure that the constructed tree is indeed a tree decomposition.

To illustrate the approach, we implement a tool that performs a code-to-code translation of C programs that do not make use of dynamic memory allocation. The output of the transformation is a program that essentially builds a tree decomposition using recursive procedure calls as in a DFS traversal of the tree. The ground program pnw-shapes are nondeterministically guessed, and the consistency of program counters and variable valuations associated to each vertex is checked for each edge (according to the semantics of the input program). Moreover, there are procedures to implement the tests, and the merging and contraction operations. Our implementation is based on the same framework we have used for CSeq [8] that translates concurrent C programs with the pthread library to standard C programs, that are then analyzed with bounded model checking tools for C.

The resulting tool is just a prototype and is essentially meant to experiment in practice our methodology on a simple class of programs. This tool is a first core of a more general tool that we plan to implement and has in it all the features to handle the sequential aspects in more complex classes of programs.

As a further contribution, we give an informal though detailed description on how this approach can be extended to handle concurrent shared-memory programs and how this relates to the sequentialization algorithms (see [19, 16, 11, 7, 3] for a sample research). We believe that our approach can be extended to many other classes of programs and systems, such as concurrent programs with a weak memory model assumption, distributed programs, and programs with dynamic data-structures, to mention some.

2 Programs with recursive procedure calls

We consider sequential programs with possibly recursive procedure calls. For the sake of simplicity and without loss of generality, we omit local variables and procedure parameters (in a procedure call, when needed, the values are passed through the global variables). Since we only admit global variables, henceforth we refer to them simply as variables.

```
Var x, y;                          procedure boo begin               procedure foo begin
                                     3: y := x;                        8: if (y > 0) then
procedure main begin                 4: call foo;                      9:     y := y − 1;
  0: assume(x=1 || x=2);             5: assert(x=1);                   A:     call foo;
  1: call boo;                       6: call foo;                      B: else skip;   fi
  2: return;                         7: return;                        C: return;
end                                end                               end
```

Fig. 1. A sample program.

In the rest of the paper, we use program P of Fig. 1 as a running example. P is a simple program with three possible behaviours depending on the initial values of the variable x being 1, 2 or an other value. In the last case, the condition of the `assume` statement does not hold and thus the computation immediately halts. In the remaining cases, the procedures *boo* and *foo* gets recursively called until the `assert` statement at program counter 5 is reached. Now, a computation with $x = 2$ violates the assertion, and thus reaches an *error* state, while a computation with $x = 1$ continues through the end of the procedure `main`.

Syntax. The BNF grammar on the right gives the formal syntax of programs. A program starts with the declaration of a finite set of variables *Var* that are visible to all procedures. We assume variables range over some (potentially infinite) data domain $\mathbb{D}$, a language of expressions $\langle expr \rangle$ interpreted over $\mathbb{D}$, and a language of predicates $\langle pred \rangle$ over the variables. Thereafter, there is a declaration of a non empty list of *procedures*, among which one called `main` that is initially executed to start the program. Each procedure is formed by a nonempty sequence of labeled statements of the form $pc : \langle stmt \rangle$ where pc is the *program counter* (or *program location*) and $\langle stmt \rangle$ defines a simple language of C-like statements. We assume that each procedure has `return` as last statement.

```
⟨prgm⟩ ::=  Var; ⟨proc⟩+
⟨proc⟩ ::=  procedure p begin ⟨pc_stmt⟩+ end
⟨pc_stmt⟩ ::=  pc : ⟨stmt⟩;
⟨stmt⟩ ::=  g :=⟨expr⟩ | skip
   |  assume(⟨pred⟩) | assert(⟨pred⟩)
   |  if ⟨pred⟩ then ⟨pc_stmt⟩+ else ⟨pc_stmt⟩+ fi
   |  while ⟨expr⟩ do ⟨pc_stmt⟩ do
   |  call p | return
```

For a program P, we denote with PC (resp., *Call*, *Ret*) the set of all program counters pc such that $pc : stmt$ (resp., $pc :$ `call` p, $pc :$ `return`) is a labeled statement of P. Furthermore, for every $pc \in Call$ we denote with $afterCall(pc)$ the (unique) program counter pc' such that $pc' : stmt$ is the statement that is executed after returning the procedure call with program counter pc.

Semantics. The semantics is given as a transition system. Each program can make procedure calls and manipulate variables. Thus, a state is a *configuration* of the form $\langle \nu, pc, St \rangle$ where ν is a valuation of the variables (i.e., $\nu : Var \mapsto \mathbb{D}$), $pc \in PC$ is a program counter, and St is the content of the call stack (i.e., the control locations of the pending procedure calls). A configuration $C = \langle \nu, pc, St \rangle$ is *initial* if pc is the program counter of the first statement of the procedure `main` and St is the empty stack.

The *transition relation*, denoted $\hookrightarrow$, is defined as usual. The control-flow statements update the program counter, possibly depending on a predicate (condition). The assignment statements update the variable valuation other than moving to the next program counter. At a procedure call, the current location of the caller (*pc*) is pushed onto the stack, and the control moves to the first location of the called procedure. At a return statement the control location at the top of the stack is popped, say *pc*, and the control moves to location *afterCall*(*pc*).

A *computation* of a program is a sequence of configurations $C_0 C_1 \ldots C_n$ such that C_0 is initial, and $C_{i-1} \hookrightarrow C_i$ for every $i \in [1, n]$.

3 Graphs representing program executions

In this section, we recall some definitions on graphs and define the notion of program nested word that we use in the rest of the paper to represent the executions of a program.

