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Electronics and Computer Science

Enriched Coalgebraic Modal Logic

by

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ABSTRACT

FACULTY OF PHYSICAL AND APPLIED SCIENCES
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ENRICHED COALGEBRAIC MODAL LOGIC

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We formalise the notion of enriched coalgebraic modal logic, and determine conditions on the category $\mathbb{V}$ (over which we enrich), that allow an enriched logical connection to be extended to a framework for enriched coalgebraic modal logic. Our framework uses $\mathbb{V}$-functors $L\colon \mathbb{A} \to \mathbb{A}$ and $T\colon \mathbb{X} \to \mathbb{X}$, where L determines the modalities of the resulting modal logics, and T determines the coalgebras that provide the semantics.

We introduce the $\mathbb{V}$-category $\mathbf{Mod}(A, \alpha)$ of models for an L-algebra (A, α), and show that the forgetful $\mathbb{V}$-functor from $\mathbf{Mod}(A, \alpha)$ to $\mathbb{X}$ creates conical colimits.

The concepts of bisimulation, simulation, and behavioural metrics (behavioural approximations), are generalised to a notion of behavioural questions that can be asked of pairs of states in a model. These behavioural questions are shown to arise through choosing the category $\mathbb{V}$ to be constructed through enrichment over a commutative unital quantale $(Q, \otimes, I)$ in the style of Lawvere (1973).

Corresponding generalisations of logical equivalence and expressivity are also introduced, and expressivity of an L-algebra (A, α) is shown to have an abstract category theoretic characterisation in terms of the existence of a so-called behavioural skeleton in the category $\mathbf{Mod}(A, \alpha)$.

In the resulting framework every model carries the means to compare the behaviour of its states, and we argue that this implies a class of systems is not fully defined until it is specified how states are to be compared or related.

Contents

Declaration of Authorship

I, Toby Wilkinson , declare that the thesis entitled *Enriched Coalgebraic Modal Logic* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: (Wilkinson, 2012b), and (Wilkinson, 2012a).

Signed:...

Date:...

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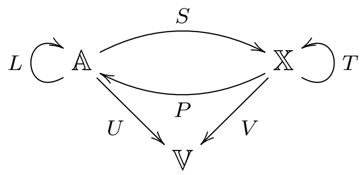
Nomenclature

$\mathbb{V}$	The symmetric monoidal closed category $(\mathbb{V}_o, \otimes, I)$.
$a_{X,Y,Z}$	The associator natural isomorphism of $\mathbb{V}$.
l_X	The left unitor natural isomorphism of $\mathbb{V}$.
r_X	The right unitor natural isomorphism of $\mathbb{V}$.
$c_{X,Y}$	The symmetry natural isomorphism of $\mathbb{V}$.
$\mathbf{elem}\lvert-\rvert$	The representable functor $\mathbb{V}_o(I, -)$.
$\widetilde{\mathbf{elem}}_{X,Y}$	The natural isomorphism $\mathbf{elem}\lvert X \otimes Y\rvert \cong \mathbf{elem}\lvert X\rvert \times \mathbf{elem}\lvert Y\rvert$.
$M_{A,B,C}$	The composition law of a $\mathbb{V}$-category.
j_A	The identity element of an object A of a $\mathbb{V}$-category.
$\mathbb{C}_o$	The underlying ordinary category of the $\mathbb{V}$-category $\mathbb{C}$.
F_o	The underlying ordinary functor of the $\mathbb{V}$-functor F.
$f \bullet g$	Composition in the underlying category of a $\mathbb{V}$-category.
$\mathbf{Set}_R$	The category of preordered sets of type R.
$\mathbf{GMet}$	The category of generalised metric spaces.
$f^\flat$	The dual adjunct of $f \in \mathbb{A}_o(A, P(X))$ under $P \dashv S$.
$g^\sharp$	The dual adjunct of $g \in \mathbb{X}_o(X, S(A))$ under $P \dashv S$.
$\Omega_\mathbb{A}$	The truth object in $\mathbb{A}$.
$\Omega_\mathbb{X}$	The truth object in $\mathbb{X}$.
$\mathbf{Alg}(L)$	The $\mathbb{V}$-category of L-algebras.
$\mathbf{CoAlg}(T)$	The $\mathbb{V}$-category of T-coalgebras.
$\mathbf{Mod}(A, \alpha)$	The $\mathbb{V}$-category of models for the L-algebra (A, α).
$U_{\mathbf{Alg}(L)}$	The forgetful $\mathbb{V}$-functor from $\mathbf{Alg}(L)$ to $\mathbb{A}$.
$U_{\mathbf{CoAlg}(T)}$	The forgetful $\mathbb{V}$-functor from $\mathbf{CoAlg}(T)$ to $\mathbb{X}$.
$U_{\mathbf{Mod}(A,\alpha)}$	The forgetful $\mathbb{V}$-functor from $\mathbf{Mod}(A, \alpha)$ to $\mathbf{CoAlg}(T)$.
Q	The commutative unital quantale $(Q, \otimes, I)$.
Q–$\mathbf{Cat}$	The category of Q-categories.
$\mathbb{V}_{Q\text{–}\mathbf{Cat}}$	A symmetric monoidal closed full subcategory of Q–$\mathbf{Cat}$.
$\mathbf{BSkel}(A, \alpha)$	The behavioural skeleton of the L-algebra (A, α).
$\mathbf{PBSkel}_M(A, \alpha)$	The parametric behavioural skeleton of (A, α) given by M.

Chapter 1

Introduction

The field of coalgebraic modal logic is now well-established in computer science, with a history dating back some 15 to 20 years. For the uninitiated, coalgebraic modal logic is the study of modal logics with semantics given by coalgebras. The coalgebras have a dynamic, "one-step" like nature, are represent generalised notions of transition system. It is these transitions that provide the "meaning" for the modalities of a modal logic. The key strength of coalgebraic modal logic is that it lends itself to an abstract way of working, that both clarifies what is really going on, and readily generalises to incorporate new ideas in a systematic way. This high level of abstraction means that the key building blocks of our framework can be summarised in the following diagram.



Here $\mathbb{A}$, $\mathbb{V}$, and $\mathbb{X}$ are categories, and L, P, S, T, U and V are functors. The basic idea is that the left-hand side is where the logics live, the right-hand side is where the coalgebras live, and the rest is plumbing that links everything together in the right way. To be a little bit more specific, the modal logics will be algebras for the functor L, or L-algebras, and the coalgebras will be coalgebras for the functor T, or T-coalgebras. The functors P and S form what is called a logical connection, which is simply a dual adjunction with a logical interpretation, and this links the modal logics to the coalgebras. Of the remaining components, the category $\mathbb{V}$ represents a base, or common level of structure that we want the other categories to share, and indeed the entire diagram is enriched over $\mathbb{V}$. Finally, the functors U and V are forgetful functors that ensure the objects of $\mathbb{A}$ and $\mathbb{X}$ can be regarded as objects of $\mathbb{V}$ with extra structure.

The above picture has started to take shape in recent years (Kupke et al., 2004b; Klin, 2007; Jacobs and Sokolova, 2010; Kurz and Velebil, 2011), though the development has been rather piecemeal, and a comprehensive unifying framework is still lacking. Some attempts have been made to start to rectify this, but much work is still to be done. Our work aims to make some progress towards this goal.

Some of the key issues that still need to be addressed include:

1. There needs to be a systematic treatment of the set of truth values of a logic, as it is increasingly clear that bivalent logics are no longer sufficient. For example, in probabilistic systems the probabilities are often only known approximately, and in such circumstances bivalent logics tend to not be robust to perturbations in the values of these probabilities (Desharnais et al., 2004).

2. There needs to be a systematic treatment of the different notions of behavioural comparability - bisimulation, simulation, approximation (behavioural metrics) - so that the relationships between them are made clear, and that a framework is in place to experiment with new notions, and the connection with the choice of truth values can be explored.

3. There needs to be a systematic treatment of semantic consequence and proof systems, as proof is the essence of logic. For example different notions of semantic consequence can be defined that are either local/global and either frame/model based, and this relates to the notions of satisfaction and validity. Therefore as validity corresponds to quantification over valuations, any framework will need to be able to handle propositional variables, their valuations, and axioms in a systematic fashion.

4. There needs to be an abstract presentation of the essence of what coalgebraic modal logic is, devoid of all computer science specific terminology, in order to facilitate the adoption of these ideas by other branches of the sciences and mathematics.

The first of these issues is addressed by what is known as a logical connection (Kurz and Velebil, 2011), and is increasingly becoming a standard foundation for work on coalgebraic modal logic. It is also formulated, as we have seen above, using the abstract mathematics of category theory, and so goes some way to addressing the fourth issue.

The second and third issues have received rather less treatment in the literature. We focus mostly on the issue of behavioural comparability, but our work also provides a foundation upon which future work can tackle the issue of propositional variables and axioms. Our approach is to exploit the power of enriched category theory, and we build upon the foundations laid in Kurz and Velebil (2011), and generalise our own work in Wilkinson (2012b) and Wilkinson (2012a).

1.1 Key Contributions

The key contributions of this thesis are as follows:

Fibrations to lift categories: In Chapter 2 we introduce the notion of the initial lift of an ordinary functor to a $\mathbb{V}$-functor as an initial lift along the 2-functor $(-)_o : \mathbb{V}-\mathbf{CAT} \to \mathbf{CAT}$ that sends a $\mathbb{V}$-category to its underlying ordinary category. Such a lift not only generates a $\mathbb{V}$-functor, but the source category is lifted to a $\mathbb{V}$-category. This lifted $\mathbb{V}$-category is more useful than the usual notion of the free $\mathbb{V}$-category over an ordinary category, and we make extensive use of it in Chapter 4, where it is used to define the enriched analogues of the standard categories of algebras and coalgebras for a functor.

Models for L-algebras: In Chapter 4 we introduce the $\mathbb{V}$-category of models for an L-algebra. This allows the clean handling of arbitrary modal logics, and thus propositional variables and axioms, and is a key building block towards our treatment of expressivity.

Behavioural questions: In Chapter 5 we show that the choice of the category $\mathbb{V}$ over which we enrich determines the type of behavioural comparisons that we can perform - bisimulation, simulation, behavioural metrics etc. Further, we show that these notions of behavioural comparability, or behavioural questions, can be generalised by enriching over symmetric monoidal closed categories constructed through enrichment over a commutative unital quantale. This then also induces a generalised notion of logical equivalence, and a generalised notion of what it means for a modal logic to be expressive with respect to the chosen notion of behavioural comparison. Together this shows that enrichment is a vital part of the general framework of coalgebraic modal logic. Moreover, it also provides a persuasive argument that a class of systems is not fully defined until it is specified how they are to be compared or related, and in our framework each model incorporates a preorder, metric, or some generalisation, for this purpose.

Behavioural skeletons: In Chapter 6 we present a systematic approach to analysing the expressivity of an L-algebra for its category of models with respect to the type of behavioural question given by the choice made for $\mathbb{V}$. This approach is a categorical one, and proceeds by examining the structure of the category of models. We introduce a structure called a behavioural skeleton, and show that the category of models for an L-algebra has such a structure if and only if the L-algebra is expressive. We also introduce parametric behavioural skeletons, and show how the parametricity can be exploited to provide a powerful tool for proving expressivity and the existence of final models.

1.2 Synopsis

A brief overview of the structure of this thesis is as follows:

Chapter 2 The structure we require of the category $\mathbb{V}$ (over which we enrich) is defined, and the categories of preordered sets and generalised metric spaces are presented as our leading examples (along with the category **Set**). We also introduce the concept of the initial lift of an ordinary functor to a $\mathbb{V}$-functor, and prove two theorems that we shall make extensive use of in Chapter 4.

Chapter 3 The enriched logical connections of Kurz and Velebil (2011) are discussed in the context of our assumptions on $\mathbb{V}$, and their logical content made explicit. Numerous examples are also demonstrated that reappear throughout subsequent chapters.

Chapter 4 The notions of algebras and coalgebras for a functor are lifted from the ordinary category theory level to the $\mathbb{V}$-category level. Coalgebraic modal logic in the $\mathbb{V}$-category setting is then introduced, and the category of models for an L-algebra defined. Finally, the forgetful functors from both the category of T-coalgebras to the base category $\mathbb{X}$, and the category of models of an L-algebra to $\mathbb{X}$, are shown to create conical colimits.

Chapter 5 Bisimulation, simulation, and behavioural approximation (metrics) are generalised to a general notion of behavioural questions that can be asked of pairs of states. These are shown to arise from different choices of a commutative unital quantale. A generalised notion of logical equivalence is also introduced, along with a generalised notion of what it means for an L-algebra to be expressive.

There is also a brief discussion raising the question of the nature of the relationship between the choice of commutative unital quantale, and the choice of truth values for the logics.

Chapter 6 A purely category theoretic characterisation of expressivity is proven in terms of the existence of a behavioural skeleton for the category of models for an L-algebra. Here a behavioural skeleton is a collection of models with certain properties for which cospans must exist, and for every other model, there must be a model in the skeleton via which it factors.

Parametric behavioural skeletons are also introduced as a tool for proving expressivity, and the cases of expressivity with respect to bisimulation and simulation are explored using the internal models of Wilkinson (2012b) and the R-models of Wilkinson (2012a).

Chapter 7 A summary of our work is presented, and possible future developments outlined.

Chapter 2

Preliminaries

Before we can begin to look at coalgebraic modal logic in an enriched setting, we need to spend some time explaining what we mean by an enriched setting, and give some indication as to why this might be a good thing to do.

We shall also need to introduce a key technical concept that underpins a lot of our future development. This is the notion of the initial lift of an ordinary functor to an enriched functor, and the subsequent lifting of an ordinary category to an enriched category.

A brief outline of this chapter is as follows:

Section 2.1 The category $\mathbb{V}$ is introduced. This is the category that we enrich over. In addition to the usual properties that are required of $\mathbb{V}$ (symmetric monoidal closed, complete and cocomplete), we also require some additional ones. These are stated and explained.

Section 2.2 The category of preordered sets is shown to satisfy the requirements we need of $\mathbb{V}$.

Section 2.3 The category of generalised metric spaces, which can be thought of as generalising the category of preorders, is also shown to meet the requirements we need of $\mathbb{V}$.

Section 2.4 Initial lifts of ordinary functors are introduced, and two theorems proven. These provide a mechanism by which we can construct $\mathbb{V}$-categories from ordinary categories that have the properties that we require, and will be key technical tools in the development of Chapter 4.

Section 2.5 Previous work by other authors using enrichment for coalgebras, and the possible connection between their work and domain theory, is discussed.

2.1 The Category $\mathbb{V}$

As already mentioned, we shall be working in an enriched setting. What this means is that we shall be using categories that are enriched over some other category. In ordinary category theory this is the category **Set**, but we shall generalise this to a category $\mathbb{V}$. This idea goes back many decades, and it is well known that such a category $\mathbb{V}$ must at a minimum carry the structure of a monoidal category, but we shall require that $\mathbb{V}$ has more structure than this.

Essentially what enriched category theory aims to do, is take the definitions and theorems of ordinary category theory, and wherever there is a hom-set, replace it with an object from the category $\mathbb{V}$. We aim to use this to pervasively sprinkle extra structure on these hom-sets. This extra structure will be fixed by specifying a particular category $\mathbb{V}$, and will be chosen depending upon the way we decide to compare the behaviour of states of coalgebras (Chapter 5).

For those readers who are unfamiliar with enriched category theory, Appendix C contains all the definitions and results we use (and a few others), but possibly the best starting point is the monograph by Kelly (Kelly, 1982).

In order to proceed we must make some basic assumptions about the category $\mathbb{V}$. These assumptions will hold throughout what follows.

Assumption 1.

1. The category $\mathbb{V} = (\mathbb{V}_o, \otimes, I)$ is symmetric monoidal closed (Appendix B).
2. The underlying category $\mathbb{V}_o$ is locally small, so there is a symmetric monoidal closed functor (Definition B.15) that extends the representable functor

$$\mathbf{elem}|-| = \mathbb{V}_o(I, -) \colon \mathbb{V}_o \to \mathbf{Set},$$

   which we assume to be faithful (Definition A.7), making $\mathbb{V}_o$ concrete over **Set** (Definition A.9).
3. The functor $\mathbf{elem}|-|$ is strong monoidal (Definition B.14), so there is a natural isomorphism

$$\widetilde{\mathbf{elem}}_{X,Y} \colon \mathbf{elem}|X \otimes Y| \cong \mathbf{elem}|X| \times \mathbf{elem}|Y| \,.$$

4. The underlying category $\mathbb{V}_o$ is complete and cocomplete.
5. The functor $\mathbf{elem}|-|$ is a fibration (Definition A.5).

That $\mathbb{V}$ is symmetric monoidal closed means that $\mathbb{V}$-categories have sufficient structure to be able to do "category theory" - specifically we have a Yoneda Lemma (Appendix C.7). It also means $\mathbb{V}$ is itself a $\mathbb{V}$-category (Appendix C.3), where each hom-object $\mathbb{V}(A, B)$ is given by the internal-hom $[A, B]$.

The functor $\mathbf{elem}|-|$ is what is traditionally (Kelly, 1982) denoted as V, but we shall use V for something else (Chapter 3). It assigns to each object of $\mathbb{V}_o$ its set of elements (Definition B.3), and by Proposition B.11 there is an isomorphism

$$\mathbf{elem}|[X, Y]| \cong \mathbb{V}_o(X, Y),$$

which means that we can freely interchange morphisms in $\mathbb{V}_o$ and elements of the corresponding internal-hom, and indeed we often blur the distinction.

The fact that we assume $\mathbf{elem}|-|$ is faithful, and thus that $\mathbb{V}_o$ is concrete over **Set** (Definition A.9), means that we can regard the objects of $\mathbb{V}$ as sets with some kind of structure. It also means by Proposition C.43, that ordinary natural transformations between the underlying functors F_o, G_o of a pair of $\mathbb{V}$-functors $F, G\colon \mathbb{C} \to \mathbb{D}$, lift to $\mathbb{V}$-natural transformations between F and G.

The natural isomorphism $\mathbf{elem}|X \otimes Y| \cong \mathbf{elem}|X| \times \mathbf{elem}|Y|$ means that we can think of elements of $X \otimes Y$ as consisting of a pair of elements, one from X, and one from Y (Definition B.15). This will be important in Section 3.4. Moreover, given a pair of $\mathbb{V}$-categories $\mathbb{B}$ and $\mathbb{C}$, this extends to an isomorphism $(\mathbb{B} \otimes \mathbb{C})_o \cong \mathbb{B}_o \times \mathbb{C}_o$ (Corollary C.15).

The underlying category $\mathbb{V}_o$ is required to be complete so that functor categories exist (Definition C.84), and cocomplete so that free $\mathbb{V}$-categories exist (Definition C.89). Both of these are prerequisites for the definition of conical colimits (Definition C.95), which we need in Chapter 6.

Finally, we require $\mathbf{elem}|-|$ to be a fibration, as this will provide the mechanism by which we perform the initial lift of ordinary functors (Section 2.4).

The category **Set** trivially satisfies the conditions of Assumption 1, but if that was the only example of interest, there would be no need to employ the machinery of enriched category theory. In order to illustrate our approach we shall therefore consider two additional examples in Section 2.2 and Section 2.3, and as we shall see in Section 2.5, these relate to previous work by other authors on coalgebras. Moreover, in Chapter 5, we shall see that they are both special cases of a more general class of categories, and that these are important for the study of the behaviours of coalgebras.

2.2 Preordered Sets

The first example of a category satisfying Assumption 1 that we shall consider is that of preordered sets.

Recall the category **Preord** of preordered sets and monotone functions, the objects of **Preord** are pairs consisting of a set, and a preorder relation on that set. Similarly, the categories **Pos** (partially ordered sets), **Setoid** (setoids), and **DiscSetoid** (discrete setoids), have for objects, pairs consisting of a set, and respectively, a partial order, equivalence relation, or the equality relation, on that set. In Levy (2011) these examples are collectively known as the preordered sets.

We can consider these examples together by means of the following definition (Wilkinson, 2012a), where by a relation of "type R", we mean either a **preorder**, **partial order**, **equivalence relation**, or **equality**. The type is fixed, and every object in the category $\mathbf{Set}_R$ (defined below) must have a relation of that type.

Definition 2.1. The category $\mathbf{Set}_R$ has for objects pairs (X, R_X), consisting of a set X, and a binary relation R_X of type R on X. The morphisms are the R-preserving functions, i.e. $f\colon (X, R_X) \to (Y, R_Y)$ is a morphism, if and only if, for all $x, x' \in X$

$$xR_Xx' \Rightarrow f(x)R_Yf(x').$$

To be explicit we have the following four cases:

1. If R is the type preorder, then $\mathbf{Set}_R$ is **Preord**.
2. If R is the type partial order, then $\mathbf{Set}_R$ is **Pos**.
3. If R is the type equivalence relation, then $\mathbf{Set}_R$ is **Setoid**.
4. If R is the type equality, then $\mathbf{Set}_R$ is **DiscSetoid**.

The category **DiscSetoid** is obviously isomorphic to **Set**, and we shall use them interchangeably.

For the category $\mathbf{Set}_R$ to be useful for our purposes, $\mathbf{Set}_R$ must satisfy the conditions of Assumption 1. It is easy to verify that the forgetful from $\mathbf{Set}_R$ to **Set** creates limits and colimits. Specifically, we have the following basic limits and colimits.

Products: the product of (X, R_X) and (Y, R_Y) is given by $(X \times Y, R_{X\times Y})$, where

$$(x, y)R_{X\times Y}(x', y') \Leftrightarrow xR_Xx' \text{ and } yR_Yy'.$$

Coproducts: the coproduct of (X, R_X) and (Y, R_Y) is given by $(X+Y, R_{X+Y})$, where

$$wR_{X+Y}w' \Leftrightarrow \begin{cases} wR_Xw' & : \text{ if } w, w' \in X \\ wR_Yw' & : \text{ if } w, w' \in Y \\ \bot & : \text{ otherwise.} \end{cases}$$

Equalisers: the equaliser of $f, g\colon (X, R_X) \to (Y, R_Y)$ is given by $e\colon (E, R_E) \to (X, R_X)$, where

$$E = \{x \in X \mid f(x) = g(x)\},$$

and

$$xR_Ex' \Leftrightarrow xR_Xx'.$$

Coequalisers: the coequaliser of $f, g\colon (X, R_X) \to (Y, R_Y)$ is given by

$$q\colon (Y, R_Y) \to (Q, R_Q),$$

where $Q = Y/\sim$, and $\sim$ is the smallest equivalence relation such that for all $x \in X$ we have $f(x) \sim g(x)$. The relation R_Q is given by

$$[q]R_Q[q'] \Leftrightarrow \text{ for all } y \sim q, \text{ and } y' \sim q', yR_Yy'.$$

Final Object: the final object is $(\mathbf{1}, R_\mathbf{1})$, where $\mathbf{1}$ is the singleton set, and $R_\mathbf{1} = \mathbf{1} \times \mathbf{1}$.

Initial Object: the initial object is $(\mathbf{0}, R_\mathbf{0})$, where both $\mathbf{0}$ and $R_\mathbf{0}$ are the empty set.

It should be clear that small products and coproducts also exist, and thus we can deduce the following proposition.

Proposition 2.2. *The category* $\mathbf{Set}_R$ *is complete and cocomplete.*

It is also easy to verify that binary products and the final object form the tensor and unit of a symmetric monoidal category. To make $\mathbf{Set}_R$ also closed we need internal-hom objects $[(X, R_X), (Y, R_Y)]$, such that $[(Y, R_Y), -]$ is right adjoint to $- \times (Y, R_Y)$ (Definition B.9). These are given as follows:

Internal-hom: the internal-hom of (X, R_X) and (Y, R_Y) is given by the set of all R-preserving functions from X to Y carrying the relation

$$fR_{[(X,R_X),(Y,R_Y)]}g \Leftrightarrow \forall x \in X, f(x)R_Yg(x).$$

Unit: the unit of the adjunction $- \times (Y, R_Y) \dashv [(Y, R_Y), -]$ is given by

$$\begin{aligned} d_{(X,R_X)} \colon (X, R_X) &\to [(Y, R_Y), (X, R_X) \times (Y, R_Y)] \\ x &\mapsto f_x \colon (Y, R_Y) \to (X, R_X) \times (Y, R_Y), \end{aligned}$$

where $f_x(y) = (x, y)$.

Counit: the counit of the adjunction $- \times (Y, R_Y) \dashv [(Y, R_Y), -]$ is given by

$$\begin{aligned} e_{(Z,R_Z)} \colon [(Y, R_Y), (Z, R_Z)] \times (Y, R_Y) &\to (Z, R_Z) \\ (g \colon (Y, R_Y) \to (Z, R_Z), y) &\mapsto g(y). \end{aligned}$$

Thus we have the following proposition.

Proposition 2.3. *The category* $\mathbf{Set}_R$ *is symmetric monoidal closed.*

Finally, the symmetric monoidal closed functor $\mathbf{elem}|-|$ (Definition B.15) is easily seen to be faithful, and strong monoidal (Definition B.14). It is also a fibration (Definition A.5), as for any function $f \colon X \to Y$, if Y carries the relation R_Y, then we can define a relation R_X on X by

$$x R_X x' \Leftrightarrow f(x) R_Y f(x').$$

This is easily shown to be universal in the sense required of an initial lift.

Therefore putting everything together we can deduce:

Proposition 2.4. *The category* $\mathbf{Set}_R$ *satisfies all the conditions of Assumption 1.*

2.3 Generalised Metric Spaces

The second category that we shall be interested in enriching over is the category of generalised metric spaces (Lawvere, 1973). Generalised metric spaces differ from the usual notion of a metric space in three ways:

1. distinct points can have zero distance between them,
2. the distance between two points can be ∞,
3. the distance between two points need not be symmetric.

The category of generalised metric spaces is defined as follows.

Definition 2.5. The category **GMet** of generalised metric spaces, has for objects pairs (X, d_X), consisting of a set X, and a function $d_X : X \times X \to [0, \infty]$, that satisfies:

1. $d_X(x, x) = 0$ for all $x \in X$,
2. $d_X(x, z) \leq d_X(x, y) + d_X(y, z)$ for all $x, y, z \in X$.

The morphisms are the non-expansive functions, i.e. $f : (X, d_X) \to (Y, d_Y)$ is a morphism, if and only if, for all $x, x' \in X$

$$d_Y(f(x), f(x')) \leq d_X(x, x').$$

It is easy to see that preorders can be regarded as generalised metric spaces, and there is a full and faithful embedding of $\mathbf{Set}_R$ in **GMet** given by

$$xR_Xy \mapsto d_X(x, y) = \begin{cases} 0 & : \text{ if } xR_Xy \\ \infty & : \text{ otherwise.} \end{cases}$$

We require that **GMet** be both complete and cocomplete, and it is easy to verify that the forgetful from **GMet** to **Set** creates limits and colimits. Specifically, we have the following basic limits and colimits.

Products: the product of (X, d_X) and (Y, d_Y) is given by $(X \times Y, d_{X \times Y})$, where

$$d_{X \times Y}((x, y), (x', y')) = \max(d_X(x, x'), d_Y(y, y')).$$

Coproducts: the coproduct of (X, d_X) and (Y, d_Y) is given by $(X + Y, d_{X+Y})$, where

$$d_{X+Y}(w, w') = \begin{cases} d_X(w, w') & : \text{ if } w, w' \in X \\ d_Y(w, w') & : \text{ if } w, w' \in Y \\ \infty & : \text{ otherwise.} \end{cases}$$

Equalisers: the equaliser of $f, g : (X, d_X) \to (Y, d_Y)$ is given by $e : (E, d_E) \to (X, d_X)$, where

$$E = \{x \in X \mid f(x) = g(x)\},$$

and

$$d_E(x, x') = d_X(x, x').$$

Coequalisers: the coequaliser of $f, g\colon (X, d_X) \to (Y, d_Y)$ is given by

$$q\colon (Y, d_Y) \to (Q, d_Q),$$

where $Q = Y/\sim$, and $\sim$ is the smallest equivalence relation such that for all $x \in X$ we have $f(x) \sim g(x)$. The metric d_Q is given by

$$d_Q([y], [y']) = \inf_{\substack{u \sim y \\ u' \sim y'}} d_Y(u, u').$$

Final Object: the final object is $(\mathbf{1}, d_{\mathbf{1}})$, where $\mathbf{1}$ is the singleton set, and $d_{\mathbf{1}}(*, *) = 0$.

Initial Object: the initial object is $(\mathbf{0}, d_{\mathbf{0}})$, where $\mathbf{0}$ is the empty set.

Since we allow distances to be infinite, small products also exist, and as all small limits and colimits can be constructed from combinations of the above, we therefore have the following proposition.

Proposition 2.6. *The category* **GMet** *is complete and cocomplete.*

We also require that **GMet** be symmetric monoidal closed, and for this we need a tensor and a unit. The obvious first choice would be to take product as the tensor, and the final object as the unit, and this indeed yields a symmetric monoidal category, but it is not closed, as in general the counits do not exist as the underlying functions are not non-expansive. So instead we define the tensor as follows.

Definition 2.7. The **tensor product** $(X, d_X) \otimes (Y, d_Y)$ of the generalised metric spaces (X, d_X) and (Y, d_Y) is given by $(X \times Y, d_{X \otimes Y})$, where

$$d_{X \otimes Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

It is easy to verify that tensor product and the final object form the tensor and unit of a symmetric monoidal category. To make **GMet** also closed we need internal-hom objects $[(X, d_X), (Y, d_Y)]$, such that $[(Y, d_Y), -]$ is right adjoint to $- \otimes (Y, d_Y)$ (Definition B.9). These are given as follows:

Internal-hom: the internal-hom of (X, d_X) and (Y, d_Y) is given by the set of all non-expansive functions from X to Y carrying the metric

$$d_{[(X,d_X),(Y,d_Y)]}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Unit: the unit of the adjunction $- \otimes (Y, d_Y) \dashv [(Y, d_Y), -]$ is given by

$$\begin{aligned} k_{(X,d_X)} \colon (X, d_X) &\to [(Y, d_Y), (X, d_X) \otimes (Y, d_Y)] \\ x &\mapsto f_x \colon (Y, d_Y) \to (X, d_X) \otimes (Y, d_Y), \end{aligned}$$

where $f_x(y) = (x, y)$.

Counit: the counit of the adjunction $- \otimes (Y, d_Y) \dashv [(Y, d_Y), -]$ is given by

$$\begin{aligned} e_{(Z,d_Z)} \colon [(Y, d_Y), (Z, d_Z)] \otimes (Y, d_Y) &\to (Z, d_Z) \\ (g \colon (Y, d_Y) \to (Z, d_Z), y) &\mapsto g(y). \end{aligned}$$

Thus we have the following proposition.

Proposition 2.8. *The category* **GMet** *is symmetric monoidal closed.*

Finally, the symmetric monoidal closed functor **elem**$|-|$ (Definition B.15) is easily seen to be faithful, and strong monoidal (Definition B.14). It is also a fibration (Definition A.5), as for any function $f \colon X \to Y$, if Y carries the metric d_Y, then we can define a metric d_X on X by

$$d_X(x, x') = d_Y(f(x), f(x')).$$

This is easily shown to be universal in the sense required of an initial lift.

Therefore putting everything together we can deduce:

Proposition 2.9. *The category* **GMet** *satisfies all the conditions of Assumption 1.*

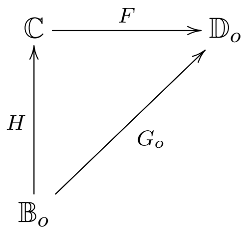
2.4 Initial Lifts of Ordinary Functors

In Chapter 4 we shall frequently find ourselves in the following situation. We have a $\mathbb{V}$-category $\mathbb{D}$, and an ordinary functor $F \colon \mathbb{C} \to \mathbb{D}_o$ to the underlying category of $\mathbb{D}$, and we would like to find a $\mathbb{V}$-category $\overline{\mathbb{C}}$ and a $\mathbb{V}$-functor $\overline{F} \colon \overline{\mathbb{C}} \to \mathbb{D}$, such that the underlying ordinary functor of $\overline{F}$ is F.

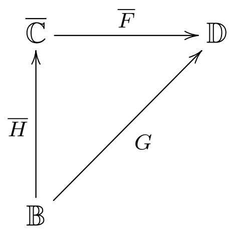
Now since we are assuming $\mathbb{V}_o$ is cocomplete, the free $\mathbb{V}$-category $\mathbb{C}_{\mathbb{V}}$ exists (Definition C.89), but often this is not the solution we are looking for. The problem is that, whilst the hom-objects of $\mathbb{C}_{\mathbb{V}}$ are indeed objects in $\mathbb{V}_o$, they are the wrong ones. What we mean by this is that since **elem**$|-|$ is faithful, $\mathbb{V}_o$ is concrete over **Set**, so the hom-objects of a $\mathbb{V}$-category are sets with some additional structure, and it is this additional structure that we are interested in. Specifically, for any given hom-set in $\mathbb{C}_o$, it may be possible to put on that set, any one of many different structures of the type specified by the category $\mathbb{V}$. So our problem becomes one of choosing the optimal such structure.

The free $\mathbb{V}$-category approach finds one extreme such solution, but usually we will be looking for a better one. But what do we mean by better? The answer is that we want it to have the following universal property.

Definition 2.10. Given a $\mathbb{V}$-category $\mathbb{D}$ and an ordinary functor $F\colon \mathbb{C} \to \mathbb{D}_o$, then an **initial lift** of F is a $\mathbb{V}$-functor $\overline{F}\colon \overline{\mathbb{C}} \to \mathbb{D}$, such that the underlying category of $\overline{\mathbb{C}}$ is $\mathbb{C}$, and the underlying functor of $\overline{F}$ is F. Moreover, it is also required that for any $\mathbb{V}$-functor $G\colon \mathbb{B} \to \mathbb{D}$, and any ordinary functor $H\colon \mathbb{B}_o \to \mathbb{C}$, such that



there exists a unique $\mathbb{V}$-functor $\overline{H}\colon \mathbb{B} \to \overline{\mathbb{C}_o}$, such that



and the underlying ordinary functor of $\overline{H}$ is H.

Remark 2.11. For readers who are familiar with such things, this is simply an initial lift along the 2-functor $(-)_o\colon \mathbb{V}{-}\mathbf{CAT} \to \mathbf{CAT}$ (Kelly, 1982, Section 1.3) that sends a $\mathbb{V}$-category to its underlying ordinary category.

So the question is how do we perform the initial lift of F? The key here is that for the ordinary functor $F\colon \mathbb{C} \to \mathbb{D}_o$, for objects A and B in $\mathbb{C}$, we have the morphism

$$F_{A,B}\colon \mathbb{C}(A,B) \to \mathbb{D}_o(F(A),F(B))$$

in **Set**. However, $\mathbb{D}_o(F(A),F(B))$ is defined to be $\mathbf{elem}|\mathbb{D}(F(A),F(B))|$, so we really have

$$F_{A,B}\colon \mathbb{C}(A,B) \to \mathbf{elem}|\mathbb{D}(F(A),F(B))|\,.$$

Now the $\mathbb{V}$-functor we are looking for would yield

$$\overline{F}_{A,B}\colon \overline{\mathbb{C}}(A,B) \to \mathbb{D}(F(A),F(B))$$

in $\mathbb{V}_o$, so what we would like to do is perform the initial lift of $F_{A,B}$ along $\mathbf{elem}|-|$ (Definition A.1). If we can do this in a coherent fashion for all objects A and B in $\mathbb{C}$, such that the lifted morphisms define a $\mathbb{V}$-functor, then we will have constructed the initial lift of F that we are looking for.

Before proceeding any further we need to address a notational issue, and make explicit a result that we use in several places.

Firstly, in any $\mathbb{V}$-category $\mathbb{C}$, we shall occasionally denote by $\bullet$ composition in the underlying ordinary category $\mathbb{C}_o$, to distinguish it from composition in $\mathbb{V}_o$. This aids clarity in the treatment of hom-functors (Section C.5) for example, where we write things like

$$\mathbb{C}(A, f) \circ u = f \bullet u.$$

Secondly, from the definition of the symmetric monoidal closed functor $\mathbf{elem}|-|$ (Definition B.15), and the definition of composition in the underlying category of a $\mathbb{V}$-category (Definition C.10) we have the following proposition.

Proposition 2.12. *Given the conditions of Assumption 1, and a $\mathbb{V}$-category $\mathbb{C}$, then the following diagram commutes.*

$$\begin{array}{ccc}
\mathbb{C}_o(B,C) \times \mathbb{C}_o(A,B) & \xrightarrow{\bullet_{A,B,C}} & \mathbb{C}_o(A,C) \\
\Big\downarrow \widetilde{\mathbf{elem}}_{\mathbb{C}(B,C),\mathbb{C}(A,B)} & \nearrow \mathbf{elem}|M_{A,B,C}| & \\
\mathbf{elem}|\mathbb{C}(B,C) \otimes \mathbb{C}(A,B)| & &
\end{array}$$

We are now ready to show that the conditions of Assumption 1 are sufficient to be able to construct the desired initial liftings of ordinary functors. The proof is quite long, as a result of the number of properties that must be proved, but hopefully Example 2.1 and Example 2.2 will show that the idea is actually quite simple.

Theorem 2.13. *Given the conditions of Assumption 1, a $\mathbb{V}$-category $\mathbb{D}$, and an ordinary functor $F\colon \mathbb{C} \to \mathbb{D}_o$, then*

1. *there is an initial lift $\overline{F}\colon \overline{\mathbb{C}} \to \mathbb{D}$ of F,*
2. *the $\mathbb{V}$-category $\overline{\mathbb{C}}$ is unique up to isomorphism.*

Proof. The proof proceeds as follows:

1. Define the objects and hom-objects of the $\mathbb{V}$-category $\overline{\mathbb{C}}$:

   $\overline{\mathbb{C}}$ has the same objects as $\mathbb{C}$, and for any pair of objects A, B in $\mathbb{C}$, and the function

$$F_{A,B}\colon \mathbb{C}(A,B) \to \mathbf{elem}|\mathbb{D}(F(A), F(B))|\,,$$

since $\mathbf{elem}|-|$ is a fibration, this has an $\mathbf{elem}|-|$-initial lift

$$\overline{F_{A,B}} \colon \overline{\mathbb{C}(A,B)} \to \mathbb{D}(F(A),F(B)),$$

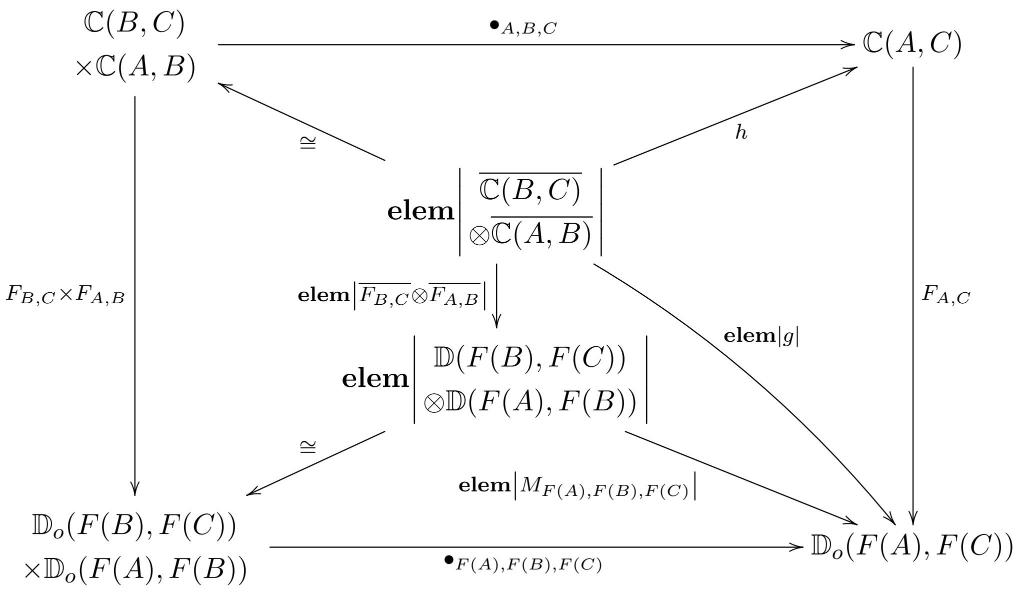
and by Corollary A.3, this is unique up to a unique isomorphism, so we can define the $\overline{\mathbb{C}}$ hom-object $\overline{\mathbb{C}}(A,B) = \overline{\mathbb{C}(A,B)}$.

2. Define the composition law for the $\mathbb{V}$-category $\overline{\mathbb{C}}$:

   We need to define a composition law

$$M_{A,B,C} \colon \overline{\mathbb{C}(B,C)} \otimes \overline{\mathbb{C}(A,B)} \to \overline{\mathbb{C}(A,C)}.$$

   If we consider the following diagram,



then the outer perimeter commutes since F is a functor. Here $\bullet$ is composition in the ordinary categories $\mathbb{C}$ and $\mathbb{D}_o$. Further, since

$$\mathbf{elem}|- \otimes -| \cong \mathbf{elem}|-| \times \mathbf{elem}|-|\,,$$

the left-hand quadrilateral commutes, and since $\mathbb{D}_o$ is the underlying ordinary category of $\mathbb{D}$, by Proposition 2.12 the bottom triangle commutes. So if we define the following morphism in $\mathbb{V}_o$

$$g = M_{F(A),F(B),F(C)} \circ \left(\overline{F_{B,C}} \otimes \overline{F_{A,B}}\right),$$

and the function $h = \bullet_{A,B,C} \circ \cong$ in **Set**, we see that we must have

$$\mathbf{elem}|g| = F_{A,C} \circ h.$$

Thus by the universal property of the **elem**$|-|$-initial lift $\overline{F_{A,C}}$, there exists a unique morphism

$$f\colon \overline{\mathbb{C}(B,C)} \otimes \overline{\mathbb{C}(A,B)} \to \overline{\mathbb{C}(A,C)}$$

such that $g = \overline{F_{A,C}} \circ f$, and $\mathbf{elem}|f| = h$.

We take f to be our composition law $M_{A,B,C}$.

3. Define the identity elements of the $\mathbb{V}$-category $\overline{\mathbb{C}}$:

   To define an identity element $j_A\colon I \to \overline{\mathbb{C}(A,A)}$, we observe that since

$$\mathbf{elem}\left|\overline{\mathbb{C}(A,A)}\right| = \mathbb{C}(A,A),$$

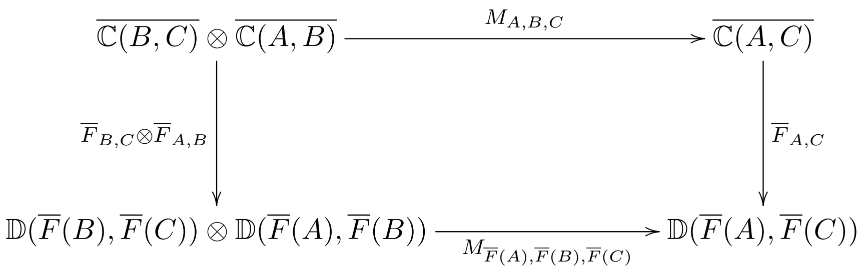
   we can take the identity morphism $1_A \in \mathbb{C}(A,A)$.

4. Show that this data defines a $\mathbb{V}$-category:

   For this collection of data to define the $\mathbb{V}$-category $\overline{\mathbb{C}}$ we require that the diagrams of Definition C.1 commute. That they do can be seen by following the following procedure. Formulate the diagram in $\mathbb{V}_o$ and apply **elem**$|-|$. Then using that **elem**$|-|$ is strong monoidal, $\mathbb{C}$ is an ordinary category, and Proposition 2.12, observe that the image of the diagram in **Set** must commute. Finally, since **elem**$|-|$ is faithful, the diagram in $\mathbb{V}_o$ must commute.

5. Show that the $\overline{F_{A,B}}$ form a $\mathbb{V}$-functor:

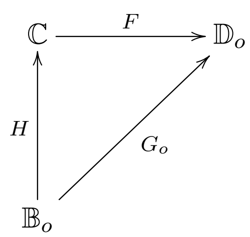
   We need to show that the $\overline{F_{A,B}}$ form a $\mathbb{V}$-functor $\overline{F}\colon \overline{\mathbb{C}} \to \mathbb{D}$. The object map of $\overline{F}$ is the same as that of F, so what is left is to show that the diagrams of Definition C.3 commute. The identity diagram is trivial, so we are left with



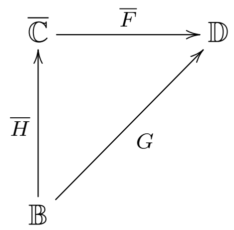
   but this clearly commutes by the construction of $M_{A,B,C}$ above.

6. Show that $\overline{F}$ has the universal property of an initial lift (step 1):

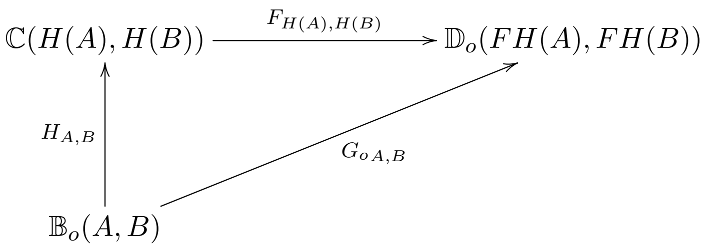
   Suppose that there is a $\mathbb{V}$-functor $G\colon \mathbb{B} \to \mathbb{D}$, and an ordinary functor $H\colon \mathbb{B}_o \to \mathbb{C}$, such that



We want to construct a $\mathbb{V}$-functor $\overline{H}\colon \mathbb{B} \to \overline{\mathbb{C}}$ such that



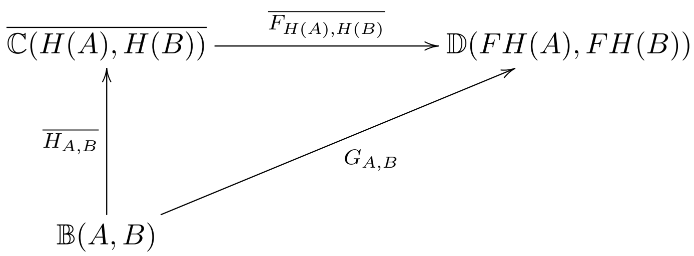
and the underlying ordinary functor of $\overline{H}$ is H. To do this we define the object map of $\overline{H}$ to be that of H, and then to define the action on hom-objects we proceed as follows. For every pair of objects $A, B \in \mathbf{obj}|\mathbb{B}|$ we can consider the diagram



where $\mathbf{elem}|G_{A,B}| = G_{oA,B}$ by Definition C.11, and so by the universal property of the $\mathbf{elem}|-|$-initial lift

$$\overline{F_{H(A),H(B)}}\colon \overline{\mathbb{C}(H(A), H(B))} \to \mathbb{D}(FH(A), FH(B)),$$

there is a unique morphism $\overline{H_{A,B}}\colon \mathbb{B}(A, B) \to \overline{\mathbb{C}(H(A), H(B))}$ such that



and $\mathbf{elem}|\overline{H_{A,B}}| = H_{A,B}$. The morphism $\overline{H_{A,B}}$ will define the action $\overline{H}_{A,B}$ of the $\mathbb{V}$-functor $\overline{H}$ on the hom-object $\mathbb{B}(A, B)$.

7. Show that $\overline{F}$ has the universal property of an initial lift (step 2):

   What remains is to show that the $\overline{H}$ we have constructed actually is a $\mathbb{V}$-functor. To do this we must show that the diagrams of Definition C.3 commute. Once again the identity diagram is trivial, and so we are left with the following diagram.

$$\begin{array}{ccc}
\mathbb{B}(B,C) \otimes \mathbb{B}(A,B) & \xrightarrow{M_{A,B,C}} & \mathbb{B}(A,C) \\
\Big\downarrow{\scriptstyle \overline{H}_{B,C} \otimes \overline{H}_{A,B}} & & \Big\downarrow{\scriptstyle \overline{H}_{A,C}} \\
\overline{\mathbb{C}}(H(B),H(C)) \otimes \overline{\mathbb{C}}(H(A),H(B)) & \xrightarrow[M_{H(A),H(B),H(C)}]{} & \overline{\mathbb{C}}(H(A),H(C))
\end{array}$$

To show that this commutes we apply the functor **elem**$|-|$ to produce its image in **Set**. This can be shown to commute using a combination of the fact that **elem**$|-|$ is strong monoidal, Proposition 2.12, and that H is an ordinary functor. Finally, since **elem**$|-|$ is faithful, the above diagram in $\mathbb{V}_o$ commutes.

□

To get a feel for how initial lifts work, we shall consider the category **Meas** of measurable spaces and measurable functions. We would like to make **Meas** into a $\mathbf{Set}_R$-category, and a **GMet**-category.

First we consider adding preorders to the objects of **Meas** and enriching over $\mathbf{Set}_R$.

Example 2.1. *The ordinary category* $\mathbf{Meas}_R$ *has the following data:*

1. *objects are triples* (X, Σ_X, R_X)*, where* (X, Σ_X) *is a measurable space, and* R_X *is a relation of type* R *on* X*,*
2. *morphisms are measurable functions that are* R*-preserving.*

There is an obvious forgetful ordinary functor $U : \mathbf{Meas}_R \to \mathbf{Set}_R$*, and* $\mathbf{Set}_R$ *is also a* $\mathbf{Set}_R$*-category, so by Proposition 2.4 and Theorem 2.13,* U *lifts to a* $\mathbf{Set}_R$*-functor, and* $\mathbf{Meas}_R$ *lifts to a* $\mathbf{Set}_R$*-category.*

The initial lift has ordered the hom-objects of $\mathbf{Meas}_R$ *pointwise, i.e. for any pair of morphisms* $f, g : (X, \Sigma_X, R_X) \to (Y, \Sigma_Y, R_Y)$*,*

$$fRg \Leftrightarrow \text{ for all } x \in X \text{ we have } f(x) R_Y g(x).$$

This is precisely what one would do, if one were to define $\mathbf{Meas}_R$ *as a* $\mathbf{Set}_R$*-category directly.*

For the other example we repeat the above, but instead add generalised metrics to the objects of **Meas** and enrich over **GMet**.

Example 2.2. *The ordinary category* **GMeas** *has the following data:*

1. *objects are triples* (X, Σ_X, d_X)*, where* (X, Σ_X) *is a measurable space, and* d_X *is a generalised metric on* X*,*

2. *morphisms are measurable functions that are non-expansive.*

There is an obvious forgetful ordinary functor $U\colon \mathbf{GMeas} \to \mathbf{GMet}$*, and* **GMet** *is also a* **GMet***-category, so by Proposition 2.9 and Theorem 2.13,* U *lifts to a* **GMet***-functor, and* **GMeas** *lifts to a* **GMet***-category.*

The initial lift has given the hom-objects a generalised metric defined pointwise, i.e. for any pair of morphisms $f, g\colon (X, \Sigma_X, d_X) \to (Y, \Sigma_Y, d_Y)$*,*

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

This is precisely what one would do, if one were to define **GMeas** *as a* **GMet***-category directly.*

In Chapter 4 we shall also need the ability to lift colimits in an ordinary category, to conical colimits (Definition C.95) in the $\mathbb{V}$-category for which the ordinary category is the underlying category. Obviously this is a bit imprecise, as an ordinary category may be the underlying category for more than one $\mathbb{V}$-category, but here we mean the $\mathbb{V}$-category constructed via the initial lift of an ordinary functor through the invocation of Theorem 2.13.

Theorem 2.14. *Given the conditions of Assumption 1, and the following:*

1. *a* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$ *that is the initial lift of an ordinary functor* $F_o\colon \mathbb{C}_o \to \mathbb{D}_o$*,*

2. *a small ordinary category* $\mathbb{J}$*, and a diagram* $D\colon \mathbb{J} \to \mathbb{C}_o$*,*

3. *the ordinary functor* $F_o\colon \mathbb{C}_o \to \mathbb{D}_o$ *creates colimits for* D *(Definition A.25),*

4. *there is a conical colimit of* $F_o D$ *in* $\mathbb{D}$*,*

then the $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$ *creates conical colimits for* D *(Definition C.99).*

Proof. The proof proceeds as follows:

1. Construct the $\mathbb{V}$-functors $\overline{D}$ and $\overline{\Delta_I}$:

   For the small ordinary category $\mathbb{J}$ we can construct the free $\mathbb{V}$-category $\mathbb{J}_{\mathbb{V}}$ (Definition C.89), which is itself small, and by Proposition C.90 and Proposition C.91, the functor categories $[\mathbb{J}_{\mathbb{V}}, \mathbb{C}]$ and $[\mathbb{J}_{\mathbb{V}}, \mathbb{V}]$ exist, and moreover there are the following isomorphisms of categories

$$[\mathbb{J}_{\mathbb{V}}, \mathbb{C}]_o \cong [\mathbb{J}, \mathbb{C}_o]$$
$$[\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o \cong [\mathbb{J}, \mathbb{V}_o],$$

   that pair the $\mathbb{V}$-functor $\overline{D} \colon \mathbb{J}_{\mathbb{V}} \to \mathbb{C}$ with D, and the $\mathbb{V}$-functor $\overline{\Delta_I} \colon \mathbb{J}_{\mathbb{V}} \to \mathbb{V}$ with the diagonal functor $\Delta_I \colon \mathbb{J} \to \mathbb{V}_o$, that maps every object in $\mathbb{J}$ to I, and every morphism in $\mathbb{J}$ to 1_I.

2. Construct the underlying colimit of F_oD in $\mathbb{D}_o$:

   By assumption there is a conical colimit $(\mathrm{colim}_{\mathbb{D}}(F_oD), \overline{\nu})$ of F_oD in $\mathbb{D}$ (Definition C.95), that is defined by the $\mathbb{V}$-natural isomorphsim (in B)

$$\mathbb{D}(\mathrm{colim}_{\mathbb{D}}(F_oD), B) \cong [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{D}(F\overline{D}(-), B)),$$

   and that has the unit

$$\overline{\nu} \colon \overline{\Delta_I} \Rightarrow \mathbb{D}(F\overline{D}(-), \mathrm{colim}_{\mathbb{D}}(F_oD)).$$

   This means there is a corresponding colimit in the underlying category $\mathbb{D}_o$, where by the isomorphism $[\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o \cong [\mathbb{J}, \mathbb{V}_o]$, there is a colimit $(\mathrm{colim}_{\mathbb{D}_o}(F_oD), \nu)$ of F_oD in $\mathbb{D}_o$, with

$$\mathrm{colim}_{\mathbb{D}_o}(F_oD) = \mathrm{colim}_{\mathbb{D}}(F_oD),$$

   and the unit

$$\nu \colon \Delta_I \Rightarrow \mathbb{D}(F_oD(-), \mathrm{colim}_{\mathbb{D}_o}(F_oD))_o$$

   has the same components as the unit $\overline{\nu}$ of the colimit in $\mathbb{D}$. There is then an isomorphism

$$\mathbb{D}_o(\mathrm{colim}_{\mathbb{D}_o}(F_oD), B) \cong [\mathbb{J}, \mathbb{V}_o](\Delta_I, \mathbb{D}(F_oD(-), B)_o),$$

   $\mathbb{V}$-natural in B.

3. Construct the colimit of D in $\mathbb{C}_o$:

   Since by assumption F_o creates colimits for D (Definition A.25), there exists a colimit $(\mathrm{colim}_{\mathbb{C}_o}(D), \mu)$ for D in $\mathbb{C}_o$, defined by

$$\mathbb{C}_o(\mathrm{colim}_{\mathbb{C}_o}(D), A) \cong [\mathbb{J}, \mathbb{V}_o](\Delta_I, \mathbb{C}(D(-), A)_o),$$

and with unit

$$\mu\colon \Delta_I \Rightarrow \mathbb{C}(D(-), \mathrm{colim}_{\mathbb{C}_o}(D))_o,$$

where

$$F_o(\mathrm{colim}_{\mathbb{C}_o}(D)) = \mathrm{colim}_{\mathbb{D}_o}(F_o D)$$
$$F_o(\mu_J) = \nu_J.$$

4. Choose a candidate for the conical colimit of D in $\mathbb{C}$:

   We need to choose a candidate for the conical colimit of D in $\mathbb{C}$, but by the isomorphism $[\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o \cong [\mathbb{J}, \mathbb{V}_o]$, the obvious choice is $(\mathrm{colim}_{\mathbb{C}}(D), \overline{\mu})$, where

   $$\mathrm{colim}_{\mathbb{C}}(D) = \mathrm{colim}_{\mathbb{C}_o}(D),$$

   and the unit $\overline{\mu}$ is given by the $\mathbb{V}$-natural transformation

   $$\overline{\mu}\colon \overline{\Delta_I} \Rightarrow \mathbb{C}(\overline{D}(-), \mathrm{colim}_{\mathbb{C}}(D)),$$

   that has the same components as the unit μ of the colimit in $\mathbb{C}_o$. To show that this is a colimit of D in $\mathbb{C}$ we must show that there is an isomorphism

   $$\mathbb{C}(\mathrm{colim}_{\mathbb{C}}(D), A) \cong [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)),$$

   $\mathbb{V}$-natural in A (Definition C.95).

5. Construct a morphism $f\colon \mathbb{C}(\mathrm{colim}_{\mathbb{C}}(D), A) \to [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A))$:

   We know that the functor category $[\mathbb{J}_{\mathbb{V}}, \mathbb{V}]$ exists, therefore we can consider the following diagram

   $$\begin{array}{ccc}
   \mathbb{C}(\mathrm{colim}_{\mathbb{C}}(D), A) & \xrightarrow{\quad f \quad} & [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)) \\
   {\scriptstyle \mathbb{C}(\overline{\mu}_J, A)}\downarrow & & \downarrow{\scriptstyle E_J} \\
   \mathbb{C}(\overline{D}(J), A) & \xrightarrow[i_{\mathbb{C}(\overline{D}(J),A)}]{} & [I, \mathbb{C}(\overline{D}(J), A)]
   \end{array}$$

   where the family of $\mathbb{V}$-natural morphisms E_J is the counit of the functor category $[\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A))$ (Definition C.84).

   Then since $i_{\mathbb{C}(\overline{D}(J),A)}$ is $\mathbb{V}$-natural in J (Section C.6), the family of morphisms

   $$i_{\mathbb{C}(\overline{D}(J),A)} \circ \mathbb{C}(\overline{\mu}_J, A)$$

   is clearly $\mathbb{V}$-natural in J, and so by the universal property of the end (Definition C.82), there exists a unique morphism f such that the diagram commutes.

6. Show that the underlying function of f is the bijection of the colimit of D in $\mathbb{C}_o$:

   The underlying function of f is given by

   $$\mathbf{elem}|f| : \mathbb{C}_o(\mathrm{colim}_{\mathbb{C}}(D), A) \to [\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o(\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)),$$

   and from the diagram above, the outer perimeter of the following diagram commutes

   $$\begin{array}{ccccc}
\mathbb{C}_o(\mathrm{colim}_{\mathbb{C}}(D), A) & & \xrightarrow{\quad\mathbf{elem}|f|\quad} & & [\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o(\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)) \\
 & \searrow^{\cong} & & \nearrow^{\cong} & \\
{\scriptstyle \mathbb{C}_o(\mu_J, A)}\downarrow & & [\mathbb{J}, \mathbb{V}_o](\Delta_I, \mathbb{C}(D(-), A)_o) & & \downarrow{\scriptstyle \mathbf{elem}|E_J|} \\
\mathbb{C}_o(D(J), A) & & \xrightarrow[\quad 1_{\mathbb{C}_o(D(J),A)}\quad]{} & & \mathbb{C}_o(D(J), A)
\end{array}$$

   where $\mathbf{elem}\left|i_{\mathbb{C}(\overline{D}(J),A)}\right| = 1_{\mathbb{C}_o(D(J),A)}$, and $\mathbf{elem}|\mathbb{C}(\overline{\mu}_J, A)| = \mathbb{C}_o(\mu_J, A)$ by Proposition C.36.

   To show that the rest of the diagram commutes, we recall that the colimit of D in $\mathbb{C}_o$ exists, and is defined by the bijection

   $$\mathbb{C}_o(\mathrm{colim}_{\mathbb{C}_o}(D), A) \cong [\mathbb{J}, \mathbb{V}_o](\Delta_I, \mathbb{C}(D(-), A)_o),$$

   that sends each mediating morphism to the corresponding cocone, and we also have the isomorphism of functor categories $[\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o \cong [\mathbb{J}, \mathbb{V}_o]$. Thus the "pentagon" via $[\mathbb{J}, \mathbb{V}_o](\Delta_I, \mathbb{C}(D(-), A)_o)$ says, that for each mediating morphism, the "arms" of the corresponding cocone, are given by the corresponding arm of the colimiting cocone composed with the mediating morphism.

   Then since E_J is a mono-source (Definition A.16), and representable functors preserve mono-sources (Proposition A.17), we have that $\mathbf{elem}|f|$ is given by the composition of isomorphisms

   $$\mathbb{C}_o(\mathrm{colim}_{\mathbb{C}}(D), A) \xrightarrow{\cong} [\mathbb{J}, \mathbb{V}_o](\Delta_I, \mathbb{C}(D(-), A)_o) \xrightarrow{\cong} [\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o(\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)).$$

7. Construct a morphism $g\colon [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)) \to \mathbb{C}(\mathrm{colim}_{\mathbb{C}}(D), A)$:

   We have one direction of the isomorphism we are trying to construct, and we now need to find an inverse to the morphism f.

   To do this we note that by Section C.6, for the functor F, we have that $F_{-,A}$ is $\mathbb{V}$-natural in the first argument

   $$F_{-,A}\colon \mathbb{C}(-, A) \Rightarrow \mathbb{D}(F(-), F(A))\colon \mathbb{C} \to \mathbb{V},$$

and thus has the underlying ordinary natural transformation

$$(F_{-,A})_o\colon \mathbb{C}(-,A)_o \Rightarrow \mathbb{D}(F_o(-),F(A))_o\colon \mathbb{C}_o \to \mathbb{V}_o.$$

Note that $(F_{-,A})_o$ is not the same as $F_{o-,A}$ (Remark C.44), indeed, the components of $F_{-,A}$ and $(F_{-,A})_o$ are exactly the same.

From this, by Proposition C.88, we have the $\mathbb{V}$-natural transformation

$$F_{\overline{D}(-),A}\colon \mathbb{C}(\overline{D}(-),A) \Rightarrow \mathbb{D}(F\overline{D}(-),F(A))\colon \mathbb{J}_{\mathbb{V}} \to \mathbb{V},$$

and by the isomorphism $[\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o \cong [\mathbb{J},\mathbb{V}_o]$, this is paired with the ordinary natural transformation

$$F_{D(-),A}\colon \mathbb{C}(D(-),A)_o \Rightarrow \mathbb{D}(F_oD(-),F(A))_o\colon \mathbb{J} \to \mathbb{V}_o.$$

Using this, and the fact that F_o creates colimits, we have the following commuting diagram.

$$\begin{array}{ccc}
\mathbb{C}_o(\operatorname{colim}_{\mathbb{C}_o}(D),A) & \xrightarrow{\cong} & [\mathbb{J},\mathbb{V}_o](\Delta_I,\mathbb{C}(D(-),A)_o) \\
\Big\downarrow {\scriptstyle F_{o\operatorname{colim}_{\mathbb{C}_o}(D),A}} & & \Big\downarrow {\scriptstyle [\mathbb{J},\mathbb{V}_o](\Delta_I,F_{D(-),A})} \\
\mathbb{D}_o(F_o(\operatorname{colim}_{\mathbb{C}_o}(D)),F_o(A)) & \xrightarrow[\cong]{} & [\mathbb{J},\mathbb{V}_o](\Delta_I,\mathbb{D}(F_oD(-),F_o(A))_o)
\end{array}$$

Now, since we have an isomorphism of categories $[\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o \cong [\mathbb{J},\mathbb{V}_o]$, and the functors defining this isomorphism must preserve composition, we also have

$$\begin{array}{ccc}
[\mathbb{J},\mathbb{V}_o](\Delta_I,\mathbb{C}(D(-),A)_o) & \xrightarrow{\cong} & [\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o(\overline{\Delta_I},\mathbb{C}(\overline{D}(-),A)) \\
\Big\downarrow {\scriptstyle [\mathbb{J},\mathbb{V}_o](\Delta_I,F_{D(-),A})} & & \Big\downarrow {\scriptstyle [\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o(\overline{\Delta_I},F_{\overline{D}(-),A})} \\
[\mathbb{J},\mathbb{V}_o](\Delta_I,\mathbb{D}(F_oD(-),F_o(A))_o) & \xrightarrow[\cong]{} & [\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o(\overline{\Delta_I},\mathbb{D}(F\overline{D}(-),F(A)))
\end{array}$$

and thus

$$\begin{array}{ccc}
\mathbb{C}_o(\operatorname{colim}_{\mathbb{C}_o}(D),A) & \xrightarrow{\cong} & [\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o(\overline{\Delta_I},\mathbb{C}(\overline{D}(-),A)) \\
\Big\downarrow {\scriptstyle F_{o\operatorname{colim}_{\mathbb{C}_o}(D),A}} & & \Big\downarrow {\scriptstyle [\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o(\overline{\Delta_I},F_{\overline{D}(-),A})} \\
\mathbb{D}_o(F_o(\operatorname{colim}_{\mathbb{C}_o}(D)),F_o(A)) & \xrightarrow[\cong]{} & [\mathbb{J}_{\mathbb{V}},\mathbb{V}]_o(\overline{\Delta_I},\mathbb{D}(F\overline{D}(-),F(A)))
\end{array}$$

So if we consider the following diagram of hom-objects in the corresponding $\mathbb{V}$-categories

$$\begin{array}{ccc}
\mathbb{C}(\text{colim}_{\mathbb{C}}(D), A) & \xleftarrow{\quad g \quad} & [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)) \\
{\scriptstyle F_{\text{colim}_{\mathbb{C}}(D),A}}\downarrow & & \downarrow{\scriptstyle [\mathbb{J}_{\mathbb{V}},\mathbb{V}](\overline{\Delta_I}, F_{\overline{D}(-),A})} \\
\mathbb{D}(F(\text{colim}_{\mathbb{C}}(D)), F(A)) & \xrightarrow[\cong]{\quad} & [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{D}(F\overline{D}(-), F(A)))
\end{array}$$

where $F_{\text{colim}_{\mathbb{C}}(D),A}$ is the **elem**$|-|$-initial lift of $F_{o\,\text{colim}_{\mathbb{C}_o}(D),A}$, then by the universal property of $F_{\text{colim}_{\mathbb{C}}(D),A}$ (Definition A.1), there is a unique morphism g making the diagram commute.

8. Show that f and g define a $\mathbb{V}$-natural isomorphism:

   We must show that f and g are inverses, and define an isomorphism. Suppose $g \circ f = h$, then **elem**$|g| \circ$ **elem**$|f| =$ **elem**$|h|$, but **elem**$|f|$ and **elem**$|g|$ are given by composites of the defining isomorphism of the colimit of D in $\mathbb{C}_o$, and the isomorphism of categories $[\mathbb{J}_{\mathbb{V}}, \mathbb{V}]_o \cong [\mathbb{J}, \mathbb{V}_o]$, thus **elem**$|h| = 1_{\mathbb{C}_o(\text{colim}_{\mathbb{C}_o}(D),A)}$. But **elem**$|-|$ is a functor, and faithful, therefore $h = 1_{\mathbb{C}(\text{colim}_{\mathbb{C}}(D),A)}$.

   Similarly $f \circ g = 1_{[\mathbb{J}_{\mathbb{V}},\mathbb{V}](\overline{\Delta_I},\mathbb{C}(\overline{D}(-),A))}$, therefore f and g define the isomorphism

   $$\mathbb{C}(\text{colim}_{\mathbb{C}}(D), A) \cong [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{D}(-), A)).$$

   To finish the proof that this is the colimit of D in $\mathbb{C}$, we must show that this isomorphism is $\mathbb{V}$-natural in A. But this follows immediately from the fact that the underlying isomorphism is natural in A and **elem**$|-|$ is faithful (Proposition C.43).

9. Show that F creates conical colimits for D:

   Finally, to show that F creates conical colimits for D (Definition C.99) we observe that by construction

   $$F(\text{colim}_{\mathbb{C}}(D)) = \text{colim}_{\mathbb{D}}(F_o D),$$

   and

   $$F(\overline{\mu}_J) = F_o(\mu_J) = \nu_J = \overline{\nu}_J,$$

   thus the unit $\overline{\mu}$ of the colimit of D in $\mathbb{C}$ is mapped to the unit $\overline{\nu}$ of the colimit of $F_o D$ in $\mathbb{D}$, and the uniqueness of such a $\mathbb{V}$-natural transformation in $\mathbb{C}$ follows from the fact that for every ordinary natural transformation $\Delta_I \Rightarrow \mathbb{C}(D(-), \text{colim}_{\mathbb{C}_o}(D))_o$, there can be at most one $\mathbb{V}$-natural transformation with the same components, and μ is the unique such ordinary natural transformation, since F_o creates colimits.

□

2.5 Discussion

In Section 2.2 and Section 2.3 we identified two categories that, in addition to the category **Set**, satisfied the conditions of Assumption 1. But why did we pick those two examples? There are two answers to this question.

The first answer is that previous authors have looked at coalgebras enriched over various categories of preorders, partial orders, or metric spaces, for example Turi and Rutten (1998); Worrell (2000a); Balan and Kurz (2011); Bílková et al. (2011). So this provides a link between our general approach and previous work in the literature.

The second answer is that both order-theoretic and metric-theoretic approaches have been taken to domain theory, and these have been shown, taking the lead from Lawvere (1973), to be related (Rutten, 1996; Wagner, 1997; Bonsangue et al., 1998). Moreover, in Balan and Kurz (2011) the authors explicitly state that they regard the categories **Preord** and **Pos** as a natural bridge between coalgebras and domain theory, and we would suggest that this extends to include metric spaces too.

Finally, as we shall see in Chapter 5, preorders and metric spaces are subsumed by the notion of a category enriched over a commutative unital quantale (Wagner, 1997).

Chapter 3

Logical Connections

At the heart of our work is the notion of a logical connection. The term itself was probably first coined in Pavlovic et al. (2006), but the idea in its most basic form goes back at least as far as Abramsky (1991).

A logical connection is a dual adjunction, or possibly even a dual equivalence, between concrete categories that arises from an object that "resides" in both categories. This object is the set of truth values for a logic - the base logic. The objects of one category are therefore logics (typically in algebraic form) of this base type, and the objects of the other category provide the semantics of these logics. The dual adjunction ties everything together in a consistent fashion.

