UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL AND HUMAN SCIENCES

Department of Mathematics

Aspects of G-Complete Reducibility

by Daniel Gold

Thesis for the degree of Doctor of Philosophy

May 2012

1 Abstract

UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL AND HUMAN SCIENCES

Department of Mathematics

Doctor of Philosophy

Aspects of G-Complete Reducibility

by Daniel Gold

Let G be a connected reductive algebraic group, and σ a Frobenius morphism of G. Corresponding to the notion of G-complete reducibility, due to J.-P. Serre, we introduce a new notion of (G, σ)-complete reducibility. We show that a σ-stable subgroup of G is (G, σ)-completely reducible if and only if it is G-completely reducible. We also strengthen this result in one direction to show that if H is a σ-stable non G-completely reducible subgroup of G, then it is contained in a proper σ-stable parabolic subgroup P of G, and in no Levi subgroup of P. We go on to introduce another new notion, that of G^σ-complete reducibility for subgroups of G^σ. We show that a subgroup of G^σ is G^σ-completely reducible if and only if it is (G, σ)-completely reducible. Finally, we introduce the notion of strong σ-reductivity in G for σ-stable subgroups of G, and show that this is an analogue to the notion of strong reductivity in G in the setting of σ-stability.

We discuss a notion of G-complete reducibility for Lie subalgebras of $\mathrm{Lie}(G)$, which was introduced by McNinch. We show that if H is a subgroup of G that is contained in $C_G(S)$, where S is a maximal torus of $C_G(\mathrm{Lie}(H))$, then H is G-completely reducible if and only if $\mathrm{Lie}(H)$ is G-completely reducible. We give criteria for a Lie subalgebra of $\mathrm{Lie}(G)$ to be G-completely reducible. For example, an ideal in $\mathrm{Lie}(G)$ is G-completely reducible if it is invariant under the adjoint action of G.

Contents

2 Declaration of Authorship

I, Daniel Gold, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research. The title of this thesis is 'Aspects of G-Complete Reducibility'. I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Either none of this work has been published before submission, or parts of this work have been published as:
    - Gold, D., Herpel, S., Röhrle, G.: *Complete Reducibility and Steinberg Endomorphisms*, Comp. Rend. Math., **349** Issues 5–6, p243–246, (2011).

Signed: Date:

3 Acknowledgements

That was tough...

Of course, I must begin by thanking the University of Southampton Department of Mathematics for hosting me during my stay, and granting me this degree. I received invaluable support from the department for which I am extremely grateful. Along the way I received guidance and academic help from many people, in particular I would like to acknowledge and thank my supervisors Gerhard Röhrle and Bernhard Köck. I am grateful to the Hebrew University of Jerusalem for giving me the opportunity to visit and study in such a great institution, and I am especially grateful to Boris Kunyavski for his selfless support.

I would also like to thank my brothers Simon, David and Michael for always doing what they do. And sometimes helping. Then there are my parents, Kathy and Alvin, who are in no small part responsible for my undertaking of this adventure, partly by having introduced me to science and mathematics at a young age which inspired me to see the beauty in it. My wife, Hadassa, was a tremendous source of strength and kept on running with me during the ups and the downs.

The theory of algebraic groups is immensely rich in its symmetries, complexity and applications. I hope I served it some small justice, and I would encourage anyone with an interest to pursue it in order to gain an appreciation of its wonder.

4 Introduction

Let G be a connected reductive algebraic group, over the algebraic closure $k = \overline{\mathbb{F}}_q$ of the field $\mathbb{F}_q$ of characteristic p with $q = p^a$ elements, for a prime p and positive integer a. The notion of G-complete reducibility, which is central to this thesis, was introduced by J.-P. Serre, see [46]. We define this notion as follows.

Definition 6.10. *Let H be a subgroup of G.*

(1) *H is called G-**irreducible** (or G-**ir**) if H is not contained in any proper parabolic subgroup of G.*

(2) *H is called G-**completely reducible** (or G-**cr**) if whenever H is contained in a proper parabolic subgroup P of G, then H is contained in a Levi subgroup of P.*

Let H be an algebraic group, then H can be embedded in the general linear group $\mathrm{GL}(V)$ for some finite dimensional vector space V, say via the map ϕ. In standard representation theory, one investigates the properties of the homomorphism $\phi : H \to \mathrm{GL}(V)$. The vector space V can be regarded as a module over the group ring kH, which we refer to simply as an H-module. In this case, H is $\mathrm{GL}(V)$-completely reducible if and only if V is a semisimple H-module. The notion of $\mathrm{GL}(V)$-complete reducibility is therefore equivalent to the notion of V being a semisimple H-module.

The notion of G-complete reducibility is defined in greater generality, and in this sense it provides results which extend those from the standard representation theory of algebraic groups. This enables a new set of tools to be employed in the study of representation theory, as well as opening a new branch of mathematics which provides its own interesting and attractive theory.

In his paper [42], Richardson introduced the notion of strong reductivity in G. A closed subgroup H of G is called strongly reductive in G if H is not contained in any proper parabolic subgroup of $C_G(S)$, where S is a maximal torus of $C_G(H)$. Suppose that H is topologically generated by the elements $x_1, \ldots, x_n$. Richardson showed that H is strongly reductive in G if and only if the orbit of G on the n-tuple $(x_1, \ldots, x_n)$ by simultaneous conjugation is closed in G^n. Since the strongly reductive subgroups of G classify the closed G-orbits in G^n, strong reductivity can be viewed as a geometric notion, see [42, Theorem 16.4]. Bate, Martin and Röhrle showed, in [1, Theorem 3.1], that the notion of G-complete reducibility is equivalent to the notion of strong reductivity in G. This result is remarkable because it provides an equivalence between the geometric notion of strong reductivity, and the group theoretic notion of complete reducibility. One implication of this result is that it enables the use of methods from the field of geometric invariant theory in the study of G-complete reducibility.

As an example of such a use of geometric invariant theory, in [35] Martin showed that a normal subgroup of a strongly reductive subgroup in G is strongly reductive in G, and from the above remarks this implies that a normal subgroup of a G-completely reducible subgroup of G is G-completely reducible. If we consider the special case where $G = \mathrm{GL}(V)$, then we see that this striking result is in fact a direct analogue of Clifford's Theory in representation theory, see [14]. Given a normal subgroup N of G, Clifford's Theory asserts that if V is a semisimple kG-module, then V is a semisimple kN-module. Since semisimplicity of the module V and complete reducibility are equivalent for subgroups of $\mathrm{GL}(V)$, and in turn as we have equivalence between complete reducibility and strong reductivity, the required equivalence between Clifford's Theory and Martin's normal subgroup result in this setting follows.

In characteristic zero, a subgroup H of G is G-completely reducible if and only if H^0 is reductive as noted in [42, §16]. By [1, Theorem 3.48], if H is connected and the characteristic of k is larger than the Coxeter number of G, then we have that H is G-completely reducible if and only if H is reductive. However, for small positive characteristic there are examples of connected reductive groups which are not G-completely reducible, for instance see [1, Example 3.45]. In this example we take the field k to have characteristic 2 and let $n \geq 4$ be even. By using a diagonal embedding of $\mathrm{Sp}_n(k)$ in $\mathrm{Sp}_n(k) \times \mathrm{Sp}_n(k)$ it is shown that $\mathrm{Sp}_n(k)$ is not $\mathrm{Sp}_{2n}(k)$-completely reducible, however $\mathrm{Sp}_n(k)$ is connected and reductive. Therefore, the study of G-complete reducibility provides some interesting examples when the characteristic of the underlying field, k, is positive and small.

Let G be a subgroup of $\mathrm{GL}_n(k)$. A homomorphism $\sigma : G \to G$ is a Frobenius morphism if some power of σ is the map which sends the matrix $(x_{ij}) \mapsto (x_{ij}^p)$. This definition can easily be extended to algebraic groups isomorphic to G. When G is simple, a surjective homomorphism of G is a Frobenius morphism if, and only if, it fixes finitely many points, that is the subgroup $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ is finite. Frobenius morphisms are of general interest because the finite groups of Lie type arise as groups of the form G^σ when G is simple.

Let σ be a Frobenius morphism of G. We say that a subgroup H of G is σ-stable if $\sigma(H) = H$. In their paper [33], Liebeck and Seitz consider the case when G is simple and of exceptional type, and $H \subseteq G^\sigma$ is a finite subgroup of σ-fixed points of G which is G-completely reducible. In this case they showed that if H is contained in a σ-stable parabolic subgroup P of G, then H is contained in a σ-stable Levi subgroup of P. This motivates the following definition, which is one of the main definitions in this thesis.

Definition 8.1. *Let H be a σ-stable subgroup of G.*

(1) *We say H is (G,σ)-**completely reducible** (or (G,σ)-**cr**) if whenever*

H is contained in a σ-stable parabolic subgroup P of G, then H is contained in a σ-stable Levi subgroup of P.

(2) *We say H is (G, σ)-**irreducible** (or (G, σ)-**ir**) if H is not contained in any proper σ-stable parabolic subgroup of G.*

The following is the first important result in this thesis, and provides one direction of our investigation of the connection between the notions of G-complete reducibility and (G, σ)-complete reducibility.

Theorem 8.6. *A σ-stable G-completely reducible subgroup of G is (G, σ)-completely reducible.*

Theorem 8.6 is an extension of part of Liebeck and Seitz's result [33, Theorem 9] in that we have removed several conditions that were imposed, namely that G is of exceptional type, and H is contained in G^{σ}; we only need that H is σ-stable.

Liebeck, Martin and Shalev showed in [31, Proposition 2.2] that in the case G is simple and not of type B_2 $(p = 2), F_4$ $(p = 2)$ or G_2 $(p = 3)$, and σ is a Frobenius morphism of G, then a finite σ-stable subgroup of G is either strongly reductive in G, or is contained in a σ-stable parabolic subgroup P of G and in no Levi subgroup of P. In other words, this result shows that if a finite σ-stable subgroup of G is not G-completely reducible, then it is not (G, σ)-completely reducible. This result provides a partial converse to Theorem 8.6.

In Section 9 and Section 10 we explore the converse to Theorem 8.6 more generally. In Section 9.2 we present Theorem 9.12 which provides a converse to Theorem 8.6 for finite σ-stable subgroups of G. The proof follows the methods of [31, Proposition 2.2], however, for the cases where G^{σ} is a Ree or Suzuki group, we need to perform a case-by-case analysis. In particular, the equivalence presented in Theorem 9.12 holds for all Frobenius morphisms of the simple group G, and is therefore a significant generalisation of [31, Proposition 2.2].

Theorem 9.12. *Let G be a simple algebraic group, and let σ be a Frobenius morphism of G. Suppose that F is a finite σ-stable subgroup of G, then*

(1) *F is G-completely reducible if and only if it is (G, σ)-completely reducible, and*

(2) *if F is not G-completely reducible, then F is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

In Section 9.3 we provide a further generalisation of [31, Proposition 2.2] by extending Theorem 9.12 to include the case where G is reductive. We present this result in Theorem 9.15.

Theorem 9.15. *Let G be a reductive algebraic group, and let σ be a Frobenius morphism of G. Suppose that F is a finite σ-stable subgroup of G, then*

(1) *F is G-completely reducible if and only if it is (G, σ)-completely reducible, and*

(2) *if F is not G-completely reducible, then F is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

Theorem 9.15 is proved by using the techniques of Liebeck, Martin and Shalev [31], in addition to a novel method of pulling back to the σ-orbits of the simple groups that occur in G. By looking at these σ-orbits we can focus on the behaviour inside each of the simple factors of G, which is well understood by Theorem 9.12. Let H be one such σ-orbit. This method allows us to switch between the H-complete reducibility and (H, σ)-complete reducibility cases inside these σ-orbits, and hence using the results of [1, §2] we provide a way to consider all the σ-orbits together to return to the situation inside G itself, and thereby obtain the result.

In Section 10 we introduce the notion of a finite-σ-structure, and using it show that an infinite σ-stable subgroup of G can be modelled as a finite subgroup, in that a finite σ-stable subgroup can be found which shares the same G-complete reducible properties as the original group. The following statement is the main result of this section and shows that Theorem 9.15 holds in the case F is replaced with an arbitrary σ-stable subgroup H of G.

Theorem 10.6. *Let G be a reductive algebraic group, and let σ be a Frobenius morphism of G. Suppose that H is a σ-stable subgroup of G, then*

(1) *H is G-completely reducible if and only if it is (G, σ)-completely reducible, and*

(2) *if H is not G-completely reducible, then H is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

Theorem 10.6 is our main theorem in the study of (G, σ)-complete reducibility, and shows that the notions of G-complete reducibility and (G, σ)-complete reducibility are equivalent for σ-stable subgroups of G. This is a startling result because neither implication is obvious, and in one direction it gives information about a subgroup H of G with respect to its containment in general parabolic and Levi subgroups of G, based only upon its containment in σ-stable parabolic and σ-stable Levi subgroups of G.

We provide examples of (G, σ)-completely reducible subgroups of G. For instance, a σ-stable Levi subgroup of G is (G, σ)-completely reducible. For a Frobenius morphism σ of G, the finite group of Lie type G^σ is (G, σ)-completely reducible.

In the case that σ is a standard Frobenius morphism, and G is a reductive group, then Theorem 10.6 is equivalent to [1, Theorem 5.8]. When σ is any Frobenius morphism of G then part (1) of Theorem 10.6 is proved in [18].

In Section 10.2 we introduce another notion related to G-complete reducibility, that of G^σ-complete reducibility. In Proposition 10.19, we show that a subgroup of G^σ is G^σ-completely reducible if and only if it is (G, σ)-completely reducible. This leads us to more examples of (G, σ)-completely reducible subgroups of G, and of G-completely reducible subgroups of G, as in Examples 10.21 and 10.22.

We conclude Section 10 by discussing an analogue in the setting of σ-stability to the notion of strong reductivity in G. We introduce the following definition.

Definition 10.23. *A σ-stable subgroup H of G is **strongly σ-reductive in G** if H is not contained in any proper σ-stable parabolic subgroup of $C_G(S)$, where S is a σ-stable maximal torus of $C_G(H)$.*

We go on to show that the notions of strong σ-reductivity in G and (G, σ)-complete reducibility are equivalent, and this is analogous in the σ-stability setting to [1, Theorem 3.1].

Theorem 10.25. *Let H be a σ-stable subgroup of G. Then, H is strongly σ-reductive in G if, and only if, it is (G, σ)-completely reducible.*

In [37], McNinch introduced the notion of G-complete reducibility for Lie subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$. This is the analogous notion in the Lie subalgebra setting to that of G-complete reducibility for subgroups of G.

Definition 11.3. *Let G be a reductive algebraic group, and let $\mathfrak{h}$ be a Lie subalgebra of $\mathrm{Lie}(G)$.*

(1) *We say that $\mathfrak{h}$ is G-**completely reducible** (or G-**cr**) if whenever $\mathfrak{h} \subseteq \mathrm{Lie}(P)$ for some parabolic subgroup P of G, then $\mathfrak{h} \subseteq \mathrm{Lie}(L)$ for some Levi subgroup L of P.*

(2) *We say that $\mathfrak{h}$ is G-**irreducible** (or G-**ir**) if $\mathfrak{h}$ is not contained in the Lie algebra of any parabolic subgroup of G.*

(3) *We say that $\mathfrak{h}$ is G-**indecomposable** (or G-**ind**) if $\mathfrak{h}$ is not contained in the Lie algebra of any proper Levi subgroup of G.*

McNinch showed, in [37, Theorem 1], that if H is a G-completely reducible subgroup of G, then $\mathfrak{h} = \mathrm{Lie}(H)$ is a G-completely reducible Lie subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. However, the converse does not hold in general for both the connected and non-connected cases. The simplest way to see this is in the non-connected case, where we take a finite non-G-completely reducible subgroup F of G. Then $\mathrm{Lie}(F)$ is trivial and so is G-completely reducible. Indeed, counterexamples to the converse of [37, Theorem 1] exist even in the connected case, and in all positive characteristics, as shown in [37, p.1].

We show, in Remark 11.5, that if $\mathfrak{h}$ is a Lie subalgebra of $\mathrm{Lie}(\mathrm{GL}(V))$, then $\mathfrak{h}$ is $\mathrm{GL}(V)$-completely reducible if and only if V is a semisimple $\mathfrak{h}$-module. Remark 11.29 shows the corresponding result holds for Lie subalgebras of $\mathrm{Lie}(\mathrm{SO}(V))$ and $\mathrm{Lie}(\mathrm{Sp}(V))$.

We obtain, in Example 11.27, that if H is a subgroup of G that is not G-completely reducible, such that $\mathrm{Lie}(H)$ is G-completely reducible, then no maximal torus of $C_G(\mathrm{Lie}(H))$ normalises H. We provide a criterion for a subgroup K of G, and a Levi subgroup L of G such that we have the following equivalences:

$$\begin{aligned} K \text{ is } G\text{-completely reducible} &\Leftrightarrow K \text{ is } L\text{-completely reducible} \\ &\Leftrightarrow \mathrm{Lie}(K) \text{ is } G\text{-completely reducible} \\ &\Leftrightarrow \mathrm{Lie}(K) \text{ is } L\text{-completely reducible.} \end{aligned}$$

As in the group case, we define a notion, that of strong reductivity in G for Lie subalgebras of $\mathfrak{g}$ and, in Corollary 11.20, we show that a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is strongly reductive in G if and only if it is G-completely reducible. This definition is important in that this approach leads to the following proposition, which gives a sufficient condition for H to be G-completely reducible if and only if $\mathfrak{h}$ is G-completely reducible.

Proposition 11.24. *Let H be a closed subgroup of G such that H is contained in $C_G(S)$ where S is a maximal torus of $C_G(\mathfrak{h})$. Then H is G-completely reducible if and only if $\mathfrak{h}$ is G-completely reducible.*

We give a number of criteria for a Lie subalgebra of $\mathfrak{g}$ to be G-completely reducible. For instance in Section 11.2, by Theorem 11.35, if $\mathfrak{h}$ is a separable Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{g}$ is a semisimple $\mathfrak{h}$-module, then $\mathfrak{h}$ is G-completely reducible. In Section 11.3, as one of the main results of this section we have the following result which is an analogue in the Lie algebra setting of Martin's result on normal subgroups of strongly reductive subgroups of G, see [35, Theorem 2].

Theorem 11.38. *Let $H \subseteq G$ and suppose that $\mathfrak{h} = \mathrm{Lie}(H)$ is a G-completely reducible Lie subalgebra of $\mathfrak{g}$. Then, any H-invariant Lie subalgebra of $\mathfrak{h}$ is G-completely reducible.*

In order to prove this result we exploit the geometric invariant theory in a similar manner to that done by Martin in [35, Theorem 2].

In Section 11.4, we present the following corollary about ideals in $\mathfrak{g}$, again giving a new criterion for a Lie subalgebra to be G-completely reducible.