Σ-labeled graphs. Fix a finite alphabet Σ of relation names. A Σ-labeled graph is a structure $G = (V, \{E\}_{E\in\Sigma})$, where V is a finite set of vertices, and for each $E \in \Sigma$, $E \subseteq V \times V$ is a set of E-labeled directed edges. An edge $(u, v) \in E$ is also denoted as uEv. A graph $G = (V, E)$ is a *line graph* if there is an ordering of all vertices of G, say $v_0, v_1, \ldots, v_n$, such that $E = \{v_{i-1}Ev_i \mid i \in [1, n]\}$.

Nested words. A *nested word*[3] is a labeled graph $(V, \rightarrow, \curvearrowright)$ where $(V, \rightarrow)$ is a line graph and $\curvearrowright$ is a matching edge relation such that for every $u, v, u', v' \in V$:

- if $u \curvearrowright v$ then $u \rightarrow^+ v$;
- if $u \curvearrowright v$, $u' \curvearrowright v'$ are distinct edges then (1) u, v, u', v' are all distinct nodes, and (2) if $u \rightarrow^+ u'$ then either $v \rightarrow^+ u'$ or $v' \rightarrow^+ v$.

Program nested words. We wish to look at the computations of a program via their behaviour graphs, i.e., finite graphs that carefully model with their edges the control-flow structure. In particular, we use as behaviour graphs the nested words annotated with the program counter (*pc*, for short) and the valuation of the variables of each *state*. We refer to such annotated nested words as *program nested words*. In Fig. 2, we give the behaviour graph of a computation of the program from Fig. 1 when $x = 1$ holds. The vertices of the nested word $v_0, \ldots, v_{17}$ are labeled with the corresponding *pc* in the program. Also, in the figure, we report the variable valuation at the beginning and update it at each node after an assignment. Moreover, the $\curvearrowright$-edges are represented as curved arrows and the $\rightarrow$-edges are represented as straight arrows. The $\rightarrow$-edges capture the linear ordering of the program states in the computation (and thus the *transitions* of the computation). The $\curvearrowright$-edges match the vertices corresponding to the *pc* of a call to a procedure to the *pc* of the next statement of the procedure that will be executed after returning the call (*return location*). Consider for example $v_1 \curvearrowright v_{17}$, v_1 corresponds to the state that precedes the call to *boo* from `main` (with *pc* 1) and v_{17} corresponds to the state after returning from this call (with *pc* 2); also we have $v_1 \rightarrow v_2$, and v_2 corresponds to the begin of the first activation of *boo* (with *pc* 3).

[3] We assume that there are no unmatched calls and returns, differently from [1].

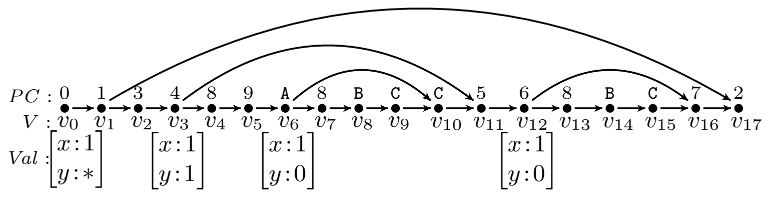

PC : 0 1 3 4 8 9 A 8 B C C 5 6 8 B C 7 2
V : v0 v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12 v13 v14 v15 v16 v17
Val : [x:1, y:*] [x:1, y:1] [x:1, y:0] [x:1, y:0]

Fig. 2. Program nested word of a run of the program in Fig. 1.

Below, we give a logical characterization of program nested words. For the ease of presentation we assume that all the procedure calls in the computations are returned. Note that this is without loss of generality, since we can always append a possibly empty sequence of additional transitions (which are not actual program transitions and thus can be recognized as such) to match all pending calls in the call stack.

Definition 1 (Program Nested Word). *A* program nested word *of a program P with set of variables Var and set of program counters PC, is a tuple $(nw, \mathit{Val}, \overline{pc})$ where*

- *$nw = (V, \to, \curvearrowright)$ is a nested word; let $V = \{v_0, v_1, \ldots, v_n\}$ such that $v_{i-1} \to v_i$ for $i \in [1, n]$;*
- *Val and $\overline{pc}$ are labeling functions that map each vertex of V respectively with a valuation of Var and a program counter in PC such that $\overline{pc}(v_0)$ is the program counter of the first statement of procedure* `main` *and for $u, v, z \in V$:*
  - *if $u \to v$ then $\langle \mathit{Val}(u), \overline{pc}(u), st\rangle \hookrightarrow \langle \mathit{Val}(v), \overline{pc}(v), st'\rangle$, for some st, st';*
  - *if $u \curvearrowright v$ then $\overline{pc}(u) \in \mathit{Call}$, $\overline{pc}(v) = \mathit{afterCall}(\overline{pc}(u))$, and $\overline{pc}(z) \in \mathit{Ret}$ where $z \to v$;*
  - *if $\overline{pc}(u) \in \mathit{Call}$, then $u \curvearrowright v$ exists;*
  - *if $u \to v$ and $\overline{pc}(u) \in \mathit{Ret}$, then $z \curvearrowright v$ exists.* □

From executions to program nested words and back: Let $\pi = C_0 C_1 \ldots C_n$ be a computation of P, where $C_i = \langle \nu_i, pc_i, St_i\rangle$ for $i \in [0, n]$. For each $i \in [0, n]$ with $pc_i \in \mathit{Call}$, we say that i *matches* j if j is the smallest index such that $j > i$ and $St_j = St_i$. We denote with $NW(\pi)$ the tuple $(nw, \mathit{Val}, \overline{pc})$ where $nw = (\{v_0, \ldots, v_n\}, \to, \curvearrowright)$ is a nested word such that (1) $v_{i-1} \to v_i$ for $i \in [1, n]$, (2) $v_i \curvearrowright v_j$ iff i matches j in π, and (3) $\mathit{Val}(v_i) = \nu_i$ and $\overline{pc}(v_i) = pc_i$, for $i \in [0, n]$. We can show that $NW(\pi)$ is indeed a program nested word of P.