A seminal paper on concrete dual adjunctions was the work of Porst and Tholen (1991), wherein the notion of a dualising object is formalised (see also Johnstone (1982, VI.4)). Recently in Kurz and Velebil (2011) these ideas have been extended to an enriched category theory setting, where the logics are also many-sorted. We shall adopt the enriched setting, but restrict ourselves to single sorted logics.

A brief outline of this chapter is as follows:

Section 3.1 The basic building blocks of a logical connection are described ($\mathbb{V}$-categories $\mathbb{A}$ and $\mathbb{X}$, $\mathbb{V}$-functors U and V, and a $\mathbb{V}$-dual adjunction $P \dashv S$), and their key properties summarised.

Section 3.2 The forgetful $\mathbb{V}$-functors U and V are discussed, and some auxiliary definitions made that will be needed in later sections.

Section 3.3 The $\mathbb{V}$-dual adjunction $P \dashv S$ is used to define the truth objects of the logical connection, and explicit forms for the unit and counit are derived.

Section 3.4 The logical interpretation of the $\mathbb{V}$-dual adjunction is given, and valuations, theory maps, and satisfaction maps are defined.

Section 3.5 A brief summary of the concept of a dualising object, and how they induce $\mathbb{V}$-dual adjunctions is presented.

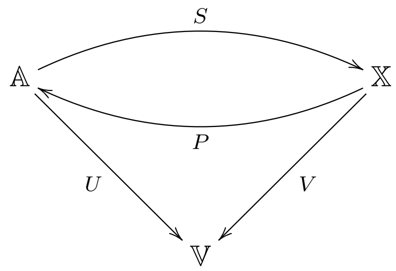
Section 3.6 A collection of both bivalent and fuzzy examples are developed, with enrichment over both preorders and generalised metric spaces.

Section 3.7 A brief review of the use of logical connections in the coalgebra literature is given.

3.1 Overview

First we shall give an overview of the basic ingredients of a logical connection, specifically an enriched logical connection. Here we follow Kurz and Velebil (2011), but restrict to a single-sorted setting.

The basic idea is that we have the following non-commuting diagram of categories and functors.



Here:

1. $\mathbb{V}$ is a symmetric monoidal closed category that satisfies Assumption 1.
2. $\mathbb{A}$ and $\mathbb{X}$ are concrete $\mathbb{V}$-categories (Definition C.25).
3. U and V are faithful and representable $\mathbb{V}$-functors (Definition C.19 and Definition C.60).
4. P and S are contravariant $\mathbb{V}$-functors that form a $\mathbb{V}$-dual adjunction (Definition C.75).

Spelling out in a little more detail what we mean by the above:

1. The category $\mathbb{V}$ is thought of as representing a kind of base-level structure that we want to be pervasive throughout the other categories and functors.
2. The objects of $\mathbb{A}$ and $\mathbb{X}$ will be thought of as base algebras and state spaces respectively. Though it must be remembered that we mean this in a very general way. Examples for $\mathbb{A}$ include Boolean algebras and distributive lattices, and for $\mathbb{X}$, sets and measurable spaces - see Section 3.6.

3. The functors U and V are forgetful functors that map algebras and state spaces to some common substrate - the category $\mathbb{V}$ representing the base-level structure.
4. The contravariant functors P and S will be thought of as mapping a state space to an algebra of generalised predicates (over that space), and an algebra to a space of generalised theories (of that algebra).

In Section 2.1 we discussed the category $\mathbb{V}$, now we shall examine the other ingredients of our framework.

The $\mathbb{V}$-Categories $\mathbb{A}$ and $\mathbb{X}$

The categories $\mathbb{A}$ and $\mathbb{X}$ are enriched over $\mathbb{V}$ (Definition C.1), and since $\mathbb{V}$ is concrete over **Set**, the hom-objects of $\mathbb{A}$ and $\mathbb{X}$ are sets with some kind of structure.

The objects of $\mathbb{A}$ are to represent logics of the type represented by $\mathbb{A}$, but typically in algebraic form. As concrete examples we could consider Boolean algebras, distributive lattices, or meet semilattices, but here we do not restrict ourselves to any particular choice.

The objects of $\mathbb{X}$ are to represent state spaces, or sets of processes, possibly with some kind of structure, for example a topology or a sigma algebra. Again we do not restrict ourselves here to any specific choices.

The $\mathbb{V}$-Functors U and V

The forgetful functors U and V are faithful (Definition C.19), and so the categories $\mathbb{A}$ and $\mathbb{X}$ are concrete over $\mathbb{V}$ (Definition C.25). The categories $\mathbb{A}$ and $\mathbb{X}$ can therefore be thought to consist of $\mathbb{V}$ objects, possibly with some additional structure, and with hom-objects given by sets of morphisms that may possibly preserve (or reflect) some, or all, of this additional structure. Moreover, the hom-objects are themselves $\mathbb{V}$ objects.

The functors U and V are also representable (Definition C.60), with representing objects A_0 and X_0, and since we will typically find the categories $\mathbb{A}$ and $\mathbb{X}$ to be categories of base algebras and state spaces, the representing objects A_0 and X_0 will correspond to the free base algebra over one generator, and the singleton state space respectively.

The $\mathbb{V}$-Dual Adjunction $P \dashv S$ [1]

The $\mathbb{V}$-dual adjunction (Definition C.75) provides the semantics for the base logics. The contravariant $\mathbb{V}$-functor P maps a state space X to an algebra (of type $\mathbb{A}$) of predicates

[1] Here and elsewhere: this symbol is reserved for the case where P is left adjoint and S is right adjoint, but in the case of contravariant functors note Remark A.45.

on X. Note, by predicate, we could mean something more general than simply a subset of X, for example a fuzzy subset. Dually, the contravariant $\mathbb{V}$-functor S maps a base logic to its set of possible theories, and assigns it whatever additional structure is required to make the set an $\mathbb{X}$ object. Again, by theory, we could mean something more general than a logically consistent set of formulae, for example with a fuzzy logic it would be a logically consistent fuzzy set of formulae.

For every $\mathbb{V}$-dual adjunction there is a $\mathbb{V}$-natural isomorphism (Proposition C.76)

$$\mathbb{A}(A, P(X)) \cong \mathbb{X}(X, S(A)).$$

On the left, the elements of $\mathbb{A}(A, P(X))$ are called valuations (Definition 3.15). They assign to each formula of A a predicate on X. On the right, the elements of $\mathbb{X}(X, S(A))$ are called theory maps (Definition 3.15). They assign to each state of X a theory of A. The dual adjunction then pairs valuations with theory maps in a consistent way.

3.2 The $\mathbb{V}$-Functors U and V

In this section we shall explore some of the consequences of making the forgetful $\mathbb{V}$-functors U and V representable, but first we shall formally state this as an assumption.

Assumption 2. We extend Assumption 1 (page 6) as follows:

6. There are faithful (Definition C.19), representable (Definition C.60), $\mathbb{V}$-functors

$$U \cong \mathbb{A}(A_0, -) \colon \mathbb{A} \to \mathbb{V}$$
$$V \cong \mathbb{X}(X_0, -) \colon \mathbb{X} \to \mathbb{V}.$$

The first observation that we can make is that the underlying ordinary functors U_o and V_o (Definition C.11) of the $\mathbb{V}$-functors U and V can be composed with $\mathbf{elem}|-|$

$$\mathbb{A}_o \xrightarrow{U_o} \mathbb{V}_o \xrightarrow{\mathbf{elem}|-|} \mathbf{Set}$$

$$\mathbb{X}_o \xrightarrow{V_o} \mathbb{V}_o \xrightarrow{\mathbf{elem}|-|} \mathbf{Set}.$$

The combined actions on the hom-sets of $\mathbb{A}_o$ and $\mathbb{X}_o$ are then seen to be

$$\mathbb{A}_o(A, B) \xrightarrow{U_{o\,A,B}} \mathbb{V}_o(U(A), U(B)) \xrightarrow{\mathbf{elem}|-|_{U(A),U(B)}} \mathbf{Set}(\mathbf{elem}|U(A)|\,, \mathbf{elem}|U(B)|)$$

$$\mathbb{X}_o(X, Y) \xrightarrow{V_{o\,X,Y}} \mathbb{V}_o(V(X), V(Y)) \xrightarrow{\mathbf{elem}|-|_{V(X),V(Y)}} \mathbf{Set}(\mathbf{elem}|V(X)|\,, \mathbf{elem}|V(Y)|).$$

Now U and V are faithful (Definition C.19), so $U_{A,B}$ and $V_{X,Y}$ are monomorphisms in $\mathbb{V}_o$, and since $\mathbf{elem}|-|$ is representable it preserves monomorphisms (Proposition A.12), and so

$$U_{oA,B} = \mathbf{elem}|U_{A,B}|$$
$$V_{oX,Y} = \mathbf{elem}|V_{X,Y}|$$

are injective. Thus since $\mathbf{elem}|-|$ is also faithful, the categories $\mathbb{A}$ and $\mathbb{X}$ are concrete over $\mathbb{V}$ (Definition C.25), and the underlying categories $\mathbb{A}_o$ and $\mathbb{X}_o$ are concrete over **Set** (Definition A.9).

The $\mathbb{V}$-functors U and V are also representable, which means that for any object A in $\mathbb{A}$, and any object X in $\mathbb{X}$,

$$\mathbf{elem}|U(A)| \cong \mathbb{A}_o(A_0, A)$$
$$\mathbf{elem}|V(X)| \cong \mathbb{X}_o(X_0, X),$$

and by Proposition C.36 we have

$$\mathbf{elem}|-|_{U(A),U(B)} \circ U_{oA,B} \cong \mathbb{A}_o(A_0, -)_{A,B}$$
$$\mathbf{elem}|-|_{V(X),V(Y)} \circ V_{oX,Y} \cong \mathbb{X}_o(X_0, -)_{X,Y},$$

and thus $\mathbf{elem}|U_o(-)|$ and $\mathbf{elem}|V_o(-)|$ are both faithful and representable.

Now faithful ordinary functors reflect monomorphisms and epimorphisms, and representable ordinary functors preserve monomorphisms (Proposition A.12 and Proposition A.14), so we have

$$\text{Inject}_{\mathbb{A}_o(A,B)} = \text{ monos in } \mathbb{A}_o(A, B)$$
$$\text{Surject}_{\mathbb{A}_o(A,B)} \subseteq \text{ epis in } \mathbb{A}_o(A, B),$$

where $\text{Inject}_{\mathbb{A}_o(A,B)}$ is the class of morphisms in $\mathbb{A}_o(A, B)$ with injective underlying functions, and $\text{Surject}_{\mathbb{A}_o(A,B)}$ those with surjective underlying functions. Similarly,

$$\text{Inject}_{\mathbb{X}_o(X,Y)} = \text{ monos in } \mathbb{X}_o(X, Y)$$
$$\text{Surject}_{\mathbb{X}_o(X,Y)} \subseteq \text{ epis in } \mathbb{X}_o(X, Y),$$

where $\text{Inject}_{\mathbb{X}_o(X,Y)}$ is the class of morphisms in $\mathbb{X}_o(X, Y)$ with injective underlying functions, and $\text{Surject}_{\mathbb{X}_o(X,Y)}$ those with surjective underlying functions.

In subsequent sections in this chapter we shall need to manipulate expressions involving the $\mathbb{V}$-functors U and V. In particular, for a morphism $f \in \mathbb{A}_o(A, B)$, and an element $a \in \mathbf{elem}|U(A)|$, we shall be interested in reversing the order of evaluation in expressions of the form $U(f)(a)$, and similarly for V.

For the functor U, the morphism $U_{A,B}$ has the transpose $U^{\dagger}_{A,B}$ under the adjunction $- \otimes U(A) \dashv [U(A), -]$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{A}(A,B) \otimes U(A) & & \\ \downarrow{\scriptstyle U_{A,B} \otimes 1_{U(A)}} & \overset{U^{\dagger}_{A,B}}{\searrow} & \\ [U(A), U(B)] \otimes U(A) & \xrightarrow[e_{U(A),U(B)}]{} & U(B) \end{array}$$

Now since the functor U is representable, by Proposition C.64, the morphism $U_{A,B}$ is given by the composite

$$U_{A,B} = [\cong_A, \cong_B^{-1}] \circ \mathbb{A}(A_0, -)_{A,B},$$

where $\cong_A$ and $\cong_B$ are the isomorphisms of the representation. Then by Definition C.32, and the $\mathbb{V}$-naturality of $\cong$, we have the following proposition.

Proposition 3.1. *Given the conditions of Assumption 2, the transpose $U^{\dagger}_{A,B}$ of $U_{A,B}$ under the adjunction $- \otimes U(A) \dashv [U(A), -]$ is given by*

$$\mathbb{A}(A,B) \otimes U(A) \xrightarrow{1_{\mathbb{A}(A,B)} \otimes \cong_A} \mathbb{A}(A,B) \otimes \mathbb{A}(A_0, A) \xrightarrow{M_{A_0,A,B}} \mathbb{A}(A_0, B) \xrightarrow{\cong_B^{-1}} U(B),$$

*and for $f \in \mathbb{A}_o(A,B)$, and $a \in$ **elem**$|U(A)|$, we have*

$$U(f)(a) = U^{\dagger}_{A,B} \circ (f \otimes a) \circ l_I^{-1} = \cong_B^{-1}(f \bullet \cong_A(a)).$$

Here $M_{A_0,A,B}$ is the composition law of $\mathbb{A}$ (Definition C.1).

We also have a dual proposition for the functor V.

Proposition 3.2. *Given the conditions of Assumption 2, the transpose $V^{\dagger}_{X,Y}$ of $V_{X,Y}$ under the adjunction $- \otimes V(X) \dashv [V(X), -]$ is given by*

$$\mathbb{X}(X,Y) \otimes V(X) \xrightarrow{1_{\mathbb{X}(X,Y)} \otimes \cong_X} \mathbb{X}(X,Y) \otimes \mathbb{X}(X_0, X) \xrightarrow{M_{X_0,X,Y}} \mathbb{X}(X_0, Y) \xrightarrow{\cong_Y^{-1}} V(Y),$$

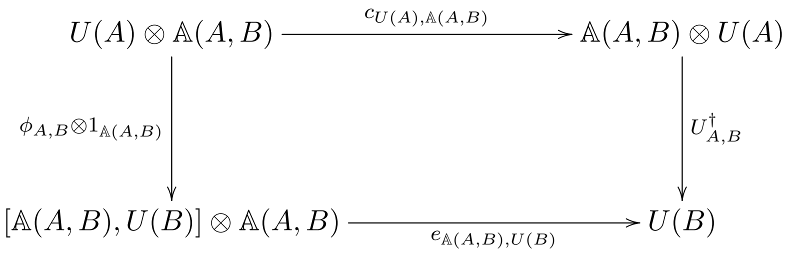
*and for $f \in \mathbb{X}_o(X,Y)$, and $x \in$ **elem**$|V(X)|$, we have*

$$V(f)(x) = V^{\dagger}_{X,Y} \circ (f \otimes x) \circ l_I^{-1} = \cong_Y^{-1}(f \bullet \cong_X(x)).$$

Next we note that using Proposition B.12, and the symmetry c of $\mathbb{V}_o$ (Definition B.6), there is a natural isomorphism

$$[U(A), [\mathbb{A}(A,B), U(B)]] \cong [\mathbb{A}(A,B), [U(A), U(B)]].$$

Specifically, if we take $U_{A,B}$, this gives a morphism $\phi_{A,B}$ such that the following diagram commutes.



Moreover, since $U_{A,B}$ and $c_{U(A),\mathbb{A}(A,B)}$ are both $\mathbb{V}$-natural in A and B, by Section C.6, so is $\phi_{A,B}$. Therefore we make the following definition.

Definition 3.3. Given the conditions of Assumption 2, we define a $\mathbb{V}$-natural transformation

$$\phi_{A,B} \colon U(A) \to [\mathbb{A}(A,B), U(B)],$$

where the component $\phi_{A,B}$ is defined to have the transpose $\phi^{\dagger}_{A,B}$ under the adjunction $- \otimes \mathbb{A}(A,B) \dashv [\mathbb{A}(A,B), -]$ given by

$$\phi^{\dagger}_{A,B} = U^{\dagger}_{A,B} \circ c_{U(A),\mathbb{A}(A,B)},$$

and for all $A, B \in \mathbf{obj}|\mathbb{A}|$, $a \in \mathbf{elem}|U(A)|$, and $f \in \mathbb{A}_o(A,B)$, we have

$$\phi_{A,B}(a)(f) = U(f)(a).$$

The $\mathbb{V}$-natural transformation ϕ thus provides the reordering of evaluation under U that we shall need later.

Similarly for the natural isomorphism

$$[V(X), [\mathbb{X}(X,Y), V(Y)]] \cong [\mathbb{X}(X,Y), [V(X), V(Y)]]$$

we make the following definition.

Definition 3.4. Given the conditions of Assumption 2, we define a $\mathbb{V}$-natural transformation

$$\psi_{X,Y} \colon V(X) \to [\mathbb{X}(X,Y), V(Y)],$$

where the component $\psi_{X,Y}$ is defined to have the transpose $\psi^{\dagger}_{X,Y}$ under the adjunction $- \otimes \mathbb{X}(X,Y) \dashv [\mathbb{X}(X,Y), -]$ given by

$$\psi^{\dagger}_{X,Y} = V^{\dagger}_{X,Y} \circ c_{V(X),\mathbb{X}(X,Y)},$$

and for all $X, Y \in \mathbf{obj}|\mathbb{X}|$, $x \in \mathbf{elem}|V(X)|$, and $f \in \mathbb{X}_o(X,Y)$, we have

$$\psi_{X,Y}(x)(f) = V(f)(x).$$

To derive explicit expressions for the components of ϕ and ψ we start with the following diagram, which commutes by the definition of the hom-functor $\mathbb{A}(-,B)$ (Definition C.34), and naturality of c and e.

$$\begin{array}{ccc}
U(A) \otimes \mathbb{A}(A,B) & \xrightarrow{c_{U(A),\mathbb{A}(A,B)}} & \mathbb{A}(A,B) \otimes U(A) \\
\downarrow {\scriptstyle \cong_A \otimes 1_{\mathbb{A}(A,B)}} & & \downarrow {\scriptstyle 1_{\mathbb{A}(A,B)} \otimes \cong_A} \\
\mathbb{A}(A_0,A) \otimes \mathbb{A}(A,B) & \xrightarrow{c_{\mathbb{A}(A_0,A),\mathbb{A}(A,B)}} & \mathbb{A}(A,B) \otimes \mathbb{A}(A_0,A) \\
\downarrow {\scriptstyle \mathbb{A}(-,B)_{A_0,A} \otimes 1_{\mathbb{A}(A,B)}} & & \downarrow {\scriptstyle M_{A_0,A,B}} \\
[\mathbb{A}(A,B),\mathbb{A}(A_0,B)] \otimes \mathbb{A}(A,B) & \xrightarrow[e_{\mathbb{A}(A,B),\mathbb{A}(A_0,B)}]{} & \mathbb{A}(A_0,B) \\
\downarrow {\scriptstyle [\mathbb{A}(A,B),\cong_B^{-1}] \otimes 1_{\mathbb{A}(A,B)}} & & \downarrow {\scriptstyle \cong_B^{-1}} \\
[\mathbb{A}(A,B),U(B)] \otimes \mathbb{A}(A,B) & \xrightarrow[e_{\mathbb{A}(A,B),U(B)}]{} & U(B)
\end{array}$$

The righthand column is the transpose of $U_{A,B}$ given by Proposition 3.1, and the lefthand column therefore gives an explicit expression for $\phi_{A,B}$.

Proposition 3.5. *Given the conditions of Assumption 2, the component $\phi_{A,B}$ of the $\mathbb{V}$-natural transformation*

$$\phi_{A,B} \colon U(A) \to [\mathbb{A}(A,B), U(B)],$$

is given by

$$U(A) \xrightarrow{\cong_A} \mathbb{A}(A_0,A) \xrightarrow{\mathbb{A}(-,B)_{A_0,A}} [\mathbb{A}(A,B),\mathbb{A}(A_0,B)] \xrightarrow{[\mathbb{A}(A,B),\cong_B^{-1}]} [\mathbb{A}(A,B),U(B)].$$

Similarly we have the corresponding result for ψ.

Proposition 3.6. *Given the conditions of Assumption 2, the component $\psi_{X,Y}$ of the $\mathbb{V}$-natural transformation*

$$\psi_{X,Y} \colon V(X) \to [\mathbb{X}(X,Y), V(Y)],$$

is given by

$$V(X) \xrightarrow{\cong_X} \mathbb{X}(X_0,X) \xrightarrow{\mathbb{X}(-,Y)_{X_0,X}} [\mathbb{X}(X,Y),\mathbb{X}(X_0,Y)] \xrightarrow{[\mathbb{X}(X,Y),\cong_Y^{-1}]} [\mathbb{X}(X,Y),V(Y)].$$

3.3 The $\mathbb{V}$-Dual Adjunction $P \dashv S$

We now spell out in more detail the $\mathbb{V}$-dual adjunction $P \dashv S$, and in particular we examine the precise form the unit and counit take as this will be useful later on. We follow the line taken in Porst and Tholen (1991, Section 1-B), but generalise to our enriched setting.

Firstly we need to make precise the assumptions we are making.

Assumption 3. We extend Assumption 2 (page 30) as follows:

7. There is a $\mathbb{V}$-dual adjunction (Definition C.75)

$$\rho, \sigma \colon P \dashv S \colon \mathbb{A} \to \mathbb{X}$$

satisfying the triangular equations

$$P\sigma \circ \rho P = 1_P$$
$$S\rho \circ \sigma S = 1_S,$$

and this yields (Proposition C.76) a $\mathbb{V}$-natural isomorphism

$$\Phi_{A,X} \colon \mathbb{A}(A, P(X)) \cong \mathbb{X}(X, S(A)).$$

We shall use the notation $f^\flat$ for the dual adjunct (Definition A.46) of

$$f \in \mathbb{A}_o(A, P(X)),$$

and $g^\sharp$ for the dual adjunct of

$$g \in \mathbb{X}_o(X, S(A)).$$

The images under P and S of the representing objects A_0 and X_0 of the functors U and V will play a vital role, and as we shall see in Section 3.4, they have a specific logical interpretation. We therefore make the following definition, and postpone the explanation of the name "truth object" until later.

Definition 3.7. Given the conditions of Assumption 3, the images under P and S of X_0 and A_0 we call the **truth objects** of $\mathbb{A}$ and $\mathbb{X}$ (respectively), and denote them as follows:

$$\Omega_{\mathbb{A}} = P(X_0)$$
$$\Omega_{\mathbb{X}} = S(A_0).$$

The truth objects $\Omega_{\mathbb{A}}$ and $\Omega_{\mathbb{X}}$ define contravariant representable $\mathbb{V}$-functors (Definition C.61) as given by the following proposition, which is a direct enrichment of Porst and Tholen (1991, Proposition 1.2).

Proposition 3.8. *Given the conditions of Assumption 3, the following hold:*

1. *The contravariant $\mathbb{V}$-functors $UP\colon \mathbb{X} \to \mathbb{V}$ and $VS\colon \mathbb{A} \to \mathbb{V}$ are representable functors, where the representing objects are the corresponding truth objects*

$$UP \cong \mathbb{X}(-, \Omega_{\mathbb{X}})$$
$$VS \cong \mathbb{A}(-, \Omega_{\mathbb{A}}).$$

2. *The truth objects $\Omega_{\mathbb{A}}$ and $\Omega_{\mathbb{X}}$ have the same underlying $\mathbb{V}$ object up to isomorphism, i.e. there exists a $\mathbb{V}$-isomorphism $\tau\colon U(\Omega_{\mathbb{A}}) \cong V(\Omega_{\mathbb{X}})$.*

Proof. For any object A in $\mathbb{A}$ we have

$$\mathbb{A}(A, \Omega_{\mathbb{A}}) = \mathbb{A}(A, P(X_0)) \cong \mathbb{X}(X_0, S(A)) \cong VS(A),$$

and thus $VS \cong \mathbb{A}(-, \Omega_{\mathbb{A}})$.

Similarly, for any object X in $\mathbb{X}$ we have

$$\mathbb{X}(X, \Omega_{\mathbb{X}}) = \mathbb{X}(X, S(A_0)) \cong \mathbb{A}(A_0, P(X)) \cong UP(X),$$

and thus $UP \cong \mathbb{X}(-, \Omega_{\mathbb{X}})$.

For the second part of the proposition, we have

$$V(\Omega_{\mathbb{X}}) = VS(A_0) \cong \mathbb{A}(A_0, \Omega_{\mathbb{A}}) \cong U(\Omega_{\mathbb{A}}).$$

□

Now, as noted in Porst and Tholen (1991, Remark 1.3), there is a special case where the isomorphisms of Proposition 3.8 are actually equalities. In such a case we use the following terminology.

Definition 3.9. Given the conditions of Assumption 3, if

$$VS = \mathbb{A}(-, \Omega_{\mathbb{A}})$$
$$UP = \mathbb{X}(-, \Omega_{\mathbb{X}}),$$

then the dual adjunction is said to be **strictly represented** by $(\Omega_{\mathbb{A}}, \Omega_{\mathbb{X}})$.

It turns out that with very little loss of generality we can always assume that we have a strict representation.

In Porst and Tholen (1991, Remark 1.3) it is noted that if the functors (ordinary functors in their case) U and V are uniquely transportable (Definition A.11), then we can always assume that the dual adjunction is strictly represented. This is also the case in our enriched setting.

To see this we first recall that if the $\mathbb{V}$-functor U is uniquely transportable (Definition C.28), then for every $\mathbb{X}$ object X, and every $\mathbb{V}_o$-isomorphism

$$f \in \mathbb{V}_o(UP(X), \mathbb{X}(X, \Omega_{\mathbb{X}})),$$

there exists a unique A_X in $\mathbb{A}$ such that $U(A_X) = \mathbb{X}(X, \Omega_{\mathbb{X}})$, and an $\mathbb{A}_o$-isomorphism

$$f_X \in \mathbb{A}_o(P(X), A_X),$$

such that $U(f_X) = f$. Thus we can define a contravariant $\mathbb{V}$-functor $P' \colon \mathbb{X} \to \mathbb{A}$ by $P'(X) = A_X$, and for every pair of objects X and Y in $\mathbb{X}$, the morphism

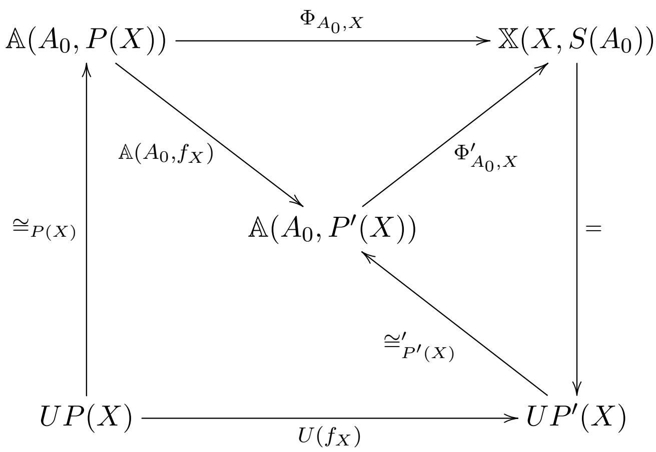
$$P'_{X,Y} \colon \mathbb{X}(X, Y) \to \mathbb{A}(P'(Y), P'(X))$$

is given by $P'_{X,Y} = \mathbb{A}(f_Y^{-1}, f_X) \circ P_{X,Y}$.

Moreover, if we consider the isomorphism

$$f = \Phi_{A_0,X} \circ \cong_{P(X)} \colon UP(X) \to \mathbb{X}(X, \Omega_{\mathbb{X}}),$$

the construction of the functor P' ensures that the outer perimeter of the following diagram commutes.



We can then define the $\mathbb{V}$-natural isomorphisms

$$\begin{aligned}\Phi'_{A_0,X} &= \Phi_{A_0,X} \circ \mathbb{A}(A_0, f_X^{-1}) \\ \cong'_{P'(X)} &= \mathbb{A}(A_0, f_X) \circ \cong_{P(X)} \circ U(f_X^{-1}),\end{aligned}$$

and since all morphisms above are isomorphisms, the rest of the diagram commutes, and in particular

$$\cong'_{P'(X)} = \Phi'^{-1}_{A_0,X}.$$

Similarly, we can define a contravariant $\mathbb{V}$-functor $S' \colon \mathbb{A} \to \mathbb{X}$, and it is easy to see that P' and S' form a $\mathbb{V}$-dual adjunction Φ', and that this dual adjunction is strictly represented by $(\Omega_{\mathbb{A}}, \Omega_{\mathbb{X}})$. Moreover, the action of the representable functors U and V on the images of objects under P' and S' gives

$$\begin{aligned}\Phi'^{-1}_{A_0,-} &\colon UP' \cong \mathbb{A}(A_0, P'(-)) \\ \Phi'_{-,X_0} &\colon VS' \cong \mathbb{X}(X_0, S'(-)).\end{aligned}$$

As unique transportability of U and V is a relatively mild condition, with minimal loss of generality we may assume that every dual adjunction is strictly represented.

Assumption 4. We extend Assumption 3 (page 35) as follows:

8. The $\mathbb{V}$-dual adjunction $\rho, \sigma \colon P \dashv S \colon \mathbb{A} \to \mathbb{X}$ is strictly represented by $(\Omega_{\mathbb{A}}, \Omega_{\mathbb{X}})$, meaning

$$\begin{aligned}VS &= \mathbb{A}(-, \Omega_{\mathbb{A}}) \\ UP &= \mathbb{X}(-, \Omega_{\mathbb{X}}),\end{aligned}$$

and

$$\begin{aligned}\Phi^{-1}_{A_0,-} &\colon UP \cong \mathbb{A}(A_0, P(-)) \\ \Phi_{-,X_0} &\colon VS \cong \mathbb{X}(X_0, S(-)).\end{aligned}$$

Under this additional assumption, it is easy to explicitly write down the isomorphism between the underlying $\mathbb{V}$ objects of the truth objects of $\mathbb{A}$ and $\mathbb{X}$ from Proposition 3.8.

Proposition 3.10. *Given the conditions of Assumption 4, then the $\mathbb{V}$-isomorphism*

$$\tau \colon U(\Omega_{\mathbb{A}}) \cong V(\Omega_{\mathbb{X}})$$

is given by

$$\tau = \Phi^{-1}_{A_0,X_0}.$$

Proof. We have that

$$U(\Omega_{\mathbb{A}}) = UP(X_0) = \mathbb{X}(X_0, \Omega_{\mathbb{X}}) = \mathbb{X}(X_0, S(A_0)),$$

and also that

$$V(\Omega_{\mathbb{X}}) = VS(A_0) = \mathbb{A}(A_0, \Omega_{\mathbb{A}}) = \mathbb{A}(A_0, P(X_0)),$$

hence since

$$\Phi_{A_0,X_0} \colon \mathbb{A}(A_0, P(X_0)) \cong \mathbb{X}(X_0, S(A_0)),$$

we can deduce $\tau = \Phi^{-1}_{A_0,X_0}$. □

What we aim to do next is to give an explicit statement of the action of the unit and counit of the dual adjunction. It will turn out that the way to do this is to use the functors U and V to translate the problem to the category $\mathbb{V}$, as all morphisms in $\mathbb{A}_o$ and $\mathbb{X}_o$ can be thought of as having an underlying $\mathbb{V}_o$ morphism.

First we need the following pair of lemmas.

Lemma 3.11. *Given the conditions of Assumption 4, for all objects A in $\mathbb{A}$, the following diagram commutes.*

$$\begin{array}{ccc} U(A) & \xrightarrow{U(\rho_A)} & UPS(A) \\ \Big\downarrow{\scriptstyle \cong_A} & & \Big\downarrow{\scriptstyle =} \\ \mathbb{A}(A_0, A) & \xrightarrow[S_{A_0,A}]{} & \mathbb{X}(S(A), \Omega_{\mathbb{X}}) \end{array}$$

Proof. By Proposition C.64, $U(\rho_A)$ is given by

$$U(A) \xrightarrow{\cong_A} \mathbb{A}(A_0, A) \xrightarrow{\mathbb{A}(A_0,\rho_A)} \mathbb{A}(A_0, PS(A)) \xrightarrow{\cong^{-1}_{PS(A)}} UPS(A),$$

but $\cong^{-1}_{PS(A)} = \Phi_{A_0,S(A)}$, and by Proposition C.81 we have

$$\begin{array}{ccc} \mathbb{A}(A_0, A) & \xrightarrow{\mathbb{A}(A_0,\rho_A)} & \mathbb{A}(A_0, PS(A)) \\ & {\scriptstyle S_{A_0,A}} \searrow & \Big\downarrow{\scriptstyle \Phi_{A_0,S(A)}} \\ & & \mathbb{X}(S(A), \Omega_{\mathbb{X}}) \end{array}$$

so the result follows. □

Lemma 3.12. *Given the conditions of Assumption 4, for all objects A in $\mathbb{A}$, the following diagram commutes.*

$$
\begin{array}{ccc}
\mathbb{A}(A_0, A) \otimes \mathbb{A}(A, \Omega_{\mathbb{A}}) & \xrightarrow{S_{A_0,A} \otimes \Phi_{A,X_0}} & \mathbb{X}(S(A), \Omega_{\mathbb{X}}) \otimes \mathbb{X}(X_0, S(A)) \\
\downarrow{\scriptstyle c_{\mathbb{A}(A_0,A),\mathbb{A}(A,\Omega_{\mathbb{A}})}} & & \\
\mathbb{A}(A, \Omega_{\mathbb{A}}) \otimes \mathbb{A}(A_0, A) & & \downarrow{\scriptstyle M_{X_0,S(A),\Omega_{\mathbb{X}}}} \\
\downarrow{\scriptstyle M_{A_0,A,\Omega_{\mathbb{A}}}} & & \\
\mathbb{A}(A_0, \Omega_{\mathbb{A}}) & \xrightarrow[\Phi_{A_0,X_0}]{} & \mathbb{X}(X_0, \Omega_{\mathbb{X}})
\end{array}
$$

Proof. From the action of the contravariant $\mathbb{V}$-functor S on composites (Definition C.3 and Definition C.7), and the naturality of M, the following diagram commutes.

$$
\begin{array}{ccc}
\mathbb{A}(A_0, A) \otimes \mathbb{A}(A, \Omega_{\mathbb{A}}) & \xrightarrow{M_{A_0,A,\Omega_{\mathbb{A}}} \circ c_{\mathbb{A}(A_0,A),\mathbb{A}(A,\Omega_{\mathbb{A}})}} & \mathbb{A}(A_0, \Omega_{\mathbb{A}}) \\
\downarrow{\scriptstyle S_{A_0,A} \otimes S_{A,\Omega_{\mathbb{A}}}} & & \downarrow{\scriptstyle S_{A_0,\Omega_{\mathbb{A}}}} \\
\mathbb{X}(S(A), \Omega_{\mathbb{X}}) \otimes \mathbb{X}(SP(X_0), S(A)) & \xrightarrow[M_{SP(X_0),S(A),\Omega_{\mathbb{X}}}]{} & \mathbb{X}(SP(X_0), \Omega_{\mathbb{X}}) \\
\downarrow{\scriptstyle 1_{\mathbb{X}(S(A),\Omega_{\mathbb{X}})} \otimes \mathbb{X}(\sigma_{X_0}, S(A))} & & \downarrow{\scriptstyle \mathbb{X}(\sigma_{X_0},\Omega_{\mathbb{X}})} \\
\mathbb{X}(S(A), \Omega_{\mathbb{X}}) \otimes \mathbb{X}(X_0, S(A)) & \xrightarrow[M_{X_0,S(A),\Omega_{\mathbb{X}}}]{} & \mathbb{X}(X_0, \Omega_{\mathbb{X}})
\end{array}
$$

Finally, by Proposition C.76, we have

$$
\begin{aligned}
\Phi_{A,X_0} &= \mathbb{X}(\sigma_{X_0}, S(A)) \circ S_{A,\Omega_{\mathbb{A}}} \\
\Phi_{A_0,X_0} &= \mathbb{X}(\sigma_{X_0}, \Omega_{\mathbb{X}}) \circ S_{A_0,\Omega_{\mathbb{A}}},
\end{aligned}
$$

and the result follows. □

The main result of this section shows that we can evaluate the unit and counit by repeated application of the functors U and V. Use is made of the $\mathbb{V}$-natural transformations ϕ (Definition 3.3) and ψ (Definition 3.4) to reorder evaluation under U and V.

This result is an extension of Porst and Tholen (1991, Proposition 1.4) to our enriched setting.

Theorem 3.13. *Given the conditions of Assumption 4, for all objects A in $\mathbb{A}$, the following diagram commutes.*

$$\begin{array}{ccc}
U(A) & \xrightarrow{\phi_{A,\Omega_{\mathbb{A}}}} & [\mathbb{A}(A,\Omega_{\mathbb{A}}),U(\Omega_{\mathbb{A}})] \\
\downarrow{\scriptstyle U(\rho_A)} & & \downarrow{\scriptstyle [\mathbb{A}(A,\Omega_{\mathbb{A}}),\tau]} \\
UPS(A) \xrightarrow[=]{} \mathbb{X}(S(A),\Omega_{\mathbb{X}}) & \xrightarrow[V_{S(A),\Omega_{\mathbb{X}}}]{} & [\mathbb{A}(A,\Omega_{\mathbb{A}}),V(\Omega_{\mathbb{X}})]
\end{array}$$

Dually, for all objects X in $\mathbb{X}$, the following diagram commutes.

$$\begin{array}{ccc}
V(X) & \xrightarrow{\psi_{X,\Omega_{\mathbb{X}}}} & [\mathbb{X}(X,\Omega_{\mathbb{X}}),V(\Omega_{\mathbb{X}})] \\
\downarrow{\scriptstyle V(\sigma_X)} & & \downarrow{\scriptstyle [\mathbb{X}(X,\Omega_{\mathbb{X}}),\tau^{-1}]} \\
VSP(X) \xrightarrow[=]{} \mathbb{A}(P(X),\Omega_{\mathbb{A}}) & \xrightarrow[U_{P(X),\Omega_{\mathbb{A}}}]{} & [\mathbb{X}(X,\Omega_{\mathbb{X}}),U(\Omega_{\mathbb{A}})]
\end{array}$$

Proof. We shall only prove the first case (for ρ_A), but the second follows in a similar fashion.

We shall proceed by writing $f = [\mathbb{A}(A,\Omega_{\mathbb{A}}),\tau] \circ \phi_{A,\Omega_{\mathbb{A}}}$ and $g = V_{S(A),\Omega_{\mathbb{X}}} \circ = \circ U(\rho_A)$, and then prove that $f = g$. To do this we will consider the transposes $f^\dagger$ and $g^\dagger$ of f and g under the adjunction $- \otimes \mathbb{A}(A,\Omega_{\mathbb{A}}) \dashv [\mathbb{A}(A,\Omega_{\mathbb{A}}),-]$. If we can show that $f^\dagger = g^\dagger$, then the bijection between morphisms and their transposes will force $f = g$.

The transpose of f is given by Definition 3.3, Proposition 3.1, and the naturality of c and e, as

$$\begin{array}{ccc}
U(A) \otimes \mathbb{A}(A,\Omega_{\mathbb{A}}) & \xrightarrow{f^\dagger} & V(\Omega_{\mathbb{X}}) \\
\downarrow{\scriptstyle \cong_A \otimes 1_{\mathbb{A}(A,\Omega_{\mathbb{A}})}} & & \uparrow{\scriptstyle \tau} \\
\mathbb{A}(A_0,A) \otimes \mathbb{A}(A,\Omega_{\mathbb{A}}) & & U(\Omega_{\mathbb{A}}) \\
\downarrow{\scriptstyle c_{\mathbb{A}(A_0,A),\mathbb{A}(A,\Omega_{\mathbb{A}})}} & & \uparrow{\scriptstyle \cong^{-1}_{\Omega_{\mathbb{A}}}} \\
\mathbb{A}(A,\Omega_{\mathbb{A}}) \otimes \mathbb{A}(A_0,A) & \xrightarrow[M_{A_0,A,\Omega_{\mathbb{A}}}]{} & \mathbb{A}(A_0,\Omega_{\mathbb{A}})
\end{array}$$

and by Proposition 3.2, and the fact that $\Phi_{A,X_0} \colon \mathbb{A}(A, \Omega_{\mathbb{A}}) = VS(A) \cong \mathbb{X}(X_0, S(A))$, the transpose of g is given by

$$
\begin{array}{ccc}
U(A) \otimes \mathbb{A}(A, \Omega_{\mathbb{A}}) & \xrightarrow{\quad g^{\dagger} \quad} & V(\Omega_{\mathbb{X}}) \\
{\scriptstyle U(\rho_A) \otimes 1_{\mathbb{A}(A,\Omega_{\mathbb{A}})}} \Big\downarrow & & \Big\uparrow {\scriptstyle \cong^{-1}_{\Omega_{\mathbb{X}}}} \\
UPS(A) \otimes \mathbb{A}(A, \Omega_{\mathbb{A}}) & & \mathbb{X}(X_0, \Omega_{\mathbb{X}}) \\
{\scriptstyle = \otimes 1_{\mathbb{A}(A,\Omega_{\mathbb{A}})}} \Big\downarrow & & \Big\uparrow {\scriptstyle M_{X_0, S(A), \Omega_{\mathbb{X}}}} \\
\mathbb{X}(S(A), \Omega_{\mathbb{X}}) \otimes \mathbb{A}(A, \Omega_{\mathbb{A}}) & \xrightarrow[1_{\mathbb{X}(S(A),\Omega_{\mathbb{X}})} \otimes \Phi_{A,X_0}]{} & \mathbb{X}(S(A), \Omega_{\mathbb{X}}) \otimes \mathbb{X}(X_0, S(A))
\end{array}
$$

Now the first thing we observe is that by Proposition 3.10 we have $\tau = \Phi^{-1}_{A_0,X_0}$, and also $\cong^{-1}_{\Omega_{\mathbb{A}}} = \Phi_{A_0,X_0} = \tau^{-1}$ and $\cong^{-1}_{\Omega_{\mathbb{X}}} = \Phi^{-1}_{A_0,X_0} = \tau$. The rest of the proof follows by applying $- \otimes \mathbb{A}(A, \Omega_{\mathbb{A}})$ to Lemma 3.11, and then using Lemma 3.12. $\square$

Finally, by Definition 3.3 and Definition 3.4, we have this simple corollary. It shows that the unit and counit of the dual adjunction are given by the evaluation of morphisms to the truth objects of $\mathbb{A}$ and $\mathbb{X}$, modulo the isomorphism τ.

Corollary 3.14. *Given the conditions of Assumption 4, for the unit and counit of the dual adjunction we have*

$$
\begin{aligned}
V(U(\rho_A)(a))(s) &= \tau(U(s)(a)) \\
U(V(\sigma_X)(x))(u) &= \tau^{-1}(V(u)(x)),
\end{aligned}
$$

for $a \in \mathbf{elem}|U(A)|$, $x \in \mathbf{elem}|V(X)|$, $s \in \mathbb{A}_o(A, \Omega_{\mathbb{A}})$, *and* $u \in \mathbb{X}_o(X, \Omega_{\mathbb{X}})$.

3.4 The Logical Interpretation

So far we have described the dual adjunction framework in which we operate purely in mathematical terms. However we intend to give this framework a logical interpretation. The key idea is that the state spaces of $\mathbb{X}$ contain states x, and the algebras of $\mathbb{A}$ contain formulas a, and we want to be able to take a pair (x, a) and assign a truth value to the formula a in the state x. These truth values we will take from the truth objects $\Omega_{\mathbb{A}}$ and $\Omega_{\mathbb{X}}$.

There is an alternative way of looking at assigning a logical interpretation. If our logics are bivalent (only two truth values), then we could instead assign to each formula a

the set of all states in which a is true. This would be a predicate on the state space. Conversely, for each state we could assign the set of all formulae that are true in that state.

Intuitively these different ways of assigning a logical interpretation ought to be equivalent, and we shall show that this is indeed the case, and that this arises from the symmetric monoidal closed structure of $\mathbb{V}$.

It should be made clear though, that our approach does not depend upon the use of bivalent logics. Our results are parametric in the truth objects $\Omega_{\mathbb{A}}$ and $\Omega_{\mathbb{X}}$.

We start with the second approach above.

Definition 3.15. Given the conditions of Assumption 4, for any object A in $\mathbb{A}$, and any object X in $\mathbb{X}$, we call any morphism

$$f \in \mathbb{A}_o(A, P(X))$$

a **valuation**, and any morphism

$$f \in \mathbb{X}_o(X, S(A))$$

a **theory map**.

The categories $\mathbb{A}_o$ and $\mathbb{X}_o$ are concrete over **Set**, since U, V, and $\mathbf{elem}|-|$ are faithful (Section 3.2), and thus the objects A and X are sets with some additional structure. A valuation then corresponds to a function mapping each formula to a predicate (set of states that satisfies it), and a theory map corresponds to a function mapping each state to a theory (set of formulae satisfied by that state). Though as already mentioned in Section 3.1, our notions of predicate and theory can be more general than mere sets, and our logics need not be restricted to the usual two truth values (true and false). For example, in Section 3.6 predicates include subsets, upsets of preorders, and fuzzy subsets, and theories include filters, prime filters, and ultrafilters.

Now since we have a dual adjunction $P \dashv S$, valuations and theory maps come in pairs. But does this pairing make sense from a logical perspective, and what about the first method of assigning a logical interpretation described above? In other words, do valuations and theory maps assign truth values to states and formulae in a consistent fashion?

Moreover, in the above definition of valuations and theory maps, the set of truth values is implicit in the logical connection. Can we be more explicit about which values are being assigned to which formulae in which states?

We aim to answer these questions by exploiting the symmetric monoidal closed structure of the category $\mathbb{V}$. Specifically, we aim to show that the following diagram commutes, where top left are valuations, top right theory maps, and at the bottom the underlying functions that assign truth values to pairs of states and formulae.

$$
\begin{array}{ccc}
\mathbb{A}(A, P(X)) & \xrightarrow{\Phi_{A,X}} & \mathbb{X}(X, S(A)) \\
\Big\downarrow{\scriptstyle U_{A,P(X)}} & & \Big\downarrow{\scriptstyle V_{X,S(A)}} \\
[U(A), UP(X)] & & [V(X), VS(A)] \\
\Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle =} \\
[U(A), \mathbb{X}(X, \Omega_{\mathbb{X}})] & & [V(X), \mathbb{A}(A, \Omega_{\mathbb{A}})] \\
\Big\downarrow{\scriptstyle [U(A), V_{X,\Omega_{\mathbb{X}}}]} & & \Big\downarrow{\scriptstyle [V(X), U_{A,\Omega_{\mathbb{A}}}]} \\
[U(A), [V(X), V(\Omega_{\mathbb{X}})]] & & [V(X), [U(A), U(\Omega_{\mathbb{A}})]] \\
\Big\downarrow{\scriptstyle p^{-1}_{U(A),V(X),V(\Omega_{\mathbb{X}})}} & & \Big\downarrow{\scriptstyle p^{-1}_{V(X),U(A),U(\Omega_{\mathbb{A}})}} \\
[U(A) \otimes V(X), V(\Omega_{\mathbb{X}})] & \xrightarrow[{[c^{-1}_{U(A),V(X)}, \tau^{-1}]}]{} & [V(X) \otimes U(A), U(\Omega_{\mathbb{A}})]
\end{array}
$$

To make the task more manageable we write

$$
\begin{aligned}
\mu &= p^{-1}_{U(A),V(X),V(\Omega_{\mathbb{X}})} \circ [U(A), V_{X,\Omega_{\mathbb{X}}}] \circ U_{A,P(X)} \\
\nu &= p^{-1}_{V(X),U(A),U(\Omega_{\mathbb{A}})} \circ [V(X), U_{A,\Omega_{\mathbb{A}}}] \circ V_{X,S(A)},
\end{aligned}
$$

then the above diagram becomes the following one.

$$
\begin{array}{ccc}
\mathbb{A}(A, P(X)) & \xrightarrow{\Phi_{A,X}} & \mathbb{X}(X, S(A)) \\
\Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \nu} \\
[U(A) \otimes V(X), V(\Omega_{\mathbb{X}})] & \xrightarrow[{[c^{-1}_{U(A),V(X)}, \tau^{-1}]}]{} & [V(X) \otimes U(A), U(\Omega_{\mathbb{A}})]
\end{array}
\tag{3.1}
$$

Then using Proposition 3.1 and Proposition 3.2, the fact that $\cong_{P(X)} = \Phi^{-1}_{A_0,X}$, and the definition of p^{-1} from Proposition B.12, it is reasonably straightforward to show that the transpose $\mu^{\dagger}$ of μ under the adjunction $- \otimes (U(A) \otimes V(X)) \dashv [U(A) \otimes V(X), -]$ is given by the following diagram.

$$
\begin{array}{ccc}
\mathbb{A}(A,P(X))\otimes(U(A)\otimes V(X)) & \xrightarrow{\mu^{\dagger}} & V(\Omega_{\mathbb{X}}) \\
\Big\downarrow {\scriptstyle 1_{\mathbb{A}(A,P(X))}\otimes(\cong_A\otimes\cong_X)} & & \Big\uparrow {\scriptstyle \tau} \\
\mathbb{A}(A,P(X))\otimes(\mathbb{A}(A_0,A)\otimes\mathbb{X}(X_0,X)) & & \\
\Big\downarrow {\scriptstyle a^{-1}_{\mathbb{A}(A,P(X)),\mathbb{A}(A_0,A),\mathbb{X}(X_0,X)}} & & \mathbb{X}(X_0,\Omega_{\mathbb{X}}) \\
(\mathbb{A}(A,P(X))\otimes\mathbb{A}(A_0,A))\otimes\mathbb{X}(X_0,X) & & \Big\uparrow {\scriptstyle M_{X_0,X,\Omega_{\mathbb{X}}}} \\
\Big\downarrow {\scriptstyle M_{A_0,A,P(X)}\otimes 1_{\mathbb{X}(X_0,X)}} & & \\
\mathbb{A}(A_0,P(X))\otimes\mathbb{X}(X_0,X) & \xrightarrow[\Phi_{A_0,X}\otimes 1_{\mathbb{X}(X_0,X)}]{} & \mathbb{X}(X,\Omega_{\mathbb{X}})\otimes\mathbb{X}(X_0,X)
\end{array}
$$

Similarly, under the adjunction $-\otimes(V(X)\otimes U(A))\dashv[V(X)\otimes U(A)),-]$, the transpose $\nu^{\dagger}$ of ν is given by

$$
\begin{array}{ccc}
\mathbb{X}(X,S(A))\otimes(V(X)\otimes U(A)) & \xrightarrow{\nu^{\dagger}} & U(\Omega_{\mathbb{A}}) \\
\Big\downarrow {\scriptstyle 1_{\mathbb{X}(X,S(A))}\otimes(\cong_X\otimes\cong_A)} & & \Big\uparrow {\scriptstyle \tau^{-1}} \\
\mathbb{X}(X,S(A))\otimes(\mathbb{X}(X_0,X)\otimes\mathbb{A}(A_0,A)) & & \\
\Big\downarrow {\scriptstyle a^{-1}_{\mathbb{X}(X,S(A)),\mathbb{X}(X_0,X),\mathbb{A}(A_0,A)}} & & \mathbb{A}(A_0,\Omega_{\mathbb{A}}) \\
(\mathbb{X}(X,S(A))\otimes\mathbb{X}(X_0,X))\otimes\mathbb{A}(A_0,A) & & \Big\uparrow {\scriptstyle M_{A_0,A,\Omega_{\mathbb{A}}}} \\
\Big\downarrow {\scriptstyle M_{X_0,X,S(A)}\otimes 1_{\mathbb{A}(A_0,A)}} & & \\
\mathbb{X}(X_0,S(A))\otimes\mathbb{A}(A_0,A) & \xrightarrow[\Phi^{-1}_{A,X_0}\otimes 1_{\mathbb{A}(A_0,A)}]{} & \mathbb{A}(A,\Omega_{\mathbb{A}})\otimes\mathbb{A}(A_0,A)
\end{array}
$$

We can therefore replace commutativity of diagram (3.1) in μ and ν, with commutativity of a diagram in $\mu^{\dagger}$ and $\nu^{\dagger}$, as given by the following proposition.

Proposition 3.16. *Given the conditions of Assumption 4, for all objects A in $\mathbb{A}$, and all objects X in $\mathbb{X}$,*

$$
\begin{array}{ccc}
\mathbb{A}(A, P(X)) & \xrightarrow{\Phi_{A,X}} & \mathbb{X}(X, S(A)) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} \\
[U(A) \otimes V(X), V(\Omega_{\mathbb{X}})] & \xrightarrow[{[c^{-1}_{U(A),V(X)}, \tau^{-1}]}]{} & [V(X) \otimes U(A), U(\Omega_{\mathbb{A}})]
\end{array}
\tag{3.2}
$$

commutes, if and only if,

$$
\begin{array}{ccc}
\mathbb{A}(A, P(X)) \otimes (U(A) \otimes V(X)) & \xrightarrow{\Phi_{A,X} \otimes c_{U(A),V(X)}} & \mathbb{X}(X, S(A)) \otimes (V(X) \otimes U(A)) \\
\downarrow{\scriptstyle \mu^{\dagger}} & & \downarrow{\scriptstyle \nu^{\dagger}} \\
V(\Omega_{\mathbb{X}}) & \xrightarrow[\tau^{-1}]{} & U(\Omega_{\mathbb{A}})
\end{array}
\tag{3.3}
$$

commutes.

Proof. Consider the following diagram.

$$
\begin{array}{ccccc}
\mathbb{A}(A, P(X)) \otimes (U(A) \otimes V(X)) & & \xrightarrow{\Phi_{A,X} \otimes c_{U(A),V(X)}} & & \mathbb{X}(X, S(A)) \otimes (V(X) \otimes U(A)) \\
\downarrow{\scriptstyle \mu \otimes 1_{U(A) \otimes V(X)}} & \searrow^{\mu^{\dagger}} & & \swarrow^{\nu^{\dagger}} & \downarrow{\scriptstyle \nu \otimes 1_{V(X) \otimes U(A)}} \\
 & & V(\Omega_{\mathbb{X}}) \xrightarrow{\tau^{-1}} U(\Omega_{\mathbb{A}}) & & \\
 & \nearrow_{e_{U(A) \otimes V(X), V(\Omega_{\mathbb{X}})}} & & \nwarrow_{e_{V(X) \otimes U(A), U(\Omega_{\mathbb{A}})}} & \\
[U(A) \otimes V(X), V(\Omega_{\mathbb{X}})] \otimes (U(A) \otimes V(X)) & & \xrightarrow[{[c^{-1}_{U(A),V(X)}, \tau^{-1}] \otimes c_{U(A),V(X)}}]{} & & [V(X) \otimes U(A), U(\Omega_{\mathbb{A}})] \otimes (V(X) \otimes U(A))
\end{array}
$$

The triangles on the left and right commute by the definitions of $\mu^{\dagger}$ and $\nu^{\dagger}$, and the bottom quadrilateral commutes by the naturality of e. So the diagram as a whole commutes if and only if the top quadrilateral commutes, which is seen to be (3.3).

Now if (3.2) commutes, applying the functor $- \otimes (U(A) \otimes V(X))$ to it, and then using the naturality of c, means the outer perimeter of the above diagram commutes, which in turn means that (3.3) must commute. Conversely, if (3.3) commutes, then the above diagram commutes, and by the uniqueness of transposes, this means that (3.2) must commute. $\square$

So using Proposition 3.16, and the definitions of $\mu^\dagger$ and $\nu^\dagger$, we have to show that the following diagram commutes.

$$
\begin{array}{ccc}
\mathbb{A}(A,P(X)) \otimes (U(A)\otimes V(X)) & \xrightarrow{\Phi_{A,X}\otimes c_{U(A),V(X)}} & \mathbb{X}(X,S(A)) \otimes (V(X)\otimes U(A)) \\
\downarrow {\scriptstyle 1_{\mathbb{A}(A,P(X))}\otimes(\cong_A\otimes\cong_X)} & & \downarrow {\scriptstyle 1_{\mathbb{X}(X,S(A))}\otimes(\cong_X\otimes\cong_A)} \\
\mathbb{A}(A,P(X)) \otimes (\mathbb{A}(A_0,A)\otimes \mathbb{X}(X_0,X)) & \xrightarrow{\Phi_{A,X}\otimes c_{\mathbb{A}(A_0,A),\mathbb{X}(X_0,X)}} & \mathbb{X}(X,S(A)) \otimes (\mathbb{X}(X_0,X)\otimes \mathbb{A}(A_0,A)) \\
\downarrow {\scriptstyle a^{-1}_{\mathbb{A}(A,P(X)),\mathbb{A}(A_0,A),\mathbb{X}(X_0,X)}} & & \downarrow {\scriptstyle a^{-1}_{\mathbb{X}(X,S(A)),\mathbb{X}(X_0,X),\mathbb{A}(A_0,A)}} \\
(\mathbb{A}(A,P(X)) \otimes (\mathbb{A}(A_0,A)) \otimes \mathbb{X}(X_0,X) & & (\mathbb{X}(X,S(A)) \otimes (\mathbb{X}(X_0,X)) \otimes \mathbb{A}(A_0,A) \\
\downarrow {\scriptstyle M_{A_0,A,P(X)}\otimes 1_{\mathbb{X}(X_0,X)}} & & \downarrow {\scriptstyle M_{X_0,X,S(A)}\otimes 1_{\mathbb{A}(A_0,A)}} \\
\mathbb{A}(A_0,P(X)) \otimes \mathbb{X}(X_0,X) & & \mathbb{X}(X_0,S(A)) \otimes \mathbb{A}(A_0,A) \\
\downarrow {\scriptstyle \Phi_{A_0,X}\otimes 1_{\mathbb{X}(X_0,X)}} & & \downarrow {\scriptstyle \Phi^{-1}_{A,X_0}\otimes 1_{\mathbb{A}(A_0,A)}} \\
\mathbb{X}(X,\Omega_{\mathbb{X}}) \otimes \mathbb{X}(X_0,X) & & \mathbb{A}(A,\Omega_{\mathbb{A}}) \otimes \mathbb{A}(A_0,A) \\
\downarrow {\scriptstyle M_{X_0,X,\Omega_{\mathbb{X}}}} & & \downarrow {\scriptstyle M_{A_0,A,\Omega_{\mathbb{A}}}} \\
\mathbb{X}(X_0,\Omega_{\mathbb{X}}) & \xrightarrow[\Phi^{-1}_{A_0,X_0}]{} & \mathbb{A}(A_0,\Omega_{\mathbb{A}}) \\
\downarrow {\scriptstyle \tau} & & \downarrow {\scriptstyle \tau^{-1}} \\
V(\Omega_{\mathbb{X}}) & \xrightarrow[\tau^{-1}]{} & U(\Omega_{\mathbb{A}})
\end{array}
$$

The top square of this diagram commutes by the naturality of c, and the bottom square commutes by Proposition 3.10, so we are left to show that the middle part of the diagram commutes. This looks strange and complicated, but in actual fact is something quite straightforward and well known in ordinary category theory.

If we consider the underlying ordinary dual adjunction $P_o \dashv S_o$ (Proposition C.70), and using the notation $f^\flat$ to represent the dual adjunct of $f\colon A \to P_o(X)$, and $f^\sharp$ to represent the dual adjunct of $f\colon X \to S_o(A)$ (Proposition A.48), then for all $f \in \mathbb{A}_o(A, P_o(X))$, $a \in \mathbb{A}_o(A_0, A)$, and $x \in \mathbb{X}_o(X_0, X)$, by Proposition A.51 we have

$$\begin{aligned}(S(a) \circ f^\flat \circ x)^\sharp &= P(x) \circ (S(a) \circ f^\flat)^\sharp \\ &= P(x) \circ (f^\flat)^\sharp \circ a \\ &= P(x) \circ f \circ a.\end{aligned}$$

This is what the middle part of the above diagram shows, except at the $\mathbb{V}$-category level. To complete the proof therefore, we must show how the ordinary category theory result implies the commutativity of the diagram in $\mathbb{V}_o$. To do this we first apply the functor **elem**$|-|$ to map the $\mathbb{V}_o$ diagram to the underlying one in **Set**. Then we use the fact that **elem**$|-|$ is strong monoidal, and Proposition 2.12, to show that the underlying diagram commutes because of the above underlying dual adjunction result. Finally, since **elem**$|-|$ is faithful, this means that the diagram in $\mathbb{V}_o$ must commute.

Thus since U and V are both faithful, every valuation or theory map corresponds to a distinct morphism of the form $U(A) \otimes V(X) \to U(\Omega_{\mathbb{A}})$ or $V(X) \otimes U(A) \to V(\Omega_{\mathbb{X}})$, and since **elem**$|-|$ is strong monoidal, these morphisms are binary maps (each element of $U(A) \otimes V(X)$ is a pair (a, x)).

Summarising all this we have the following theorem.

Theorem 3.17. *Given the conditions of Assumption 4, and any objects A in $\mathbb{A}$ and X in $\mathbb{X}$, then for any valuation $f \in \mathbb{A}_o(A, P(X))$, and its dual adjunct theory map $f^\flat \in \mathbb{X}_o(X, S(A))$, we have*

$$V(U(f)(a))(x) = \tau(U(V(f^\flat)(x))(a))$$

*for all $a \in$ **elem**$|U(A)|$ and $x \in$ **elem**$|V(X)|$. Also, representing each f, and $f^\flat$, is a distinct **satisfaction map***

$$\begin{aligned}\vDash_f \colon U(A) \otimes V(X) &\to V(\Omega_{\mathbb{X}}) \\ a \otimes x &\mapsto V(U(f)(a))(x)\end{aligned}$$

$$\begin{aligned}\vDash_{f^\flat} \colon V(X) \otimes U(A) &\to U(\Omega_{\mathbb{A}}) \\ x \otimes a &\mapsto U(V(f^\flat)(x))(a),\end{aligned}$$

and moreover we have that

$$\vDash_f = \tau \circ \vDash_{f^\flat} \circ c_{U(A),V(X)}.$$

What this theorem shows, is that as stated at the start of this section, valuations and theory maps correspond to the possibly more intuitive notion of taking a pair consisting of a formula and a state, and assigning a truth value to that pair - the truth status of that formula in that state.

However, at the level of the satisfaction maps (the category $\mathbb{V}_o$), the algebraic properties of the logical connectives of objects in $\mathbb{A}$, or the topologies or other structure of objects in $\mathbb{X}$, have been forgotten (by the functors U and V). Thus not every possible choice of map $U(A) \otimes V(X) \to U(\Omega_{\mathbb{A}})$ or $V(X) \otimes U(A) \to V(\Omega_{\mathbb{X}})$ corresponds to a valuation or theory map.

Usually in the literature when a map of the form $\vDash : U(A) \otimes V(X) \to V(\Omega_{\mathbb{X}})$ is defined, it is done so inductively on the structure of the formula a, but this is needed precisely because this information is not present at the level of the category $\mathbb{V}_o$. At the level of the category $\mathbb{A}_o$ this structure is built in, and all morphisms must preserve it. Thus valuations, or dually theory maps, are a mathematically cleaner way to give the semantics of a base logic in $\mathbb{A}$.

3.5 Dualising Objects

In the previous sections we have examined the properties of a logical connection, but now we want to concentrate on finding logical connections. In the next section we shall produce a collection of example logical connections that will be used in subsequent chapters. To do this we will require a technical tool, and that is what we shall cover in this section.

The technical tool we shall use is what we shall call a dualising object. This is not a new idea (for example see Johnstone (1982, VI.4)), and we shall only present a brief summary of the material in Porst and Tholen (1991); Kurz and Velebil (2011), wherein a dualising object is known as a schizophrenic object.

To formulate the notion of a dualising object we shall need the concept of an F-initial lift, where here $F : \mathbb{C} \to \mathbb{V}$ is a $\mathbb{V}$-functor. In Chapter 2 we introduced the idea of the initial lift of an ordinary functor along the functor $\mathbf{elem}|-| : \mathbb{V}_o \to \mathbf{Set}$ to create a $\mathbb{V}$-functor (Definition 2.10), here we lift families of morphisms in $\mathbb{V}_o$ to families of morphisms in $\mathbb{C}_o$. This concept will only be used in the remainder of this chapter.

Our approach is to present material from Kurz and Velebil (2011), but restricted to the single sorted case. We do this because we are only working in a single sorted framework, but this also has the effect of significantly simplifying the presentation.

Definition 3.18. Given a $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$ we have the following definitions:

1. An F-**structured source** is a morphism

$$\lambda\colon W \to [Z, F(C)]$$

in $\mathbb{V}_o$.

2. An F-**lift** of λ is a morphism

$$\overline{\lambda}\colon W \to \mathbb{C}(\overline{Z}, C)$$

in $\mathbb{V}_o$ such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\overline{\lambda}} & \mathbb{C}(\overline{Z}, C) \\ & \searrow^{\lambda} & \downarrow{\scriptstyle F_{\overline{Z},C}} \\ & & [Z, F(C)] \end{array}$$

commutes.

3. An F-**initial lift** of λ is an F-lift $\overline{\lambda}$ such that

$$\begin{array}{ccc} \mathbb{C}(C', \overline{Z}) & \xrightarrow{\hom_{\mathbb{C}}(C',\overline{\lambda})} & [W, \mathbb{C}(C', C)] \\ {\scriptstyle F_{C',\overline{Z}}}\downarrow & & \downarrow{\scriptstyle [W, F_{C',C}]} \\ [F(C'), Z] & \xrightarrow[\hom_{\mathbb{V}}(F(C'),\lambda)]{} & [W, [F(C'), F(C)]] \end{array}$$

is a pullback in $\mathbb{V}_o$ for all $C' \in \mathbf{obj}|\mathbb{C}|$.

Here, $\hom_{\mathbb{C}}(C', \overline{\lambda})$ is defined such that

$$\hom_{\mathbb{C}}(C', \overline{\lambda})(f)(w) = \overline{\lambda}(w) \bullet f,$$

for $f \in \mathbb{C}_o(C', \overline{Z})$, and $w \in \mathbf{elem}|W|$.

Similarly, $\hom_{\mathbb{V}}(F(C'), \lambda)$ is defined such that

$$\hom_{\mathbb{V}}(F(C'), \lambda)(g)(w) = \lambda(w) \circ g,$$

for $g \in \mathbb{V}_o(F(C'), Z)$, and $w \in \mathbf{elem}|W|$.

An F-structured source can be thought of as a family of $\mathbb{V}_o$ morphisms from Z to $F(C)$ indexed by W. An F-lift of λ is then a W-indexed family of $\mathbb{C}_o$ morphisms from $\overline{Z}$ to

C. Here $\overline{Z}$ is an object of $\mathbb{C}$ such that $F(\overline{Z}) = Z$. It can be thought of as somehow putting the additional structure of a $\mathbb{C}$ object onto an underlying $\mathbb{V}$ object, such that each $\mathbb{V}_o$ morphism $\lambda(w) = F_{\overline{Z},C}(\overline{\lambda}(w))$.

An F-initial lift is then an optimal choice of $\overline{Z}$, one such that for any $\mathbb{V}_o$ morphism $h \colon F(C') \to Z$, for some object C' of $\mathbb{C}$, and any W-indexed family of morphisms $\mu(w) \in \mathbb{C}_o(C', C)$, such that for all $w \in \mathbf{elem}|W|$,

$$F_{C',C}(\mu(w)) = \lambda(w) \circ h,$$

there exists a unique $\mathbb{C}_o$ morphism $\overline{h} \colon C' \to \overline{Z}$, such that for all $w \in \mathbf{elem}|W|$,

$$\mu(w) = \overline{\lambda}(w) \bullet \overline{h},$$

and

$$F_{C',\overline{Z}}(\overline{h}) = h.$$

Using the above, in conjunction with the definitions of ϕ and ψ from Definition 3.3 and Definition 3.4, we can define a dualising object as follows.

Definition 3.19. Given the conditions of Assumption 2, a triple $(\Omega_{\mathbb{A}}, \Omega_{\mathbb{X}}, \tau)$, consisting of an object $\Omega_{\mathbb{A}}$ in $\mathbb{A}$, an object $\Omega_{\mathbb{X}}$ in $\mathbb{X}$, and an isomorphism $\tau \colon U(\Omega_{\mathbb{A}}) \to V(\Omega_{\mathbb{X}})$ in $\mathbb{V}_o$, is called a **dualising object** if the following hold:

1. For every A in $\mathbb{A}$, the V-structured source

$$\mu_A \colon U(A) \xrightarrow{\phi_{A,\Omega_{\mathbb{A}}}} [\mathbb{A}(A, \Omega_{\mathbb{A}}), U(\Omega_{\mathbb{A}})] \xrightarrow{[\mathbb{A}(A,\Omega_{\mathbb{A}}),\tau]} [\mathbb{A}(A, \Omega_{\mathbb{A}}), V(\Omega_{\mathbb{X}})]$$

   has a V-initial lift

$$\overline{\mu}_A \colon U(A) \to \mathbb{X}(S(A), \Omega_{\mathbb{X}}).$$

2. For every X in $\mathbb{X}$, the U-structured source

$$\nu_X \colon V(X) \xrightarrow{\psi_{X,\Omega_{\mathbb{X}}}} [\mathbb{X}(X, \Omega_{\mathbb{X}}), V(\Omega_{\mathbb{X}})] \xrightarrow{[\mathbb{X}(X,\Omega_{\mathbb{X}}),\tau^{-1}]} [\mathbb{X}(X, \Omega_{\mathbb{X}}), U(\Omega_{\mathbb{A}})]$$

   has a U-initial lift

$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), \Omega_{\mathbb{A}}).$$

In the above definition we have used the suggestive notation of $S(A)$ and $P(X)$ for the initial lifts of $\mathbb{A}(A, \Omega_{\mathbb{A}})$ and $\mathbb{X}(X, \Omega_{\mathbb{X}})$. This is in anticipation of the following result, which is Kurz and Velebil (2011, Theorem 4.16), and which we state without proof.

Theorem 3.20. *Given the conditions of Assumption 2, every dualising object* $(\Omega_{\mathbb{A}}, \Omega_{\mathbb{X}}, \tau)$ *induces a* $\mathbb{V}$*-dual adjunction* $P \dashv S \colon \mathbb{A} \to \mathbb{X}$ *strictly represented by* $(\Omega_{\mathbb{A}}, \Omega_{\mathbb{X}})$.

3.6 Examples

In this section we shall establish a collection of different logical connections that we shall build upon in subsequent chapters. These examples are by no means exhaustive, and indeed, in the case of enrichment over **Set** (i.e. ordinary category theory), there are many examples to be found in the literature (Section 3.7).

It should also be noted that these examples derive from those of Jacobs and Sokolova (2010), but extended and enriched where appropriate.