Corollary 11.40. *Let G be a simple algebraic group over k. Let $\mathfrak{m}$ be an ideal in $\mathfrak{g}$. If $\mathfrak{m}$ is G-invariant, then $\mathfrak{m}$ is G-completely reducible. In particular, if $char(k) \geq 3$, then any ideal in $\mathfrak{g}$ is G-completely reducible.*

Using Hogeweij's list of G-invariant ideals in $\mathfrak{g}$ for simple G, see [21], Corollary 11.40 gives a new method of finding G-completely reducible subalgebras of $\mathfrak{g}$.

This thesis is divided into three parts. Part I provides an introduction to the theory of affine varieties and algebraic groups and includes an outline of the classification of simple algebraic groups into the classical and exceptional types. Later in Part I we discuss the structure of reductive algebraic groups. Reductive groups are interesting and important as they possess a rich structure. In a reductive group we can ask whether a subgroup is G-completely reducible by looking at its properties of containment within parabolic subgroups and Levi subgroups. We also provide a short survey of some of the relevant and interesting results in the theory of G-complete reducibility to emerge over the past 10 years. We go on to discuss Frobenius morphisms of algebraic groups, and remark that the finite groups of Lie type arise as the fixed point groups of Frobenius morphisms.

In Part II we introduce the notion of (G, σ)-complete reducibility and discuss it in depth. We draw some parallels with the theory of G-complete reducibility. We go on to introduce the notion of G^σ-complete reducibility, again drawing parallels with G-complete reducibility, and show how these notions provide examples of G-completely reducible subgroups of G.

In Part III we discuss the notion of G-complete reducibility for Lie subalgebras of $\mathrm{Lie}(G)$. We show that G-complete reducibility has an analogue in the form of strong reductivity in G for Lie algebras. We use this to provide a criterion for a subgroup to be G-completely reducible if and only if its Lie algebra is.

Part I

Background Material

5 Introduction to Algebraic Groups

In this section we introduce algebraic varieties and define what an algebraic group is. We go on to sketch the classification of the simple algebraic groups. Throughout we let k be an algebraically closed field of arbitrary characteristic, unless stated otherwise.

5.1 Affine Sets and the Zariski Topology

View k^n as an n-dimensional vector space, and let $k[T] = k[T_1, \ldots, T_n]$ be the polynomial algebra of k-valued functions in n variables. Elements of $k[T]$ may be viewed as functions $f : k^n \to k$, that is k-valued functions on k^n. We define a point $x \in k^n$ to be a **zero of the function** f if $f(x) = 0$. We say that x is a **zero of the ideal** $I \subset k[T]$ if $f(x) = 0$ for all $f \in I$. We denote the set of zeros of the ideal I by $\mathscr{V}(I)$, that is $\mathscr{V}(I) := \{x \in k^n \mid f(x) = 0 \text{ for all } f \in I\}$. If $X \subseteq k^n$, then we denote the ideal of all $f \in k[T]$ whose zero set contains X by $\mathcal{I}(X)$, that is $\mathcal{I}(X) := \{f \in k[T] \mid f(x) = 0 \text{ for all } x \in X\}$.

For an ideal $I \subseteq k[T]$, the **radical of** I, denoted $\sqrt{I}$, is defined to be the set of all $f \in k[T]$ such that $f^m \in I$ for some $m \geq 1$.

There is a topology on k^n, called the **Zariski topology**, whose closed sets are the $\mathscr{V}(I)$. We call the closed sets in this topology **affine sets**. If we take $I = \{0\}$, then $\mathscr{V}(I) = k^n$, hence k^n is a closed set and its complement $\emptyset$ is open. Similarly, if we take $I = k[T]$, then $\mathscr{V}(I) = \emptyset$, hence the empty set is a closed set and its complement k^n is open. These are important examples of affine sets. We denote k^n when viewed as an affine set by the symbol $\mathbb{A}^n$.

For a subset $X \subseteq \mathbb{A}^n$ we have $\mathscr{V}(\mathcal{I}(X)) \supseteq X$, and for an ideal $I \subseteq k[T]$ we have $\mathcal{I}(\mathscr{V}(I)) \supseteq I$. A famous theorem called Hilbert's Nullstellensatz ('Theorem of Zeros'), see for example [29, Theorem 1.5] or [23, Theorem 1.1], provides the equality $\mathcal{I}(\mathscr{V}(I)) = \sqrt{I}$, giving a bijective correspondence between the affine sets in $\mathbb{A}^n$ and the set of radical ideals of $k[T]$. Examples of radical ideals in $k[T]$ are prime ideals. One corollary is that if I is a proper ideal in $k[T]$, then $\mathscr{V}(I)$ is not empty, and this is the motivation for the name of the theorem.

Let I be a maximal ideal in $k[T]$. From Hilbert's Nullstellensatz, we can conclude that $I \subset \mathcal{I}(\{x\}) \subsetneq k[T]$ for some $x \in \mathbb{A}^n$. Hence, $I = \mathcal{I}(\{x\})$. Conversely, if $x \in \mathbb{A}^n$, then $f(T) \mapsto f(x)$ is a surjective homomorphism $k[T] \to k$ with kernel $\mathcal{I}(\{x\})$. Therefore, to each point in $\mathbb{A}^n$ there exists a corresponding unique maximal ideal in $k[T]$.

Definition 5.1. *A topological space is called* ***irreducible*** *if it is not the union of two proper closed subsets.*

A point $(a_1, \ldots, a_n)$ in $\mathbb{A}^n$, is closed in the Zariski topology, being the unique zero of the polynomials $x_1 - a_1, \ldots, x_n - a_n$. In $\mathbb{A}^n$, finite sets of at least two points are reducible, being the union of finitely many points. The zero set of $x_1^2 + x_2^2 = c$, for a constant $c \in k^*$ (a circle), or a line in $\mathbb{A}^2$ are irreducible. The union of two intersecting but non-parallel affine lines in $\mathbb{A}^2$, however, is a reducible topological space, since it is the union of two different lines each of which is an affine set.

Definition 5.2. *A topological space is* ***connected*** *if it is not the union of two proper closed disjoint subsets.*

Immediately from Definition 5.2, we see that an irreducible space is connected. Therefore, the examples for irreducible sets are connected. However, in the converse, the union of two intersecting but non-parallel affine lines in $\mathbb{A}^2$ is connected but not irreducible.

The following is [23, Proposition 1.3 C].

Lemma 5.3. *A subset X of $\mathbb{A}^n$ is irreducible if and only if its ideal $\mathcal{I}(X)$ is prime. In particular, $\mathbb{A}^n$ is irreducible.*

Proof. Let $I = \mathcal{I}(X)$. Suppose X is irreducible. Let $f_1(T)f_2(T) \in I$. Then, each $x \in X$ is a zero of $f_1(T)$ or $f_2(T)$, so that X is covered by $\mathscr{V}(I_1) \cup \mathscr{V}(I_2)$, for I_i the ideal generated by $f_i(T)$. As X is irreducible, it lies completely within one of these two sets, so that $f_1(T) \in I$, or $f_2(T) \in I$. Thus, I is prime.

Conversely, suppose that I is prime, and that $X = X_1 \cup X_2$ for X_i a closed subset of X. If both X_i are properly contained within X, then there exists some $f_i(T) \in \mathcal{I}(X_i)$, such that $f_i(T) \notin I$. However, $f_1(T)f_2(T)$ vanishes on all of X, contradicting the primeness of I. $\square$

We now consider how to construct products of affine sets. Let $\{f_i\}$ be a set of polynomials in n variables that generate the ideal $(f_1, f_2, \ldots)$ in the ring $k[T_1, \ldots, T_n]$, and $\{g_j\}$ be a set of polynomials in m variables that generate the ideal $(g_1, g_2, \ldots)$ in the ring $k[T_{n+1}, \ldots, T_{n+m}]$. If $X = \mathscr{V}(f_1, f_2, \ldots) \subseteq \mathbb{A}^n$ and $Y = \mathscr{V}(g_1, g_2, \ldots) \subseteq \mathbb{A}^m$, then it is natural to consider $X \times Y$ to be the zero set of all the f_i and g_j viewed as polynomials in $n + m$ variables in $\mathbb{A}^n \times \mathbb{A}^m := \mathbb{A}^{n+m}$. The following is [23, Proposition 1.4].

Proposition 5.4. *If $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are closed irreducible sets, then $X \times Y$ is closed and irreducible in $\mathbb{A}^{n+m}$.*

The Zariski topology on $\mathbb{R}^n$ differs from euclidean topology in that far fewer sets are closed. For example, in $\mathbb{R}$ the only Zariski closed sets are $\mathbb{R}$ and finite sets of points, since points are the common zeros of linear polynomials. However, unlike in euclidean topology the interval $[0, 1]$, for example, is not closed.

In the Zariski topology every non-empty open set is infinite, and in an irreducible variety a non-trivial open set intersects non-trivially with every other non-trivial open set. Therefore, non-empty open sets are dense in their ambient space. For more information see, for instance, [23, §1.3]. Examples of open sets in $\mathbb{A}^n$ are $\emptyset$, $\mathbb{A}^n$ itself and the so-called principal open sets $\mathbb{A}^n_f$, which are defined as the non-vanishing of a single polynomial $f \in k[T]$ (this notion is formalised in Definition 5.8), that is $\mathbb{A}^n_f := \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$ for some $f \in k[T]$. For further discussion on these fact see [23, §1.2], for instance.

Definition 5.5. *A topological space is said to satisfy the* ***descending chain condition*** *(or* ***DCC****) if each non-empty chain of inclusions of closed subsets* $V_1 \supseteq V_2 \supseteq \cdots$ *stabilises.*

Definition 5.6. *A topological space is* ***Noetherian*** *if each non-empty collection of closed subspaces has a minimal element relative to inclusion.*

An affine set is Noetherian if and only if it satisfies the DCC. Let $X = \mathscr{V}(I)$ for some $I \subseteq k[T]$. Hilbert's Basis Theorem ([23, §0 0.1], for instance) asserts that $k[T]$ is **Noetherian**, that is it satisfies the ascending chain condition on ideals (each non-empty chain of inclusions of ideals in $k[T]$ has a maximal element), or equivalently that each ideal in $k[T]$ has a finite set of generators. Therefore, as the radical ideals correspond to the affine sets in $\mathbb{A}^n$, each non-empty collection of closed subsets of X contains a minimal element. Hence, X is Noetherian.

The following is [23, Proposition 1.3 B], where a proof of this result can be found.

Theorem 5.7. *Let X be a Noetherian topological space. Then X has finitely many maximal irreducible closed subspaces whose union is X.*

Consider the union $X = X_1 \cup \cdots \cup X_n$, where the X_i are the irreducible affine sets such that there are no inclusions within the set $\{X_1, \ldots, X_n\}$, and which exists by Theorem 5.7. The X_i are uniquely determined, and are called the **irreducible components** of X. They are the maximal irreducible subspaces of X. For example, the group $M_n(k)$, of monomial $n \times n$ matrices over k, consists of the matrices (x_{ij}) with exactly one non-zero entry in each row and column. This is an affine set because its underlying set determined by the polynomial conditions as follows:

$$\begin{array}{ll} x_{ij}x_{ik} = 0 & \text{if } j \neq k \\ x_{ij}x_{kj} = 0 & \text{if } i \neq k \\ \sum_i x_{ij} = c_1 & \text{for each } j, \text{ and } c_1 \in k^* \\ \sum_j x_{ij} = c_2 & \text{for each } i, \text{ and } c_2 \in k^*. \end{array}$$

The group $M_n(k)$ has $n!$ irreducible components. If $n = 2$, these irreducible components are comprised of the sets of the diagonal matrices and the anti-diagonal matrices over k.

The polynomials in $k[T]$, which when restricted to the set X are distinct, are in one-to-one correspondence with the k-algebra $k[T]/\mathcal{I}(X)$. We denote this algebra $k[X]$ and call it the **affine algebra** of X. We have that whenever X is irreducible $k[X]$ is an integral domain, since $\mathcal{I}(X)$ is a prime ideal. We form the field of fractions of $k[X]$, that is the smallest field containing $k[X]$ as a sub-ring, and denote it $k(X)$. The field $k(X)$ is the **function field** of X, and consists of **rational functions** of the form $f = g/h$ for $g, h \in k[X]$ such that $h(x) \neq 0$ for some $x \in X$.

Definition 5.8. *Let X be a closed affine set, and let f be a function in $k[X]$. The set $X_f := \{x \in X \mid f(x) \neq 0\}$ is called a **principal open set in** X.*

The principal open subsets form a basis of the Zariski topology on X. This can be seen because $\mathscr{V}(I + J) = \mathscr{V}(I) \cap \mathscr{V}(J)$ and so $\mathscr{V}(I + J)^c = \mathscr{V}(I)^c \cup \mathscr{V}(J)^c$, where the c denotes the complement.

Hilbert's Nullstellensatz can be adapted to $k[X]$, giving that closed subsets of X correspond one-to-one with the radical ideals in $k[X]$, and the irreducible subsets of X correspond to the prime ideals in $k[X]$. Furthermore, points of X correspond one-to-one with maximal ideals in $k[X]$, see [48, §1.3.2.]. In this sense, all the geometric information about X is transferred to $k[X]$.

5.2 Affine Varieties and Morphisms

Let X be an irreducible affine set with field of functions $k(X)$. For $x \in X$ define the ring,

$$\mathcal{O}_x := \{f \in k(X) \mid f = g/h, \text{ where } g, h \in k[X] \text{ and } h(x) \neq 0\}.$$

Associated to $\mathcal{O}_x$, is an evaluation function ϕ given by $\phi(f) = f(x)$ for all $f \in \mathcal{O}_x$. Therefore, ϕ is a surjective ring homomorphism from $\mathcal{O}_x$ onto k whose kernel is the ideal $\mathfrak{m}_x$ of all polynomial quotients $g/h \in \mathcal{O}_x$, with $g(x) = 0$. Hence, we have that $\mathcal{O}_x/\mathfrak{m}_x \cong k$. Therefore, $\mathfrak{m}_x$ is a maximal ideal in $\mathcal{O}_x$. A **local ring** is one which has a unique maximal ideal. In fact, $\mathfrak{m}_x$ is the unique maximal ideal in $\mathcal{O}_x$, see [23, §2.1] for instance. The ring $\mathcal{O}_x$ is called the **local ring** of x on X.

Let V be an open neighbourhood of x in X, and let $f : V \to k$ be a function. Then f is called **regular at** x if there exists $g, h \in k[X]$ and an open set $U \subset V$ containing x such that $f(y) = g(y)/h(y)$ and $h(y) \neq 0$, for all $y \in U$. Furthermore, f is called **regular on** V if it is regular at all $x \in V$. The ring of functions regular on V is denoted $\mathcal{O}_X(V)$.

Every polynomial $f \in k[X]$ is a regular function on X, in particular the zero polynomial is regular.

We can view $\mathcal{O}_X$ as a function assigning to each open subset $U \subset X$ a k-algebra $\mathcal{O}_X(U)$ of k-valued functions on U, which is non-trivial by the last remark. In fact $\mathcal{O}_X$ is a **sheaf of functions** on X in that:

(1) if $U \subset V$ are open sets and $f \in \mathcal{O}_X(V)$ then $f|_U \in \mathcal{O}_X(U)$, and

(2) if U is covered by open sets U_i, given $f_i \in \mathcal{O}_X(U_i)$ such that $f_i = f_j$ on $U_i \cap U_j$, then there exists a unique $f \in \mathcal{O}_X(U)$ such that $f|_{U_i} = f_i$.

The notion of a sheaf of functions for affine sets is well defined for arbitrary topological spaces. Let X be a topological space, then we call the pair $(X, \mathcal{O}_X)$ a **ringed space**.

Definition 5.9. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces.*

(1) *We call the map $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a* ***morphism*** *if*

 (a) *$\phi : X \to Y$ is continuous, and*

 (b) *whenever $V \subset Y$ is open and $U = \phi^{-1}(V)$, then $f \circ \phi|_U \in \mathcal{O}_X(U)$ for any $f \in \mathcal{O}_Y(V)$.*

(2) *We say $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are* ***isomorphic*** *if there are morphisms $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and $\phi^{-1} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ such that $\phi \circ \phi^{-1} = id_{(Y, \mathcal{O}_Y)}$ and $\phi^{-1} \circ \phi = id_{(X, \mathcal{O}_X)}$.*

Definition 5.10. *The ringed space $(X, \mathcal{O}_X)$ is called an* ***affine variety*** *if it is isomorphic to $(Y, \mathcal{O}_Y)$ where Y is an affine set. We frequently denote this affine variety $(X, \mathcal{O}_X)$ by X, and implicitly have in mind its sheaf of functions.*

The group $\mathrm{GL}_n(k)$ is the set of all $n \times n$ matrices with entries in the field k and whose determinant is non-zero, and can be identified with the principal open set in $\mathbb{A}^{n^2}$ determined by the non-vanishing of the determinant function det. For each matrix $g \in \mathrm{GL}_n(k)$ we set $a_g = \frac{1}{\det(g)}$. Then $\mathrm{GL}_n(k)$ can be embedded in $\mathbb{A}^{n^2+1}$ via the map $g \mapsto (g, a_g)$, for $g \in \mathrm{GL}_n(k), a_g \in k^*$. We then identify $\mathrm{GL}_n(k)$ with the zero set in $\mathbb{A}^{n^2+1}$ of the ideal of polynomials $(\det(g)a_g - 1)$. This is a polynomial map, and so sends regular functions to regular functions. The inverse of g is defined by $g^{-1} = \frac{1}{\det(g)}(\mathrm{Adj}(g))$ where $\mathrm{Adj}(g)$ is the adjugate matrix of g, which is determined by the matrix of minors of g, and the matrix of minors is determined by polynomial conditions. We have that $\frac{1}{\det(g)} = a_g$ is a polynomial in the $n^2 + 1$th coordinate, so this shows that $g \mapsto g^{-1}$ is a polynomial map, and hence a morphism. Therefore, the map $(g, a_g) \mapsto g^{-1}$ is a polynomial map and is the required inverse to show that $\mathrm{GL}_n(k)$ is identified as an affine variety.

Another example of non-isomorphic affine varieties is $\mathbb{A}^1$ and k^*, where the latter is the multiplicative group of the field k and is identified with the affine variety $\mathrm{GL}_1(k)$.

At this point we mention that the affine variety $\mathrm{SL}_n(k)$ is defined by $\mathrm{SL}_n(k) = \{g \in \mathrm{GL}_n(k) \mid \deg(g) = 1\}$, and is a normal subgroup of $\mathrm{GL}_n(k)$. We introduce some more examples in §5.6.

For any function $f \in k[Y]$, and a morphism $\phi : X \to Y$ of varieties, the second condition in Definition 5.9 implies that the function $f \circ \phi \in k[X]$ is regular on X. The map $\phi^* : k[Y] \to k[X]$, defined by $\phi^*(f) = f \circ \phi$, is a k-algebra homomorphism, called the **comorphism** of ϕ. If $\phi(X)$ is dense in Y, then $\phi^* : k[Y] \to k[X]$ is injective, see [23, §1.5].

Let $\phi : X \to Y$ be a morphism of affine varieties. The affine varieties X and Y are defined by the zeros of polynomial functions, and ϕ maps each polynomial function $f \in k[Y]$ to the polynomial function $f \circ \phi \in k[X]$. Therefore, when we consider morphisms between varieties, we have in mind functions defined by polynomial conditions, that is a polynomial function ϕ that act on the polynomial f.