Vice-versa, consider a program nested word $pnw = (nw, \mathit{Val}, \overline{pc})$ of P and let $\{v_0, \ldots, v_n\}$ be the set of vertices of nw such that $v_{i-1} \to v_i$ for $i \in [1, n]$. We denote with $RUN(pnw)$ the sequence of configurations $C_0 C_1 \ldots C_n$ where denoting $C_i = \langle \mathit{Val}(v_i), \overline{pc}(v_i), St_i\rangle$, St_0 is the empty stack and for $i \in [1, n]$: (1) if $\overline{pc}(v_{i-1}) \in \mathit{Call}$ then $St_i = \overline{pc}(v_{i-1}).St_{i-1}$ (procedure call), (2) if $v_j \curvearrowright v_i$ then $St_{i-1} = \overline{pc}(v_j).St_i$ (return from a call), (3) otherwise $St_i = St_{i-1}$ (internal move). We can show that $RUN(\pi)$ is indeed a computation of P.

Thus the following holds:

Theorem 1. *Given a program P there is a one-to-one mapping (modulo a vertex renaming) between the computations and the program nested words of P.*

4 Bounded tree-width analysis of programs

The main intuition behind our methodology is to use the tree decompositions of the behaviour graphs of a program, to guide the exploration of its computations.

Informally, a *tree decomposition* of a graph G is a binary tree whose nodes are labeled with sets of G vertices, which are called *bags*, such that every edge or vertex of G is covered by at least one bag, and if a vertex v belongs to two bags labeling two different nodes then all the bags on the unique path connecting such nodes also contain v. Fig. 3(a) gives a tree decomposition of the program nested word of Fig. 2 where each bag is implicitly defined by the vertices of the graph that labels the node. A formal definition of tree decomposition is given at the end of this section.

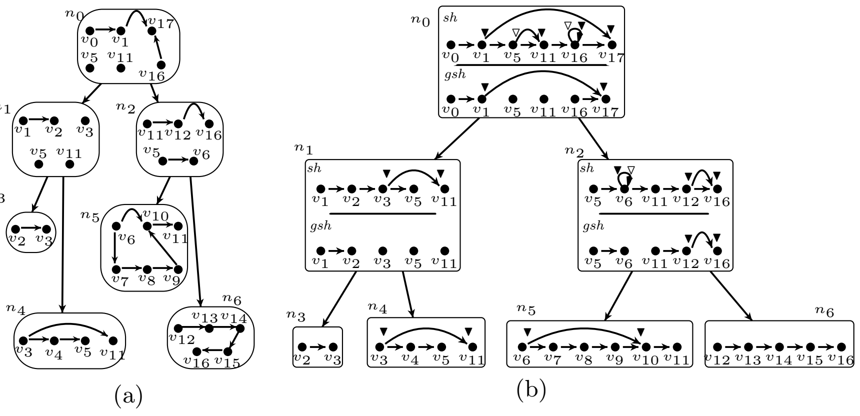
We illustrate the role played by tree decompositions in our methodology on the sample program nested word of Fig. 2 and with respect to the above mentioned tree decomposition.

We augment the tree decompositions by adding to each node some edges of the graph such that each edge is mapped exactly to a node whose bag contains both of its endpoints (note that such a labeling is always possible since by definition each edge is covered by at least one bag). The tree decomposition of Fig. 3(a) is augmented in such a way.

Now, by assuming that we have an augmented tree decomposition for a program nested word of a program P (we will discuss in the next section how to generate efficiently such tree decompositions), we check that a labeling of each vertex in each bag with a program counter and a variable valuation of P forms a computation of P: that is, we can reconstruct a program nested word of P from the additional labeling and the tree decomposition.

Starting from the leaves, we locally check the consistency of the transitions captured by the edges in the bag. For example, in node n_5, we can check for the consistency of the program counters and variable evaluations of the vertices $v_6, v_7, v_8, v_9, v_{10}, v_{11}$ according to (1) the transitions of P corresponding to the $\rightarrow$-edges (v_6, v_7), (v_7, v_8), (v_9, v_{10}) and (v_{10}, v_{11}) and (2) the $\curvearrowright$-edge (v_6, v_{10}). Then, moving up to the parent of n_5, i.e., n_2, we do not need to keep the information for v_7, v_8, v_9, v_{10}, since all the edges involving them have already been examined (this is carefully captured by the tree decomposition that does not contain these vertices in the bag of n_2). However, we need to check that the program counter and the variable valuation associated with the vertices that are kept, i.e., v_6 and v_{11}, are the same as in n_2 and n_5.