The logical connections to follow will be built from a common collection of components, and we shall consider three cases: enrichment over **Set**, enrichment over $\mathbf{Set}_R$ (Definition 2.1), and enrichment over **GMet** (Definition 2.5).

The categories **MSL**, **DL**, and **BA** consist, respectively, of meet semilattices with top, distributive lattices with top, and Boolean algebras. As defined they are ordinary categories, but each object A (of any of these categories) can be given a natural order: $a \leq b \Leftrightarrow a = a \wedge b$, and so can be thought of as a set with a preorder, or a partial order. Alternatively, using the equality relation on A, A can be thought of as a set with an equivalence relation, or equality relation. It is not hard to see that the morphisms of **MSL**, **DL**, and **BA** preserve these order relations, and are themselves ordered pointwise.

Definition 3.21. The categories **MSL**, **DL**, and **BA** are enriched over $\mathbf{Set}_R$, with each object A carrying the relation R_A defined as follows:

1. if the type R represents preorders or partial orders,

$$aR_Ab \Leftrightarrow a = a \wedge b,$$

2. if the type R represents equivalence relations or equality,

$$aR_Ab \Leftrightarrow a = b.$$

The hom-objects are ordered pointwise, i.e. for $f, g \colon A \to B$,

$$fRg \Leftrightarrow \text{ for all } a \in A \text{ we have } f(a)R_Bg(a).$$

Similarly, objects of the categories **MSL**, **DL**, and **BA** can be given a generalised metric in a natural way, and once again the morphisms preserve the metrics, and can themselves be given a metric.

Definition 3.22. The categories **MSL**, **DL**, and **BA** are enriched over **GMet**, with each object A carrying the generalised metric d_A defined by

$$d_A(a,b) = \begin{cases} 0 & : \text{ if } a = a \wedge b \\ \infty & : \text{ otherwise.} \end{cases}$$

The generalised metric on each hom-object is defined for $f, g \colon A \to B$ as

$$d(f,g) = \sup_{a \in A} d_B(f(a), g(a)).$$

The categories **MSL**, **DL**, and **BA** form our base logics, and give us respectively: conjunction and true; conjunction, disjunction, and true; and conjunction, disjunction, negation, true, and false. To this we need to add a set of truth values. We shall consider two cases. The first is the usual case of bivalent logic, where truth values come from the set $\mathbf{2} = \{0,1\}$. The second is the case of fuzzy logic, where truth values are taken from the unit interval $[0,1]$.

These sets of truth values need to be given preorders and metrics in the case of enrichment over $\mathbf{Set}_R$ or **GMet**.

In the case of bivalent logics we make the following definition.

Definition 3.23. The set $\mathbf{2} = \{0,1\}$, as an object in $\mathbf{Set}_R$, is defined to have the following preorder relation:

1. if the type R represents preorders or partial orders,

$$R_{\mathbf{2}} = \{(0,0),(0,1),(1,1)\},$$

2. if the type R represents equivalence relations or equality,

$$R_{\mathbf{2}} = \{(0,0),(1,1)\},$$

and as an object in **GMet**, is defined to have the generalised metric given by

$$d_{\mathbf{2}}(a,b) = \begin{cases} 0 & : \text{ if } (a,b) \in \{(0,0),(0,1),(1,1)\} \\ \infty & : \text{ otherwise.} \end{cases}$$

Similarly, in the case of fuzzy logics we make the following definition.

Definition 3.24. The unit interval $[0,1]$, as an object in $\mathbf{Set}_R$, is defined to have the following preorder relation:

1. if the type R represents preorders or partial orders,

$$xR_{[0,1]}y \Leftrightarrow x \leq y,$$

2. if the type R represents equivalence relations or equality,

$$xR_{[0,1]}y \Leftrightarrow x = y,$$

and as an object in **GMet**, is defined to have the generalised metric given by

$$d_{[0,1]}(x,y) = \begin{cases} y - x & : \text{ if } x \leq y \\ \infty & : \text{ otherwise.} \end{cases}$$

The truth set **2** needs to be given an algebraic structure in order to make it into an object in the categories **MSL**, **DL**, and **BA**.

Definition 3.25. For each of the categories **MSL**, **DL**, and **BA**, the set $\mathbf{2} = \{0,1\}$ has the corresponding subset of the following operations:

1. $\top = 1$,
2. $a \wedge b = \min(a,b)$,
3. $a \vee b = \max(a,b)$,
4. $\neg a = 1 - a$.

The truth set $[0,1]$ however, can only be made into an object in the categories **MSL** and **DL**, and not into a Boolean algebra, since $a \vee \neg a = \top$ is not valid in fuzzy logic. Note, here we are defining what in Łukasiewicz logic are known as weak conjunction and disjunction.

Definition 3.26. For each of the categories **MSL** and **DL**, the unit interval $[0,1]$ has the corresponding subset of the following operations:

1. $\top = 1$,
2. $x \wedge y = \min(x,y)$,
3. $x \vee y = \max(x,y)$.

For state spaces we will use the categories **Set**, $\mathbf{Set}_R$, **GMet**, and **Meas**, the category of measurable spaces. These are all ordinary categories, but $\mathbf{Set}_R$ and **GMet** are also enriched over themselves, and from Example 2.1 and Example 2.2, we also have the $\mathbf{Set}_R$-category $\mathbf{Meas}_R$, and the **GMet**-category **GMeas**.

The truth sets **2** and $[0, 1]$ can be made into objects of **Meas**, $\mathbf{Meas}_R$ and **GMeas** by giving them sigma algebra structures. For our purposes the following choices suffice.

Definition 3.27. The set **2** carries the sigma algebra defined by

$$\Sigma_{\mathbf{2}} = \{\emptyset, \{0\}, \{1\}, \mathbf{2}\},$$

and the unit interval $[0, 1]$ carries the sigma algebra given by the Borel sets of $[0, 1]$.

We shall now introduce some terminology that will make our work easier.

Definition 3.28.

1. A morphism $u \in \mathbf{Set}(X, \mathbf{2})$ defines a **subset** of X.
2. A morphism $u \in \mathbf{Set}(X, [0, 1])$ defines a **fuzzy subset** of X.
3. A morphism $u \in \mathbf{Set}_R(X, \mathbf{2})$ defines a **right R-closed subset** of X.

   In detail this corresponds to:

   $$\text{if } x \in u \text{ and } xR_Xy \text{ then } y \in u.$$

4. A morphism $u \in \mathbf{Set}_R(X, [0, 1])$ defines a **right R-closed fuzzy subset** of X.

   In detail this corresponds to:

   $$\text{if } xR_Xy \text{ then } u(x) \leq u(y).$$

5. A morphism $u \in \mathbf{GMet}(X, \mathbf{2})$ defines a **right d-closed subset** of X.

   In detail this corresponds to:

   $$\text{if } x \in u \text{ and } d_X(x, y) < \infty \text{ then } y \in u.$$

6. A morphism $u \in \mathbf{GMet}(X, [0, 1])$ defines a **right d-closed fuzzy subset** of X.

   In detail this corresponds to:

   $$\text{if } d_X(x, y) < \infty \text{ then } u(x) \leq u(y).$$

A right R-closed subset is the generalisation of an upset (Davey and Priestley, 2002). Moreover, because $\{1\} \in \Sigma_{\mathbf{2}}$, a morphism $u \in \mathbf{Meas}(X, \mathbf{2})$ is a measurable subset of

X, and this combines in the obvious way with the notions of right R-closed, and right d-closed, in the categories $\mathbf{Meas}_R$ and **GMeas**.

We can use similar terminology on the algebra side, starting with standard definitions from order theory (Davey and Priestley, 2002), and then introducing their obvious fuzzy analogues.

Definition 3.29.

1. A morphism $s \in \mathbf{MSL}(A, \mathbf{2})$ defines a **filter** of A.

   In detail this corresponds to:

   $$s \text{ is an upset, and if } a, b \in s \text{ then } a \wedge b \in s.$$

2. A morphism $s \in \mathbf{DL}(A, \mathbf{2})$ defines a **prime filter** of A.

   In detail this corresponds to:

   $$s \text{ is a filter, and if } a \vee b \in s \text{ then either } a \in s \text{ or } b \in s.$$

3. A morphism $s \in \mathbf{BA}(A, \mathbf{2})$ defines a **ultrafilter** of A.

   In detail this corresponds to:

   $$s \text{ is a filter, and for all } a \in A, \text{ either } a \in s \text{ or } \neg a \in s.$$

4. A morphism $s \in \mathbf{MSL}(A, [0, 1])$ defines a **fuzzy filter** of A.

   In detail this corresponds to:

   $$s \text{ is a fuzzy upset, and } s(a \wedge b) = \min(s(a), s(b)).$$

5. A morphism $s \in \mathbf{DL}(A, [0, 1])$ defines a **fuzzy prime filter** of A.

   In detail this corresponds to:

   $$s \text{ is a fuzzy filter, and } s(a \vee b) = \max(s(a), s(b)).$$

Now we know from Definition 3.21 and Definition 3.22 that the categories **MSL**, **DL**, and **BA** can be enriched over $\mathbf{Set}_R$ and **GMet**, and that this introduces the notions of right R-closed sets and right d-closed sets. However, in these cases this adds nothing new. Thus when talking about the various flavours of filters, we shall not use the right R-closed and right d-closed terminology.

So far we have described the different categories that we shall use to form our examples, and we have also described two different sets of truth values, **2** and $[0, 1]$, and how to make these sets into objects of the chosen categories. Next we need to define the forgetful functors U and V.

Definition 3.30.

1. For the categories **MSL**, **DL**, and **BA**, and the three cases of enrichment over **Set**, $\mathbf{Set}_R$, and **GMet**, the functor U is the obvious forgetful functor that takes each object and simply forgets some of the structure.

   This definition of U is obviously faithful, and it is also representable, with the representing object being the free algebra over one generator in the respective category **MSL**, **DL**, or **BA**.

2. For the categories $\mathbf{Set}_R$ and **GMet**, since we are taking them to be enriched over themselves, for the forgetful functor V we simply take the identity functor.

   The functor V is then representable, with the representing object in each case being the corresponding final object, which is the singleton $\mathbf{1}$ (with additional structure).

3. For the categories **Meas**, $\mathbf{Meas}_R$, and **GMeas**, since we consider them to be enriched over **Set**, $\mathbf{Set}_R$, and **GMet** respectively, we define the forgetful functor V to simply forget the sigma algebra associated with each object.

   The functor V is clearly faithful, and is representable, with the representing object in each case being the corresponding final object, which is the singleton $\mathbf{1}$ (with additional structure).

We are now ready to construct a series of logical connections from the different components described above. To do this, the main technical tool we shall use is Theorem 3.20, which requires that we establish the existence of initial lifts of certain morphisms.

To use Theorem 3.20 we must first find for each example of a logical connection a triple $(\Omega_{\mathbb{A}}, \Omega_{\mathbb{X}}, \tau)$. In our examples we intend $\Omega_{\mathbb{A}}$ and $\Omega_{\mathbb{X}}$ to be $\mathbf{2}$ or $[0,1]$ with the appropriate additional structures given by Definition 3.23, Definition 3.24, Definition 3.25, Definition 3.26, and Definition 3.27. This means in all our examples the isomorphism

$$\tau\colon U(\Omega_{\mathbb{A}}) \to V(\Omega_{\mathbb{X}})$$

will be the identity.

Finally, the underlying functions of the unit and counit in each example are given by Corollary 3.14.

Bivalent Logical Connections

Example 3.1 (Bivalent **MSL**, **DL**, or **BA**, and **Set** enriched over **Set**).

1. *The morphism*
$$\mu_A \colon U(A) \to [\mathbb{A}(A, \mathbf{2}), V(\mathbf{2})]$$
assigns to each $a \in U(A)$*, the set of filters/prime filters/ultrafilters of* A *that contain* a*. Since* V *is the identity functor, the* V*-initial lift*
$$\overline{\mu}_A \colon U(A) \to \mathbf{Set}(S(A), \mathbf{2})$$
assigns to each a *the corresponding subset of* $S(A)$*, where*
$$S(A) = \mathbb{A}(A, \mathbf{2}).$$

2. *The morphism*
$$\nu_X \colon V(X) \to [\mathbf{Set}(X, \mathbf{2}), U(\mathbf{2})]$$
assigns to each $x \in V(X)$*, the set of subsets of* X *that contain* x*. The* U*-initial lift*
$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), \mathbf{2})$$
assigns to each x *the corresponding filter/prime filter/ultrafilter of* $P(X)$*, where*
$$UP(X) = \mathbf{Set}(X, \mathbf{2}),$$
and $P(X)$ *is defined to be* $\mathbf{Set}(X, \mathbf{2})$ *with the relevant subset of the following operations:*

   $\top$**:** $\top$ *is defined to be the set* X*,*

   $\wedge$**:** *for* $u, v \in P(X)$ *define* $u \wedge v = u \cap v$ *(set intersection),*

   $\vee$**:** *for* $u, v \in P(X)$ *define* $u \vee v = u \cup v$ *(set union),*

   $\neg$**:** *for* $u \in P(X)$ *define* $\neg u = u^c$ *(set complement in* X*).*

3. *The unit is given by*
$$\rho_A(a) = \{s \in S(A) \mid a \in s\},$$
and the counit by
$$\sigma_X(x) = \{u \in P(X) \mid x \in u\}.$$

Example 3.2 (Bivalent **MSL**, **DL**, or **BA**, and $\mathbf{Set}_R$ enriched over $\mathbf{Set}_R$).

1. *The morphism*
$$\mu_A \colon U(A) \to [\mathbb{A}(A, \mathbf{2}), V(\mathbf{2})]$$
assigns to each $a \in U(A)$*, the right* R*-closed set of filters/prime filters/ultrafilters of* A *that contain* a*. Since* V *is the identity functor, the* V*-initial lift*
$$\overline{\mu}_A \colon U(A) \to \mathbf{Set}_R(S(A), \mathbf{2})$$
assigns to each a *the corresponding right* R*-closed subset of* $S(A)$*, where*
$$S(A) = \mathbb{A}(A, \mathbf{2}).$$
Here $S(A)$ *is ordered by inclusion if the type* R *represents preorders or partial orders, and by equality if the type* R *represents equivalence relations or equality.*

2. *The morphism*
$$\nu_X \colon V(X) \to [\mathbf{Set}_R(X, \mathbf{2}), U(\mathbf{2})]$$
assigns to each $x \in V(X)$*, the right* R*-closed set of right* R*-closed subsets of* X *that contain* x*. The* U*-initial lift*
$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), \mathbf{2})$$
assigns to each x *the corresponding filter/prime filter/ultrafilter of* $P(X)$*, where*
$$UP(X) = \mathbf{Set}_R(X, \mathbf{2}),$$
and $P(X)$ *is defined to be* $\mathbf{Set}_R(X, \mathbf{2})$ *with the relevant subset of the following operations:*

   $\top$**:** $\top$ *is defined to be the set* X*,*

   $\wedge$**:** *for* $u, v \in P(X)$ *define* $u \wedge v = u \cap v$ *(set intersection),*

   $\vee$**:** *for* $u, v \in P(X)$ *define* $u \vee v = u \cup v$ *(set union),*

   $\neg$**:** *for* $u \in P(X)$ *define* $\neg u = u^c$ *(set complement in* X*).*

   Note: for u *right* R*-closed,* $\neg u$ *is only right* R*-closed if the type* R *represents equivalence relations or equality. Thus only in these cases do we have a logical connection between* **BA** *and* $\mathbf{Set}_R$*.*

3. *The unit is given by*
$$\rho_A(a) = \{s \in S(A) \mid a \in s\},$$
and the counit by
$$\sigma_X(x) = \{u \in P(X) \mid x \in u\}.$$

Example 3.3 (Bivalent **MSL** or **DL**, and **GMet** enriched over **GMet**).

1. *The morphism*

$$\mu_A \colon U(A) \to [\mathbb{A}(A, \mathbf{2}), V(\mathbf{2})]$$

assigns to each $a \in U(A)$, the right d-closed set of filters/prime filters of A that contain a. Since V is the identity functor, the V-initial lift

$$\overline{\mu}_A \colon U(A) \to \mathbf{GMet}(S(A), \mathbf{2})$$

assigns to each a the corresponding right d-closed subset of $S(A)$, where

$$S(A) = \mathbb{A}(A, \mathbf{2}).$$

Here $S(A)$ has the metric $d(s, s') = 0$, if $s \subseteq s'$, and $d(s, s') = \infty$ otherwise.

2. *The morphism*

$$\nu_X \colon V(X) \to [\mathbf{GMet}(X, \mathbf{2}), U(\mathbf{2})]$$

assigns to each $x \in V(X)$, the right d-closed set of right d-closed subsets of X that contain x. The U-initial lift

$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), \mathbf{2})$$

assigns to each x the corresponding filter/prime filter of $P(X)$, where

$$UP(X) = \mathbf{GMet}(X, \mathbf{2}),$$

and $P(X)$ is defined to be $\mathbf{GMet}(X, \mathbf{2})$ *with the relevant subset of the following operations:*

$\top$**:** *$\top$ is defined to be the set X,*

$\wedge$**:** *for $u, v \in P(X)$ define $u \wedge v = u \cap v$ (set intersection),*

$\vee$**:** *for $u, v \in P(X)$ define $u \vee v = u \cup v$ (set union).*

3. *The unit is given by*

$$\rho_A(a) = \{s \in S(A) \mid a \in s\},$$

and the counit by

$$\sigma_X(x) = \{u \in P(X) \mid x \in u\}.$$

Example 3.4 (Bivalent **MSL**, **DL**, or **BA**, and **Meas** enriched over **Set**).

1. *The morphism*
$$\mu_A\colon U(A) \to [\mathbb{A}(A, \mathbf{2}), V(\mathbf{2})]$$
assigns to each $a \in U(A)$, the set of filters/prime filters/ultrafilters of A that contain a. The V-initial lift
$$\overline{\mu}_A\colon U(A) \to \mathbf{Meas}(S(A), \mathbf{2})$$
assigns to each a the corresponding measurable subset of $S(A)$, where
$$VS(A) = \mathbb{A}(A, \mathbf{2}),$$
and $S(A)$ is defined to be $\mathbb{A}(A, \mathbf{2})$ with the sigma algebra generated by the family of sets $(\{s \in S(A) \mid a \in s\})_{a \in A}$.

2. *The morphism*
$$\nu_X\colon V(X) \to [\mathbf{Meas}(X, \mathbf{2}), U(\mathbf{2})]$$
assigns to each $x \in V(X)$, the set of measurable subsets of X that contain x. The U-initial lift
$$\overline{\nu}_X\colon V(X) \to \mathbb{A}(P(X), \mathbf{2})$$
assigns to each x the corresponding filter/prime filter/ultrafilter of $P(X)$, where
$$UP(X) = \mathbf{Meas}(X, \mathbf{2}),$$
and $P(X)$ is defined to be $\mathbf{Meas}(X, \mathbf{2})$ with the relevant subset of the following operations:

   $\top$**:** *$\top$ is defined to be the set X,*

   $\wedge$**:** *for $u, v \in P(X)$ define $u \wedge v = u \cap v$ (set intersection),*

   $\vee$**:** *for $u, v \in P(X)$ define $u \vee v = u \cup v$ (set union),*

   $\neg$**:** *for $u \in P(X)$ define $\neg u = u^c$ (set complement in X).*

3. *The unit is given by*
$$\rho_A(a) = \{s \in S(A) \mid a \in s\},$$
and the counit by
$$\sigma_X(x) = \{u \in P(X) \mid x \in u\}.$$

Example 3.5 (Bivalent **MSL**, **DL**, or **BA**, and $\mathbf{Meas}_R$ enriched over $\mathbf{Set}_R$).

1. *The morphism*
$$\mu_A \colon U(A) \to [\mathbb{A}(A,\mathbf{2}), V(\mathbf{2})]$$
assigns to each $a \in U(A)$*, the right* R*-closed set of filters/prime filters/ultrafilters of* A *that contain* a*. The* V*-initial lift*
$$\overline{\mu}_A \colon U(A) \to \mathbf{Meas}_R(S(A), \mathbf{2})$$
assigns to each a *the corresponding right* R*-closed measurable subset of* $S(A)$*, where*
$$VS(A) = \mathbb{A}(A, \mathbf{2}),$$
and $S(A)$ *is defined to be* $\mathbb{A}(A,\mathbf{2})$ *with the sigma algebra generated by the family of sets* $(\{s \in S(A) \mid a \in s\})_{a\in A}$*, and ordered by inclusion if the type* R *represents preorders or partial orders, and by equality if the type* R *represents equivalence relations or equality.*

2. *The morphism*
$$\nu_X \colon V(X) \to [\mathbf{Meas}_R(X,\mathbf{2}), U(\mathbf{2})]$$
assigns to each $x \in V(X)$*, the right* R*-closed set of right* R*-closed measurable subsets of* X *that contain* x*. The* U*-initial lift*
$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), \mathbf{2})$$
assigns to each x *the corresponding filter/prime filter/ultrafilter of* $P(X)$*, where*
$$UP(X) = \mathbf{Meas}_R(X, \mathbf{2}),$$
and $P(X)$ *is defined to be* $\mathbf{Meas}_R(X,\mathbf{2})$ *with the relevant subset of the following operations:*

$\top$**:** $\top$ *is defined to be the set* X*,*

$\wedge$**:** *for* $u, v \in P(X)$ *define* $u \wedge v = u \cap v$ *(set intersection),*

$\vee$**:** *for* $u, v \in P(X)$ *define* $u \vee v = u \cup v$ *(set union),*

$\neg$**:** *for* $u \in P(X)$ *define* $\neg u = u^c$ *(set complement in* X*).*

Note: for u *right* R*-closed,* $\neg u$ *is only right* R*-closed if the type* R *represents equivalence relations or equality. Thus only in these cases do we have a logical connection between* **BA** *and* $\mathbf{Meas}_R$*.*

3. *The unit is given by*
$$\rho_A(a) = \{s \in S(A) \mid a \in s\},$$

and the counit by

$$\sigma_X(x) = \{u \in P(X) \mid x \in u\}.$$

Example 3.6 (Bivalent **MSL** or **DL**, and **GMeas** enriched over **GMet**)**.**

1. *The morphism*

$$\mu_A \colon U(A) \to [\mathbb{A}(A, \mathbf{2}), V(\mathbf{2})]$$

assigns to each $a \in U(A)$, the right d-closed set of filters/prime filters of A that contain a. The V-initial lift

$$\overline{\mu}_A \colon U(A) \to \mathbf{GMeas}(S(A), \mathbf{2})$$

assigns to each a the corresponding right d-closed measurable subset of $S(A)$, where

$$VS(A) = \mathbb{A}(A, \mathbf{2}),$$

and $S(A)$ is defined to be $\mathbb{A}(A, \mathbf{2})$ with the sigma algebra generated by the family of sets $(\{s \in S(A) \mid a \in s\})_{a \in A}$, and with the metric $d(s, s') = 0$, if $s \subseteq s'$, and $d(s, s') = \infty$ otherwise.

2. *The morphism*

$$\nu_X \colon V(X) \to [\mathbf{GMeas}(X, \mathbf{2}), U(\mathbf{2})]$$

assigns to each $x \in V(X)$, the right d-closed set of right d-closed measurable subsets of X that contain x. The U-initial lift

$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), \mathbf{2})$$

assigns to each x the corresponding filter/prime filter of $P(X)$, where

$$UP(X) = \mathbf{GMeas}(X, \mathbf{2}),$$

and $P(X)$ is defined to be $\mathbf{GMeas}(X, \mathbf{2})$ with the relevant subset of the following operations:

$\top$**:** *$\top$ is defined to be the set X,*

$\wedge$**:** *for $u, v \in P(X)$ define $u \wedge v = u \cap v$ (set intersection),*

$\vee$**:** *for $u, v \in P(X)$ define $u \vee v = u \cup v$ (set union).*

3. *The unit is given by*

$$\rho_A(a) = \{s \in S(A) \mid a \in s\},$$

and the counit by

$$\sigma_X(x) = \{u \in P(X) \mid x \in u\}.$$

Fuzzy Logical Connections

Example 3.7 (Fuzzy **MSL** or **DL**, and **Set** enriched over **Set**).

1. *The morphism*
$$\mu_A \colon U(A) \to [\mathbb{A}(A, [0,1]), V([0,1])]$$
assigns to each $a \in U(A)$, the fuzzy set u, of fuzzy filters/fuzzy prime filters s (of A), such that $u(s) = s(a)$. Since V is the identity functor, the V-initial lift
$$\overline{\mu}_A \colon U(A) \to \mathbf{Set}(S(A), [0,1])$$
assigns to each a the corresponding fuzzy subset of $S(A)$, where
$$S(A) = \mathbb{A}(A, [0,1]).$$

2. *The morphism*
$$\nu_X \colon V(X) \to [\mathbf{Set}(X, [0,1]), U([0,1])]$$
assigns to each $x \in V(X)$, the fuzzy set s, of fuzzy subsets u (of X), such that $s(u) = u(x)$. The U-initial lift
$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), [0,1])$$
assigns to each x the corresponding fuzzy filter/fuzzy prime filter of $P(X)$, where
$$UP(X) = \mathbf{Set}(X, [0,1]),$$
and $P(X)$ is defined to be $\mathbf{Set}(X, [0,1])$ with the relevant subset of the following operations:

$\top$**:** *$\top$ is defined by $\top(x) = 1$,*

$\wedge$**:** *for $u, v \in P(X)$ define $(u \wedge v)(x) = \min(u(x), v(x))$,*

$\vee$**:** *for $u, v \in P(X)$ define $(u \vee v)(x) = \max(u(x), v(x))$.*

3. *The unit is given by*
$$\rho_A(a)(s) = s(a),$$
and the counit by
$$\sigma_X(x)(u) = u(x).$$

Example 3.8 (Fuzzy **MSL** or **DL**, and $\mathbf{Set}_R$ enriched over $\mathbf{Set}_R$)**.**

1. *The morphism*

$$\mu_A \colon U(A) \to [\mathbb{A}(A,[0,1]), V([0,1])]$$

assigns to each $a \in U(A)$, the right R-closed fuzzy set u, of fuzzy filters/fuzzy prime filters s (of A), such that $u(s) = s(a)$. Since V is the identity functor, the V-initial lift

$$\overline{\mu}_A \colon U(A) \to \mathbf{Set}_R(S(A),[0,1])$$

assigns to each a the corresponding right R-closed fuzzy subset of $S(A)$, where

$$S(A) = \mathbb{A}(A,[0,1]).$$

Here $S(A)$ is ordered pointwise if the type R represents preorders or partial orders, and by equality if the type R represents equivalence relations or equality.

2. *The morphism*

$$\nu_X \colon V(X) \to [\mathbf{Set}_R(X,[0,1]), U([0,1])]$$

assigns to each $x \in V(X)$, the right R-closed fuzzy set s, of right R-closed fuzzy subsets u (of X), such that $s(u) = u(x)$. The U-initial lift

$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X),[0,1])$$

assigns to each x the corresponding fuzzy filter/fuzzy prime filter of $P(X)$, where

$$UP(X) = \mathbf{Set}_R(X,[0,1]),$$

and $P(X)$ is defined to be $\mathbf{Set}_R(X,[0,1])$ with the relevant subset of the following operations:

$\top$**:** *$\top$ is defined by $\top(x) = 1$,*

$\wedge$**:** *for $u, v \in P(X)$ define $(u \wedge v)(x) = \min(u(x), v(x))$,*

$\vee$**:** *for $u, v \in P(X)$ define $(u \vee v)(x) = \max(u(x), v(x))$.*

3. *The unit is given by*

$$\rho_A(a)(s) = s(a),$$

and the counit by

$$\sigma_X(x)(u) = u(x).$$

Example 3.9 (Fuzzy **MSL** or **DL**, and **GMet** enriched over **GMet**).

1. *The morphism*
$$\mu_A \colon U(A) \to [\mathbb{A}(A, [0,1]), V([0,1])]$$
assigns to each $a \in U(A)$, the right d-closed fuzzy set u, of fuzzy filters/fuzzy prime filters s (of A), such that $u(s) = s(a)$. Since V is the identity functor, the V-initial lift
$$\overline{\mu}_A \colon U(A) \to \mathbf{GMet}(S(A), [0,1])$$
assigns to each a the corresponding right d-closed fuzzy subset of $S(A)$, where
$$S(A) = \mathbb{A}(A, [0,1]).$$
Here $S(A)$ has the generalised metric $d(s,s') = 0$, if $s(a) \leq s'(a)$ for all $a \in A$, and $d(s,s') = \infty$ otherwise.

2. *The morphism*
$$\nu_X \colon V(X) \to [\mathbf{GMet}(X, [0,1]), U([0,1])]$$
assigns to each $x \in V(X)$, the right d-closed fuzzy set s, of right d-closed fuzzy subsets u (of X), such that $s(u) = u(x)$. The U-initial lift
$$\overline{\nu}_X \colon V(X) \to \mathbb{A}(P(X), [0,1])$$
assigns to each x the corresponding fuzzy filter/fuzzy prime filter of $P(X)$, where
$$UP(X) = \mathbf{GMet}(X, [0,1]),$$
and $P(X)$ is defined to be $\mathbf{GMet}(X, [0,1])$ with the relevant subset of the following operations:

   $\top$**:** *$\top$ is defined by $\top(x) = 1$,*

   $\wedge$**:** *for $u, v \in P(X)$ define $(u \wedge v)(x) = \min(u(x), v(x))$,*

   $\vee$**:** *for $u, v \in P(X)$ define $(u \vee v)(x) = \max(u(x), v(x))$.*

3. *The unit is given by*
$$\rho_A(a)(s) = s(a),$$
and the counit by
$$\sigma_X(x)(u) = u(x).$$

3.7 Discussion

Logical connections are rapidly becoming the standard base for formulating coalgebraic modal logic, however to date most work has been done using ordinary dual adjunctions, for example Kurz and Pattinson (2002, 2005); Kupke et al. (2004b); Bonsangue and Kurz (2005, 2006); Kurz (2006); Pavlovic et al. (2006); Kurz and Rosický (2007, 2012); Klin (2007); Kurz and Petrişan (2008); Jacobs and Sokolova (2010), although there are undoubtedly many others.

Many of the above authors worked with simple examples of dual adjunctions between **BA** and **Set**, but others considered state spaces with some kind of topology. Kurz and Pattinson (2002, 2005) considered the topology induced by observation of only a finite number of transition steps, and Kupke et al. (2004b) looked at coalgebras on Stone spaces (compact, totally disconnected, Hausdorff spaces), as by the Stone Representation Theorem, **BA** is dual to **Stone** (Stone Duality). The work of Bonsangue and Kurz (2005, 2006) then applied the generalisations of Stone Duality of Johnstone (1982), to look at coalgebras on general categories of topological spaces, and modal logics constructed from (dually equivalent) categories of lattices.

Other general classes of dual adjunctions have been investigated. In Kurz and Rosický (2007, 2012); Kurz and Petrişan (2008) the logical connections employed arise from base categories that are constructed from two different completions of a common category with finite limits and colimits.

Recently some work has begun that utilises logical connections enriched over **Preord** and **Pos** to extend coalgebraic modal logic (Kapulkin et al., 2010, 2012; Bílková et al., 2011), but no one as far as we are aware has attempted to do coalgebraic modal logic enriched over anything else, for example metric spaces.

Finally, the duality between real C^*-algebras and compact Hausdorff spaces has been used to investigate Markov Processes (Mislove et al., 2004), although the authors did not make use of coalgebras, nor modal logic. However, other authors have treated Markov Processes coalgebraically (Jacobs and Sokolova, 2010), although in this case the logical connection used was between **MSL** and **Meas**.

Chapter 4

Coalgebraic Modal Logics

In Chapter 3 we described the concept of a logical connection. This forms the static base of our framework. What we mean by this, is that the formulae are given meaning by way of generalised predicates on a state space, but there is no notion of transition from one state to another. We shall address this in this chapter by adding dynamics to our state spaces in the form of coalgebras, and we shall add modalities to our logics to capture, or model, the dynamics introduced by these coalgebras.

A brief outline of this chapter is as follows:

Section 4.1 The standard notion of an algebra or coalgebra for a functor is extended to the enriched setting, and the $\mathbb{V}$-categories $\mathbf{Alg}(L)$ and $\mathbf{CoAlg}(T)$ defined through the initial lifts of the relevant forgetful functors.

Section 4.2 Abstract and concrete coalgebraic modal logics are discussed, where the latter constitute presentations of the former, and the abstract modal logics are given by L-algebras.

Section 4.3 The semantics of coalgebraic modal logics are given, and the $\mathbb{V}$-category $\mathbf{Mod}(A, \alpha)$ of models for an L-algebra (A, α) is defined.

Section 4.4 The forgetful functors that define $\mathbf{CoAlg}(T)$ and $\mathbf{Mod}(A, \alpha)$ via initial lifts are shown to create conical colimits.

Section 4.5 A brief summary of some of the related work in the coalgebraic modal logic literature is given, including issues of soundness and completeness that we do not pursue in our work. Alternatives to coalgebraic modal logic, such as Moss' coalgebraic logic, and various coequational logics are also briefly mentioned.

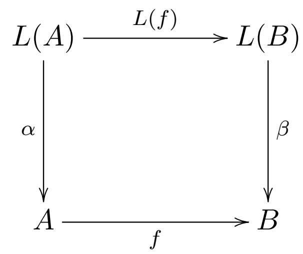
4.1 Algebras and Coalgebras for a Functor

We start by recalling the standard definitions of algebras and coalgebras for an ordinary functor, and extend them in an obvious way to the case of a $\mathbb{V}$-functor. It should be readily apparent that since algebras and coalgebras for a functor are pairs of carrier objects and morphisms to/from the carrier, that the correct definition will be at the level of the underlying category.

Definition 4.1. Given a $\mathbb{V}$-functor $L\colon \mathbb{A} \to \mathbb{A}$, an **algebra for** L, or an L**-algebra**, is a pair (A, α), where A is an object in $\mathbb{A}$, and $\alpha \in \mathbb{A}_o(L(A), A)$.

L-algebras form an ordinary category.

Definition 4.2. The ordinary category $\mathbf{Alg}(L)_o$ has L-algebras as objects, and a morphism $f\colon (A, \alpha) \to (B, \beta)$ is given by an $f \in \mathbb{A}_o(A, B)$ such that the following diagram commutes in $\mathbb{A}_o$.

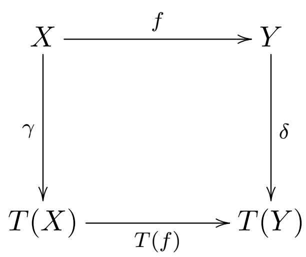


Similarly we have coalgebras for a functor.

Definition 4.3. Given a $\mathbb{V}$-functor $T\colon \mathbb{X} \to \mathbb{X}$, a **coalgebra for** T, or a T**-coalgebra**, is a pair (X, γ), where X is an object in $\mathbb{X}$, and $\gamma \in \mathbb{X}_o(X, T(X))$.

T-coalgebras form an ordinary category.

Definition 4.4. The ordinary category $\mathbf{CoAlg}(T)_o$ has T-coalgebras as objects, and a morphism $f\colon (X, \gamma) \to (Y, \delta)$ is given by an $f \in \mathbb{X}_o(X, Y)$ such that the following diagram commutes in $\mathbb{X}_o$.



The obvious question to ask at this point is, do L-algebras and T-coalgebras form $\mathbb{V}$-categories?

Well, since $\mathbb{V}_o$ is cocomplete, by Definition C.89 every locally small ordinary category yields a free $\mathbb{V}$-category with the same objects. So the answer is clearly yes. However, the hom-objects of such a free $\mathbb{V}$-category are given as copowers of I. What this means is, that in the case of enrichment over preordered sets (Definition 2.1), the hom-objects have the discrete preorder. Now this is perfectly valid, but we know that L-algebra and T-coalgebra morphisms form a subset of the morphisms between the carrier objects, and that these morphisms are ordered. So intuitively we should be able to simply restrict the relevant orders when constructing the hom-objects of the $\mathbb{V}$-categories $\mathbf{Alg}(L)$ and $\mathbf{CoAlg}(T)$. Specifically, the hom-object $\mathbf{Alg}(L)((A,\alpha),(B,\beta))$ would be the hom-set $\mathbf{Alg}(L)_o((A,\alpha),(B,\beta))$ supplied with the largest preorder consistent with the preorder on B. In other words, the restriction of the preorder on $\mathbb{A}(A,B)$ to the subset of morphisms that are L-algebra morphisms from (A,α) to (B,β).

To formalise this intuition we shall form the initial lifts (Definition 2.10) of the relevant forgetful functors.

Definition 4.5. Given a $\mathbb{V}$-functor $L\colon \mathbb{A} \to \mathbb{A}$, there is a forgetful (faithful) functor

$$\begin{aligned} U_{\mathbf{Alg}(L)_o}\colon \mathbf{Alg}(L)_o &\to \mathbb{A}_o \\ (A,\alpha) &\mapsto A \\ f\colon (A,\alpha)\to(B,\beta) &\mapsto f\colon A\to B. \end{aligned}$$

Definition 4.6. Given a $\mathbb{V}$-functor $T\colon \mathbb{X} \to \mathbb{X}$, there is a forgetful (faithful) functor

$$\begin{aligned} U_{\mathbf{CoAlg}(T)_o}\colon \mathbf{CoAlg}(T)_o &\to \mathbb{X}_o \\ (X,\gamma) &\mapsto X \\ f\colon (X,\gamma)\to(Y,\delta) &\mapsto f\colon X\to Y. \end{aligned}$$

Now invoking Theorem 2.13, we can form the initial lifts of the ordinary functors $U_{\mathbf{Alg}(L)_o}$ and $U_{\mathbf{CoAlg}(T)_o}$, and since $\mathbf{elem}|-|$ is faithful, by Proposition C.21 so are the initial lifts.

Proposition 4.7. *Given the conditions of Assumption 4, the forgetful ordinary functors $U_{\mathbf{Alg}(L)_o}$ and $U_{\mathbf{CoAlg}(T)_o}$ have initial lifts*

$$\begin{aligned} \overline{U_{\mathbf{Alg}(L)_o}}\colon \overline{\mathbf{Alg}(L)_o} &\to \mathbb{A} \\ \overline{U_{\mathbf{CoAlg}(T)_o}}\colon \overline{\mathbf{CoAlg}(T)_o} &\to \mathbb{X}, \end{aligned}$$

where the $\mathbb{V}$-functors $\overline{U_{\mathbf{Alg}(L)_o}}$ and $\overline{U_{\mathbf{CoAlg}(T)_o}}$ are faithful, and the $\mathbb{V}$-categories $\overline{\mathbf{Alg}(L)_o}$ and $\overline{\mathbf{CoAlg}(T)_o}$ are unique up to isomorphism.

We take these initial lifts to be the definitions of the $\mathbb{V}$-categories of L-algebras and T-coalgebras that we are looking for.

Definition 4.8. Given the conditions of Assumption 4, the $\mathbb{V}$-categories $\mathbf{Alg}(L)$ and $\mathbf{CoAlg}(T)$, and the forgetful $\mathbb{V}$-functors

$$U_{\mathbf{Alg}(L)} \colon \mathbf{Alg}(L) \to \mathbb{A}$$
$$U_{\mathbf{CoAlg}(T)} \colon \mathbf{CoAlg}(T) \to \mathbb{X},$$

are the initial lifts of the forgetful functors $U_{\mathbf{Alg}(L)_o}$ and $U_{\mathbf{CoAlg}(T)_o}$.

L-Algebra Examples

As mentioned in the introduction to this chapter, and more properly discussed in Section 4.2, the L-algebras are intended to represent base logics (objects in the category $\mathbb{A}$) that have been augmented with additional modal operators (introduced by the $\mathbb{V}$-functor L). This means we are intending that there exist L-algebras that are in fact the Lindenbaum-Tarski algebras (term algebras) of the modal logics. This in turn implies that the functor L often has a special form. Specifically, it is often of the form $L = F_{\mathbb{B}} M_{\mathbb{B}} U_{\mathbb{B}}$ (Kupke et al., 2004b; Jacobs and Sokolova, 2010). Here $U_{\mathbb{B}} \colon \mathbb{A} \to \mathbb{B}$ is a forgetful functor from the category $\mathbb{A}$ to a category $\mathbb{B}$, where the objects of $\mathbb{B}$ are algebras with only a subset of the operations of the corresponding algebras of $\mathbb{A}$. For example $U_{\mathbf{MSL}} \colon \mathbf{BA} \to \mathbf{MSL}$. The functor $F_{\mathbb{B}} \colon \mathbb{B} \to \mathbb{A}$ is then the left adjoint to $U_{\mathbb{B}}$, and creates the free $\mathbb{A}$ algebras over $\mathbb{B}$ objects. Finally, the functor $M_{\mathbb{B}} \colon \mathbb{B} \to \mathbb{B}$ is typically of the form

$$M_{\mathbb{B}}(-) = \coprod_{\lambda \in \Lambda_{\mathbb{B}}} (-)^{\mathrm{ar}(\lambda)}$$

where $\Lambda_{\mathbb{B}}$ is a set of modalities λ of arity $\mathrm{ar}(\lambda) \in \mathbb{N}$.

What is going on here, is that each L-algebra (A, α) is required to have a function $\lambda_A \colon A^{\mathrm{ar}(\lambda)} \to A$ for each of the modalities $\lambda \in \Lambda_{\mathbb{B}}$. However, they need not be required to preserve all the structure of A (which is an object of $\mathbb{A}$). For example, when constructing the category of modal algebras $\mathbf{MA}$, one adds to each Boolean algebra a modality $\Box$ that is required to preserve finite meets but not joins (Kupke et al., 2004b, Definition 3.1). Thus the forgetful functor $U_{\mathbb{B}}$ is chosen to forget the structure of A that λ does not preserve, λ is then defined as a morphism in the category $\mathbb{B}$ (the structure that it must preserve), and then the functor $F_{\mathbb{B}}$ creates the free $\mathbb{A}$ object over the result.

Now, since $F_{\mathbb{B}}$ is the left adjoint to $U_{\mathbb{B}}$ it preserves colimits, so we can write L as

$$L(-) = \coprod_{\lambda \in \Lambda_{\mathbb{B}}} F_{\mathbb{B}}(U_{\mathbb{B}}(-))^{\mathrm{ar}(\lambda)},$$

and if modalities are required that preserve two different substructures of the objects of $\mathbb{A}$, we can introduce separate categories $\mathbb{B}$ and $\mathbb{C}$ and define L as

$$L(-) = \coprod_{\lambda \in \Lambda_{\mathbb{B}}} F_{\mathbb{B}}(U_{\mathbb{B}}(-))^{\mathrm{ar}(\lambda)} + \coprod_{\lambda \in \Lambda_{\mathbb{C}}} F_{\mathbb{C}}(U_{\mathbb{C}}(-))^{\mathrm{ar}(\lambda)}.$$

This obviously generalises further, but we shall only look at examples where the modalities all preserve the same structure, moreover, our modalities will all be unary operators. This means our functor L will be of the form

$$L(-) = \coprod_{\lambda \in \Lambda_{\mathbb{B}}} F_{\mathbb{B}} U_{\mathbb{B}}(-).$$

For our examples we shall look at enrichment over **Set** (ordinary category theory), $\mathbf{Set}_R$ (Definition 2.1), and **GMet** (Definition 2.5). We shall primarily be concerned with adding modalities of the form $\langle l \rangle$ or $[l]$ for some $l \in \Sigma$, or L_r for some $r \in \mathbb{Q} \cap [0, 1]$. Here Σ is some set of labels, and $\langle l \rangle a$ has an intended reading of "can make a transition with label l to a state where a is true", $[l]a$ has the intended reading "every transition with label l leads to a state where a is true", and $L_r a$ has the intended reading "in the next step a is true with probability at least r".

The examples when enriching over **Set** are well known, for example see Jacobs and Sokolova (2010).

Example 4.1 (Adding $\Box$ to **BA** enriched over **Set**)**.** *This is the classic example of adding the $\Box$ operator to propositional logic to yield the basic modal logic. The algebras of the basic modal logic are the modal algebras* **MA**. *For this we use the adjunction*

$$F_{\mathbf{MSL}} \dashv U_{\mathbf{MSL}} \colon \mathbf{BA} \to \mathbf{MSL},$$

and define $L = F_{\mathbf{MSL}} U_{\mathbf{MSL}}$, giving $\mathbf{Alg}(L) \cong \mathbf{MA}$. The basic modal logic is usually the starting point for modal logics with semantics given by Kripke frames (Blackburn et al., 2001).

Example 4.2 (Adding $[l]$ to **BA** enriched over **Set**)**.** *This is a variant of Example 4.1, and we take the same adjunction*

$$F_{\mathbf{MSL}} \dashv U_{\mathbf{MSL}} \colon \mathbf{BA} \to \mathbf{MSL},$$

but we define

$$L(-) = \coprod_{l \in \Sigma} F_{\mathbf{MSL}} U_{\mathbf{MSL}}(-).$$

The category $\mathbf{Alg}(L)$ is then isomorphic to the category of Boolean algebras each with a set of finite meet preserving operators $[l]$ index by $l \in \Sigma$. These logics have found use characterising bisimulation of Labelled Transition Systems (Hennessy and Milner, 1980, 1985).

Example 4.3 (Adding L_r to **MSL** enriched over **Set**). *Using the adjunction*

$$F_{\mathbf{Set}} \dashv U_{\mathbf{Set}} \colon \mathbf{MSL} \to \mathbf{Set},$$

we define

$$L(-) = \coprod_{r \in \mathbb{Q} \cap [0,1]} F_{\mathbf{Set}} U_{\mathbf{Set}}(-).$$

The category $\mathbf{Alg}(L)$ *is then isomorphic to the category of meet semilattices (with top) each with a set of operators* L_r *indexed by* $r \in \mathbb{Q} \cap [0,1]$. *These logics have found use characterising bisimulation of Markov Chains (Larsen and Skou, 1991).*

The examples in the case of enrichment over $\mathbf{Set}_R$ are straightforward variations of the last two examples above. The first appeared in Wilkinson (2012a), and the second is new.

Example 4.4 (Adding $\langle l \rangle$ to **MSL** enriched over $\mathbf{Set}_R$). *This is a variant of Example 4.2, but without negation, and we add the modalities* $\langle l \rangle$ *rather than* $[l]$. *The categories are also enriched over* $\mathbf{Set}_R$, *and we make use of the fact that* **MSL** *is naturally so (Definition 3.21).*

We take the adjunction

$$F_{\mathbf{Set}_R} \dashv U_{\mathbf{Set}_R} \colon \mathbf{MSL} \to \mathbf{Set}_R,$$

where $F_{\mathbf{Set}_R}(X, R_X)$ *is the usual free meet semilattice* $F(X)$ *over the set of variables* X, *and the relation* $R_{F(X)}$ *is given by* $[x]R_{F(X)}[y] \Leftrightarrow xR_X y$, *for* $x, y \in X$. *The functor* L *is then defined by*

$$L(-) = \coprod_{l \in \Sigma} F_{\mathbf{Set}_R} U_{\mathbf{Set}_R}(-).$$

The category $\mathbf{Alg}(L)$ *is isomorphic to the category of meet semilattices (with top) each with a set of operators* $\langle l \rangle$ *indexed by* $l \in \Sigma$. *These logics have found use characterising simulation of Labelled Transition Systems (van Glabbeek, 2001).*

Example 4.5 (Adding L_r to **DL** enriched over $\mathbf{Set}_R$). *This is a variant of Example 4.3, but with joins and enriched over* $\mathbf{Set}_R$. *Here we use the fact that each object of the category* **DL** *has a natural order relation (Definition 3.21).*

We take the adjunction

$$F_{\mathbf{Set}_R} \dashv U_{\mathbf{Set}_R} \colon \mathbf{DL} \to \mathbf{Set}_R,$$

where $F_{\mathbf{Set}_R}(X, R_X)$ *is the usual free distributive lattice* $F(X)$ *over the set of variables* X, *and the relation* $R_{F(X)}$ *is given by* $[x]R_{F(X)}[y] \Leftrightarrow xR_X y$, *for* $x, y \in X$. *The functor* L *is then defined by*

$$L(-) = \coprod_{r \in \mathbb{Q} \cap [0,1]} F_{\mathbf{Set}_R} U_{\mathbf{Set}_R}(-).$$

The category $\mathbf{Alg}(L)$ *is isomorphic to the category of distributive lattices (with top) each with a set of operators* L_r *indexed by* $r \in \mathbb{Q} \cap [0, 1]$*. These logics have found use characterising simulation of Markov Chains (Desharnais et al., 2003).*

Finally, since $\mathbf{Set}_R$ can be embedded in $\mathbf{GMet}$ (Section 2.3), enrichment over $\mathbf{GMet}$ yields examples that correspond to those above. These are both new results.

Example 4.6 (Adding $\langle l \rangle$ to **MSL** enriched over **GMet**)**.** *This is a variant of Example 4.4, but enriched over* **GMet***. Here we use the fact that the objects of the category* **MSL** *have a natural metric (Definition 3.22).*

We take the adjunction

$$F_{\mathbf{GMet}} \dashv U_{\mathbf{GMet}} \colon \mathbf{MSL} \to \mathbf{GMet},$$

where $F_{\mathbf{GMet}}(X, d_X)$ *is the usual free meet semilattice* $F(X)$ *over the set of variables* X*, and the metric* $d_{F(X)}$ *is given by* $d_{F(X)}([x], [y]) = d_X(x, y)$*, for* $x, y \in X$*. The functor* L *is then defined by*

$$L(-) = \coprod_{l \in \Sigma} F_{\mathbf{GMet}} U_{\mathbf{GMet}}(-).$$

The category $\mathbf{Alg}(L)$ *is isomorphic to the category of meet semilattices (with top) each with a set of operators* $\langle l \rangle$ *indexed by* $l \in \Sigma$*. These are the same logics as Example 4.4, but equipped with a metric rather than an order relation.*

Example 4.7 (Adding L_r to **DL** enriched over **GMet**)**.** *This is a variant of Example 4.5, but enriched over* **GMet***. Here we use the fact that the objects of the category* **DL** *have a natural metric (Definition 3.22).*

We take the adjunction

$$F_{\mathbf{GMet}} \dashv U_{\mathbf{GMet}} \colon \mathbf{DL} \to \mathbf{GMet},$$

where $F_{\mathbf{GMet}}(X, d_X)$ *is the usual free distributive lattice* $F(X)$ *over the set of variables* X*, and the metric* $d_{F(X)}$ *is given by* $d_{F(X)}([x], [y]) = d_X(x, y)$*, for* $x, y \in X$*. The functor* L *is then defined by*

$$L(-) = \coprod_{r \in \mathbb{Q} \cap [0,1]} F_{\mathbf{GMet}} U_{\mathbf{GMet}}(-).$$

The category $\mathbf{Alg}(L)$ *is isomorphic to the category of distributive lattices (with top) each with a set of operators* L_r *indexed by* $r \in \mathbb{Q} \cap [0, 1]$*. These are the same logics as Example 4.5, but equipped with a metric rather than an order relation.*

T-Coalgebra Examples

There are many examples of T-coalgebras in the literature, especially in the case of enrichment over **Set** (ordinary category theory), but other authors have also looked at coalgebras in an enriched setting, though not in the full generality above (Turi and Rutten, 1998; Worrell, 2000a; Balan and Kurz, 2011; Bílková et al., 2011).

We shall now introduce a number of examples of T-coalgebras, and the most obvious place to start, is the well-known coalgebraic formulation of Kripke frames (Blackburn et al., 2001).

Example 4.8 (Powerset on **Set** enriched over **Set**). *The functor T is defined as*

$$T(X) = \mathcal{P}(X),$$

and for any function $f \colon X \to Y$, the action of T on f is the function

$$\begin{aligned} T(f) \colon \mathcal{P}(X) &\to \mathcal{P}(Y) \\ u &\mapsto \{f(x) \mid x \in u\}. \end{aligned}$$

The powerset functor corresponds to unbounded non-determinism, but we could make an entirely analogous definition for the finite powerset functor, which would correspond to finite branching non-determinism. In fact, this applies to the other examples below that incorporate the powerset functor.

The next step is to consider the powerset functor in the cases of enrichment over $\mathbf{Set}_R$ and **GMet**. The first case appeared in Wilkinson (2012a), and the second is the analogous result for **GMet**.

Example 4.9 (Powerset on $\mathbf{Set}_R$ enriched over $\mathbf{Set}_R$). *The functor T is defined as*

$$T(X, R_X) = (\mathcal{P}(X), R_{\mathcal{P}(X)}),$$

where assuming the type R represents preorders,

$$uR_{\mathcal{P}(X)}v \Leftrightarrow \forall x \in u \; \exists y \in v.\; xR_Xy,$$

and for any function $f \colon (X, R_X) \to (Y, R_Y)$, the action of T on f is the function

$$\begin{aligned} T(f) \colon \mathcal{P}(X) &\to \mathcal{P}(Y) \\ u &\mapsto \{f(x) \mid x \in u\}. \end{aligned}$$

The preorder $R_{\mathcal{P}(X)}$ is the one-sided counterpart of the Egli-Milner order used in the study of powerdomains (Plotkin, 1976).

Example 4.10 (Powerset on **GMet** enriched over **GMet**). *The functor T is defined as*

$$T(X, d_X) = (\mathcal{P}(X), d_{\mathcal{P}(X)}),$$

where

$$d_{\mathcal{P}(X)}(u, v) = \sup_{x \in u} \left(\inf_{y \in v} d_X(x, y) \right),$$

and for any function $f : (X, d_X) \to (Y, d_Y)$, the action of T on f is the function

$$\begin{aligned} T(f) : \mathcal{P}(X) &\to \mathcal{P}(Y) \\ u &\mapsto \{f(x) \mid x \in u\}. \end{aligned}$$

The metric $d_{\mathcal{P}(X)}$ is the one-sided counterpart of the Hausdorff distance (Rutten, 1998).

The particular choice of preorder or metric in these two extensions to the standard powerset example, corresponds to how we intend to compare behaviours of states. This will be explored in detail in Section 5.1.

We can also have finite powerset versions of these examples, where the preorder and metric remain unchanged.

It should also be noted that T-coalgebra structure maps $\gamma : (X, R_X) \to T(X, R_X)$ and $\gamma : (X, d_X) \to T(X, d_X)$ must be R-preserving, or non-expansive, respectively. This might seem like a strong constraint, but it is not. This is because we are free to give any set X the discrete order or metric, and if we do so, we place no restrictions on the possible choices of γ. Again this will be explained in Section 5.1.

From the powerset functor we can model Labelled Transition Systems (LTSs). Below we shall look at unbounded branching LTSs, but using the finite powerset functor we could also describe finite branching LTSs.

We proceed by taking a set of labels Σ, and giving it the preorder

$$l R_\Sigma l' \Leftrightarrow l = l',$$

and the metric

$$d_\Sigma(l, l') = \begin{cases} 0 & : \text{ if } l = l' \\ \infty & : \text{ otherwise.} \end{cases}$$

Now using this set of labels, and the definitions of products in $\mathbf{Set}_R$ (Section 2.2), and **GMet** (Section 2.3), we can model Labelled Transition Systems as follows.

Example 4.11 (LTS enriched over **Set**)**.** *The functor T is defined as*

$$T(X) = \mathcal{P}(\Sigma \times X),$$

and for any function $f\colon X \to Y$, the action of T on f is the function

$$\begin{aligned} T(f)\colon \mathcal{P}(\Sigma \times X) &\to \mathcal{P}(\Sigma \times Y) \\ u &\mapsto \{(l, f(x)) \mid (l, x) \in u\}. \end{aligned}$$

Example 4.12 (LTS enriched over $\mathbf{Set}_R$)**.** *The functor T is defined as*

$$T(X, R_X) = (\mathcal{P}(\Sigma \times X), R_{\mathcal{P}(\Sigma\times X)}),$$

where assuming the type R represents preorders,

$$\begin{aligned} uR_{\mathcal{P}(\Sigma\times X)}v &\Leftrightarrow \forall (l, x) \in u\ \exists (l', x') \in v.\ (l, x)R_{\Sigma\times X}(l', x') \\ &\Leftrightarrow \forall (l, x) \in u\ \exists (l', x') \in v.\ l = l' \text{ and } xR_X x', \end{aligned}$$

and for any function $f\colon (X, R_X) \to (Y, R_Y)$, the action of T on f is the function

$$\begin{aligned} T(f)\colon \mathcal{P}(\Sigma \times X) &\to \mathcal{P}(\Sigma \times Y) \\ u &\mapsto \{(l, f(x)) \mid (l, x) \in u\}. \end{aligned}$$

Example 4.13 (LTS enriched over **GMet**)**.** *The functor T is defined as*

$$T(X, d_X) = (\mathcal{P}(\Sigma \times X), d_{\mathcal{P}(\Sigma\times X)}),$$

where

$$\begin{aligned} d_{\mathcal{P}(\Sigma\times X)}(u, v) &= \sup_{(l,x)\in u} \left(\inf_{(l',x')\in v} d_{\Sigma\times X}((l, x), (l', x')) \right) \\ &= \sup_{(l,x)\in u} \left(\inf_{(l',x')\in v} \max(d_\Sigma(l, l'), d_X(x, x')) \right), \end{aligned}$$

and for any function $f\colon (X, d_X) \to (Y, d_Y)$, the action of T on f is the function

$$\begin{aligned} T(f)\colon \mathcal{P}(\Sigma \times X) &\to \mathcal{P}(\Sigma \times Y) \\ u &\mapsto \{(l, f(x)) \mid (l, x) \in u\}. \end{aligned}$$

Once again the preorder $R_{\mathcal{P}(\Sigma\times X)}$ and metric $d_{\mathcal{P}(\Sigma\times X)}$ govern how the behaviour of states will be compared, and constrain the choice of possible T-coalgebra structure maps. For example, for $\gamma\colon (X, R_X) \to T(X, R_X)$, if xR_Xy, then since morphisms in $\mathbf{Set}_R$ must be R-preserving, we must have $\gamma(x)R_{\mathcal{P}(\Sigma\times X)}\gamma(y)$. Looking at the above definition we then see that this means that x must be simulated by y (using the standard notion of simulation for LTSs (van Glabbeek, 2001)). It will turn out that $d_{\mathcal{P}(\Sigma\times X)}$ allows us to talk of approximations of LTSs, and both of these will be discussed in detail in Section 5.1.

The above examples have covered the basics of non-determinism, the other main class of transition systems in computer science are the probabilistic transition systems. The simplest examples we can consider are coalgebras for the finite subprobability distribution functor $\mathcal{D}\colon \mathbf{Set} \to \mathbf{Set}$ (de Vink and Rutten, 1999; Jacobs and Sokolova, 2010), which is defined as

$$\mathcal{D}(X) = \{\phi\colon X \to [0,1] \mid \mathrm{supp}(\phi) \text{ is finite and } \textstyle\sum_{x\in X}\phi(x) \leq 1\}.$$

Here, ϕ is a subprobability distribution (total probability may be less than 1), with finite support, i.e. $\mathrm{supp}(\phi) = \{x \in X \mid \phi(x) \neq 0\}$ is finite. Each distribution $\phi\colon X \to [0,1]$ extends to a function

$$\begin{aligned}\phi\colon \mathcal{P}(X) &\to [0,1]\\ u &\mapsto \sum_{x\in u}\phi(x),\end{aligned}$$

and for any function $f\colon X \to Y$, the action of $\mathcal{D}$ on f is given by

$$\mathcal{D}(f)(\phi)(y) = \phi(f^{-1}[\{y\}]).$$

Example 4.14 (Distribution functor on **Set** enriched over **Set**)**.** *The functor T is defined as*

$$T(X) = \mathcal{D}(X),$$

and for any function $f\colon X \to Y$, the action of T on f is given by

$$T(f)(\phi)(y) = \sum_{x\in f^{-1}[\{y\}]}\phi(x).$$

For the cases of enrichment over $\mathbf{Set}_R$ and **GMet** we need to give the interval $[0,1]$ a preorder and a metric, as we wish to compare probabilities. For this we take the usual linear order on $[0,1]$, and the metric given, as in Definition 3.24, by

$$d_{[0,1]}(x,y) = \begin{cases} y - x & : \text{ if } x \leq y \\ \infty & : \text{ otherwise.}\end{cases}$$

Using this we then have the following two examples, where the first is related to standard notions of simulation for Markov Chains (Desharnais et al., 2003), and the second is suggested as a plausible metric analogue.

Example 4.15 (Distribution functor on $\mathbf{Set}_R$ enriched over $\mathbf{Set}_R$)**.** *The functor T is defined as*

$$T(X, R_X) = (\mathcal{D}(X), R_{\mathcal{D}(X)}),$$

where assuming the type R represents preorders,

$$\phi R_{\mathcal{D}(X)} \psi \Leftrightarrow \forall u \subseteq X \ (u \text{ right } R\text{-closed} \Rightarrow \phi(u) \leq \psi(u)),$$

and for any function $f\colon (X, R_X) \to (Y, R_Y)$, the action of T on f is given by

$$T(f)(\phi)(y) = \sum_{x \in f^{-1}[\{y\}]} \phi(x).$$

The preorder $R_{\mathcal{D}(X)}$ corresponds to the definition of a simulation relation for Markov Chains (Desharnais et al., 2003).

Example 4.16 (Distribution functor on **GMet** enriched over **GMet**)**.** *The functor T is defined as*

$$T(X, d_X) = (\mathcal{D}(X), d_{\mathcal{D}(X)}),$$

where

$$d_{\mathcal{D}(X)}(\phi, \psi) = \sup_{\substack{u \subseteq X \\ \text{right } d\text{-closed}}} d_{[0,1]}(\phi(u), \psi(u)),$$

and for any function $f\colon (X, d_X) \to (Y, d_Y)$, the action of T on f is given by

$$T(f)(\phi)(y) = \sum_{x \in f^{-1}[\{y\}]} \phi(x).$$

The metric $d_{\mathcal{D}(X)}$ does not to our knowledge appear in the literature, but is a possible analogue of the preorder $R_{\mathcal{D}(X)}$ from the example above. It represents a notion of distance between distributions that compares, not the value of the distributions at individual states, but the value on sets of states that are closed under finite distance (recall the definition of right d-closed from Definition 3.28).

The probabilistic examples above are Markov Chains with discrete probability distributions. Markov Processes are a continuous generalisation of Markov Chains formulated using measurable spaces. The standard way to do this coalgebraically is via the Giry functor $\mathcal{G}\colon \mathbf{Meas} \to \mathbf{Meas}$ (de Vink and Rutten, 1999; Panangaden, 1999; Jacobs and Sokolova, 2010). This is defined as

$$\mathcal{G}(X, \Sigma_X) = (\mathcal{G}(X), \Sigma_{\mathcal{G}(X)}),$$

where

$$\mathcal{G}(X) = \{\phi\colon \Sigma_X \to [0, 1] \mid \phi \text{ a subprobability measure}\},$$

and $\Sigma_{\mathcal{G}(X)}$ is the sigma algebra on $\mathcal{G}(X)$ generated by sets of the form

$$\{\phi \in \mathcal{G}(X) \mid \phi(M) \geq r\},$$

for $M \in \Sigma_X$, and $r \in \mathbb{Q} \cap [0,1]$. A subprobability measure ϕ must satisfy $\phi(\emptyset) = 0$, and

$$\phi\left(\bigcup_i M_i\right) = \sum_i \phi(M_i),$$

for countable unions of pairwise disjoint $M_i \in \Sigma_X$.

For every measurable function $f\colon (X, \Sigma_X) \to (Y, \Sigma_Y)$, there is an inverse function $f^{-1}\colon \Sigma_Y \to \Sigma_X$, and the action of $\mathcal{G}$ on f is defined by

$$\begin{aligned} \mathcal{G}(f)\colon \mathcal{G}(X) &\to \mathcal{G}(Y) \\ \phi &\mapsto \phi \circ f^{-1}. \end{aligned}$$

Example 4.17 (Giry functor on **Meas** enriched over **Set**)**.** *The functor T is defined as*

$$T(X, \Sigma_X) = (\mathcal{G}(X), \Sigma_{\mathcal{G}(X)}),$$

and for any measurable function $f\colon (X, \Sigma_X) \to (Y, \Sigma_Y)$, the action of T on f is given by

$$\begin{aligned} T(f)\colon \mathcal{G}(X) &\to \mathcal{G}(Y) \\ \phi &\mapsto \phi \circ f^{-1}. \end{aligned}$$

To enrich over $\mathbf{Set}_R$ and **GMet** we extend the definition of the Giry functor to the categories $\mathbf{Meas}_R$ (Example 2.1) and **GMeas** (Example 2.2).

Example 4.18 (Giry functor on $\mathbf{Meas}_R$ enriched over $\mathbf{Set}_R$)**.** *The functor T is defined as*

$$T(X, \Sigma_X, R_X) = (\mathcal{G}(X), \Sigma_{\mathcal{G}(X)}, R_{\mathcal{G}(X)}),$$

where assuming the type R represents preorders,

$$\phi R_{\mathcal{G}(X)} \psi \Leftrightarrow \forall M \in \Sigma_X \ (M \text{ right } R\text{-closed} \Rightarrow \phi(M) \leq \psi(M)),$$

and for any measurable function $f\colon (X, \Sigma_X, R_X) \to (Y, \Sigma_Y, R_Y)$, the action of T on f is given by

$$\begin{aligned} T(f)\colon \mathcal{G}(X) &\to \mathcal{G}(Y) \\ \phi &\mapsto \phi \circ f^{-1}. \end{aligned}$$

The preorder $R_{\mathcal{G}(X)}$ corresponds to the definition of a simulation relation for Markov Processes (Desharnais et al., 2003).

Example 4.19 (Giry functor on **GMeas** enriched over **GMet**). *The functor T is defined as*

$$T(X, \Sigma_X, d_X) = (\mathcal{G}(X), \Sigma_{\mathcal{G}(X)}, d_{\mathcal{G}(X)}),$$

where

$$d_{\mathcal{G}(X)}(\phi, \psi) = \sup_{\substack{M \in \Sigma_X \\ \text{right } d\text{-closed}}} d_{[0,1]}(\phi(M), \psi(M)),$$

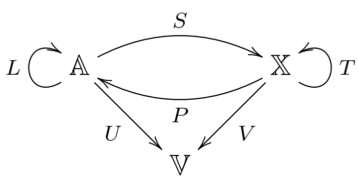
and for any measurable function $f\colon (X, \Sigma_X, d_X) \to (Y, \Sigma_Y, d_Y)$, the action of T on f is given by

$$\begin{aligned} T(f)\colon \mathcal{G}(X) &\to \mathcal{G}(Y) \\ \phi &\mapsto \phi \circ f^{-1}. \end{aligned}$$

The metric $d_{\mathcal{G}(X)}$ is the generalisation to Markov Processes of the metric from Example 4.16.

4.2 Coalgebraic Modal Logics

In Section 3.1 we described the $\mathbb{V}$-categories and functors that form the base level of our framework - something we called a logical connection. To this we now add $\mathbb{V}$-functors $L\colon \mathbb{A} \to \mathbb{A}$ and $T\colon \mathbb{X} \to \mathbb{X}$ as shown in the following non-commuting diagram.



The $\mathbb{V}$-functor T will introduce dynamics to the semantics via T-coalgebras, and the $\mathbb{V}$-functor L will extend the base logics by adding modalities.

This approach follows that of Kupke et al. (2004b); Klin (2007); Jacobs and Sokolova (2010), and is increasingly the standard approach to formulating coalgebraic modal logic.

Recall that the $\mathbb{V}$-category $\mathbb{X}$ represents a collection of state spaces, the $\mathbb{V}$-functor T then defines a collection of generalised transition systems on these state spaces as T-coalgebras. Similarly, the $\mathbb{V}$-category $\mathbb{A}$ represents a collection of base logics to which modal operators are to be added. These are introduced via the $\mathbb{V}$-functor L, and the corresponding modal logics are the L-algebras.

The semantics of the modal logics represented by the L-algebras are given in two stages. First the logical connection gives a semantics for the base logics in terms of the state spaces, and then secondly, a $\mathbb{V}$-natural transformation

$$\delta\colon LP \Rightarrow PT$$

gives the semantics of the modal operators in terms of the transition structures introduced by T (Kupke et al., 2004a, 2005).

We summarise the assumptions underlying the above as follows.

Assumption 5. We extend Assumption 4 (page 38) as follows:

9. There is a $\mathbb{V}$-functor $L\colon \mathbb{A} \to \mathbb{A}$.
10. There is a $\mathbb{V}$-functor $T\colon \mathbb{X} \to \mathbb{X}$.
11. There is a $\mathbb{V}$-natural transformation $\delta\colon LP \Rightarrow PT$.

We shall talk more about the $\mathbb{V}$-natural transformation δ in Section 4.3, but first we say more about the $\mathbb{V}$-functor L.

In Bonsangue and Kurz (2006); Kurz (2006) a distinction is drawn between an abstract modal logic and a concrete modal logic. A concrete modal logic is what a logician would call a modal logic. It consists of a syntax of propositional variables, connectives, and modal operators, and the terms in this syntax are related by a class of equations that are derived from a base collection of equations called axioms.

An abstract modal logic is an L-algebra (A, α) for some $\mathbb{V}$-functor L. The idea is that we have abstracted away the particular choice of syntax of a concrete modal logic. What matters is the collection of terms and their interrelations, not the exact choice of primitives for a modal logic. A particular choice of primitives and axioms is called a presentation, and a modal logic may have more than one, so why privilege one over the others? Obviously it makes sense to do so in some situations, but for what we are doing it makes sense to abstract all this away and hide it in the $\mathbb{V}$-functor L.

In Section 4.1 we described a general process whereby a presentation of a concrete modal logic could be turned into a $\mathbb{V}$-functor L, and numerous examples were given. To go in the other direction is to seek a presentation of a $\mathbb{V}$-functor by operations and equations. See for example Bonsangue and Kurz (2006).

One final observation is that, not only need the presentation for a given L not be unique, but the choice of L itself need not be unique. The choice of L is often guided by a particular presentation that one has in mind, but one also has the freedom to choose exactly how much of the dynamical behaviour introduced by T is to be captured by the L-algebras. One could for example decide to take the identity functor for L, and simply not add any modalities at all. Obviously such a choice would yield logics that do not capture all (or any!) of the behaviour in the T-coalgebras, but a less extreme example may be to only attempt to capture some of the behaviour. This would be an engineering choice, guided by practical considerations.

4.3 Models for L-Algebras

As promised in Section 4.2, in this section we shall aim to explain, and then explore, the action of the $\mathbb{V}$-natural transformation $\delta\colon LP \Rightarrow PT$ from Assumption 5. In essence all that δ does is provide the semantics of the modal operators implicit in the $\mathbb{V}$-functor L, but using it we can make an elegant definition of the notion of a model for an L-algebra.