For two irreducible affine varieties X and Y, with coordinate rings $k[X]$ and $k[Y]$, the product variety is endowed with the Zariski topology and is irreducible as described in Proposition 5.4. By [23, §2.4], the coordinate ring $k[X \times Y]$ of $X \times Y$ is $k[X] \otimes_k k[Y]$, and the function field of $X \times Y$ is the field of fractions of the integral domain $k[X] \otimes_k k[Y]$ (see [17, p.182] for a proof that $k[X] \otimes_k k[Y]$ is an integral domain).

5.3 Projective Varieties and Complete Varieties

In this section, we introduce the notion of projective and complete varieties. More information on the following can be found in [23, §1.6] and [6, §AG.4,§AG.7].

A **prevariety** over k is a topological space X, which has a sheaf $\mathcal{O}_X$ of k-valued functions, so that $(X, \mathcal{O}_X)$ is a ringed space, and is the union of finitely many open subsets U_i, where each U_i is isomorphic to an affine variety whose sheaf of functions is $\mathcal{O}_X(U_i)$. We have a notion of a morphism between prevarieties that is analogous to that in the case of affine varieties. A prevariety need not be an affine variety. We proceed by introducing the notion of projective varieties, and these are important examples of prevarieties that are not affine varieties. However, we do have the converse, that an affine variety is a prevariety.

Projective n-space $\mathbb{P}^n(k)$ (or $\mathbb{P}^n$) over k is defined to be the set of equivalence classes in $k^{n+1} - \{0\}$ under the equivalence relation $\sim$ defined by $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if and only if $(x_0, \ldots, x_n) = (ay_0, \ldots, ay_n)$, for some $a \in k^*$. We denote the equivalence class of $x = (x_0, \ldots, x_n)$ by $[x]$. The underlying set of the projective n-space $\mathbb{P}^n(k)$ can be identified with the set of 1-dimensional subspaces of k^{n+1}.

Let $\pi : k^n - \{0\} \to \mathbb{P}^n(k)$ be the map $x \mapsto [x]$, for each $x \in k^n$. A subset $U \subset \mathbb{P}^n(k)$ is declared **open** if $\pi^{-1}(U)$ is open in k^n. This defines a topology on $\mathbb{P}^n(k)$. The projective n-space $\mathbb{P}^n(k)$ can be covered by the open sets

$$P_i := \{[x] \mid x = (x_0, \ldots, x_n) \in k^{n+1} \text{ with } x_i \neq 0\},$$

for each $i = 0, \ldots, n$. Moreover, there is a bijection between the underlying set of each P_i and the affine variety $\mathbb{A}^n$. For each $(x_0, \ldots, x_n) \in P_i$, this bijection is given by the map

$$[(x_0, ..., x_n)] \mapsto (x_i^{-1}x_0, \ldots, x_i^{-1}x_{i-1}, x_i^{-1}x_{i+1} \ldots, x_i^{-1}x_n).$$

A **monomial** in $n+1$ variables $x_0, x_1, \ldots, x_n$ is a product $x_0^{a_0}x_1^{a_1} \cdots x_n^{a_n}$ where the indices a_i are all non-negative integers, and their sum $a_0 + a_1 + \cdots + a_n$ is called the **degree** of the monomial. Let f be a homogeneous polynomial in $k[X_0, \ldots, X_n]$, that is a polynomial in $n+1$ variables whose monomials that have non-zero coefficients all have the same degree.

Consider an ideal I of homogeneous polynomials in $k[X_0,\ldots,X_n]$. We use the notation $f[x]$ when considering the zero set of the homogeneous polynomial $f \in I$ since $f(x_0,\ldots,x_n) = 0$ if, and only if $f(ax_0,\ldots,ax_n) = 0$, for all non-zero scalars a.

The set $\mathscr{V}(I) = \{[x] \in \mathbb{P}^n \mid f[x] = 0 \text{ for all } f \in I\}$ is closed in $\mathbb{P}^n$. A closed subset in $\mathbb{P}^n$ is called a **projective set**. If X is a projective set defined by the ideal I of homogeneous polynomials in $k[X_0,\ldots,X_n]$, then the coordinate ring of X is $k[X] = k[X_0,\ldots,X_n]/I$. Let $L = \{f/g \mid f,g \in k[X]$ are homogeneous polynomials of the same degree, and $g \neq 0\}$. For $[x] \in X$ we write $\mathcal{O}_{[x]} = \{f/g \in L \mid g(x) \neq 0\}$. For an open set $U \subseteq X$ define $\mathcal{O}_X(U) = \bigcap_{[x]\in U} \mathcal{O}_{[x]}$, and for open subsets of U the restriction maps are taken to be inclusions. This defines a sheaf on the projective set X. For further details see, for instance [6, §AG.7].

Definition 5.11. *The ringed space $(X, \mathcal{O}_X)$ is called a **projective variety** if it is isomorphic to $(Y, \mathcal{O}_Y)$ where Y is an projective set. We frequently denote the projective variety $(X, \mathcal{O}_X)$ by X, and implicitly have in mind its sheaf of functions.*

We have that a projective variety is a prevariety, but not an affine variety (except in the trivial case). According to [10, p.1], the product of two prevarieties is again a prevariety.

Definition 5.12. (1) *A prevariety X is called an **algebraic variety** if the diagonal map $\Delta(X) = \{(x,x) \mid x \in X\}$ is closed in the prevariety $X \times X$.*

(2) *An algebraic variety X is said to be **complete** if for any algebraic variety Y the projection map $\pi_Y : Y \times X \to Y$ sends closed sets to closed sets, i.e. π_Y is a closed map.*

The notion of a *completeness* is an analogue for varieties to the notion of *compactness* for topological spaces, see [6, §7] for instance, for further discussion.

The following theorem is given in [6, Theorem 7.4].

Theorem 5.13. *A projective variety is a complete variety.*

In the following we will be interested in linear algebraic groups, and these arise from affine varieties. The notions of projective and complete varieties are needed to describe the structure of reductive algebraic groups. In particular we use these notions to describe the parabolic subgroups of a reductive algebraic group, see §6.1.

5.4 Dimension

The **dimension** $\dim(X)$ of an irreducible variety X is the transcendence degree of $k(X)$ over k, it is equal to the maximum number of algebraically independent rational functions on X (that is, the rational functions that satisfy no non-trivial polynomial with coefficients in k), see [23, §3.1]. In general, the dimension of a variety X is defined to be the supremum of the dimensions of the irreducible components of X. The dimension of affine line $\mathbb{A}^1$ is 1. The union of two different affine lines which intersect in $\mathbb{A}^2$ is 1-dimensional because its irreducible components are 1-dimensional affine lines. A finite set of points is 0-dimensional.

A **hypersurface** is the zero set in $\mathbb{A}^n$ of a single non-scalar polynomial. For instance in $\mathbb{A}^{n^2}$ the hypersurface defined by the polynomial $\det(x_{ij}) = 1$ is $\mathrm{SL}_n(k)$.

The following result is [23, Proposition 3.2].

Proposition 5.14. *Let X be an irreducible variety, Y a proper closed irreducible subvariety of X. Then $\dim Y < \dim X$.*

The dimension of $\mathbb{A}^n$ is equal to the transcendence degree of $k(\mathbb{A}^n) = k(X_1, \ldots, X_n)$ over k. This field consists of rational functions generated by the n independent variables $X_1, \ldots, X_n$, and hence the degree of $\mathbb{A}^n$ is equal to n.

From Proposition 5.14 we see that if Y is an irreducible subvariety of the irreducible variety X and $\dim Y = \dim X$, then $Y = X$.

5.5 Tangent Spaces

Intuitively, the tangent space of a curve at a point is a line passing through that point that is tangential to the curve at that point, and the tangent space to a surface at a point is a plane passing through that point that is tangential to the surface at that point. For example, a sphere resting on a plane C at the origin O has as a tangent space at O the plane C. Geometrically, the tangent space of a variety X at a point x is given by the vanishing of all partial derivatives of the functions f at x as f ranges over $\mathcal{I}(X)$. Algebraically, this is expressed in the following way.

Let X be an irreducible variety over k. Let $x \in X$, and recall the definition of the local ring of x on X. A **point derivation** of $\mathcal{O}_x$ is a k-linear map $\delta : \mathcal{O}_x \to k$ satisfying $\delta(fg) = f(x)\delta(g) + \delta(f)g(x)$, for all $f, g \in \mathcal{O}_x$. The k-vector space of all point derivations of $\mathcal{O}_x$ is the **tangent space** of X at x, denoted by $T_x(X)$. Recall, given a morphism $\phi : X \to Y$ of varieties, the comorphism of ϕ is the map $\phi^* : k[Y] \to k[X]$ defined by $\phi^*(f) = f \circ \phi$. Given a derivation $\delta \in T_x(X)$, by [48, §4.1], $\delta \circ \phi^*$ is a derivation in $T_{\phi(x)}(Y)$. This map is into the tangent space of Y at $\phi(x)$ because if f is regular at x, then $\phi^* \circ f$ is regular at $\phi(x)$. So we obtain a linear map $\partial_x\phi : T_x(X) \to T_{\phi(x)}(Y)$ of tangent spaces, given by $\partial_x\phi(\delta) = \delta \circ \phi^*$ for all $\delta \in T_x(X)$, and this map is called the **differential of ϕ at x**.

For any variety X, by [48, Theorem 4.3.3.(iii)], we have that $\dim T_x(X) \geq \dim X$. A point $x \in X$ is called **simple** if $\dim T_x(X) = \dim X$, and [23, Theorem 5.2] shows that simple points exist in all irreducible varieties. If every point of X is simple then we say that X is **smooth**. For example, affine n-space $\mathbb{A}^n$ is smooth, however, the union Y of two non-parallel affine lines in $\mathbb{A}^2$ is not smooth because at the intersection point $y \in Y$ of the two lines we have $2 = \dim T_y(Y) \geq \dim Y = 1$.

5.6 Affine Algebraic Groups

Throughout this section we assume that k is an algebraically closed field of arbitrary characteristic, unless stated otherwise.

An **affine algebraic group** G is a group whose underlying set is an affine variety over k, such that the product map $\pi : G \times G \to G$ given by $(x, y) \mapsto xy$, and inverse map $\iota : G \to G$ given by $x \to x^{-1}$, for $x, y \in G$, are both morphisms of the underlying varieties.

Given algebraic groups G and H, a map $f : G \to H$ is a **morphism of algebraic groups** if f is both a group homomorphism and a morphism of the underlying varieties of G and H. The map f is an **isomorphism of the algebraic groups** G and H if there exists an inverse morphism $f^{-1} : H \to G$, such that $f^{-1} \circ f = \mathrm{id}_G$, and $f \circ f^{-1} = \mathrm{id}_H$.

A **G-variety** is a variety X defined over the field k equipped with a G-action $G \times X \to X$ which is a morphism of varieties. The **G-orbit** of a point $x \in X$ is the set $\{g \cdot x \mid g \in G\}$, denoted by $G \cdot x$.

It is shown in [40, §3] that, for a G-variety X and some $x \in X$, the orbit $G \cdot x$ is open in its closure $\overline{G \cdot x}$, and the boundary of this closure $\overline{G \cdot x} \backslash G \cdot x$ is a union of G-orbits each of which has dimension strictly less than $\dim(G)$. Furthermore, by [39, No.8], there is a unique closed orbit in $\overline{G \cdot x}$.

We call a subvariety Y of X, G**-invariant** if $G \times Y \to Y$. The set of fixed points of X under the action of G is denoted X^G. Clearly, X^G is G-invariant.

Let H be a subgroup of G. We wish to define the structure of an algebraic variety on the coset space G/H. We view G as an H-variety, and we let Y be an algebraic variety for which there is a surjective morphism $\pi : G \to Y$ of varieties. We define the **fibre of** π **over** $y \in Y$ to be the subvariety $\pi^{-1}(\{y\})$ in G. We say that π is a **quotient morphism** if π is surjective and open (that is, the image of every G-invariant open subset of G is open), and if $U \subseteq G$ is open then the comorphism π^* of π induces an isomorphism from $k[\pi(U)]$ onto the set $\{f \in k[U] \mid f \text{ is constant on the fibres of } \pi|_U\}$.

A **quotient of** G **by** H, denoted G/H, is a surjective morphism $\pi : G \to Y$ of varieties such that the fibres of π are the orbits of H in G, and such that π is a quotient morphism. By [6, §6.3], the quotient is uniquely determined up to isomorphism, if it exists.

If H is a normal subgroup of G, then the quotient has the structure of an algebraic group, see [6, Theorem 6.8] for instance. For more details on quotients of varieties, see [6, §6.3] for instance.

Many of the affine varieties encountered so far are also algebraic groups, for example affine space $\mathbb{A}^n$ with respect to coordinatewise addition of points. In §5.2, we saw that the underlying set of $\mathrm{GL}_n(k)$ forms an affine variety. The product map $\mathrm{GL}_n(k) \times \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ is clearly a morphism. To show that the inverse map is a morphism, recall that we identify the underlying set of $\mathrm{GL}_n(k)$ with an affine set in $\mathbb{A}^{n^2+1}$ via the map $g \mapsto (g, a_g)$, for $g \in \mathrm{GL}_n(k), a_g \in k^*$. We have seen that the inverse of g is defined by $g^{-1} = a_g(\mathrm{Adj}(g))$, which is a matrix determined by polynomial conditions, and so $g \mapsto g^{-1}$ is a morphism. Therefore, $\mathrm{GL}_n(k)$ is an example of an algebraic group.

Clearly, a closed subgroup of an algebraic group is again an algebraic group, for example the closed subgroup $\mathrm{SL}_n(k)$ of $\mathrm{GL}_n(k)$ is an algebraic group. Since any finite set of points is closed, a finite subgroup of $\mathrm{GL}_n(k)$ is an algebraic group.

The following is [23, Corollary 8.2].

Proposition 5.15. *Let H be a closed subgroup of the algebraic group G. Then, both the centraliser $C_G(H)$ and normaliser $N_G(H)$ in G of H are closed subgroups of G. In addition the centraliser $C_G(x)$ for all $x \in G$ is also a closed subgroup of G.*

Using Proposition 5.15 we can construct many more algebraic groups. For instance, we now have a method to construct infinite algebraic groups from finite ones. For an example of this consider the element $x = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(k)$. Then x generates a finite subgroup of $\mathrm{GL}_2(k)$. By Proposition 5.15 the subgroup $C_{\mathrm{GL}_2(k)}(x)$ of $\mathrm{GL}_2(k)$ is closed. It is straightforward to verify this as $C_{\mathrm{GL}_2(k)}(x)$ is comprised of matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-zero determinant, such that $c = 0, d = a - 2b$. The fact that $C_{\mathrm{GL}_n(k)}(x)$ is infinite also follows from [30, Theorem 1.2].

We present other important examples of algebraic groups. For example, $\mathrm{T}_n(k)$, the group of upper triangular matrices in $\mathrm{GL}_n(k)$, and $\mathrm{U}_n(k)$ called the group of upper unitriangular matrices in $\mathrm{GL}_n(k)$, consisting of the elements of $\mathrm{T}_n(k)$ whose diagonal entries are all $1s$.

We denote by " + " and "." the additive and multiplicative field operations on k, and by "0" and "1" the additive and multiplicative identities, respectively.

The **additive group** $\mathbb{G}_a$ is the affine line $\mathbb{A}^1$ with group operation " + " and identity element "0", and is isomorphic to the subgroup $\mathrm{U}_2(k)$ of upper unitriangular 2×2 matrices in $\mathrm{GL}_2(k)$.

The **multiplicative group** $\mathbb{G}_m$ is the affine open subset $k^* \subset \mathbb{A}^1$, with group operation "$\cdot$" and identity element "1", and is isomorphic to $\mathrm{GL}_1(k)$.

Note that the results already given allow us to construct new algebraic groups from old ones. For instance, Proposition 5.4 shows that the direct product of two algebraic groups A and B forms an affine variety $A \times B$. As described in Section 5.2, the function field of $A \times B$ is the field of fractions of the integral domain $k[A] \otimes_k k[B]$. Since the inverse maps on A and B are individually morphisms, it is clear that so is the inverse map $A \times B \to A \times B$ given by $(a, b) \mapsto (a^{-1}, b^{-1})$. Also the product maps on A and B are individually morphisms, and so the product map $(A \times B) \times (A \times B) \to A \times B$ given by $(a, b) \times (a', b') = (aa', bb')$ is a morphism. Therefore, since $A \times B$ is a group, it is an algebraic group. For example, we may take the direct product of n copies of $\mathbb{G}_m$. The resulting group is isomorphic to the subgroup $\mathrm{D}_n(k)$ of $\mathrm{GL}_n(k)$ consisting of diagonal matrices.

We refer to an affine algebraic group as a **linear algebraic group**, and the motivation for this terminology is given by the following theorem, which is [23, Theorem 8.6].

Theorem 5.16. *Let G be an affine algebraic group over k, then G is isomorphic to a closed subgroup of $\mathrm{GL}_n(k)$, for n a positive integer.*

Let G be an algebraic group and $X_1, \ldots, X_n$ be those irreducible components of G that contain e, the identity of G. The product $X_1 \times \cdots \times X_n$ is irreducible by Proposition 5.4. Let $f_i : X_i \to G$ be the inclusion map from each X_i into G. Consider the map $f : X_1 \times \cdots \times X_n \to G$ given by $(x_1, \ldots, x_n) \mapsto f_1(x_1) \cdots f_n(x_n)$, for $x_i \in X_i$. By [6, Proposition 14.10] this map is a homomorphism, therefore the image under f of $X_1 \times \cdots \times X_n$ in G is irreducible. We will comment further on this map in Remark 5.22. This image contains e, so $X_1 \cdots X_n \subseteq X_i$ for some i. Conversely $X_i \subseteq X_1 \cdots X_n$ for all i, therefore we conclude that $n = 1$. From this we imply that there is a unique irreducible component of G containing e, which we denote by G^0. In fact G^0 is a normal subgroup of finite index in G whose cosets are the connected and irreducible components of G. Furthermore any closed subgroup of G of finite index contains G^0. We say G is **connected** if $G = G^0$. Proofs of these facts can be found in [23, Proposition 7.3], for instance.

The groups $\mathbb{G}_a, \mathbb{G}_m, \mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k)$ are connected. For $\mathbb{G}_a$, we have that $\mathbb{G}_a = \mathbb{A}^1$, which is clearly irreducible as a variety. We have that $\mathrm{GL}_n(k)$ is identified with the principal open set $\{x \in \mathbb{A}^{n^2} \mid \det(x) \neq 0\}$ in the irreducible variety $\mathbb{A}^{n^2}$. Thus, the closure in $\mathbb{A}^{n^2}$ of $\mathrm{GL}_n(k)$ is the whole space, and by [48, Lemma 1.2.3 (i)] $\mathrm{GL}_n(k)$ is an irreducible variety, and hence a connected algebraic group. The connectedness of $\mathbb{G}_m$ follows since $\mathbb{G}_m = \mathrm{GL}_1(k)$.