In this use of tree decompositions, a bag associated with a node n is the *interface*, or the sticking vertices, of the portion of nested word corresponding to the subtree rooted at n with the rest of the computation. Thus, if we restrict to tree decompositions with bags of size bounded by a parameter k (the width), at each node we only need to track $O(k)$ information. By choosing $k \geq 3$, for the class of programs that we have defined, we can explore the whole set of computations of a program, as stated by Theorem 2 at the end of this section.



n_0
v_0 v_1 v_{17} v_5 v_{11} v_{16}
n_1
v_1 v_2 v_3 v_5 v_{11}
n_2
v_{11} v_{12} v_{16} v_5 v_6
n_3
v_2 v_3
n_5
v_{10} v_6 v_{11} v_7 v_8 v_9
n_4
v_3 v_4 v_5 v_{11}
n_6
v_{13} v_{14} v_{12} v_{16} v_{15}
(a)
sh
v_0 v_1 v_5 v_{11} v_{16} v_{17}
gsh
v_0 v_1 v_5 v_{11} v_{16} v_{17}
sh
v_1 v_2 v_3 v_5 v_{11}
gsh
v_1 v_2 v_3 v_5 v_{11}
sh
v_5 v_6 v_{11} v_{12} v_{16}
gsh
v_5 v_6 v_{11} v_{12} v_{16}
v_2 v_3
v_3 v_4 v_5 v_{11}
v_6 v_7 v_8 v_9 v_{10} v_{11}
v_{12} v_{13} v_{14} v_{15} v_{16}
(b)

Fig. 3. Example of a tree decomposition (a) and a nw-shape tree (b) for the program nested word of Fig. 2.

Tree decompositions and tree-width. A *tree-decomposition* of a Σ-labeled graph $G = (V, \{E\}_{E\in\Sigma})$ is a pair $D = (T, \{bag_t\}_{t\in N})$, where T is a binary tree with set of nodes N, and $bag_t \subseteq V$ satisfying the following:

- for every $v \in V$, there is a node $t \in N$ such that $v \in bag_t$;
- for every $E \in \Sigma$ and $(u, v) \in E$, there is a node $t \in N$ such that $u, v \in bag_t$;
- if $u \in (bag_t \cap bag_{t'})$ then $u \in bag_{t''}$ for every T node t'' that lies on the unique path connecting t to t' in T.

The *width* of a tree decomposition $(T, \{bag_t\}_{t\in N})$ is the size of the largest bag in it, minus one; i.e. $max_{t\in N}\{|bag_n|\} - 1$. The *tree-width* of a graph is the *smallest* of the widths overall its tree decompositions.

Theorem 2 ([17]). *Any nested word has tree-width at most* 2.

5 Getting tree decompositions for program nested words

In the description of our approach from the previous section, we assume that a tree-decomposition of a program nested word is given. Indeed, we compute such decompositions on the fly. For this, we need to carry in our *summaries* (at each node) some structural information about the portion of the nested word so far explored, such that we can correctly check portions of nested words separately, then combine them in the same structure, and in the end claim that we have constructed a valid tree decomposition for a computation of the program.

The informal scenario. Our notion of *summary* for nested words is a shape of a nested word, *nw-shape* for short. Informally, a nw-shape is either a fragment of a nested word (*ground nw-shape*), or a *merge* of two "compatible"nw-shapes, or a *contraction* of a nw-shape on a set of vertices. By compatible we essentially mean that the nw-shapes can share nodes but the edges do not overlap. For example, each of the graphs labeling the nodes of the tree in Fig. 3(b) is a nw-shape.

In addition to the notation used for nested words, we also use the symbols $\blacktriangledown$ and $\triangledown$ to annotate respectively that an end-point of a $\curvearrowright$-edge is the actual one or it replaces one that has been abstracted away. In particular, for a ground nw-shape all the endpoints of a $\curvearrowright$-edge are marked with $\blacktriangledown$.

To generate a tree decomposition, we construct a nw-shape tree. In Fig. 3, we give a nw-shape tree for the nested word from Fig. 2. We start from the leaves that are assigned each with a ground nw-shape. For each internal node n, we add a ground nw-shape (marked with *gsh* below the line in each node) and compute a summary of the ground nw-shapes in the subtree rooted at n (marked with *sh* above the line). The summary is computed by merging the summaries at the children and the ground nw-shape of the current node, and then contracting the resulting nw-shape on the vertices of the ground nw-shape (note that at each node the nw-shape and the ground nw-shape have the same set of vertices).

For example, consider the node n_1. The nw-shapes from nodes n_3 and n_4 just share the vertex v_3 and therefore they can be merged (in fact they are compatible since when glued on v_3 no pair of edges overlaps). Similarly, the resulting nw-shape can be merged with the ground nw-shape of n_1, and the resulting nw-shape has vertices v_1, v_2, v_3, v_4, v_5 and v_{11}. In the contraction, only v_4 gets abstracted away. The effect of the contraction is thus to remove v_4 and connect v_3 to v_5 to store the information that all in between v_3 and v_5 has been already explored.

The contraction is slightly more intricate when an endpoint of a $\curvearrowright$-edge is abstracted away. In fact, in this case, the new endpoint (if any) is selected, among those that have not been removed, as the closest one which is included in the portion of the graph below the edge. In particular, the $\curvearrowright$-edge from v_3 to v_{11} in n_1 is replaced (in the contraction) by the $\curvearrowright$-edge from v_5 to v_{11} in n_0, and since v_5 is not the actual left endpoint of this edge, we annotate the left end of $v_5 \curvearrowright v_{11}$ with $\triangledown$. Also observe that in case there is no such vertex (both the endpoints are abstracted away and no vertices in the between are kept in the new set of vertices) the edge is canceled. This is in fact the case, for the self-loop on v_6 in node n_2, that does not appear in the nw-shape of n_0.

There are two more conditions to ensure in order to get a nw-shape tree. First, the vertices of a ground shape at an internal node must contain all the vertices at the "borders"of the maximal lines defined by $\rightarrow$-edges in the nw-shapes of its children and the endpoints of the $\curvearrowright$-edges that have not been added yet in the shapes of the subtree (see for example vertex v_{11} in n_2). Second, the nw-shape of the root must be entirely connected through the $\rightarrow$-edges (*linerarly connected*) and should have all the endpoints of $\curvearrowright$ matched (*fully matched*).