Recall from Definition 3.15 the definition of a valuation $f \in \mathbb{A}_o(A, P(X))$. Now since under Assumption 5 the ordinary category $\mathbb{A}_o$ is concrete over **Set** (Section 3.2), f has an underlying function that maps formulae in A to predicates on X.

Intuitively therefore, in order to give semantics to an L-algebra (A, α), we need to find an $f \in \mathbb{A}_o(A, P(X))$ that also respects the additional structure (modal operators and axioms) that L adds to A. In other words we want f to be an L-algebra morphism, but between which L-algebras? Well obviously the domain must be (A, α), but what about the codomain? For that we should construct an L-algebra with the carrier object $P(X)$.

So how do we do this? First we note that the modalities introduced by L are intended to capture the dynamics introduced by T, so we pick a T-coalgebra (X, γ). Now

$$\gamma \in \mathbb{X}_o(X, T(X)),$$

so under the $\mathbb{V}$-functor P we have

$$P(\gamma) \in \mathbb{A}_o(PT(X), P(X)),$$

so if we had a morphism $g \in \mathbb{A}_o(LP(X), PT(X))$, we could take the composite to give an L-algebra. This is precisely what the $\mathbb{V}$-natural transformation $\delta\colon LP \Rightarrow PT$ gives.

Summarising the above, we can follow the approach in the literature (see for example Bonsangue and Kurz (2006); Kurz (2006)), and make the following definition.

Definition 4.9. Given the conditions of Assumption 5, there is an ordinary functor

$$\begin{aligned}
\tilde{P}_o\colon \mathbf{CoAlg}(T)_o &\to \mathbf{Alg}(L)_o \\
(X, \gamma) &\mapsto (P(X), P(\gamma) \circ \delta_X) \\
f\colon (X, \gamma) \to (Z, \xi) &\mapsto P(f)\colon \tilde{P}(Z, \xi) \to \tilde{P}(X, \gamma).
\end{aligned}$$

Now the above definition is standard for coalgebraic modal logic in an ordinary category theory setting, but we are working in an enriched setting, so can we lift $\tilde{P}_o$ to a $\mathbb{V}$-functor? The following proposition shows that this is the case, and relies upon the universal property of the construction of $\mathbf{Alg}(L)$.

Proposition 4.10. *Given the conditions of Assumption 5, there exits a unique* $\mathbb{V}$-*functor*

$$\tilde{P} \colon \mathbf{CoAlg}(T) \to \mathbf{Alg}(L)$$

with the underlying ordinary functor $\tilde{P}_o$.

Proof. The $\mathbb{V}$-category $\mathbf{Alg}(L)$ is defined via the initial lift (Definition 2.10) of the forgetful ordinary functor $U_{\mathbf{Alg}(L)_o} \colon \mathbf{Alg}(L)_o \to \mathbb{A}_o$ (Definition 4.8), and since we have

$$\mathbf{CoAlg}(T) \xrightarrow{U_{\mathbf{CoAlg}(T)}} \mathbb{X} \xrightarrow{P} \mathbb{A},$$

and by the definition of $\tilde{P}_o$ the following diagram commutes

$$\begin{array}{ccc} \mathbf{CoAlg}(T)_o & \xrightarrow{\tilde{P}_o} & \mathbf{Alg}(L)_o \\ {\scriptstyle U_{\mathbf{CoAlg}(T)_o}}\downarrow & & \downarrow{\scriptstyle U_{\mathbf{Alg}(L)_o}} \\ \mathbb{X}_o & \xrightarrow[P_o]{} & \mathbb{A}_o \end{array}$$

then there exists a unique $\mathbb{V}$-functor

$$\tilde{P} \colon \mathbf{CoAlg}(T) \to \mathbf{Alg}(L)$$

with the underlying ordinary functor $\tilde{P}_o$. □

We are now ready to define a valuation for an L-algebra. The definition mirrors that of Definition 3.15, and is a direct generalisation to the enriched setting of the usual approach in the literature (see for example Bonsangue and Kurz (2006); Kurz (2006)).

Definition 4.11. Given the conditions of Assumption 5, for any L-algebra (A, α), and any T-coalgebra (X, γ), a **valuation** is any

$$f \in \mathbf{Alg}(L)_o((A, \alpha), \tilde{P}(X, \gamma)).$$

At this point we should unpack this definition to see what it means in practice. To do this we shall look at the case of an L-algebra corresponding to a concrete modal logic.

Given a presentation for L (Section 4.2), the free L-algebras are the Lindenbaum-Tarski algebras for the corresponding concrete modal logics. Specifically, for a logic $\mathcal{L}$ given by a syntax of propositional variables, connectives, and modal operators, and a proof system that induces a derivability relation $\Gamma \vdash \phi$ for $\phi \in \mathcal{L}$ and $\Gamma \subseteq \mathcal{L}$, the Lindenbaum-Tarski algebra of $\mathcal{L}$ is an L-algebra with the carrier set $\mathcal{L}/\!\approx$, where

$$\phi \approx \psi \Leftrightarrow \phi \vdash \psi \text{ and } \psi \vdash \phi$$

is an equivalence relation since the base logics of $\mathbb{A}$ are represented by algebras. If $\mathcal{L}$ has an $\rightarrow$ operator, then the proof system is usually such that $\phi \approx \psi \Leftrightarrow \vdash \phi \leftrightarrow \psi$. To define the necessary operations on $\mathcal{L}/\!\approx$ to make it an L-algebra, the proof system of $\mathcal{L}$ must satisfy certain constraints in order that $\approx$ is a congruence. For example, for a modality λ the proof system should have a rule

$$\phi \vdash \psi \Rightarrow [\lambda]\phi \vdash [\lambda]\psi,$$

or, $\vdash \phi \leftrightarrow \psi \Rightarrow \vdash [\lambda]\phi \leftrightarrow [\lambda]\psi$ (Kupke et al., 2004a), for then we can define

$$[\lambda]_{\mathcal{L}/\approx}[\phi]_{\mathcal{L}/\approx} = [[\lambda]\phi]_{\mathcal{L}/\approx},$$

where $[\phi]_{\mathcal{L}/\approx}$ is an equivalence class of $\mathcal{L}$ under $\approx$, and $[\lambda]_{\mathcal{L}/\approx}$ is the operation on $\mathcal{L}/\!\approx$ for the modality λ. Finally, we can define $[\phi]_{\mathcal{L}/\approx} \leq [\psi]_{\mathcal{L}/\approx} \Leftrightarrow \phi \vdash \psi$.

If the proof system contains no axioms beyond those required to make $\mathcal{L}/\!\approx$ into an L-algebra, then the Lindenbaum-Tarski algebra of $\mathcal{L}$ is the free L-algebra over the set of propositional variables of $\mathcal{L}$, and if $\mathcal{L}$ has no propositional variables, then the Lindenbaum-Tarski algebra is the initial L-algebra.

If now we write $|A| = \mathbf{elem}|U(A)|$ for the underlying set of an object A in $\mathbb{A}$, then for a T-coalgebra (X, γ) the underlying set of $\tilde{P}(X, \gamma)$ is $|P(X)|$, and we can define a function

$$f\colon \mathcal{L}/\!\approx \rightarrow |P(X)|,$$

and this gives

$$[\![-]\!]_f = f \circ q\colon \mathcal{L} \rightarrow |P(X)|,$$

where $q\colon \mathcal{L} \rightarrow \mathcal{L}/\!\approx$ is the quotient map of $\approx$. The valuation given by f is the unique L-algebra homomorphism that extends the function f, and indeed, the function f need only be given for the equivalence classes $[p]_{\mathcal{L}/\approx}$, where p is a propositional variable, as the extension to $\mathcal{L}/\!\approx$ follows by induction. In the case of the initial L-algebra, since there are no variables, f is unique, and we simply write $[\![-]\!]$ for $[\![-]\!]_f$.

From the above, somewhat informal discussion, we can see that our definition of a valuation captures the concrete notion of assigning a predicate on X to each formula of a logic. Though, it should be noted, that the formulas of $\mathcal{L}/\!\approx$ are in fact equivalence classes of terms in $\mathcal{L}$. We shall sometimes blur this distinction, and use a valuation f, and the function $[\![-]\!]_f$, interchangeably.

So now that we know how to define a valuation for an L-algebra via the $\mathbb{V}$-natural transformation $\delta\colon LP \Rightarrow PT$, how do we find δ? The answer is that we can construct δ from predicate liftings.

Given a concrete presentation of L in terms of a specific set of modal operators, then the $\mathbb{V}$-natural transformation $\delta\colon LP \Rightarrow PT$ corresponds to a set of predicate liftings (Jacobs, 2000; Pattinson, 2001, 2003; Kurz and Pattinson, 2002, 2005; Schröder, 2008).

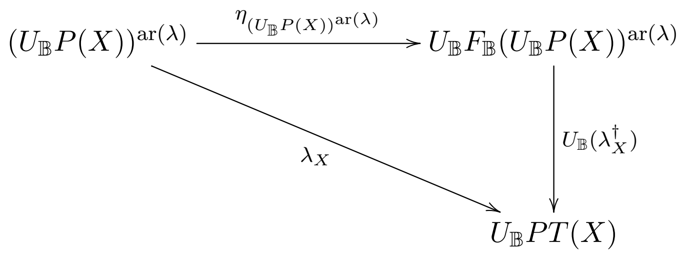
For a modality λ in a set of modal operators $\Lambda_{\mathbb{B}}$, if the arity $\mathrm{ar}(\lambda) \in \mathbb{N}$, the general form of the predicate lifting for λ is a so called polyadic predicate lifting (Schröder, 2008), which is a $\mathbb{V}$-natural transformation of the form

$$\lambda_X\colon (U_{\mathbb{B}}P(X))^{\mathrm{ar}(\lambda)} \to U_{\mathbb{B}}PT(X).$$

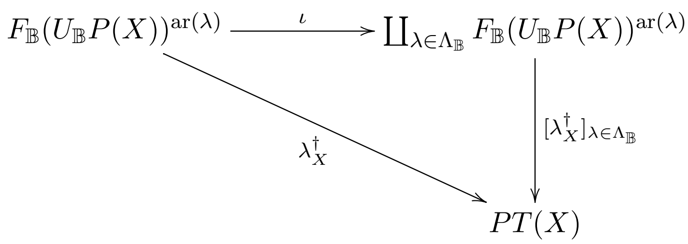
Here $\Lambda_{\mathbb{B}}$ is as defined in Section 4.1, and L is given by

$$L(-) = \coprod_{\lambda \in \Lambda_{\mathbb{B}}} F_{\mathbb{B}}(U_{\mathbb{B}}(-))^{\mathrm{ar}(\lambda)}.$$

Using the unit of the adjunction $F_{\mathbb{B}} \dashv U_{\mathbb{B}}\colon \mathbb{A} \to \mathbb{B}$, we have



and the coproduct then gives



from which we see that we can take δ_X to be given by

$$\delta_X = [\lambda_X^{\dagger}]_{\lambda \in \Lambda_{\mathbb{B}}}.$$

By way of illustrating this, we shall consider four variants of Hennessy-Milner logic for Labelled Transition Systems. For a set of labels Σ, the standard formulation of Hennessy-Milner logic (Hennessy and Milner, 1980, 1985) is given by the following syntax:

$$\mathcal{L}_1 \ni \phi ::= \mathrm{tt} \mid p \mid \phi \wedge \phi \mid \neg\phi \mid [l]\phi \quad \text{where } l \in \Sigma \text{ and } p \in \mathrm{Var}.$$

The first variant we shall consider is the standard bivalent formulation of Hennessy-Milner logic, as used to characterise bisimulation of Labelled Transition Systems (Hennessy and Milner, 1980, 1985).

Example 4.20 (Bivalent logic for bisimulation of LTSs)**.** *The logical connection is the dual adjunction between* **BA** *and* **Set** *from Example 3.1, to which we add the functor L from Example 4.2 which adds the modal operators $[l]$, and the functor $T(X) = \mathcal{P}(\Sigma \times X)$ from Example 4.11.*

The predicate liftings for the modalities are then the family of natural transformations

$$\begin{aligned}
[l]_X \colon U_{\mathbf{MSL}}P(X) &\to U_{\mathbf{MSL}}P\mathcal{P}(\Sigma \times X) \\
\top &\mapsto \mathcal{P}(\Sigma \times X) \\
u &\mapsto \{w \in \mathcal{P}(\Sigma \times X) \mid \forall (l', x) \in w,\ l' = l \Rightarrow x \in u\} \\
u \wedge v &\mapsto [l]_X(u) \cap [l]_X(v),
\end{aligned}$$

giving ($[u_l]$ is the equivalence class in $LP(X)$ of an element of the l^{th} copy of $P(X)$)

$$\begin{aligned}
\delta_X \colon LP(X) &\to PT(X) \\
\top &\mapsto \mathcal{P}(\Sigma \times X) \\
[u_l] &\mapsto \{w \in \mathcal{P}(\Sigma \times X) \mid \forall (l', x) \in w,\ l' = l \Rightarrow x \in u\} \\
[u_{l_1}] \wedge [v_{l_2}] &\mapsto \delta_X([u_{l_1}]) \cap \delta_X([v_{l_2}]) \\
\neg [u_l] &\mapsto \delta_X([u_l])^c.
\end{aligned}$$

So for the L-algebra given by the Lindenbaum-Tarski algebra of $\mathcal{L}_1$, a T-coalgebra (X, γ), and a function $f \colon \mathrm{Var} \to |P(X)|$, there is a unique function $[\![-]\!]_f$ given by

$$\begin{aligned}
[\![-]\!]_f \colon \mathcal{L}_1 &\to |P(X)| \\
tt &\mapsto X \\
p &\mapsto f(p) \\
[l]\phi &\mapsto \{x \in X \mid \forall (l', x') \in \gamma(x),\ l' = l \Rightarrow x' \in [\![\phi]\!]_f\} \\
\phi \wedge \psi &\mapsto [\![\phi]\!]_f \cap [\![\psi]\!]_f \\
\neg \phi &\mapsto [\![\phi]\!]_f^c.
\end{aligned}$$

As a variant of this we can look at a fuzzy version of Hennessy-Milner logic, again aimed at bisimulation of Labelled Transition Systems. However, since $\phi \vee \neg\phi = tt$ is not valid in fuzzy logic (Definition 3.26), we can no longer use Boolean algebras as our starting point. We could at this point investigate the use of MV-algebras, which provide an algebraic semantics of many-valued logics (Chang, 1958, 1959), but instead, since this is purely for illustrative purposes, we drop negation from our logic:

$$\mathcal{L}_2 \ni \phi ::= \mathrm{tt} \mid p \mid \phi \wedge \phi \mid [l]\phi \quad \text{where } l \in \Sigma \text{ and } p \in \mathrm{Var}.$$

Example 4.21 (Fuzzy logic for bisimulation of LTSs)**.** *The logical connection is the dual adjunction between* **MSL** *and* **Set** *from Example 3.7, to which we add the functor* $L = \coprod_{l\in\Sigma}(-)$ *which adds the modal operators* $[l]$*, and the functor* $T(X) = \mathcal{P}(\Sigma \times X)$ *from Example 4.11.*

The predicate liftings for the modalities are then the family of natural transformations

$$\begin{aligned}
[l]_X \colon P(X) &\to P\mathcal{P}(\Sigma \times X)\\
\top &\mapsto \mathcal{P}(\Sigma \times X)\\
u &\mapsto \lambda \colon \mathcal{P}(\Sigma \times X) \to [0,1]\\
&\qquad\qquad w \mapsto \inf_{\substack{(l',x)\in w\\ l'=l}} u(x)\\
u \wedge v &\mapsto [l]_X(u) \wedge [l]_X(v),
\end{aligned}$$

giving ($[u_l]$ is the equivalence class in $LP(X)$ of an element of the l^{th} copy of $P(X)$)

$$\begin{aligned}
\delta_X \colon LP(X) &\to PT(X)\\
\top &\mapsto \mathcal{P}(\Sigma \times X)\\
[u_l] &\mapsto \lambda \colon \mathcal{P}(\Sigma \times X) \to [0,1]\\
&\qquad\qquad w \mapsto \inf_{\substack{(l',x)\in w\\ l'=l}} u(x)\\
[u_{l_1}] \wedge [v_{l_2}] &\mapsto \delta_X([u_{l_1}]) \wedge \delta_X([v_{l_2}]).
\end{aligned}$$

So for the L-algebra given by the Lindenbaum-Tarski algebra of $\mathcal{L}_2$, a T-coalgebra (X,γ), and a function $f\colon \mathrm{Var} \to |P(X)|$, there is a unique function $[\![-]\!]_f$ given by

$$\begin{aligned}
[\![-]\!]_f \colon \mathcal{L}_2 &\to |P(X)|\\
tt &\mapsto X\\
p &\mapsto f(p)\\
[l]\phi &\mapsto \lambda \colon X \to [0,1]\\
&\qquad x \mapsto \inf_{\substack{(l',x')\in\gamma(x)\\ l'=l}} [\![\phi]\!]_f(x')\\
\phi \wedge \psi &\mapsto [\![\phi]\!]_f \wedge [\![\psi]\!]_f.
\end{aligned}$$

The semantics of the propositional variables, true, and conjunction are standard from fuzzy logic. For the modal operator $[l]$ we see that at each $x \in X$, the fuzzy degree of $[l]\phi$ is the smallest fuzzy degree of ϕ that is directly accessible from x. In the case where the fuzzy degrees for ϕ are restricted to 0 or 1 for all $x \in X$, the semantics of $[l]\phi$ coincides with that from the bivalent case (Example 4.20).

We now look at bivalent and fuzzy variants of Hennessy-Milner logic for simulation. For this we shall enrich over the category $\mathbf{Set}_R$ for the case where the type R is preorders. The reasons for doing so will be explained in Section 5.1.

The syntax of our logic will be the following standard formulation (van Glabbeek, 2001)

$$\mathcal{L}_3 \ni \phi ::= \mathrm{tt} \mid p \mid \phi \wedge \phi \mid \langle l \rangle \phi \quad \text{where } l \in \Sigma \text{ and } p \in \mathrm{Var}.$$

Example 4.22 (Bivalent logic for simulation of LTSs)**.** *The logical connection is the dual adjunction between* **MSL** *and* $\mathbf{Set}_R$ *from Example 3.2 with the type R set to preorders. To this we add the functor L from Example 4.4, which adds the modal operators $\langle l \rangle$, and the functor*

$$T(X, R_X) = (\mathcal{P}(\Sigma \times X), R_{\mathcal{P}(\Sigma \times X)})$$

from Example 4.12.

The predicate liftings for the modalities are then the family of natural transformations

$$\begin{aligned} \langle l \rangle_{(X,R_X)} \colon U_{\mathbf{Set}_R} P(X, R_X) &\to U_{\mathbf{Set}_R} P(\mathcal{P}(\Sigma \times X), R_{\mathcal{P}(\Sigma \times X)}) \\ u &\mapsto \{w \in \mathcal{P}(\Sigma \times X) \mid \exists (l', x) \in w,\ l' = l \textit{ and } x \in u\}, \end{aligned}$$

where u, and the set it is mapped to, are both right R-closed (upsets). This gives

$$\begin{aligned} \delta_{(X,R_X)} \colon LP(X, R_X) &\to PT(X, R_X) \\ \top &\mapsto \mathcal{P}(\Sigma \times X) \\ [u_l] &\mapsto \{w \in \mathcal{P}(\Sigma \times X) \mid \exists (l', x) \in w,\ l' = l \textit{ and } x \in u\} \\ [u_{l_1}] \wedge [v_{l_2}] &\mapsto \delta_X([u_{l_1}]) \cap \delta_X([v_{l_2}]). \end{aligned}$$

So for the L-algebra given by the Lindenbaum-Tarski algebra of $\mathcal{L}_3$, a T-coalgebra $((X, R_X), \gamma)$, and a function $f \colon \mathrm{Var} \to |P(X, R_X)|$, there is a unique function $[\![-]\!]_f$ given by

$$\begin{aligned} [\![-]\!]_f \colon \mathcal{L}_3 &\to |P(X, R_X)| \\ tt &\mapsto X \\ p &\mapsto f(p) \\ \langle l \rangle \phi &\mapsto \{x \in X \mid \exists (l', x') \in \gamma(x),\ l' = l \textit{ and } x' \in [\![\phi]\!]_f\} \\ \phi \wedge \psi &\mapsto [\![\phi]\!]_f \cap [\![\psi]\!]_f. \end{aligned}$$

Again, for all $\phi \in \mathcal{L}_3$, the set $[\![\phi]\!]_f$ is right R-closed. What this means is, if x satisfies ϕ, and y simulates x, then y satisfies ϕ.

Example 4.23 (Fuzzy logic for simulation of LTSs)**.** *The logical connection is the dual adjunction between* **MSL** *and* $\mathbf{Set}_R$ *from Example 3.8 with the type R set to preorders. To this we add the functor L from Example 4.4, which adds the modal operators $\langle l \rangle$, and the functor*

$$T(X, R_X) = (\mathcal{P}(\Sigma \times X), R_{\mathcal{P}(\Sigma \times X)})$$

from Example 4.12.

The predicate liftings for the modalities are then the family of natural transformations

$$\begin{aligned}\langle l \rangle_{(X,R_X)} \colon U_{\mathbf{Set}_R} P(X, R_X) \to U_{\mathbf{Set}_R} P(\mathcal{P}(\Sigma \times X), R_{\mathcal{P}(\Sigma \times X)})\\ u \mapsto \lambda \colon \mathcal{P}(\Sigma \times X) \to [0,1]\\ w \mapsto \sup_{\substack{(l',x) \in w\\ l'=l}} u(x),\end{aligned}$$

where u, and the fuzzy set it is mapped to, are both right R-closed (fuzzy upsets). This gives

$$\begin{aligned}\delta_{(X,R_X)} \colon LP(X, R_X) &\to PT(X, R_X)\\ \top &\mapsto \mathcal{P}(\Sigma \times X)\\ [u_l] &\mapsto \lambda \colon \mathcal{P}(\Sigma \times X) \to [0,1]\\ &\qquad w \mapsto \sup_{\substack{(l',x) \in w\\ l'=l}} u(x)\\ [u_{l_1}] \wedge [v_{l_2}] &\mapsto \delta_X([u_{l_1}]) \wedge \delta_X([v_{l_2}]).\end{aligned}$$

So for the L-algebra given by the Lindenbaum-Tarski algebra of $\mathcal{L}_3$, a T-coalgebra $((X, R_X), \gamma)$, and a function $f \colon \mathrm{Var} \to |P(X, R_X)|$, there is a unique function $[\![-]\!]_f$ given by

$$\begin{aligned}[\![-]\!]_f \colon \mathcal{L}_3 &\to |P(X, R_X)|\\ tt &\mapsto X\\ p &\mapsto f(p)\\ \langle l \rangle \phi &\mapsto \lambda \colon X \to [0,1]\\ &\qquad x \mapsto \sup_{\substack{(l',x') \in \gamma(x)\\ l'=l}} [\![\phi]\!]_f(x')\\ \phi \wedge \psi &\mapsto [\![\phi]\!]_f \wedge [\![\psi]\!]_f.\end{aligned}$$

Again, for all $\phi \in \mathcal{L}_3$, the fuzzy set $[\![\phi]\!]_f$ is right R-closed. What this means is, if ϕ is true to a particular degree at x, and y simulates x, then ϕ is true to at least the same degree at y.

Once again the semantics of the propositional variables, true, and conjunction are standard from fuzzy logic. For the modal operator $\langle l\rangle$ we see that at each $x \in X$, the fuzzy degree of $\langle l\rangle\phi$ is the largest fuzzy degree of ϕ that is directly accessible from x. Again, in the case where the fuzzy degrees for ϕ are restricted to 0 or 1 for all $x \in X$, the semantics of $\langle l\rangle\phi$ coincides with that from the bivalent case (Example 4.22).

Returning to our discussion of the coalgebraic semantics of L-algebras, we see that we can regard the whole of the coalgebraic modal logic project as a massive generalisation of Kripke semantics for modal logic (Blackburn et al., 2001). Now in Kripke semantics the concepts of frame, valuation, and model are introduced, where a model is a pair consisting of a frame and a valuation. Above we have generalised the notion of a valuation to our enriched coalgebraic framework, so what of frames and models?

Definition 4.12. Given the conditions of Assumption 5, for any L-algebra (A, α), any T-coalgebra (X, γ), and any valuation

$$f \in \mathbf{Alg}(L)_o((A, \alpha), \tilde{P}(X, \gamma)),$$

the pair $((X, \gamma), f)$ is called a **model** for (A, α).

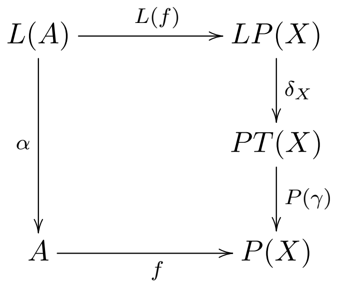
Clearly, if (A, α) is the initial L-algebra, then for every T-coalgebra (X, γ) the unique morphism $!\colon (A, \alpha) \to \tilde{P}(X, \gamma)$ makes the pair $((X, \gamma), !)$ a model for (A, α). Similarly, for a free L-algebra (A, α) given by the Lindenbaum-Tarski algebra for some logic $\mathcal{L}$, for every T-coalgebra (X, γ), the valuation f given by the function $f\colon \mathcal{L}/{\approx} \to |P(X)|$ makes the pair $((X, \gamma), f)$ a model for (A, α). However, if (A, α) is an arbitrary L-algebra, then it may be the case for some T-coalgebras that no valuation exists.

Following the conventions of Kripke semantics (Blackburn et al., 2001) we could now call a T-coalgebra a frame, but we have no need for this extra terminology. Moreover, in Kripke semantics frames are intimately related to the notion of validity, a topic that we will not pursue in this thesis. Therefore we shall restrict ourselves to talking about models.

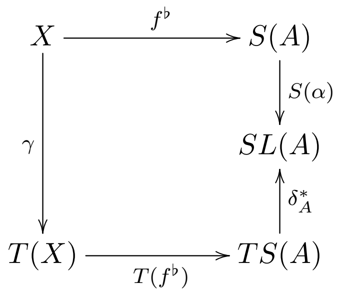
Remark 4.13. Validity, like satisfaction, is about the truth of a formula, either at an individual state, or at all states in a T-coalgebra. The distinction is that for validity we quantify over all valuations of the propositional variables. What we mean by truth though, is implicitly a bivalent concept. Or at least in the general case, appears to require the existence of a largest truth value - for example, a formula ϕ is valid in fuzzy logic if ϕ has the truth value 1 under all valuations of the propositional variables of ϕ (Chang, 1958). This is not something that we assume (see Section 5.3).

The alert reader may have noticed that when we extended the definition of a valuation from the base level (Definition 3.15) to that of L-algebras (Definition 4.11) we made no mention of theory maps. This is something we shall address next.

If we unpack Definition 4.11 we see that we require that the following diagram commutes in $\mathbb{A}_o$.



Now, as observed in Pavlovic et al. (2006, Theorem 1(b)), the logical connection allows every such diagram in $\mathbb{A}_o$ to be redrawn in $\mathbb{X}_o$ as



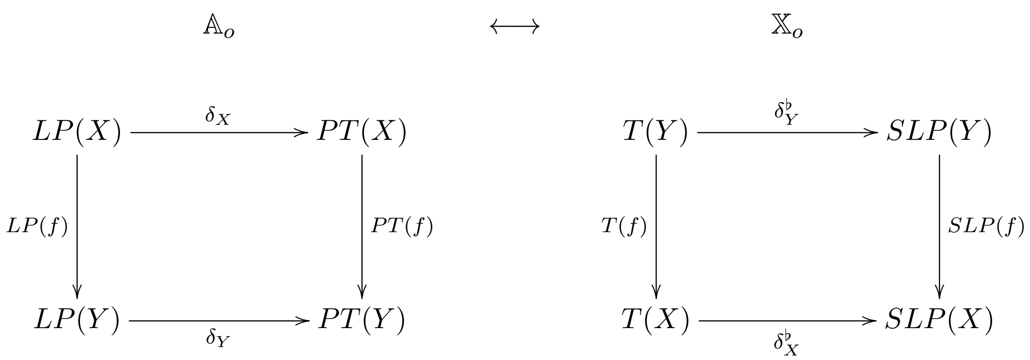
and moreover, this relationship is a bijection. Here $f^\flat$ is the transpose of f under the logical connection, and $\delta^* \colon TS \Rightarrow SL$ is defined following Klin (2007) as follows.

Definition 4.14. Given the conditions of Assumption 5, define the $\mathbb{V}$-natural transformation $\delta^* \colon TS \Rightarrow SL$ by

$$\delta^* = SL\rho \circ \delta^\flat S,$$

where ρ is the unit of the logical connection, and $\delta^\flat$ is the transpose of δ under the logical connection.

Note that the transpose $\delta^\flat \colon T \Rightarrow SLP$ of $\delta \colon LP \Rightarrow PT$ is constructed at the level of the underlying categories, where for all $f \in \mathbb{X}_o(Y, X)$, there is a bijection between diagrams on the left, and diagrams on the right.



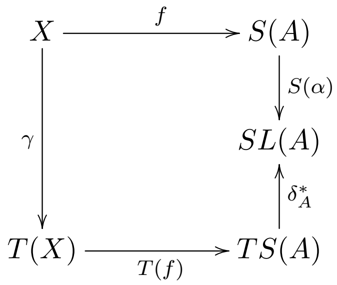
This defines an ordinary natural transformation $\delta_o^\flat$, and then since **elem**$|-|$ is faithful, by Proposition C.43, this lifts to a $\mathbb{V}$-natural transformation $\delta^\flat$.

Using this we can now extended the definition of a theory map from the base level (Definition 3.15) to that of L-algebras and T-coalgebras.

Definition 4.15. Given the conditions of Assumption 5, for any L-algebra (A, α), and any T-coalgebra (X, γ), a **theory map** is any

$$f \in \mathbb{X}_o(X, S(A)),$$

such that the following commutes



Remark 4.16. If the L-algebra (A, α) is a free L-algebra, and therefore the Lindenbaum-Tarski algebra for some concrete modal logic $\mathcal{L}$, then for any x in X, the theory $f(x)$ will typically be a filter/prime filter/ultrafilter of A, where the formulae of A are in actual fact equivalence classes of terms from $\mathcal{L}$. The union $\bigcup f(x)$ then gives the set of terms of $\mathcal{L}$ satisfied at x, and is logically consistent with respect to the proof system of $\mathcal{L}$.

It must be emphasised that the bijection between valuations and theory maps at the base level, provided by the logical connection, lifts to a bijection at the level of L-algebras and T-coalgebras.

Proposition 4.17. *Given the conditions of Assumption 5, for any L-algebra (A, α), and any T-coalgebra (X, γ), there is a bijection between valuations and theory maps that is the restriction of the bijection*

$$\mathbb{A}_o(A, P(X)) \cong \mathbb{X}_o(X, S(A))$$

given by the logical connection.

This bijection between valuations and theory maps means that the definition of a model from Definition 4.12 can be reformulated in terms of a theory map, and moreover, for each L-algebra we can construct a category of such models.

Definition 4.18. Given the conditions of Assumption 5, for any L-algebra (A, α), the ordinary category $\mathbf{Mod}(A, \alpha)_o$ has objects given by pairs

$$((X, \gamma), f),$$

where (X, γ) is a T-coalgebra, and $f \in \mathbb{X}_o(X, S(A))$ is a theory map, and morphisms

$$g\colon ((X_1, \gamma_1), f_1) \to ((X_2, \gamma_2), f_2),$$

given by $g \in \mathbf{CoAlg}(T)_o((X_1, \gamma_1), (X_2, \gamma_2))$ such that $f_1 = f_2 \circ g$.

In the above definition, the requirement on model morphisms that $f_1 = f_2 \circ g$ arises from the fact that theory maps need not be unique. In simple terms, we have to ensure that any propositional variables are given interpretations in the two models that are compatible with the T-coalgebra morphism.

In Doberkat (2009) a similar definition of a category of models for an L-algebra is made, however this is done in terms of diagrams in $\mathbb{A}_o$ i.e. pairs of T-coalgebras and valuations. In following chapters we prefer to work in $\mathbb{X}_o$, but as already noted above, the logical connection allows us to move freely backwards and forwards between the two definitions.

Like in Section 4.1, we intend to perform an initial lift of the forgetful functor of the ordinary category $\mathbf{Mod}(A, \alpha)_o$, in order to create a $\mathbb{V}$-category. To do this we need to define a forgetful functor from $\mathbf{Mod}(A, \alpha)_o$ to the underlying category of some $\mathbb{V}$-category. There is an obvious choice for this.

Definition 4.19. Given the conditions of Assumption 5, for any L-algebra (A, α), there is a forgetful (faithful) functor

$$\begin{aligned} U_{\mathbf{Mod}(A,\alpha)_o}\colon \mathbf{Mod}(A, \alpha)_o &\to \mathbf{CoAlg}(T)_o \\ ((X, \gamma), f) &\mapsto (X, \gamma) \\ g\colon ((X_1, \gamma_1), f_1) \to ((X_2, \gamma_2), f_2) &\mapsto g\colon (X_1, \gamma_1) \to (X_2, \gamma_2). \end{aligned}$$

Using Theorem 2.13, and that $\mathbf{elem}|-|$ is faithful, by Proposition C.21 so is the initial lift, and so we have the following proposition.

Proposition 4.20. *Given the conditions of Assumption 5, the forgetful ordinary functor* $U_{\mathbf{Mod}(A,\alpha)_o}$ *has the initial lift*

$$\overline{U_{\mathbf{Mod}(A,\alpha)_o}}\colon \overline{\mathbf{Mod}(A, \alpha)_o} \to \mathbf{CoAlg}(T)$$

where the $\mathbb{V}$*-functor* $\overline{U_{\mathbf{Mod}(A,\alpha)_o}}$ *is faithful, and the* $\mathbb{V}$*-category* $\overline{\mathbf{Mod}(A, \alpha)_o}$ *is unique up to isomorphism.*

We take this initial lift to be the definition of the $\mathbb{V}$-category of models for the L-algebra (A, α).

Definition 4.21. Given the conditions of Assumption 5, the $\mathbb{V}$-category $\mathbf{Mod}(A, \alpha)$, and the forgetful $\mathbb{V}$-functor

$$U_{\mathbf{Mod}(A,\alpha)} \colon \mathbf{Mod}(A, \alpha) \to \mathbf{CoAlg}(T),$$

are the initial lift of the forgetful functor $U_{\mathbf{Mod}(A,\alpha)_o}$.

Of course, when defining the category $\mathbf{Mod}(A, \alpha)$ we could have considered the obvious forgetful ordinary functor from $\mathbf{Mod}(A, \alpha)_o$ to $\mathbb{X}_o$. But it is easy to see that this is just the composite $U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o}$, and by Proposition A.2, its initial lift is just the composite $U_{\mathbf{CoAlg}(T)} U_{\mathbf{Mod}(A,\alpha)}$ up to a unique isomorphism. Hence we can use either forgetful functor, and everything is consistent.

Remark 4.22. If the L-algebra (A, α) is the initial L-algebra, then it is easy to see that $\mathbf{Mod}(A, \alpha)_o \cong \mathbf{CoAlg}(T)_o$, and using both the initial lifts of $U_{\mathbf{Mod}(A,\alpha)_o}$ and the above forgetful ordinary functor from $\mathbf{Mod}(A, \alpha)_o$ to $\mathbb{X}_o$, this lifts to $\mathbf{Mod}(A, \alpha) \cong \mathbf{CoAlg}(T)$.

4.4 Colimits in $\mathbf{CoAlg}(T)$ and $\mathbf{Mod}(A, \alpha)$

In Chapter 6 we shall see that one of the most important aspects of the structure of the category $\mathbf{Mod}(A, \alpha)$ is the presence, or otherwise, of colimits. To be more precise, we shall be interested in what are known as conical colimits. In enriched category theory the notion of colimits is generalised to what are variously known as indexed, or weighted, colimits (Definition C.93). The conical colimits (Definition C.95) are then a special case, and correspond, as their name suggests, to the usual ordinary category theory notion of colimits based upon cocones.

In this section we shall prove that the forgetful functors $U_{\mathbf{CoAlg}(T)} \colon \mathbf{CoAlg}(T) \to \mathbb{X}$ and $U_{\mathbf{CoAlg}(T)} U_{\mathbf{Mod}(A,\alpha)} \colon \mathbf{Mod}(A, \alpha) \to \mathbb{X}$ create small conical colimits (Definition C.99), and the main technical tool we shall use is Theorem 2.14, which relies in an essential way upon the fact that these forgetful functors are initial lifts.

In the proofs that follow, we shall use the following notation:

1. $\mathbb{J}$ will be a small ordinary category that specifies the type, or shape, of colimit we are dealing with.

2. Δ_C denotes the diagonal functor $\Delta_C \colon \mathbb{J} \to \mathbb{C}_o$ that sends every object of $\mathbb{J}$ to C, and every morphism of $\mathbb{J}$ to 1_C.

3. Δ_f denotes the natural transformation $\Delta_f \colon \Delta_A \Rightarrow \Delta_B \colon \mathbb{J} \to \mathbb{C}_o$, for which the component $(\Delta_f)_J = f \colon A \to B$, for all objects J in $\mathbb{J}$.

First we extend to the enriched setting the well known result (see for example Rutten (2000) for the case in **Set**) that the forgetful functor $U_{\mathbf{CoAlg}(T)_o} \colon \mathbf{CoAlg}(T)_o \to \mathbb{X}_o$ creates small colimits.

Theorem 4.23. *Given the conditions of Assumption 5, the forgetful* $\mathbb{V}$*-functor*

$$U_{\mathbf{CoAlg}(T)} \colon \mathbf{CoAlg}(T) \to \mathbb{X}$$

creates small conical colimits.

Proof. Consider a small ordinary category $\mathbb{J}$ and a functor $D \colon \mathbb{J} \to \mathbf{CoAlg}(T)_o$, and suppose that $\mathbb{X}$ has conical colimits of shape $\mathbb{J}$. Then we have that $U_{\mathbf{CoAlg}(T)_o}D$ has a colimit $(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),\ \phi)$, where the unit ϕ is the ordinary natural transformation

$$\phi \colon \Delta_I \Rightarrow \mathbb{X}\big(U_{\mathbf{CoAlg}(T)_o}D(-),\ \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D)\big)_o \colon \mathbb{J} \to \mathbb{V}_o,$$

the components of which are the cocone

$$\phi_J \colon U_{\mathbf{CoAlg}(T)_o}D(J) \to \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),$$

in the ordinary category $\mathbb{X}_o$. We shall therefore view the unit ϕ as the following natural transformation

$$\phi \colon U_{\mathbf{CoAlg}(T)_o}D \Rightarrow \Delta_{\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D)} \colon \mathbb{J} \to \mathbb{X}_o.$$

In order to simplify what comes next, we shall define the following ordinary natural transformation

$$\gamma_D \colon U_{\mathbf{CoAlg}(T)_o}D \Rightarrow T_o U_{\mathbf{CoAlg}(T)_o}D \colon \mathbb{J} \to \mathbb{X}_o,$$

the component $\gamma_{D(J)}$ of which, is the structure map of the T-coalgebra indexed by J.

Next we need to show that $U_{\mathbf{CoAlg}(T)_o} \colon \mathbf{CoAlg}(T)_o \to \mathbb{X}_o$ creates colimits of shape $\mathbb{J}$. We proceed as follows:

1. Put a T-coalgebra structure map on $\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D)$:

   $T\phi \circ \gamma_D \colon U_{\mathbf{CoAlg}(T)_o}D \Rightarrow \Delta_{T(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D))}$ is a cocone for $U_{\mathbf{CoAlg}(T)_o}D$, therefore there exists a unique

$$\chi \colon \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D) \to T(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D))$$

that gives a natural transformation

$$\Delta_\chi \colon \Delta_{\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} D)} \Rightarrow \Delta_{T(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} D))}$$

such that $T\phi \circ \gamma_D = \Delta_\chi \circ \phi$. This yields a T-coalgebra $(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} D),\ \chi)$, and the ϕ_J become T-coalgebra morphisms.

2. Construct a cocone for D from $(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} D),\ \chi)$:

   We have a natural transformation

$$\theta \colon D \Rightarrow \Delta_{(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} D), \chi)} \colon \mathbb{J} \to \mathbf{CoAlg}(T)_o,$$

   where $U_{\mathbf{CoAlg}(T)_o}\theta = \phi$, and this is a cocone for D.

3. For another cocone of D construct a unique mediating morphism between the carrier objects:

   If we consider any other cocone $((Z, \xi),\ \psi \colon D \Rightarrow \Delta_{(Z,\xi)})$ for D, then we clearly have that

$$(Z,\ U_{\mathbf{CoAlg}(T)_o}\psi \colon U_{\mathbf{CoAlg}(T)_o} D \Rightarrow \triangle_Z)$$

   is a cocone of $U_{\mathbf{CoAlg}(T)_o} D$, and thus there exists a unique

$$\mu \colon \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} D) \to Z,$$

   giving a natural transformation

$$\Delta_\mu \colon \triangle_{\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} D)} \Rightarrow \Delta_Z \colon \mathbb{J} \to \mathbb{X}_o,$$

   such that $U_{\mathbf{CoAlg}(T)_o}\psi = \Delta_\mu \circ \phi$.

4. Show that the mediating morphism is a T-coalgebra morphism:

   The ψ_J are T-coalgebra morphisms which means that

$$\Delta_\xi \circ U_{\mathbf{CoAlg}(T)_o}\psi \colon U_{\mathbf{CoAlg}(T)_o} D \Rightarrow \Delta_{T(Z)}$$
$$T_o U_{\mathbf{CoAlg}(T)_o}\psi \circ \gamma_D \colon U_{\mathbf{CoAlg}(T)_o} D \Rightarrow \Delta_{T(Z)}$$

   represent the same cocone for $U_{\mathbf{CoAlg}(T)_o} D$. Further, we have that

$$\Delta_\xi \circ U_{\mathbf{CoAlg}(T)_o}\psi = \Delta_\xi \circ \Delta_\mu \circ \phi = \Delta_{\xi \circ \mu} \circ \phi,$$

   and also that

$$T_o U_{\mathbf{CoAlg}(T)_o}\psi \circ \gamma_D = \Delta_{T(\mu)} \circ T\phi \circ \gamma_D = \Delta_{T(\mu)} \circ \Delta_\chi \circ \phi = \Delta_{T(\mu) \circ \chi} \circ \phi.$$

So by the universal property of the colimit $(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),\ \phi)$, we have $\xi \circ \mu = T(\mu) \circ \chi$. Thus μ is a T-coalgebra morphism, and we have a natural transformation

$$\Delta_\mu \colon \Delta_{(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),\chi)} \Rightarrow \Delta_{(Z,\xi)} \colon \mathbb{J} \to \mathbf{CoAlg}(T)_o.$$

Therefore $\psi = \Delta_\mu \circ \theta$, and

$$\Big((\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),\chi),\ \theta\colon D \Rightarrow \Delta_{(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),\chi)}\Big)$$

is a colimit of D.

5. Deduce that $U_{\mathbf{CoAlg}(T)_o}$ creates colimits of shape $\mathbb{J}$:

   It is clear that $((\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),\chi),\ \theta)$ is the unique cocone for D that is mapped by $U_{\mathbf{CoAlg}(T)_o}$ to the colimit $(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}D),\ \phi)$ of $U_{\mathbf{CoAlg}(T)_o}D$. Thus we can conclude that $U_{\mathbf{CoAlg}(T)_o}$ creates colimits of shape $\mathbb{J}$.

Finally by Theorem 2.14, we can deduce that the forgetful $\mathbb{V}$-functor

$$U_{\mathbf{CoAlg}(T)} \colon \mathbf{CoAlg}(T) \to \mathbb{X}$$

creates small conical colimits. □

The case for the composite forgetful functor $U_{\mathbf{CoAlg}(T)}U_{\mathbf{Mod}(A,\alpha)} \colon \mathbf{Mod}(A,\alpha) \to \mathbb{X}$ follows in a similar fashion, with the additional detail that a theory map must be constructed for the colimit.

Theorem 4.24. *Given the conditions of Assumption 5, the forgetful $\mathbb{V}$-functor*

$$U_{\mathbf{CoAlg}(T)}U_{\mathbf{Mod}(A,\alpha)} \colon \mathbf{Mod}(A,\alpha) \to \mathbb{X}$$

creates small conical colimits.

Proof. Consider a small ordinary category $\mathbb{J}$ and a functor $D\colon \mathbb{J} \to \mathbf{Mod}(A,\alpha)_o$, and suppose that $\mathbb{X}$ has conical colimits of shape $\mathbb{J}$. Then we have that $U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D$ has a colimit $(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D),\ \phi)$, where the unit ϕ is the ordinary natural transformation

$$\phi\colon \Delta_I \Rightarrow \mathbb{X}\big(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D(-),\ \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D)\big)_o \colon \mathbb{J} \to \mathbb{V}_o,$$

the components of which are the cocone

$$\phi_J \colon U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D(J) \to \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D),$$

in the ordinary category $\mathbb{X}_o$. We shall therefore view the unit ϕ as the following natural transformation

$$\phi\colon U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D \Rightarrow \Delta_{\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D)}\colon \mathbb{J} \to \mathbb{X}_o.$$

In order to simplify what comes next, we shall define the following ordinary natural transformation

$$\gamma_D\colon U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D \Rightarrow T_oU_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D\colon \mathbb{J} \to \mathbb{X}_o,$$

the component $\gamma_{D(J)}$ of which, is the structure map of the T-coalgebra indexed by J, and the ordinary natural transformation

$$f_D\colon U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D \Rightarrow \Delta_{S(A)}\colon \mathbb{J} \to \mathbb{X}_o,$$

the component $f_{D(J)}$ of which, is the theory map of the T-coalgebra indexed by J.

Next we need to show that $U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}\colon \mathbf{Mod}(A,\alpha)_o \to \mathbb{X}_o$ creates colimits of shape $\mathbb{J}$. We proceed as follows:

1. Use the functor $U_{\mathbf{CoAlg}(T)_o}$ to construct a colimiting T-coalgebra on $\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D)$:

   By Theorem 4.23 there is a unique T-coalgebra

   $$\chi\colon \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D) \to T(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D))$$

   making the ϕ_J into T-coalgebra morphisms, and thus a natural transformation

   $$\theta\colon U_{\mathbf{Mod}(A,\alpha)_o}D \Rightarrow \Delta_{(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D),\chi)}\colon \mathbb{J} \to \mathbf{CoAlg}(T)_o,$$

   such that

   $$\big((\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D),\chi),\ \theta\big)$$

   is the colimit of $U_{\mathbf{Mod}(A,\alpha)_o}D$. Furthermore, $U_{\mathbf{CoAlg}(T)_o}\theta = \phi$.

2. Construct a morphism g from $\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D)$ to $S(A)$:

   The theory maps associated with the diagram D define a natural transformation $f_D\colon U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D \Rightarrow \Delta_{S(A)}$ that is a cocone for $U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D$. Therefore there exists a unique morphism

   $$g\colon \mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D) \to S(A)$$

   giving a natural transformation

   $$\Delta_g\colon \Delta_{\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o}U_{\mathbf{Mod}(A,\alpha)_o}D)} \Rightarrow \triangle_{S(A)}\colon \mathbb{J} \to \mathbb{X}_o,$$

such that $f_D = \Delta_g \circ \phi$.

3. Show that g is a theory map:

   The natural transformations

$$\Delta_{S(\alpha)} \circ f_D \colon U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D \Rightarrow \Delta_{SL(A)}$$
$$\Delta_{\delta_A^*} \circ T f_D \circ \gamma_D \colon U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D \Rightarrow \Delta_{SL(A)}$$

   are cocones for $U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D$. Indeed, they represent the same cocone, and we have

$$\Delta_{S(\alpha)} \circ f_D = \Delta_{S(\alpha)} \circ \Delta_g \circ \phi = \Delta_{S(\alpha)\circ g} \circ \phi,$$

   and

$$\begin{aligned} \Delta_{\delta_A^*} \circ T f_D \circ \gamma_D &= \Delta_{\delta_A^*} \circ \Delta_{T(g)} \circ T\phi \circ \gamma_D \\ &= \Delta_{\delta_A^*} \circ \Delta_{T(g)} \circ \Delta_\chi \circ \phi \\ &= \Delta_{\delta_A^* \circ T(g) \circ \chi} \circ \phi. \end{aligned}$$

   The universal property of the colimit $(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D),\ \phi)$ then yields $S(\alpha) \circ g = \delta_A^* \circ T(g) \circ \chi$. Thus g is a theory map and

$$\big((\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D), \chi),\ g\big)$$

   is a model in $\mathbf{Mod}(A,\alpha)_o$, and since $f_D = \Delta_g \circ \phi$, the ϕ_J are model morphisms.

4. Construct a cocone for D from $((\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D), \chi),\ g)$:

   We have a natural transformation

$$\tau \colon D \Rightarrow \Delta_{((\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D), \chi),\ g)} \colon \mathbb{J} \to \mathbf{Mod}(A,\alpha)_o,$$

   where $U_{\mathbf{Mod}(A,\alpha)_o} \tau = \theta$, and this is a cocone for D.

5. For another cocone of D construct a unique mediating morphism between the T-coalgebras:

   If we consider any other cocone $(((Z,\xi),h),\ \psi \colon D \Rightarrow \Delta_{((Z,\xi),h)})$ for D, then we clearly have that $((Z,\xi),\ U_{\mathbf{Mod}(A,\alpha)_o}\psi \colon U_{\mathbf{Mod}(A,\alpha)_o} D \Rightarrow \Delta_{(Z,\xi)})$ is a cocone for $U_{\mathbf{Mod}(A,\alpha)_o} D$, and thus there exists a unique

$$\mu \colon (\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D),\ \chi) \to (Z,\xi),$$

   giving a natural transformation

$$\Delta_\mu \colon \Delta_{(\mathrm{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D),\ \chi)} \Rightarrow \Delta_{(Z,\xi)} \colon \mathbb{J} \to \mathbf{CoAlg}(T)_o,$$

   such that $U_{\mathbf{Mod}(A,\alpha)_o}\psi = \Delta_\mu \circ \theta$.

6. Show that the mediating morphism is a model morphism:

   In choosing the cocone $\big(((Z,\xi),h),\ \psi\colon D \Rightarrow \Delta_{((Z,\xi),h)}\big)$ we are imposing the constraint
$$f_D = \Delta_h \circ U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} \psi,$$
   as this is required for the ψ_J to be morphisms in $\mathbf{Mod}(A,\alpha)_o$. Now
$$U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} \psi = \Delta_{U_{\mathbf{CoAlg}(T)_o}\mu} \circ \phi,$$
   so
$$f_D = \Delta_h \circ \Delta_{U_{\mathbf{CoAlg}(T)_o}\mu} \circ \phi,$$
   and by the uniqueness of g, we therefore have
$$g = h \circ U_{\mathbf{CoAlg}(T)_o}\mu.$$
   This means that μ is also a morphism in $\mathbf{Mod}(A,\alpha)_o$, and thus we now have a natural transformation
$$\Delta_\mu\colon \Delta_{((\operatorname{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D),\chi),g)} \Rightarrow \Delta_{((Z,\xi),h)}\colon \mathbb{J} \to \mathbf{Mod}(A,\alpha)_o,$$
   and with this $\psi = \Delta_\mu \circ \tau$. This completes the proof that the cocone
$$\big(((\operatorname{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D),\chi),g),\ \tau\big)$$
   is a colimit of D.

7. Deduce that $U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o}$ creates colimits of shape $\mathbb{J}$:

   It is clear that $(((\operatorname{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D),\chi),g),\ \tau)$ is the unique cocone for D that is mapped by $U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o}$ to the colimit
$$(\operatorname{colim}_{\mathbb{X}_o}(U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o} D),\ \phi)$$
   of $U_{\mathbf{CoAlg}(T)_o} D$. Thus we can conclude that $U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o}$ creates colimits of shape $\mathbb{J}$.

Finally by Theorem 2.14, we can deduce that the forgetful $\mathbb{V}$-functor
$$U_{\mathbf{CoAlg}(T)} U_{\mathbf{Mod}(A,\alpha)}\colon \mathbf{Mod}(A,\alpha) \to \mathbb{X}$$
creates small conical colimits. $\square$

4.5 Discussion

In this chapter we have formalised a general notion of enriched coalgebraic modal logic. This extends previous work, both in the extensive literature on coalgebraic modal logic in ordinary category theory, and the more recent work of Kapulkin et al. (2010, 2012); Bílková et al. (2011), where the enrichment is over **Preord** and **Pos**.

We also provide a framework for the study of the modal logic counterpart to the work of Turi and Rutten (1998); Worrell (2000a); Balan and Kurz (2011) on coalgebras enriched over various categories of preorders, partial orders, or metric spaces. Moreover, we have extended the systematic study of models for a modal logic that we introduced in Wilkinson (2012b) to the enriched setting.

There is however something missing from the presentation of coalgebraic modal logics in this chapter, and indeed in this thesis - our modal logics could be more accurately described as coalgebraic modal languages, as we make no mention of proof systems. Indeed, when we are working with abstract modal logics (L-algebras for a general functor L), we do not even have a syntax. However, if there is a presentation for L (Section 4.2), then the free L-algebras are the Lindenbaum-Tarski algebras for concrete modal logics (Section 4.3), and the presentation also gives a proof system. Various authors have looked at such proof systems, and attempted to tackle the question of completeness of modal logics with coalgebraic semantics.

A common technique is to use induction along the terminal sequence (Worrell, 1999, 2005) in conjunction with predicate liftings. In Pattinson (2003) the weak completeness of a local consequence relation is investigated for **Set** coalgebras and modal logics that extend the propositional logics in **BA**, and in Kupke et al. (2004a) the results are extended to weak completeness for coalgebras on a general category (though still with modal logics built upon **BA**). The work of Kupke et al. (2004a) also makes use of the algebraic semantics of $\mathbf{Alg}(L)$, and the fact that the modal proof system is equivalent to equational logic.

The algebraic semantics approach to completeness is well known from Kripke semantics, where the key result is known as the Jónsson-Tarski Theorem (a good introduction can be found in Blackburn et al. (2001, Chapter 5)). Various authors have produced coalgebraic versions of the Jónsson-Tarski Theorem (Jacobs, 2001; Kupke et al., 2005; Kurz and Rosický, 2012). In simple terms a T-coalgebra structure is constructed on $S(A)$ by defining a morphism $h\colon SL(A) \to TS(A)$ such that $h \circ \delta^*_A = 1_{TS(A)}$, and this yields a weak completeness result, though the results of Kurz and Rosický (2012) are more general, and yield a strong completeness result for a global consequence relation.

In contrast to the approach via algebraic semantics, in Schröder and Pattinson (2009), strong completeness is shown for modal logics built on **BA** and coalgebras on **Set**, for a local consequence relation based on models. This result requires that the functor T satisfies certain conditions.

The axioms of coalgebraic modal logics have also been studied, and it has been found that they can be grouped into those that are rank 1 (have precisely one level of nesting of the modal operators), and those that are not.

In Schröder (2007) it was shown that for any functor T on **Set**, that $\mathbf{CoAlg}(T)$ could be axiomatised by a weakly complete rank 1 modal logic built upon **BA**. Then in Schröder and Pattinson (2007b) it was shown that every rank 1 modal logic built upon **BA** has a sound and strongly complete coalgebraic semantics with respect to $\mathbf{CoAlg}(T)$, for a specially constructed functor T on **Set**. Some work has also been done to look at modal logics that include axioms that are not rank 1 (Pattinson and Schröder, 2008; Schröder and Pattinson, 2009). In Pattinson and Schröder (2008), for an arbitrary collection of additional non rank 1 axioms (called frame conditions), the full subcategory of $\mathbf{CoAlg}(T)$ is considered where in each T-coalgebra these additional axioms are valid. Such a category is similar in spirit to our category of models, in that both incorporate the idea that going beyond the initial L-algebra requires a corresponding restriction in the T-coalgebras that should be considered. However, their work is again built upon **BA** and **Set**, and uses the notion of validity, which we avoid (Remark 4.13). Our definition of $\mathbf{Mod}(A, \alpha)$ on the other hand works at the level of abstract L-algebras, and is therefore in that respect more general. Fundamentally however, the approaches are different; we work with models, and their approach is more analogous to Kripke frames.

There has also been work to generalise theorems from the Kripke semantics of modal logic (Blackburn et al., 2001) to the coalgebraic setting. In Kupke et al. (2005) the "bisimulation somewhere else" theorem, and in Kurz and Rosický (2007) the Goldblatt-Thomason Theorem, are recast into the framework of coalgebras on **Set**, and modal logics built upon **BA**.

Finally, it should be noted that a special type of modal logic has been studied as a logic for coalgebras. Prior to the work by Kurz on coalgebraic modal logic (Kurz, 2001), Moss introduced coalgebraic logic (Moss, 1999), which is a special type of modal logic where the syntax is derived from the functor T (that specifies the coalgebras). Many researchers continue to work on Moss' logic, and recent work (Kupke et al., 2008, 2012) has shown that it too can be given by a functor L and thus incorporated into our framework, but we shall not discuss it further in this thesis.

Other alternatives to coalgebraic modal logics are the so called coequational logics (Kurz, 2000; Rutten, 2000; Kurz and Rosický, 2002, 2005; Awodey and Hughes, 2003; Adámek, 2005; Schwencke, 2008). This line of work was inspired by Birkhoff's variety and completeness theorems of universal algebra, and various notions of coequation have been introduced, along with logics to reason about them. A very general notion of a predicate that is invariant under bisimulation has also been defined, arising from the idea that this captures the essence of modal logic, and these are found to have the same expressive power as coequations (Kurz and Rosický, 2002, 2005). As these logics constitute a very different approach to the one we follow, we shall not mention them again in the remainder of this thesis.

Chapter 5

Behavioural Questions

In the previous chapters we have created a logical connection framework based upon a dual adjunction enriched over a symmetric monoidal closed category $\mathbb{V}$, and shown how to extend this framework to incorporate algebras and coalgebras for $\mathbb{V}$-functors defined on the two base categories $\mathbb{A}$ and $\mathbb{X}$.

Now we shall explore how we can compare the behaviour of states of coalgebras, and in doing so we shall make clear the role the category $\mathbb{V}$ plays in these comparisons.

A brief outline of this chapter is as follows:

Section 5.1 The T-coalgebra examples from Section 4.1 are discussed and the role of the preorders and generalised metrics explained.

Section 5.2 The idea of what a general notion of behavioural question for a pair of states might be is examined, and it is proposed that the answers should form a commutative unital quantale. From this general notions of behavioural and logical adjacency are defined, as well as a general definition of what it means for an L-algebra to be expressive.

Section 5.3 It is observed that each choice of quantale yields a candidate for the set of truth values of the logical connection. In the case of generalised metric spaces this suggests that real-valued logics may be the "correct" choice.

Section 5.4 Previous approaches to bisimulation, simulation, and behavioural metrics are compared to our approach.

5.1 Bisimulation, Simulation, and Approximation

In Section 4.1 we gave numerous examples of T-coalgebras for enrichment over **Set**, $\mathbf{Set}_R$, and **GMet**. We shall now look at those examples in more detail.

First of all we shall look at the examples where we enrich over **Set**. These are the familiar examples that anyone with exposure to coalgebras are likely to be familiar with.

The first example is the powerset functor on **Set** (Example 4.8), which we then extend to Labelled Transition Systems (Example 4.11). The other two examples are probabilistic, corresponding to Markov Chains (Example 4.14), and Markov Processes (Example 4.17).

The key point to note is that, given a set X (or a measurable space (X, Σ_X) in the case of Markov Processes), there are no constraints on the choice of T-coalgebra structure map $\gamma\colon X \to T(X)$, other than that γ must be a function (a measurable function for $\gamma\colon (X, \Sigma_X) \to T(X, \Sigma_X)$).

Now we know from Section 2.2 that the category **Set** is isomorphic to the category $\mathbf{Set}_R$ if the type R represents equality. In other words, every set can be considered to have the discrete preorder, where two states are related if and only if they are equal. We can therefore think of γ as being constrained to preserve equality. In this sense, the equality relation R_X is required to be a bisimulation on X under γ, as every state is bisimilar to itself.

Next we consider the examples where we enriched over $\mathbf{Set}_R$, but with the type R chosen to be preorders. For the powerset functor (Example 4.9), given a preorder (X, R_X), and $x, y \in X$, our choice of T-coalgebra structure map $\gamma\colon (X, R_X) \to T(X, R_X)$ is constrained by the requirement that γ be R-preserving, and therefore must satisfy

$$xR_X y \Rightarrow \forall x' \in \gamma(x)\ \exists y' \in \gamma(y).\ x' R_X y'.$$

Similarly, for Labelled Transition Systems (Example 4.12), γ is required to satisfy

$$xR_X y \Rightarrow \forall (l, x') \in \gamma(x)\ \exists (l', y') \in \gamma(y).\ l = l' \text{ and } x' R_X y'.$$

This means that the preorder R_X is required to be a Labelled Transition System simulation on X for γ (van Glabbeek, 2001).

The two probabilistic examples, of Markov Chains (Example 4.15), and Markov Processes (Example 4.18), are similar to the above, with γ required to satisfy

$$xR_X y \Rightarrow \forall u \subseteq X\ (u \text{ right } R\text{-closed} \Rightarrow \gamma(x)(u) \leq \gamma(y)(u)),$$

and

$$xR_X y \Rightarrow \forall M \in \Sigma_X\ (M \text{ right } R\text{-closed} \Rightarrow \gamma(x)(M) \leq \gamma(y)(M)),$$

respectively. In these cases the preorder R_X is required to be a Markov Chain or Markov Process simulation (Desharnais et al., 2003).

The remaining examples were enriched over **GMet**. Like the case for enrichment over $\mathbf{Set}_R$, additional constraints are placed on the choice of a T-coalgebra structure map γ.

Specifically, for the powerset functor (Example 4.10), γ is required to satisfy

$$d_X(x,y) \geq \sup_{x' \in \gamma(x)} \left(\inf_{y' \in \gamma(y)} d_X(x',y') \right),$$

and for Labelled Transition Systems (Example 4.13), γ is required to satisfy

$$d_X(x,y) \geq \sup_{(l,x') \in \gamma(x)} \left(\inf_{(l',y') \in \gamma(y)} \max(d_\Sigma(l,l'), d_X(x',y')) \right).$$

Here the generalised metric d_X is required to be an approximate, or quantitive, simulation metric on X for γ. What this means, is that for every transition that x can make y can match it, and in doing so, y moves to a successor state that is at least as close to the successor state of x, as y was to x. This is related to the notion of branching distance in de Alfaro et al. (2004).

The final two examples are Markov Chains (Example 4.16), where γ is required to satisfy

$$d_X(x,y) \geq \sup_{\substack{u \subseteq X \\ \text{right } d\text{-closed}}} d_{[0,1]}(\gamma(x)(u), \gamma(y)(u)),$$

and Markov Processes (Example 4.19), where γ is required to satisfy

$$d_X(x,y) \geq \sup_{\substack{M \in \Sigma_X \\ \text{right } d\text{-closed}}} d_{[0,1]}(\gamma(x)(M), \gamma(y)(M)).$$

These two conditions again require d_X to be an approximate simulation metric, though whether they correctly capture what is required of such approximate simulations is uncertain. The work of Desharnais et al. (2004) on metrics for Markov Processes takes a different approach, and conflates the approximation of the Markov Processes, with the valuations of the corresponding real-valued logic. Also in de Vink and Rutten (1999) an ultrametric is defined which differs from the generalised metric of Example 4.19, but is used to study probabilistic bisimulation, not approximate simulation.

Since the analysis of Markov Processes is not our main concern, we shall not pursue the question of how best to augment the Distribution functor and the Giry functor for enrichment over **GMet**.

In all the discussion above we have alternated between saying that a particular constraint is placed on the choice of a T-coalgebra structure map γ, and that a corresponding constraint is placed upon the preorder R_X, or metric d_X, that a state space X carries. This is no accident.

Traditionally when transition systems have been discussed, a class of transition systems is defined, and only afterwards is the corresponding notion of bisimulation or simulation specified. Thus it is common to find many different types of bisimulation defined for a given type of transition system. Our approach is different. We first appropriately choose the category $\mathbb{V}$ over which we enrich, depending upon whether we are interested in bisimulation, simulation, approximate simulation, or possibly some other notion of behavioural comparability. Then we choose our functor T such that it simultaneously defines both the transition structure, and how states are to be compared. In other words, our view is that a transition system type is not fully defined until it is specified how to compare the behaviour of states. This idea is consistent with the spirit of category theory, where the relationships between objects are considered as important, if not more so, than the structure of the objects themselves.

In the remainder of this chapter we shall formalise the idea that through enriching over different choices of the category $\mathbb{V}$, we can endow our coalgebras with different notions of behavioural comparability. In doing so, we shall take an idea from Worrell (2000a) and Worrell (2000b), and adapt it to our framework.

5.2 Behavioural Questions

In the previous section we saw how enriching over the categories **Set**, $\mathbf{Set}_R$, and **GMet**, could lead to different notions of what it meant to compare the behaviour of states: bisimulation, simulation, and approximate simulation. But are these the only ways in which the behaviour of states can be compared? Given a pair of states, what might we want to say about the behaviour of one with respect to the behaviour of the other? Or put another way, what behavioural questions can we ask of this pair of states?

Obviously, this is all rather vague and open ended, so we need to make things more concrete. The first thing we can say is that we have seen that the choice of category over which we enrich appears to play a big part. The second is that Lawvere observed that preorders and generalised metric spaces were in actual fact categories enriched over $\mathbf{2}$ and $[0, \infty]$ respectively (Lawvere, 1973). Let us look at this in more detail.

Every preorder or generalised metric space consists of a pair (X, q_X), where X is a set, $q_X \colon X \times X \to Q$ is a function, and Q is either $\mathbf{2}$ or $[0, \infty]$ respectively. So if X is the underlying set of the carrier object of some coalgebra, then the sets $\mathbf{2}$ and $[0, \infty]$ can be thought of as the set of possible answers to questions that can be asked of two states of that coalgebra. Specifically, given states $x, y \in X$, then $q_X(x, y) \in Q$ is the answer to the question asked of x and y. For example, if the question was "do x and y have the same behaviour?" we would expect the answer yes (1) or no (0), whereas if the question was "how close is the behaviour of y to that of x?" we would expect the answer to lie in the interval $[0, \infty]$.

It would seem therefore that we should consider other possible choices for the set Q. But which ones? Do we have a free hand, or are there constraints on what we can choose?

Looking at the examples of preorders and generalised metric spaces, we note that the functions between preorders are required to be order preserving, and the functions between metric spaces are required to be non-expansive. This means that in these two cases the set Q carries an order, and any coalgebra morphism must respect that order. In particular, if there is a chain of coalgebra morphisms leading from an arbitrary coalgebra (X, γ) to the final coalgebra, then for $x, y \in X$, the chain of "answers" should be seen to monotonically approach the answer given by the final coalgebra, and that this answer is the definitive answer.

This property is an appealing one, as it extends the notion of final coalgebra semantics to say that, not only is the behaviour of a state given by its image in the final coalgebra, but if we want to compare the behaviour of two states, we should do this by comparing their images in the final coalgebra. Moreover, as we move along a chain of coalgebra morphisms our answers can only improve.

There are two final properties of preorders and generalised metric spaces that we have not discussed. Specifically, reflexivity and transitivity for preorders, and $d_X(x, x) = 0$ and the triangle inequality for metrics. Famously, Lawvere observed that these correspond to the existence of identities, and composition of morphisms, thus making preorders and generalised metric spaces into categories (Lawvere, 1973). But what does this mean in our context?

The existence of identities corresponds to the fact that we always know that a state has the same behaviour as itself. Again this is a desirable property. Composition of morphisms is more difficult to understand. It would seem that if we have a coalgebra (X, γ), and states $x, y, z \in X$, and we know how the behaviour of y compares to x, and how the behaviour of z compares to y, then composition allows us to compute an estimate, or bound, for how the behaviour of z compares to x. It is hard to imagine that having this capability would ever be a problem, but equally, it is not clear why we should always desire this property. However, as we shall see in Proposition 5.7, this property turns out to be vital.

To summarise, we should choose a set Q and supply it with an order relation and sufficient additional structure that pairs (X, q_X) are in fact categories, and morphisms $f\colon (X, q_X) \to (Y, q_Y)$ should be functions $f\colon X \to Y$ that respect the order of Q. Then following Lawvere (1973), for a pair (X, q_X), the value $q_X(x, y) \in Q$ is the hom-object of x and y, making (X, q_X) a category enriched over Q.

To do this we will require that Q have at the bare minimum the structure of a monoidal category, with a tensor $\otimes$, and a unit I. In fact, in order to satisfy the conditions of Assumption 1, we require that Q be a commutative unital quantale (Wagner, 1997; Worrell, 2000a). The reasons for this will be explained in due course, but first we start with a definition.

Definition 5.1. A **quantale** $(Q, \otimes)$ is a complete lattice Q with an associative operation $\otimes$, such that $\otimes$ preserves all joins:

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \otimes b_i)$$
$$\bigvee_{i \in I} b_i \otimes a = \bigvee_{i \in I} (b_i \otimes a).$$

If in addition $\otimes$ is commutative, then $(Q, \otimes)$ is a **commutative quantale**, and if there exists $I \in Q$, such that for all $a \in Q$,

$$I \otimes a = a = a \otimes I,$$

then $(Q, \otimes, I)$ is a **unital quantale**.

Now any lattice is a partial order, with

$$a \leq b \Leftrightarrow a = a \wedge b \Leftrightarrow a \vee b = b,$$

and it is well-known that partial orders are categories. So we can ask what the additional structure of a commutative unital quantale means from a category theory perspective.

Firstly let us look at the operation $\otimes$. Take $a, b, c \in Q$ such that $a \leq b$, then

$$c \otimes b = c \otimes (a \vee b) = (c \otimes a) \vee (c \otimes b),$$

thus $c \otimes a \leq c \otimes b$. Similarly, $a \otimes c \leq b \otimes c$, and it is easy to see that $\otimes$ is a functor, and since by assumption it is commutative and associative, together with I, this makes $(Q, \otimes, I)$ into a symmetric monoidal category.

Now if we consider the functors $a \otimes -$ and $- \otimes a$, from the definition of a quantale these preserve joins, but in a partial order joins are colimits, indeed the only colimits. Thus by the Adjoint Functor Theorem for partial orders, there exist right adjoints to $a \otimes -$ and $- \otimes a$, which we shall denote $[a, -]_L$ and $[a, -]_R$ respectively, such that

$$b \leq [a, c]_L \Leftrightarrow a \otimes b \leq c$$
$$b \leq [a, c]_R \Leftrightarrow b \otimes a \leq c.$$

However, $\otimes$ is commutative, and Q is a partial order, thus $[a, c]_L = [a, c]_R$.