It can be shown that $\mathrm{SL}_n(k)$ is generated by groups $U_{i,j}$ (for $i \neq j$) having 1s on the main diagonal, an arbitrary entry in the (i,j)-th position and zeros everywhere else. There are finitely many such groups $U_{i,j}$, each isomorphic to $\mathbb{G}_a$, and for each we have a morphism $f_{i,j} : U_{i,j} \to \mathrm{GL}_n(k)$ such that $e \in f_{i,j}(U_{i,j})$. If we set $M = \bigcup_{i,j} f_{i,j}(U_{i,j})$, then we can apply [23, Proposition 7.5] to obtain that the intersection of all the closed subgroups of $\mathrm{GL}_n(k)$ containing M is connected. This shows that $\mathrm{SL}_n(k)$ is connected.

Any non-trivial finite subgroup of $\mathrm{GL}_n(k)$ is disconnected. The identity component of the group $M_n(k)$ of $n \times n$ monomial matrices is the group of $n \times n$ diagonal matrices. In this case, $M_n(k)/M_n(k)^0 \cong S_n$.

We call a group that is isomorphic to a direct product of n copies of $\mathbb{G}_m$ a **torus of rank** n. For example, $\mathrm{D}_m(k)$ is isomorphic to a torus of rank m in $\mathrm{GL}_n(k)$, for all $m \leq n$. A torus is called a **maximal torus** if it is not contained in any other torus. A maximal torus in $\mathrm{GL}_n(k)$ is the group of diagonal matrices $\mathrm{D}_n(k)$.

The following is [23, Corollary 16.3].

Lemma 5.17. *Let S be a torus in an algebraic group G. Then, $N_G(S)^0 = C_G(S)^0$.*

The following description of the Jordan decomposition is from [48, §2.4].

Let V be a finite dimensional vector space over k, and let 1 represent the identity map in $\mathrm{GL}(V)$. An element x_s of $\mathrm{GL}(V)$ is called **semisimple** if V admits a basis consisting of eigenvectors of x_s. An element x_u of $\mathrm{GL}(V)$ is called **unipotent** if $x_u - 1$ is nilpotent, that is, there exists some positive integer k, for which $(x_u - 1)^k = 0$.

In $\mathrm{GL}(V)$ any element x can be written uniquely as a commuting product of a unipotent element x_u and of a semisimple element x_s, see [48, Theorem 2.4.5.]. This is known as the **abstract Jordan decomposition** of x.

In the linear algebraic group G, let $\rho(g) : k[G] \to k[G]$ denote the right translation by the element g given by

$$(\rho(g)f)(x) = f(xg)$$

for each $f \in k[G], x \in G$. Then $\rho(g)$ can be viewed as an element of $\mathrm{GL}(k[G])$. Hence, $\rho(g)$ has the Jordan decomposition $\rho(g) = \rho(g)_s\rho(g)_u$.

There is an equivalent notion, the **Jordan decomposition in** G, to that of endomorphisms of a vector space, see [48, Theorem 2.4.8.]. This states that there exist unique elements g_u and g_s in G such that $\rho(g)_s = \rho(g_s)$ and $\rho(g)_u = \rho(g_u)$, and g is equal to the commuting product of g_u and g_s. We say that g_u is the **unipotent** part of g, and g_s is the **semisimple** part of g.

The homomorphic image of a unipotent element (resp. semisimple element) in G is unipotent (resp. semisimple) in the homomorphic image of G. An algebraic group is called **unipotent** if all its elements are unipotent.

For any group G, the **commutator** $[g, h]$ of the elements $g, h \in G$ is defined by $[g, h] = ghg^{-1}h^{-1}$. We define the **commutator** $[H, K]$ for subgroups H, K of G to be the group generated by all commutators of the form

$[h,k]$ for $h \in H$ and $k \in K$. An important example of the commutator of two groups is $[G,G]$, the so-called **derived subgroup** of G which, by [23, Proposition 17.2] is a closed normal subgroup of G (and connected if G is).

The **descending central series** of a group G is defined to be the series $\mathscr{C}^0(G) \supseteq \mathscr{C}^1(G) \supseteq \cdots$, where $\mathscr{C}^0(G) = G$, and $\mathscr{C}^{i+1}(G) = [G, \mathscr{C}^i(G)]$. The **derived series** of a group G is defined to be the series $\mathscr{D}^0(G) \supseteq \mathscr{D}^1(G) \supseteq \cdots$, where $\mathscr{D}^0(G) = G$, and $\mathscr{D}^{i+1}(G) = [\mathscr{D}^i(G), \mathscr{D}^i(G)]$. Clearly $\mathscr{D}^i(G) \subseteq \mathscr{C}^i(G)$, for all i.

Let G be an algebraic group. Then G is called **nilpotent** if its descending central series reaches e in finitely many steps, and **solvable** if its derived series reaches e in finitely many steps. Clearly, a nilpotent group is solvable. The homomorphic image of a solvable (resp. nilpotent) group is solvable (resp. nilpotent). The product of two normal solvable subgroups of a group is also solvable. For proofs of these results see [23, §17].

Theorem 5.18 (Borel's fixed point theorem)**.** *Let G be a connected solvable algebraic group, and let X be a complete G-variety. Then X^G is non-empty.*

A proof of Borel's Fixed Point Theorem can be found in, for example, [23, Theorem 21.2]. As we shall see, this result is crucial in showing that the maximal closed connected solvable subgroups of G (the so called "Borel subgroups") are all conjugate in G.

Theorem 5.19 (Lie-Kolchin theorem)**.** *Let $G = \mathrm{GL}_n(k)$ and let H be a unipotent subgroup of G. Then there exists some $x \in \mathrm{GL}_n(k)$ such that xHx^{-1} is a subgroup of $U_n(k)$.*

In fact, any closed connected solvable subgroup of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of $\mathrm{T}_n(k)$. This is sometimes what is known as the Lie-Kolchin Thoerem, see for instance [10, §1.6].

The groups $\mathrm{U}_n(k)$ and $\mathrm{T}_n(k)$ are solvable. To see this first observe that $[\mathrm{T}_n(k), \mathrm{T}_n(k)] = \mathrm{U}_n(k)$. The commutator of two upper triangular matrices with the same number of non-zero diagonals above the lead diagonal, is an upper triangular matrix with at least one more diagonal above the lead diagonal that contains all zero entries, thus we have the solvability of $\mathrm{U}_n(k)$ and $\mathrm{T}_n(k)$. Examples of nilpotent groups are commutative group because the commutator of two commuting matrices results in the identity matrix. The group $\mathrm{U}_n(k)$ is nilpotent, but $\mathrm{T}_n(k)$ is not nilpotent for $n \geq 2$. This shows that a solvable group need not be nilpotent. For further discussion and proofs see [23, §17].

We now discuss some important types of algebraic group. For a more extensive account of the following material, see [23, Chapter VII].

We call G **simple** if every proper closed normal subgroup of G is finite. For example, for $n > 1$, we have $\mathrm{GL}_n(k)$ is not simple since it has $\mathrm{SL}_n(k)$ as a normal subgroup, being the kernel of the det homomorphism. Clearly any 1-dimensional group is simple, hence $\mathbb{G}_m = k^*$ is simple. From the discussion in [23, p.164] we see that $\mathrm{SL}_n(k)$ is simple, although it does contain the finite normal subgroup of scalar multiples of the identity aI_n such that $a^n = 1$.

Evidently a simple group is connected, however the converse does not hold as is shown, for example, with the group $\mathrm{GL}_n(k)$.

Let G be an algebraic group.

(1) The closed connected normal solvable subgroup $R(G)$ of G containing any other such subgroup is called the **radical** of G.

(2) The closed connected normal unipotent subgroup $R_u(G)$ of G containing any other such subgroup is called the **unipotent radical** of G.

Given two normal solvable subgroups A and B of G, their product AB is also normal and solvable, by [23, Lemma 17.3]. Therefore, we have that $R(G)$, the radical of G, is uniquely defined. The group $R_u(G)$ is the subgroup of $R(G)$ consisting of all its unipotent elements, see [23, §19.5]. This shows that $R(G)$ and $R_u(G)$ are well-defined.

If G is connected and $R(G)$ is trivial we call G **semisimple**. For example, $\mathrm{SL}_n(k)$ is semisimple because it is simple. If G is connected and $R_u(G)$ is trivial we call G **reductive**. For example, as $\mathrm{GL}_n(k)$ is the product of $\mathrm{SL}_n(k)$ and its centre Z, we have $R(\mathrm{GL}_n(k)) = Z$ and hence, $R_u(\mathrm{GL}_n(k)) = e$. Therefore, $\mathrm{GL}_n(k)$ is reductive. Since $R_u(G) \subseteq R(G)$, a semisimple group is reductive. By [23, Lemma 19.5] the derived subgroup $[G, G]$ of a reductive group G is semisimple.

Although the notion of a reductive group G is well-defined for non-connected G, much of our work holds only for connected G. Therefore, we shall assume that by a *reductive group* we mean a connected reductive group.

A simple algebraic group is reductive since its only proper normal subgroups are disconnected (not connected). If G is a simple algebraic group, then $G/Z(G)$ is a simple abstract group, by [23, Corollary 29.5].

Reductive groups are of particular interest so we record some important properties. Let G be reductive with centre $Z(G)$, then $G = Z(G)[G, G]$ and $Z(G)^0 = R(G)$ is a torus and $[G, G]$ is a semisimple group with a finite intersection with $Z(G)$. Furthermore, we can decompose G as $G = G_1 \cdots G_n Z(G)$ with G_i simple so that $G_i \cap G_j$ is finite for all $i \neq j$, and $[G_i, G_j] = e$. The G_i occurring in this decomposition are called the **simple components** of G. For an account of this see [10, §1].

Definition 5.20. *Let G be a reductive algebraic group, and let $X_1, \ldots, X_n$ be subgroups of G. Let $f_i : X_i \to G$ be maps, then the map $f : X_1 \times \cdots \times X_n \to G$ given by $(x_1, \ldots, x_n) \mapsto f_1(x_1) \cdots f_n(x_n)$, for $x_i \in X_i$ is called the* ***product map of the*** *f_is.*

Definition 5.21. *Let G be a reductive algebraic group, and let $G_1, \ldots, G_n$ be normal subgroups of G. We say that G is an* ***almost direct product of***

the *G_is if the product map of the inclusions $G_i \to G$ is a homomorphism of the direct product $G_1 \times \cdots \times G_n$ onto G, with a finite kernel.*

Remark 5.22. Let G be a semisimple algebraic group. According to [6, Proposition 14.10] we have that G is an almost direct product of its simple components $G_1, \ldots, G_n$. In particular, if the $f_i : G_i \to G$ are inclusion maps, then the product map $G_1 \times \cdots \times G_n \to G$ of the f_is given by $(g_1, \ldots, g_n) \mapsto f_1(g_1) \cdots f_n(g_n)$ is a homomorphism with finite kernel.

Definition 5.23. *We call the group G* ***topologically finitely generated*** *if it is the Zariski closure of a group generated by finitely many elements, that is $G = \overline{\langle x_1, \ldots, x_n \rangle}$ for a finite list $x_1, \ldots, x_n$.*

For any subgroup S of G we have that the closure $\overline{S}$ in the Zariski topology is a subgroup of G. In particular if $x_1, \ldots, x_l \in G$ then the Zariski closure $\overline{H}$ of the group $H := \langle x_1, \ldots, x_l \rangle$ generated by the elements $x_1, \ldots, x_l$ is a closed subgroup of G. Suppose $k = \overline{\mathbb{F}}_p$ for some prime p. By Theorem 5.16, each x_i may be viewed as a matrix with entries in $\overline{\mathbb{F}}_p$. Since $\overline{\mathbb{F}}_p = \cup_{m \geq 1} \mathbb{F}_{p^{m!}}$, we have that each x_i is a matrix with entries in $\mathbb{F}_{p^{m!}}$, for m large enough. Since each element of the group H generated by $x_1, \ldots, x_l$ is a matrix with entries $\mathbb{F}_{p^{m!}}$, the group H lies in $\mathrm{GL}_n(\mathbb{F}_{p^{m!}})$, by Theorem 5.16. Since $\mathrm{GL}_n(\mathbb{F}_{p^{m!}})$ is a finite group, $\langle x_1, \ldots, x_l \rangle$ is a finite group. That is, $H = \overline{H}$ is finite.

As just seen if $k = \overline{\mathbb{F}}_p$, then every topologically finitely generated group is finite. However, as we will see, it is crucial to some of the results in this exposition to be able to work with topologically finitely generated subgroups of G when G is an algebraic group over the field $\overline{\mathbb{F}}_p$. Therefore an alternative approach is required, and is given by [1, Lemma 2.10]. This lemma allows us to reduce to the case that a subgroup H of G is topologically finitely generated within the field of study of this thesis. We discuss this result further in Lemma 6.24.

5.7 The Lie Algebra of an Algebraic Group

The study of Lie algebras is extensive, and the field is often treated as a self-contained subject. However, its relevance to other areas, in particular to Lie groups and algebraic groups, is fundamental. Here we provide an introduction relevant for our purposes. For a more extensive account, see [6, §3].

A **Lie algebra** is a vector space $\mathfrak{g}$ over a field k with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted $(x, y) \mapsto [x, y]$ such that:

L1 The bracket operation $[\ ,\ ]$ is bilinear.

L2 $[x, x] = 0$ for all $x \in \mathfrak{g}$.

L3 $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Property L1 ensures that $\mathfrak{g}$ is a k-algebra with respect to $[\ ,\ ]$.

Given a k-algebra A, we say that the linear map $D : A \to A$ is a **derivation** of A if $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. Let $\mathcal{D}(A)$ denote the space of all derivations of A. Let G be an algebraic group with affine algebra $A = k[G]$, then we may consider the space of all derivations of A. For $g \in G$ define $\lambda_g : A \to A$ via

$$(\lambda_g a)(x) = a(g^{-1}x)$$

for each $a \in A, x \in G$. Now set

$$\mathfrak{g} = \{D \in \mathcal{D}(A) \mid D \circ \lambda_g = \lambda_g \circ D, \text{ for all } g \in G\}$$

We define $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ for $D_1, D_2 \in \mathfrak{g}$, and this defines a Lie algebra structure on $\mathfrak{g}$. We call $\mathfrak{g}$ the **Lie algebra** of G, denoted by $\mathrm{Lie}(G)$.

As an example, for $\mathrm{char}(k) \neq 2$ consider the following standard basis of the Lie algebra $\mathfrak{sl}_2(k)$, which is the Lie algebra of the group $\mathrm{SL}_2(k)$, that is $\mathfrak{sl}_2(k) = \mathrm{Lie}(\mathrm{SL}_2(k))$. See [23, §9.4] for details. Set

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to check that under the bracket operation these basis vectors obey $[H, X] = 2X, [H, Y] = -2Y$ and $[X, Y] = H$. It is also straightforward to check that these vectors satisfy $L1, L2$ and $L3$ above, thus $\mathfrak{sl}_2(k)$ is a Lie algebra.

By [6, Theorem 3.4], we have an isomorphism δ of vector spaces $\mathfrak{g} \to T_e(G)$ given by the map $\delta : D(f) \mapsto D(f)(e) \in T_e(G)$ for any $f \in \mathcal{O}_e, D \in \mathfrak{g}$. Since $D(fg)(e) = f(e)D((g)(e)) + D((f)(e))g(e) \in k$, we have that $\delta(D(f))$ is a point derivation of $\mathcal{O}_e$. In fact the tangent space at any point of G is isomorphic to this Lie algebra because if f is regular at e, then $\lambda_x f$ is regular at x, so a point derivation at e becomes a point derivation at x under the translation map λ_x.

For example, the tangent space $T_{I_n}(\mathrm{GL}_n(k))$ to $\mathrm{GL}_n(k)$ at the point I_n is the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{GL}_n(k)) = \mathrm{Mat}_n(k)$ of $n \times n$ matrices over k. For a proof of this fact see, for example, [6, Examples 3.9 (c)]. The Lie algebra of $\mathrm{GL}_n(k)$ is denoted $\mathfrak{gl}_n(k)$.

We now have associated to each algebraic group G a Lie algebra $\mathfrak{g}$. Next we associate to the Lie algebra of each reductive group a root system.

5.8 The Adjoint Representation

In this section we let G be a reductive algebraic group. By [23, §5.4], the differential map (introduced in §5.5) has functorial properties. Let $g \in G$, and $\mathfrak{g} = \mathrm{Lie}(G)$. Consider the automorphism $\mathrm{Int}(g) : G \to G$ given by $\mathrm{Int}(g)(x) = gxg^{-1}$. Its differential $\partial_e \mathrm{Int}(g)$ is an automorphism of the Lie algebra, by [6, §3.12], and is denoted $\mathrm{Ad}(g)$. That is, $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$. We call the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ the **adjoint representation** of G. The differential of Ad at e is the endomorphism $\partial_e \mathrm{Ad} : \mathfrak{g} \to \mathrm{Lie}(\mathrm{GL}(\mathfrak{g})) =$

End($\mathfrak{g}$) given by $\partial_e \operatorname{Ad}(X) : Y \mapsto [X, Y]$ of $\mathfrak{g}$ for all $X, Y \in \mathfrak{g}$, see [6, §3.14], and we denote this endomorphism by ad. We call $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the **adjoint representation** of the Lie algebra $\mathfrak{g}$.

Given a torus T in G, we have that T acts on $\mathfrak{g}$ via the adjoint representation Ad, that is $\operatorname{Ad}(t) : \mathfrak{g} \to \mathfrak{g}$ for all $t \in T$.

Since T is diagonalisable, so is its homomorphic image under Ad and so we can write the vector space $\mathfrak{g}$ as a direct sum of **weight spaces**:

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \operatorname{Ad}(t)(X) = \alpha(t)X, \text{ for all } t \in T\},$$

where $\alpha \in \operatorname{Hom}(T, \mathbb{G}_m) =: X(T)$ - the group of algebraic group morphisms from T to $\mathbb{G}_m$. The group $X(T)$ is called the **character group** of T, its elements are called **characters** of T. The α for which $\mathfrak{g}_\alpha \neq 0$ are called **weights** of T in $\mathfrak{g}$. The non-zero weights are called **roots of G relative to T**, and the set of these is called the **root system of G relative to T**, denoted $\Phi(G, T)$.

The set $Y(G) := \operatorname{Hom}(\mathbb{G}_m, G)$ of algebraic group morphisms from $\mathbb{G}_m$ to G is called the set of **one-parameter subgroups** of G.

Let $\alpha \in X(T)$ and let $\beta \in Y(T)$. Since $\alpha \circ \beta \in \operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ there is some $[\beta, \alpha] \in \mathbb{Z}$ such that $\alpha \circ \beta(x) = x^{[\beta,\alpha]}$ for all $x \in \mathbb{G}_m$.

5.9 More on Lie Algebras

A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Lie subalgebra** of $\mathfrak{g}$ if $\mathfrak{h}$ is closed under the Lie bracket operation. A Lie subalgebra $\mathfrak{i}$ of $\mathfrak{g}$ is called an **ideal** of $\mathfrak{g}$ if $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$, that is $[I, X] \in \mathfrak{i}$ for all $I \in \mathfrak{i}, X \in \mathfrak{g}$. A **homomorphism** $\phi : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras is a linear map ϕ such that $\phi[X, Y] = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$.