An augmented tree decomposition is easily obtained from an nw-shape tree by retaining for each node just its ground shape. Since also the vice-versa holds, i.e. for each augmented tree decomposition there is a corresponding nw-shape tree, our method captures all the augmented tree decompositions of a program. Moreover, it can be implemented on the fly, being all the operations local.

Nested word shapes. Let $T = \{\triangledown, \blacktriangledown\}$ be an alphabet, where the symbol $\triangledown$ stands for *abstract*, and $\blacktriangledown$ stands for *concrete*.

Definition 2 (Nested Word Shapes). *A* nested word shape *(nw-shape) is a tuple $S = (V, \Rightarrow, \rightarrow, \{\overset{t,z}{\curvearrowright}\}_{t,z \in T})$ where*

– *V is a finite set of vertices;*
– *$(V, \Rightarrow)$ is a line graph;*
– *the set of* linear edges *$\rightarrow$ is a subset of $\Rightarrow$;*
– *the set of the* matching edges *$\curvearrowright = (\bigcup_{t,z \in T} \overset{t,z}{\curvearrowright})$ where $\overset{t,z}{\curvearrowright} \subseteq V \times V$ and is such that (where $a, b, c, d, \in T$ and $u, v, x, y \in V$):*
  - *if $u \curvearrowright v$ then also $u \Rightarrow^* v$;*
  - *if $u \curvearrowright v$ and $x \curvearrowright y$, then the following does not hold:*
    * *$u \Rightarrow^+ x \Rightarrow^+ v \Rightarrow^+ y$ (matching edges do not cross);*
    * *$(u, v) \neq (x, y)$ and $v = x$ (call and return of distinct matching edges must not coincide);*
  - *$u \overset{\blacktriangledown,\blacktriangledown}{\curvearrowright} u$ does not hold;*
  - *at most one among $u \overset{\blacktriangledown,\triangledown}{\curvearrowright} v$, $u \overset{\triangledown,\blacktriangledown}{\curvearrowright} v$ and $u \overset{\blacktriangledown,\blacktriangledown}{\curvearrowright} v$ holds;*
  - *if $u \overset{a,b}{\curvearrowright} v$, $u \overset{c,d}{\curvearrowright} y$ and $y \Rightarrow^+ v$ then $a = \triangledown$;*
  - *if $u \overset{a,b}{\curvearrowright} v$, $x \overset{c,d}{\curvearrowright} v$ and $u \Rightarrow^+ x$ then $b = \triangledown$.*

S is ground *if all of its matching edges are concrete, i.e. $\curvearrowright$ is exactly $\overset{\blacktriangledown,\blacktriangledown}{\curvearrowright}$. S is* linearly connected *if $\rightarrow$ is exactly $\Rightarrow$.* □

A *linear border* of a shape S is a vertex $u \in V$ without a linear successor or a linear predecessor, i.e., either $u \not\rightarrow v$ for each $v \in V$ or $v \not\rightarrow u$ for each $v \in V$.

Operations on shapes. In the following, we fix $S = (V, \Rightarrow, \rightarrow, \{\overset{t,z}{\curvearrowright}\}_{t,z \in T})$, $S' = (V', \Rightarrow', \rightarrow', \{\overset{t,z}{\curvearrowright}'\}_{t,z \in T})$, and $S_i = (V_i, \Rightarrow_i, \rightarrow_i, \{\overset{t,z}{\curvearrowright}_i\}_{t,z \in T})$ for $i = 1, 2$.

An nw-shape S' is the *contraction* of an nw-shape S on a set of vertices $V' \subseteq V$, denoted $S' = \mathit{contraction}(S, V')$, if the following holds:

– $\Rightarrow'^*$ is the total order on V' induced by $\Rightarrow^*$;
– $\rightarrow'$ is the set of all pairs $(x, y) \in V' \times V'$ such that either (1) $x \rightarrow y$, or (2) there is a sequence of vertices $u_1, u_2, \ldots, u_m \in (V \setminus V')$ such that $x \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_m \rightarrow y$.
– for each matching edge (u, v) of S, denote with $\mathit{contraction}_{\curvearrowright}(u, v)$ the pair (x, y) if the following holds:
  - $u \Rightarrow^* x \Rightarrow^* y \Rightarrow^* v$;
  - x is the smallest vertex $x' \in V'$ with $u \Rightarrow^* x'$; and
  - y is the greatest vertex $y' \in V'$ with $y' \Rightarrow^* v$.

  For every $t, z \in T$, $\overset{t,z}{\curvearrowright}'$ is the minimal set containing all pairs (x, y) such that there exist $u, v \in V$ where (1) $u \overset{p,s}{\curvearrowright} v$ for some p, s, (2) $\mathit{contraction}_{\curvearrowright}(u, v) = (x, y)$, (3) $t = \blacktriangledown$ iff $u = x$ and $p = \blacktriangledown$, and (4) $z = \blacktriangledown$ iff $v = y$ and $s = \blacktriangledown$.

S is the *merge* of two nw-shapes S_1 and S_2, denoted $S = \mathit{merge}(S_1, S_2)$, if S is an nw-shape and the following holds:

- $V = V_1 \cup V_2$;
- $\Rightarrow_1, \Rightarrow_2 \subseteq \Rightarrow$;
- $(\rightarrow_1 \cap \rightarrow_2) = \emptyset$, and $\rightarrow = (\rightarrow_1 \cup \rightarrow_2)$;
- For every $t, z \in T$, $\overset{t,z}{\curvearrowright} = (\overset{t,z}{\curvearrowright}_1 \cup \overset{t,z}{\curvearrowright}_2)$.