Putting this all together, and noting that left adjoints preserve colimits, and meets and joins are limits and colimits (respectively) in a partial order, we have the following result.

Proposition 5.2. *A partial order is a symmetric monoidal closed category that is both complete and cocomplete, if and only if, it is a commutative unital quantale.*

A commutative unital quantale $(Q, \otimes, I)$ therefore has all the structure necessary to define categories enriched over Q (Wagner, 1997; Worrell, 2000a).

Definition 5.3. Given a commutative unital quantale $(Q, \otimes, I)$, a **Q-category** is a pair (X, q_X), where X is a set, and q_X is a function $q_X \colon X \times X \to Q$ called the **hom-functor**, such that the following hold:

Reflexivity: $I \leq q_X(x, x)$ for all $x \in X$,

Transitivity: $q_X(x, y) \otimes q_X(y, z) \leq q_X(x, z)$ for all $x, y, z \in X$.

A **Q-functor** $f \colon (X, q_X) \to (Y, q_Y)$ is a function $f \colon X \to Y$ such that for all $x, x' \in X$

$$q_X(x, x') \leq q_Y(f(x), f(x')).$$

Q-categories can be thought of as generalisations of preorders or generalised metric spaces in the spirit of Lawvere (1973), and the condition on Q-functors is the generalisation of the order preserving or non-expanding properties of the morphisms between preorders or generalised metric spaces.

Now just like the case of $\mathbf{Set}_R$ (Definition 2.1) and **GMet** (Definition 2.5), we are interested in the category of all Q-categories.

Definition 5.4. Given a commutative unital quantale $(Q, \otimes, I)$, the category Q–**Cat** has for objects Q-categories, and for morphisms Q-functors.

As ultimately we want to enrich over Q–**Cat**, we would like Q–**Cat** to satisfy the conditions of Assumption 1.

The first property of Q–**Cat** that we require, is that Q–**Cat** is complete and cocomplete. This can be seen to follow from the fact that Q is a complete lattice.

Products: the product of (X, q_X) and (Y, q_Y) is given by $(X \times Y, q_{X \times Y})$, where

$$q_{X \times Y}((x, y), (x', y')) = q_X(x, x') \wedge q_Y(y, y').$$

Coproducts: the coproduct of (X, q_X) and (Y, q_Y) is given by $(X + Y, q_{X+Y})$, where

$$q_{X+Y}(w, w') = \begin{cases} q_X(w, w') & : \text{ if } w, w' \in X \\ q_Y(w, w') & : \text{ if } w, w' \in Y \\ \bot & : \text{ otherwise.} \end{cases}$$

Equalisers: the equaliser of $f, g\colon (X, q_X) \to (Y, q_Y)$ is given by $e\colon (E, q_E) \to (X, q_X)$, where

$$E = \{x \in X \mid f(x) = g(x)\},$$

and

$$q_E(x, x') = q_X(x, x').$$

Coequalisers: the coequaliser of $f, g\colon (X, q_X) \to (Y, q_Y)$ is given by

$$h\colon (Y, d_Y) \to (H, d_H),$$

where $H = Y/\sim$, and $\sim$ is the smallest equivalence relation such that for all $x \in X$ we have $f(x) \sim g(x)$. The hom-functor q_H is given by

$$q_H([y], [y']) = \bigvee_{\substack{u \sim y \\ u' \sim y'}} q_Y(u, u').$$

Final Object: the final object is $(\mathbf{1}, q_{\mathbf{1}})$, where $\mathbf{1}$ is the singleton set, and $q_{\mathbf{1}}(*, *) = \top$.

Initial Object: the initial object is $(\mathbf{0}, q_{\mathbf{0}})$, where $\mathbf{0}$ is the empty set.

Since Q is a complete lattice, small products also exist, as do small coproducts, and so we have the following proposition.

Proposition 5.5. *The category Q–**Cat** is complete and cocomplete.*

We also require that Q–**Cat** be symmetric monoidal closed, and for this we need a tensor and a unit. We define the tensor as follows. Note, this defines a functor since $\otimes$ is a functor on Q.

Definition 5.6. Given a commutative unital quantale $(Q, \otimes, I)$, the **tensor product** $(X, q_X) \otimes (Y, q_Y)$ of the Q-categories (X, q_X) and (Y, q_Y) is given by $(X \times Y, q_{X \otimes Y})$, where

$$q_{X \otimes Y}((x, y), (x', y')) = q_X(x, x') \otimes q_Y(y, y'),$$

and the **unit** Q-category is the singleton set $(\mathbf{1}, q_I)$ with $q_I(*, *) = I$.

Note that in general the unit Q-category is not the final Q-category.

It is easy to verify that tensor product and the unit Q-category form the tensor and unit of a symmetric monoidal category. To make Q–**Cat** also closed we need internal-hom objects $[(X, q_X), (Y, q_Y)]$, such that $[(Y, q_Y), -]$ is right adjoint to $- \otimes (Y, q_Y)$ (Definition B.9). These are given as follows:

Internal-hom: the internal-hom of (X, q_X) and (Y, q_Y) is given by the set of all Q-functors from (X, q_X) to (Y, q_Y) with the hom-functor

$$q_{[(X,q_X),(Y,q_Y)]}(f, g) = \bigwedge_{x \in X} q_Y(f(x), g(x)).$$

Unit: the unit of the adjunction $- \otimes (Y, q_Y) \dashv [(Y, q_Y), -]$ is given by

$$\begin{aligned} d_{(X,q_X)} \colon (X, q_X) &\to [(Y, q_Y), (X, q_X) \otimes (Y, q_Y)] \\ x &\mapsto f_x \colon (Y, q_Y) \to (X, q_X) \otimes (Y, q_Y), \end{aligned}$$

where $f_x(y) = (x, y)$.

Counit: the counit of the adjunction $- \otimes (Y, q_Y) \dashv [(Y, q_Y), -]$ is given by

$$\begin{aligned} e_{(Z,q_Z)} \colon [(Y, q_Y), (Z, q_Z)] \otimes (Y, q_Y) &\to (Z, q_Z) \\ (g \colon (Y, q_Y) \to (Z, q_Z), y) &\mapsto g(y). \end{aligned}$$

To show that these do indeed make Q–**Cat** closed, we are required to use that Q is both closed and complete, and also make use of all the defining properties of Q-categories and Q-functors (thus explaining why we said at the beginning of this section that it was vital that each pair (X, q_X) was a category). This gives the following proposition, which also appears as Wagner (1997, Proposition 1.14) and Worrell (2000b, Definition 4.3.8).

Proposition 5.7. *The category* Q–**Cat** *is symmetric monoidal closed.*

In Wagner (1997) it is remarked that Q–**Cat** is Cartesian closed if and only if Q is a complete Heyting algebra with $\otimes$ given by meet.

Finally, the symmetric monoidal closed functor **elem**$|-|$ (Definition B.15) is easily seen to be faithful, and strong monoidal (Definition B.14). It is also a fibration (Definition A.5), as for any function $f \colon X \to Y$, if Y carries the hom-functor q_Y, then we can define a hom-functor q_X on X by

$$q_X(x, x') = q_Y(f(x), f(x')).$$

This is easily shown to be universal in the sense required of an initial lift.

Therefore putting everything together we can deduce:

Proposition 5.8. *The category* Q–**Cat** *satisfies all the conditions of Assumption 1.*

The above results show, that based on the assumption that one should enrich over a category with a structure analogous to **Preord** or **GMet**, that in order to satisfy the basic assumptions on $\mathbb{V}_o$ we used to develop logical connections (Chapter 3) and coalgebraic modal logic (Chapter 4), we require that Q be a commutative unital quantale.

The eagle-eyed reader will have spotted though, that for the category $\mathbf{Set}_R$, the objects satisfy additional axioms beyond those of a Q-category in the cases where the type R does not represent preorders. For example, in the case of equivalence relations we require symmetry in addition to reflexivity and transitivity. Thus in the general case we are forced to consider full subcategories of Q–**Cat**.

Definition 5.9. Given a commutative unital quantale $(Q, \otimes, I)$, we shall use the notation $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ for any full subcategory of Q–**Cat** that satisfies the conditions of Assumption 1.

To show that our generalisation genuinely subsumes bisimulation, simulation, and approximate simulation, we should be able to recreate the categories **Set**, $\mathbf{Set}_R$, and **GMet** via instances of Q–**Cat** for appropriate choices of Q.

Example 5.1. *If we take Q to be the set* **2** *with the usual order, and take $\otimes$ to be meet, and I to be 1, then Q–**Cat** is the category **Preord**, and for each of the four choices of the type R, the category $\mathbf{Set}_R$ (and thus **Set**) is a full subcategory of Q–**Cat**.*

Example 5.2. *If we take Q to be the set $[0, \infty]$ with the opposite order, and take $\otimes$ to be $+$, and I to be 0, then Q–**Cat** is the category **GMet**.*

Note, **Set** cannot be recovered from **2** with the discrete order, as **2** is not then complete.

The above examples show that we can recreate all of our previously discussed notions of behavioural comparability through an appropriate choice of a commutative unital quantale, but do we get anything more? Can we find new ways of comparing the behaviour of states, new behavioural questions that we can ask?

The following simple example from Wagner (1997) is a possibility.

Example 5.3. *If we have a commutative monoid $(M, +, 0)$, then we can take Q to be $\mathcal{P}(M)$, the powerset of M, with the order given by inclusion, meet and join given by intersection and union, and take $\otimes$ to be defined by*

$$u \otimes v = \{m + n \mid m \in u, n \in v\},$$

*and I to be $\{0\}$. Then Q–**Cat** is the category $\mathbf{WDGraph}_M$ of weighted directed graphs (X, e_X), where the weights are elements of M, every vertex has a self-loop of weight 0, and to compose edges we add the weights. Here e_X assigns the set of edges to every ordered pair of points. Note, between any ordered pair of points there can be multiple edges, but only one with each weight. The morphisms of $\mathbf{WDGraph}_M$ are functions $f \colon X \to Y$ such that*

$$e_X(x, x') \subseteq e_Y(f(x), f(x')).$$

We can try out $\mathbf{WDGraph}_M$ to see what it might give us, by defining the powerset functor on $\mathbf{WDGraph}_M$.

Example 5.4 (Powerset on $\mathbf{WDGraph}_M$ enriched over $\mathbf{WDGraph}_M$)**.** *The functor T is defined as*

$$T(X, e_X) = (\mathcal{P}(X), e_{\mathcal{P}(X)}),$$

where

$$e_{\mathcal{P}(X)}(u, v) = \bigcap_{\substack{x \in u \\ y \in v}} e_X(x, y),$$

and for any function $f\colon (X, e_X) \to (Y, e_Y)$, the action of T on f is the function

$$\begin{aligned} T(f)\colon \mathcal{P}(X) &\to \mathcal{P}(Y) \\ u &\mapsto \{f(x) \mid x \in u\}. \end{aligned}$$

To see what this might actually mean we need to consider what happens to a T-coalgebra $\gamma\colon (X, e_X) \to T(X, e_X)$. The structure map γ is required to satisfy

$$e_X(x, y) \subseteq \bigcap_{\substack{x' \in \gamma(x) \\ y' \in \gamma(y)}} e_X(x', y'),$$

which says that for any pair of states x and y, the smallest common set of edge weights between every possible successor of x, and every possible successor of y, must contain the set of edge weights between x and y. So if we regard the set of edges from x to y as signifying some set of properties that y has with respects to x, then when x and y transition to successor states, the successor of y must have at least the same properties with respect to the successor of x, as y had to x.

In this example there is no observable behaviour, so it is not obvious that the set of properties that y has with respect to x has anything to do with behaviour. However, we can extend this example to Labelled Transition Systems by taking a set of labels Σ, and defining $e_\Sigma(l, l') = M$ if $l = l'$, and $e_\Sigma(l, l') = \emptyset$ otherwise.

Example 5.5 (LTS enriched over $\mathbf{WDGraph}_M$)**.** *The functor T is defined as*

$$T(X, e_X) = (\mathcal{P}(\Sigma \times X), e_{\mathcal{P}(\Sigma \times X)}),$$

where

$$\begin{aligned} e_{\mathcal{P}(\Sigma \times X)}(u, v) &= \bigcap_{\substack{(l,x) \in u \\ (l',x') \in v}} e_{\Sigma \times X}((l, x), (l', x')) \\ &= \bigcap_{\substack{(l,x) \in u \\ (l',x') \in v}} (e_\Sigma(l, l') \cap e_X(x, x')), \end{aligned}$$

and for any function $f\colon (X, e_X) \to (Y, e_Y)$*, the action of* T *on* f *is the function*

$$\begin{aligned} T(f)\colon \mathcal{P}(\Sigma \times X) &\to \mathcal{P}(\Sigma \times Y) \\ u &\mapsto \{(l, f(x)) \mid (l, x) \in u\}. \end{aligned}$$

Now when we consider a T-coalgebra, we find that the structure map γ must satisfy

$$e_X(x, y) \subseteq \bigcap_{\substack{(l,x')\in\gamma(x) \\ (l',y')\in\gamma(y)}} (e_\Sigma(l, l') \cap e_X(x', y')).$$

This has the same constraint as in the simple powerset case, but in addition, for every transition that x can make with label l, y must also be able to make a transition with label l that preserves the set of properties that y has with respect to x.

In this formulation of Labelled Transition Systems we chose a particular graph e_Σ on the set of labels Σ that only distinguished whether two labels were equal, but this is obviously not the only choice we could have made. This particular choice was made to force y to have to be able to match the label chosen by x, but if there was some kind of relationship between the different labels, where l' had a particular set of properties (from M) with respect to l, then the choices y would have to be able to make to follow x would be different. Specifically, y would have to be able to choose a transition (l', y') such that both the set of properties that l' had with respect to l, and the set of properties that y' had with respect to x', contained the set of properties that y had with respect to x.

It is clear from the above that we can define many different notions of behavioural comparability, or behavioural questions, but we still have not addressed how to compare states from different T-coalgebras.

In actual fact we are not really interested in comparing states from completely arbitrary T-coalgebras, but rather from T-coalgebras that are models for some L-algebra, since our interest is in coalgebraic modal logic.

First we instantiate our running assumptions by fixing the category $\mathbb{V}$ to be of the form $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ for some commutative unital quantale Q.

Assumption 6. We extend Assumption 5 (page 83) as follows:

12. We fix the category $\mathbb{V}$ to be $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ (Definition 5.9), where $(Q, \otimes, I)$ is a commutative unital quantale.

Now we are finally ready to write down precisely what we mean when we talk of comparing the behaviour of two states.

Definition 5.10. Given the conditions of Assumption 6, for any two models X_1, X_2 in $\mathbf{Mod}(A, \alpha)$, and any two states $x_1 \in X_1$, $x_2 \in X_2$, if there exists in $\mathbf{Mod}(A, \alpha)_o$ a cospan

$$X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2,$$

then we say x_2 **has a behavioural adjacency bound of**

$$q_{X_3}(f_1(x_1), f_2(x_2))$$

with respect to x_1.

Here the model morphisms f_1 and f_2 are T-coalgebra morphisms, and thus transport states x_1 and x_2 bisimilarly to the model X_3 where their images $f_1(x_1)$ and $f_2(x_2)$ are compared. The resulting answer $q_{X_3}(f_1(x_1), f_2(x_2))$ is a lower bound to the definitive answer (given by the final model), since for any morphism $f_3 \in \mathbf{Mod}(A, \alpha)_o(X_3, X_4)$, the answer $q_{X_4}(f_3 \circ f_1(x_1), f_3 \circ f_2(x_2))$ is at least as good, and possibly better, as f_3 has an underlying Q-functor (Definition 5.3).

In the concrete examples of enrichment over the categories **Set**, $\mathbf{Set}_R$, and **GMet**, Definition 5.10 takes on the following familiar forms.

Example 5.6 (Enrichment over **Set**). *In this case $q_X(x, y)$ simply determines if $x = y$ (Example 5.1), and so we therefore look for a cospan where the model morphisms f_1 and f_2 identify x_1 and x_2. We then say x_1 and x_2 are behaviourally equivalent (Kurz, 2000).*

Example 5.7 (Enrichment over $\mathbf{Set}_R$). *In this case $q_X(x, y)$ determines membership in the relation R_X (Example 5.1), and so we therefore look for cospans where*

$$f_1(x_1) R_{X_3} f_2(x_2).$$

In Wilkinson (2012a) x_1 and x_2 are then said to be behaviourally R-related.

There are actually four cases to consider depending upon the type R:

1. *If R is the type preorder, then we have simulation.*
2. *If R is the type partial order, then we have simulation where mutual simulation implies bisimulation.*
3. *If R is the type equivalence relation, then we have mutual simulation.*
4. *If R is the type equality, then we have bisimulation.*

Example 5.8 (Enrichment over **GMet**). *In this case $q_X(x, y)$ is the metric $d_X(x, y)$ (Example 5.2), and so we say that x_2 is at most $d_{X_3}(f_1(x_1), f_2(x_2))$ from x_1, or x_2 approximately simulates x_1 to within at least the accuracy $d_{X_3}(f_1(x_1), f_2(x_2))$.*

We now need to consider the logical counterpart to Definition 5.10. Given a pair of models in $\mathbf{Mod}(A, \alpha)$, and a state from each model, we want to compare the logical theories of these two states. Now, the logical theory for each state is an element of $S(A)$, so we should use $q_{S(A)}$ to compare the theories.

Definition 5.11. Given the conditions of Assumption 6, for any two models X_1, X_2 in $\mathbf{Mod}(A, \alpha)$, and any two states $x_1 \in X_1$, $x_2 \in X_2$, we say x_2 **has a logical adjacency of**

$$q_{S(A)}(f_1(x_1), f_2(x_2))$$

with respect to x_1, where f_1 and f_2 are the theory maps of X_1 and X_2 respectively.

Note here, unlike in Definition 5.10, we do not have a lower bound on the logical adjacency, but the actual definitive value. This is because for each model the theory map is part of the definition, and therefore unique.

To understand this definition we should look at some examples. Now we know by Assumption 6 that $VS(A) = \mathbb{A}(A, \Omega_{\mathbb{A}})$, and that $U_{A,\Omega_{\mathbb{A}}} \colon \mathbb{A}(A, \Omega_{\mathbb{A}}) \to [U(A), U(\Omega_{\mathbb{A}})]$, and since the $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ functors U and V are forgetful functors, we typically find that $q_{S(A)}$ is the same as $q_{[U(A),U(\Omega_{\mathbb{A}})]}$. Thus in many cases

$$q_{S(A)}(s, s') = \bigwedge_{a \in A} q_{\Omega_{\mathbb{A}}}(s(a), s'(a)).$$

Example 5.9 (Example 3.1)**.** *In this example $q_{S(A)}(s, s')$ tests for equality of the filters/prime filters/ultrafilters s and s'. This is the usual notion of logical equivalence (Kurz, 2001).*

Example 5.10 (Example 3.2)**.** *In this example $q_{S(A)}(s, s')$ tests for inclusion of the filter/prime filter s in s', or in the case that the type R is equality, it reverts to the example above. This example captures the notion that two states may be logically R-related of Wilkinson (2012a).*

Example 5.11 (Example 3.9)**.** *In this example $q_{S(A)}(s, s')$ is given by the metric*

$$d_{S(A)}(s, s') = \sup_{a \in A} d_{[0,1]}(s(a), s'(a)),$$

which gives the distance between the fuzzy filters/fuzzy prime filters s and s'.

The following result is a simple consequence of the fact that the theory maps of models are morphisms in $\mathbb{X}$, and have underlying morphisms in $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$, which are Q-functors (Definition 5.3).

Proposition 5.12. *Given the conditions of Assumption 6, for any two models X_1, X_2 in $\mathbf{Mod}(A, \alpha)$, and any two states $x_1 \in X_1$, $x_2 \in X_2$, if x_2 has a behavioural adjacency bound of b with respect to x_1, then*

$$b \leq q_{S(A)}(f_1(x_1), f_2(x_2)),$$

where $q_{S(A)}(f_1(x_1), f_2(x_2))$ is the logical adjacency of x_2 with respect to x_1, and f_1 and f_2 are the theory maps of X_1 and X_2 respectively.

If it is possible to find a cospan in $\mathbf{Mod}(A, \alpha)_o$ such that the inequality of Proposition 5.12 can be made into equality, then the L-algebra (A, α) is said to be expressive for $\mathbf{Mod}(A, \alpha)$.

Definition 5.13. Given the conditions of Assumption 6, an L-algebra (A, α) is **expressive for $\mathbf{Mod}(A, \alpha)$**, if for any two models X_1, X_2 in $\mathbf{Mod}(A, \alpha)$, and any two states $x_1 \in X_1$, $x_2 \in X_2$, there exists in $\mathbf{Mod}(A, \alpha)_o$ a cospan

$$X_1 \xrightarrow{g_1} X_3 \xleftarrow{g_2} X_2,$$

such that

$$q_{X_3}(g_1(x_1), g_2(x_2)) = q_{S(A)}(f_1(x_1), f_2(x_2)),$$

where f_1 and f_2 are the theory maps of X_1 and X_2 respectively.

This definition is slightly stronger than the standard cospan based definition of expressivity for bisimulation, and the definition of expressivity for simulation in Wilkinson (2012a). The difference is that in these two cases if x_1 and x_2 are not logically equivalent (not logically R-related) i.e. $f_1(x_1) \not{R}_{S(A)} f_2(x_2)$, then there is no requirement that there exist a cospan such that $g_1(x_1) \not{R}_{X_3} g_2(x_2)$. However the above definition requires this. In practice this is not a problem, since if $\mathbb{X}$ has coproducts, then by Theorem 4.24, so does $\mathbf{Mod}(A, \alpha)$, and thus such a cospan always exists.

In the case of approximate simulation however, we need to take into account the full range of values that the metric can take, and we want the distance between the behaviours of two states to equal the distance between their logical theories. This forces the above, slightly stronger, definition.

5.3 Behavioural Questions and Truth Values

The definition of expressivity of an L-algebra (A, α) for its category of models $\mathbf{Mod}(A, \alpha)$ with respect to $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ (Definition 5.13) makes use of the Q-category structure on $S(A)$, the collection of theories of (A, α). Moreover, as already noted in the previous section,

$q_{S(A)}$ is often given by

$$q_{S(A)}(s, s') = \bigwedge_{a \in A} q_{\Omega_{\mathbb{A}}}(s(a), s'(a)),$$

where $q_{\Omega_{\mathbb{A}}}$ is the Q-category structure on the truth object $\Omega_{\mathbb{A}}$.

An obvious question then is how do we choose a set of truth values? Specifically, should it relate in some way to Q, and if so, how?

The first observation we can make is that since $(Q, \otimes, I)$ is a commutative unital quantale, it is a symmetric monoidal closed category (Proposition 5.2), and thus it enriches over itself (Section C.3).

Proposition 5.14. *If $(Q, \otimes, I)$ is a commutative unital quantale, then Q is itself a Q-category, with $q_Q(a, b) = [a, b]$.*

In the case of the examples that have motivated our study, we get the following.

Example 5.12 (Example 5.1)**.** *If we take Q to be the set* **2** *with the usual order, and take $\otimes$ to be meet, and I to be* 1*, then*

$$[a, b] = \begin{cases} 1 & : \text{ if } a \leq b \\ 0 & : \text{ otherwise.} \end{cases}$$

Example 5.13 (Example 5.2)**.** *If we take Q to be the set $[0, \infty]$ with the opposite order, and take $\otimes$ to be $+$, and I to be* 0*, then*

$$[a, b] = \begin{cases} b - a & : \text{ if } a \leq b \\ 0 & : \text{ otherwise.} \end{cases}$$

So in the case of enrichment over $\mathbf{Set}_R$, and with the type R representing preorders or partial orders, we find that $(\mathbf{2}, q_{\mathbf{2}})$ is the same as the truth object in our bivalent examples (Definition 3.23). However, in the case of enrichment over **Set**, $(\mathbf{2}, q_{\mathbf{2}})$ is not an object in the full subcategory of Q−**Cat** that is **Set**, and indeed in our bivalent examples we have taken the truth object to be **2** with the discrete order. Finally, in the case of enrichment over **GMet**, none of our examples have taken the interval $[0, \infty]$ as the set of truth values.

So does this mean that the choice of Q has nothing to do with the choice of truth values? Not necessarily. In Example 4.20 and Example 4.21 we enrich over **Set** and take the truth object to be **2** or $[0, 1]$ respectively. In both cases the set of truth values is just a set, i.e. carries the discrete order. However, whilst constructing the predicate liftings and the natural transformation $\delta \colon LP \Rightarrow PT$ we make explicit use of the fact that both **2** and $[0, 1]$ have meets given by the usual orders. Similarly, in Example 4.22 and Example 4.23 we use the joins of **2** and $[0, 1]$.

So it would appear that the truth values get used in at least two different ways. Firstly in the construction of the logical connection, and then secondly in the definition of the predicate liftings and the natural transformation $\delta\colon LP \Rightarrow PT$. In the latter case the set of truth values may carry additional structure not required to define the logical connection, and this may be the structure of a lattice, or even a commutative unital quantale.

Also, in several of our examples we have used fuzzy logics with truth values taken from $[0,1]$ purely for illustrative purposes, but are there ever compelling reasons to do so? Starting with Kozen (1981, 1985) and then Panangaden (1999); Desharnais et al. (2004); van Breugel et al. (2005), it has been argued that probabilistic systems should be modelled with a logic of real-valued functions taking their values in the interval $[0,1]$. One of the motivations for doing so, is that bivalent logics are not robust to small changes in the probabilities, and so "close" approximations to a probabilistic system may have radically different logical theories.

However, Example 5.13 above suggests that to study approximations we perhaps should use a logic with truth values from $[0,\infty]$, though perhaps this should be just thought of as a rescaling of the interval $[0,1]$? Having said that, in Mislove et al. (2004) Markov Processes are studied using a duality between real C^*-algebras and compact Hausdorff spaces where the duality arises from the set $\mathbb{R}$, so it is far from clear what the correct approach should be.

In general therefore, at this stage it is unclear what the correct choice of truth values should be, nor how they should relate to the choice of category $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$. Further study of examples along the lines of the real-valued logics for Markov Processes may provide clues as to what the correct approach, if such a general approach exists, should be.

5.4 Discussion

In the literature there have been many different approaches to bisimulation and simulation for coalgebras, but essentially they can be split into two distinct groups - those that use spans, and those that use cospans. Our work falls squarely in the latter camp.

The first approaches, starting with Aczel and Mendler (1989) [1], were interested in bisimulation and were span based. The aim was to construct a relation $R \subseteq X \times Y$, the bisimulation, on the carriers X and Y of a pair of coalgebras. This approach was subsequently generalised to that of a relation lifting, first through the use of fibrations (Jacobs, 1995; Hermida and Jacobs, 1998; Klin, 2005), and then via relators (Rutten, 1998).

[1] Note that in the same paper, Aczel and Mendler also introduce a notion of (pre)congruence, which is essentially a cospan approach (Kurz, 2000, Section 1.2).

In the relator approach of Rutten (1998) a functor T on **Set** extends to a unique functor on **Rel** [2], the relator, if and only if T preserves weak pullbacks (Carboni et al., 1991). This condition on T is also required (in general) to construct an Aczel-Mendler style bisimulation (Aczel and Mendler, 1989).

The relator approach was then generalised in two different ways. Firstly, relators were applied to simulation and metric bisimulation through generalising relations to the enriched equivalent - bimodules (Rutten, 1998; Turi and Rutten, 1998; Worrell, 2000a,b). Here in the case of the approach by Worrell, the functor T on Q–**Cat** is extended to a graph homomorphism [3] on the corresponding category of bimodules, and this is a lax functor [4], if and only if T preserves Q-embeddings (Definition 6.1) (Worrell, 2000a, Theorem 4.5). Then using the graph homomorphism extending T, a notion of a T-simulation is defined, and if the extension of T is a lax functor, the composition of T-simulations is also a T-simulation.

In the approach to bisimulation of Rutten (1998) the extension of T to **Rel** is fixed, so the second generalisation was to take a separate relator Γ on **Rel** that was not derived from T (Hughes and Jacobs, 2004; Cîrstea, 2006; Levy, 2011). This then allows the notion of a Γ-simulation for T-coalgebras to be defined, and for different choices of Γ, this yields different notions of simulation.

The strength of the span based approach is that when one thinks about the states of two coalgebras, and one wants to compare the behaviours of pairs of states, one's intuitive response is to think of constructing a relation on the two sets of states. The weakness though of this approach, is that not only does one have to find the relation, but to show that it is the relation one is looking for, one has to put a coalgebra structure on it. In general, if we desire that such a relation be transitive, and this is indeed what we expect for standard notions of bisimulation or simulation, this can only be done if T preserves weak pullbacks (Rutten, 1998), or some generalisation of this in the enriched case (Worrell, 2000a; Bílková et al., 2011).

The alternative approach to bisimulation using cospans originated in the PhD thesis of Kurz (Kurz, 2000), and has the key advantage that it does not need the functor T to preserve weak pullbacks. The two are easily seen to be equivalent in most cases. Specifically, if the category $\mathbb{X}$ has pushouts, then any span based bisimulation yields a cospan based bisimulation, and if $\mathbb{X}$ has weak pullbacks and T preserves them, then any cospan based bisimulation yields a span based bisimulation.

This removal of the requirement that T preserves weak pullbacks has practical consequences. For example, in Danos et al. (2006) it is shown that the cospan approach

[2] There are two formulations of the category **Rel** that appear in the coalgebra literature: the first has sets as objects and binary relations as morphisms, and the second has binary relations as objects and pairs of relation preserving functions as morphisms.

[3] A mapping of objects and morphisms that need not preserve composition and identities.

[4] $1_{T(X)} \leq T(1_X)$ for all objects X, and $T(f) \circ T(g) \leq T(f \circ g)$ for all composable morphisms f and g.

greatly simplifies the analysis of Markov Process viewed as coalgebras for the Giry functor (Example 4.17) which does not preserve weak pullbacks. Moreover, the authors point out that in their earlier work (Desharnais et al., 2002) the proofs made explicit use of cospans, but that at the time they regarded them as merely an intermediate step towards the construction of an appropriate span. Also, the cospan approach greatly extended their results to general measurable spaces, whereas the original work was restricted to analytic spaces by the preservation of weak pullbacks requirement.

In a similar vein, whilst the approach of Worrell (2000a) is nominally span based through the use of bimodules and the desire to construct simulation relations (or their generalisations), in the detailed proofs explicit use is made of the collage of a bimodule, which is a cospan. Indeed, this is contrasted (Worrell, 2000a, Section 4) with the span based approach underlying the corresponding result in Carboni et al. (1991) which forms the basis of Rutten (1998). However, once again, in order that the required relation be transitive, we require that the composite of T-simulations be a T-simulation, and this means that the functor T must preserve Q-embeddings. We side step this requirement by working with an explicit cospan based notion of simulation and bisimulation. Also, our work greatly extends that of Worrell by enriching over Q–**Cat**, which means the objects of the category $\mathbb{X}$ upon which T is defined, can also carry additional structure (sigma algebras for example). Whereas in Worrell (2000a) and Worrell (2000b) the functor T is constrained to act directly on Q–**Cat**.

Recently (Kapulkin et al., 2010, 2012) the cospan approach has been extended to simulation through enriching over **Pos** and looking at cospans to the final coalgebra. This work also relates to that of Levy (Levy, 2011), who takes a relator approach to simulation, but links it to final coalgebras over the categories we subsumed into $\mathbf{Set}_R$, though he does not work in an enriched setting. Our work extends this to general cospans, not just those to the final coalgebra, and to other notions of behavioural comparability beyond bisimulation and simulation. We also work with models, and not just with coalgebras. This means we also have a generalised notion of logical comparability, and our notions of behavioural comparability correctly handle propositional variables (cf. the definition of bisimulation in Blackburn et al. (2001)), so we can work with arbitrary L-algebras, not just the initial one.

Finally, we can give the following slightly more detailed account, taken from our earlier work Wilkinson (2012a), of how our approach to simulation relates to the more standard relator based approach mentioned above.

For any functor $F\colon \mathbf{Set} \to \mathbf{Set}$, an F-relator is defined as a functor $\Gamma\colon \mathbf{Rel} \to \mathbf{Rel}$ that satisfies certain additional properties, and then using this, it is standard to define a notion of Γ-simulation for F-coalgebras. Now, associated with F and Γ is a functor

$T\colon \mathbf{Preord} \to \mathbf{Preord}$ (Hughes and Jacobs, 2004, Lemma 5.5) (Levy, 2011, Definition 11) given by

$$T(X, R_X) = (F(X), \Gamma(R_X)),$$

and under certain conditions (Hughes and Jacobs, 2004, Theorem 9.4) the final T-coalgebra is the final F-coalgebra with the preorder given by the Γ-similarity relation. This final T-coalgebra characterises Γ-similarity of F-coalgebras as every set carries a discrete preorder (equality). Thus for every F-coalgebra there is a corresponding T-coalgebra, and given two F-coalgebras, the Γ-similarity relation on those two F-coalgebras is given by the preorder on the images of states under the corresponding unique cospan of morphisms to the final T-coalgebra (Cîrstea, 2006, Remark 21).

Now in our general framework, for the initial L-algebra, every T-coalgebra has a unique theory map making it a model. Therefore if there exists a final T-coalgebra, it is a model, and moreover every other model factors uniquely via it. It is thus the final model Z. So for any cospan of models

$$X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$$

such that $f_1(x_1)\, R_{X_3}\, f_2(x_2)$, there exists a unique model morphism $g\colon X_3 \to Z$, and this gives $g \circ f_1(x_1)\, R_Z\, g \circ f_2(x_2)$. So if T is given by an F-relator as above, our notion of similarity coincides with Γ-similarity.

Our notion of simulation can thus be seen as taking the F-relator notion of simulation and extending it to arbitrary cospans in $\mathbf{Mod}(A, \alpha)_o$, not just those with the final T-coalgebra as the target, and also to an arbitrary functor T, rather than one arising from a functor F on $\mathbf{Set}$ and an F-relator Γ.

Chapter 6

Expressivity

In Chapter 5 we introduced a generalised notion of what it means for an L-algebra (A, α) to be expressive for its category of models $\mathbf{Mod}(A, \alpha)$ (Definition 5.13). In this chapter we shall show that whether (A, α) is expressive can be characterised by the structure of the category $\mathbf{Mod}(A, \alpha)$, and we shall explore how this may be used to prove expressivity.

A brief outline of this chapter is as follows:

Section 6.1 The concept of a behavioural skeleton is introduced, and a theorem proved that says that an L-algebra (A, α) is expressive if and only if $\mathbf{Mod}(A, \alpha)$ has a behavioural skeleton.

Section 6.2 Parametric behavioural skeletons are introduced as a flexible tool for proving expressivity, and a result proved that shows that through the careful choice of a factorisation system for $\mathbb{X}_o$, expressivity of an L-algebra (A, α) follows from a condition on δ^*_A. Conditions are also given for the existence of final models.

Section 6.3 The specific case of expressivity with respect to bisimulation is examined using the internal models of Wilkinson (2012b).

Section 6.4 The specific case of expressivity with respect to simulation is examined using the R-models of Wilkinson (2012a).

Section 6.5 A brief discussion of different approaches for proving expressivity from the literature is given.

6.1 Behavioural Skeletons

In Wilkinson (2012b) we introduced the notion of an internal model, and in Wilkinson (2012a) we extended this notion to an R-model, and showed how these two notions can be used to give a characterisation of expressivity, that in the former cases is with respect to bisimulation, and in the latter with respect to simulation. These two cases correspond to enrichment over **Set** and $\mathbf{Set}_R$ respectively.

In this section we shall generalise still further to the case of enrichment over $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ (Definition 5.9). We shall proceed by defining the abstract notion of a behavioural skeleton, and then show how they can be used to give a characterisation of expressivity. First though we need to define a piece of terminology we will make use of later.

Definition 6.1. Given the conditions of Assumption 6, a morphism

$$f\colon (X, q_X) \to (Y, q_Y)$$

in $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ is said to be **Q-preserving** if

$$q_X(x, x') = q_Y(f(x), f(x')),$$

and a morphism in $\mathbb{X}_o$ is **Q-preserving** if its underlying morphism in $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ is. A model $((X, \gamma), f)$ in $\mathbf{Mod}(A, \alpha)$ is said to be **Q-preserving** if its theory map is Q-preserving, and a model morphism

$$h\colon ((X, \gamma), f) \to ((Y, \zeta), g)$$

is said to be **Q-preserving** if the morphism $h\colon X \to Y$ in $\mathbb{X}_o$ is. If in addition a Q-preserving morphism also has an injective underlying function, then it is said to be a **Q-embedding**.

Note by Section 3.2, Q-embeddings are precisely those morphism of $\mathbb{V}_{Q\text{-}\mathbf{Cat}}$ or $\mathbb{X}_o$ that are monomorphisms and Q-preserving.

Definition 6.2. Given the conditions of Assumption 6, the (unique up to isomorphism) skeleton (Definition C.31) of the full subcategory (Definition C.22) of Q-preserving models of $\mathbf{Mod}(A, \alpha)$ is a **behavioural skeleton** of $\mathbf{Mod}(A, \alpha)$, and denoted $\mathbf{BSkel}(A, \alpha)$, if it has the following properties:

1. For every model X in $\mathbf{Mod}(A, \alpha)$, there exists a model Y in $\mathbf{BSkel}(A, \alpha)$, and a morphism $f\colon X \to I_{\mathbf{BSkel}(A,\alpha)}(Y)$ in $\mathbf{Mod}(A, \alpha)_o$. Here the functor

   $$I_{\mathbf{BSkel}(A,\alpha)}\colon \mathbf{BSkel}(A, \alpha) \to \mathbf{Mod}(A, \alpha)$$

   is the inclusion functor. We say that X **factors via** Y.

2. For every pair of models X_1 and X_2 in $\mathbf{BSkel}(A, \alpha)$ there exists a cospan

   $$X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$$

   in $\mathbf{BSkel}(A, \alpha)_o$.

It is relatively easy to prove that the existence of a behavioural skeleton ensures that an L-algebra is expressive for its models (Definition 5.13).

Proposition 6.3. *Given the conditions of Assumption 6, if* $\mathbf{Mod}(A, \alpha)$ *has a behavioural skeleton* $\mathbf{BSkel}(A, \alpha)$*, then* (A, α) *is expressive for* $\mathbf{Mod}(A, \alpha)$*.*

Proof. Take any pair of models X_1 and X_2 in $\mathbf{Mod}(A, \alpha)$. These factor via the models Y_1 and Y_2 in $\mathbf{BSkel}(A, \alpha)$, and there also exists a model Y_3 in $\mathbf{BSkel}(A, \alpha)$ such that there exists a cospan $Y_1 \to Y_3 \leftarrow Y_2$. Thus both X_1 and X_2 factor via Y_3.

Spelling this out in more detail, the models $((X_1, \gamma_1), f_1)$ and $((X_2, \gamma_2), f_2)$ factor via the model $((Y_3, \zeta_3), h_3)$ by way of T-coalgebra morphisms $g_1 \colon (X_1, \gamma_1) \to (Y_3, \zeta_3)$ and $g_2 \colon (X_2, \gamma_2) \to (Y_3, \zeta_3)$, such that $f_1 = h_3 \circ g_1$ and $f_2 = h_3 \circ g_2$.

Now if we consider two states $x_1 \in X_1$ and $x_2 \in X_2$, then since h_3 is Q-preserving, we have

$$\begin{aligned} q_{S(A)}(f_1(x_1), f_2(x_2)) &= q_{S(A)}(h_3 \circ g_1(x_1), h_3 \circ g_2(x_2)) \\ &= q_{Y_3}(g_1(x_1), g_2(x_2)) \end{aligned}$$

as required. □

We are interested in conditions where the converse is true, i.e. under which conditions is the existence of a behavioural skeleton necessary for expressivity?

To answer this we need to think a bit about what the definition of a behavioural skeleton actually says, and how it relates to expressivity. Expressivity says that any pair of states can be mapped bisimilarly to a model where their behavioural adjacency is equal to their logical adjacency. So if we could bisimilarly map any two states to a Q-preserving model we would be done. But is it realistic to expect Q-preserving models to exist, and is it reasonable to expect that any state, in any model, can be mapped bisimilarly to a state in a Q-preserving model?

It turns out that the key question is whether a given model factors via a Q-preserving model, for if that is the case, then provided the category $\mathbb{X}$ has binary coproducts, and thus by Theorem 4.24, $\mathbf{Mod}(A, \alpha)$ also has binary coproducts, any two states, no matter which models they are in, can be bisimilarly mapped to a Q-preserving model.

This constraint that $\mathbb{X}$ should have binary coproducts is very mild, however in the rare cases where $\mathbb{X}$ does not have binary coproducts, it should be noted that this is only a sufficient condition anyway, it may not be a necessary one. If binary coproducts exist they provide an easy way to generate the required cospan in $\mathbf{Mod}(A, \alpha)_o$, but it is not necessarily the case that all such cospans derive from coproducts in $\mathbb{X}$.

So how do we determine whether a given model factors via a Q-preserving model? More precisely, does expressivity require that this be the case?

One way to answer this is to take a model, and then look at the image of its theory map. If we could put a T-coalgebra structure map on the image (assumed to carry a restriction of $q_{S(A)}$), such that the surjective function from the carrier of the model to the theory map image is a T-coalgebra morphism, then we would have constructed such a factorisation via a Q-preserving model.

However, in the case of logics that are expressive for simulation, the above procedure is often found to be too aggressive. For example, in the case of simulation of Labelled Transition Systems (Example 4.12), the logic that is usually chosen ($\mathrm{tt} \mid \wedge \mid \langle l \rangle$) is unable to distinguish mutually similar states that are not bisimilar. Thus two states in a model can have the same theory, i.e. be identified by the theory map in $S(A)$, but not be bisimilar (Example 6.4). Therefore attempting to put a T-coalgebra structure map on the image of the theory map, as above, will fail, as the resulting surjective function could not be a T-coalgebra morphism as it will identify states that are not bisimilar.

The way to proceed therefore is to "work from the other direction". For each model we look to create its smallest bisimilar quotient, in other words, we identify all pairs of bisimilar states. We do this by looking at factorisations of model morphisms, and by assuming the existence of a factorisation system in $\mathbf{Mod}(A, \alpha)_o$, not $\mathbb{X}_o$, we ensure that all the operations we perform result in another model.

Ultimately we shall relate the factorisation system on $\mathbf{Mod}(A, \alpha)_o$ to more primitive notions, but for the purposes of the next few results we make the following assumptions.

Assumption 7. We extend Assumption 6 (page 118) as follows:

13. The category $\mathbf{Mod}(A, \alpha)$ has small pushouts.
14. The category $\mathbf{Mod}(A, \alpha)_o$ has a factorisation system (E, M) (Definition A.18).
15. M is a subclass of those morphisms in $\mathbf{Mod}(A, \alpha)_o$ that are Q-preserving.
16. E is a subclass of those morphisms in $\mathbf{Mod}(A, \alpha)_o$ with surjective underlying functions.
17. The category $\mathbf{Mod}(A, \alpha)_o$ is E-cowellpowered (Definition A.21).

Note that there is a forgetful functor

$$VU_{\mathbf{CoAlg}(T)}U_{\mathbf{Mod}(A,\alpha)} \colon \mathbf{Mod}(A, \alpha) \to \mathbb{V}_{Q\text{-}\mathbf{Cat}},$$

and since the underlying functor is also faithful (Proposition C.20), it can be composed with $\mathbf{elem}|-|$ to yield a faithful functor to $\mathbf{Set}$. Then since faithful functors reflect monomorphisms and epimorphisms (Proposition A.12 and Proposition A.14), we have

$$\begin{aligned}\text{Inject}_{\mathbf{Mod}(A,\alpha)_o} &\subseteq \text{ monos in } \mathbf{Mod}(A,\alpha)_o\\ \text{Surject}_{\mathbf{Mod}(A,\alpha)_o} &\subseteq \text{ epis in } \mathbf{Mod}(A,\alpha)_o,\end{aligned}$$

where $\text{Inject}_{\mathbf{Mod}(A,\alpha)_o}$ is the class of morphisms in $\mathbf{Mod}(A,\alpha)_o$ with injective underlying functions, and $\text{Surject}_{\mathbf{Mod}(A,\alpha)_o}$ those with surjective underlying functions.

Using the above assumptions we can show that for an expressive logic, every model must factor via a Q-preserving model.

Theorem 6.4. *Given the conditions of Assumption 7, if the L-algebra (A,α) is expressive for $\mathbf{Mod}(A,\alpha)$, then every model in $\mathbf{Mod}(A,\alpha)$ factors via a model that is Q-preserving.*

Proof. We proceed as follows:

1. All model morphisms have an (E,M)-factorisation:

   Since $\mathbf{Mod}(A,\alpha)_o$ has a factorisation system (E,M), any model morphism

   $$g\colon ((X,\gamma),f) \to ((X',\gamma'),f')$$

   factors via a model $((I,\zeta),f'\circ m)$, where $g = m\circ e$, and

   $$e\colon ((X,\gamma),f) \to ((I,\zeta),f'\circ m)$$

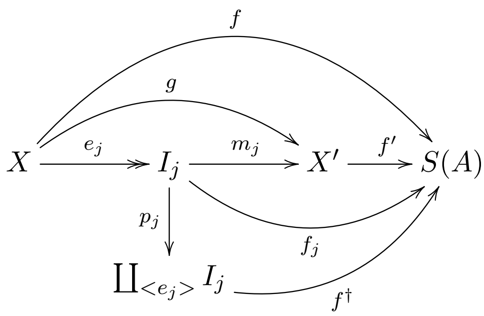
   is in E, and

   $$m\colon ((I,\zeta),f'\circ m) \to ((X',\gamma'),f')$$

   is in M.

2. Take the pushout of the E-quotient objects of $((X,\gamma),f)$:

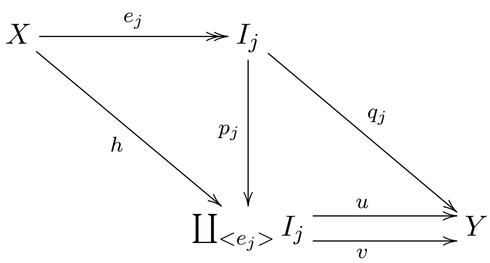
   Given a model $((X,\gamma),f)$, since $\mathbf{Mod}(A,\alpha)_o$ is E-cowellpowered, the collection of equivalence classes of E-quotient objects is indexed by a set J, and we can therefore take the pushout of a representative from each equivalence class $\coprod_{<e_j>}((I_j,\zeta_j),f_j)$, which by Theorem 4.24, we can write as $((\coprod_{<e_j>} I_j,\zeta),f^\dagger)$ for some ζ and $f^\dagger$. This gives the following diagram



where any $g\colon ((X,\gamma),f) \to ((X',\gamma'),f')$ factors via a representative of one of the equivalence classes.

3. Construct a model epimorphism $h\colon ((X,\gamma),f) \to ((\coprod_{<e_j>} I_j, \zeta), f^\dagger)$:

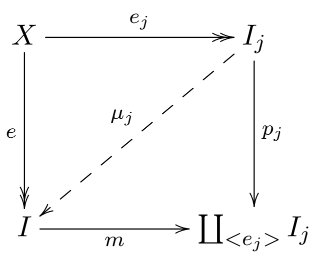
   By the definition of a pushout there is a morphism $h = p_j \circ e_j$ for all $j \in J$ in $\mathbf{Mod}(A,\alpha)_o$. To show that this is an epimorphism we use the fact that the forgetful functor $U_{\mathbf{CoAlg}(T)_o} U_{\mathbf{Mod}(A,\alpha)_o}\colon \mathbf{Mod}(A,\alpha)_o \to \mathbb{X}_o$ reflects epimorphisms (Proposition A.14). Given any parallel pair of morphisms u and v in $\mathbf{Mod}(A,\alpha)_o$, where for the underlying morphisms $u,v\colon \coprod_{<e_j>} I_j \to Y$, if $u \circ h = v \circ h$, then $u \circ p_j \circ e_j = v \circ p_j \circ e_j$, but since e_j is an epimorphism, we must therefore have $u \circ p_j = v \circ p_j = q_j$, as in the following diagram



   Clearly the q_j form a cocone for the pushout, so by the universal property of the pushout $u = v$, and thus h is an epimorphism.

4. Show $h, p_j \in E$ for all $j \in J$:

   If we take the (E,M)-factorisation of h in $\mathbf{Mod}(A,\alpha)_o$ given by e and m, then by the diagonalisation property of the factorisation system, there exists a unique $\mathbf{Mod}(A,\alpha)_o$ morphism μ_j for each $j \in J$ such that the following diagram commutes



   Once again the μ_j form a cocone for the pushout, so there exists a unique morphism

$$\eta\colon \coprod_{<e_j>} I_j \to I$$

   in $\mathbf{Mod}(A,\alpha)_o$ such that $\mu_j = \eta \circ p_j$. Now trivially $1_I \circ e = e$, and also

$$\eta \circ m \circ e = \eta \circ m \circ \mu_j \circ e_j = \eta \circ p_j \circ e_j = \mu_j \circ e_j = e,$$

so since e is an epimorphism, we must have $\eta \circ m = 1_I$. Similarly, we have $1_{\coprod_{<e_j>} I_j} \circ h = h$, and

$$m \circ \eta \circ h = m \circ \eta \circ p_j \circ e_j = m \circ \mu_j \circ e_j = p_j \circ e_j = h,$$

and since h is also an epimorphism, we must have $m \circ \eta = 1_{\coprod_{<e_j>} I_j}$. From this we deduce that m is an isomorphism, and therefore $h \in E$, and so by Proposition A.20, $p_j \in E$ for all $j \in J$.

5. Show that the theory map $f^\dagger$ is Q-preserving:

   Since $\coprod_{<e_j>} I_j$ has an underlying set we can pick a pair of states $w_1, w_2 \in \coprod_{<e_j>} I_j$. Now since $h \in E$ is a surjective function, there exists states $x_1, x_2 \in X$ such that $w_1 = h(x_1)$ and $w_2 = h(x_2)$. Thus

   $$q_{S(A)}(f(x_1), f(x_2)) = q_{S(A)}(f^\dagger(w_1), f^\dagger(w_2)),$$

   and by expressivity there must exist a model morphism

   $$g \colon ((X, \gamma), f) \to ((X', \gamma'), f')$$

   such that
   $$q_{X'}(g(x_1), g(x_2)) = q_{S(A)}(f^\dagger(w_1), f^\dagger(w_2)),$$

   and therefore a $j \in J$ such that

   $$q_{X'}(m_j \circ e_j(x_1), m_j \circ e_j(x_2)) = q_{S(A)}(f^\dagger(w_1), f^\dagger(w_2)).$$

   However, since $m_j \in M$, we have that m_j is Q-preserving, therefore

   $$q_{I_j}(e_j(x_1), e_j(x_2)) = q_{S(A)}(f^\dagger(w_1), f^\dagger(w_2)).$$

   Thus since p_j has an underlying Q-functor

   $$q_{\coprod_{<e_j>} I_j}(p_j \circ e_j(x_1), p_j \circ e_j(x_2)) \geq q_{S(A)}(f^\dagger(w_1), f^\dagger(w_2)),$$

   but $f^\dagger$ also has an underlying Q-functor and Q is a partial order, so

   $$q_{\coprod_{<e_j>} I_j}(w_1, w_2) = q_{S(A)}(f^\dagger(w_1), f^\dagger(w_2)),$$

   from which we deduce that $f^\dagger$ is Q-preserving.

$\square$

As already alluded to, given the above result, with the additional assumption that coproducts of models exist, it is easy to show that cospans of Q-preserving models also exist.

Assumption 8. We extend Assumption 7 (page 130) as follows:

18. The category $\mathbf{Mod}(A, \alpha)$ has binary coproducts.

The following easy result is a direct consequence of Theorem 6.4.

Corollary 6.5. *Given the conditions of Assumption 8, if the L-algebra (A, α) is expressive for $\mathbf{Mod}(A, \alpha)$, then for every pair of Q-preserving models, there exists a cospan of Q-preserving models in $\mathbf{Mod}(A, \alpha)_o$.*

Proof. Given two Q-preserving models X_1 and X_2, by assumption their coproduct exists, and by Theorem 6.4 the coproduct factors via a Q-preserving model, say X_3, and this induces an obvious cospan between X_1 and X_2. □

From Proposition 6.3, Theorem 6.4, and Corollary 6.5, we obtain our main expressivity result - an abstract, category theoretic, characterisation of expressivity.

Theorem 6.6. *Given the conditions of Assumption 8, an L-algebra (A, α) is expressive for $\mathbf{Mod}(A, \alpha)$, if and only if, $\mathbf{Mod}(A, \alpha)$ has a behavioural skeleton $\mathbf{BSkel}(A, \alpha)$.*

The conditions of Assumption 8 are precisely those required to prove our characterisation result (Theorem 6.6), and may appear slightly strange, or awkward to use. However, it is possible to show that they follow from appropriate conditions on the category $\mathbb{X}$ and the functor T. Essentially what is required is that $\mathbb{X}$ has enough colimits, and that $\mathbb{X}_o$ has a proper factorisation system (Definition A.19), the monomorphisms of which are preserved by T.

Assumption 9. We extend Assumption 6 (page 118) as follows:

13. The category $\mathbb{X}$ has small conical colimits.
14. The category $\mathbb{X}_o$ has a factorisation system (E, M) (Definition A.18).
15. M is a subclass of those morphisms in $\mathbb{X}_o$ that are Q-embeddings.
16. E is a subclass of those morphisms in $\mathbb{X}_o$ with surjective underlying functions.
17. The category $\mathbb{X}_o$ is E-cowellpowered (Definition A.21).
18. T preserves M, i.e. $m \in M \Rightarrow T(m) \in M$.

Using these assumptions Theorem 6.6 can be restated as follows. Here it should be noted that, even though the morphisms in the class M of the factorisation system are Q-embeddings, the models in the behavioural skeleton need only be Q-preserving. This is because the factorisation system in $\mathbb{X}_o$ is not directly used to construct the models of the behavioural skeleton, but rather to induce the factorisation system of $\mathbf{Mod}(A, \alpha)_o$.

Corollary 6.7. *Given the conditions of Assumption 9, an L-algebra (A, α) is expressive for $\mathbf{Mod}(A, \alpha)$, if and only if, $\mathbf{Mod}(A, \alpha)$ has a behavioural skeleton $\mathbf{BSkel}(A, \alpha)$.*

Proof. We have to show that the premises of Theorem 6.6 hold. Firstly we observe that by Theorem 4.24, $\mathbf{Mod}(A, \alpha)$ has small conical colimits.

To show that the factorisation system of $\mathbb{X}_o$ lifts to $\mathbf{Mod}(A, \alpha)_o$ we note that in Jacobs and Sokolova (2010) it is observed that if T preserves M, and the members of M are monomorphisms, then the factorisation system of $\mathbb{X}_o$ lifts to $\mathbf{CoAlg}(T)_o$, and it is easy to see that this extends to $\mathbf{Mod}(A, \alpha)_o$.

Finally, since the morphisms in E are epimorphisms, given a span in $\mathbf{Mod}(A, \alpha)_o$ where the underlying morphisms are in E, there is an isomorphism between the two so defined E-quotient objects in $\mathbf{Mod}(A, \alpha)_o$, if and only if, there is an isomorphism between the underlying E-quotient objects in $\mathbb{X}_o$. Therefore $\mathbf{Mod}(A, \alpha)_o$ is E-cowellpowered. □

Remark 6.8. The lifting of a factorisation system for $\mathbb{X}_o$ to the category $\mathbf{CoAlg}(T)_o$ (as in the above proof) is also examined in Kurz (2000, Section 1.3), and this follows previous work on the application of factorisation systems to the study of categories of algebras, for example see Adámek et al. (1990).

6.2 Parametric and Strong Behavioural Skeletons

So far we have looked at behavioural skeletons $\mathbf{BSkel}(A, \alpha)$ where the objects are Q-preserving models of $\mathbf{Mod}(A, \alpha)$, and seen that under certain mild assumptions on the category $\mathbb{X}$, that $\mathbf{BSkel}(A, \alpha)$ characterises expressivity of (A, α). However, it turns out in practice that often we want to work with models that have additional properties beyond being Q-preserving. We therefore introduce the notion of a parametric behavioural skeleton, where the parametricity is in the subclass of models of $\mathbf{Mod}(A, \alpha)$ that define the subcategory for which we take the skeleton.

Definition 6.9. Given the conditions of Assumption 6, and a subclass M of the models of $\mathbf{Mod}(A, \alpha)$ that are Q-preserving, then the skeleton of the full subcategory of $\mathbf{Mod}(A, \alpha)$ given by M is a **parametric behavioural skeleton** of $\mathbf{Mod}(A, \alpha)$, and denoted $\mathbf{PBSkel}_M(A, \alpha)$, if it has the following properties:

1. For every model X in $\mathbf{Mod}(A, \alpha)$, there exists a model Y in $\mathbf{PBSkel}_M(A, \alpha)$, and a morphism $f: X \to I_{\mathbf{PBSkel}_M(A,\alpha)}(Y)$ in $\mathbf{Mod}(A, \alpha)_o$. Here the functor

$$I_{\mathbf{PBSkel}_M(A,\alpha)}: \mathbf{PBSkel}_M(A, \alpha) \to \mathbf{Mod}(A, \alpha)$$

is the inclusion functor. We say that X **factors via** Y.

2. For every pair of models X_1 and X_2 in $\mathbf{PBSkel}_M(A, \alpha)$ there exists a cospan

$$X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$$

in $\mathbf{PBSkel}_M(A, \alpha)_o$.

If in addition the theory map of every model in M is a monomorphism, then the category $\mathbf{PBSkel}_M(A, \alpha)$ is said to be a **strong behavioural skeleton** of $\mathbf{Mod}(A, \alpha)$.

The first thing to note, is that in the proof of Proposition 6.3, no assumption was made that $\mathbf{BSkel}(A, \alpha)$ contained a representative from all equivalence classes of isomorphic Q-preserving models, thus the result also holds for parametric behavioural skeletons.

Proposition 6.10. *Given the conditions of Assumption 6, if* $\mathbf{Mod}(A, \alpha)$ *has a parametric behavioural skeleton* $\mathbf{PBSkel}_M(A, \alpha)$*, for some class* M*, then* (A, α) *is expressive for* $\mathbf{Mod}(A, \alpha)$*.*

Theorem 6.4 on the other hand, clearly does not hold in general for parametric behavioural skeletons, as expressivity is only strong enough to force the existence of a Q-preserving model, it cannot impose any additional structure that might be required of some arbitrary subclass M of Q-preserving models. For example, if the carriers of our T-coalgebras had a topology, and the Q-preserving models were those with continuous injective theory maps, and the subclass M consisted of models with theory maps that were topological embeddings, then expressivity is only strong enough to construct a model with a continuous injective theory map, and in general this need not be a topological embedding.

So what use are parametric behavioural skeletons, if they only characterise expressivity when M is the class of all Q-preserving models of $\mathbf{Mod}(A, \alpha)$? Well, Proposition 6.10 says that if a class M can be found such that $\mathbf{PBSkel}_M(A, \alpha)$ is a parametric behavioural skeleton of $\mathbf{Mod}(A, \alpha)$, then (A, α) is expressive for $\mathbf{Mod}(A, \alpha)$. To find such a class M, one is primarily faced with the task of showing that every model factors via a model in M, and this is often easier if the models of M have additional properties (see for example Example 6.3).

To proceed we shall consider the class M to be defined to consist of those models with theory maps taken from a subclass of the Q-preserving morphisms of $\mathbb{X}_o$ that we shall also refer to as M. It should be noted though, that not every morphism in M need be the theory map of a model. For example it may not have the target $S(A)$.

We now choose M to be a subclass of the Q-preserving morphisms of $\mathbb{X}_o$ such that there exists a class E of morphisms in $\mathbb{X}_o$, and together (E, M) is a factorisation system for $\mathbb{X}_o$. Typically the morphisms of M will also be monomorphisms to ensure the unique diagonalisation property of the factorisation system, but at this stage we do not require this, so we do not assume it.

Assumption 10. We extend Assumption 6 (page 118) as follows:

13. The category $\mathbb{X}_o$ has a factorisation system (E, M) (Definition A.18).

14. M is a subclass of those morphisms in $\mathbb{X}_o$ that are Q-preserving.

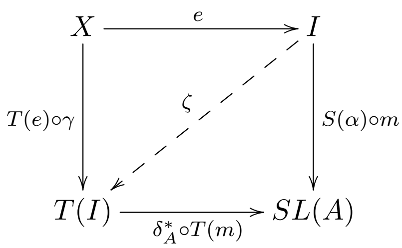
Under these assumptions we find, given a particular technical condition involving M, T, and δ^*_A (Definition 4.14), that models factor via models with theory maps in M.

Proposition 6.11. *Given the conditions of Assumption 10, if*

$$m \in M \Rightarrow \delta^*_A \circ T(m) \in M,$$

then every model in $\mathbf{Mod}(A, \alpha)$ *factors via a model whose theory map is in* M*.*

Proof. Consider a model $((X, \gamma), f)$ in $\mathbf{Mod}(A, \alpha)$. Then by the factorisation system there exists $e \in E$ and $m \in M$ such that $f = m \circ e$, and by the definition of a model, the perimeter of the following diagram commutes



Then by assumption $\delta^*_A \circ T(m) \in M$, so by the diagonalisation property of the factorisation system, there exists a unique $\zeta \colon I \to T(I)$ making the diagram commute.

Thus $((I, \zeta), m)$ is a model in $\mathbf{Mod}(A, \alpha)$ with theory map $m \in M$, and e is the model morphism by which $((X, \gamma), f)$ factors via $((I, \zeta), m)$. □

Now that we have conditions that yield Q-preserving models (possibly with additional properties) via which other models factor, we also need cospans of such models in order to create a parametric behavioural skeleton. This is easy to do, and we use the same technique that we used for Corollary 6.5.

Assumption 11. We extend Assumption 10 (page 137) as follows:

15. The category $\mathbb{X}$ has binary coproducts.

With this additional assumption we get the expressivity result we are looking for.

Corollary 6.12. *Given the conditions of Assumption 11, if*

$$m \in M \Rightarrow \delta_A^* \circ T(m) \in M,$$

then the models of $\mathbf{Mod}(A, \alpha)$ *with theory maps in* M*, define a parametric behavioural skeleton* $\mathbf{PBSkel}_M(A, \alpha)$*, and* (A, α) *is expressive for* $\mathbf{Mod}(A, \alpha)$*.*

Proof. By Proposition 6.11 every model in $\mathbf{Mod}(A, \alpha)$ factors via a model with a theory map in M, and following our slight abuse of notation, we also use M to describe the class of models with theory maps in M. Then since $\mathbb{X}$ has binary coproducts, by Theorem 4.24 the coproduct of every pair of models in M exists, and by Proposition 6.11 again, factors via a model in M. Thus we have cospans of models in M.

Hence the full subcategory of $\mathbf{Mod}(A, \alpha)$ given by the models in M defines a parametric behavioural skeleton $\mathbf{PBSkel}_M(A, \alpha)$, and so by Proposition 6.10, we have that (A, α) is expressive for $\mathbf{Mod}(A, \alpha)$. □

This result is a generalisation of Wilkinson (2012b, Corollary 35), which in turn closely follows Klin (2007, Theorem 4.2) and Jacobs and Sokolova (2010, Theorem 4).

To apply Corollary 6.12 one typically uses the fact that M is closed under composition (Proposition A.20), and splits the condition

$$m \in M \Rightarrow \delta_A^* \circ T(m) \in M$$

into $m \in M \Rightarrow T(m) \in M$, and $\delta_A^* \in M$. The former is often very easy to show, and guides the choice of M, and the latter is often quite difficult, and is where the bulk of the work lies.

In many cases though, Corollary 6.12 is not applicable. This is because the unique diagonalisation property of the factorisation system (E, M) typically forces the morphisms of M to have injective underlying functions, and as discussed in Section 6.1, this is sometimes too strong a condition to ask of the theory maps of the models of a behavioural skeleton. In this case what is likely to happen is that δ_A^* fails to be in M (Example 6.4).

As well as providing a tool for proving expressivity, parametric behavioural skeletons also provide a way of showing that final models exist.

Assumption 12. We extend Assumption 6 (page 118) as follows:

13. M is a subclass of those morphisms in $\mathbb{X}_o$ that are Q-preserving.

14. The category $\mathbb{X}_o$ is M-wellpowered (Definition A.21).

15. The category $\mathbb{X}$ has small coproducts.

Proposition 6.13. *Given the conditions of Assumption 12, if* $\mathbf{Mod}(A, \alpha)$ *has a strong parametric behavioural skeleton* $\mathbf{PBSkel}_M(A, \alpha)$*, then* $\mathbf{Mod}(A, \alpha)_o$ *has a final object.*

Proof. Since $\mathbb{X}_o$ is M-wellpowered, $\mathbf{PBSkel}_M(A, \alpha)$ is small, and thus by Theorem 4.24, the coproduct of all objects in $\mathbf{PBSkel}_M(A, \alpha)$ exists as an object in $\mathbf{Mod}(A, \alpha)$. But then since every object in $\mathbf{Mod}(A, \alpha)$ factors via an object in $\mathbf{PBSkel}_M(A, \alpha)$, so does the coproduct. Call this object Z. For any other object in $\mathbf{PBSkel}_M(A, \alpha)$, the inclusion morphism in the coproduct composes with the factoring morphism from the coproduct to Z, to give a morphism to Z.

Now given any object in $\mathbf{Mod}(A, \alpha)$ it will factor via an object in $\mathbf{PBSkel}_M(A, \alpha)$, and thus also via Z, and since $\mathbf{PBSkel}_M(A, \alpha)$ is strong, the theory map of Z is a monomorphism, and so the morphism to Z will be unique. Therefore Z is a final object in $\mathbf{Mod}(A, \alpha)_o$. □

If in Proposition 6.13 the parametric behavioural skeleton $\mathbf{PBSkel}_M(A, \alpha)$ is not strong, then the above proof only allows us to infer that $\mathbf{Mod}(A, \alpha)_o$ has a weakly final object. However, in some cases it actually has a final object. For example in the case of finitely branching Labelled Transition Systems there is a final coalgebra, and it is a model for the initial algebra of the logic given by $(\mathrm{tt} \mid \wedge \mid \langle l \rangle)$, however the theory map is not injective, as non-bisimilar states can have the same theory under this logic.

6.3 Bisimulation via Internal Models

As we have seen, if we are interested in bisimulation we should enrich over the category **Set**, and the Q-preserving models will then be those with injective theory maps. Also from Section 3.2 we know that the monomorphisms in $\mathbb{X}_o$ are precisely those with injective underlying functions. Therefore we should look to construct a parametric behavioural skeleton from a subclass of the models with theory maps that are monomorphisms. In Wilkinson (2012b) such models were called internal models.

Definition 6.14. Given a class M of monomorphisms in $\mathbb{X}_o$, we define the category $\mathbf{IntMod}_M(A,\alpha)$ of **internal models** of (A,α) to be the full subcategory of $\mathbf{Mod}(A,\alpha)$ where the theory maps are in M, and write

$$I_{\mathbf{IntMod}_M(A,\alpha)} \colon \mathbf{IntMod}_M(A,\alpha) \to \mathbf{Mod}(A,\alpha)$$

for the corresponding inclusion functor.

We parameterise by the class M as we hope to apply Corollary 6.12, and this is typically done by requiring that the members of M are preserved by T. In Example 4.17 the Giry functor does not preserve all monomorphisms, but does preserve a particular subclass of them, and we shall exploit this in Example 6.3.

Internal models can be thought of as a generalisation of the canonical models of Kripke semantics (Blackburn et al., 2001). A canonical model is a model of a modal logic constructed from the syntax itself. The idea is that when trying to prove completeness, by the way the canonical model is constructed from the syntax, for every formula that is not derivable, one can find a state that witnesses that the formula is not valid.

In such a canonical model the possible worlds are the theories of the logic. In our setup $S(A)$ is the collection of all possible theories of (A,α), so an obvious question is when can we construct a model from $S(A)$, i.e. when can we put a T-coalgebra structure on $S(A)$ such that it becomes a model for (A,α)?

In general this cannot be done. However in Schröder and Pattinson (2009) (following Jacobs (2001); Kupke et al. (2005); Kurz and Rosický (2012) - see also Section 4.5), for the standard logical connection between **BA** and **Set** (Example 3.1), conditions are given for the existence of a (not necessarily unique) model with carrier set $S(A)$. From this they derive a strong completeness result.

Internal models extend this idea, and are models built over subsets of $S(A)$, moreover, if there is a largest internal model, then it can be regarded as the canonical model (in the Kripke sense). However, if the carrier set of the largest internal model is a strict subset of $S(A)$, it may fail to yield any kind of completeness result.

Our present interest in internal models is not with regards completeness though, for we have made no mention of proof systems etc., but rather as a means to address expressivity by way of Corollary 6.12.

In Section 4.3 we looked at the standard bivalent formulation of Hennessy-Milner logic for Labelled Transition systems (Example 4.20), and a fuzzy variant from which we removed negation (Example 4.21). We shall now proceed to investigate the expressivity of these two logics.

Example 6.1 (Bivalent logic for bisimulation of LTSs). *Continuing Example 4.20 we observe that the category* **Set** *can be given the factorisation system* (E, M)*, where* E *is the class of all surjective functions, and* M *is the class of all injective functions. Moreover, the functor* $T(X) = \mathcal{P}(\Sigma \times X)$ *preserves injective functions.*

So if for an L-algebra (A, α) δ^*_A *is injective, then by Corollary 6.12,* (A, α) *is expressive for bisimulation of those Labelled Transition Systems that are models for* (A, α)*.*

From the definition of δ^*_A *(Definition 4.14), we see that* $\delta^*_A = S(\delta_{S(A)} \circ L\rho_A) \circ \sigma_{TS(A)}$*, and from the counit of the logical connection (Example 3.1)*

$$\delta^*_A(v) = \{[a_l] \in L(A) \mid v \in (\delta_{S(A)} \circ L\rho_A)([a_l])\},$$

and so from the unit of the logical connection and the form of $\delta_{S(A)}$ *from Example 4.20, we finally have*

$$\delta^*_A(v) = \{[a_l] \in L(A) \mid \forall (l', s) \in v, l' = l \text{ and } a \in s\}.$$

To show that δ^*_A *is injective we consider* $v, v' \in TS(A)$ *such that* $v \neq v'$*. We need to show that there exists a formula* $a \in A$*, and a label* $l \in \Sigma$*, such that (without loss of generality) there exists an ultrafilter* $s \in \mathbf{BA}(A, \mathbf{2})$*, with* $(l, s) \in v$ *and* $a \in s$*, but for all* $(l', s') \in v'$*, either* $l' \neq l$ *or* $a \notin s$*.*

In the case of finitely branching Labelled Transition Systems (finite powerset functor), using the fact that all $s \in \mathbf{BA}(A, \mathbf{2})$ *are ultrafilters, it is indeed possible to find an* $[a_l]$ *to distinguish* $\delta^*_A(v)$ *and* $\delta^*_A(v')$ *(Jacobs and Sokolova, 2010, Theorem 9). However, in the case of unbounded non-determinisim, since our logic only has finite conjunctions, this is not possible.*

Hence we can deduce that Hennessy-Milner logic is expressive for bisimulation of finitely branching Labelled Transition Systems (Hennessy and Milner, 1980, 1985).