Let $\mathfrak{h}$ be a Lie subalgebra of the Lie algebra $\mathfrak{g}$, then $\operatorname{ad}(X)(Y) \in \mathfrak{g}$ for all $X \in \mathfrak{h}, Y \in \mathfrak{g}$. Therefore, $\operatorname{ad}(\mathfrak{h}) \subseteq \operatorname{End}(\mathfrak{g})$. Thus $\mathfrak{g}$ can be viewed as an **$\mathfrak{h}$-module** (see [22, §6.1]).

Similarly, if $\mathfrak{g} = \mathrm{Lie}(G)$, then we have an action of G on $\mathfrak{g}$ via the adjoint map $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called **G-invariant** if it is $\mathrm{Ad}(G)$-invariant, that is if $\mathrm{Ad}(g)\mathfrak{h} \subseteq \mathfrak{h}$ for all $g \in G$.

5.10 Root Systems

The study of root systems is, like that of Lie algebras, a self-contained subject; for which we give a brief introduction. More details can be found in [22, §9].

Let E be an l-dimensional vector-space over $\mathbb{R}$ (for l a positive integer) together with an inner product $(\ ,\ ) : E \times E \to \mathbb{R}$. For any vectors $u, v \in E$ the **magnitude** of u is given by $||u|| = \sqrt{(u,u)}$, and the **angle** between u and v is given by $\theta = \arccos\frac{(u,v)}{||u||||v||}$.

For each non-zero vector $v \in E$ define a **reflection** relative to v to be a linear transformation from E to itself sending v to $-v$ which fixes the subspace $P_v := \{u \in E \mid (u,v) = 0\}$ of codimension 1 orthogonal to v. Now, for any $u \in E$ the reflection relative to v is given by the formula

$$\sigma_v(u) = u - \frac{2(u,v)}{(u,u)}v.$$

As a matter of convenience, we write

$$\langle u, v\rangle = \frac{2(u,v)}{(u,u)}. \tag{1}$$

Definition 5.24. *A **root system** in the real vector space E is a subset Φ of E, whose elements are called **roots**, satisfying the following conditions:*

(1) *Φ is finite, spans E, and does not contain 0.*

(2) *If $v \in \Phi$, then the only multiples of v in Φ are $\pm v$.*

(3) *If $v \in \Phi$, the orthogonal reflection with respect to v leaves Φ invariant.*

(4) *If $v, u \in \Phi$ then $\sigma_v(u) - u$ is an integer multiple of v.*

By Definition 5.24 (4), we have $\langle u, v\rangle \in \mathbb{Z}$.

Note that since σ_v leaves the finite generating set Φ of E stable, the reflection is uniquely determined by v. Denote by $l = \dim E$ the **rank** of Φ.

The group $W(\Phi) \subset \mathrm{GL}(E)$ generated by the σ_v for $v \in \Phi$ is finite by [23, p.229], and is called the **Weyl group of** Φ.

Two root systems Φ, Φ' in the real vector spaces E, E' respectively are said to be **isomorphic** if there is an isomorphism of $\alpha : E \to E'$ of vector spaces with $\alpha(\Phi) = \Phi'$, such that $\langle \alpha(u), \alpha(v)\rangle = \langle u, v\rangle$, for all $u, v \in \Phi$.

The root system Φ is called **irreducible** if it cannot be partitioned into the union of two mutually orthogonal proper subsets. Every root system is the disjoint union of irreducible root systems.

A subset Δ of Φ is called a **base** if Δ is a basis of E, and if each root in Φ can be written as a sum $\Sigma_{\alpha\in\Delta} n_\alpha \alpha$ of roots in Φ, where the n_α are integers, all non-positive, or all non-negative. The roots in Δ are called **simple**. Suppose that the root ξ is equal to the sum $\sum_\alpha n_\alpha \alpha$, of the simple roots $\alpha \in \Delta$. If all the coefficients n_α in the sum are non-negative, then ξ is called **positive**, otherwise ξ is called **negative**. By [22, Theorem 10.1], every root system Φ has a base.

The subset of Φ consisting of positive roots is denoted Φ^+, and the subset of Φ consisting of negative roots is denoted Φ^-. The root system Φ can be written as the disjoint union $\Phi = \Phi^+ \cup \Phi^-$ of positive and negative roots.

The following is [23, Theorem 27.1]. This result shows that every reductive algebraic group has a root system.

Theorem 5.25. *Let G be a semisimple algebraic group. Let T be a maximal torus of G and set $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$ together with an inner product $(\ ,\ )$ on E. Then the root system $\Phi = \Phi(G, T)$ of G relative to T is a root system in E in the sense defined above, and the Wely group of Φ is isomorphic to $N_G(T)/C_G(T)$.*

By [23, Proposition 24.1 B], we see that the root systems of G and (G,G) are in one-to-one correspondence.

Theorem 5.25 provides a link between the abstract notion of a root system and the notion of the root system of an algebraic group introduced in §5.8 by showing that they coincide. This link is crucial in section §5.11 because it enables us to classify the simple algebraic groups by looking at their root systems.

Let $\Phi = \Phi(G,T)$ be the root system of G relative to the maximal torus T of G. By Theorem 5.25 we have that the Weyl group $W(\Phi)$ of Φ is isomorphic to $N_G(T)/C_G(T)$, so from now on we denote the Weyl group by W. As we shall see in §6.1, W is independent of Φ and T, and so this notation is justified.

The following is [23, Corollary 27.5].

Lemma 5.26. *Let G be a semisimple algebraic group. The decomposition $G = G_1 \cdots G_n$ of G into its simple components (as in §5.6) corresponds precisely to the decomposition of Φ into its irreducible components.*

We have that if G is a reductive group, then we can decompose G as $G = G_1 \cdots G_n Z(G)^0$, where the $G_1, \ldots, G_n$ are the simple components of (G,G) as in Lemma 5.26, and the root system of each G_i is irreducible.

5.11 Classification of Simple Algebraic Groups

See Chapter XI of [23] for a more extensive account of the following.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ for G a reductive algebraic group, and let T be a maximal torus of G. As introduced in §5.10, consider the real vector space $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$, with an inner product $(\ ,\ )$ on E, and let $\Phi = \Phi(G,T)$ be the root system of G with respect to T.

A **lattice** in E is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis of E, and its rank is the dimension of E. Define the **root lattice** Λ_r be the $\mathbb{Z}$-span of the elements of Φ. Define the **weight lattice** Λ to be the $\mathbb{Z}$-span of all the vectors $\gamma \in E$, such that $\langle \gamma, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$. The $\gamma \in \Lambda$ are called **abstract weights**. Both Λ and Λ_r are lattices in E of finite rank, and Λ contains Λ_r as a subgroup of finite index.

Given a base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of the root system Φ, the elements $\{2\alpha_i/(\alpha_i, \alpha_i) \mid 1 \leq i \leq l\}$ form a basis of E, called the **dual basis** of Φ. Let $\gamma_1, \ldots, \gamma_l$ be the dual basis of E relative to $(\ ,\ )$ such that $\langle \gamma_i, \alpha_j \rangle = \delta_{ij}$. It can be shown (see [22, p. 67]) that Λ is a lattice with basis consisting of the γ_i.

The **Cartan matrix** of Φ is defined to be the matrix $(\langle \alpha_i, \alpha_j \rangle)$, with $\langle \alpha_i, \alpha_j \rangle$ in its (i, j)-th entry. As described in [22, §13.1], the dual basis of Λ can be obtained by multiplying the original basis $\{\alpha_i\}$ by the inverse of the Cartan matrix. This inverse introduces a denominator, which is the determinant of the Cartan matrix, and which measures the index of Λ_r in Λ. In fact Λ/Λ_r is a cyclic group whose structure is described in [22, §13.1].

We call the invariant $\Lambda/X(T)$ the **fundamental group** of G. If $X(T) = \Lambda_r$ we say that G is **adjoint**, and if $X(T) = \Lambda$ we say that G is **simply connected**. For example, $\mathrm{SL}_2(k)$ is a simply connected group, and $\mathrm{PGL}_2(k) := \mathrm{GL}_2(k)/Z(\mathrm{GL}_2(k))$ is an adjoint group. For any semisimple group G, the adjoint representation $\mathrm{Ad}(G)$ of G in $\mathrm{GL}(\mathfrak{g})$ is an adjoint group, see [23, §31.1].

For any irreducible root system, [23, §33.6] shows that there exists a simple algebraic group having that root system. However, each root system does not necessarily give rise to a unique algebraic group. For instance, as we remark after Theorem 5.27, $\mathrm{SL}_{n+1}(k)$ and $\mathrm{PGL}_{n+1}(k) = \mathrm{GL}_{n+1}(k)/Z(\mathrm{GL}_{n+1}(k))$ have the same root system, but the former is simply connected and the latter is adjoint. As we are about to see in Theorem 5.29, in most cases the root system and fundamental group of a simple algebraic group G are enough to uniquely determine it up to isomorphism. The simple algebraic groups are classified by their root systems, and we determine all the irreducible root systems below.

Given the base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of $\Phi(G)$, we consider the range of possible values that $\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ can take in $\mathbb{Z}$. We have $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle = 4\cos^2\theta$. This number is a positive integer and each factor on the left has like sign so the only possibilities are shown in [22, p45, Table 1]. This number is therefore 0, 1, 2 or 3. If we draw a graph with l vertices corresponding to the roots α_i of the base Δ, and join the i-th vertex to the j-th vertex by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges, the result is known as a **Coxeter graph**. By [22, Lemma 10.4 C], for Φ irreducible, at most two different root lengths can occur in Φ. By putting an arrow pointing to the shorter root on any double or triple edge, then the graph is called a **Dynkin diagram**. There are nine types of connected Dynkin diagram, and each one corresponds to a particular class of irreducible root system.

In order to classify the possible Coxeter graphs we assume that $\{v_1, \ldots, v_l\}$ is a set of l linearly independent unit-vectors in E for which $(v_i, v_j) \leq 0$, for $i \neq j$ and $4(v_i, v_j)^2$ is equal to $0, 1, 2$ or 3. The elements $\alpha/||\alpha||$ for $\alpha \in \Delta$ satisfy these criteria. From these assumptions we are able to classify the Coxeter graphs, and then the classification of possible Dynkin diagrams follows easily as they have the same shapes as the Coxeter graphs, but by putting in the relevant arrows we see that a double or triple edge occurs, we obtain the Dynkin diagrams.

Working out all the irreducible (i.e. connected) Coxeter graphs uses mainly euclidean geometric ideas. Details are given in [22, §11.4], for instance.

Theorem 5.27. *The connected Dynkin diagrams are classified by the four classical types A_n, B_n, C_n, D_n and the five exceptional types E_6, E_7, E_8, F_4 and G_2. The subscript is the number of roots in a basis of the corresponding root system.*

What is remarkable about Theorem 5.27 is that the classification of the simple algebraic groups, which are topological and group theoretic constructions, is achieved using euclidean geometry. This is one example of the elegance of the theory of algebraic groups.

Terminology 5.28. A simple algebraic group is said to be of **classical type** if its root system is of type A_n, B_n, C_n or D_n, and is said to be of **exceptional type** otherwise.

With reference to [23, §33.6] and [28, §25.A.], we list some examples of semisimple and adjoint classical simple algebraic groups.

In type $A_n, n \geq 1$, we have the **special linear group** $\mathrm{SL}_{n+1}(k)$ of determinant 1 matrices in $\mathrm{GL}_{n+1}(k)$ is a simply connected group of this type. The adjoint representation $\mathrm{Ad} : \mathrm{SL}_{n+1}(k) \to \mathrm{GL}(\mathfrak{sl}_{n+1}(k))$, where $\mathfrak{sl}_{n+1}(k) = \mathrm{Lie}(\mathrm{SL}_n(k))$, has as its kernel μ_{n+1}, the group of $n+1$-th roots of unity. The group $\mathrm{SL}_{n+1}(k)/\mu_{n+1} := \mathrm{PGL}_{n+1}(k)$ is an adjoint group of this type.

In type $B_n, n \geq 1$, a simply connected group of this type is the **spinor group** $\mathrm{Spin}_{2n+1}(k)$, and an adjoint group of this type is the **special orthogonal group** $\mathrm{SO}_{2n+1}(k)$.

According to [23, §7.2], if $\mathrm{char}(k) \neq 2$, the group SO_{2n+1} can be defined as the matrices in $x \in \mathrm{SL}_{2n+1}(k)$ such that $x^T s x = s$, where x^T is the transpose of x, and $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix}$ for J the $n \times n$ matrix consisting of 1s on the antidiagonal, and zeros elsewhere. A definition of this group in characteristic two can be found in [11, §1], for instance.

In general, the orthogonal groups, denoted $\mathrm{O}_n(k)$, are defined in [28, p.348]. These groups preserve a non-degenerate quadratic form on an n-dimensional k-vector space. The normal subgroup of determinant 1 matrices is the special orthogonal group $\mathrm{SO}_n(k)$. The definition of $\mathrm{Spin}_n(k)$ is given in [28, p.349].

In type $C_n, n \geq 1$, a simply connected group of this type is the **symplectic group** $\mathrm{Sp}_{2n}(k)$, and an adjoint group of this type is the **projective symplectic group** $\mathrm{PGSp}_{2n}(k)$.

The symplectic group $\mathrm{Sp}_{2n}(k)$ is defined in [23, §7.2] as the $x \in \mathrm{GL}_{2n}(k)$ such that $x^T \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} x = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$ where J is the n by n matrix consisting 1s on its antidiagonal and zeros elsewhere. The definition $\mathrm{PGSp}_{2n}(k)$ is given in [28, p.347], for instance, where we are given the isomorphism $\mathrm{PGSp}_{2n}(k) \cong \mathrm{Sp}_{2n}(k)/\mu_2$.

In type $D_n, n \geq 2$, a simply connected group of this type is the spinor group $\mathrm{Spin}_{2n}(k)$, and an adjoint group of this type is the **special orthogonal group** $\mathrm{SO}_{2n}(k)$. If n is odd we have the intermediate group the orthogonal group $\mathrm{O}_{2n}(k)$, and if n is even there are two more intermediate groups, the **half-spinor groups** $\mathrm{Spin}^{\pm}_{2n}(k)$.

If $\mathrm{char}(k) \neq 2$ we have that $\mathrm{SO}_{2n}(k)$ is defined as the $x \in \mathrm{SL}_{2n}(k)$ such that $x^T s x = s$ where $s = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$. As before, the definition of this group in characteristic two can be found in [11, §1], for instance.

The groups $\mathrm{Spin}^{\pm}_{2n}(k)$ are defined in [28, p.359], and these two groups are isomorphic to each other, but are not isomorphic to $\mathrm{O}_{2n}(k)$.

Now we state [23, Theorem 32.1].

Theorem 5.29. *If G, G' are simple algebraic groups having isomorphic root systems and isomorphic fundamental groups, then G and G' are isomorphic as algebraic groups with the exception of when the root system is of type D_l, where $l \geq 6$ is even and the fundamental group has order two, then there may be two distinct isomorphism types.*

Remark 5.30. In type D_l for $l \geq 6$ and even, the two non-isomorphic groups which arise are $\mathrm{O}_{2n}(k)$ and $\mathrm{Spin}^{\pm}_{2n}(k)$.

The connected Dynkin diagrams associated to simple groups are presented in [22, p.58], for instance.

Remark 5.31. Let G be a reductive group. By Theorem 5.25, each root $\alpha \in \Phi(G,T)$ of G with respect to T can be written as an integral sum of simple roots. Let p be a prime number. If p does not divide any of the coefficients in this sum, p is said to be **good** for G, and otherwise is said to be **bad** for G. If G is simple, then the bad primes p are as follows; for type A_n there are no bad primes, $p = 2$ for types B_n, C_n and D_n, $p = 2, 3$ for types G_2, F_4, E_6 and E_7, $p = 2, 3$ and 5 for type E_8.

By Lemma 5.26, the root system Φ of $G = G_1 \cdots G_n$ decomposes into a disjoint union $\Phi = \Phi_1 \cup \cdots \cup \Phi_n$, where Φ_i is a root systems of G_i. Therefore, the set of bad primes for a reductive group G is the union of the sets of bad primes for the G_i.

The prime p is said to be **very good** for G if p is good for G and p does not divide $n + 1$ for any of the simple components of type A_n that occur in the decomposition of G into its simple factors.

6 G-Complete Reducibility

6.1 The Structure of Reductive Groups

Let G be a connected algebraic group over the algebraically closed field k.

A **Borel subgroup** of G is defined to be a maximal connected solvable subgroup of G. One-dimensional connected subgroups of G are isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$, and so are commutative and hence solvable. Therefore, Borel subgroups exist in G. The Lie-Kolchin Theorem gives that every Borel subgroup of $\mathrm{GL}_n(k)$ is conjugate to $\mathrm{T}_n(k)$. Furthermore, by Theorem 5.16 we have that every Borel subgroup of the algebraic group $\mathrm{GL}_n(k)$ is conjugate to a subgroup of $\mathrm{T}_n(k)$.

The following is [6, Theorem 11.1].

Theorem 6.1. *Let B be a Borel subgroup of G, then G/B is a projective variety.*

Our next result shows that all the Borel subgroups of G are conjugate, and that all the maximal tori of G are conjugate. This is an important result and is used many times throughout this thesis, for example in Corollary 7.14.

Proposition 6.2. *The set of Borel subgroups of G forms one G-conjugacy class, and the set of maximal tori of G forms one G-conjugacy class.*

Proof. Let B be a Borel subgroup of G and suppose that G is not solvable to avoid the trivial case, so $G \neq B$. By Theorem 6.1, the quotient G/B is thus a non-trivial projective variety, and hence non-affine. Suppose that B' is another Borel subgroup of G. In particular, B' is a connected solvable algebraic group which acts on the complete variety G/B. Applying Theorem 5.18, the action of B' on G/B leaves a point, gB say, of G/B fixed. Therefore $B'gB = gB$. Hence, $g^{-1}B'g \subseteq B$. By the maximality of B', we have the equality $g^{-1}B'g = B$.

Since a torus is connected and solvable it lies in a Borel subgroup. Let T be a maximal torus G, and without loss suppose T is contained in B (so T is a maximal torus of B). Suppose T' is another maximal torus of G', contained in the Borel subgroup $B' = gBg^{-1}$ for some $g \in G$. Then, $g^{-1}T'g$ is a maximal torus of G contained in B and by [23, Theorem 19.3] $gT'g^{-1}$ (and hence T') is conjugate to T, giving the result. $\square$

We define a **parabolic subgroup** to be a subgroup P for which G/P is a projective variety. By [23, Corollary B 21.3], a subgroup P of G is a parabolic subgroup of G if and only if it contains a Borel subgroup of G. Note that this means G itself and B are examples of parabolic subgroups of G.

In the case $G = \mathrm{GL}(V)$, by the Lie-Kolchin Theorem 5.19 we see that a Borel subgroup of G is conjugate to the group $\mathrm{T}_n(k)$, and is hence the stabiliser of a complete flag $\{0\} \neq V_1 \subset \cdots \subset V_m = V$ of V, where the V_i are subspaces of V and $m = \dim V$. In this case, a parabolic subgroup $P \subset G$ is the stabiliser of a partial flag $\{0\} \neq V_1 \subset \cdots \subset V_{m'} \subseteq V$ of V, for $m' \leq \dim V$.