Tree decompositions via nw-shapes. In an nw-shape tree $\mathcal{T}$, the vertices of the nw-shape are typed as either left ($\mathcal{L}$) or right ($\mathcal{R}$) endpoint of a matching edge, or none of them, with the meaning that $u \in \mathcal{L}$, resp. $v \in \mathcal{R}$, iff there is a ground nw-shape of $\mathcal{T}$ that contains an edge $u \overset{\blacktriangledown,\blacktriangledown}{\curvearrowright} v$.

Definition 3 (NW-SHAPE TREE). *An* nw-shape tree $\mathcal{T}$ *is a triple* (T, sh, gsh) *where* T *is a binary tree with set of nodes* N*, and* sh *and* gsh *label each node of* T *with respectively an nw-shape and a ground nw-shape such that for each* $n \in N$ *the following holds:*

- *if* n *is a leaf, then* $sh(n) = gsh(n)$*;*
- *if* n *is an internal node, denoting with* n_1 *and* n_2 *its left and right children and with* V_n *the set of the vertices of* $gsh(n)$*:*
  - $sh(n) = contraction(S(n), V_n)$ *where* $S(n)$ *is the merge of* $sh(n_1)$*,* $sh(n_2)$ *and* $gsh(n)$*;*
  - *denoting with* $\mathcal{LR}$ *the set of all vertices from* $\mathcal{L} \cup \mathcal{R}$ *that are not concrete endpoints of* $\curvearrowright$*-edges of* $sh(n_1)$ *and* $sh(n_2)$*,* V_n *contains* $\mathcal{LR}$ *and all the linear borders of* $sh(n_1)$ *and* $sh(n_2)$*;*
  - *if* n *is the root, then additionally* $sh(n)$ *is also linearly connected and* fully matched*, that is, all its vertices from* $\mathcal{L} \cup \mathcal{R}$ *are concrete endpoints of some* $\curvearrowright$*-edge.*

We denote with $G(\mathcal{T})$ *the graph* $\bigcup_{n \in N} gsh(n)$*.* □

Note that in the above definition for each $n \in N$, $gsh(n)$ and $sh(n)$ have the same vertices and each edge of $gsh(n)$ is also an edge of $sh(n)$. By structural induction, it is possible to show that the graph obtained by the union of the ground nw-shapes of the leaves of a subtree is a ground nw-shape corresponding to a fragment of a nested word. When the nw-shape associated with the root of the subtree is also linearly connected and fully matched, then the resulting ground nw-shape corresponds to a nested word.

Lemma 1. *For any nw-shape tree* $\mathcal{T}$*,* $G(\mathcal{T})$ *is a nested word.*

For a nested word w denote with $\bullet\!\!-\!w\!-\!\!\bullet$ the nested word obtained from w by adding two new vertices v_L and v_R along with the edges $v_L \rightarrow v$ and $v' \rightarrow v_R$ where v denotes the first vertex and v' the last vertex of w according to $\rightarrow^*$.

From a nw-shape tree $\mathcal{T} = (T, sh, gsh)$ such that $G(\mathcal{T}) = \bullet\!\!-\!w\!-\!\!\bullet$, define $D(\mathcal{T}) = (T, \{V_n\}_{n \in N})$ where N is the set of nodes of T and V_n is the set of all the vertices of $sh(n)$ but v_L and v_R. By the definitions of nw-shape tree and tree decomposition, we get that $D(\mathcal{T})$ is a tree decomposition of w.

Vice-versa, consider a tree decomposition $D = (T, \{B_n\}_{n \in N})$ of a nested word $w = (V, \rightarrow, \curvearrowright)$. Add to each bag B_n the vertices v_L and v_R and define:

- $\{\rightarrow_n\}_{n\in N}$ such that each edge $u \rightarrow v$ of w, belongs to exactly one $\rightarrow_n$ such that $u, v \in B_n$, and each of $v_L \rightarrow v$ and $v' \rightarrow v_R$ belongs to exactly one $\rightarrow_n$;
- $\{\curvearrowright_n\}_{n\in N}$ such that for each edge $u \curvearrowright v$ of w, $u \overset{\blacktriangledown,\blacktriangledown}{\curvearrowright} v$ belongs to exactly one $\curvearrowright_n$ such that $u, v \in B_n$;
- each $\Rightarrow_n$ is the total order on B_n induced by $\rightarrow^*$;
- $\mathit{gsh}(n) = (B_n, \Rightarrow_n, \rightarrow_n, \curvearrowright_n)$, $\mathcal{L} = \{u \mid u \curvearrowright v\}$ and $\mathcal{R} = \{v \mid u \curvearrowright v\}$.

Starting from the parents of the leaves of T we compute for each node n, $\mathit{sh}(n)$ as the contraction on V_n of the merge of $\mathit{gsh}(n)$, $\mathit{sh}(n_1)$ and $\mathit{sh}(n_2)$ where n_1 and n_2 are the children of n. By the definition, we can show that $\mathcal{T} = (T, \mathit{sh}, \mathit{gsh})$ is an nw-shape and $G(\mathcal{T}) = \bullet\text{-}w\text{-}\bullet$. Therefore, the following theorem holds.

Theorem 3. *For any nested word w, there exists a tree decomposition D of width k iff there exists a nw-shape tree $\mathcal{T} = (T, \mathit{sh}, \mathit{gsh})$ such that $\bullet\text{-}w\text{-}\bullet = G(\mathcal{T})$ and each $\mathit{gsh}(n)$ has at most k vertices of w.*

Note that the additional vertices v_L and v_R do not correspond to any state of the program and are not really needed to have the above theorem. In fact, it would be sufficient to modify the definition of nw-shape tree such that a left (respectively right) linear border can be abstracted away as soon as a prefix (resp. a suffix) of the nested word has been explored.