Example 6.2 (Fuzzy logic for bisimulation of LTSs). *Continuing Example 4.21 we observe that, as in Example 6.1, the category* **Set** *can be given the factorisation system* (E, M)*, where* E *is the class of all surjective functions, and* M *is the class of all injective functions, and the functor* $T(X) = \mathcal{P}(\Sigma \times X)$ *preserves the injective functions.*

This time we find that

$$\delta^*_A(v)([a_l]) = (\delta_{S(A)} \circ L\rho_A)([a_l])(v),$$

and this means

$$\delta^*_A(v)([a_l]) = \inf_{\substack{(l', s) \in v \\ l' = l}} s(a).$$

In this case δ_A^* is unlikely to be injective, even in the finitely branching case, since the property we relied upon in Example 6.1 was that the s were ultrafilters. In other words, for all $a \in A$, either $a \in s$ or $\neg a \in s$, and this allowed us to assert the existence of the element of A we required. We have no equivalent property in our formulation of the fuzzy case. However this should not be surprising, as whilst we were forced to discard $a \vee \neg a = tt$ as this is not valid in fuzzy logic, we took the stronger action of discarding negation completely (Example 4.21). We had no justification for doing this other than expediency.

There are many other examples in the literature that are directly ammenable to the internal models approach to proving expressivity for bisimulation. A good source of examples can be found in Jacobs and Sokolova (2010), and we shall briefly cover one of them. The significance of this example is that it illustrates why the category $\mathbf{IntMod}_M(A, \alpha)$ is parameterised by the class of morphisms M.

Example 6.3 (Bivalent logic for bisimulation of Markov Processes)**.** *Markov Processes are given by coalgebras for the Giry functor on measurable spaces (Example 4.17). For the logic we take the logical connection between* **MSL** *and* **Meas** *(Example 3.4), and add modalities of the form L_r, indexed by $r \in \mathbb{Q} \cap [0, 1]$ (Example 4.3).*

To apply Corollary 6.12, first we note that since sigma algebras are closed under intersection **Meas** *is topological over* **Set** *(Adámek et al., 1990, Definition 21.1), so by Adámek et al. (1990, Theorem 21.16)* **Meas** *is cocomplete.*

Then in Jacobs and Sokolova (2010, Section 3.1) it is observed that morphisms with surjective underlying functions, and morphisms with injective underlying functions and surjective inverse image functions, form a factorisation system (E, M) for **Meas***. Moreover, the Giry functor $\mathcal{G}$ is observed to preserve M.*

For the modalities given in Example 4.3 there is a natural choice for the natural transformation δ, and in Jacobs and Sokolova (2010, Theorem 17) it is shown that δ^* is componentwise in M.

Thus Corollary 6.12 allows us to conclude that the logic given by the syntax

$$\phi ::= \mathrm{tt} \mid \phi \wedge \phi \mid L_r\phi \quad \textit{where } r \in \mathbb{Q} \cap [0, 1]$$

is expressive for bisimulation of Markov Processes (Desharnais et al., 2002).

6.4 Simulation via R-Models

In the previous section we looked at using internal models to prove expressivity for bisimulation. If on the other hand we are interested in simulation, then instead we must enrich over the category $\mathbf{Set}_R$ for some choice of the type R. This corresponds to taking Q to be $\mathbf{2}$, the two element set with the usual order (Example 5.1). The Q-preserving models will then be those with R-reflecting theory maps, and in Wilkinson (2012a) such models were called R-models.

Definition 6.15. The category $\mathbf{R\text{–}Mod}(A, \alpha)$ of ***R*-models** of (A, α) is the full subcategory of $\mathbf{Mod}(A, \alpha)$ where the theory maps are R-reflecting. A function $f\colon X \to Y$ is R-reflecting, if for all $x, y \in X$, if $f(x)R_Y f(y)$ then $xR_X y$. We write

$$I_{\mathbf{R\text{–}Mod}(A,\alpha)}\colon \mathbf{R\text{–}Mod}(A, \alpha) \to \mathbf{Mod}(A, \alpha)$$

for the corresponding inclusion functor.

To demonstrate the use of R-models we shall continue the study of simulation for Labelled Transition Systems from Example 4.22 and Example 4.23.

First we recall that, as mentioned in Section 6.1 already, for Labelled Transition Systems, mutual simulation does not imply bisimulation. In other words, given states x and y, it may be the case that x simulates y, and y simulates x, but this does not mean that x and y need be bisimilar. Thus we should not expect the R-reflecting models to have injective theory maps, and this probably precludes the use, à la Proposition 6.11, of a factorisation system in $\mathbb{X}_o$ to show that all models factor via an R-reflecting model (the unique diagonalisation property of a factorisation system (E, M) typically forces the morphisms of M to be monomorphisms).

In this example we shall therefore proceed differently (Wilkinson, 2012a).

Example 6.4 (Bivalent logic for simulation of LTSs)**.** *We recall from Example 4.22 that we have a logical connection given by the dual adjunction between* $\mathbf{MSL}$ *and* $\mathbf{Set}_R$ *from Example 3.2, with the type R set to preorders. To this we have added the functor L from Example 4.4, which adds the modal operators* $\langle l \rangle$*, and the functor*

$$T(X, R_X) = (\mathcal{P}(\Sigma \times X), R_{\mathcal{P}(\Sigma \times X)})$$

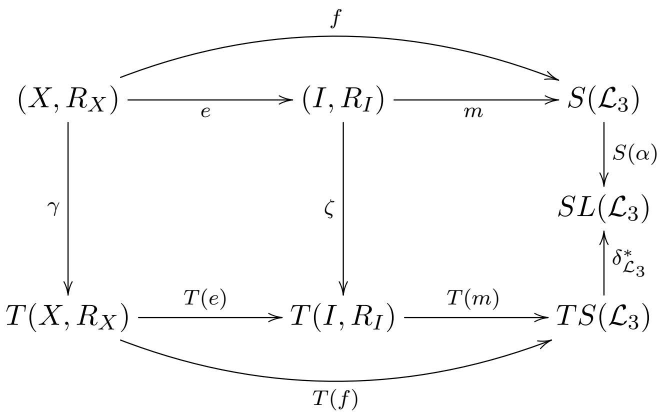
from Example 4.12. This then yielded

$$\begin{aligned}
\delta_{(X,R_X)}\colon LP(X, R_X) &\to PT(X, R_X) \\
\top &\mapsto \mathcal{P}(\Sigma \times X) \\
[u_l] &\mapsto \{w \in \mathcal{P}(\Sigma \times X) \mid \exists (l', x) \in w,\ l' = l \text{ and } x \in u\} \\
[u_{l_1}] \wedge [v_{l_2}] &\mapsto \delta_X([u_{l_1}]) \cap \delta_X([v_{l_2}]),
\end{aligned}$$

and from this, following a similar line of reasoning as for Example 6.1, we get

$$\delta^*_A(v) = \{[a_l] \in L(A) \mid \exists (l', s) \in v, l' = l \text{ and } a \in s\}.$$

To proceed we shall consider the following commuting diagram



where we first choose $(((X, R_X), \gamma), f)$ *to be any model of the logic*

$$\mathcal{L}_3 \ni \phi ::= \mathrm{tt} \mid p \mid \phi \wedge \phi \mid \langle l \rangle \phi \quad \textit{where } l \in \Sigma \textit{ and } p \in \mathit{Var},$$

and then construct $(((I, R_I), \zeta), m)$ *such that it is an* R*-model for the* L*-algebra* $(\mathcal{L}_3, \alpha)$*. Here, in a slight abuse of notation,* $\mathcal{L}_3$ *refers both to the logic and its Lindenbaum-Tarski algebra. We can do this as there is an obvious bijection between filters/ultrafilters of a logic, and the corresponding filters/ultrafilters of its Lindenbaum-Tarski algebra.*

In actual fact, we now choose to restrict ourselves to finitely branching Labelled Transition Systems (the finite powerset functor), for then we can use the result from Example 6.1 that Hennessy-Milner logic ($\mathcal{L}_3$ *with negation) is expressive for bisimulation of finitely branching Labelled Transition Systems.*

Specifically, using a functor $U_{\mathrm{disc}} \colon \mathbf{Set}_R \to \mathbf{Set}_R$ *that assigns to every object the discrete preorder (in other words, forgets the current preorder), any model* $(((X, R_X), \gamma), f)$ *can be quotiented via a surjective* T*-coalgebra morphism* $e \colon (X, \gamma) \to (I, \zeta)$*, where* I *is a subset of the ultrafilters of Hennessy-Milner logic. There is then an obvious function* $m \colon I \to U_{\mathrm{disc}} S(\mathcal{L}_3)$ *that maps an ultrafilter in Hennessy-Milner logic to the corresponding filter in* $\mathcal{L}_3$ *by throwing out all the formulae that contain negation, and moreover,* $U_{\mathrm{disc}}(f) = m \circ e$*. The way to think of this, is that a filter in* $\mathcal{L}_3$ *lists all the possible future things a state in a transition system can do, and an ultrafilter in Hennessy-Milner logic explicitly adds all the things it cannot do.*

Now $S(\mathcal{L}_3)$ *is ordered by inclusion, and it is easy to see that* I *can be given a preorder* R_I *such that* e *is* R*-preserving, and* m *is both* R*-preserving and* R*-reflecting. Specifically, we can order the ultrafilters of* I *by the inclusion order on their negation free subsets.*

Further, since e *is surjective,* $((I, U_{\mathrm{disc}}(\zeta)), U_{\mathrm{disc}}(m))$ *is a model for* $\mathcal{L}_3$*. What remains to be shown is that* ζ *preserves the preorder* R_I*, for if that is the case, then* $(((I, R_I), \zeta), m)$ *is an* R*-model for* $\mathcal{L}_3$*. It is easily seen that this is the case if* T *preserves* R*-reflecting morphisms, and* $\delta^*_{\mathcal{L}_3}$ *is* R*-reflecting. The former is not very hard to show, so what* *remains is to show that* $\delta^*_{\mathcal{L}_3}$ *is R-reflecting. In fact we shall show this for an arbitrary L-algebra* (A, α)*.*

To do this suppose $v \not{R}_{TS(A)} v'$*, then*

$$v \not{R}_{TS(A)} v' \Leftrightarrow \exists (l, s) \in v.\ \forall (l, s') \in v' \text{ either } l \neq l' \text{ or } s \not{R}_{S(A)} s'.$$

Now, our plan is to find an $[a_l] \in L(A)$ *such that* $a \in s$*, and for all* $(l', s') \in v'$*, either* $l \neq l'$*, or* $a \notin s'$*.*

The first case to consider is if there is no $(l', s') \in v'$ *such that* $l = l'$*, for then we can take* $a = \top$*. If that is not the case, then there is a finite set of pairs* $(l, s') \in v'$ *such that* $s \not{R}_{S(A)} s'$*. Now* $s \not{R}_{S(A)} s'$ *means* $s \not\subseteq s'$*, so it is possible to find an element of* s *that is not in any of the* s' *(do it pairwise and then take the meet - we can do this as* v' *is finite). Therefore* $\delta^*_A(v) \not\subseteq \delta^*_A(v')$*, which means* $\delta^*_A(v) \not{R}_{SL(A)} \delta^*_A(v')$*, and thus* δ^*_A *is R-reflecting.*

We have thus shown that every model for $\mathcal{L}_3$ *factors via an R-model. Further, since* $\mathbf{Set}_R$ *has coproducts, by Theorem 4.24 the coproduct of any pair of R-models, as models, exists, and since any model factors via an R-model, this yields a cospan of R-models. Therefore the R-models of* $\mathcal{L}_3$ *form a parametric behavioural skeleton, and so by Proposition 6.10, the logic* $\mathcal{L}_3$ *is expressive for simulation of finitely branching Labelled Transition Systems (van Glabbeek, 2001).*

Note, it is easy to see that δ^*_A *is not injective, since if* v *and* v' *differ only in that for some* $s \in S(A)$*, we have* $(l, s) \in v$ *and* $(l, s) \notin v'$*, but there exists an* $s' \in S(A)$ *such that* $s \subset s'$*, and both* $(l, s') \in v$ *and* $(l, s') \in v'$*, then clearly* $\delta^*_A(v) = \delta^*_A(v')$*.*

Thus an attempt to use the factorisation system of Example 6.1 to invoke Corollary 6.12 would have failed.

The fuzzy logic version is less interesting, but only because we have not properly considered the role of negation.

Example 6.5 (Fuzzy logic for simulation of LTSs)**.** *Example 4.23 is similar to Example 4.22, however, the failure to have a corresponding expressivity result in the bisimulation case (Example 6.2), means we cannot repeat the procedure of Example 6.4.*

This is not to say that an expressivity result cannot be proven in the fuzzy case, but only that our decision to remove negation from the logic in Example 4.21, without properly considering what to put in its place, means that we do not have the tools we need to hand.

6.5 Discussion

As we have seen, expressivity of a modal logic can be characterised by the existence of a particular structure to the category of its models. However, as we have also seen, determining the existence of this structure can be far from easy. For expressivity with respect to bisimulation, the existence of a factorisation system for the category $\mathbb{X}_o$ can prove very useful (Corollary 6.12), but in the case of simulation, this is often not the case (Example 6.4).

However what Example 6.4 does show, is that proving expressivity for bisimulation may be a stepping-stone to proving expressivity for simulation, or expressivity for some other form of behavioural comparability. This is consistent with the proof of Theorem 6.4, where a Q-preserving model is constructed by "quotienting out" bisimilar states. The question then is how to turn this into a general technique for proving expressivity, and more work needs to be done to understand this.

The first step is probably to look at existing expressivity proofs in the literature, especially those not formulated in terms of coalgebras, and try to recast them into our framework - for example the simulation result for Markov Processes of Desharnais et al. (2003), or the approximation results for Markov Processes of Desharnais et al. (2004).

There is also a body of work in the literature (Klin, 2005, 2007; Schröder, 2008), where given certain conditions, any functor T admits a modal logic that is expressive for all T-coalgebras. This is typically phrased in terms of the existence of a collection of polyadic modalities and their corresponding predicate liftings, but from this we can construct a functor L and a natural transformation $\delta\colon LP \Rightarrow PT$. This approach is different in spirit from the approach taken in this chapter. Here the authors prove the existence of an expressive logic, whereas our approach is to try to establish whether a given logic is expressive. Finally, the above results only work for bisimulation, however recent work has started to extend this to simulation as well (Kapulkin et al., 2010, 2012).

Chapter 7

Conclusions and Future Work

In this thesis we have presented our contribution towards a framework for the systematic study of coalgebraic modal logic, where we have particularly emphasised the role that enrichment should play in this framework.

The main technical conclusions from our work are as follows:

1. Enrichment is an essential part of the framework of coalgebraic modal logic, and it controls the choice of behavioural questions that a modal logic is intended to capture.
2. The choice of behavioural questions is limited only by our imagination, as it is determined by the choice of a commutative unital quantale.
3. Expressivity of an L-algebra is determined by the structure of its category of models. This then provides an avenue by which powerful tools of category theory like factorisation systems, can be brought to bear when trying to prove expressivity.

However, the main conclusion that we feel should be drawn, is that category theory is the natural language in which to frame modal logic. As a consequence, we feel it will prove fruitful to further investigate which ready-made tools in the mathematical toolbox of category theory can be applied to the study of modal logic.

7.1 Future Work

Many-Sorted Logics

The first possible extension to our work would be to look at the full many-sorted enriched logical connections of Kurz and Velebil (2011), and try to extend this to many-sorted coalgebraic modal logic. Some work has already been done on many-sorted coalgebraic modal logics, but not as far as we are aware in an enriched setting - for example Jacobs (2001); Schröder and Pattinson (2007a); Kurz and Petrişan (2008). We would therefore look to combine these approaches with the work presented in this thesis.

Traces

It may not have escaped the observant reader that there is one class of behavioural comparisons that we have not mentioned - traces.

The first thing we can say is that besides asking whether two states have the same trace (or set of traces in the case of non-determinism), we can also ask if one trace is a prefix of another, or ask the distance between two traces, if there is a metric on the set of labels say. Therefore we can ask of pairs of traces all the same sorts of questions we can ask of pairs of states - "Are they equal?", "Is one greater than the other?", "How far apart are they?". So perhaps we can handle traces directly in our framework by treating them as the "states" to be compared, and not the actual states themselves?

The coalgebraic approach to finite traces (Hasuo et al., 2007) replaces our functor T with the composite functor TF and a distributive law $\lambda\colon FT \Rightarrow TF$, here T is in actual fact a monad and represents the branching type, and F represents the transition type. Then the initial F-algebra (the elements of which are the finite traces) lifts to a final $\overline{F}$-coalgebra, where $\overline{F}$ is the lifting of F to the Kleisli category $\mathcal{K}\ell(T)$ of T, and any TF-coalgebra corresponds to a $\overline{F}$-coalgebra. Thus for any TF-coalgebra there is a unique $\overline{F}$-coalgebra morphism between the corresponding $\overline{F}$-coalgebra and the final $\overline{F}$-coalgebra. This is called the finite trace map of the TF-coalgebra (Hasuo et al., 2007).

This suggests that perhaps we should simply try to instantiate the category $\mathbb{X}$ in our framework with the Kleisli category for T, however in the case of infinite traces the situation is more complicated. In this case one uses the final F-coalgebra (the elements of which are the maximal, possibly infinite traces), but in general this does not lead to a final $\overline{F}$-coalgebra in $\mathcal{K}\ell(T)$, and the resulting trace map is an op-lax $\overline{F}$-coalgebra morphism (Cîrstea, 2010). Lax and op-lax $\overline{F}$-coalgebra morphisms also appear in Hasuo (2006, 2010), where they are related to forward and backward simulations respectively.

Finally, the path based modal logics that are typically used when reasoning about infinite traces are 2-sorted - there are formulae that represent states, and formulae that represent paths (Cîrstea, 2010). Therefore a full treatment of traces and path based modal logics is likely to require a many-sorted variant of our framework.

Modularity

Various authors have looked at modularity of coalgebras and coalgebraic modal logics, for example Cîrstea (2006) and Cîrstea and Pattinson (2007), and in Kurz and Petrişan (2008) it is shown that even if the resulting composite logic is single-sorted, a many-sorted logic is required during the construction of this composite logic from the component logics. Some work has also been done to explore the decidability of such modularly defined coalgebraic modal logics (Schröder and Pattinson, 2007a).

More generally, in a monoidal category the tensor $\otimes$ is often regarded as "parallel composition", and some authors take the view that composition should be regarded as a colimit (Goguen, 1991). We therefore propose investigating how our framework could be extended to incorporate some of these ideas. For example, we could assume that the categories $\mathbb{A}$ and $\mathbb{X}$ are also monoidal categories (in addition to $\mathbb{V}$), and that the functors U, V, P, S, L and T are monoidal functors.

There is also the dual notion of forgetting, or hiding, parts of a system's structure - putting the lid on the box so we cannot see the internal workings. This introduces the notion of τ transitions - transitions that we cannot observe - and weak bisimulation, something that as far as the author is aware, has not been given a satisfactory coalgebraic treatment.

Approximations of Probabilistic Systems

In Section 5.3 we discussed several papers on Probabilistic PDL and approximations of Markov Processes, and we believe that translating this work to our framework would form an interesting case study, and help to clarify some of the questions raised in Section 5.3.

General Proof Method for Expressivity

As was discussed in Section 6.5, we are currently lacking a general method for proving expressivity in cases other than bisimulation. This warrants further investigation.

Proof Systems, Soundness, and Completeness

Finally, as discussed in Section 4.5, a coalgebraic modal logic can be given a proof system, and then questions of soundness and completeness arise. Our systematic approach to handling models of coalgebraic modal logics may provide tools for answering these questions.

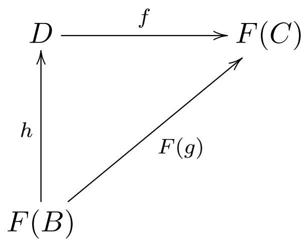
Appendix A

Category Theory

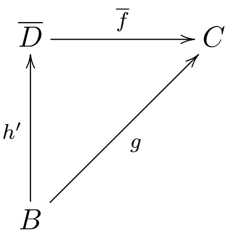
This chapter is not intended to cover the basics of category theory, for that the reader is advised to try any of the many very good books on the subject, for example Adámek et al. (1990) or Mac Lane (1997). Instead this chapter summarises some of the more advanced topics we use in the rest of the text.

A.1 Initial Lifts and Fibrations

Definition A.1. Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, and a morphism $f\colon D \to F(C)$ in $\mathbb{D}$, then an F**-initial lift** of f is a morphism $\overline{f}\colon \overline{D} \to C$ in $\mathbb{C}$ such that $F(\overline{f}) = f$, and for any other pair of morphisms $g\colon B \to C$ in $\mathbb{C}$, and $h\colon F(B) \to D$ in $\mathbb{D}$ such that

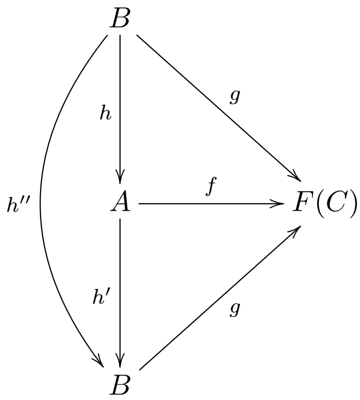


there exists a unique morphism $h'\colon B \to \overline{D}$ such that



and $F(h') = h$.

Proposition A.2. *Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, the following commuting diagram in $\mathbb{D}$*



and the F-initial lifts $\overline{f}\colon \overline{A} \to C$, and $\overline{g}\colon \overline{B} \to C$, then

$$\overline{h''} = \overline{h'} \circ \overline{h},$$

and

$$\overline{h''} = 1_{\overline{B}} \iff h'' = 1_B.$$

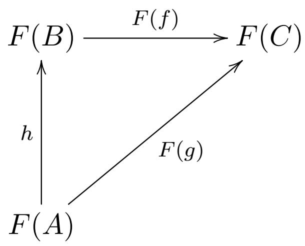
Proof. The morphisms $\overline{h}$, $\overline{h''}$, and $\overline{h''}$ clearly exist by the universal property of the F-initial lifts $\overline{f}$ and $\overline{g}$. That $\overline{h''} = \overline{h'} \circ \overline{h}$ follows from the uniqueness property associated with the F-initial lift of g.

Now if $\overline{h''} = 1_{\overline{B}}$, since $h'' = F(\overline{h''})$, we must have $h'' = 1_B$. Conversely, if $h'' = 1_B$, then clearly $1_{\overline{B}}$ is a possible choice for $\overline{h''}$, and by uniqueness, it is the only one. □

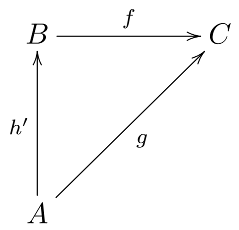
If in Proposition A.2, $A = B$, and h' and h'' equal 1_A, but $\overline{f}$ and $\overline{g}$ remain distinct F-initial liftings, then $\overline{h}$ and $\overline{h'}$ define an isomorphism between $\overline{A}$ and $\overline{B}$.

Corollary A.3. *Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, F-initial liftings are unique up to a unique isomorphism.*

Definition A.4. Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, a morphism $f\colon B \to C$ in $\mathbb{C}$ is **cartesian**, if for all pairs of morphisms $g\colon A \to C$ in $\mathbb{C}$, and $h\colon F(A) \to F(B)$ in $\mathbb{D}$ such that



there exists a unique morphism $h' \colon A \to B$ such that



and $F(h') = h$.

Definition A.5. A functor $F \colon \mathbb{C} \to \mathbb{D}$ is called a **fibration** if for every $C \in \mathbf{obj}|\mathbb{C}|$, and every morphism $f \colon D \to F(C)$ in $\mathbb{D}$, there exists a cartesian morphism $f' \colon D' \to C$ in $\mathbb{C}$, such that $F(f') = f$.

We have the following trivial proposition.

Proposition A.6. *A functor $F \colon \mathbb{C} \to \mathbb{D}$ is a fibration, if and only if, every morphism $f \colon D \to F(C)$ in $\mathbb{D}$ has an F-initial lifting.*

A.2 Concrete Categories

For many categories $\mathbb{A}$ the objects can be viewed as sets with some additional structure, and the morphisms as functions that preserve that structure. Thus by considering $\mathbb{A}$ simply as a category we lose this additional information. The way to retain access to this additional information is through a construction known as a concrete category.

The material in this section is taken from Adámek et al. (1990).

First we need a few preliminary definitions.

Definition A.7 (Adámek et al. (1990), Definition 3.27)**.** Let $F \colon \mathbb{A} \to \mathbb{B}$ be a functor.

1. F is called an **embedding** provided that F is injective on morphisms.
2. F is called **faithful** provided that all the hom-set restrictions

   $$F_{A,A'} \colon \mathbb{A}(A, A') \to \mathbb{B}(F(A), F(A'))$$

   are injective.
3. F is called **full** provided that all hom-set restrictions are surjective.
4. F is called **amnestic** provided that an $\mathbb{A}$-isomorphism f is an identity whenever $F(f)$ is an identity.

Remark A.8 (Adámek et al. (1990), Remark 3.28). Notice that a functor is:

1. an embedding if and only if it is faithful and injective on objects,
2. an isomorphism if and only if it is full, faithful, and bijective on objects.

Now that the preliminaries are out of the way we can provide the general definition of a concrete category.

Definition A.9 (Adámek et al. (1990), Definition 5.1)**.**

1. Let $\mathbb{X}$ be a category. A **concrete category** over $\mathbb{X}$ is a pair $(\mathbb{A}, U)$, where $\mathbb{A}$ is a category and $U\colon \mathbb{A} \to \mathbb{X}$ is a faithful functor. Sometimes U is called the **forgetful** (or **underlying**) functor of the concrete category and $\mathbb{X}$ is called the **base category** for $(\mathbb{A}, U)$.
2. A concrete category over **Set** is called a **construct**.

Remark A.10. For a pair of categories $\mathbb{A}$ and $\mathbb{X}$ there may be more than one choice of faithful functor $U\colon \mathbb{A} \to \mathbb{X}$ giving a concrete category over $\mathbb{X}$.

Definition A.11 (Adámek et al. (1990), Definition 5.28)**.** A concrete category $(\mathbb{A}, U)$ over $\mathbb{X}$ is said to be **(uniquely) transportable** provided that for every $\mathbb{A}$-object A and every $\mathbb{X}$-isomorphism $f\colon U(A) \to X$ there exists a (unique) $\mathbb{A}$-object B with $U(B) = X$ such that $f\colon A \to B$ is an $\mathbb{A}$-isomorphism.

In the category **Set**, monomorphisms are injective functions, and epimorphisms are surjective functions. This leads to the following results.

Proposition A.12 (Adámek et al. (1990), Proposition 7.37)**.**

1. *Every representable functor preserves monomorphisms, i.e., if $F\colon \mathbb{A} \to$ **Set** is representable and if f is a monomorphism in $\mathbb{A}$, then $F(f)$ is a monomorphism in* **Set** *(i.e., an injective function).*
2. *Every faithful functor reflects monomorphisms, i.e., if $F\colon \mathbb{A} \to \mathbb{B}$ is faithful and $F(f)$ is a $\mathbb{B}$-monomorphism, then f is an $\mathbb{A}$-monomorphism.*

Corollary A.13 (Adámek et al. (1990), Corollary 7.38)**.** *In any construct all morphisms with injective underlying functions are monomorphisms. When the underlying functor is representable, the monomorphisms are precisely the morphisms with injective underlying functions.*

Proposition A.14 (Adámek et al. (1990), Proposition 7.44)**.** *Every faithful functor reflects epimorphisms.*

Corollary A.15 (Adámek et al. (1990), Corollary 7.45)**.** *In any construct all morphisms with surjective underlying functions are epimorphisms.*

We shall also need the following generalisations of some of the above results.

Definition A.16 (Adámek et al. (1990), Definition 10.5)**.** A pair $(A, (f_i)_I)$, consisting of an object A, and a family of morphisms $f_i \colon A \to A_i$ indexed by some class I, is called a **mono-source**, if for any pair of morphisms $r, s \colon B \to A$, if $f_i \circ r = f_i \circ s$ for all i, then $r = s$.

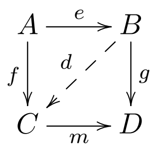
Proposition A.17 (Adámek et al. (1990), Definition 10.7)**.** *Representable functors preserve mono-sources (i.e., if $G \colon \mathbb{A} \to \mathbf{Set}$ is a representable functor, and $(A, (f_i)_I)$ is a mono-source in $\mathbb{A}$, then $(G(A), (G(f_i))_I)$ is a mono-source in* **Set***).*

A.3 Factorisation Systems

Often we need to be able to factorise morphisms. The standard approach to this is via a factorisation system (Adámek et al., 1990).

Definition A.18. In a category $\mathbb{C}$, a pair (E, M) of classes of morphisms is called a **factorisation system** for $\mathbb{C}$, if the following hold:

1. If $e \in E$, and h an isomorphism in $\mathbb{C}$, then if $h \circ e$ exists, $h \circ e \in E$.
2. If $m \in M$, and h an isomorphism in $\mathbb{C}$, then if $m \circ h$ exists, $m \circ h \in M$.
3. $\mathbb{C}$ has (E, M)**-factorisations**; i.e. every morphism f in $\mathbb{C}$ factors as $f = m \circ e$, with $m \in M$ and $e \in E$.
4. $\mathbb{C}$ has the **unique** (E, M)**-diagonalisation property**; i.e. every commuting square in $\mathbb{C}$, with $e \in E$ and $m \in M$, has a unique diagonal d such that the following commutes



Definition A.19. In a category $\mathbb{C}$ a factorisation system (E, M) is called **proper**, if E is a subclass of the epimorphisms of $\mathbb{C}$, and if M is a subclass of the monomorphisms of $\mathbb{C}$.

The classes E and M of a factorisation system are closed under composition. We formalise this in the following proposition, which is a statement of parts of Adámek et al. (1990, Propositions 14.6, 14.9).

Proposition A.20. *Let $\mathbb{C}$ be a category with a factorisation system (E, M).*

1. *Each of E and M is closed under composition.*
2. *If $f \circ g \in M$ and $f \in M$, then $g \in M$.*
3. *If $f \circ g \in E$ and $g \in E$, then $f \in E$.*

A class of monomorphisms defines a notion of subobject in a category, and it is often important that for every object in a category its collection of subobjects is a set. The following definitions are standard (Adámek et al., 1990).

Definition A.21. Given a class M of monomorphisms in a category $\mathbb{C}$ we define the following:

1. An M**-subobject** of an object A in $\mathbb{C}$ is a pair (S, m), where $m\colon S \to A$ is in M.
2. Two M-subobjects (S, m) and (S', m') of A are **isomorphic** if there exists an isomorphism $h\colon S \to S'$ such that $m = m' \circ h$.
3. $\mathbb{C}$ is M**-wellpowered** if no object in $\mathbb{C}$ has a proper class of pairwise non-isomorphic M-subobjects. Here by pairwise non-isomorphic we mean that any pair of distinct subobjects are non-isomorphic.

Dually, for a class E of epimorphisms we can define an E**-quotient object** of an object A as a pair (e, Q), where $e\colon A \to Q$ is in E. The obvious dual notion to $\mathbb{C}$ being M-wellpowered is that $\mathbb{C}$ is E**-cowellpowered**.

A.4 Preservation and Creation of Limits and Colimits

The following definitions are standard (Adámek et al., 1990).

Definition A.22. Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, let $\mathbb{J}$ denote any small category, and $D\colon \mathbb{J} \to \mathbb{C}$ any functor, then we say that:

1. F **preserves limits of** D, if and only if, $(L, \phi_j)_{j \in \mathbb{J}}$ is a limit of D implies $(FL, F(\phi_j))_{j \in \mathbb{J}}$ is a limit of FD.
2. F **preserves limits of type** $\mathbb{J}$, if and only if, F preserves limits of D for all $D\colon \mathbb{J} \to \mathbb{C}$.
3. F **preserves limits**, or is **continuous**, if and only if, F preserves limits of type $\mathbb{J}$ for all small categories $\mathbb{J}$.

Definition A.23. Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, let $\mathbb{J}$ denote any small category, and $D\colon \mathbb{J} \to \mathbb{C}$ any functor, then we say that:

1. F **creates limits of** D, if and only if, $(L, \phi_j)_{j\in\mathbb{J}}$ is a limit of FD implies there exists a unique cone $(L', \phi'_j)_{j\in\mathbb{J}}$ of D such that $F(L', \phi'_j)_{j\in\mathbb{J}} = (L, \phi_j)_{j\in\mathbb{J}}$, and moreover, $(L', \phi'_j)_{j\in\mathbb{J}}$ is a limit of D.

2. F **creates limits of type** $\mathbb{J}$, if and only if, F creates limits of D for all $D\colon \mathbb{J} \to \mathbb{C}$.

3. F **creates limits**, if and only if, F creates limits of type $\mathbb{J}$ for all small categories $\mathbb{J}$.

Definition A.24. Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, let $\mathbb{J}$ denote any small category, and $D\colon \mathbb{J} \to \mathbb{C}$ any functor, then we say that:

1. F **preserves colimits of** D, if and only if, $(L, \phi_j)_{j\in\mathbb{J}}$ is a colimit of D implies $(FL, F(\phi_j))_{j\in\mathbb{J}}$ is a colimit of FD.

2. F **preserves colimits of type** $\mathbb{J}$, if and only if, F preserves colimits of D for all $D\colon \mathbb{J} \to \mathbb{C}$.

3. F **preserves colimits**, or is **cocontinuous**, if and only if, F preserves colimits of type $\mathbb{J}$ for all small categories $\mathbb{J}$.

Definition A.25. Given a functor $F\colon \mathbb{C} \to \mathbb{D}$, let $\mathbb{J}$ denote any small category, and $D\colon \mathbb{J} \to \mathbb{C}$ any functor, then we say that:

1. F **creates colimits of** D, if and only if, $(L, \phi_j)_{j\in\mathbb{J}}$ is a colimit of FD implies there exists a unique cocone $(L', \phi'_j)_{j\in\mathbb{J}}$ of D such that $F(L', \phi'_j)_{j\in\mathbb{J}} = (L, \phi_j)_{j\in\mathbb{J}}$, and moreover, $(L', \phi'_j)_{j\in\mathbb{J}}$ is a colimit of D.

2. F **creates colimits of type** $\mathbb{J}$, if and only if, F creates colimits of D for all $D\colon \mathbb{J} \to \mathbb{C}$.

3. F **creates colimits**, if and only if, F creates colimits of type $\mathbb{J}$ for all small categories $\mathbb{J}$.

A.5 Natural Transformations in Several Variables

The notion of a natural transformation between two functors is well known, but it is usually only presented in its most basic form, where the components are only indexed by a single variable. However the definition can be readily extended to a many variable form.

Recall the definition from Mac Lane (1997, II.3) of the product of two categories.

Definition A.26. Given two categories $\mathbb{B}$ and $\mathbb{C}$, the **product** of $\mathbb{B}$ and $\mathbb{C}$ is a category $\mathbb{B} \times \mathbb{C}$ with the following data:

1. $\mathbf{obj}|\mathbb{B} \times \mathbb{C}| = \mathbf{obj}|\mathbb{B}| \times \mathbf{obj}|\mathbb{C}|$ i.e. objects in $\mathbb{B} \times \mathbb{C}$ are pairs consisting of an object from $\mathbb{B}$ and an object from $\mathbb{C}$.
2. A morphism $(B, C) \to (B', C')$ is a pair (f, g) of arrows $f: B \to B'$ and $g: C \to C'$, and composition of two morphisms

$$(B, C) \xrightarrow{(f,g)} (B', C') \xrightarrow{(f',g')} (B'', C'')$$

is defined by the composites in $\mathbb{B}$ and $\mathbb{C}$ as

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g).$$

Remark A.27. If we construct the product $\mathbb{B}^{op} \times \mathbb{C}$, then the objects are still pairs from $\mathbb{B}$ and $\mathbb{C}$, but a morphism $(f, g): (B, C) \to (B', C')$ is given by the pair $f: B' \to B$ and $g: C \to C'$, and the composite

$$(B, C) \xrightarrow{(f,g)} (B', C') \xrightarrow{(f',g')} (B'', C'')$$

is defined as

$$(f', g') \circ (f, g) = (f \circ f', g' \circ g).$$

Since $\mathbb{B} \times \mathbb{C}$ is a category, we can define functors $F, G: \mathbb{B} \times \mathbb{C} \to \mathbb{D}$, and natural transformations $\alpha: F \Rightarrow G$, with components $\alpha_{(B,C)}$ for every object $(B, C) \in \mathbf{obj}|\mathbb{B} \times \mathbb{C}|$. However, we can also consider naturality in B or C separately. The proposition below shows that naturality can be examined variable-by-variable.

Definition A.28. Given a pair of functors $F, G: \mathbb{B} \times \mathbb{C} \to \mathbb{D}$, a collection of morphisms $\alpha_{B,C}: F(B, C) \to G(B, C)$, one for every pair $(B, C) \in \mathbf{obj}|\mathbb{B} \times \mathbb{C}|$, is **natural** in B, if for each $C \in \mathbf{obj}|\mathbb{C}|$, the components $\alpha_{B,C}$ for all $B \in \mathbf{obj}|\mathbb{B}|$ define a natural transformation

$$\alpha_{-,C}: F(-, C) \Rightarrow G(-, C).$$

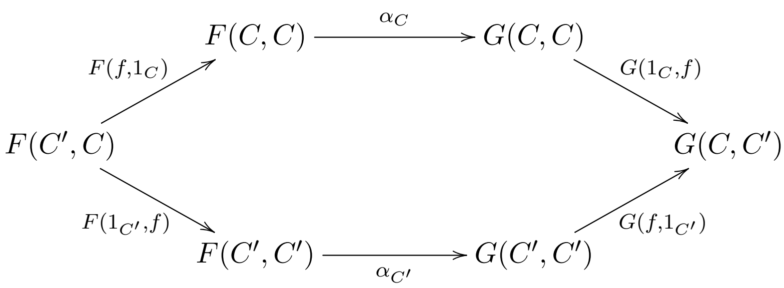
Similarly for α natural in C.

Proposition A.29 (Mac Lane (1997), II.3 Proposition 2)**.** *Given a pair of functors $F, G: \mathbb{B} \times \mathbb{C} \to \mathbb{D}$, a collection of morphisms $\alpha_{B,C}: F(B, C) \to G(B, C)$ is a natural transformation $\alpha: F \Rightarrow G$, if and only if, α is natural in B for each $C \in$ **obj**$|\mathbb{C}|$, and natural in C for each $B \in$ **obj**$|\mathbb{B}|$.*

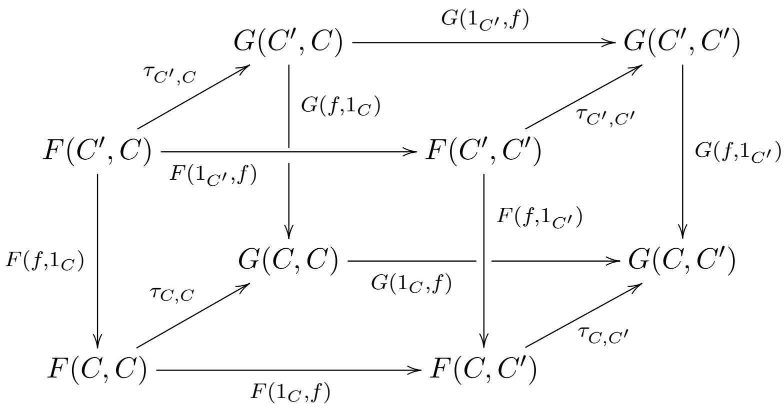
A.6 Dinatural Transformations

The material in this section comes from Mac Lane (1997, IX.4).

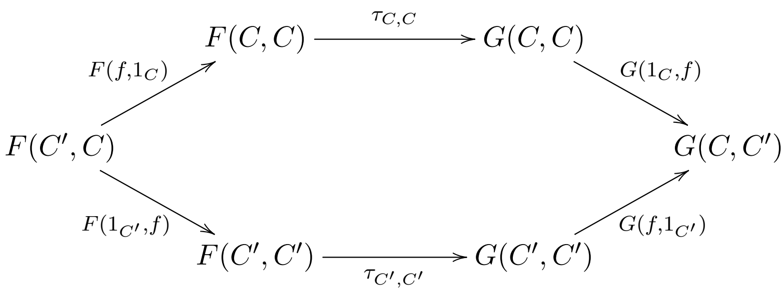
Definition A.30. Given a pair of functors $F, G\colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$, we define a **dinatural transformation** $\alpha\colon F \Rightarrow G$ as a collection of components $\alpha_C\colon F(C,C) \to G(C,C)$, one for each $C \in \mathbf{obj}|\mathbb{C}|$, such that for all morphisms $f\colon C \to C'$ in $\mathbb{C}$ the following diagram commutes



Now noting Remark A.27, we could consider any natural transformation $\tau\colon F \Rightarrow G$ for functors $F, G\colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$. Then for any morphism $f\colon C \to C'$ we have the following commuting cube



from which we see that the following two paths from opposing corners $F(C', C)$ and $G(C, C')$ commute



Thus τ defines a dinatural transformation α, where the component $\alpha_C = \tau_{C,C}$.

Not every dinatural transformation arises from an ordinary natural transformation though.

In addition to the general definition of a dinatural transformation, it makes sense to consider several common special cases where F and G are "dummy" in one or more variables.

1. If both F and G are dummy in the first variable, then α is simply a natural transformation between functors from $\mathbb{C}$ to $\mathbb{D}$.
2. If both F and G are dummy in the second variable, then α is simply a natural transformation between functors from $\mathbb{C}^{op}$ to $\mathbb{D}$, and this can be thought of as a natural transformation between contravariant functors from $\mathbb{C}$ to $\mathbb{D}$.
3. If F is dummy in the first variable and G dummy in the second variable then the following must commute

$$\begin{array}{ccc} F(C) & \xrightarrow{\alpha_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \uparrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\alpha_{C'}]{} & G(C') \end{array}$$

   which is a natural transformation between a covariant F and a contravariant G.
4. If F is dummy in the second variable and G dummy in the first variable then the following must commute

$$\begin{array}{ccc} F(C) & \xrightarrow{\alpha_C} & G(C) \\ {\scriptstyle F(f)}\uparrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\alpha_{C'}]{} & G(C') \end{array}$$

   which is a natural transformation between a contravariant F and a covariant G.
5. If F is dummy in both variables then the following must commute

$$\begin{array}{ccc} D & \xrightarrow{\alpha_C} & G(C,C) \\ {\scriptstyle \alpha_{C'}}\downarrow & & \downarrow{\scriptstyle G(1_C,f)} \\ G(C',C') & \xrightarrow[G(f,1_{C'})]{} & G(C,C') \end{array}$$

   and α is called an **extranatural transformation** from D to G.

6. If G is dummy in both variables then the following must commute

$$\begin{array}{ccc} F(C',C) & \xrightarrow{F(1_{C'},f)} & F(C',C') \\ {\scriptstyle F(f,1_C)}\downarrow & & \downarrow{\scriptstyle \alpha_{C'}} \\ F(C,C) & \xrightarrow[\alpha_C]{} & D \end{array}$$

and α is called an **extranatural transformation** from F to D.

Extranatural transformations occur in combinations with ordinary natural transformations, so we make the following general definition.

Definition A.31. Given a pair of functors

$$F\colon \mathbb{C}^{op} \times \mathbb{C} \times \mathbb{A} \to \mathbb{B} \qquad\qquad G\colon \mathbb{A} \times \mathbb{D}^{op} \times \mathbb{D} \to \mathbb{B}$$

we define a **natural transformation** $\alpha\colon F \Rightarrow G$ as a collection of components

$$\alpha_{C,A,D}\colon F(C,C,A) \to G(A,D,D),$$

one for each triple of objects $(C, A, D) \in \mathbf{obj}|\mathbb{C} \times \mathbb{A} \times \mathbb{D}|$, such that the following hold:

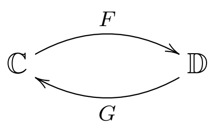
1. for C and D fixed, $\alpha_{C,-,D}$ is natural (in the ordinary sense) in A,
2. for A and D fixed, $\alpha_{-,A,D}$ is extranatural in C,
3. for C and A fixed, $\alpha_{C,A,-}$ is extranatural in D.

Remark A.32. Any of the categories $\mathbb{A}$, $\mathbb{C}$, and $\mathbb{D}$ can be replaced by a product of several categories, and in each case naturality in a product argument may be replaced by naturality in each argument of the tuple (that makes up the product argument) where the others are fixed. The ordinary natural transformation case is covered by Definition A.28 and Proposition A.29, but the extranatural case is analogous.

A.7 Adjunctions

A very powerful idea in category theory is that of an adjunction. Here we summarise the basic idea (and results) as typically given for pairs of covariant functors (Mac Lane, 1997), and then in the next section, present the corresponding results for pairs of contravariant functors - a so called dual adjunction, or "adjunction on the right".

Definition A.33. Given a pair of categories $\mathbb{C}$ and $\mathbb{D}$, an **adjunction from $\mathbb{C}$ to $\mathbb{D}$** is given by a triple (F, G, Φ), where F and G are covariant functors as follows



and

$$\Phi_{C,D} \colon \mathbb{D}(F(C), D) \Rightarrow \mathbb{C}(C, G(D))$$

is a natural isomorphism. We write $F \dashv G$, or $(F \dashv G, \Phi)$ if we want to be explicit about the choice of Φ.

Definition A.34. Given an adjunction $(F \dashv G, \Phi)$, we say that F is the **left adjoint of** G, and G is the **right adjoint of** F. Further, we say $\Phi^{-1}_{C,D}(f) \colon F(C) \to D$ is the **left adjunct of** $f \colon C \to G(D)$, and $\Phi_{C,D}(g) \colon C \to G(D)$ is the **right adjunct of** $g \colon F(C) \to D$.

Adjunctions have lots of interesting properties, many of which are themselves sufficient to define the concept of an adjunction. We start with the following.

Definition A.35. Given an adjunction $(F \dashv G, \Phi)$, the **unit** is a natural transformation $\eta \colon 1_{\mathbb{C}} \Rightarrow GF$ given by

$$\eta_C = \Phi_{C,F(C)}(1_{F(C)}),$$

and the **counit** is a natural transformation $\varepsilon \colon FG \Rightarrow 1_{\mathbb{D}}$ given by

$$\varepsilon_D = \Phi^{-1}_{G(D),D}(1_{G(D)}).$$

The next three propositions correspond to Mac Lane (1997, IV.1 Theorem 1).

Proposition A.36. *Given an adjunction $(F \dashv G, \Phi)$ then:*

1. *The right adjunct of any $g \colon F(C) \to D$ is given by*

$$\Phi_{C,D}(g) = G(g) \circ \eta_C.$$

2. *The left adjunct of any $f \colon C \to G(D)$ is given by*

$$\Phi^{-1}_{C,D}(f) = \varepsilon_D \circ F(f).$$

Proposition A.37. *Given an adjunction* $(F \dashv G, \Phi)$ *then the following hold:*

1. η_C *is a universal morphism from* C *to* G*, i.e. any other morphism* $f\colon C \to G(D)$ *from* C *to* G *factors as*
$$f = G(g) \circ \eta_C,$$
for a unique $g\colon F(C) \to D$ *(the left adjunct of* f*).*

2. ε_D *is a universal morphism from* F *to* D *i.e. any other morphism* $g\colon F(C) \to D$ *from* F *to* D *factors as*
$$g = \varepsilon_D \circ F(f),$$
for a unique $f\colon C \to G(D)$ *(the right adjunct of* g*).*

Proposition A.38. *Given an adjunction* $(F \dashv G, \Phi)$ *then the following hold:*

$$\begin{aligned} G\varepsilon \circ \eta G &= 1_G \\ \varepsilon F \circ F\eta &= 1_F. \end{aligned}$$

The following proposition, giving left and right adjuncts for composite morphisms, follows from the naturality of Φ.

Proposition A.39. *Given an adjunction* $(F \dashv G, \Phi)$*, and morphisms* $f\colon C \to G(D)$*,* $g\colon F(C) \to D$*,* $h\colon C' \to C$ *and* $k\colon D \to D'$*, then the following hold:*

$$\begin{aligned} \Phi_{C,D}(k \circ g) &= G(k) \circ \Phi_{C,D}(g) \\ \Phi_{C,D}(g \circ F(h)) &= \Phi_{C,D}(g) \circ h \\ \Phi^{-1}_{C,D}(f \circ h) &= \Phi^{-1}_{C,D}(f) \circ F(h) \\ \Phi^{-1}_{C,D}(G(k) \circ f) &= k \circ \Phi^{-1}_{C,D}(f). \end{aligned}$$

The following theorem collects together the different alternative definitions of an adjunction, and is very useful when trying to construct an adjunction.

Theorem A.40 (Mac Lane (1997), IV.1 Theorem 2)**.** *Any adjunction* $(F \dashv G, \Phi)$ *is completely determined by any of the following:*

1. *Functors* F*,* G*, and a natural transformation* $\eta\colon 1_{\mathbb{C}} \Rightarrow GF$*, such that each* η_C *is universal from* C *to* G*. Then* Φ *is defined by* $\Phi_{C,D}(g) = G(g) \circ \eta_C$*.*

2. *The functor* G*, and for each* C *in* $\mathbb{C}$*, an object* D_F *in* $\mathbb{D}$*, and a universal morphism* $\eta_C\colon C \to G(D_F)$ *from* C *to* G*. Then the functor* F *has object mapping* $F(C) = D_F$*, and is defined on morphisms* $h\colon C \to C'$*, by* $GF(h) \circ \eta_C = \eta_{C'} \circ h$*.*

3. *Functors* F*,* G*, and a natural transformation* $\varepsilon\colon FG \Rightarrow 1_{\mathbb{D}}$*, such that each* ε_D *is universal from* F *to* D*. Then* Φ^{-1} *is defined by* $\Phi^{-1}_{C,D}(f) = \varepsilon_D \circ F(f)$*.*

4. *The functor F, and for each D in $\mathbb{D}$, an object C_G in $\mathbb{C}$, and a universal morphism $\varepsilon_D\colon F(C_G) \to D$ from F to D. Then the functor G has object mapping $G(D) = C_G$, and is defined on morphisms $k\colon D \to D'$, by $k \circ \varepsilon_D = \varepsilon_{D'} \circ FG(k)$.*

5. *Functors F, G, and natural transformations $\eta\colon 1_{\mathbb{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathbb{D}}$, such that $G\varepsilon \circ \eta G = 1_G$ and $\varepsilon F \circ F\eta = 1_F$. Then Φ is defined by $\Phi_{C,D}(g) = G(g) \circ \eta_C$, and Φ^{-1} by $\Phi^{-1}_{C,D}(f) = \varepsilon_D \circ F(f)$.*

The next theorem is the very useful result that left adjoints preserve colimits (Definition A.24), and right adjoints preserve limits (Definition A.22).

Theorem A.41 (Mac Lane (1997), V.5 Theorem 1)**.** *Given an adjunction $(F \dashv G, \Phi)$, and functors $S\colon \mathbb{I} \to \mathbb{C}$ and $T\colon \mathbb{J} \to \mathbb{D}$, then:*

1. *If S has the colimiting cone $\tau\colon S \Rightarrow \Delta(C)$ in $\mathbb{C}$, where Δ is the diagonal functor $\Delta\colon \mathbb{C} \to \mathbb{C}^{\mathbb{I}}$, then FS has the colimiting cone $F\tau\colon FS \Rightarrow F\Delta(C)$ in $\mathbb{D}$.*

2. *If T has the limiting cone $\tau\colon \Delta(D) \Rightarrow T$ in $\mathbb{D}$, where Δ is the diagonal functor $\Delta\colon \mathbb{D} \to \mathbb{D}^{\mathbb{J}}$, then GT has the limiting cone $G\tau\colon G\Delta(D) \Rightarrow GT$ in $\mathbb{C}$.*

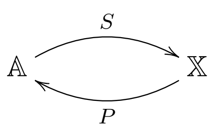
If both the unit and counit of an adjunction are natural isomorphisms, then we can make the following stronger definition.

Definition A.42. Given an adjunction $(F \dashv G, \Phi)$, if the unit $\eta\colon 1_{\mathbb{C}} \Rightarrow GF$, and counit $\varepsilon\colon FG \Rightarrow 1_{\mathbb{D}}$, are both natural isomorphisms, then $(F \dashv G, \Phi)$ is an **equivalence between $\mathbb{C}$ and $\mathbb{D}$**.

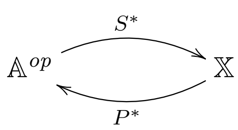
A.8 Dual Adjunctions

In the definition of an adjunction the two functors are covariant, however it is often the case that we have a similar situation, except that the two functors are contravariant. This will lead to the definition of what is known as a dual adjunction, or an "adjunction on the right" (Mac Lane, 1997, IV.2).

First though it should be noted, that if we are given a pair of contravariant functors as follows

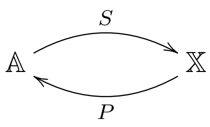


then we can take the opposite of one of the categories and replace the two functors with their covariant equivalents



Then using Definition A.33 we can write $(P^* \dashv S^*, \Phi)$, where Φ is the natural transformation $\Phi\colon \mathbb{A}^{op}(P^*(-), -) \Rightarrow \mathbb{X}(-, S^*(-))$. Thus we can make the following definition.

Definition A.43. Given a pair of categories $\mathbb{A}$ and $\mathbb{X}$, a **dual adjunction from $\mathbb{A}$ to $\mathbb{X}$** is given by a triple (P, S, Φ), where P and S are contravariant functors as follows



and

$$\Phi_{A,X}\colon \mathbb{A}(A, P(X)) \Rightarrow \mathbb{X}(X, S(A))$$

is a natural isomorphism. We write $P \dashv S$, or $(P \dashv S, \Phi)$ if we want to be explicit about the choice of Φ.

Unlike the covariant case, for dual adjunctions we have the following result.

Proposition A.44. *For dual adjunctions the following holds*

$$(P \dashv S, \Phi) \Leftrightarrow (S \dashv P, \Phi^{-1})$$

Proof. By Definition A.43 we have $(P \dashv S, \Phi)$ corresponds to a natural isomorphism $\Phi_{A,X}\colon \mathbb{A}(A, P(X)) \Rightarrow \mathbb{X}(X, S(A))$, and $(S \dashv P, \Psi)$ corresponds to a natural isomorphism $\Psi_{X,A}\colon \mathbb{X}(X, S(A)) \Rightarrow \mathbb{A}(A, P(X))$, but clearly $\Phi^{-1}_{A,X}$ is a suitable choice for $\Psi_{X,A}$ (or $\Psi^{-1}_{X,A}$ for $\Phi_{A,X}$), and thus we have $(P \dashv S, \Phi) \Leftrightarrow (S \dashv P, \Phi^{-1})$. $\square$

Remark A.45. As a result of Proposition A.44, dual adjunctions do not have the concept of a left or right adjoint. In this respect they are symmetric.

Definition A.46. Given a dual adjunction $(P \dashv S, \Phi)$, we say $\Phi_{A,X}(f)\colon X \to S(A)$ is the **dual adjunct of** $f\colon A \to P(X)$, and $\Phi^{-1}_{A,X}(g)\colon \to P(X)$ is the **dual adjunct of** $g\colon X \to S(A)$. We write $f^\flat$ for $\Phi_{A,X}(f)$, and $g^\sharp$ for $\Phi^{-1}_{X,A}(g)$, and note $(f^\flat)^\sharp = f$ and $(g^\sharp)^\flat = g$.

Following Definition A.35, for a dual adjunction we can also define the unit and counit. Once again, as a result of Proposition A.44 there is no real distinction between which is which, though they remain distinct natural transformations. We adopt the following convention.

Definition A.47. Given a dual adjunction $(P \dashv S, \Phi)$, the **unit** is a natural transformation $\rho\colon 1_{\mathbb{A}} \Rightarrow PS$ given by

$$\rho_A = \Phi^{-1}_{S(A),A}(1_{S(A)}),$$

and the **counit** is a natural transformation $\sigma\colon 1_{\mathbb{X}} \Rightarrow SP$ given by

$$\sigma_X = \Phi_{P(X),X}(1_{P(X)}).$$

Dual adjunctions have an analogue to Proposition A.36 as follows.

Proposition A.48. *Given a dual adjunction $(P \dashv S, \Phi)$ then:*

1. *The dual adjunct of any $f\colon A \to P(X)$ is given by*

$$f^\flat = S(f) \circ \sigma_X.$$

2. *The dual adjunct of any $g\colon X \to S(A)$ is given by*

$$g^\sharp = P(g) \circ \rho_A.$$

And an analogue to Proposition A.37.

Proposition A.49. *Given a dual adjunction $(P \dashv S, \Phi)$ then the following hold:*

1. *ρ_A is a universal morphism from A to P, i.e. any other morphism $f\colon A \to P(X)$ from A to P factors as*
$$f = P(f^\flat) \circ \rho_A,$$
for a unique $f^\flat\colon X \to S(A)$ (the dual adjunct of f).

2. *σ_X is a universal morphism from X to S, i.e. any other morphism $g\colon X \to S(A)$ from X to S factors as*
$$g = S(g^\sharp) \circ \sigma_X,$$
for a unique $g^\sharp\colon A \to P(X)$ (the dual adjunct of g).

And also an analogue to Proposition A.38.

Proposition A.50. *Given a dual adjunction $(P \dashv S, \Phi)$ then the following hold:*

$$P\sigma \circ \rho P = 1_P$$
$$S\rho \circ \sigma S = 1_S.$$

And an analogue to Proposition A.39.

Proposition A.51. *Given a dual adjunction $(P \dashv S, \Phi)$, and morphisms $f\colon A \to P(X)$, $g\colon X \to S(A)$, $h\colon A' \to A$ and $k\colon X' \to X$, then the following hold:*

$$\begin{aligned}
(f \circ h)^\flat &= S(h) \circ f^\flat \\
(P(k) \circ f)^\flat &= f^\flat \circ k \\
(g \circ k)^\sharp &= P(k) \circ g^\sharp \\
(S(h) \circ g)^\sharp &= g^\sharp \circ h.
\end{aligned}$$

Just as in Theorem A.40, there is a theorem that collects together the different alternative definitions of a dual adjunction, and is very useful when trying to construct a dual adjunction.

Theorem A.52. *Any dual adjunction $(P \dashv S, \Phi)$ is completely determined by any of the following:*

1. *Contravariant functors P, S, and a natural transformation $\rho\colon 1_{\mathbb{A}} \Rightarrow PS$, such that each ρ_A is universal from A to P. Then Φ^{-1} is defined by $\Phi^{-1}_{X,A}(g) = P(g) \circ \rho_A$.*

2. *The contravariant functor P, and for each A in $\mathbb{A}$, an object X_S in $\mathbb{X}$, and a universal morphism $\rho_A\colon A \to P(X_S)$ from A to P. Then the contravariant functor S has object mapping $S(A) = X_S$, and is defined on morphisms $h\colon A \to A'$, by $PS(h) \circ \rho_A = \rho_{A'} \circ h$.*

3. *Contravariant functors P, S, and a natural transformation $\sigma\colon 1_{\mathbb{X}} \Rightarrow SP$, such that each σ_X is universal from X to S. Then Φ is defined by $\Phi_{A,X}(f) = S(f) \circ \sigma_X$.*

4. *The contravariant functor S, and for each X in $\mathbb{X}$, an object A_P in $\mathbb{A}$, and a universal morphism $\sigma_X\colon X \to S(A_P)$ from X to S. Then the contravariant functor P has object mapping $P(X) = A_P$, and is defined on morphisms $k\colon X \to X'$, by $SP(k) \circ \sigma_X = \sigma_{X'} \circ k$.*

5. *Contravariant functors P, S, and natural transformations $\rho\colon 1_{\mathbb{A}} \Rightarrow PS$ and $\sigma\colon 1_{\mathbb{X}} \Rightarrow SP$, such that $P\sigma \circ \rho P = 1_P$ and $S\rho \circ \sigma S = 1_S$. Then Φ is defined by $\Phi_{A,X}(f) = S(f) \circ \sigma_X$, and Φ^{-1} by $\Phi^{-1}_{X,A}(g) = P(g) \circ \rho_A$.*

Theorem A.41 states that left adjoints preserve colimits, and right adjoints preserve limits. For a dual adjunction we make no distinction between left and right adjoints, and since the functors P and S are contravariant, we have that both P and S map colimits to limits.

Theorem A.53. *Given a dual adjunction* $(P \dashv S, \Phi)$*, and functors* $F\colon \mathbb{I} \to \mathbb{A}$ *and* $G\colon \mathbb{J} \to \mathbb{X}$*, then:*

1. *If* F *has the colimiting cone* $\tau\colon F \Rightarrow \Delta(A)$ *in* $\mathbb{A}$*, where* Δ *is the diagonal functor* $\Delta\colon \mathbb{A} \to \mathbb{A}^{\mathbb{I}}$*, then* SF *has the limiting cone* $S\tau\colon S\Delta(A) \Rightarrow SF$ *in* $\mathbb{X}$*.*

2. *If* G *has the colimiting cone* $\tau\colon G \Rightarrow \Delta(X)$ *in* $\mathbb{X}$*, where* Δ *is the diagonal functor* $\Delta\colon \mathbb{X} \to \mathbb{X}^{\mathbb{J}}$*, then* PG *has the limiting cone* $P\tau\colon P\Delta(X) \Rightarrow PG$ *in* $\mathbb{A}$*.*

Following Definition A.42, if both the unit and counit of a dual adjunction are natural isomorphisms then we can make the following stronger definition.

Definition A.54. Given a dual adjunction $(P \dashv S, \Phi)$, if the unit $\rho\colon 1_{\mathbb{A}} \Rightarrow PS$, and counit $\sigma\colon 1_{\mathbb{X}} \Rightarrow SP$, are both natural isomorphisms, then $(P \dashv S, \Phi)$ is a **dual equivalence between** $\mathbb{A}$ **and** $\mathbb{X}$.

Appendix B

Monoidal Categories

The idea of a monoidal category is that we make abstract some of the properties of the category **Set** that are found to be so useful in mathematics. In this way we can make it explicit when we use these properties, and moreover, we can prove results that only use these properties, and thus that will hold for categories other than **Set**.

The following definition is standard material, see for example Mac Lane (1997, VII.1) or Kelly (1982, Section 1.1).

Definition B.1. A **monoidal category** $\mathbb{V} = (\mathbb{V}_o, \otimes, I, a, l, r)$ has the following data:

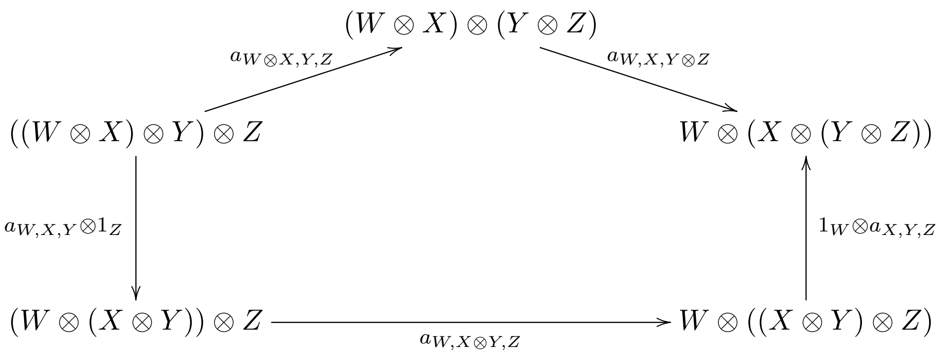
1. a category $\mathbb{V}_o$,
2. a functor $\otimes \colon \mathbb{V}_o \times \mathbb{V}_o \to \mathbb{V}_o$ called the **tensor product**,
3. an object I of $\mathbb{V}_o$ called the **unit**,
4. a natural isomorphism

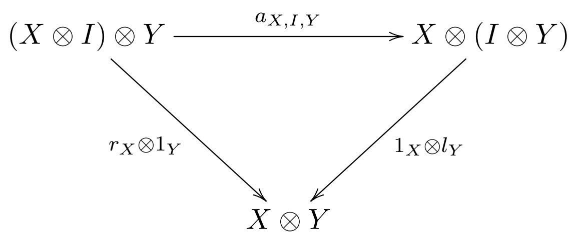
   $$a_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

   called the **associator**,
5. natural isomorphisms

   $$l_X \colon I \otimes X \to X$$
   $$r_X \colon X \otimes I \to X$$

   called the **left unitor** and **right unitor**, such that the following diagrams commute





Remark B.2. In Mac Lane (1997) an additional axiom is required, specifically that $l_I = r_I$, but this can be shown to follow from the other axioms (Eilenberg and Kelly, 1966, II Proposition 1.1), and so most authors do not make it part of the definition.

There is a famous result, a coherence theorem (Mac Lane, 1997, VII.2), that says every diagram of natural transformations formed from $\otimes$, I, 1_{-}, the natural transformations a, l, r, and their inverses, commutes.

Like in the case of the category **Set**, we frequently would like to talk about the elements of and object in a monoidal category. The following definition has been found to be the best statement of this notion, as the subsequent proposition generalises the fact that in **Set** two functions $f, g \colon X \to Y$ are equal if they are the same on all elements of X.

Definition B.3. Given a monoidal category $\mathbb{V}$, if $\mathbb{V}_o$ is locally small, we can define the representable functor

$$\mathbf{elem}|-| = \mathbb{V}_o(I, -) \colon \mathbb{V}_o \to \mathbf{Set},$$

and for any object X of $\mathbb{V}_o$, we say f is an **element of** X, if and only if, $f \in \mathbf{elem}|X|$, i.e. $f \colon I \to X$.

Proposition B.4. *For a monoidal category* $\mathbb{V}$*, with* $\mathbb{V}_o$ *locally small, if* $\mathbf{elem}|-|$ *is faithful, then a pair of morphisms* $f, g \in \mathbb{V}_o(A, B)$ *are equal, if and only if they are the same on all elements of* A*.*

We also frequently want to think of an element of the tensor of two objects X and Y to consist of a pair of elements, one from X, and one from Y. We can do this if the canonical natural transformation below is a natural isomorphism.

Proposition B.5. *For a monoidal category* $\mathbb{V}$*, with* $\mathbb{V}_o$ *locally small, there exits a canonical natural transformation*

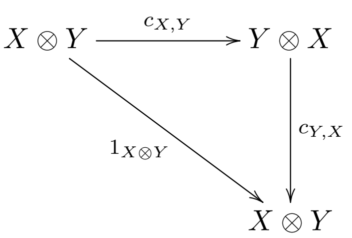
$$\mathbf{elem}|X| \times \mathbf{elem}|Y| \to \mathbf{elem}|X \otimes Y|$$

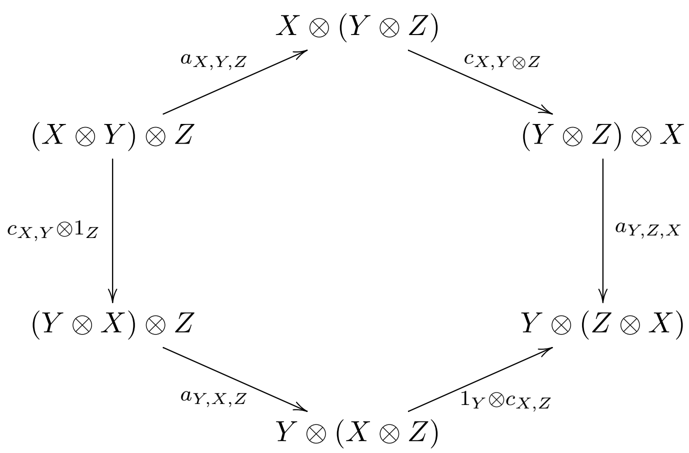
natural in X *and* Y*, given by*

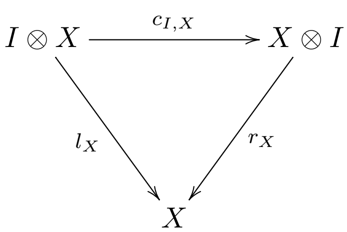
$$I \xrightarrow{l_I^{-1}} I \otimes I \xrightarrow{f \otimes g} X \otimes Y.$$

So far we have only captured the structure of **Set** that corresponds to the formation of the cartesian product of sets. Another important property is that of a symmetry (Mac Lane, 1997; Kelly, 1982).

Definition B.6. Given a monoidal category $\mathbb{V}$, a **symmetry** is a natural isomorphism $c_{X,Y} \colon X \otimes Y \to Y \otimes X$ such that the following diagrams commute







Remark B.7. A monoidal category may have more than one symmetry.

Definition B.8. A monoidal category with a symmetry is called a **symmetric monoidal category**.

Once again, there is a coherence theorem (Mac Lane, 1997, XI.1) for symmetric monoidal categories.

The final bit of structure that **Set** possesses that we require, is the existence of function spaces - given two sets X and Y, the collection of all functions from X to Y is also a set, and this set has certain properties. To do this we define what we mean for a symmetric monoidal category to be closed (Mac Lane, 1997; Kelly, 1982).

Definition B.9. A symmetric monoidal category $\mathbb{V}$ is **closed** if for every object Y in $\mathbb{V}_o$ the functor $-\otimes Y$ has a right adjoint $[Y,-]$, where $[-,-]\colon \mathbb{V}_o \times \mathbb{V}_o \to \mathbb{V}_o$ is a functor called the **internal-hom**, and the unit and counit are denoted

$$d_{X,Y}\colon X \to [Y, X\otimes Y]$$
$$e_{Y,Z}\colon [Y,Z]\otimes Y \to Z$$

with e called **evaluation**.

Proposition B.10 (Kelly (1982), Section 1.8)**.** *Given a symmetric monoidal closed category $\mathbb{V}$, for every object Y in $\mathbb{V}_o$, the morphisms*

$$d_{X,Y}\colon X \to [Y, X\otimes Y]$$
$$e_{Y,Z}\colon [Y,Z]\otimes Y \to Z$$

are natural in X and Z, and extranatural in Y.