Suppose that G is a reductive group. We can now show that the Weyl group of G is independent of the choice of maximal torus of G. Let T be a maximal torus in G. Then [6, §13.17 Corollary 2(c), and Corollary 11.19] give that $C_G(T) = T = N_G(T)^0$. Thus the Weyl group is equal to $W = N_G(T)/T$. Suppose that S is another maximal torus of G. By Proposition 6.2 $T^g = S$, for some $g \in G$. We have an isomorphism $N_G(T^g) \cong N_G(T)^g$ given by $h \to h^{g^{-1}}$ for $h \in N_G(T)$. Hence, $N_G(S)/S = N_G(T^g)/T^g \cong N_G(T)^g/T^g$. Furthermore, $N_G(T)/T$ is isomorphic to $N_G(T)^g/T^g$ via the map $hT \mapsto h^gT^g$. The Weyl group W of the root system Φ, as defined in §5.10, acts on the root system $\Phi = \Phi(G,T)$. Let $\alpha \in \Phi, t \in T, n \in N_G(T)$ then $n \cdot \alpha(t) = \alpha(ntn^{-1})$ is again a root since $n \cdot \alpha$ is not the zero map and its weight space is non-zero.

Let T be a maximal torus of G, so that we have the root system $\Phi = \Phi(G,T)$. Let B be a Borel subgroup of G containing T. Let Δ be a base of Φ. As remarked earlier, such bases exist and each root of Φ can be written as a linear combination of elements of Δ with all non-negative or all non-positive coefficients. The roots which can be written with non-negative coefficients are called *positive* and the others *negative*.

We have that [6, Theorem 14.1] implies that there exists a unique Borel subgroup of G **opposite** to B with respect to T, which we denote by B^-, for which $B \cap B^- = T$. Also, the system of roots $\Phi(B,T)$ consists of positive roots which we denote Φ^+, and $\Phi(B^-,T)$ consists of negative roots denoted Φ^-, and by [48, 7.4.5.(b)] $\Phi^- = -\Phi^+$.

For example, in $\mathrm{GL}_n(k)$ the group of upper triangular matrices $\mathrm{T}_n(k)$ is a Borel subgroup and so is its opposite, the group of lower triangular matrices $\mathrm{T}_n(k)^-$. The intersection of these two Borel subgroups is the group $\mathrm{D}_n(k)$ of diagonal matrices, and is a maximal torus of $\mathrm{GL}_n(k)$.

This is indicative of the more general situation. For any Borel subgroup B of G, $B \cap B^- = T$ where T is a maximal torus of G. We also have that $B = UT$ where $U = R_u(B)$ is the unipotent radical of B, and $B^- = U^-T$ where $U^- = R_u(B^-)$ and $U \cap U^- = e$. Again, using the case of $G = \mathrm{GL}_n(k)$ as an example clarifies the situation. Then the Borel subgroup $\mathrm{T}_n(k)$ of upper-triangular matrices clearly has such a decomposition as $\mathrm{U}_n(k)\mathrm{D}_n(k)$, where $\mathrm{U}_n(k) = R_u(\mathrm{T}_n(k))$ is the group of upper unitriangular matrices, and $\mathrm{D}_n(k)$ is the group of diagonal matrices. Clearly, we also have $\mathrm{T}_n^-(k) = \mathrm{U}_n^-(k)\mathrm{D}_n(k)$, where $\mathrm{U}_n^-(k) = R_u(\mathrm{T}_n^-(k))$ is the group of lower unitriangular matrices, and $\mathrm{T}_n(k) \cap \mathrm{T}_n^-(k) = \mathrm{D}_n(k)$, and also $\mathrm{U}_n(k) \cap \mathrm{U}_n^-(k) = \mathrm{I}_n$.

The following is [48, 8.1.1 (i)].

Lemma 6.3. *Let G be a reductive group and T a maximal torus of G. For any root $\alpha \in \Phi(G,T)$ there exists an isomorphism u_α from $\mathbb{G}_a$ onto* *a uniquely determined closed subgroup* U_α *of* G *such that* $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ *for all* $t \in T, x \in k$.

By Lemma 6.3, each root $\alpha \in \Phi$ gives rise to a **root subgroup of** G **relative to** T, denoted U_α, and contained in U. Similarly, $-\alpha$ gives rise to the opposite root subgroup $U_{-\alpha}$ in U^-. We have that $G = \langle T, U_\alpha \mid \alpha \in \Phi \rangle$, for a proof see for instance [48, Proposition 8.1.1 (ii)].

The U_α and $U_{-\alpha}$ are the minimal proper subgroups of U and U^- which are normalised by T. They are one dimensional subgroups of G, each isomorphic to $\mathbb{G}_a$. Each such root subgroup is determined by a distinct root, by [48, Proposition 8.1.1].

Let P be a parabolic subgroup of G. Then, by [48, Theorem 8.4.3], P can be written as a semi-direct product $P = L \ltimes R_u(P)$, where L is a reductive group called a **Levi subgroup of** G. The Levi subgroup in this decomposition of P is unique up to conjugation by an element of P, by [15, Proposition 1.22]. If L is a Levi subgroup of G such that $P = L \ltimes R_u(P)$, for some parabolic subgroup P of G, then L is called a **Levi subgroup of** P. A parabolic subgroup P^- of G is said to be **opposite** to the parabolic subgroup P of G if $P \cap P^-$ is a Levi subgroup of both P and P^-. For each parabolic subgroup P of G, and a Levi subgroup L of P, there exists a unique opposite parabolic subgroup P^- of G such that $P^- \cap P = L$ by [6, Proposition 14.21].

We continue our illustration of the general situation using $G = \mathrm{GL}_n(k)$, with maximal torus $T = \mathrm{D}_n(k)$. Then, $N_{\mathrm{GL}_n(k)}(T)$ is the group of monomial matrices in $\mathrm{GL}_n(k)$, and the Weyl group W is the group $N_{\mathrm{GL}_n(k)}(\mathrm{D}_n(k))/\mathrm{D}_n(k)$, which is isomorphic to the symmetric group S_n on n letters. Given the Borel subgroup $B = \mathrm{T}_n(k)$, then any subgroup containing B is a parabolic subgroup P of G. With respect to a suitable basis of V, where V is the natural module for $\mathrm{GL}_n(k)$, a parabolic subgroup P of G is of block diagonal form having arbitrary entries above the blocks and zeros below. For example, if the diagonal part of P consists of s blocks, then the matrices in P have the form:

$$\begin{pmatrix} \boxed{\mathrm{GL}_{m_1}(k)} & & * \\ & \ddots & \\ 0 & & \boxed{\mathrm{GL}_{m_s}(k)} \end{pmatrix}.$$

Then, a Levi subgroup L of P consists of blocks on the diagonal, with each block isomorphic to some $\mathrm{GL}_{m_i}(k)$, for m_i corresponding to the i-th block, where $i \leq s$. The entries above and below these blocks are all zero.

The unipotent radical $R_u(P)$ of P is then of the following form:

$$\begin{pmatrix} \boxed{I_{m_1}} & & * \\ & \ddots & \\ 0 & & \boxed{I_{m_s}} \end{pmatrix}$$

where I_{m_i} for $i \leq s$ is the $m_i \times m_i$ identity matrix in $\mathrm{GL}_{m_i}(k)$.

The root subgroups in $G = \mathrm{GL}_n(k)$ are the subgroups $U_{\alpha_{ij}} = \{I_n + aE_{ij} \mid a \in k\}$, where E_{ij} is the matrix with a 1 in the (i,j)-position and zeros everywhere else. The α_{ij} are the roots of $\mathrm{GL}_n(k)$ relative to the maximal torus $\mathrm{D}_n(k)$, and are elements of $\mathrm{Hom}(\mathrm{D}_n(k), \mathbb{G}_m)$ of the form:

$$\alpha_{ij} : \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \mapsto a_i a_j^{-1}, \qquad \text{for } i \neq j.$$

We present some useful results about these groups.

Proposition 6.4. *For a reductive algebraic group G, and a parabolic subgroup P of G, the following hold:*

(1) *Any Levi subgroup $L \subseteq P$ is of the form $C_G(S)$ where S is a maximal torus of $R(P)$, the radical of P (see §5.6). Furthermore, $S = Z(L)^0$.*

(2) *For any torus S of G, the group $C_G(S)$ is a Levi subgroup of some parabolic subgroup of G. This implies that $C_G(S)$ is reductive. Every Levi subgroup of G has this form.*

(3) *The set of Levi subgroups of P forms one conjugacy class under the action of $R_u(P)$.*

(4) *Every parabolic subgroup P of G is connected, and self-normalising, i.e. $N_G(P) = P$.*

These are all standard results. By the definition of Levi subgroup of P we see that it is isomorphic to $P/R_u(P)$, hence is reductive. Observe that part (4) is [6, Theorem 11.16]. Parts (1) - (3) follow from [6, Proposition 11.23].

6.2 Standard Parabolic and Levi Subgroups

We now proceed with a characterisation of parabolic and Levi subgroups of the reductive group G.

Let T be a maximal torus of G, and let W be the Weyl group $N_G(T)/T$. Suppose that $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ is a base for $\Phi = \Phi(G, T)$, and let $K = \{1, \ldots, l\}$. For $I \subseteq K$ let $\Delta_I := \{\alpha_i \mid \alpha_i \in \Delta, i \in I\}$. Let $W_I := \langle \sigma_{\alpha_i} \mid \sigma_{\alpha_i} \in W, i \in I\rangle$ be the subgroup of the Weyl group W generated by the σ_{α_i}, where the σ_{α_i} are elements of a generating set of W as defined in §5.10, labeled such that σ_{α_i} corresponds to a reflection sending α_i to $-\alpha_i$. Let Φ_I be the set of roots that are linear combinations of the roots in Δ_I. Define N_I as the pre-image of W_I with respect to the projection $N_G(T) \to W$, that is $N_I/T = W_I$. Then, according to [10, §2.1], we define P_I to be $P_I = BN_IB$, where B is the Borel subgroup of G determined by Δ (for the uniqueness of B, see for instance [23, §27.3]). By [48, Lemma 8.4.3], P_I is a parabolic subgroup of G, called the **standard parabolic subgroup** of G relative to I (with respect to B). Then $P_K = G$ and $P_\emptyset = B$. By the proof of [10, Proposition 2.8.4], we have that $R_u(P_I) = \langle U_\alpha \mid \alpha \in \Phi^+, \alpha \notin \Phi_I^+\rangle$. The P_I are the subgroups of G containing B. By [23, Theorem 30.1], every parabolic subgroup of G is conjugate to some P_I.

Recall that a G-variety is a variety V over k equipped with an action $G \times V \to V$ of the group G on the variety V, such that this action is a morphism of varieties, and the G-orbit of a point $v \in V$ is the set $\{g \cdot v \mid g \in G\}$.

Let V be an affine G-variety, and let λ be an element of the set of one-parameter subgroups $Y(G)$ of G. Let $v \in V$. We say that the **limit** $\lim_{x\to 0} \lambda(x) \cdot v$ exists and is equal to u if there is a morphism $M_v(\lambda) : k \to V$ such that $M_v(\lambda)(x) = \lambda(x) \cdot v$ for all $x \neq 0$ and $M_v(\lambda)(0) = u$.

The following lemma is obtained from [38, p11], for instance.

Lemma 6.5. *The set $P_\lambda := \{g \in G \mid \lim_{x\to 0} \lambda(x) g \lambda(x)^{-1} exists\}$ is a subgroup of G.*

Proof. For $g \in P_\lambda$, set $\phi_g(\lambda) : k^* \to G$ to be the map $x \mapsto \lambda(x) g \lambda(x)^{-1}$ for all $x \in k^*$, and define the map $M_g(\lambda)(x) : k \to G$ by:

$$M_g(\lambda)(x) = \begin{cases} \phi_g(\lambda(x)) & \text{if } x \in k^* \\ \lim_{x\to 0} \lambda(x) g \lambda(x)^{-1} & \text{if } x = 0 \end{cases}$$

Let $g_1, g_2 \in P_\lambda$. Define $(M_{g_1}(\lambda), M_{g_2}(\lambda)) : k \to G \times G$ to be map $x \mapsto (M_{g_1}(\lambda)(x), M_{g_2}(\lambda)(x))$ for all $x \in k$. Define the multiplication map $\pi : G \times G \to G$ by $\pi(g, g') = gg'$.

Clearly we have $M_{g_1 g_2}(\lambda)(x) = \pi(M_{g_1}(\lambda)(x), M_{g_2}(\lambda)(x))$, for all $x \in k^*$. If $x = 0$, we have

$$\begin{aligned} M_{g_1 g_2}(\lambda)(0) &= \lim_{x\to 0} \lambda(x) g_1 g_2 \lambda(x)^{-1} \\ &= \lim_{x\to 0} \lambda(x) g_1 \lambda(x)^{-1} \lambda(x) g_2 \lambda(x)^{-1} \\ &= (\lim_{x\to 0} \lambda(x) g_1 \lambda(x)^{-1})(\lim_{x\to 0} \lambda(x) g_2 \lambda(x)^{-1}) \\ &= \pi(M_{g_1}(\lambda)(0), M_{g_2}(\lambda)(0)), \end{aligned}$$

where the product of limits can be taken for the g_i individually in the third equality above because of the continuity of the morphism $\lambda : k^* \to G$.

Hence, we obtain $M_{g_1g_2}(\lambda)(x) = \pi(M_{g_1}(\lambda)(x), M_{g_2}(\lambda)(x))$, for all $x \in k$. Therefore $\lim_{x\to 0} \lambda(x)g_1g_2\lambda(x)^{-1}$ exists and so $g_1g_2 \in P_\lambda$. A similar argument shows $g_1^{-1} \in P_\lambda$. So P_λ is a group and now we have to show that it is a parabolic subgroup of G. $\square$

The following lemma is [48, Proposition 8.4.5], and shows that for each $\lambda \in Y(G)$, there is a corresponding parabolic subgroup P_λ of G.

Lemma 6.6. *The group $P_\lambda := \{g \in G \mid \lim_{x\to 0} \lambda(x)g\lambda(x)^{-1} \mathit{exists}\}$ is a parabolic subgroup of G.*

Proof. A proof is provided in [48, Proposition 8.4.5], in which it is shown that it is possible to pick a Borel subgroup B, all of whose generators (namely, a maximal torus T of G and the root subgroups U_α for $\alpha \in \Phi(G,T)^+$) are contained in P_λ. Hence $B \subseteq P_\lambda$ and is therefore P_λ is a parabolic subgroup of G. $\square$

According to [48, Theorem 8.4.3, Theorem 8.4.5], for any $\lambda \in Y(G)$, there is a unique subset $I \subseteq K$ such that $P_\lambda = P_I$. We define $\Phi(P_I, T)$ to be the union of the positive roots and the negative roots which come from Δ_I. By [23, Theorem 30.1 (b)], this is the root system of P_I relative to T. Suppose that B is a Borel subgroup of G contained in P_I and containing T, then P_I is generated by B and the $U_{-\alpha}$ for $\alpha \in I$. This is also seen intuitively, since B is generated by all the positive root groups, therefore a group containing B must be generated by the positive root groups in addition to some other generators, which can be any set of negative root groups.

By [6, Proposition 14.18] we have that $P_I = L_I \ltimes R_u(P_I)$, where L_I is called the **standard Levi subgroup** of G relative to I, and $L_I = C_G(\cap_{\alpha\in I} \ker\alpha)^0$. By [42, 2.3], $L_I = C_G(\lambda(k^*))$, so we write $L_I := L_\lambda$.

The map $c_\lambda : P_\lambda \longrightarrow L_\lambda$ given by $g \mapsto \lim_{x\to 0} \lambda(x)g\lambda(x)^{-1}$ is a surjective homomorphism of algebraic groups with kernel $R_u(P_\lambda)$.

For any reductive subgroup H of G we have a natural inclusion of sets of one-parameter subgroups $Y(H) \subseteq Y(G)$. Therefore, for $\lambda \in Y(H)$ we obtain a parabolic subgroup of H and one of G by using the construction given in Lemma 6.6. We denote these by $P_\lambda(H)$ and $P_\lambda(G)$ respectively. Similarly, we have corresponding Levi subgroups of H and G, denoted $L_\lambda(H)$ and $L_\lambda(G)$, respectively.

Clearly we have that $P_\lambda(H) = P_\lambda(G) \cap H$, and so every parabolic subgroup of H is the intersection of a parabolic subgroup of G with H. Furthermore, $R_u(P_\lambda(H)) = R_u(P_\lambda(G)) \cap H$, and $L_\lambda(H) = L_\lambda(G) \cap H$. Let M be any Levi subgroup of $P_\lambda(H)$. Then, by Proposition 6.4 (3), there exists some $u \in R_u(P_\lambda(H))$ such that $uL_\lambda(H)u^{-1} = M$. If we set $L := uL_\lambda(G)u^{-1}$, then $M = L \cap H$ and so every Levi subgroup of H is the intersection of a Levi subgroup of G with H. Combining these results, we have the following corollary.

Corollary 6.7. *Let H be a reductive subgroup of G. Then for each parabolic subgroup Q of H, there exists a parabolic subgroup P of G such that $Q = P \cap H$, and $R_u(Q) = R_u(P) \cap H$. Moreover for any Levi subgroup M of Q, there exists a Levi subgroup L of P such that $M = L \cap H$.*

The following is a result of Borel and Tits, see [7, Proposition 3.1].

Theorem 6.8. *Let U be a closed unipotent subgroup of G. Then there exists a parabolic subgroup P of G such that $N_G(U) \subseteq P$ and $U \subseteq R_u(P)$.*

The following lemma is a standard result about parabolic subgroups of connected reductive algebraic groups which can be inferred from [8, Proposition 4.10].

Lemma 6.9. *Let G be a connected reductive algebraic group, and let P and P' be two parabolic subgroups of G. Then the following are equivalent.*

(1) *$P \cap P'$ is reductive, and*

(2) *P and P' are opposite parabolic subgroups of G.*

6.3 G-Complete Reducibility

Let G be a connected reductive algebraic group over the algebraically closed field k. We introduce the notion of G-complete reducibility and we will show, using [1, Theorem 3.1], that it is equivalent to Richardson's geometric notion of strong reductivity, see [42].

The notion of G-complete reducibility was introduced by J.-P. Serre in [46]. The equivalence between the notions of G-complete reducibility and strong reductivity in G is significant because the former is a group theoretic notion, while the latter is geometric. This equivalence enables new methods to be employed in the theory of G-complete reducibility.

Let V be a kG-module (or a G-module for short).

- V is **irreducible** if no proper subspace of V, other than the trivial subspace, is G-stable.
- V is **semisimple** if it is a direct sum of irreducible submodules.

The following definition is due to J-P. Serre, see [45].