Shapes and shape trees for programs. We augment nw-shape and nw-shape trees with program counters and variable valuations. In particular, we define a *pnw-shape* inductively from portions of program nested words and with merging and contraction operations. The merging requires that a same vertex is labeled with the same program counter and the same variable valuation. Analogously to nw-shape trees, we define the *pnw-shape tree* with respect to the notion of pnw-shape. More details can be found in Appendix A.

6 Implementation

In this section, we report on the prototype tool that we have implemented. Our implementation is targeted to use a verifier of sequential programs (with recursive procedure calls), though also fixed-point translations in the style of [9, 10, 13] are possible.

The input program P is transformed into a program P' that is essentially composed of procedure `main` and five more procedures: `ShapeTree()`, `check(S)`, `CreateGroundShape(k)`, `merge(S1,S2)` and `contraction(S1,S2)`. Except for `ShapeTree()`, all the other procedures do not contain recursive calls. Procedure `check` verifies that `S` is indeed a pnw-shape that can label the root of a pnw-shape tree, i.e., it is linearly connected and fully matched. (Observe that fully matched within a program nw-shape-tree refers to all the vertices that are marked with a program counter from *Call* or correspond to the return states after a call.) If this is the case and the last vertex (according to the linear order) corresponds to an error state, then a statement `assert`(0) is reachable (which defines the error state in P'). Procedures merge and contraction implement the corresponding operations on pnw-shapes. Additionally, contraction also ensures that S2 has as vertices all the linear borders of S1 along with the vertices corresponding to calls and return states that have not been yet matched with $\curvearrowright$-edges (which is required by Definition 3). Procedure CreateGroundShape nondeterministically generates a ground pnw-shape with k vertices and for each edge of the nw-shape it ensures that the program counters of its endpoints conform to the meaning of the edge in the program (i.e., a transition or a matching of a call and a return state). Moreover, if an edge corresponds to a transition, it ensures also that the values of the variables at its endpoints are consistent with the semantics of the transition.

Procedure ShapeTree (on the side) computes the pnw-shape at the nodes of a possible pnw-shape tree. When invoked from main, it starts from the root. At each node, it guesses a ground nw-shape S by invoking CreateShapeGround.Then, nondeterministically it decides whether the current node is a leaf or is internal. In the first case, S is returned, otherwise two recursive calls to ShapeTree are done (one for the left child and the other for the right one). The pnw-shapes returned by these calls are meant to label the two children, then according to the definition of pnw-shape tree these are merged and contracted, thus obtaining the pnw-shape for the current node, that is returned.

```
const k
shape ShapeTree() {
  shape S;
  S = CreateGroundShape(k);
  if (nondet()) {
    S1 = contraction(ShapeTree(),S);
    S2 = contraction(ShapeTree(),S);
    S  = merge(S, merge(S1,S2));
 }
 return S;
}

int main(){ shape S=ShapeTree();
            check(S);}
```

Observe that ShapeTree exactly implements the properties of Definition 3. To reduce the memory usage, we do the contraction of the nw-shape of the children of a node before merging them. This avoids the construction of an intermediate nw-shape with 2k nodes.

We have implemented this schema for C programs as a code-to-code translation that serves as a preprocessor for an off-the-shelf tool that automatically verify C programs. Our implementation is based on the framework we have used for CSeq [8] and that translates concurrent C programs with the pthread library to standard C programs, that can be then analyzed with model checking tools for C (e.g., CBMC [4], ESBMC [5], LLBMC [18]).

Our tool is meant to illustrate our methodology and to make a first step towards its implementation for more complex classes of programs. Also, note that although the target tool can analyze sequential programs by itself, we force it to explore the state space according to all the possible tree decompositions of the program behaviour graphs. We have tried an handful of toy examples with CBMC as backend and we have registered that our transformation slows down the tool used in the experiments. The problem seems to be that the implementation of the nw-shape procedures spans $\sim$500 l.o.c. in addition to the code of the original program, which makes the resulting program substantially bigger. One possible fix for this is to build the shapes nondeterministically and then constrain them logically by using assume statements. Further information on our tool, along with the programs and the corresponding translated programs we used, are available `http://users.ecs.soton.ac.uk/gp4/cav13.html`.

7 Discussion

In this paper, we have presented a new methodology to perform software analysis. The main idea is to transform the input programs such that the exploration of the computations is guided by the tree decompositions of their behaviour graphs. We have developed in details our methodology for sequential programs with recursive procedure calls and without dynamic data structures, and implemented such transformation as a code-to-code translation of C programs. The resulting tool is just a prototype and is essentially a proof of concept where we experiment in practice our methodology on a simple class of programs.

In this section, we discuss how to extend our approach to concurrent programs and how it relates to sequentialization of concurrent programs. We then conclude with some remarks and future work.

Concurrent programs. Concurrent programs consists of a finite number of threads where each of them is defined by a sequential program. All threads run in parallel and communicate through a finite number of shared variables according to the sequential consistency memory model (SC). A natural graph encoding for the computations of concurrent programs is the following. The behaviour of each thread is modeled with a nested word. Further, the behaviour of the shared memory is represented by a line graph capturing the sequence of memory operations, where each vertex represents an unique read or write operation. Vertices of the nested words are labeled, as usual, with a program counter and a valuation of the global variables, while memory vertices are labeled with a valuation of the shared variables. A vertex u of a nested word that "reads" a shared variable for executing the local transition, it is linked through a *memory edge* to the memory vertex representing that operation. The direction of this edge is reversed if the vertex "writes" to a shared variable. Since each memory vertex u represents exactly one memory operation, u has exactly one memory edge incident on it. Of course memory edges will never cross w.r.t. temporal events (as we assume SC). Let us call these behaviour graphs *concurrent nested words* (*cnw* for short).