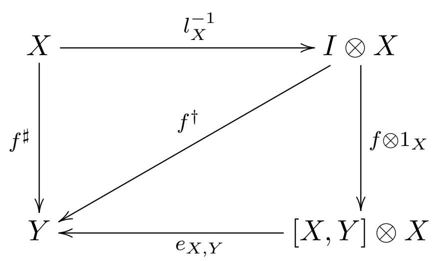
For a symmetric monoidal closed category $\mathbb{V}$, where $\mathbb{V}_o$ is locally small, we have that

$$\mathbb{V}_o(X\otimes Y, Z) \cong \mathbb{V}_o(X, [Y,Z]),$$

and thus

$$\mathbb{V}_o(X,Y) \cong \mathbb{V}_o(I\otimes X, Y) \cong \mathbb{V}_o(I, [X,Y]) = \mathbf{elem}|[X,Y]|\,,$$

which corresponds to the following diagram



where for $f \in \mathbf{elem}|[X,Y]|$, $f^\dagger$ is the transpose of f under the adjunction $-\otimes X \dashv [X,-]$, and $f^\sharp = f^\dagger \circ l_X^{-1}$.

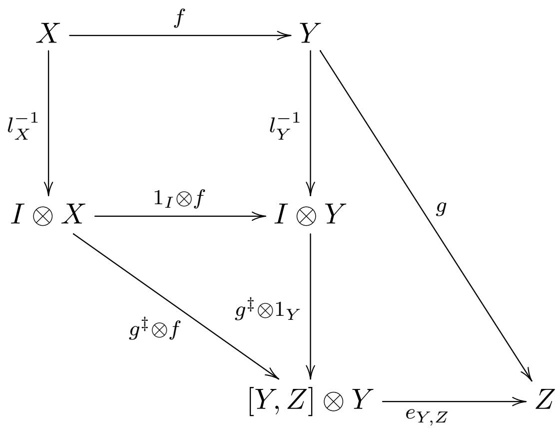
We can summarise this in the following important proposition.

Proposition B.11 (Kelly (1982), Section 1.5)**.** *Given a symmetric monoidal closed category $\mathbb{V}$, with $\mathbb{V}_o$ locally small, for every pairs of objects X and Y in $\mathbb{V}_o$, we have a natural isomorphism*

$$\begin{aligned} \mathbf{elem}|[X,Y]| &\cong \mathbb{V}_o(X,Y) \\ f &\mapsto f^\dagger \circ l_X^{-1} \\ (g \circ l_X)^\ddagger &\leftarrow\!\shortmid g \end{aligned}$$

where $-^\dagger$ and $-^\ddagger$ denote the transposes (in the two directions respectively) under the adjunction $- \otimes X \dashv [X,-]$.

This bijection between the morphisms from X to Y and the elements of $[X,Y]$, means that for $f\colon X \to Y$, $g\colon Y \to Z$, and $g^\ddagger\colon I \to [Y,Z]$, the following commutes



and we see that

$$g \circ f = e_{Y,Z} \circ (g^\ddagger \otimes f) \circ l_X^{-1}.$$

In particular, if f is an element $y\colon I \to Y$ of Y, we write $g(y)$ for $g \circ y$, and we see that $g^\ddagger$ can be regarded as actually being g, and e then evaluates g at y.

For this interpretation to make sense, we need e to also capture associativity of evaluation, but that is precisely what the extranaturality of e from Proposition B.10 guarantees.

Often we will blur the distinction between elements of $\mathbf{elem}|[X,Y]|$ and elements of $\mathbb{V}_o(X,Y)$, and use them interchangeably.

The following natural isomorphisms appear frequently, and are very useful.

Proposition B.12 (Kelly (1982), Section 1.5)**.** *Given a symmetric monoidal closed category $\mathbb{V}$, for every object X in $\mathbb{V}_o$, there is a natural isomorphism*

$$i_X\colon X \to [I,X]$$

given by $i_X = r_Z^\ddagger$ and $i_X^{-1} = 1_{[I,Z]}^\dagger \circ r_{[I,Z]}^{-1}$, where $\dagger$ and $\ddagger$ are transposes under the adjunction $- \otimes X \dashv [X,-]$.

Also, for all objects X, Y, and Z in $\mathbb{V}_o$, there is a natural isomorphism

$$p_{X,Y,Z} \colon [X \otimes Y, Z] \to [X, [Y, Z]]$$

given by

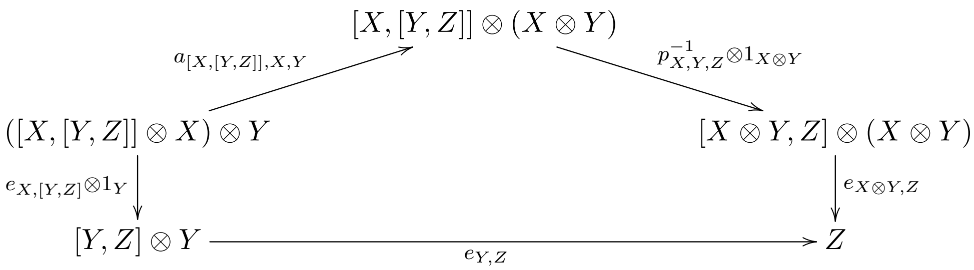
$$p_{X,Y,Z} = ((e_{X\otimes Y,Z} \circ a_{[X\otimes Y,Z],X,Y})^{\ddagger})^{\ddagger},$$

where the inner $\ddagger$ *is the transpose under* $- \otimes Y \dashv [Y, -]$, *and the outer* $\ddagger$ *is the transpose under* $- \otimes X \dashv [X, -]$, *and*

$$p^{-1}_{X,Y,Z} = (e^{\dagger}_{X,[Y,Z]} \circ a^{-1}_{[X,[Y,Z]],X,Y})^{\ddagger},$$

where the inner $\dagger$ *is the transpose under* $- \otimes Y \dashv [Y, -]$, *and the outer* $\ddagger$ *is the transpose under* $- \otimes (X \otimes Y) \dashv [X \otimes Y, -]$.

Further, the following commutes

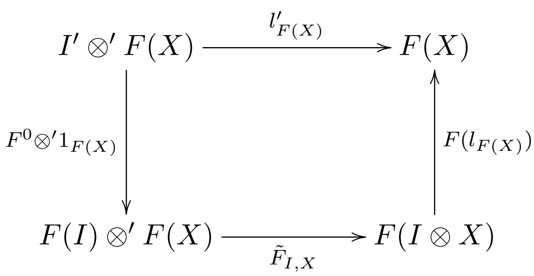


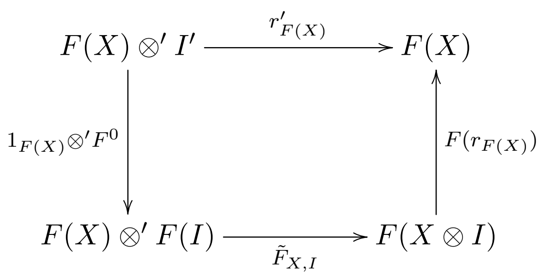
There is also the concept of a functor between monoidal categories that preserves the monoidal structure (Mac Lane, 1997; Eilenberg and Kelly, 1966).

Definition B.13. Given a pair of symmetric monoidal closed categories $\mathbb{V}$ and $\mathbb{V}'$ a **symmetric monoidal closed functor** $F \colon \mathbb{V} \to \mathbb{V}'$ has the following data:

1. a functor $F \colon \mathbb{V}_o \to \mathbb{V}'_o$,
2. a natural transformation $\tilde{F} \colon F(-) \otimes' F(-) \Rightarrow F(- \otimes -)$,
3. a natural transformation $\hat{F} \colon F([-, -]) \Rightarrow [F(-), F(-)]'$,
4. a moprhism $F^0 \colon I' \to F(I)$,

such that all the following diagrams commute.





$$
\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
{\scriptstyle \tilde{F}_{X,Y} \otimes' 1_{F(Z)}} \downarrow & & \downarrow {\scriptstyle 1_{F(X)} \otimes' \tilde{F}_{Y,Z}} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
{\scriptstyle \tilde{F}_{X \otimes Y, Z}} \downarrow & & \downarrow {\scriptstyle \tilde{F}_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(a_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z))
\end{array}
$$

$$
\begin{array}{ccc}
F(X) \otimes' F(Y) & \xrightarrow{c'_{F(X),F(Y)}} & F(Y) \otimes' F(X) \\
{\scriptstyle \tilde{F}_{X,Y}} \downarrow & & \downarrow {\scriptstyle \tilde{F}_{Y,X}} \\
F(X \otimes Y) & \xrightarrow[F(c_{X,Y})]{} & F(Y \otimes X)
\end{array}
$$

$$
\begin{array}{ccc}
I' & \xrightarrow{j'_{F(X)}} & [F(X), F(X)]' \\
{\scriptstyle F^0} \downarrow & & \uparrow {\scriptstyle \hat{F}_{X,X}} \\
F(I) & \xrightarrow[F(j_{F(X)})]{} & F([X, X])
\end{array}
\qquad
\begin{array}{ccc}
F(X) & \xrightarrow{i'_{F(X)}} & [I', F(X)]' \\
{\scriptstyle F(i_{F(X)})} \downarrow & & \uparrow {\scriptstyle [F^0, 1_{F(X)}]} \\
F([I, X]) & \xrightarrow[\hat{F}_{I,X}]{} & [F(I), F(X)]'
\end{array}
$$

$$
\begin{array}{ccc}
F([Y, Z]) & \xrightarrow{F([X,-]_{Y,Z})} & F([[X, Y], [X, Z]]) \\
{\scriptstyle \hat{F}_{Y,Z}} \downarrow & & \downarrow {\scriptstyle \hat{F}_{[X,Y],[X,Z]}} \\
[F(Y), F(Z)]' & & [F([X, Y]), F([X, Z])]' \\
{\scriptstyle [F(X),-]'_{F(Y),F(Z)}} \downarrow & & \downarrow {\scriptstyle [1_{F([X,Y])}, \hat{F}_{X,Z}]'} \\
[[F(X), F(Y)]', [F(X), F(Z)]']' & \xrightarrow[{[\hat{F}_{X,Y}, 1_{[F(X),F(Z)]'}]'}]{} & [F([X, Y]), [F(X), F(Z)]']'
\end{array}
$$

$$
\begin{array}{ccc}
F([X \otimes Y, Z]) & \xrightarrow{F(p_{X,Y,Z})} & F([X,[Y,Z]]) \\
\Big\downarrow {\scriptstyle \hat{F}_{X \otimes Y, Z}} & & \Big\downarrow {\scriptstyle \hat{F}_{X,[Y,Z]}} \\
[F(X \otimes Y), F(Z)]' & & [F(X), F([Y,Z])]' \\
\Big\downarrow {\scriptstyle [\tilde{F}_{X,Y}, 1_{F(Z)}]'} & & \Big\downarrow {\scriptstyle [1_{F(X)}, \hat{F}_{Y,Z}]'} \\
[F(X) \otimes' F(Y), F(Z)]' & \xrightarrow[p'_{F(X),F(Y),F(Z)}]{} & [F(X),[F(Y),F(Z)]']'
\end{array}
$$

There are also various strengthenings of a symmetric monoidal closed functor (Mac Lane, 1997, XI.2), the terminology can be somewhat confusing however. We make the following, possibly non-standard, definitions, as they are the appropriate ones for our needs.

Definition B.14. A symmetric monoidal closed functor $(F, \tilde{F}, \hat{F}, F^0)$ is:

1. a **strong monoidal functor** if $\tilde{F}$ and F^0 are isomorphisms,
2. a **strong closed functor** if $\hat{F}$ and F^0 are isomorphisms,
3. a **strong monoidal closed functor** if $\tilde{F}$, $\hat{F}$ and F^0 are isomorphisms.

If in the above, the isomorphisms are strengthened further to identities, then we have a **strict monoidal functor**, **strict closed functor**, or **strict monoidal closed functor** respectively.

For a symmetric monoidal closed functor $(F, \tilde{F}, \hat{F}, F^0)$, it is possible to define $\hat{F}$ in a canonical way in terms of $\tilde{F}$ as a transpose under the adjunction $- \otimes' F(X) \dashv [F(X), -]'$.

$$
\begin{array}{ccc}
F([X,Y]) \otimes' F(X) & \xrightarrow{\tilde{F}_{[X,Y],X}} & F([X,Y] \otimes X) \\
\Big\downarrow {\scriptstyle \hat{F}_{X,Y} \otimes' 1_{F(X)}} & & \Big\downarrow {\scriptstyle F(e_{X,Y})} \\
[F(X), F(Y)]' \otimes' F(X) & \xrightarrow[e'_{F(X),F(Y)}]{} & F(Y)
\end{array}
$$

As the category **Set** was our prototype symmetric monoidal closed category, we can extend the functor $\mathbf{elem}|-| : \mathbb{V}_o \to \mathbf{Set}$ to a symmetric monoidal closed functor (Eilenberg and Kelly, 1966, I.3, Prop 3.11), (Eilenberg and Kelly, 1966, II.8, Prop 8.1), and (Eilenberg and Kelly, 1966, III.1, Prop 1.3).

Definition B.15. Given a symmetric monoidal closed category $\mathbb{V}$, if $\mathbb{V}_o$ is locally small, we define a symmetric monoidal closed functor $\mathbf{elem}|{-}| : \mathbb{V} \to \mathbf{Set}$ as follows:

1. take Definition B.3 and define

$$\mathbf{elem}|{-}| = \mathbb{V}_o(I, -),$$

2. take Proposition B.5 and define

$$\begin{aligned} \widetilde{\mathbf{elem}}_{X,Y} : \mathbf{elem}|X| \times \mathbf{elem}|Y| &\to \mathbf{elem}|X \otimes Y| \\ (f, g) &\mapsto (f \otimes g) \circ l_I^{-1}, \end{aligned}$$

3. define the natural transformation

$$\widehat{\mathbf{elem}}_{X,Y} : \mathbf{elem}|[X,Y]| \to \mathbf{Set}(\mathbf{elem}|X|\,, \mathbf{elem}|Y|)$$

as the transpose of

$$\mathbf{elem}|[X,Y]| \times \mathbf{elem}|X| \xrightarrow{\widetilde{\mathbf{elem}}_{[X,Y],X}} \mathbf{elem}|[X,Y] \otimes X| \xrightarrow{\mathbf{elem}|e_{X,Y}|} \mathbf{elem}|Y|$$

under the adjunction $- \times \mathbf{elem}|X| \dashv \mathbf{Set}(\mathbf{elem}|X|\,, -)$,

4. define the morphism

$$\begin{aligned} \mathbf{elem}^0 : \{*\} &\to \mathbf{elem}|I| \\ * &\mapsto 1_I. \end{aligned}$$

Appendix C

Enriched Category Theory

In its most simple terms, enriched category theory can be thought of as ordinary category theory where the hom-sets have additional structure, for example morphisms can be ordered pointwise. However, this is not the fully story. What is really going on in enriched category theory, is that those properties of the category **Set** that are implicitly assumed in ordinary category theory (hom-sets etc), are made explicit through the use of a symmetric monoidal closed category (Definition B.9).

We will denote this symmetric monoidal closed category $\mathbb{V}$, and in addition, throughout we shall assume that $\mathbb{V}_o$ is both complete and cocomplete, and $\mathbb{V}_o$ is locally small.

The material in this chapter closely follows that in the first few chapters of Kelly (1982), but with occasional reference to the original material summarised therein.

C.1 Enriched Categories

Definition C.1. A $\mathbb{V}$**-category** $\mathbb{C}$ has the following data:

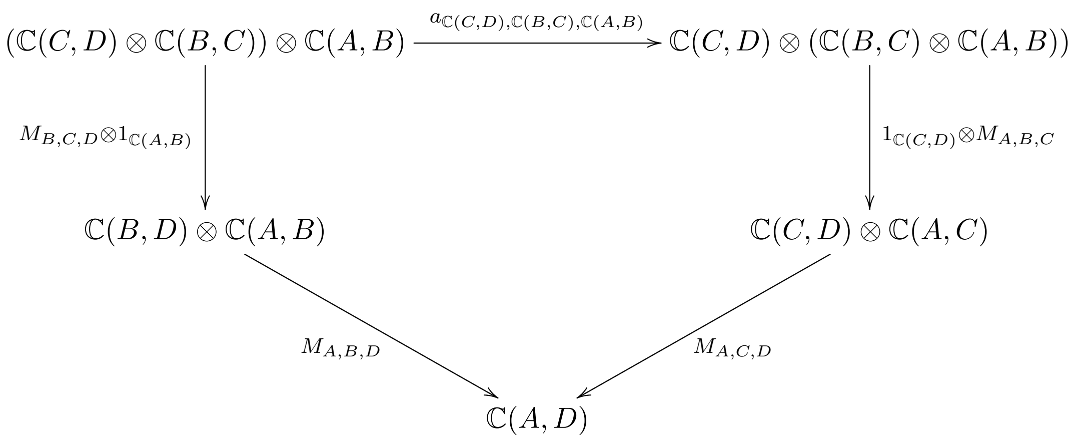
1. a collection of objects $\mathbf{obj}|\mathbb{C}|$,
2. for each pair $A, B \in \mathbf{obj}|\mathbb{C}|$ a **hom-object** $\mathbb{C}(A, B)$ in $\mathbb{V}_o$,
3. for each triple $A, B, C \in \mathbf{obj}|\mathbb{C}|$ a **composition law**

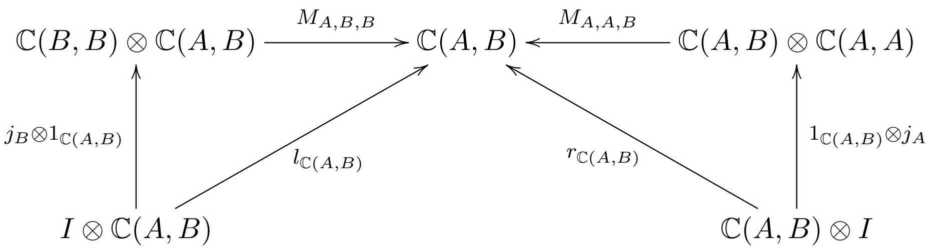
$$M_{A,B,C}\colon \mathbb{C}(B, C) \otimes \mathbb{C}(A, B) \to \mathbb{C}(A, C),$$

4. for every $A \in \mathbf{obj}|\mathbb{C}|$ an **identity element**

$$j_A\colon I \to \mathbb{C}(A, A),$$

subject to the following diagrams commuting





The idea is that a morphism between objects A and B in a $\mathbb{V}$-category $\mathbb{C}$, is an element (Definition B.3) of the hom-object $\mathbb{C}(A,B)$. The first diagram then ensures that composition of morphisms is associative, and the second diagram ensures that the identity elements really are identities under composition.

Proposition C.2 (Kelly (1982), Section 1.8)**.** *In any $\mathbb{V}$-category $\mathbb{C}$, the composition law*

$$M_{A,B,C}\colon \mathbb{C}(B,C)\otimes\mathbb{C}(A,B)\to\mathbb{C}(A,C),$$

is natural in A and C, and extranatural in B, and the identity elements

$$j_A\colon I\to\mathbb{C}(A,A),$$

are extranatural in A.

Now that we have defined enriched categories we can proceed to define enriched functors between them.

Definition C.3. A **$\mathbb{V}$-functor** $F\colon \mathbb{C}\to\mathbb{D}$ is defined as:

1. an object function $F\colon \mathbf{obj}|\mathbb{C}|\to\mathbf{obj}|\mathbb{D}|$,
2. for every pair $A,B\in\mathbf{obj}|\mathbb{C}|$ there is a morphism

$$F_{A,B}\colon \mathbb{C}(A,B)\to\mathbb{D}(F(A),F(B))$$

subject to the following diagrams commuting

$$\begin{array}{ccc}
\mathbb{C}(B,C) \otimes \mathbb{C}(A,B) & \xrightarrow{M_{A,B,C}} & \mathbb{C}(A,C) \\
\Big\downarrow {\scriptstyle F_{B,C} \otimes F_{A,B}} & & \Big\downarrow {\scriptstyle F_{A,C}} \\
\mathbb{D}(F(B),F(C)) \otimes \mathbb{D}(F(A),F(B)) & \xrightarrow[M_{F(A),F(B),F(C)}]{} & \mathbb{D}(F(A),F(C))
\end{array}$$

$$\begin{array}{ccc}
 & & \mathbb{C}(A,A) \\
 & \nearrow^{j_A} & \Big\downarrow {\scriptstyle F_{A,A}} \\
I & \searrow_{j_{F(A)}} & \mathbb{D}(F(A),F(A))
\end{array}$$

This definition is the obvious one to make, where the first diagram ensures that the image of the composite of a pair of morphisms, is the composite of their images, and the second diagram ensures that image of an identity is an identity.

Similarly we can define enriched natural transformations. Initially we shall only consider the enriched version of an ordinary natural transformation, though subsequently (Section C.6) we shall extend this to enriched extranatural transformations (Section A.6).

Definition C.4. A $\mathbb{V}$**-natural transformation** $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ is defined as an **obj**$|\mathbb{C}|$ indexed family of components

$$\alpha_A\colon I \to \mathbb{D}(F(A), G(A))$$

such that the following diagram commutes

$$\begin{array}{ccc}
I \otimes \mathbb{C}(A,B) & \xrightarrow{\alpha_B \otimes F_{A,B}} & \mathbb{D}(F(B),G(B)) \otimes \mathbb{D}(F(A),F(B)) \\
\Big\uparrow {\scriptstyle l^{-1}_{\mathbb{C}(A,B)}} & & \Big\downarrow {\scriptstyle M_{F(A),F(B),G(B)}} \\
\mathbb{C}(A,B) & & \mathbb{D}(F(A),G(B)) \\
\Big\downarrow {\scriptstyle r^{-1}_{\mathbb{C}(A,B)}} & & \Big\uparrow {\scriptstyle M_{F(A),G(A),G(B)}} \\
\mathbb{C}(A,B) \otimes I & \xrightarrow[G_{A,B} \otimes \alpha_A]{} & \mathbb{D}(G(A),G(B)) \otimes \mathbb{D}(F(A),G(A))
\end{array}$$

We can also define the enriched analogue of composition of natural transformations.

Definition C.5. The vertical composite $\beta \circ \alpha$ of the pair $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ and $\beta\colon G \Rightarrow H\colon \mathbb{C} \to \mathbb{D}$ has the component $(\beta \circ \alpha)_A$ given by

$$\begin{array}{ccc} I & \xrightarrow{(\beta\circ\alpha)_A} & \mathbb{D}(F(A), H(A)) \\ {\scriptstyle l_I^{-1}}\downarrow & & \uparrow{\scriptstyle M_{F(A),G(A),H(A)}} \\ I \otimes I & \xrightarrow[\beta_A \otimes \alpha_A]{} & \mathbb{D}(G(A), H(A)) \otimes \mathbb{D}(F(A), G(A)) \end{array}$$

The composite of $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ and $H\colon \mathbb{D} \to \mathbb{E}$ has the component $(H\alpha)_A$ given by

$$I \xrightarrow{\alpha_A} \mathbb{D}(F(A), G(A)) \xrightarrow{H_{F(A),G(A)}} \mathbb{E}(HF(A), HG(A)),$$

and the composite of $F\colon \mathbb{C} \to \mathbb{D}$ and $\alpha\colon G \Rightarrow H\colon \mathbb{D} \to \mathbb{E}$ has the component $(\alpha F)_A$ given by $\alpha_{F(A)}$.

The enriched equivalent of the notion of the product of two ordinary categories (Definition A.26), is that of the tensor product of two $\mathbb{V}$-categories.

Definition C.6. Given two $\mathbb{V}$-categories $\mathbb{B}$ and $\mathbb{C}$ the **tensor product** of $\mathbb{B}$ and $\mathbb{C}$ is a $\mathbb{V}$-category $\mathbb{B} \otimes \mathbb{C}$ with the following data:

1. $\mathbf{obj}|\mathbb{B} \otimes \mathbb{C}| = \mathbf{obj}|\mathbb{B}| \times \mathbf{obj}|\mathbb{C}|$ i.e. objects in $\mathbb{B} \otimes \mathbb{C}$ are pairs consisting of an object from $\mathbb{B}$ and an object from $\mathbb{C}$.

2. For each pair of pairs $(B, C), (B', C') \in \mathbf{obj}|\mathbb{B} \otimes \mathbb{C}|$ the hom-object

$$(\mathbb{B} \otimes \mathbb{C})((B, C), (B', C')) = \mathbb{B}(B, B') \otimes \mathbb{C}(C, C'),$$

and the composition law

$$M_{(B,C),(B',C'),(B'',C'')}\colon \begin{array}{c} (\mathbb{B} \otimes \mathbb{C})((B', C'), (B'', C'')) \\ \otimes(\mathbb{B} \otimes \mathbb{C})((B, C), (B', C')) \end{array} \to (\mathbb{B} \otimes \mathbb{C})((B, C), (B'', C''))$$

is given by

$$\begin{array}{ccc} \begin{array}{c} (\mathbb{B} \otimes \mathbb{C})((B', C'), (B'', C'')) \\ \otimes(\mathbb{B} \otimes \mathbb{C})((B, C), (B', C')) \end{array} & \xrightarrow{M_{(B,C),(B',C'),(B'',C'')}} & \mathbb{B}(B, B'') \otimes \mathbb{C}(C, C'') \\ {\scriptstyle m}\downarrow & \nearrow{\scriptstyle M_{B,B',B''} \otimes M_{C,C',C''}} & \\ \begin{array}{c} (\mathbb{B}(B', B'') \otimes \mathbb{B}(B, B')) \\ \otimes(\mathbb{C}(C', C'') \otimes \mathbb{C}(C, C')) \end{array} & & \end{array}$$

where $m\colon (W \otimes X) \otimes (Y \otimes Z) \cong (W \otimes Y) \otimes (X \otimes Z)$ is defined in terms of a and c.

3. For each object $(B, C) \in \mathbf{obj}|\mathbb{B} \otimes \mathbb{C}|$ the identity element

$$j_{(B,C)}\colon I \to (\mathbb{B} \otimes \mathbb{C})((B, C), (B, C))$$

is given by the composite

$$I \xrightarrow{l_I^{-1}} I \otimes I \xrightarrow{j_B \otimes j_C} \mathbb{B}(B, B) \otimes \mathbb{C}(C, C).$$

This definition is quite general, but if the functor $\mathbf{elem}|-|$ is a strong monoidal functor, then the natural transformation (Definition B.15)

$$\widetilde{\mathbf{elem}}_{X,Y}\colon \mathbf{elem}|X| \times \mathbf{elem}|Y| \to \mathbf{elem}|X \otimes Y|$$

is a natural isomorphism. This means that the elements of $(\mathbb{B} \otimes \mathbb{C})((B, C), (B', C'))$ are pairs of elements from $\mathbb{B}(B, B')$ and $\mathbb{C}(C, C')$, and composition is then the direct generalisation of composition for the product of ordinary categories given in Definition A.26.

Another basic construct that can be defined is that of an opposite $\mathbb{V}$-category, and from this we can defined contravariant $\mathbb{V}$-functors.

Definition C.7. Given a $\mathbb{V}$-category $\mathbb{C}$ we can define the **opposite** $\mathbb{V}$-category $\mathbb{C}^{op}$ by the following data:

1. $\mathbf{obj}|\mathbb{C}^{op}| = \mathbf{obj}|\mathbb{C}|$.

2. For each pair $(A, B) \in \mathbf{obj}|\mathbb{C}^{op}|$ the hom-object $\mathbb{C}^{op}(A, B) = \mathbb{C}(B, A)$, and the composition law

$$M_{A,B,C}\colon \mathbb{C}^{op}(B, C) \otimes \mathbb{C}^{op}(A, B) \to \mathbb{C}^{op}(A, C)$$

is the composite

$$\mathbb{C}(C, B) \otimes \mathbb{C}(B, A) \xrightarrow{c_{\mathbb{C}(C,B),\mathbb{C}(B,A)}} \mathbb{C}(B, A) \otimes \mathbb{C}(C, B) \xrightarrow{M_{C,B,A}} \mathbb{C}(C, A).$$

3. For each object $A \in \mathbf{obj}|\mathbb{C}^{op}|$ the identity element

$$j_A\colon I \to \mathbb{C}^{op}(A, A)$$

is given by that from $\mathbb{C}$.

Definition C.8. A $\mathbb{V}$-functor $F\colon \mathbb{C}^{op} \to \mathbb{D}$ is called a **contravariant $\mathbb{V}$-functor** from $\mathbb{C}$ to $\mathbb{D}$, and a $\mathbb{V}$-functor $F\colon \mathbb{B} \otimes \mathbb{C} \to \mathbb{D}$ is a **$\mathbb{V}$-functor of two variables**.

We now have the definitions we need to handle enriched natural transformations of several variables, and state an enriched version of Proposition A.29.

Proposition C.9 (Kelly (1982), Section 1.4)**.** *Given a pair of* $\mathbb{V}$*-functors*

$$F, G\colon \mathbb{B} \otimes \mathbb{C} \to \mathbb{D},$$

a collection of morphisms

$$\alpha_{B,C}\colon I \to \mathbb{D}(F(B,C), G(B,C))$$

is a $\mathbb{V}$*-natural transformation* $\alpha\colon F \Rightarrow G$*, if and only if,* α *is* $\mathbb{V}$*-natural in* B *for each* $C \in \mathbf{obj}|\mathbb{C}|$*, and* $\mathbb{V}$*-natural in* C *for each* $B \in \mathbf{obj}|\mathbb{B}|$*.*

C.2 Underlying Ordinary Categories

Recall from Definition B.3 that for every monoidal category $\mathbb{V}$ there is a notion of element, and a functor

$$\mathbf{elem}|-| = \mathbb{V}_o(I, -)\colon \mathbb{V}_o \to \mathbf{Set},$$

that gives the set of elements of any object of $\mathbb{V}_o$. We can use this to define for every $\mathbb{V}$-category $\mathbb{C}$, an underlying ordinary category $\mathbb{C}_o$.

Definition C.10. For any $\mathbb{V}$-category $\mathbb{C}$ the **underlying category** $\mathbb{C}_o$ (an ordinary category) has the following data:

1. The same objects as $\mathbb{C}$, i.e. $\mathbf{obj}|\mathbb{C}_o| = \mathbf{obj}|\mathbb{C}|$.
2. For any pair $A, B \in \mathbf{obj}|\mathbb{C}_o|$ we define the hom-set

   $$\mathbb{C}_o(A, B) = \mathbf{elem}|\mathbb{C}(A, B)|\,,$$

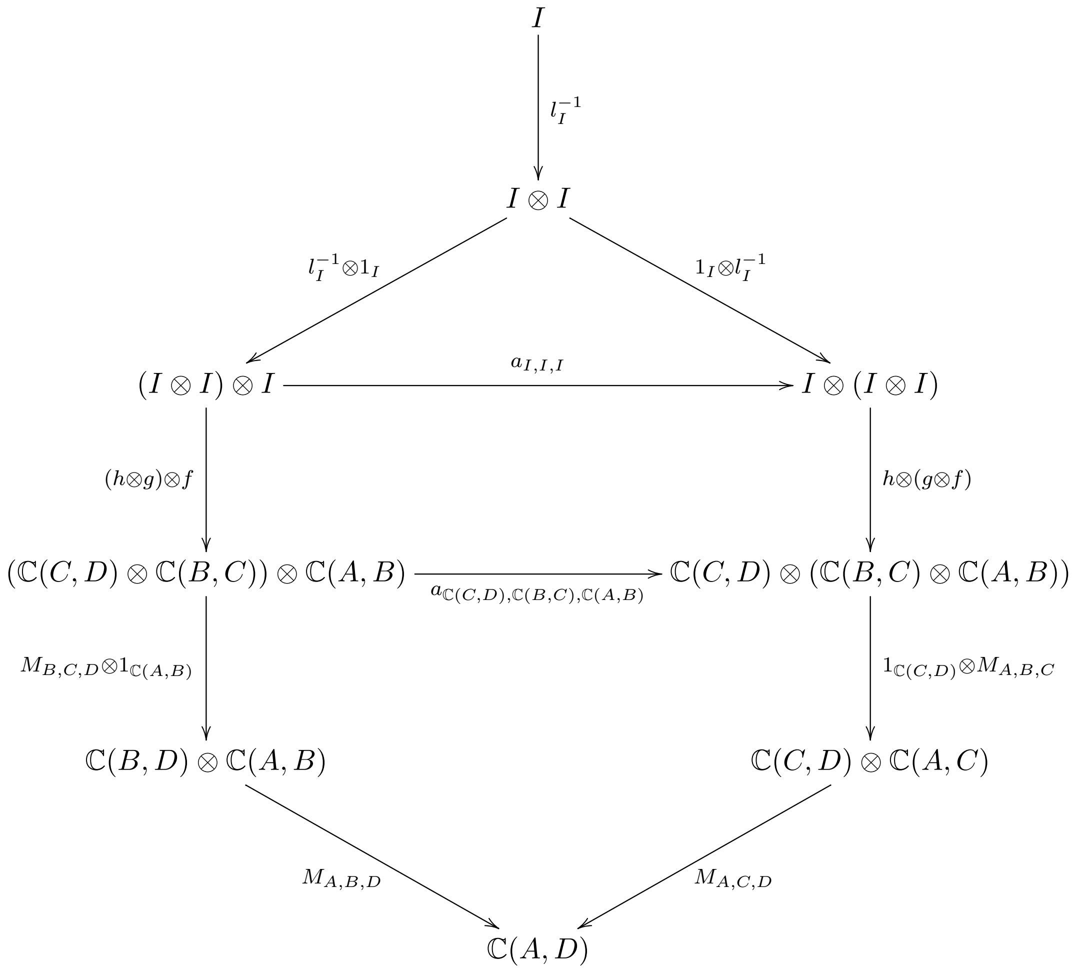
   i.e. for any element $f \in \mathbf{elem}|\mathbb{C}(A, B)|$ given by $f\colon I \to \mathbb{C}(A, B)$, there is a morphism $f\colon A \to B$ in $\mathbb{C}_o$.
3. For any pair $f\colon A \to B$ and $g\colon B \to C$ of morphisms in $\mathbb{C}_o$, the composite $g \bullet f$ is defined by the element

   $$I \xrightarrow{l_I^{-1}} I \otimes I \xrightarrow{g \otimes f} \mathbb{C}(B, C) \otimes \mathbb{C}(A, B) \xrightarrow{M_{A,B,C}} \mathbb{C}(A, C).$$

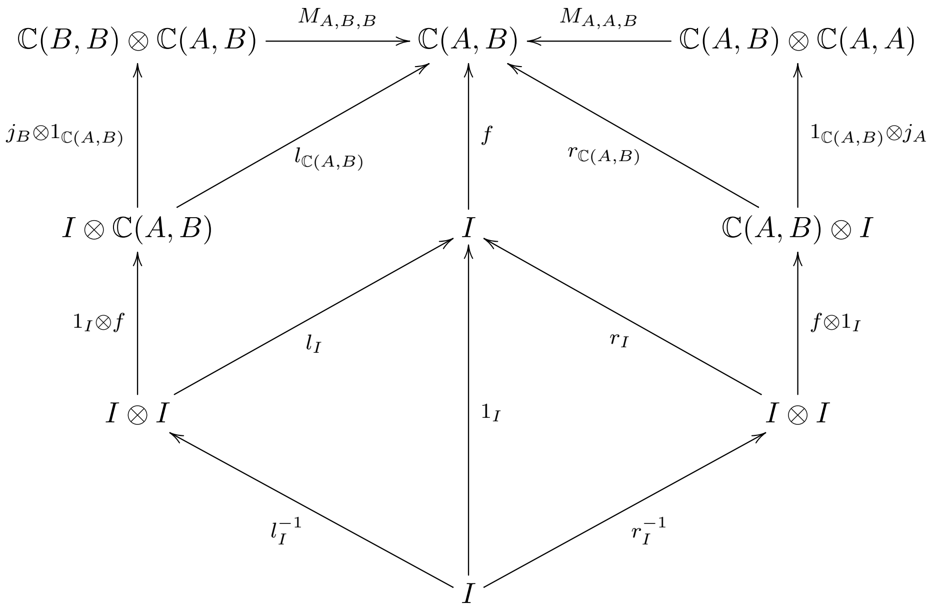
4. For every $A \in \mathbf{obj}|\mathbb{C}_o|$ the identity morphism is given by the identity element

   $$j_A\colon I \to \mathbb{C}(A, A).$$

To see that this actually is a category, observe that associativity of composition follows from the following diagram



and the unit laws from



If every $\mathbb{V}$-category has an underlying ordinary category, then it makes sense to ask if every $\mathbb{V}$-functor has an underlying ordinary functor. This turns out to be the case.

Definition C.11. For any $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$ the **underlying functor** $F_o\colon \mathbb{C}_o \to \mathbb{D}_o$ (an ordinary functor) is defined as follows:

1. F_o has the same object function as F, i.e. $F_o(A) = F(A)$.
2. For any pair $A, B \in \mathbf{obj}|\mathbb{C}_o|$ the function

$$F_{oA,B}\colon \mathbb{C}_o(A,B) \to \mathbb{D}_o(F(A), F(B))$$

is given by

$$\mathbf{elem}|F_{A,B}| : \mathbf{elem}|\mathbb{C}(A,B)| \to \mathbf{elem}|\mathbb{D}(F(A), F(B))|\,,$$

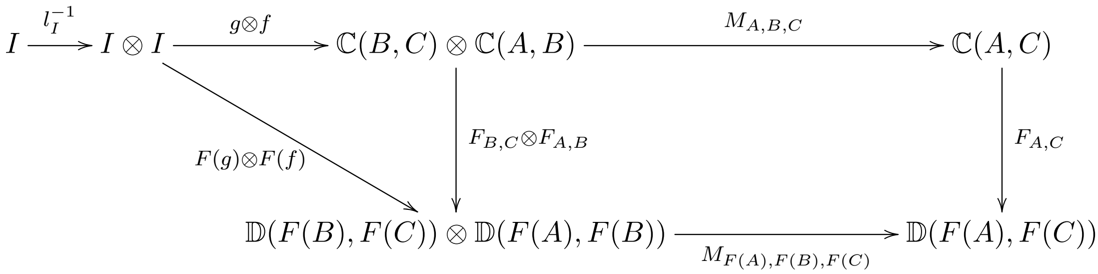
where for $f \in \mathbb{C}_o(A,B)$, we write $F(f) \in \mathbb{D}_o(F(A), F(B))$ for the composite

$$I \xrightarrow{\;f\;} \mathbb{C}(A,B) \xrightarrow{\;F_{A,B}\;} \mathbb{D}(F(A), F(B)),$$

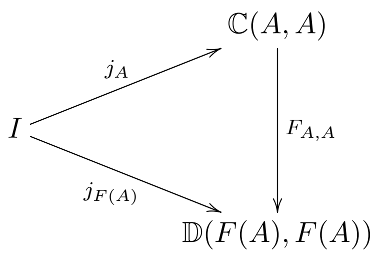
and then since $\mathbf{elem}|-|$ is the hom-functor $\mathbb{V}_o(I, -)$, we have

$$F_{oA,B}(f) = \mathbf{elem}|F_{A,B}|\,(f) = F_{A,B} \circ f = F(f).$$

To see that this actually is a functor, observe that preservation of composition follows from the diagram



and preservation of identities is simply



from Definition C.3.

It should be noted though, that whilst every $\mathbb{V}$-functor has a unique underlying ordinary functor, unless **elem**$|-|$ is faithful, two $\mathbb{V}$-functors can have the same underlying ordinary functor.

Since every $\mathbb{V}$-functor has an underlying ordinary functor we can define an ordinary natural transformation underlying every $\mathbb{V}$-natural transformation.

Definition C.12. For any $\mathbb{V}$-natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ the **underlying natural transformation** $\alpha_o\colon F_o \Rightarrow G_o\colon \mathbb{C}_o \to \mathbb{D}_o$ (an ordinary natural transformation) has the component $\alpha_{oA}\colon F_o(A) \to G_o(A)$ given by the element

$$\alpha_A\colon I \to \mathbb{D}(F(A), G(A)).$$

That this indeed gives an ordinary natural transformation follows from the following diagram

$$\begin{array}{ccccc}
I \otimes I & \xrightarrow{1_I \otimes f} & I \otimes \mathbb{C}(A,B) & \xrightarrow{\alpha_B \otimes F_{A,B}} & \mathbb{D}(F(B),G(B)) \otimes \mathbb{D}(F(A),F(B)) \\
{\scriptstyle l_I^{-1}}\uparrow & & \uparrow{\scriptstyle l_{\mathbb{C}(A,B)}^{-1}} & & \downarrow{\scriptstyle M_{F(A),F(B),G(B)}} \\
I & \xrightarrow{f} & \mathbb{C}(A,B) & & \mathbb{D}(F(A),G(B)) \\
{\scriptstyle r_I^{-1}}\downarrow & & \downarrow{\scriptstyle r_{\mathbb{C}(A,B)}^{-1}} & & \uparrow{\scriptstyle M_{F(A),G(A),G(B)}} \\
I \otimes I & \xrightarrow[f \otimes 1_I]{} & \mathbb{C}(A,B) \otimes I & \xrightarrow[G_{A,B} \otimes \alpha_A]{} & \mathbb{D}(G(A),G(B)) \otimes \mathbb{D}(F(A),G(A))
\end{array}$$

In the converse direction, given $\mathbb{V}$-functors $F, G\colon \mathbb{C} \to \mathbb{D}$, and an ordinary natural transformation $\alpha_o\colon F_o \Rightarrow G_o\colon \mathbb{C}_o \to \mathbb{D}_o$ in the underlying categories, then α_o lifts to a $\mathbb{V}$-natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$, if

$$\begin{array}{c}
\mathbb{V}_o(\mathbb{C}(A,B), \mathbb{D}(F(A),G(B))) \\
{\scriptstyle \mathbf{elem}|-|_{\mathbb{C}(A,B),\mathbb{D}(F(A),G(B))}}\downarrow \\
\mathbf{Set}(\mathbb{C}_o(A,B), \mathbb{D}_o(F(A),G(B)))
\end{array}$$

is injective. For then the right-hand "hexagon" in the above diagram commutes if the outer perimeter commutes. Thus we have the following proposition.

Proposition C.13. *Given $\mathbb{V}$-functors $F, G\colon \mathbb{C} \to \mathbb{D}$, if the functor* **elem**$|-|$ *is faithful, then every ordinary natural transformation $\alpha_o\colon F_o \Rightarrow G_o\colon \mathbb{C}_o \to \mathbb{D}_o$ lifts to a $\mathbb{V}$-natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$.*

For the tensor product of a pair of $\mathbb{V}$-categories we need to be careful. It is easy to see that $(\mathbb{B} \otimes \mathbb{C})_o$ and $\mathbb{B}_o \times \mathbb{C}_o$ must have the same objects, but they do not necessarily have the same morphisms.

For any $\mathbb{V}$-category $\mathbb{C}$, the underlying hom-sets $\mathbb{C}_o(A, B) = \mathbf{elem}|\mathbb{C}(A, B)|$, so using Definition B.15, we have the following result.

Proposition C.14 (Kelly (1982), Section 1.4)**.** *Given two $\mathbb{V}$-categories $\mathbb{B}$ and $\mathbb{C}$, the natural transformation*

$$\widetilde{\mathbf{elem}}_{X,Y} \colon \mathbf{elem}|X| \times \mathbf{elem}|Y| \to \mathbf{elem}|X \otimes Y|$$

given by

$$I \xrightarrow{l_I^{-1}} I \otimes I \xrightarrow{f \otimes g} X \otimes Y$$

yields a canonical functor

$$H \colon \mathbb{B}_o \times \mathbb{C}_o \to (\mathbb{B} \otimes \mathbb{C})_o.$$

Furthermore, for a $\mathbb{V}$-functor $F \colon \mathbb{B} \otimes \mathbb{C} \to \mathbb{D}$, the partial functors of the composite ordinary functor

$$\mathbb{B}_o \times \mathbb{C}_o \xrightarrow{H} (\mathbb{B} \otimes \mathbb{C})_o \xrightarrow{F_o} \mathbb{D}_o$$

are precisely $F(A, -)_o$ and $F(-, B)_o$.

Corollary C.15. *Given two $\mathbb{V}$-categories $\mathbb{B}$ and $\mathbb{C}$, if the functor* $\mathbf{elem}|-|$ *is a strong monoidal functor, then the natural transformation*

$$\widetilde{\mathbf{elem}}_{X,Y} \colon \mathbf{elem}|X| \times \mathbf{elem}|Y| \to \mathbf{elem}|X \otimes Y|$$

given by

$$I \xrightarrow{l_I^{-1}} I \otimes I \xrightarrow{f \otimes g} X \otimes Y$$

is a natural isomorphism, and the category $\mathbb{B}_o \times \mathbb{C}_o$ is isomorphic to the category $(\mathbb{B} \otimes \mathbb{C})_o$.

For the opposite category of any $\mathbb{V}$-category $\mathbb{C}$, the underlying ordinary category is given by the opposite category of $\mathbb{C}_o$.

Proposition C.16 (Kelly (1982), Section 1.4)**.** *For any $\mathbb{V}$-category $\mathbb{C}$*

$$(\mathbb{C}^{op})_o = (\mathbb{C}_o)^{op}.$$

C.3 $\mathbb{V}$ is Enriched over Itself

Since $\mathbb{V}$ is a symmetric monoidal closed category, it carries sufficient structure to be able to construct a $\mathbb{V}$-category with the same objects as $\mathbb{V}_o$, and who's hom-objects have the same elements as the hom-sets of $\mathbb{V}_o$. The key to doing this are the internal hom-objects.

In a slight abuse of notation we also call this $\mathbb{V}$-category $\mathbb{V}$.

Definition C.17. We define the $\mathbb{V}$-category $\hat{\mathbb{V}}$ by the following data:

1. $\hat{\mathbb{V}}$ has the same objects as $\mathbb{V}_o$, i.e. $\mathbf{obj}|\hat{\mathbb{V}}| = \mathbf{obj}|\mathbb{V}_o|$.
2. For every pair $A, B \in \mathbf{obj}|\hat{\mathbb{V}}|$ the hom-object $\hat{\mathbb{V}}(A, B) = [A, B]$.
3. For all $A, B, C \in \mathbf{obj}|\hat{\mathbb{V}}|$ the composition law

$$M_{A,B,C} \colon [B, C] \otimes [A, B] \to [A, C]$$

is given by the transpose under the adjunction $- \otimes A \dashv [A, -]$ of the composite morphism $M^{\dagger}_{A,B,C}$ defined by

$$\begin{array}{ccc} ([B,C] \otimes [A,B]) \otimes A & \xrightarrow{M^{\dagger}_{A,B,C}} & C \\ \Big\downarrow{\scriptstyle a_{[B,C],[A,B],A}} & & \Big\uparrow{\scriptstyle e_{B,C}} \\ [B,C] \otimes ([A,B] \otimes A) & \xrightarrow[1_{[B,C]} \otimes e_{A,B}]{} & [B,C] \otimes B \end{array}$$

4. For every $A \in \mathbf{obj}|\hat{\mathbb{V}}|$ the identity element $j_A \colon I \to [A, A]$ is given by the transpose under the adjunction $- \otimes A \dashv [A, -]$ of the morphism $l_A \colon I \otimes A \to A$.

The above $\mathbb{V}$-category $\hat{\mathbb{V}}$ we actually want to call $\mathbb{V}$, so why are we justified in doing so? Well the underlying category $\hat{\mathbb{V}}_o$ of $\hat{\mathbb{V}}$ has the same objects as $\mathbb{V}_o$, and by Proposition B.11,

$$\hat{\mathbb{V}}_o(A, B) = \mathbf{elem}|[A, B]| \cong \mathbb{V}_o(A, B).$$

This therefore yields the following proposition.

Proposition C.18 (Kelly (1982), Section 1.6)**.** *For the $\mathbb{V}$-category $\hat{\mathbb{V}}$, the underlying category $\hat{\mathbb{V}}_o$, is isomorphic to the category $\mathbb{V}_o$.*

We henceforth identify $\hat{\mathbb{V}}_o$ and $\mathbb{V}_o$, and simply refer to $\hat{\mathbb{V}}$ as $\mathbb{V}$.

C.4 Subcategories and Concrete $\mathbb{V}$-categories

The material in this section does not follow Kelly (1982), but the definitions are reasonably obvious extensions of the corresponding notions from ordinary category theory.

We can define an enriched analogue of faithful functors (Definition A.7).

Definition C.19. Let $F\colon \mathbb{C} \to \mathbb{D}$ be a $\mathbb{V}$-functor.

1. F is **full**, if for all $A, B \in \mathbf{obj}|\mathbb{C}|$, $F_{A,B}$ is an epimorphism in $\mathbb{V}_o$.
2. F is **faithful**, if for all $A, B \in \mathbf{obj}|\mathbb{C}|$, $F_{A,B}$ is a monomorphism in $\mathbb{V}_o$.
3. F is **fully faithful**, if for all $A, B \in \mathbf{obj}|\mathbb{C}|$, $F_{A,B}$ is an isomorphism in $\mathbb{V}_o$.

Note that in $\mathbb{V}_o$ it is in general not the case that bimorphisms are isomorphisms, therefore unlike the case in ordinary category theory, a $\mathbb{V}$-functor that is both full and faithful, is not necessarily fully faithful.

Now since $\mathbf{elem}|-|$ is representable, by Proposition A.12 it preserves monomorphisms, and since all functors preserve isomorphisms, we have the following result.

Proposition C.20. *Given a $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$ we have the following:*

1. *If F is faithful, then the underlying ordinary functor $\mathbb{F}_o$ is faithful.*
2. *If F is fully faithful, then the underlying ordinary functor $\mathbb{F}_o$ is full and faithful.*

If in addition $\mathbf{elem}|-|$ is faithful, then by Proposition A.12 and Proposition A.14, we have the following result.

Proposition C.21. *Given a $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$, if* $\mathbf{elem}|-|$ *is faithful, we have the following:*

1. *If the underlying ordinary functor $\mathbb{F}_o$ is full, then F is full.*
2. *If the underlying ordinary functor $\mathbb{F}_o$ is faithful, then F is faithful.*

Note that faithfulness of $\mathbf{elem}|-|$ is not enough to lift a full and faithful F_o to a fully faithful F.

Using the above definitions we can define the notion of a subcategory for $\mathbb{V}$-categories.

Definition C.22. Given a $\mathbb{V}$-category $\mathbb{C}$, a **subcategory** $\mathbb{B}$ of $\mathbb{C}$ is a $\mathbb{V}$-category $\mathbb{B}$, where the objects of $\mathbb{B}$ are a subclass of the objects of $\mathbb{C}$, and where there exists a faithful $\mathbb{V}$-functor $I\colon \mathbb{B} \to \mathbb{C}$ called the **inclusion functor**, that is the identity on objects. If I is fully faithful, then $\mathbb{B}$ is a **full subcategory** of $\mathbb{C}$.

Proposition C.20 then immediately gives the following.

Proposition C.23. *Given* $\mathbb{V}$*-categories* $\mathbb{B}$ *and* $\mathbb{C}$ *then the following hold:*

1. *If* $\mathbb{B}$ *is a subcategory of* $\mathbb{C}$*, then* $\mathbb{B}_o$ *is a subcategory of* $\mathbb{C}_o$*.*
2. *If* $\mathbb{B}$ *is a full subcategory of* $\mathbb{C}$*, then* $\mathbb{B}_o$ *is a full subcategory of* $\mathbb{C}_o$*.*

Conversely, by Proposition C.21 we have a partial dual result.

Proposition C.24. *Given* $\mathbb{V}$*-categories* $\mathbb{B}$ *and* $\mathbb{C}$*, if* $\mathbf{elem}|-|$ *is faithful, and if* $\mathbb{B}_o$ *is a subcategory of* $\mathbb{C}_o$*, then* $\mathbb{B}$ *is a subcategory of* $\mathbb{C}$*.*

Note again, faithfulness of $\mathbf{elem}|-|$ is not enough to ensure a full subcategory at the underlying ordinary category level, lifts to a full subcategory at the enriched level.

Just like in ordinary category theory (Definition A.9), the notion of a faithful functor captures what it means for one category to be concrete over another.

Definition C.25. Let $\mathbb{D}$ be a $\mathbb{V}$-category. A **concrete** $\mathbb{V}$**-category** over $\mathbb{D}$ is a pair $(\mathbb{C}, U)$, where $\mathbb{C}$ is a $\mathbb{V}$-category and $U : \mathbb{C} \to \mathbb{D}$ is a faithful $\mathbb{V}$-functor. Sometimes U is called the **forgetful** (or **underlying**) $\mathbb{V}$-functor of the concrete $\mathbb{V}$-category and $\mathbb{D}$ is called the **base** $\mathbb{V}$**-category** for $(\mathbb{C}, U)$.

Since the underlying functors of faithful $\mathbb{V}$-functors are faithful, we have the following results.

Proposition C.26. *Given a* $\mathbb{V}$*-category* $\mathbb{D}$*, if* $(\mathbb{C}, U)$ *is concrete over* $\mathbb{D}$*, then* $(\mathbb{C}_o, U_o)$ *is concrete over* $\mathbb{D}_o$*.*

Proposition C.27. *Given a* $\mathbb{V}$*-category* $\mathbb{D}$ *and a* $\mathbb{V}$ *functor* $U : \mathbb{C} \to \mathbb{D}$*, if* $\mathbf{elem}|-|$ *is faithful, and if* $(\mathbb{C}_o, U_o)$ *is concrete over* $\mathbb{D}_o$*, then* $(\mathbb{C}, U)$ *is concrete over* $\mathbb{D}$*.*

We also have an enriched version of unique transportability (Definition A.11).

Definition C.28. A concrete category $(\mathbb{C}, U)$ over $\mathbb{D}$ is **(uniquely) transportable**, if for every isomorphism $f \in \mathbb{D}_o(U(C), D)$, there exists a (unique) $C' \in \mathbf{obj}|\mathbb{C}|$ such that $U(C') = D$ and $f : C \to C'$ is an isomorphism in $\mathbb{C}_o$.

It is easy to see that this corresponds to unique transportability of $(\mathbb{C}_o, U_o)$ over $\mathbb{D}_o$, and if $\mathbf{elem}|-|$ is faithful we have the converse result.

Proposition C.29. *Given a* $\mathbb{V}$*-category* $\mathbb{D}$*, if the concrete category* $(\mathbb{C}, U)$ *over* $\mathbb{D}$ *is (uniquely) transportable, then* $(\mathbb{C}_o, U_o)$ *is (uniquely) transportable.*

Proposition C.30. *Given a* $\mathbb{V}$*-category* $\mathbb{D}$ *and a* $\mathbb{V}$*-functor* $U\colon \mathbb{C} \to \mathbb{D}$*, if* $\mathbf{elem}|-|$ *is faithful, and if the concrete category* $(\mathbb{C}_o, U_o)$ *over* $\mathbb{D}_o$ *is (uniquely) transportable, then* $(\mathbb{C}, U)$ *is (uniquely) transportable.*

Finally, we can define what we mean by the skeleton of a $\mathbb{V}$-category.

Definition C.31. A **skeleton** of a $\mathbb{V}$-category $\mathbb{C}$ is any full subcategory $\mathbb{B}$ such that each object of $\mathbb{C}$ is isomorphic to exactly one object of $\mathbb{B}$.

C.5 Hom-Functors

One of the most useful tools in ordinary category theory are hom-functors, as from these we can develop the notion of a representable functor. Enriched category theory is no different in this respect.

We start with the definition of a covariant hom-functor.

Definition C.32. Given a $\mathbb{V}$-category $\mathbb{C}$, and an object A in $\mathbb{C}$, we can define a **covariant hom-functor** $\mathbb{C}(A,-)\colon \mathbb{C} \to \mathbb{V}$, as follows:

1. For any $B \in \mathbf{obj}|\mathbb{C}|$ the action of $\mathbb{C}(A,-)$ on B is given by $\mathbb{C}(A,B)$.
2. For any pair $B, C \in \mathbf{obj}|\mathbb{C}|$ the morphism

   $$\mathbb{C}(A,-)_{B,C}\colon \mathbb{C}(B,C) \to [\mathbb{C}(A,B), \mathbb{C}(A,C)]$$

   is the transpose of $M_{A,B,C}$ under the adjunction $- \otimes A \dashv [A,-]$.

Now for $f \in \mathbf{elem}|\mathbb{C}(B,C)| = \mathbb{C}_o(B,C)$ we can form the following diagram

$$\begin{array}{ccc}
I \otimes \mathbb{C}(A,B) & & \\
\downarrow{\scriptstyle f \otimes 1_{\mathbb{C}(A,B)}} & & \\
\mathbb{C}(B,C) \otimes \mathbb{C}(A,B) & \xrightarrow{\quad M_{A,B,C} \quad} & \mathbb{C}(A,C) \\
\downarrow{\scriptstyle \mathbb{C}(A,-)_{B,C} \otimes 1_{\mathbb{C}(A,B)}} & & \| \\
[\mathbb{C}(A,B), \mathbb{C}(A,C)] \otimes \mathbb{C}(A,B) & \xrightarrow[e_{\mathbb{C}(A,B),\mathbb{C}(A,C)}]{} & \mathbb{C}(A,C)
\end{array}$$

and using $\mathbb{C}(A,-)_{B,C} \circ f = \mathbb{C}(A,f)$ (Definition C.11), and Proposition B.11, we have the following result.

Proposition C.33 (Kelly (1982), Section 1.6)**.** *Given a* $\mathbb{V}$*-category* $\mathbb{C}$*, and the covariant hom-functor* $\mathbb{C}(A,-)$*, then for any* $f \in \mathbf{elem}|\mathbb{C}(B,C)| = \mathbb{C}_o(B,C)$*, the morphsim* $\mathbb{C}(A,f)$ *is given by*

$$\mathbb{C}(A,B) \xrightarrow{l^{-1}_{\mathbb{C}(A,B)}} I \otimes \mathbb{C}(A,B) \xrightarrow{f \otimes 1_{\mathbb{C}(A,B)}} \mathbb{C}(B,C) \otimes \mathbb{C}(A,B) \xrightarrow{M_{A,B,C}} \mathbb{C}(A,C).$$

Similarly we can define a contravariant hom-functor.

Definition C.34. Given a $\mathbb{V}$-category $\mathbb{C}$, and an object C in $\mathbb{C}$, we can define a **contravariant hom-functor** $\mathbb{C}(-,C) \colon \mathbb{C} \to \mathbb{V}$, as follows:

1. For any $A \in \mathbf{obj}|\mathbb{C}|$ the action of $\mathbb{C}(-,C)$ on A is given by $\mathbb{C}(A,C)$.
2. For any pair $A, B \in \mathbf{obj}|\mathbb{C}|$ the morphism

   $$\mathbb{C}(-,C)_{A,B} \colon \mathbb{C}(A,B) \to [\mathbb{C}(B,C), \mathbb{C}(A,C)]$$

   is the transpose of $M_{A,B,C} \circ c_{\mathbb{C}(A,B),\mathbb{C}(B,C)}$ under the adjunction $- \otimes A \dashv [A,-]$.

Now for $f \in \mathbf{elem}|\mathbb{C}(A,B)| = \mathbb{C}_o(A,B)$ we can form the following diagram

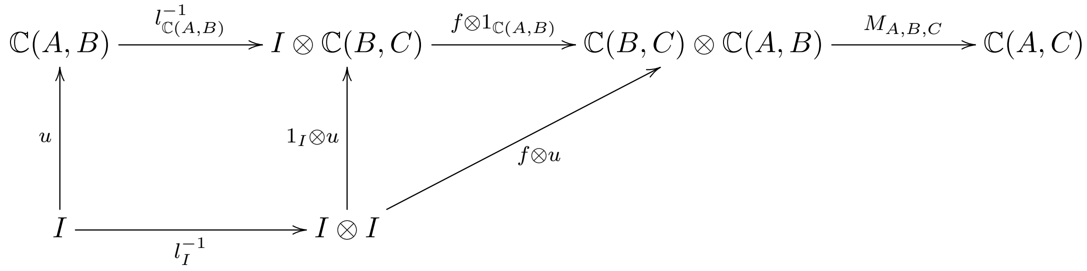
$$\begin{array}{ccc}
I \otimes \mathbb{C}(B,C) & & \\
\Big\downarrow{\scriptstyle f \otimes 1_{\mathbb{C}(B,C)}} & & \\
\mathbb{C}(A,B) \otimes \mathbb{C}(B,C) & \xrightarrow{c_{\mathbb{C}(A,B),\mathbb{C}(B,C)}} & \mathbb{C}(B,C) \otimes \mathbb{C}(A,B) \\
\Big\downarrow{\scriptstyle \mathbb{C}(-,C)_{A,B} \otimes 1_{\mathbb{C}(B,C)}} & & \Big\downarrow{\scriptstyle M_{A,B,C}} \\
[\mathbb{C}(B,C), \mathbb{C}(A,C)] \otimes \mathbb{C}(B,C) & \xrightarrow[e_{\mathbb{C}(A,B),\mathbb{C}(A,C)}]{} & \mathbb{C}(A,C)
\end{array}$$

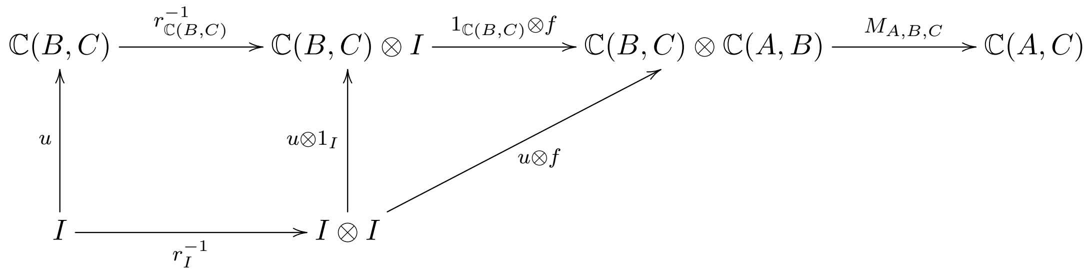
and using $\mathbb{C}(-,C)_{A,B} \circ f = \mathbb{C}(f,C)$ (Definition C.11), Proposition B.11, and Definition B.6, we have the following result.

Proposition C.35 (Kelly (1982), Section 1.6)**.** *Given a* $\mathbb{V}$*-category* $\mathbb{C}$*, and the contravariant hom-functor* $\mathbb{C}(-,C)$*, then for any* $f \in \mathbf{elem}|\mathbb{C}(A,B)| = \mathbb{C}_o(A,B)$*, the morphsim* $\mathbb{C}(f,C)$ *is given by*

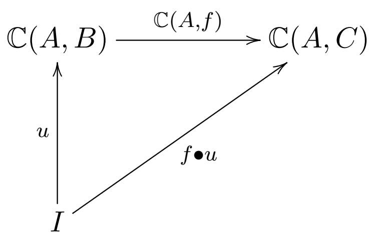
$$\mathbb{C}(B,C) \xrightarrow{r^{-1}_{\mathbb{C}(B,C)}} \mathbb{C}(B,C) \otimes I \xrightarrow{1_{\mathbb{C}(B,C)} \otimes f} \mathbb{C}(B,C) \otimes \mathbb{C}(A,B) \xrightarrow{M_{A,B,C}} \mathbb{C}(A,C).$$

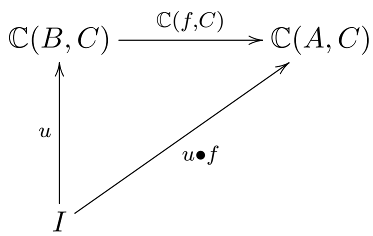
It therefore follows from Proposition C.33 and Proposition C.35 that the following diagrams commute.





In both cases it should be observed that the lower path from I to $\mathbb{C}(A,C)$ correspond to the composites $f \bullet u$ and $u \bullet f$ (in the underlying category $\mathbb{C}_o$) respectively, which gives the following commuting diagrams.





Now it should be noted that for the covariant hom-functor $\mathbb{C}(A,-)\colon \mathbb{C} \to \mathbb{V}$, that the underlying functor $\mathbb{C}(A,-)_o\colon \mathbb{C}_o \to \mathbb{V}_o$ is not the same as the ordinary hom-functor $\mathbb{C}_o(A,-)\colon \mathbb{C}_o \to \mathbf{Set}$. Specifically we have

$$\begin{aligned} \mathbb{C}(A,-)_{o\,B,C}\colon \mathbb{C}_o(B,C) &\to \mathbb{V}_o(\mathbb{C}(A,B),\mathbb{C}(A,C)) \\ f &\mapsto \mathbb{C}(A,f)\colon \mathbb{C}(A,B) \to \mathbb{C}(A,C) \end{aligned}$$

$$\begin{aligned} \mathbb{C}_o(A,-)_{B,C}\colon \mathbb{C}_o(B,C) &\to \mathbf{Set}(\mathbb{C}_o(A,B),\mathbb{C}_o(A,C)) \\ f &\mapsto \mathbb{C}_o(A,f)\colon \mathbb{C}_o(A,B) \to \mathbb{C}_o(A,C). \end{aligned}$$

However, since $\mathbf{elem}|-|$ is the ordinary hom-functor $\mathbb{V}_o(I,-)$ (Definition B.3), we have

$$\begin{aligned}\mathbf{elem}|\mathbb{C}(A,f)| : \mathbf{elem}|\mathbb{C}(A,B)| &\to \mathbf{elem}|\mathbb{C}(A,C)| \\ u &\mapsto \mathbb{C}(A,f)\circ u\end{aligned}$$

where the composition $\mathbb{C}(A,f)\circ u$ is in $\mathbb{V}_o$. But as shown above, $\mathbb{C}(A,f)\circ u = f \bullet u$, and since $\mathbf{elem}|\mathbb{C}(A,B)| = \mathbb{C}_o(A,B)$ (Definition C.10), we have

$$\begin{aligned}\mathbf{elem}|\mathbb{C}(A,f)| : \mathbb{C}_o(A,B) &\to \mathbb{C}_o(A,C) \\ u &\mapsto f \bullet u\end{aligned}$$

but this is just the definition of $\mathbb{C}_o(A,f)$.

The contravariant case proceeds similarly, and we have the following proposition.

Proposition C.36. *Given a $\mathbb{V}$-category $\mathbb{C}$, then for all $A,B,C \in \mathbf{obj}|\mathbb{C}|$*

$$\begin{aligned}\mathbb{C}_o(A,-) &= \mathbf{elem}|\mathbb{C}(A,-)_o| \\ \mathbb{C}_o(-,C) &= \mathbf{elem}|\mathbb{C}(-,C)_o|\,,\end{aligned}$$

with

$$\begin{aligned}\mathbb{C}_o(A,-)_{B,C} &= \mathbf{elem}|-|_{\mathbb{C}(A,B),\mathbb{C}(A,C)} \circ \mathbb{C}(A,-)_{o\,B,C} \\ \mathbb{C}_o(-,C)_{A,B} &= \mathbf{elem}|-|_{\mathbb{C}(B,C),\mathbb{C}(A,C)} \circ \mathbb{C}(-,C)_{o\,A,B}.\end{aligned}$$

C.6 Extranatural Transformations

As is the case in ordinary category theory (Section A.6) there is a more general notion of naturality in enriched category theory than that of basic $\mathbb{V}$-natural transformations (Definition C.4).

We start by using the hom-functors of the previous section to redraw the commutativity diagram from the definition of a $\mathbb{V}$-natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ as follows.

$$\begin{array}{ccc}\mathbb{C}(A,B) & \xrightarrow{F_{A,B}} & \mathbb{D}(F(A),F(B)) \\ {\scriptstyle G_{A,B}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}(F(A),\alpha_B)} \\ \mathbb{D}(G(A),G(B)) & \xrightarrow[\mathbb{D}(\alpha_A,G(B))]{} & \mathbb{D}(F(A),G(B))\end{array}$$

It is this diagram that we shall proceed to generalise.

Definition C.37. Given a $\mathbb{V}$-functor $F\colon \mathbb{C}^{op} \otimes \mathbb{C} \to \mathbb{D}$ we define an **extranatural transformation** from D to F by an $\mathbf{obj}|\mathbb{C}|$ indexed collection of components

$$\alpha_A\colon D \to F(A,A)$$

in $\mathbb{D}_o$, such that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{C}(A,B) & \xrightarrow{F(A,-)_{A,B}} & \mathbb{D}(F(A,A),F(A,B)) \\
{\scriptstyle F(-,B)_{A,B}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}(\alpha_A,F(A,B))} \\
\mathbb{D}(F(B,B),F(A,B)) & \xrightarrow[\mathbb{D}(\alpha_B,F(A,B))]{} & \mathbb{D}(D,F(A,B))
\end{array}$$

Similarly we have a dual notion.

Definition C.38. Given a $\mathbb{V}$-functor $F\colon \mathbb{C}^{op} \otimes \mathbb{C} \to \mathbb{D}$ we define an **extranatural transformation** from F to D by an $\mathbf{obj}|\mathbb{C}|$ indexed collection of components

$$\alpha_A\colon F(A,A) \to D$$

in $\mathbb{D}_o$, such that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{C}(A,B) & \xrightarrow{F(B,-)_{A,B}} & \mathbb{D}(F(B,A),F(B,B)) \\
{\scriptstyle F(-,A)_{A,B}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}(F(B,A),\alpha_B)} \\
\mathbb{D}(F(B,A),F(A,A)) & \xrightarrow[\mathbb{D}(F(B,A),\alpha_A)]{} & \mathbb{D}(F(B,A),D)
\end{array}$$

Like the case for $\mathbb{V}$-natural transformations (Definition C.5), we can define the composition of a $\mathbb{V}$-extranatural transformation with a $\mathbb{V}$-functor to yield another $\mathbb{V}$-extranatural transformation.

Definition C.39. Given the $\mathbb{V}$-functors $F\colon \mathbb{C}^{op} \otimes \mathbb{C} \to \mathbb{D}$, $G\colon \mathbb{D} \to \mathbb{E}$, and $H\colon \mathbb{B} \to \mathbb{C}$, the composite of the $\mathbb{V}$-extranatural transformation $\alpha_A\colon D \to F(A,A)$ and G, has the component $(G\alpha)_A$ given by

$$G(\alpha_A)\colon G(D) \to GF(A,A),$$

and the composite with H, has the component $(\alpha H)_A$ given by

$$\alpha_{H(A)}\colon D \to F(H(A),H(A)).$$

Dually, the composite of the $\mathbb{V}$-extranatural transformation $\beta_A \colon F(A,A) \to D$ and G, has the component $(G\beta)_A$ given by

$$G(\beta_A) \colon GF(A,A) \to G(D),$$

and the composite with H, has the component $(\beta H)_A$ given by

$$\beta_{H(A)} \colon F(H(A), H(A)) \to D.$$

Using the tensor product of two $\mathbb{V}$-categories we can handle $\mathbb{V}$-extranatural transformations of several variables, and extend the $\mathbb{V}$-natural transformation result of Proposition C.9.