Definition 6.10. *Let H be a subgroup of G.*

(1) *H is called **G-irreducible** (or **G-ir**) if H is not contained in any proper parabolic subgroup of G.*

(2) *H is called **G-completely reducible** (or **G-cr**) if whenever H is contained in a proper parabolic subgroup P of G, then H is contained in a Levi subgroup of P.*

Note that G itself is trivially G-completely reducible. However, a reductive subgroup of G is not necessarily G-completely reducible, as shown by the counter-example given in [1, Example 3.45] which we will discuss again in Example 10.16.

To determine whether a subgroup of G is G-completely reducible, we need to examine its containment inside various parabolic subgroups and Levi subgroups of G. Let P be a parabolic subgroup of G, and L a Levi subgroup of G. For a subgroup H of G, we note that $H \subseteq P$ if and only if the Zariski closure $\overline{H}$ of H is contained in P, since P is closed. Similarly, $H \subseteq L$ if and only if $\overline{H}$ is contained in L. Therefore, we may assume without loss that H is closed.

Lemma 6.11. *Let $G = \mathrm{GL}(V)$, where V is finite dimensional, and let H be a subgroup of G, so that V is an H-module. Then V is a semisimple H-module if and only if H is G-completely reducible.*

Proof. A parabolic subgroup P of $\mathrm{GL}(V)$ is the stabiliser of a flag $\mathcal{F} := (V_1, \ldots, V_m)$ of subspaces $\{0\} \neq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m$ of V, where $m \leq \dim(V)$.

For each $g \in P$ we have $gV_i = V_i$. Hence, g induces an automorphism of V_i/V_{i-1}, for $i = 1, \ldots, m$. We may choose a complement W_i to V_{i-1} in V_i such that $V_i = V_{i-1} \oplus W_i$. Then a Levi subgroup L of P consists of the $g \in P$ that stabilise each of the W_i. We have L is isomorphic to $\mathrm{GL}_{n_1}(k) \times \cdots \times \mathrm{GL}_{n_m}(k)$, where each $n_i = \dim(W_i)$. To each subspace U of V that is stabilised by L, there is a complement to U in V that is also stabilised by L. Thus V is a semisimple L-module.

Suppose that H is G-completely reducible, and that H stabilises a subspace U of V. Then H is contained in a parabolic subgroup P of G that also stabilises U. Each Levi subgroup of P stabilises U, and a complement to U. As H is G-completely reducible, H is contained in some Levi subgroup L

of P. As $H \subseteq L$, and U has an L-stable complement, then U has the same H-stable complement. Thus, V is a semisimple H-module.

Conversely, suppose that V is a semisimple H-module and that H is contained in a parabolic subgroup P of G. Since P acts on V by stabilising a flag $(V_1, \ldots, V_n)$, we have that H also stabilises $(V_1, \ldots, V_n)$. Since V is a semisimple H-module H is of block diagonal form, and so is contained in a Levi subgroup of P. $\square$

Remark 6.12. It is not difficult to show that since $\mathrm{SL}(V) = [\mathrm{GL}(V), \mathrm{GL}(V)]$ we have that a subgroup of $\mathrm{SL}(V)$ is $\mathrm{SL}(V)$-completely reducible if and only if it is $\mathrm{GL}(V)$-completely reducible, see for instance Lemma 9.7 later in this thesis for more details. Therefore, we have that Lemma 6.11 holds for $G = \mathrm{SL}(V)$.

The corresponding situation, when the characteristic of k is different from 2 and $G = \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$, holds analogously, and is discussed in [1, Example 3.23], and [45, Example 3.2.2(b)].

This result shows that the notions of G-complete reducibility and semisimplicity of modules coincide when working in $\mathrm{GL}(V)$, however the notion of G-complete reducibility clearly extends to an arbitrary reductive group G, and is in this sense a generalisation of the notion of reducibility of linear representations.

We now introduce an important invariant for simple algebraic groups.

Definition 6.13. *Let G be a simple algebraic group, with maximal torus T. Let $\Phi(G,T)$ be the root system of G relative to T with a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ for $\alpha_i \in \Phi(G,T)$. Let $\sigma_{\alpha_i} \in W(\Phi)$ be the reflection corresponding to α_i. Let Γ be the group generated by the element $\sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$. The* ***Coxeter number*** *h of G is the order of the group Γ.*

The Coxeter number of G is an invariant of G, and this can be seen by

[9, Theorem 1(ii) Ch.V, §6] which shows that the number of roots of G is equal to hl where l is the rank of G.

Let V be a finite dimensional vector space, and let G be $\mathrm{GL}(V)$. Then, as we noted in Lemma 6.11, a subgroup H of G is G-completely reducible if, and only if, V is a semisimple H-module. By Remark 6.12 we can replace $\mathrm{GL}(V)$ with $\mathrm{SL}(V)$ and the corresponding result holds. Now let G be either $\mathrm{Sp}(V)$, or $\mathrm{SO}(V)$, where as above V is a finite dimensional vector space. In this case if $\mathrm{char}(k) > 2$, then a subgroup H of G is G-completely reducible if, and only if, V is a semisimple H-module, see [45, Example 3.2.2(b)].

There is a counterexample for $\mathrm{char}(k) = 2$ which is described in [1, Example 3.45]. In this example we take n to be an even integer, greater than or equal to 4. By embedding $\mathrm{Sp}_n(k)$ diagonally in $\mathrm{Sp}_n(k) \times \mathrm{Sp}_n(k)$ it is shown that $\mathrm{Sp}_n(k)$ is not $\mathrm{Sp}_{2n}(k)$-completely reducible, however $\mathrm{Sp}_n(k)$ is connected and reductive. Consider $\mathrm{Sp}(V \oplus V')$, where V, V' are natural modules for the $\mathrm{Sp}_n(k)$ factors of $\mathrm{Sp}_n(k)\times\mathrm{Sp}_n(k)$, then $V\oplus V'$ is a semisimple $\mathrm{Sp}_n(k)$-module.

Let G be an exceptional group, then G can be embedded in $\mathrm{GL}(\mathfrak{g})$. Let h be the Coxeter number of G, and let H be a subgroup of G. If $\mathrm{char}(k) > 2h - 2$, then the Lie algebra $\mathfrak{g}$ of G is a semisimple H-module (via $\mathrm{Ad}\,|_H$) if and only if H is G-completely reducible, for details see [45, Corollary 5.5]. The bound on the forward implication can be improved. In the case G is an adjoint simple group of exceptional type, by [1, Remark 3.43], we only need $\mathrm{char}(k)$ to be good for G.

The following is a result of Serre, see [47, Property 4].

Lemma 6.14. *If $H \subseteq G$ is G-completely reducible, then H^0 is reductive.*

Proof. Suppose by way of contradiction that $R_u(H) \neq e$. Then, by Theorem 6.8, we have $R_u(H) \subseteq R_u(P)$ for some parabolic subgroup P of G with $H \subseteq N_G(R_u(H)) \subseteq P$. Now H is G-cr, so it is contained in some Levi subgroup L of P. Then $R_u(H) \subseteq H \cap R_u(P) \subseteq L \cap R_u(P) = e$. This is a contradiction, therefore $R_u(H) = e$ and H^0 is reductive. $\square$

We have the reverse implication to Lemma 6.14 under certain circumstances. If the characteristic of k is larger than the Coxeter number of G, then a closed connected subgroup H of G is G-completely reducible if and only if H is reductive, see [1, Theorem 3.48]. If $\mathrm{char}(k) = 0$, then a closed subgroup H of G is G-completely reducible if and only if H^0 is reductive, by [45, Proposition 4.2]. To avoid the non-interesting case, from now on we assume k has positive characteristic.

We conclude this section with a result about reductive subgroups of parabolic subgroups.

Lemma 6.15. *Let P be a parabolic subgroup of G. Suppose that H is a connected reductive subgroup of G contained in P. Then, H intersects $R_u(P)$ trivially.*

Proof. Let $U = H \cap R_u(P)$. Then, U is a normal unipotent subgroup of H. Since H is reductive, U must be finite.

Then U is a finite normal subgroup of the connected group H. By [23, Proposition 8.1], H acts trivially on U, and so U is central in H. However, the centre of H is a torus consisting of only semisimple elements. Thus, $H \cap R_u(P) = e$. $\square$

6.4 Strong Reductivity

In this section we introduce the notion of strong reductivity in G, which is due to Richardson [42]. We also describe some important work of Bate, Martin and Röhrle [1], which provides a link between Richardson's geometric notion of strong reductivity and the group theoretic notion of G-complete reducibility.

In much of the following we have a group G acting on the variety G^n for some integer n in the following manner. We let G act on itself by conjugation, and extend this action to G^n by considering the action by simultaneous conjugation defined by $g \cdot (x_1, \ldots, x_n) = (gx_1g^{-1}, \ldots, gx_ng^{-1})$ for $g \in G, (x_1, \ldots, x_n) \in G^n$. In this sense, the variety G^n is a G-variety.

Set $\mathbf{x} := (x_1, \ldots, x_n)$. For a one-parameter subgroup $\lambda \in Y(G)$, we say that the limit $\lim_{x\to 0} \lambda(x) \cdot \mathbf{x}$ **exists** if $\lim_{x\to 0} \lambda(x) \cdot x_i$ exists for each i.

The following definition is due to Richardson, see [42, Definition 16.1].

Definition 6.16. *Let G be a reductive group. A subgroup H of G is said to be **strongly reductive in** G if H is not contained in any proper parabolic subgroup of $C_G(S)$ where S is a maximal torus of $C_G(H)$.*

Remark 6.17. It is clear that Definition 6.16 does not depend on the choice of S. To see this, fix the maximal torus S of $C_G(H)$, and suppose that H is contained in a proper parabolic subgroup P of $C_G(S)$. Let S_1 be another maximal torus of $C_G(H)$. As tori are connected, and both S and S_1 are maximal tori in $C_G(H)^0$, by Proposition 6.2 $S_1 = S^g$ for some $g \in C_G(H)$. Therefore, $H = H^g \subset P^g \subset C_G(S)^g = C_G(S^g) = C_G(S_1)$, where P^g is a proper parabolic subgroup of $C_G(S_1)$. The same argument works in the other direction, hence H is contained in a proper parabolic subgroup of $C_G(S)$ if, and only if, it is contained in a proper parabolic subgroup of $C_G(S_1)$.

The following result provides an equivalence between the notions of strong reductivity in G and G-complete reducibility. This link was proved in [1, Theorem 3.1]. We restate this result and sketch its proof below and note that the methods employed are similar to those used in other results that follow, for example in Theorem 10.25.

Theorem 6.18. *Let G be a reductive group and H a closed subgroup of G. Then H is G-completely reducible if and only if H is strongly reductive in G.*

Proof. For the forward direction we have that H is G-cr and suppose by way of contradiction that H is contained in a proper parabolic subgroup Q of $C_G(S)$, where S is a maximal torus of $C_G(H)$. by Corollary 6.7, there exists a parabolic subgroup P of G such that $Q = C_G(S) \cap P$, where P contains S and H. Since H is G-cr it is contained in a Levi subgroup L of P, where $L = C_G(T)$ for some torus $T \in C_P(H)$. Since S is a maximal torus of $C_P(H)$, there is some $g \in C_P(H)$ such that T^g is contained in S. Therefore, $C_G(S)$ is contained in L^g. Thus, $C_G(S) \subseteq P$, however this implies that $Q = C_G(S)$ which is a contradiction. We conclude that H is strongly reductive in G.

For the reverse, suppose that H is strongly reductive in G, and let S be a maximal torus of $C_G(H)$. Then, H is not contained in any proper parabolic subgroup of $C_G(S)$. By Proposition 6.4 (2), $C_G(S) := L$ is a Levi subgroup of G. Let Q be a parabolic subgroup of G containing L as a Levi subgroup. By [8, Proposition 4.4(c)], Q is a minimal parabolic subgroup of G with respect to containing H.

Let P be a parabolic subgroup of G containing H. By [8, Proposition 4.4(b)] and the minimality of Q we have $Q = (P \cap Q)R_u(Q)$. Similarly, we have $(P \cap Q)R_u(P)$ is a parabolic subgroup of G contained in P. For any parabolic subgroup $P' \subseteq P$, and a Levi subgroup M' of P', by [8, Proposition 4.4], there is a Levi subgroup M of P such that $M' \subseteq M$. Therefore, we may assume that P is minimal with respect to containing H, and thus $P = (P \cap Q)R_u(P)$.

By [8, Proposition 4.4(b)], P contains a Levi subgroup, M_Q say, of Q, and Q contains a Levi subgroup, M_P say, of P. By choosing Levi subgroups L_P and L_Q of P and Q respectively such that $L_P \cap L_Q$ contains a maximal torus of G, we have the standard decomposition of $P \cap Q$, as given by in the proof of [1, Theorem 3.1]:

$$P \cap Q = (L_P \cap L_Q)(L_P \cap R_u(Q))(R_u(P) \cap L_Q)(R_u(P) \cap R_u(Q)). \qquad (2)$$

Then $R_u(P \cap Q)$ is the product of the last three factors of equation (2). Since M_P is reductive and is contained in $P \cap Q$, we have $M_P \cap R_u(P \cap Q)$ is trivial. Therefore, M_P is isomorphic to a subgroup of $L_P \cap L_Q$. It then follows that $M := L_P = L_Q$ is a common Levi subgroup of both P and Q.

Let P^- be the opposite parabolic subgroup to P with respect to M. We have that L and M are both Levi subgroups of Q, hence are $R_u(Q)$-conjugate. Our goal is to show that L is $R_u(P^-)$-conjugate to a Levi subgroup of P^-, for then we could conclude that $H \subset P^-$, and hence $H \subseteq P \cap P^- = M$. This would give that H is G-cr.

We have that

$$R_u(Q) = (R_u(Q) \cap R_u(P^-))(R_u(Q) \cap R_u(P)).$$

Therefore, $yzMz^{-1}y^{-1} = L$ for $y \in R_u(Q) \cap R_u(P^-)$ and $z \in R_u(Q) \cap R_u(P)$. Since $zMz^{-1} \subset P$, we can take $z = e$ without loss. This gives that $L = yMy^{-1} \subseteq P^-$. Hence, $H \subseteq P \cap P^- = M$, as required. $\square$

We see from Lemma 6.14 and Theorem 6.18 that a strongly reductive subgroup of G is reductive. Without appealing to the notion of G-complete reducibility, Richardson proves that a strongly reductive subgroup of G is reductive in [42, Lemma 16.3].

Strong reductivity is a geometric notion, in that strongly reductive subgroups correspond to closed orbits, as the following theorem, [42, Theorem 16.4] due to Richardson, shows.

Theorem 6.19. *Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in G^n$. Then the orbit $G \cdot \boldsymbol{x}$ is closed in G^n if and only if $H = \overline{\langle x_1, \ldots, x_n \rangle}$ is strongly reductive in G.*

In order to prove Theorem 6.19 we need the following, which is the Hilbert–Mumford Theorem as presented in [26, Theorem 1.4].

Theorem 6.20. *Let G be a reductive group and V a G-variety and let $v \in V$. Let U be a closed G-subvariety of V which meets the closure of $G \cdot v$. Then there exists a one-parameter subgroup $\lambda \in Y(G)$ such that $\lim_{x \to 0} \lambda(x) \cdot v$ exists and belongs to U.*

Definition 6.21. *Let X be a G-variety. Let $Z = \cap_{x \in X} C_G(x)$ be the kernel of the action of G on X. We say $x \in X$ is a **stable point** for the action of G if the orbit $G \cdot x$ is closed in X and $C_G(x)/Z$ is finite.*

Lemma 6.22. *Let $x \in X$ and let S be a maximal torus of $C_G(x)$. Then $G \cdot x$ is closed if and only if x is a stable point for the action of $C_G(S)$ on X^S.*

Proof. If $G \cdot x$ is closed then $C_G(S) \cdot x$ is closed by [43, Theorem C], and is therefore an affine variety. Let $H = C_{C_G(S)}(x)$. As $C_G(S) \cdot x$ is closed, [43, Lemma 10.1.3] implies that $C_G(S)/H$ is an affine variety. Therefore, by [41, Theorem A], H^0 is reductive. We have that S is central in H and therefore also in H^0, since S is connected. The group S is a maximal torus of H^0 as S is a maximal torus of $C_G(x)$, and $H \subseteq C_G(x)$. Hence, $C_{H^0}(S) = S = H^0$. Finally, H/S is finite and $S \subseteq Z$, for Z the kernel of the action of $C_G(S)$ on X^S, so x is a stable point.

Conversely if x is a stable point for the action of $C_G(S)$ on X^S, then $C_G(S) \cdot x$ is closed and, by [43, Theorem C], $G \cdot x$ is closed. □

The following proposition, due to Richardson [43], is essential in proving Theorem 6.19.

Proposition 6.23. *Let $(x_1, \ldots, x_n) \in G^n$. Then $(x_1, \ldots, x_n)$ is a stable point of G^n if and only if $\overline{\langle x_1, \ldots, x_n \rangle}$ is not contained in any proper parabolic subgroup of G.*

Proof. Set $\mathbf{x} := (x_1, \ldots, x_n)$ and $H := \overline{\langle x_1, \ldots, x_n \rangle}$. Suppose that H is not contained in any proper parabolic subgroup of G. Suppose $G \cdot \mathbf{x}$ is not closed. Let $\mathbf{y} = (y_1, \ldots, y_n) \in \overline{G \cdot \mathbf{x}} \backslash G \cdot \mathbf{x}$ such that $G \cdot \mathbf{y}$ is the unique closed orbit in this boundary, which exists by [39, No.8]. Then by Theorem 6.20 there exists $\lambda \in Y(G)$ such that $\lim_{x \to 0} \lambda(x) \cdot \mathbf{x}$ exists and belongs to $G \cdot \mathbf{y}$ (which is closed by hypothesis). But then $H \subseteq P_\lambda$. If P_λ were proper in G this would be a contradiction, therefore λ must be central in G giving $\lim_{x \to 0} \lambda(x) \cdot \mathbf{x} = \mathbf{x}$. Thus $\mathbf{x} \in G \cdot \mathbf{y}$, so $G \cdot \mathbf{x}$ is closed and therefore affine.

As $G \cdot \mathbf{x}$ is closed, [43, Lemma 10.1.3] implies that $G/C_G(\mathbf{x})$ is an affine variety. Therefore, by [41, Theorem A], $C_G(\mathbf{x})^0$ is reductive. Let S be a maximal torus of $C_G(\mathbf{x})$, and let λ be a one-parameter subgroup of S. Then $\mathbf{x} \in C_G(S)^n \subseteq C_G(\lambda(k^*))^n$. So $H \subseteq C_G(\lambda(k^*)) \subseteq P_\lambda$. Therefore $P_\lambda = G$, hence all the one-parameter subgroups of S are central in G. Therefore, $S \subseteq Z(G)^0$. The group $C_G(\mathbf{x})^0$ is reductive and S is a central maximal torus in $C_G(\mathbf{x})^0$, so $C_{C_G(\mathbf{x})^0}(S) = S = C_G(\mathbf{x})^0$.