Concurrent nested words admit natural summaries that reflect their composition. A *concurrent nw-shape* (*cnw-shape* for short) is formed by a distinct nw-shape for each component nested word, and an additional *memory-shape* that is a nw-shapes without matching edges. Additional care should be taken for the memory edges to avoid crossing. We have indeed worked out the details of this representation. For example, a *contraction* operation on a cnw-shape can be accomplished by executing a contraction on each component nw-shape and the memory-shape. Furthermore, memory edges are contracted similarly to matching edges of nw-shapes. The *merge* operation is defined exactly as for nw-shapes. By defining *cnw-shape trees* using the same combination of operations seen for nw-shape trees, we can show an equivalent of Theorem 3 for the concurrent setting. In addition, a code-to-code translation for concurrent programs is again possible (it is similar to that described in Section 6). The important fact here is that we can reuse verification tools designed for sequential programs for analyzing concurrent programs.

Here we convey the idea that our approach actually leads to a sort of *sequentialization* when applied to concurrent programs and implemented as a code-to-code translation to sequential programs. A sequentialization translates a concurrent program P into a nondeterministic sequential program P' that (under certain assumptions) behaves equivalently [19, 16, 11]. To make the approach effective, P' should not track the whole state space of the concurrent program, as in the cross product of the thread states. All sequentializations that have been proposed in literature only track one local state at a time and use k copies of the shared variables, for a given parameter k. Under these restrictions, such approaches can only cover a strict subset of computations in which each thread can at most interact k times with the other threads. These features are indeed desirable as we get a parameterized analysis technique that aims at exploiting as much as possible by tuning k for the underlying sequential verification tool. By increasing the parameter k, we can capture more computations, but this of course comes with a cost in terms of computational resources.

Our analysis schema shares all the good features of the sequentializations with the advantage of covering even more computations for the same parameter k. In fact, by considering cnw-shapes with at most k nodes we also track k copies of the variables (either global or shared), but cover all cnw of tree-width k vs. existing sequentializations being only able to intercept a very small subset of them. Moreover, a different sequentialization needs to be designed to capture new classes of behaviours (parameterized programs [12], thread creation [7, 3], scope bounded [14, 15], etc.), while our schema is uniform for all of them.

Future work. We believe that obtaining scalable solutions for sequential programs based on our approach will pave the way to lift such results to the concurrent settings. On the theoretical side, it would be interesting to study how computations of concurrent programs running under weak memory models can be modeled with behaviour graphs. Similarly, for distributed programs where the communication among threads happens through FIFO channels (see [17] for behaviour graphs of these programs). Further, we believe that our approach could be useful to reason about programs manipulating heaps. The intuition is that concurrent and distributed programs can be seen as sequential programs that use stacks for recursion and queues to simulate FIFO channels. We thus wonder whether our approach can be lifted to more general data structures.

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A Shapes and shape trees for programs

As described in Section 4, we aim to use augmented tree decompositions annotated with the program counters and the variable valuations. This corresponds to adding this annotations to nw-shapes and nw-shape trees.

For a mapping $f : A \to B$ and $C \subseteq A$, we denote with $f_{|C}$ the restriction of f to C. For mappings $f_i : A_i \to B_i$ $i = 1, 2$, we denote with $f_1 \cup f_2$ the mapping defined as $f_1(x)$ for each $x \in A_1$ and $f_2(x)$ for each $x \in A_2 \setminus A_1$.

Fix a program P.

A *ground pnw-shape* $\mathcal{S}$ of P is $(S, \mathit{Val}, \overline{pc})$ such that S is a ground nw-shape and there exists a program nested word $(nw, \mathit{Val}', \overline{pc}')$ of P such that S is a subgraph of nw, $\mathit{Val} = \mathit{Val}'_{|V}$ and $\overline{pc} = \overline{pc}'_{|V}$.

A *pnw-shape* $\mathcal{S}$ of P is either a ground pnw-shape or $\mathcal{S} = (S, \mathit{Val}, \overline{pc})$ is:

- the *contraction* of a pnw-shape, that is, there is pnw-shape $\mathcal{S}' = (S', \mathit{Val}', \overline{pc}')$ and denoting with V the set of vertices of S, $S = \mathit{contraction}(S', V)$, $\mathit{Val} = \mathit{Val}'_{|V}$ to V and $\overline{pc} = \overline{pc}'_{|V}$, or
- the *merging* of two pnw-shapes, that is, there are two pnw-shapes $\mathcal{S}_1 = (S_1, \mathit{Val}_1, \overline{pc}_1)$ and $\mathcal{S}_2 = (S_2, \mathit{Val}_2, \overline{pc}_2)$ for which, denoting with V the intersection of the sets of vertices of S_1 and S_2, $\mathit{Val}_1(v) = \mathit{Val}_2(v)$ and $\overline{pc}_1(v) = \overline{pc}_2(v)$ for every $v \in V$, then $S = \mathit{merge}(S_1, S_2)$, $\mathit{Val} = \mathit{Val}_1 \cup \mathit{Val}_2$ and $\overline{pc} = \overline{pc}_1 \cup \overline{pc}_2$.

Analogously to nw-shape tree, we define the *pnw-shape tree* with respect to the notion of pnw-shape. The definition is exactly that given there except that the merging and contraction operations are on pnw-shape, and thus we omit further details.