Proposition C.40 (Kelly (1982), Section 1.7)**.** *Given a $\mathbb{V}$-functor*

$$F \colon (\mathbb{B} \otimes \mathbb{C})^{op} \otimes (\mathbb{B} \otimes \mathbb{C}) \to \mathbb{D},$$

a collection of morphisms

$$\alpha_{B,C} \colon K \to F(B,C,B,C)$$

is $\mathbb{V}$-extranatural in (B,C), if and only if, α is $\mathbb{V}$-extranatural in B for each $C \in \mathbf{obj}|\mathbb{C}|$, and $\mathbb{V}$-extranatural in C for each $B \in \mathbf{obj}|\mathbb{B}|$.

It is not usually necessary to make an explicit distinction between $\mathbb{V}$-naturality and $\mathbb{V}$-extranaturality, and indeed, in line with the ordinary category theory case (Definition A.31), both can be combined into a general notion of $\mathbb{V}$-natural transformation.

Definition C.41. Given a pair of $\mathbb{V}$-functors

$$F \colon \mathbb{C}^{op} \otimes \mathbb{C} \otimes \mathbb{A} \to \mathbb{B} \qquad\qquad G \colon \mathbb{A} \otimes \mathbb{D}^{op} \otimes \mathbb{D} \to \mathbb{B}$$

we define a $\mathbb{V}$-**natural transformation** $\alpha \colon F \Rightarrow G$ as a collection of components

$$\alpha_{C,A,D} \colon F(C,C,A) \to G(A,D,D),$$

one for each triple of objects $(C,A,D) \in \mathbf{obj}|\mathbb{C} \times \mathbb{A} \times \mathbb{D}|$, such that the following hold:

1. for C and D fixed, $\alpha_{C,-,D}$ is $\mathbb{V}$-natural (in the ordinary sense) in A,
2. for A and D fixed, $\alpha_{-,A,D}$ is $\mathbb{V}$-extranatural in C,
3. for C and A fixed, $\alpha_{C,A,-}$ is $\mathbb{V}$-extranatural in D.

Remark C.42. Any of the $\mathbb{V}$-categories $\mathbb{A}$, $\mathbb{C}$, and $\mathbb{D}$ can be replaced by a tensor product of several $\mathbb{V}$-categories, and in each case $\mathbb{V}$-naturality in a tensor product argument may be replaced by $\mathbb{V}$-naturality in each argument of the tuple (that makes up the tensor product argument) where the others are fixed. The ordinary $\mathbb{V}$-natural transformation case is covered by Proposition C.9, and the $\mathbb{V}$-extranatural case by Proposition C.40.

Proposition C.13 also extends to cover this expanded notion of $\mathbb{V}$-natural transformation.

Proposition C.43. *Given a pair of* $\mathbb{V}$*-functors*

$$F\colon \mathbb{C}^{op} \otimes \mathbb{C} \otimes \mathbb{A} \to \mathbb{B} \qquad\qquad G\colon \mathbb{A} \otimes \mathbb{D}^{op} \otimes \mathbb{D} \to \mathbb{B}$$

if the functor **elem**$|-|$ *is faithful, then every ordinary natural transformation*

$$\alpha_o\colon F_o \Rightarrow G_o$$

lifts to a $\mathbb{V}$*-natural transformation* $\alpha\colon F \Rightarrow G$*.*

We have come across many such $\mathbb{V}$-natural transformations so far, without realising (Kelly, 1982, Section 1.8):

1. For a $\mathbb{V}$-category $\mathbb{C}$, the composition law and identity elements

$$M_{A,B,C}\colon \mathbb{C}(B,C) \otimes \mathbb{C}(A,B) \to \mathbb{C}(A,C)$$

$$j_A\colon I \to \mathbb{C}(A,A)$$

   are $\mathbb{V}$-natural in every variable.

2. For a $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$, the family of morphisms

$$F_{A,B}\colon \mathbb{C}(A,B) \to \mathbb{D}(F(A),F(B))$$

   are $\mathbb{V}$-natural in A and B.

3. For the $\mathbb{V}$-category $\mathbb{V}$, the unit and counit

$$d_{X,Y}\colon Y \to [Y, X \otimes Y]$$

$$e_{Y,Z}\colon [Y,Z] \otimes Y \to Z$$

   are $\mathbb{V}$-natural in every variable, and the isomorphisms

$$i_X\colon X \to [I,X]$$

$$p_{X,Y,Z}\colon [X \otimes Y, Z] \to [X,[Y,Z]]$$

   are also $\mathbb{V}$-natural in every variable.

Remark C.44. The case of the $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$ is worth further discussion. The $\mathbb{V}$-natural transformation $F_{A,-}$ has the underlying ordinary natural transformation $(F_{A,-})_o$, which is not the same as $F_{oA,-}$, as can be seen if we spell out the signatures

$$(F_{A,-})_o\colon \mathbb{C}(A,-)_o \Rightarrow \mathbb{D}(F(A),F(-))_o\colon \mathbb{C}_o \to \mathbb{V}_o$$
$$F_{oA,-}\colon \mathbb{C}_o(A,-) \Rightarrow \mathbb{D}_o(F(A),F(-))\colon \mathbb{C}_o \to \mathbf{Set}.$$

Proposition C.45 (Kelly (1982), Section 1.8(m))**.** *A family of morphisms*

$$f_{D,A,B,E,C,F}\colon F(D,D,A,B) \otimes G(E,E,A,C) \to H(F,F,B,C)$$

is $\mathbb{V}$-natural in any of its variables, if and only if, the corresponding transpose

$$f^{\ddagger}_{D,A,B,E,C,F}\colon F(D,D,A,B) \to [G(E,E,A,C),H(F,F,B,C)]$$

is so.

C.7 The Yoneda Lemma

We now state (without proof) from Kelly (1982, Section 1.9) a weak form of the Yoneda Lemma for $\mathbb{V}$-categories. It is a weak form because the isomorphism is a bijection of sets, not an isomorphism of $\mathbb{V}_o$ objects.

Lemma C.46 (Yoneda (weak form))**.** *Given a covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$, and $K \in \mathbf{obj}|\mathbb{C}|$, if we write $\mathbb{V}{-}\mathbf{nat}(\mathbb{C}(K,-),F)$ for the set of all $\mathbb{V}$-natural transformations from the hom-functor $\mathbb{C}(K,-)$ to F, then we have an isomorphism*

$$\mathbb{V}{-}\mathbf{nat}(\mathbb{C}(K,-),F) \cong \mathbf{elem}|F(K)|\,,$$

where any $\alpha\colon \mathbb{C}(K,-) \Rightarrow F$ is mapped to $\eta\colon I \to F(K)$ given by

$$I \xrightarrow{\;j_K\;} \mathbb{C}(K,K) \xrightarrow{\;\alpha_K\;} F(K),$$

and any $\eta\colon I \to F(K)$ is mapped to $\alpha\colon \mathbb{C}(K,-) \Rightarrow F$ given by

$$\mathbb{C}(K,A) \xrightarrow{\;F_{K,A}\;} [F(K),F(A)] \xrightarrow{\;[\eta,F(A)]\;} [I,F(A)] \xrightarrow{\;i^{-1}\;} F(A).$$

Though not in Kelly (1982), there is also a contravariant form of the Yoneda Lemma.

Lemma C.47 (Contravariant Yoneda (weak form)). *Given a contravariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$, and $K \in \mathbf{obj}|\mathbb{C}|$, if we write $\mathbb{V}{-}\mathbf{nat}(\mathbb{C}(-,K),F)$ for the set of all $\mathbb{V}$-natural transformations from the hom-functor $\mathbb{C}(-,K)$ to F, then we have an isomorphism*

$$\mathbb{V}{-}\mathbf{nat}(\mathbb{C}(-,K),F) \cong \mathbf{elem}|F(K)|\,,$$

where any $\alpha\colon \mathbb{C}(-,K) \Rightarrow F$ is mapped to $\eta\colon I \to F(K)$ given by

$$I \xrightarrow{\;j_K\;} \mathbb{C}(K,K) \xrightarrow{\;\alpha_K\;} F(K),$$

and any $\eta\colon I \to F(K)$ is mapped to $\alpha\colon \mathbb{C}(-,K) \Rightarrow F$ given by

$$\mathbb{C}(A,K) \xrightarrow{\;F_{A,K}\;} [F(K),F(A)] \xrightarrow{\;[\eta,F(A)]\;} [I,F(A)] \xrightarrow{\;i^{-1}\;} F(A).$$

We shall also need the following special cases. See Kelly (1982, Section 1.9) for the first, the others are simple variants.

Proposition C.48.

1. *Given the covariant $\mathbb{V}$-functors $F\colon \mathbb{C} \to \mathbb{D}$, and $G\colon \mathbb{D} \to \mathbb{C}$, for all $C \in \mathbf{obj}|\mathbb{C}|$, there exists a bijection*

   $$\{\alpha_{C,-}\colon \mathbb{D}(F(C),-) \Rightarrow \mathbb{C}(C,G(-))\} \cong \{\eta_C\colon C \to GF(C)\},$$

   such that for every $\mathbb{V}$-natural transformation $\alpha_{C,-}$, there is a unique η_C, $\mathbb{V}$-natural in C, given by the image of $1_{F(C)}$ under

   $$\mathbf{elem}\left|\alpha_{C,F(C)}\right| : \mathbb{D}_o(F(C),F(C)) \to \mathbb{C}_o(C,GF(C)),$$

   and for every η_C, $\mathbb{V}$-natural in C, there is a unique $\mathbb{V}$-natural transformation $\alpha_{C,-}$, where the component $\alpha_{C,A}$ is given by

   $$\mathbb{D}(F(C),A) \xrightarrow{\;G_{F(C),A}\;} \mathbb{C}(GF(C)),G(A)) \xrightarrow{\;\mathbb{C}(\eta_C,G(A))\;} \mathbb{C}(C,G(A)).$$

2. *Given the covariant $\mathbb{V}$-functors $F\colon \mathbb{C} \to \mathbb{D}$, and $G\colon \mathbb{D} \to \mathbb{C}$, for all $D \in \mathbf{obj}|\mathbb{D}|$, there exists a bijection*

   $$\{\alpha_{-,D}\colon \mathbb{C}(-,G(D)) \Rightarrow \mathbb{D}(F(-),D)\} \cong \{\eta_D\colon FG(D) \to D\},$$

   such that for every $\mathbb{V}$-natural transformation $\alpha_{-,D}$, there is a unique η_D, $\mathbb{V}$-natural in D, given by the image of $1_{G(D)}$ under

   $$\mathbf{elem}\left|\alpha_{G(D),D}\right| : \mathbb{C}_o(G(D),G(D)) \to \mathbb{D}_o(FG(D),D),$$

and for every η_D, $\mathbb{V}$-natural in D, there is a unique $\mathbb{V}$-natural transformation $\alpha_{-,D}$, where the component $\alpha_{A,D}$ is given by

$$\mathbb{C}(A, G(D)) \xrightarrow{F_{A,G(D)}} \mathbb{D}(F(A), FG(D)) \xrightarrow{\mathbb{D}(F(A),\eta_D)} \mathbb{D}(F(A), D).$$

3. *Given the contravariant $\mathbb{V}$-functors $F\colon \mathbb{C} \to \mathbb{D}$, and $G\colon \mathbb{D} \to \mathbb{C}$, for all $C \in \mathbf{obj}|\mathbb{C}|$, there exists a bijection*

$$\{\alpha_{-,C}\colon \mathbb{D}(-, F(C)) \Rightarrow \mathbb{C}(C, G(-))\} \cong \{\eta_C\colon C \to GF(C)\},$$

such that for every $\mathbb{V}$-natural transformation $\alpha_{-,C}$, there is a unique η_C, $\mathbb{V}$-natural in C, given by the image of $1_{F(C)}$ under

$$\mathbf{elem}\big|\alpha_{F(C),C}\big| : \mathbb{D}_o(F(C), F(C)) \to \mathbb{C}_o(C, GF(C)),$$

and for every η_C, $\mathbb{V}$-natural in C, there is a unique $\mathbb{V}$-natural transformation $\alpha_{-,C}$, where the component $\alpha_{A,C}$ is given by

$$\mathbb{D}(A, F(C)) \xrightarrow{G_{A,F(C)}} \mathbb{C}(GF(C), G(A)) \xrightarrow{\mathbb{C}(\eta_C,G(A))} \mathbb{C}(C, G(A)).$$

4. *Given the contravariant $\mathbb{V}$-functors $F\colon \mathbb{C} \to \mathbb{D}$, and $G\colon \mathbb{D} \to \mathbb{C}$, for all $D \in \mathbf{obj}|\mathbb{D}|$, there exists a bijection*

$$\{\alpha_{D,-}\colon \mathbb{C}(G(D), -) \Rightarrow \mathbb{D}(F(-), D)\} \cong \{\eta_D\colon FG(D) \to D\},$$

such that for every $\mathbb{V}$-natural transformation $\alpha_{D,-}$, there is a unique η_D, $\mathbb{V}$-natural in D, given by the image of $1_{G(D)}$ under

$$\mathbf{elem}\big|\alpha_{D,G(D)}\big| : \mathbb{C}_o(G(D), G(D)) \to \mathbb{D}_o(FG(D), D),$$

and for every η_D, $\mathbb{V}$-natural in D, there is a unique $\mathbb{V}$-natural transformation $\alpha_{D,-}$, where the component $\alpha_{D,A}$ is given by

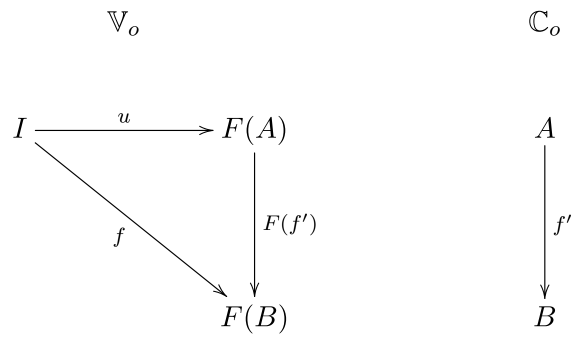
$$\mathbb{C}(G(D), A) \xrightarrow{F_{G(D),A}} \mathbb{D}(F(A), FG(D)) \xrightarrow{\mathbb{D}(F(A),\eta_D)} \mathbb{D}(F(A), D).$$

C.8 Universal Elements and Universal Morphisms

The material in this section follows that of Mac Lane (1997, III.1–2), adapted to the enriched setting.

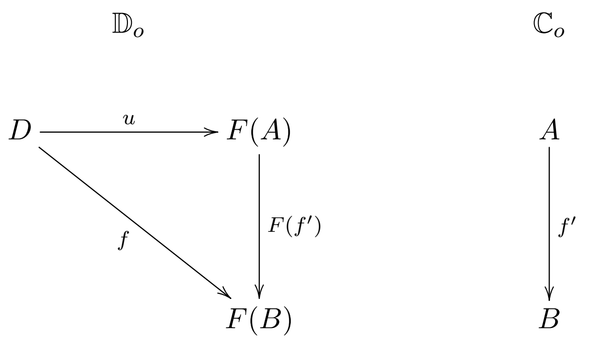
We start with the idea of a universal element. This is an idea that possibly finds its true home in enriched category theory, as it is in this setting that we have formalised the notion of element.

Definition C.49. For a covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$, a **universal element of** F is a pair (A, u), where $A \in \mathbf{obj}|\mathbb{C}|$, and $u \in \mathbf{elem}|F(A)|$ (i.e. $u\colon I \to F(A)$), such that for all pairs (B, f), with $B \in \mathbf{obj}|\mathbb{C}|$ and $f \in \mathbf{elem}|F(B)|$, there exists a unique $f' \in \mathbb{C}_o(A, B)$ with $F(f') \bullet u = f$.



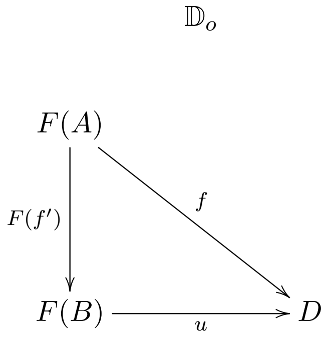
We can also define universal morphisms to or from a $\mathbb{V}$-functor. These will be familiar to any student of ordinary category theory, and since formally we don't have morphisms in a $\mathbb{V}$-category, the correct place to define them is in the corresponding underlying categories. However, as we shall see in Proposition C.59, since universal morphisms are defined at the level of the underlying categories, they are really too weak a concept.

Definition C.50. For a covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$, and an object D in $\mathbb{D}$, a pair (A, u), where $A \in \mathbf{obj}|\mathbb{C}|$, and $u \in \mathbb{D}_o(D, F(A))$, is a **universal morphism from** D **to** F, if for all $B \in \mathbf{obj}|\mathbb{C}|$ and $f \in \mathbb{D}_o(D, F(B))$, there exists a unique $f' \in \mathbb{C}_o(A, B)$ with $F(f') \bullet u = f$.

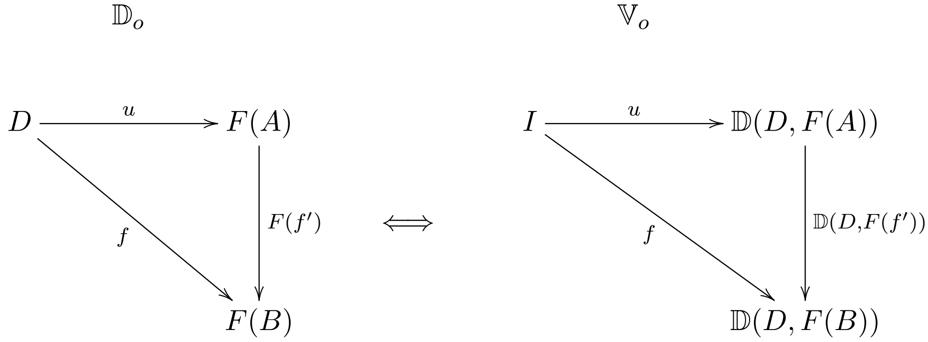


The dual definition is a universal morphism from a functor.

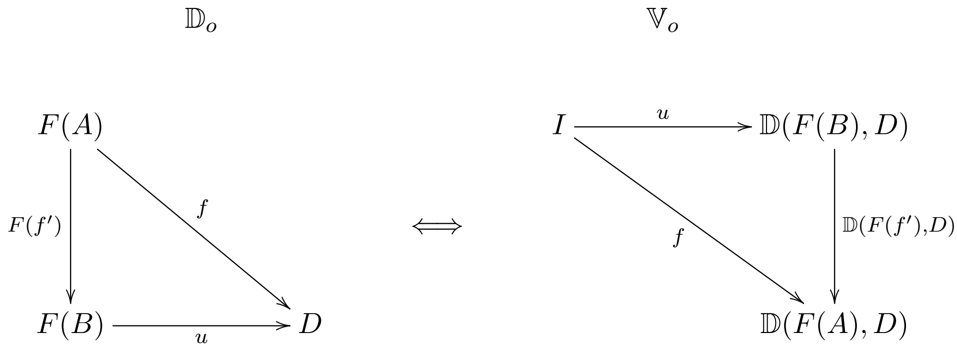
Definition C.51. For a covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$, and an object D in $\mathbb{D}$, a pair (B, u), where $B \in \mathbf{obj}|\mathbb{C}|$, and $u \in \mathbb{D}_o(F(B), D)$, is a **universal morphism from** F **to** D, if for all $A \in \mathbf{obj}|\mathbb{C}|$ and $f \in \mathbb{D}_o(F(A), D)$, there exists a unique $f' \in \mathbb{C}_o(A, B)$ with $u \bullet F(f') = f$.



Now we know from Section C.5 on hom-functors, that if $F\colon \mathbb{C} \to \mathbb{D}$ is covariant, then for $f' \in \mathbb{C}_o(A, B)$ we have the following bijection of diagrams



and similarly



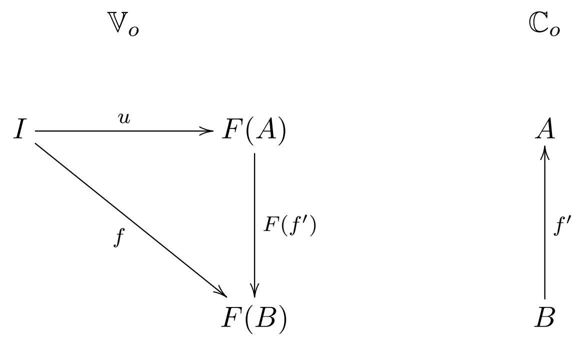
This then leads us to the following propositions.

Proposition C.52. *For a covariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, a pair* $(A, u\colon D \to F(A))$ *is a universal morphism from* D *to* F*, if and only if,* $(A, u\colon I \to \mathbb{D}(D, F(A)))$ *is a universal element of* $\mathbb{D}(D, F(-))$*.*

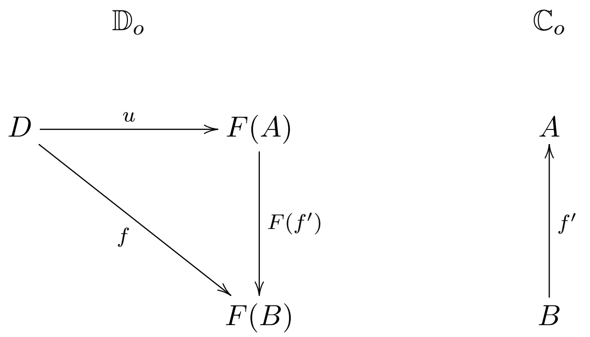
Proposition C.53. *For a covariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, a pair* $(B, u\colon F(B) \to D)$ *is a universal morphism from* F *to* D*, if and only if,* $(B, u\colon I \to \mathbb{D}(F(B), D))$ *is a universal element of* $\mathbb{D}(F(-), D)$*.*

As usual there are contravariant versions of universal elements and universal morphisms.

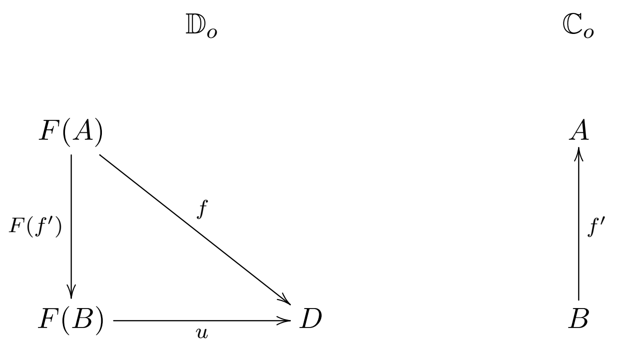
Definition C.54. For a contravariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$, a **universal element of** F is a pair (A, u), where $A \in \mathbf{obj}|\mathbb{C}|$, and $u \in \mathbf{elem}|F(A)|$ (i.e. $u\colon I \to F(A)$), such that for all pairs (B, f), with $B \in \mathbf{obj}|\mathbb{C}|$ and $f \in \mathbf{elem}|F(B)|$, there exists a unique $f' \in \mathbb{C}_o(B, A)$ with $F(f') \bullet u = f$.



Definition C.55. For a contravariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$, and an object D in $\mathbb{D}$, a pair (A, u), where $A \in \mathbf{obj}|\mathbb{C}|$, and $u \in \mathbb{D}_o(D, F(A))$, is a **universal morphism from** D **to** F, if for all $B \in \mathbf{obj}|\mathbb{C}|$ and $f \in \mathbb{D}_o(D, F(B))$, there exists a unique $f' \in \mathbb{C}_o(B, A)$ with $F(f') \bullet u = f$.



Definition C.56. For a contravariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$, and an object D in $\mathbb{D}$, a pair (B, u), where $B \in \mathbf{obj}|\mathbb{C}|$, and $u \in \mathbb{D}_o(F(B), D)$, is a **universal morphism from** F **to** D, if for all $A \in \mathbf{obj}|\mathbb{C}|$ and $f \in \mathbb{D}_o(F(A), D)$, there exists a unique $f' \in \mathbb{C}_o(B, A)$ with $u \bullet F(f') = f$.



Just like in the covariant case, universal elements and universal morphisms are essentially the same thing.

Proposition C.57. *For a contravariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, a pair* $(A, u\colon D \to F(A))$ *is a universal morphism from* D *to* F*, if and only if,* $(A, u\colon I \to \mathbb{D}(D, F(A)))$ *is a universal element of* $\mathbb{D}(D, F(-))$*.*

Proposition C.58. *For a contravariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, a pair* $(B, u\colon F(B) \to D)$ *is a universal morphism from* F *to* D*, if and only if,* $(B, u\colon I \to \mathbb{D}(F(B), D))$ *is a universal element of* $\mathbb{D}(F(-), D)$*.*

For a covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$, the Yoneda Lemma (Proposition C.48), when applied to the functor $\mathbb{D}(D, F(-))$, gives a bijection between $\mathbb{V}$-natural transformations of the form $\alpha_B\colon \mathbb{C}(A, B) \to \mathbb{D}(D, F(B))$, and morphisms $u\colon D \to F(A)$.

So given a morphism $u\colon D \to F(A)$, by Proposition C.48, we have that

$$\alpha_B = \mathbb{D}(u, F(B)) \circ F_{A,B},$$

and so if u is universal from D to F, $\mathbf{elem}|\alpha_B| : \mathbb{C}_o(A, B) \to \mathbb{D}_o(D, F(B))$ is a bijection of hom-sets.

Conversely, if α_B is a natural isomorphism, then for any $f\colon D \to F(B)$, there is a unique $f'\colon A \to B$ such that $\alpha_B(f') = f$. So by the naturality of α, we have $f = F(f') \bullet \alpha_A(1_A)$, but by Yoneda, $u = \alpha_A(1_A)$, and so $f = F(f') \bullet u$. Thus u is universal from D to F.

We can follow a similar argument for the other cases in Proposition C.48, and this yields the following proposition, which is essentially Mac Lane (1997, III.2 Proposition 1).

Proposition C.59.

1. *Given a covariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, and* $A \in \mathbf{obj}|\mathbb{C}|$ *and* $D \in \mathbf{obj}|\mathbb{D}|$*, then there is a bijection between natural isomorphisms of the form*

$$\alpha_B\colon \mathbb{C}_o(A, B) \to \mathbb{D}_o(D, F(B)),$$

   and morphisms $u\colon D \to F(A)$ *that are universal from* D *to* F*.*

2. *Given a covariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, and* $A \in \mathbf{obj}|\mathbb{C}|$ *and* $D \in \mathbf{obj}|\mathbb{D}|$*, then there is a bijection between natural isomorphisms of the form*

$$\alpha_B\colon \mathbb{C}_o(B, A) \to \mathbb{D}_o(F(B), D),$$

   and morphisms $u\colon F(A) \to D$ *that are universal from* F *to* D*.*

3. *Given a contravariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, and* $A \in \mathbf{obj}|\mathbb{C}|$ *and* $D \in \mathbf{obj}|\mathbb{D}|$*, then there is a bijection between natural isomorphisms of the form*

$$\alpha_B\colon \mathbb{C}_o(B, A) \to \mathbb{D}_o(D, F(B)),$$

and morphisms $u\colon D \to F(A)$ *that are universal from* D *to* F*.*

4. *Given a contravariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{D}$*, and* $A \in \mathbf{obj}|\mathbb{C}|$ *and* $D \in \mathbf{obj}|\mathbb{D}|$*, then there is a bijection between natural isomorphisms of the form*

$$\alpha_B\colon \mathbb{C}_o(A, B) \to \mathbb{D}_o(F(B), D),$$

and morphisms $u\colon F(A) \to D$ *that are universal from* F *to* D*.*

C.9 Representable Functors

Just like in ordinary category theory, from the definition of an enriched hom-functor we can define what it means for a $\mathbb{V}$-functor to be representable.

Definition C.60. A covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$ is **representable**, if there exists $K \in \mathbf{obj}|\mathbb{C}|$, and a $\mathbb{V}$-natural isomorphism

$$\alpha\colon \mathbb{C}(K, -) \Rightarrow F\colon \mathbb{C} \to \mathbb{V}.$$

The pair (K, α) is a **representation** of F, and the corresponding element $\eta\colon I \to F(K)$ given by the Yoneda Lemma is called the **unit** of the representation.

Definition C.61. A contravariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$ is **representable**, if there exists $K \in \mathbf{obj}|\mathbb{C}|$, and a $\mathbb{V}$-natural isomorphism

$$\alpha\colon \mathbb{C}(-, K) \Rightarrow F\colon \mathbb{C} \to \mathbb{V}.$$

The pair (K, α) is a **representation** of F, and the corresponding element $\eta\colon I \to F(K)$ given by the Yoneda Lemma is called the **counit** of the representation.

Suppose we have two representations (K, α) and (K', α') for a covariant $\mathbb{V}$-functor F. Then clearly since α and α' are isomorphisms, for all $A \in \mathbf{obj}|\mathbb{C}|$, there exists an isomorphism between $\mathbb{C}(K, A)$ and $\mathbb{C}(K', A)$. Now by the Yoneda Lemma (Proposition C.48), this means there exists a unique $k\colon K' \to K$ in $\mathbb{C}_o$, such that the isomorphism between $\mathbb{C}(K, A)$ and $\mathbb{C}(K', A)$ is

$$\mathbb{C}(k, A)\colon \mathbb{C}(K, A) \to \mathbb{C}(K', A).$$

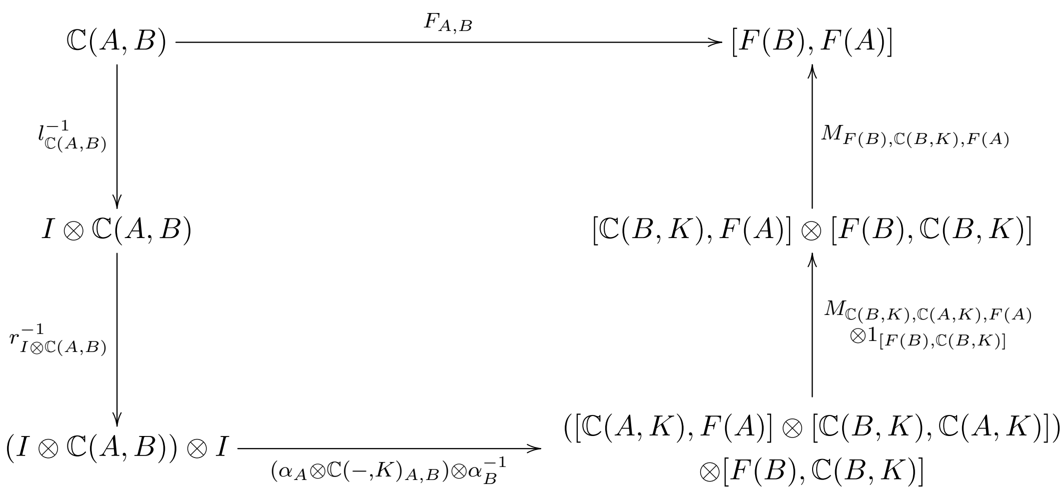
Moreover, it is clear that $\mathbb{C}(k, A)$ is an isomorphism if and only if k is, which yields the following proposition.

Proposition C.62 (Kelly (1982), Section 1.10)**.** *A representation* (K, α) *of a covariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{V}$ *is unique up to isomorphism, i.e. given another representation* (K', α')*, there exists a unique isomorphism* $k\colon K' \to K$*, such that for all* $A \in \mathbf{obj}|\mathbb{C}|$*,* $\alpha_A = \alpha'_A \circ \mathbb{C}(k, A)$*.*

Similarly for a contravariant $\mathbb{V}$-functor.

Proposition C.63. *A representation* (K, α) *of a contravariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{V}$ *is unique up to isomorphism, i.e. given another representation* (K', α')*, there exists a unique isomorphism* $k\colon K \to K'$*, such that for all* $A \in \mathbf{obj}|\mathbb{C}|$*,* $\alpha_A = \alpha'_A \circ \mathbb{C}(A, k)$*.*

Given a representation (K, α) for a contravariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$, it is straightforward to show that the following diagram commutes.



A similar diagram can be constructed for the covariant case, and together they yield the following propositions.

Proposition C.64. *Given a covariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{V}$*, and a representation* (K, α)*, for any* $f \in \mathbb{C}_o(A, B)$*,*

$$F(f) = \alpha_B \circ \mathbb{C}(K, f) \circ \alpha_A^{-1}.$$

Proposition C.65. *Given a contravariant* $\mathbb{V}$*-functor* $F\colon \mathbb{C} \to \mathbb{V}$*, and a representation* (K, α)*, for any* $f \in \mathbb{C}_o(A, B)$*,*

$$F(f) = \alpha_A \circ \mathbb{C}(f, K) \circ \alpha_B^{-1}$$

Suppose we have a covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$, and a representation $\mathbb{C}(K, -) \cong F$. Now we know from Proposition B.12, that for all $X \in \mathbf{obj}|\mathbb{V}|$, we have $X \cong [I, X]$, so

$$\mathbb{C}(K, -) \cong F \cong [I, F(-)].$$

Similarly, if F is contravariant, we have

$$\mathbb{C}(-, K) \cong F \cong [I, F(-)].$$

Proposition C.59 then says that there exists a unique morphism $u\colon I \to F(K)$ that is universal from I to F (the unit or counit of the representation to be precise).

We can summarise this as the following proposition, that can be seen to be one direction of Mac Lane (1997, III.2 Proposition 2).

Proposition C.66. *Given a $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{V}$, for every representation (K, α), there exists a unique morphism $u\colon I \to F(K)$ that is universal from I to F.*

Remark C.67. It should be noted that Proposition C.66 is one directional, unlike Mac Lane (1997, III.2 Proposition 2), where each universal morphism from I to F yields a representation. The reason for this is simple, universal morphisms are defined at the level of the underlying categories, and can only induce bijections of hom-sets, not isomorphisms in $\mathbb{V}$ of hom-objects.

C.10 Adjunctions

We continue our enrichment of ordinary category theory notions by looking at adjunctions.

Definition C.68. A **$\mathbb{V}$-adjunction** $\eta, \varepsilon\colon F \dashv G\colon \mathbb{D} \to \mathbb{C}$, between covariant $\mathbb{V}$-functors $F\colon \mathbb{C} \to \mathbb{D}$ (the **left adjoint**), and $G\colon \mathbb{D} \to \mathbb{C}$ (the **right adjoint**), consists of $\mathbb{V}$-natural transformations $\eta\colon 1 \Rightarrow GF$ (the **unit**), and $\varepsilon\colon FG \Rightarrow 1$ (the **counit**), satisfying the **triangular equations**

$$\varepsilon F \circ F\eta = 1_F$$
$$G\varepsilon \circ \eta G = 1_G.$$

Now it is well known in ordinary category theory that an adjunction corresponds to a bijection of hom-sets (Definition A.33), so we can consider a $\mathbb{V}$-natural transformation of the form

$$\Phi_{C,D}\colon \mathbb{D}(F(C), D) \to \mathbb{C}(C, G(D)).$$

By the Yoneda Lemma (Proposition C.48), $\Phi_{C,D} = \mathbb{C}(\eta_C, G(D)) \circ G_{F(C),D}$ for a unique $\eta_C\colon C \to GF(C)$, $\mathbb{V}$-natural in C, and similarly for $\Psi_{C,D}\colon \mathbb{C}(C, G(D)) \to \mathbb{D}(F(C), D)$, we have that $\Psi_{C,D} = \mathbb{D}(F(C), \varepsilon_D) \circ F_{C,G(D)}$ for a unique $\varepsilon_D\colon FG(D) \to D$, $\mathbb{V}$-natural in D.

We would like $\Phi_{C,D}$ to be an isomorphism with $\Psi_{C,D}$ its inverse, so we want

$$\Psi_{C,D} \circ \Phi_{C,D} = 1_{\mathbb{D}(F(C),D)}$$
$$\Phi_{C,D} \circ \Psi_{C,D} = 1_{\mathbb{C}(C,G(D))}.$$

Setting $D = F(C)$ into the first equation, and considering the action on $1_{F(C)}$, we recover the first of the triangular equations. Similarly, setting $C = G(D)$ in the second equation recovers the second triangular equation.

Thus we have established the following result.

Proposition C.69 (Kelly (1969), Proposition 3.1)**.** *There is a bijection between* $\mathbb{V}$-*adjunctions*

$$\eta, \varepsilon \colon F \dashv G \colon \mathbb{D} \to \mathbb{C},$$

and $\mathbb{V}$*-natural isomorphisms*

$$\Phi_{C,D} \colon \mathbb{D}(F(C), D) \cong \mathbb{C}(C, G(D)),$$

where

$$\Phi_{C,D} = \mathbb{C}(\eta_C, G(D)) \circ G_{F(C),D}$$
$$\Phi^{-1}_{C,D} = \mathbb{D}(F(C), \varepsilon_D) \circ F_{C,G(D)}.$$

From the definition of the underlying ordinary natural transformation of a $\mathbb{V}$-natural transformation (Definition C.12), it is clear that any $\mathbb{V}$-adjunction $\eta, \varepsilon \colon F \dashv G \colon \mathbb{D} \to \mathbb{C}$ has an underlying ordinary adjunction $\eta_o, \varepsilon_o \colon F_o \dashv G_o \colon \mathbb{D}_o \to \mathbb{C}_o$, and that the corresponding isomorphism of hom-sets is $\mathbf{elem}|\Phi_{C,D}| : \mathbb{D}_o(F(C), D) \cong \mathbb{C}_o(C, G(D))$.

Proposition C.70 (Kelly (1982), Section 1.11)**.** *Given a* $\mathbb{V}$*-adjunction*

$$\eta, \varepsilon \colon F \dashv G \colon \mathbb{D} \to \mathbb{C},$$

there is an underlying ordinary adjunction

$$\eta_o, \varepsilon_o \colon F_o \dashv G_o \colon \mathbb{D}_o \to \mathbb{C}_o.$$

Moreover, for the isomorphism of hom-objects

$$\Phi_{C,D} \colon \mathbb{D}(F(C), D) \cong \mathbb{C}(C, G(D)),$$

the corresponding isomorphism of hom-sets is

$$\mathbf{elem}|\Phi_{C,D}| : \mathbb{D}_o(F(C), D) \cong \mathbb{C}_o(C, G(D)).$$

If **elem**$|-|$ is faithful, the existence of an ordinary adjunction between the ordinary functors underlying a pair of $\mathbb{V}$-functors, is enough to guarantee a $\mathbb{V}$-adjunction. Faithfulness of **elem**$|-|$ ensures that the unit and the counit are $\mathbb{V}$-natural (Proposition C.43), and since a $\mathbb{V}$-natural transformation and its underlying ordinary natural transformation have the same components, the triangular equations hold automatically.

Proposition C.71. *Given covariant $\mathbb{V}$-functors $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{C}$ such that the underlying ordinary functors form an ordinary adjunction $\eta_o, \varepsilon_o\colon F_o \dashv G_o\colon \mathbb{D}_o \to \mathbb{C}_o$, then if the functor* **elem**$|-|$ *is faithful, this lifts to a $\mathbb{V}$-adjunction $\eta, \varepsilon\colon F \dashv G\colon \mathbb{D} \to \mathbb{C}$.*

From Proposition C.69 it is clear that their is a tight relationship between $\mathbb{V}$-adjunctions and representable $\mathbb{V}$-functors, and this can be made precise as follows.

Proposition C.72 (Kelly (1982), Section 1.11)**.**

1. *A covariant $\mathbb{V}$-functor $G\colon \mathbb{D} \to \mathbb{C}$ has a left adjoint exactly when each $\mathbb{C}(C, G(-))$ is representable.*
2. *A covariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$ has a right adjoint exactly when each $\mathbb{D}(F(-), D)$ is representable.*

The notion of a $\mathbb{V}$-adjunction can be strengthen to that of an equivalence of $\mathbb{V}$-categories.

Definition C.73 (Kelly (1982), Section 1.11)**.** Given a $\mathbb{V}$-adjunction

$$\eta, \varepsilon\colon F \dashv G\colon \mathbb{D} \to \mathbb{C},$$

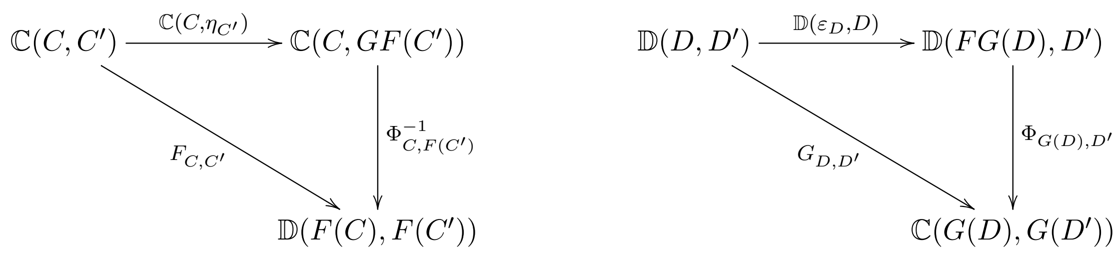
if the unit η and counit ε are $\mathbb{V}$-natural isomorphisms, then $\eta, \varepsilon\colon F \dashv G\colon \mathbb{D} \to \mathbb{C}$ is an **equivalence between** $\mathbb{C}$ **and** $\mathbb{D}$.

The following result is often useful in computations involving $\mathbb{V}$-adjunctions.

Proposition C.74 (Kelly (1969), Proposition 3.2)**.** *Given a $\mathbb{V}$-adjunction*

$$\eta, \varepsilon\colon F \dashv G\colon \mathbb{D} \to \mathbb{C},$$

*then for all $C, C' \in$ **obj**$|\mathbb{C}|$, and all $D, D' \in$ **obj**$|\mathbb{D}|$, the morphisms $F_{C,C'}$ and $G_{D,D'}$ are given by*



For contravariant functors we can develop a dual formulation of the above, though in this case we lose the obvious distinction between the left and right adjoints (Remark A.45).

Definition C.75. A $\mathbb{V}$**-dual adjunction** $\eta, \varepsilon \colon G \dashv F \colon \mathbb{C} \to \mathbb{D}$, between contravariant $\mathbb{V}$-functors $F \colon \mathbb{C} \to \mathbb{D}$, and $G \colon \mathbb{D} \to \mathbb{C}$, consists of $\mathbb{V}$-natural transformations $\eta \colon 1 \Rightarrow GF$ (the **unit**), and $\varepsilon \colon 1 \Rightarrow FG$ (the **counit**), satisfying the **triangular equations**

$$F\eta \circ \varepsilon F = 1_F$$
$$G\varepsilon \circ \eta G = 1_G.$$

Proposition C.76. *There is a bijection between* $\mathbb{V}$*-dual adjunctions*

$$\eta, \varepsilon \colon G \dashv F \colon \mathbb{C} \to \mathbb{D},$$

and $\mathbb{V}$*-natural isomorphisms*

$$\Phi_{C,D} \colon \mathbb{C}(C, G(D)) \cong \mathbb{D}(D, F(C)),$$

where

$$\Phi_{C,D} = \mathbb{D}(\varepsilon_D, F(C)) \circ F_{C,G(D)}$$
$$\Phi^{-1}_{C,D} = \mathbb{C}(\eta_C, G(D)) \circ G_{D,F(C)}.$$

Proposition C.77. *Given a* $\mathbb{V}$*-dual adjunction*

$$\eta, \varepsilon \colon G \dashv F \colon \mathbb{C} \to \mathbb{D},$$

there is an underlying ordinary dual adjunction

$$\eta_o, \varepsilon_o \colon G_o \dashv F_o \colon \mathbb{C}_o \to \mathbb{D}_o.$$

Moreover, for the isomorphism of hom-objects

$$\Phi_{C,D} \colon \mathbb{C}(C, G(D)) \cong \mathbb{D}(D, F(C)),$$

the corresponding isomorphism of hom-sets is

$$\mathbf{elem}|\Phi_{C,D}| : \mathbb{C}_o(C, G(D)) \cong \mathbb{D}_o(D, F(C)).$$

Proposition C.78. *Given contravariant* $\mathbb{V}$*-functors* $F \colon \mathbb{C} \to \mathbb{D}$ *and* $G \colon \mathbb{D} \to \mathbb{C}$ *such that the underlying functors form an ordinary dual adjunction* $\eta_o, \varepsilon_o \colon G_o \dashv F_o \colon \mathbb{C}_o \to \mathbb{D}_o$*, then if the functor* **elem**$|-|$ *is faithful, this lifts to a* $\mathbb{V}$*-dual adjunction* $\eta, \varepsilon \colon G \dashv F \colon \mathbb{C} \to \mathbb{D}$.

Since there is no real distinction between the left and right adjoints in a dual adjunction, Proposition C.72 collapses to the following.

Proposition C.79. *A contravariant $\mathbb{V}$-functor $F\colon \mathbb{C} \to \mathbb{D}$ has a dual adjoint exactly when each $\mathbb{D}(D, F(-))$ is representable.*

Definition C.80. Given a $\mathbb{V}$-dual adjunction

$$\eta, \varepsilon\colon G \dashv F\colon \mathbb{C} \to \mathbb{D},$$

if the unit η and counit ε are $\mathbb{V}$-natural isomorphisms, then $\eta, \varepsilon\colon G \dashv F\colon \mathbb{C} \to \mathbb{D}$ is a **dual equivalence between $\mathbb{C}$ and $\mathbb{D}$.**

Proposition C.81. *Given a $\mathbb{V}$-dual adjunction*

$$\eta, \varepsilon\colon G \dashv F\colon \mathbb{C} \to \mathbb{D},$$

then for all $C, C' \in \mathbf{obj}|\mathbb{C}|$, and all $D, D' \in \mathbf{obj}|\mathbb{D}|$, the morphisms $F_{C,C'}$ and $G_{D,D'}$ are given by

$$\begin{array}{ccc} \mathbb{C}(C,C') & \xrightarrow{\mathbb{C}(C,\eta_{C'})} & \mathbb{C}(C,GF(C')) \\ & \searrow^{F_{C,C'}} & \downarrow{\scriptstyle \Phi_{C,F(C')}} \\ & & \mathbb{D}(F(C'),F(C)) \end{array} \qquad \begin{array}{ccc} \mathbb{D}(D,D') & \xrightarrow{\mathbb{D}(D,\varepsilon_{D'})} & \mathbb{D}(D,FG(D')) \\ & \searrow^{G_{D,D'}} & \downarrow{\scriptstyle \Phi^{-1}_{G(D'),D}} \\ & & \mathbb{C}(G(D'),G(D)) \end{array}$$

C.11 Functor Categories

In order to develop the notion of limits and colimits for $\mathbb{V}$-categories, we need the concept of a functor category, but before we can define these, we need to define the concept of an end.

Definition C.82 (Kelly (1982), Section 2.1)**.** Given a $\mathbb{V}$-functor $F\colon \mathbb{C}^{op} \otimes \mathbb{C} \to \mathbb{V}$, if there exists a $\mathbb{V}$-natural family of morphisms

$$\lambda_A\colon K \to F(A, A),$$

such that for every $\mathbb{V}$-natural $\alpha_A\colon X \to F(A, A)$ there exists a unique $f\colon X \to K$ such that $\alpha_A = \lambda_A \circ f$, then the pair (K, λ) is called the **end** of F, and we write

$$\int_A F(A, A)$$

for K, and λ is called the **counit** of the end.

Now, since we are assuming that $\mathbb{V}_o$ is complete, we have the following result which ensures the existence of ends if $\mathbb{C}$ is small.

Proposition C.83 (Kelly (1982), Section 2.1)**.** *If the* $\mathbb{V}$*-category* $\mathbb{C}$ *is small, then for all* $\mathbb{V}$*-functors* $F\colon \mathbb{C}^{op} \otimes \mathbb{C} \to \mathbb{V}$*, the end*

$$\lambda_A\colon \int_A F(A, A) \to F(A, A)$$

exists.

Using the notion of an end we define functor categories as follows.

Definition C.84 (Kelly (1982), Section 2.2)**.** Given the $\mathbb{V}$-categories $\mathbb{C}$ and $\mathbb{D}$, the **functor category** $[\mathbb{C}, \mathbb{D}]$ has the following data:

1. The objects of $[\mathbb{C}, \mathbb{D}]$ are the $\mathbb{V}$-functors $F\colon \mathbb{C} \to \mathbb{D}$.
2. For every pair $F, G \in \mathbf{obj}|[\mathbb{C}, \mathbb{D}]|$, the hom-object $[\mathbb{C}, \mathbb{D}](F, G)$ is given by the end

   $$[\mathbb{C}, \mathbb{D}](F, G) = \int_A \mathbb{D}(F(A), G(A)),$$

   with the counit

   $$E_A = E_{A,F,G}\colon [\mathbb{C}, \mathbb{D}](F, G) \to \mathbb{D}(F(A), G(A)).$$

3. For all $F, G, H \in \mathbf{obj}|[\mathbb{C}, \mathbb{D}]|$, the composition law

   $$M_{F,G,H}\colon [\mathbb{C}, \mathbb{D}](G, H) \otimes [\mathbb{C}, \mathbb{D}](F, G) \to [\mathbb{C}, \mathbb{D}](F, H)$$

   is given by the universal property of $E_{A,F,H}$, such that

   $$\begin{array}{ccc} [\mathbb{C}, \mathbb{D}](G, H) \otimes [\mathbb{C}, \mathbb{D}](F, G) & \xrightarrow{M_{F,G,H}} & [\mathbb{C}, \mathbb{D}](F, H) \\ \downarrow {\scriptstyle E_{A,G,H} \otimes E_{A,F,G}} & & \downarrow {\scriptstyle E_{A,F,H}} \\ \mathbb{D}(G(A), H(A)) \otimes \mathbb{D}(F(A), G(A)) & \xrightarrow[M_{F(A),G(A),H(A)}]{} & \mathbb{D}(F(A), H(A)) \end{array}$$

4. For every $F \in \mathbf{obj}|[\mathbb{C}, \mathbb{D}]|$, the identity element j_F is given by the universal property of $E_{A,F,F}$, such that

   $$E_{A,F,F}(j_F) = j_{F(A)}.$$

If the $\mathbb{V}$-category $\mathbb{C}$ is small, then by Proposition C.83, the hom-objects all exist, and thus so does the functor category $[\mathbb{C}, \mathbb{D}]$.

Proposition C.85 (Kelly (1982), Section 2.2)**.** *Given the* $\mathbb{V}$*-categories* $\mathbb{C}$ *and* $\mathbb{D}$*, if* $\mathbb{C}$ *is small, then the functor category* $[\mathbb{C}, \mathbb{D}]$ *exists.*

The hom-set $[\mathbb{C},\mathbb{D}]_o(F,G)$ of the underlying category $[\mathbb{C},\mathbb{D}]_o$, corresponding to the $\mathbb{V}$-functors $F,G\colon \mathbb{C}\to\mathbb{D}$, is the set of $\mathbb{V}$-natural transformations $\alpha\colon F\Rightarrow G$.

The $E_{A,F,G}$ form a $\mathbb{V}$-functor $E_A\colon [\mathbb{C},\mathbb{D}]\to\mathbb{D}$ as given by the following definition.

Definition C.86 (Kelly (1982), Section 2.2)**.** Given the $\mathbb{V}$-categories $\mathbb{C}$ and $\mathbb{D}$, if the functor category $[\mathbb{C},\mathbb{D}]$ exists, then the family of morphisms

$$E_{A,F,G}\colon [\mathbb{C},\mathbb{D}](F,G)\to\mathbb{D}(F(A),G(A))$$

defines for all $A\in\mathbf{obj}|\mathbb{C}|$, a $\mathbb{V}$-functor called **evaluation at** A

$$E_A\colon [\mathbb{C},\mathbb{D}]\to\mathbb{D},$$

where $E_A(F)=F(A)$, and for every pair $F,G\in\mathbf{obj}|[\mathbb{C},\mathbb{D}]|$, $E_{A,F,G}(\alpha)=\alpha_A$.

Given a $\mathbb{V}$-functor $F\colon\mathbb{C}\to\mathbb{D}$ it straightforward to define a $\mathbb{V}$-functor from $[\mathbb{B},\mathbb{C}]$ to $[\mathbb{B},\mathbb{D}]$, that corresponds to post-composition with F.

Proposition C.87. *Given a $\mathbb{V}$-category $\mathbb{B}$ and a $\mathbb{V}$-functor $F\colon\mathbb{C}\to\mathbb{D}$, if the functor categories $[\mathbb{B},\mathbb{C}]$ and $[\mathbb{B},\mathbb{D}]$ exist, then we can define a $\mathbb{V}$-functor*

$$\hat{F}\colon [\mathbb{B},\mathbb{C}]\to[\mathbb{B},\mathbb{D}],$$

where $\hat{F}(G)=FG$, and for all $G,H\in\mathbf{obj}|[\mathbb{B},\mathbb{C}]|$,

$$(\hat{F})_{oG,H}\colon [\mathbb{B},\mathbb{C}]_o(G,H)\to[\mathbb{B},\mathbb{D}]_o(FG,FH)$$
$$\alpha\mapsto F\alpha.$$

Similarly for pre-composition with F.

Proposition C.88. *Given a $\mathbb{V}$-category $\mathbb{D}$ and a $\mathbb{V}$-functor $F\colon\mathbb{B}\to\mathbb{C}$, if the functor categories $[\mathbb{C},\mathbb{D}]$ and $[\mathbb{B},\mathbb{D}]$ exist, then we can define a $\mathbb{V}$-functor*

$$\hat{F}\colon [\mathbb{C},\mathbb{D}]\to[\mathbb{B},\mathbb{D}],$$

where $\hat{F}(G)=GF$, and for all $G,H\in\mathbf{obj}|[\mathbb{C},\mathbb{D}]|$,

$$(\hat{F})_{oG,H}\colon [\mathbb{C},\mathbb{D}]_o(G,H)\to[\mathbb{B},\mathbb{D}]_o(GF,HF)$$
$$\alpha\mapsto \alpha F.$$

C.12 Free $\mathbb{V}$-Categories

Since we are assuming that $\mathbb{V}_o$ is cocomplete, it has small coproducts, so the functor $\mathbf{elem}|-|$ has a left adjoint $\coprod_{-} I\colon \mathbf{Set}\to\mathbb{V}_o$, sending the set E to the coproduct $\coprod_E I$

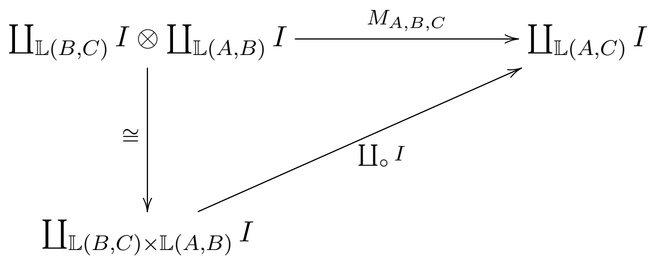
of E copies of I. Moreover, since $\mathbb{V}_o$ is closed, $\otimes$ preserves colimits, so

$$\begin{aligned}\coprod_E I \otimes \coprod_F I &\cong \coprod_E \left(I \otimes \coprod_F I\right)\\ &\cong \coprod_E \left(\coprod_F I\right)\\ &\cong \coprod_{E\times F} I.\end{aligned}$$

From this we are able to define for every ordinary category $\mathbb{L}$, the free $\mathbb{V}$-category over $\mathbb{L}$. This has the same objects as $\mathbb{L}$, but "promotes" the hom-sets of $\mathbb{L}$ to objects in $\mathbb{V}_o$ using the above left adjoint to $\mathbf{elem}|-|$.

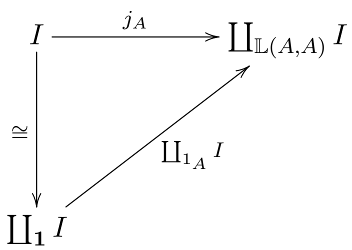
Definition C.89 (Kelly (1982), Section 2.5)**.** Given a locally small ordinary category $\mathbb{L}$, the $\mathbb{V}$-category $\mathbb{L}_{\mathbb{V}}$ is called the **free $\mathbb{V}$-category** on $\mathbb{L}$, and has the following data:

1. The objects of $\mathbb{L}_{\mathbb{V}}$ are precisely the same as those of $\mathbb{L}$, i.e. $\mathbf{obj}|\mathbb{L}_{\mathbb{V}}| = \mathbf{obj}|\mathbb{L}|$.
2. For every pair $A, B \in \mathbf{obj}|\mathbb{L}_{\mathbb{V}}|$, the hom-object $\mathbb{L}_{\mathbb{V}}(A, B)$ is given by
$$\mathbb{L}_{\mathbb{V}}(A, B) = \coprod_{\mathbb{L}(A,B)} I.$$
3. For all pair $A, B, C \in \mathbf{obj}|\mathbb{L}_{\mathbb{V}}|$, the composition law
$$M_{A,B,C} \colon \mathbb{L}_{\mathbb{V}}(B, C) \otimes \mathbb{L}_{\mathbb{V}}(A, B) \to \mathbb{L}_{\mathbb{V}}(A, C)$$
is given by



where $\coprod_\circ I$ is $\coprod_- I$ acting on composition in $\mathbb{L}$.

4. For every $A \in \mathbf{obj}|\mathbb{L}_{\mathbb{V}}|$, the identity element j_A is given by



Using the above definition of a free $\mathbb{V}$-category, it is also possible to lift an ordinary functor $F\colon \mathbb{L} \to \mathbb{C}_o$, where $\mathbb{C}_o$ is the underlying category of a $\mathbb{C}$-catgeory, to a $\mathbb{V}$-functor $\overline{F}\colon \mathbb{L}_{\mathbb{V}} \to \mathbb{C}$. Similarly, ordinary natural transformations can be lifted too.

Proposition C.90 (Kelly (1982), Section 2.5)**.** *Given a locally small ordinary category $\mathbb{L}$, there is an ordinary functor $\psi\colon \mathbb{L} \to (\mathbb{L}_{\mathbb{V}})_o$ defined by:*

1. *On objects ψ is the identity.*
2. *For every pair $A, B \in \mathbf{obj}|\mathbb{L}|$, the morphism $\psi_{A,B}\colon \mathbb{L}(A,B) \to \mathbf{elem}\left|\coprod_{\mathbb{L}(A,B)} I\right|$ is defined in the obvious way.*

Given a $\mathbb{V}$-category $\mathbb{C}$, then the following are true:

1. *If $F\colon \mathbb{L} \to \mathbb{C}_o$ is an ordinary functor, then there exists a $\mathbb{V}$-functor*

$$\overline{F}\colon \mathbb{L}_{\mathbb{V}} \to \mathbb{C},$$

such that $(\overline{F})_o \circ \psi = F$. Moreover, $\overline{F}$ is defined as follows:

   (a) *$\overline{F}(A) = F(A)$, for any $A \in \mathbf{obj}|\mathbb{L}_{\mathbb{V}}|$,*

   (b) *for every pair $A, B \in \mathbf{obj}|\mathbb{L}_{\mathbb{V}}|$, the morphism*

$$\overline{F}_{A,B}\colon \coprod_{\mathbb{L}(A,B)} I \to \mathbb{C}(F(A), F(B))$$

is the transpose of $F_{A,B}\colon \mathbb{L}(A,B) \to \mathbb{C}_o(F(A), F(B))$ under the adjunction $\coprod_{-} I \dashv \mathbf{elem}|-| : \mathbb{V}_o \to \mathbf{Set}$.

2. *If $\alpha\colon F \Rightarrow G\colon \mathbb{L} \to \mathbb{C}_o$ is an ordinary natural transformation, then there exists a $\mathbb{V}$-natural transformation*

$$\overline{\alpha}\colon \overline{F} \Rightarrow \overline{G}\colon \mathbb{L}_{\mathbb{V}} \to \mathbb{C},$$

such that $(\overline{\alpha})_o\psi = \alpha$. Moreover, the component $\overline{\alpha}_A$ is $\alpha_A \in \mathbb{C}_o(F(A), G(A))$.

If the ordinary category $\mathbb{L}$ is small, then since $\mathbb{L}_{\mathbb{V}}$ has precisely the same objects as $\mathbb{L}$, it too must be small, and hence by Proposition C.85, the functor category $[\mathbb{L}_{\mathbb{V}}, \mathbb{C}]$ exists. The objects of the underlying ordinary category $[\mathbb{L}_{\mathbb{V}}, \mathbb{C}]_o$ are the $\mathbb{V}$-functors from $\mathbb{L}_{\mathbb{V}}$ to $\mathbb{C}$, and the construction of Proposition C.90 yields a bijection between them and the ordinary functors from $\mathbb{L}$ to $\mathbb{C}_o$. Moreover, this extends to an isomorphism of ordinary categories.

Proposition C.91 (Kelly (1982), Section 2.5)**.** *Given a small ordinary category $\mathbb{L}$, and a $\mathbb{V}$-category $\mathbb{C}$, then $\mathbb{L}_{\mathbb{V}}$ is a small $\mathbb{V}$-category, and $[\mathbb{L}_{\mathbb{V}}, \mathbb{C}]$ exists. Moreover,*

$$[\mathbb{L}_{\mathbb{V}}, \mathbb{C}]_o \cong [\mathbb{L}, \mathbb{C}_o].$$

C.13 Limits and Colimits

Enriched category theory has a more general notion of limits and colimits than is standard in ordinary category theory. These are the so called indexed, or weighted, limits and colimits. The standard "cone" based limits and colimits of ordinary category theory then acquire the name conical limits and colimits. It turns out, though we shall not discuss this further, that in ordinary category theory all indexed limits can be constructed from conical limits, and similarly for colimits. However, in the general enriched case this is no longer true.

Definition C.92 (Kelly (1982), Section 3.1)**.** Given the covariant $\mathbb{V}$-functor $F\colon \mathbb{K} \to \mathbb{V}$ and the covariant $\mathbb{V}$-functor $G\colon \mathbb{K} \to \mathbb{C}$, if the contravariant $\mathbb{V}$-functor $H\colon \mathbb{C} \to \mathbb{V}$ given by

$$H(A) = [\mathbb{K}, \mathbb{V}](F, \mathbb{C}(A, G(-)))$$

is defined for all A, and has a representation

$$\mathbb{C}(A, \{F, G\}) \cong [\mathbb{K}, \mathbb{V}](F, \mathbb{C}(A, G(-))),$$

with counit

$$\mu\colon F \Rightarrow \mathbb{C}(\{F, G\}, G(-)),$$

then the representation $(\{F, G\}, \mu)$ is called the **limit of G indexed by F**. The $\mathbb{V}$-functor F is called the **indexing type**, and the $\mathbb{V}$-functor G is called the **diagram in $\mathbb{C}$ of type F**.

Definition C.93 (Kelly (1982), Section 3.1)**.** Given the contravariant $\mathbb{V}$-functor $F\colon \mathbb{K} \to \mathbb{V}$ and the covariant $\mathbb{V}$-functor $G\colon \mathbb{K} \to \mathbb{C}$, if the covariant $\mathbb{V}$-functor $H\colon \mathbb{C} \to \mathbb{V}$ given by

$$H(A) = [\mathbb{K}, \mathbb{V}](F, \mathbb{C}(G(-), A))$$

is defined for all A, and has a representation

$$\mathbb{C}(F * G, A) \cong [\mathbb{K}, \mathbb{V}](F, \mathbb{C}(G(-), A))$$

with unit

$$\mu\colon F \Rightarrow \mathbb{C}(G(-), F * G)$$

then the representation $(F * G, \mu)$ is called the **colimit of G indexed by F**. The $\mathbb{V}$-functor F is called the **indexing type**, and the $\mathbb{V}$-functor G is called the **diagram in $\mathbb{C}$ of type F**.

Conical limits and colimits correspond to the special case where the diagrams are given by an ordinary category $\mathbb{J}$, and all "cone" vertices have the same weight or index. We do this by defining the diagonal functor $\Delta_I\colon \mathbb{J} \to \mathbb{V}_o$ that maps every object of $\mathbb{J}$ to I, and every morphism of $\mathbb{J}$ to 1_I. Then by Proposition C.90, Δ_I and the diagram $G\colon \mathbb{J} \to \mathbb{C}_o$ are paired with $\mathbb{V}$-functors $\overline{\Delta_I}\colon \mathbb{J}_{\mathbb{V}} \to \mathbb{V}$ and $\overline{G}\colon \mathbb{J}_{\mathbb{V}} \to \mathbb{C}$, for which we take the indexed limits and colimits.

Definition C.94 (Kelly (1982), Section 3.8)**.** Given a $\mathbb{V}$-category $\mathbb{C}$, a locally small ordinary category $\mathbb{J}$, and an ordinary functor $G\colon \mathbb{J} \to \mathbb{C}_o$, then the limit $(\{\overline{\Delta_I}, \overline{G}\}, \mu)$, if it exists, of $\overline{G}\colon \mathbb{J}_{\mathbb{V}} \to \mathbb{C}$ indexed by $\overline{\Delta_I}\colon \mathbb{J}_{\mathbb{V}} \to \mathbb{V}$, is called the **conical limit in** $\mathbb{C}$ **of** G, and we write

$$\lim_{\mathbb{C}}(G) = \{\overline{\Delta_I}, \overline{G}\},$$

giving the representation

$$\mathbb{C}(A, \lim_{\mathbb{C}}(G)) \cong [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(A, \overline{G}(-))).$$

The counit

$$\mu\colon \overline{\Delta_I} \Rightarrow \mathbb{C}(\lim_{\mathbb{C}}(G), \overline{G}(-))$$

has components

$$\mu_J\colon \lim_{\mathbb{C}}(G) \to G(J),$$

which give the limiting cone of G in $\mathbb{C}_o$.

Definition C.95 (Kelly (1982), Section 3.8)**.** Given a $\mathbb{V}$-category $\mathbb{C}$, a locally small ordinary category $\mathbb{J}$, and an ordinary functor $G\colon \mathbb{J} \to \mathbb{C}_o$, then the colimit $(\overline{\Delta_I} * \overline{G}, \mu)$, if it exists, of $\overline{G}\colon \mathbb{J}_{\mathbb{V}} \to \mathbb{C}$ indexed by $\overline{\Delta_I}\colon \mathbb{J}_{\mathbb{V}} \to \mathbb{V}$, is called the **conical colimit in** $\mathbb{C}$ **of** G, and we write

$$\operatorname{colim}_{\mathbb{C}}(G) = \overline{\Delta_I} * \overline{G},$$

giving the representation

$$\mathbb{C}(\operatorname{colim}_{\mathbb{C}}(G), A) \cong [\mathbb{J}_{\mathbb{V}}, \mathbb{V}](\overline{\Delta_I}, \mathbb{C}(\overline{G}(-), A)).$$

The unit

$$\mu\colon \overline{\Delta_I} \Rightarrow \mathbb{C}(\overline{G}(-), \operatorname{colim}_{\mathbb{C}}(G))$$

has components

$$\mu_J\colon G(J) \to \operatorname{colim}_{\mathbb{C}}(G),$$

which give the colimiting cocone of G in $\mathbb{C}_o$.

Just like in ordinary category theory, $\mathbb{V}$-functors can be said to preserve and create limits and colimits.

Given a $\mathbb{V}$-functor $H\colon \mathbb{C} \to \mathbb{D}$, then for $\mathbb{V}$-functors $F\colon \mathbb{K} \to \mathbb{V}$ and $G\colon \mathbb{K} \to \mathbb{C}$, any $\mathbb{V}$-natural transformation

$$\alpha\colon F \Rightarrow \mathbb{C}(A, G(-)),$$

yields a $\mathbb{V}$-natural transformation

$$\beta\colon F \Rightarrow \mathbb{D}(H(A), HG(-)),$$

given by $\beta = H_{A,G(-)} \circ \alpha$. Thus the limit $(\{F, G\}, \mu)$ of G indexed by F, can be mapped to the pair $(H(\{F, G\}), H_{\{F,G\},G(-)} \circ \mu)$. Similarly, if F is contravariant, the colimit $(F * G, \mu)$ of G indexed by F, can be mapped to the pair $(H(F * G), H_{G(-),F*G} \circ \mu)$.

Using this we can define what is meant by preservation and creation of limits and colimits in the enriched setting.

Definition C.96. Given the covariant $\mathbb{V}$-functor $F\colon \mathbb{K} \to \mathbb{V}$, and the $\mathbb{V}$-functors $G\colon \mathbb{K} \to \mathbb{C}$, and $H\colon \mathbb{C} \to \mathbb{D}$, then H **preserves limits** of G indexed by F, if for any limit $(\{F, G\}, \mu)$ of G indexed by F, we have that $(H(\{F, G\}), H_{\{F,G\},G(-)} \circ \mu)$ is a limit of HG indexed by F.

Definition C.97. Given the contravariant $\mathbb{V}$-functor $F\colon \mathbb{K} \to \mathbb{V}$, and the $\mathbb{V}$-functors $G\colon \mathbb{K} \to \mathbb{C}$, and $H\colon \mathbb{C} \to \mathbb{D}$, then H **preserves colimits** of G indexed by F, if for any colimit $(F * G, \mu)$ of G indexed by F, we have that $(H(F * G), H_{G(-),F*G} \circ \mu)$ is a colimit of HG indexed by F.

Definition C.98. Given the covariant $\mathbb{V}$-functor $F\colon \mathbb{K} \to \mathbb{V}$, and the $\mathbb{V}$-functors $G\colon \mathbb{K} \to \mathbb{C}$, and $H\colon \mathbb{C} \to \mathbb{D}$, then H **creates limits** of G indexed by F, if for any limit $(\{F, HG\}, \mu)$ of HG indexed by F, there exists a unique pair (A, ν), where ν is a $\mathbb{V}$-natural transformation

$$\nu\colon F \Rightarrow \mathbb{C}(A, G(-)),$$

and such that $H(A) = \{F, HG\}$, $H_{A,G(-)} \circ \nu = \mu$, and (A, ν) is a limit of G indexed by F.

Definition C.99. Given the contravariant $\mathbb{V}$-functor $F\colon \mathbb{K} \to \mathbb{V}$, and the $\mathbb{V}$-functors $G\colon \mathbb{K} \to \mathbb{C}$, and $H\colon \mathbb{C} \to \mathbb{D}$, then H **creates colimits** of G indexed by F, if for any colimit $(F * HG, \mu)$ of HG indexed by F, there exists a unique pair (A, ν), where ν is a $\mathbb{V}$-natural transformation

$$\nu\colon F \Rightarrow \mathbb{C}(G(-), A),$$

and such that $H(A) = F * HG$, $H_{G(-),A} \circ \nu = \mu$, and (A, ν) is a colimit of G indexed by F.

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