Set $Z = \cap_{\mathbf{x} \in G^n} C_G(\mathbf{x})$. For $\mathbf{x}$ to be a stable point we require $C_G(\mathbf{x})/Z$ to be finite. We have that $C_G(\mathbf{x})/S$ is finite. However $Z \supseteq S$. This can be seen since $S \subseteq Z(G)^0$ and so S commutes with all $\mathbf{z} \in G^n$ under the diagonal action. Therefore $S \subseteq C_G(\mathbf{z})$ for each $\mathbf{z} \in G^n$, and so S is contained in their intersection. Hence $C_G(\mathbf{x})/Z$ is no bigger than $C_G(\mathbf{x})/S$, and so is also finite.

Conversely, suppose there exists some non-central one-parameter subgroup $\lambda \in Y(G)$ with $H \subseteq P_\lambda$. Set $\mathbf{y} = (c_\lambda(x_1), \ldots, c_\lambda(x_n))$, where c_λ is the map defined in §6.2. Then, each $c_\lambda(x_i) \in L_\lambda = C_G(\lambda(k^*))$. So $c_\lambda(x_i) \in C_G(\lambda(k^*))$ for each i, and so $C_G(\mathbf{y}) \supseteq C_G(C_G(\lambda(k^*))) \supseteq \lambda(k^*)$. This means that $\lambda \in Y(C_G(\mathbf{y}))$. However, $\lambda \notin Z(G)$, so $Z(G)^0 \subsetneq C_G(\mathbf{y})^0$. The two groups $Z(G)^0$ and $C_G(\mathbf{y})^0$ must therefore have different dimensions because they are connected, and so $C_G(\mathbf{y})/Z(G)$ is infinite. Since $\cap_{\mathbf{y} \in G^n} C_G(\mathbf{y})$ is the kernel of the action of G on G^n, it is contained in $Z(G)$. Thus, $C_G(\mathbf{y})/\cap_{\mathbf{y} \in G^n} C_G(\mathbf{y})$ is infinite and $\mathbf{y}$ is not a stable point.

If $\mathbf{y} \in G \cdot \mathbf{x}$, then $\mathbf{y} = g \cdot \mathbf{x}$ for some $g \in G$. Let $h \in C_G(\mathbf{x})$. Then

$ghg^{-1} \cdot \mathbf{y} = gh \cdot \mathbf{x} = g \cdot \mathbf{x} = \mathbf{y}$ so $h^g \in C_G(\mathbf{y})$. Hence, $C_G(\mathbf{x})$ and $C_G(\mathbf{y})$ are conjugate groups. The quotient $C_G(\mathbf{x})/Z(G)$ is thus infinite and $\mathbf{x}$ is also not a stable point. Now suppose $\mathbf{y} \notin G \cdot \mathbf{x}$, then the orbit of $\mathbf{x}$ is not closed because it does not contain the limit $\mathbf{y}$. Again, we conclude that $\mathbf{x}$ is not stable. $\square$

We now present a proof of Theorem 6.19.

Proof. Set $X = G^n$, and let $\mathbf{x} = (x_1, \ldots, x_n) \in X$. Recall that $H = \overline{\langle x_1, \ldots, x_n \rangle}$. Fix a maximal torus S of $C_G(H) = C_G(\mathbf{x})$.

Suppose $G \cdot \mathbf{x}$ is closed. Then, by Lemma 6.22, $\mathbf{x}$ is a stable point for the action of $C_G(S)$ on X^S. But then, by Proposition 6.23, H is not contained in any proper parabolic subgroup of $C_G(S)$, because $C_G(H) \subseteq \cap_i C_G(x_i)$ and clearly $S \subseteq C_G(H)$. So H is strongly reductive in G.

Conversely, let S be a maximal torus of $\cap_i C_G(x_i)$. Assume H is strongly reductive in G. By Proposition 6.23, $\mathbf{x}$ is a stable point for the action of $C_G(S)$ on $(C_G(S))^n$. The orbit $C_G(S) \cdot \mathbf{x}$ is closed in X and hence, by [43, Theorem C], $G \cdot \mathbf{x}$ is closed. $\square$

6.5 Topologically Finitely Generated Groups

The following is [1, Lemma 2.10], and we include its proof here because we use it as the basis for the proof of Lemma 10.4 later on.

Lemma 6.24. *Let H be a closed subgroup of G. Then, there exists a topologically finitely generated subgroup Γ of H such that for any parabolic subgroup P of G and any Levi subgroup L of P, P contains H if and only if P contains Γ, and L contains H if and only if L contains Γ.*

Proof. Recall that each parabolic subgroup containing a Borel subgroup B of G is of the form P_I for some subset I of an indexing set $\{1, \ldots, l\}$ of the l roots in $\Delta \subseteq \Phi(B, T)$ and each subset I of $\{1, \ldots, l\}$ gives rise to a parabolic subgroup containing B. Moreover, P_I is conjugate to P_J if and only if $I = J$. Then since each Borel subgroup of G is conjugate to B, any parabolic subgroup of G has a conjugate containing B and is therefore conjugate to a parabolic subgroup of the form P_I for some I. Since Δ is a finite set there are only finitely many conjugacy classes of parabolic subgroups of G with representatives $P_1, \ldots, P_m$, say. Each P_i has one conjugacy class of Levi subgroups since all Levi subgroups of a parabolic P_i are conjugate by $R_u(P)$. Let $L_1, \ldots, L_n$ be representatives of the set of conjugacy classes of Levi subgroups. Note that although to each parabolic subgroup P_i there is one conjugacy class of Levi subgroups, by Proposition 6.4, some of these classes may coincide so that in general $n \leq m$.

Now for any subgroup $H' \subseteq H$ set

$$C_i(H') := \{g \in G \mid H' \subseteq gP_ig^{-1}\}$$

and

$$D_j(H') := \{g \in G \mid H' \subseteq gL_jg^{-1}\}.$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$. By [23, Proposition 8.2 (a)], each of these sets is closed and, for any subgroup H'' containing H', we have $C_i(H'') \subseteq C_i(H')$ and $D_j(H'') \subseteq D_j(H')$. For each i, j set $\mathcal{C}_i, \mathcal{D}_j$ to be the set of all $C_i(H'), D_j(H')$ for $H' \subseteq H$ topologically finitely generated. For an increasing chain of topologically finitely generated subgroups of H, we get that $\mathcal{C}_i, \mathcal{D}_j$ contain decreasing chains of closed sets for each i and j, each of which must terminate by the descending chain condition on closed sets. By Zorn's Lemma we can find minimal elements in $\mathcal{C}_i, \mathcal{D}_j$, for each i, j, and these minimal elements arise from finitely generated subgroups of H. Take Γ to be the subgroup of H generated by all of these finitely generated subgroups.

We have that $C_i(\Gamma) \subseteq C_i(H')$ and $D_j(\Gamma) \subseteq D_j(H')$ for all i, j and all topologically finitely generated subgroups $H' \subseteq H$. Then, $C_i(\Gamma) = C_i(H)$

and $D_j(\Gamma) = D_j(H)$ for all i, j since if $C_i(H) \subsetneq C_i(\Gamma)$, then there exists some $g \in C_i(\Gamma)$ and $h \in H$ with $ghg^{-1} \notin P_i$. But then the group Γ' topologically generated by Γ and h has $C_i(\Gamma') \subsetneq C_i(\Gamma)$, which is a contradiction.

Now suppose $H \subseteq P$. As mentioned P is G-conjugate to some P_i, so for some $g \in G$ we have $gPg^{-1} = P_i$ and $gHg^{-1} \subseteq P_i$, so $g \in C_i(H) = C_i(\Gamma)$ and so $g\Gamma g^{-1} \subseteq P_i$ and so $\Gamma \subseteq P$. Conversely, suppose that $\Gamma \subseteq P$. Then $g\Gamma g^{-1} \subseteq P_i$. Hence, $g \in C_i(\Gamma) = C_i(H)$. Therefore, $gHg^{-1} \subseteq P_i$, and so $H \subseteq P$. The argument is similar for Levi subgroups. □

Lemma 6.24 shows that for the purposes of studying G-complete reducibility, we only need to consider topologically finitely generated subgroups of G. This is because we may replace a subgroup H of G with the group Γ, as in Lemma 6.24, which has exactly the same properties as H in terms of its G-complete reducibility. This is an important observation because in order to investigate whether H is G-completely reducible, we examine the G-orbit of a tuple $(g_1, \ldots, g_n)$ in G^n which generates Γ. As described by Theorem 6.19 this orbit determines whether Γ is strongly reductive in G, and hence, by Theorem 6.18, whether Γ is G-completely reducible. Therefore, in order to exploit the benefits of this geometric approach to G-complete reducibility, we rely on Lemma 6.24.

6.6 Normal Subgroups

In this section we follow the argument of Martin to prove [35, Theorem 2], which shows that a normal subgroup of a G-completely reducible subgroup of G is itself G-completely reducible. We introduce some terminology and initial results.

Let V be an affine G-variety, and let λ be an element of the set of one-parameter subgroups $Y(G)$ of G. Let $v \in V$. Recall that the limit $\lim_{x\to 0} \lambda(x)\cdot v$ is said to exist and equal u if there is a morphism $M_v(\lambda) : k \to$

V such that $M_v(\lambda)(x) = \lambda(x) \cdot v$ for every $x \in k^*$ and $M_v(\lambda)(0) = u$. Denote by $|V, v|$ the set of one-parameter subgroups λ such that $\lim_{x\to 0} \lambda(x) \cdot v$ exists.

Let U be a closed G-invariant subvariety of V such that $v \notin U$. Then $M_v(\lambda)(k^*) \cap U = \emptyset$. Therefore, $M_v(\lambda)(k) \cap U$ is non-empty if and only if $M_v(\lambda)(0) \in U$.

For a commutative ring A we denote the set of all proper prime ideals of A by $\mathrm{Spec}(A)$. Suppose that $M_v(\lambda)(0) \in U$. We have that $U \subseteq V$ and so $k[U] = k[V]/I$ for some ideal $I \subseteq k[V]$. Hence, we have that the comorphism $M_v(\lambda)^*$ maps $k[V]$ to $k[\mathbb{A}^1]$. Therefore, the ideal generated by $M_v(\lambda)^*(I)$, denoted $[I]$, is an ideal in $k[\mathbb{A}^1]$. The pre-image $M_v(\lambda)^{-1}(U)$ of U under the map $M_v(\lambda)$ is a closed subvariety of $\mathbb{A}^1$ determined by $\mathrm{Spec}(k[\mathbb{A}^1]/[I])$, and which as a set is just $\{0\}$. Therefore, there is some positive integer m such that $M_v(\lambda)^{-1}(U)$ is determined by $\mathrm{Spec}(k[T]/(T^m))$ for an indeterminate T. The integer m depends on U, v and λ, so we denote it by $\alpha_{U,v}(\lambda)$. Note this is non-negative in general, and positive if and only if $M_v(\lambda)(0) \in U$.

A **length function** on $Y(G)$ is a G-invariant function $||-|| : Y(G) \to \mathbb{R}_{\geq 0}$ such that for any maximal torus T of G there is a positive definite $\mathbb{Z}$-valued bilinear form $\langle \cdot, \cdot \rangle$ on $Y(T)$ such that for all $\lambda \in Y(T)$ we have $||\lambda|| = \sqrt{\langle \lambda, \lambda \rangle}$. It is shown in [26] that length functions on $Y(G)$ exist.

Let W be the Weyl group $N_G(T)/T$ of G, in particular W is finite. We have $N_G(T)$ acts on T by conjugation, and so we have an action of W on $Y(T)$ defined by $nT \cdot \lambda(x) = n\lambda(x)n^{-1}$ for $\lambda \in Y(T), n \in N_G(T), x \in k^*$. For any positive definite $\mathbb{Z}$-valued bilinear form $\langle \cdot, \cdot \rangle$ on $Y(T)$, we can form a W-invariant length function $|| \cdot ||_W$ in the following way. Since W is finite, we define a W-invariant positive definite $\mathbb{Z}$-valued bilinear form $\langle \cdot, \cdot \rangle_W$ on $Y(T)$ by setting:

$$\langle \lambda, \mu \rangle_W = \Sigma_{w \in W} \langle w \cdot \lambda, w \cdot \mu \rangle, \quad \text{for } \lambda, \mu \in Y(T).$$

By [26, Lemma 2.1 (a)], there is a bijective correspondence between G-orbits of $Y(G)$ and W-orbits of $Y(T)$. For any $g \in G, \lambda \in Y(T)$, we have $g \cdot \lambda \in Y(G)$ can be identified with some $n \cdot \lambda \in Y(T)$ for $n \in N_G(T)$. Since $\langle \cdot, \cdot \rangle_W$ is W-invariant, we can set $||g \cdot \lambda||_W = \langle \lambda, \lambda \rangle_W$.

Let $|V, v|_U$ denote the set of one-parameter subgroups λ of G, such that $\lim_{x \to 0} \lambda(x) \cdot v$ exists and belongs to U.

We call a one-parameter subgroup $\lambda \in Y(G)$ **indivisible** if it is not of the form $n\mu$ for any $\mu \in Y(G)$ and any integer n greater than 1.

The Hilbert–Mumford Theorem, Theorem 6.20, asserts that, for V, v and U as above, a one-parameter subgroup $\lambda \in Y(G)$ can be chosen so that the integer $\alpha_{U,v}(\lambda)$ is non-zero. The following theorem of Kempf is [26, Theorem 3.4], and shows that the $\alpha_{U,v}(\lambda)$ reach a certain upper bound, and that the λ reaching this upper bound give rise to a particular class of parabolic subgroups of G, see Notation 6.26.

Theorem 6.25. *Let G be a reductive algebraic group, V a G-variety containing v and U a closed G-stable subvariety of V which does not contain v, and satisfies $U \cap \overline{G \cdot v} \neq \emptyset$. Then the following hold:*

(1) *The function $|V, v|_U \to \mathbb{R}$ given by $\lambda \mapsto \alpha_{U,v}(\lambda)/||\lambda||_W$ reaches an upper bound.*

(2) *There exists a non-trivial indivisible one-parameter subgroup $\lambda \in |V, v|_U$ which attains this upper bound. For any other one-parameter subgroup μ with this property we have:*

    (a) *$P_\lambda = P_\mu$;*

    (b) *λ and μ are conjugate by some element of $R_u(P_\lambda)$.*

Notation 6.26. The parabolic subgroup P_λ of G arising in this theorem is determined uniquely by V, v and U, so we can denote P_λ by $P_{U,v}$, and $P_{U,v}$ is called the **destabilising** parabolic subgroup of G with respect to U and v. We call the indivisible one-parameter subgroup λ **optimal** with respect to V, v and U. Define $\Lambda_{U,v}$ to be the subset of $|V, v|$ containing the indivisible optimal one-parameter subgroups of G with respect to U and v.

Lemma 6.27. *We have $gP_{U,v}g^{-1} = P_{U,g\cdot v}$ for all $g \in G$.*

Proof. Let $* : G \times G \to G$ denote the action of conjugation, given by $g * h = ghg^{-1}$. We first show that for any $g \in G$ we have $g * \Lambda_{U,v} = \Lambda_{U,g\cdot v}$. Note that by the calculation given in the proof of [42, Lemma 2.7], the limit $g \cdot (\lim_{x\to 0} \lambda(x) \cdot v)$ exists if and only if the limit $\lim_{x\to 0} g \cdot (\lambda(x) \cdot v)$ exists. Let $g \in G$, we have:

$$\begin{aligned}
|V, g \cdot v| &= \{\lambda : \mathbb{G}_m \to G \mid \lim_{x\to 0} \lambda(x) \cdot (g \cdot v) \text{ exists}\} \\
&= \{\lambda : \mathbb{G}_m \to G \mid g^{-1} \cdot (\lim_{x\to 0} \lambda(x) \cdot (g \cdot v)) \text{ exists}\} \\
&= \{\lambda : \mathbb{G}_m \to G \mid \lim_{x\to 0} g^{-1} \cdot (\lambda(x) \cdot (g \cdot v)) \text{ exists}\} \\
&= \{\lambda : \mathbb{G}_m \to G \mid \lim_{x\to 0} (g^{-1}\lambda(x)g) \cdot v \text{ exists}\} \\
&= \{\lambda : \mathbb{G}_m \to G \mid \lim_{x\to 0} (g^{-1} * \lambda(x)) \cdot v \text{ exists}\} \\
&= \{g * \lambda' : \mathbb{G}_m \to G \mid \lim_{x\to 0} \lambda'(x) \cdot v \text{ exists}\} \\
&= g * |V, v|.
\end{aligned}$$

By the proof of [26, Corollary 3.5] we have $\alpha_{U,g\cdot v}(g * \lambda) = \alpha_{U,v}(\lambda)$. As $||\cdot||_W$ is G-invariant we have $\Lambda_{U,g\cdot v} = g * \Lambda_{U,v}$. Now we can show that $g * P_{U,v} = g * P_\lambda = P_{g*\lambda}$ for $\lambda \in \Lambda_{U,v}$. But since $g * \Lambda_{U,v} = \Lambda_{U,g\cdot v}$, we have that $g * \lambda \in \Lambda_{U,g\cdot v}$. So $P_{g*\lambda} = P_{U,g\cdot v}$. ☐

If we consider the induced action of G on the G-variety V^n, for some n, then the obvious action of S_n on V^n commutes with the G-action. As noted in [35], this can be used to show that $\Lambda_{U,\omega\cdot\mathbf{v}} = \Lambda_{U,\mathbf{v}}$, and $P_{U,\omega\cdot\mathbf{v}} = P_{U,\mathbf{v}}$, for $\omega \in S_n, \mathbf{v} \in V^n$.

Theorem 6.28. *Let H be a closed subgroup of the reductive group G, and let N be a normal subgroup of H. Suppose H is G-completely reducible, then N is also G-completely reducible.*

Proof. By Theorem 6.18 all we need to show is that if H is strongly reductive in G, then so is N. Therefore, suppose by way of contradiction that H is strongly reductive in G, and that N is not. By Lemma 6.24, we may assume without loss that H^0 and N are topologically finitely generated by $h_1, \dots, h_m$ and $n_1, \dots, n_r$ respectively.

By [36, Proposition 3.2] there exists a finite subgroup $F = \{f_1, \dots, f_s\}$ of H such that $H = H^0F$. Define the tuple $\mathbf{n} \in N^{rs}$ by

$$\mathbf{n} = (n_1, \dots, n_r, f_2n_1f_2^{-1}, \dots, f_2n_rf_2^{-1}, \dots, f_sn_1f_s^{-1}, \dots, f_sn_rf_s^{-1})$$

and the tuple $\mathbf{h} \in H^{rs+m}$ by

$$\mathbf{h} = (n_1, \dots, n_r, f_2n_1f_2^{-1}, \dots, f_2n_rf_2^{-1}, \dots, f_sn_1f_s^{-1}, \dots, f_sn_rf_s^{-1}, h_1, \dots, h_m).$$

Since N is not strongly reductive in G, by Theorem 6.19, the orbit $G \cdot \mathbf{n}$ is not closed in G^{rs}. By [39, No.8] the closure of this orbit contains a unique closed orbit, O_1 say. Set $O := \cup_{\omega \in S_{rs}} \omega \cdot O_1$, where S_{rs} acts on G^{rs} in the obvious way. Then O is a union of finitely many closed G-orbits, hence is closed. Furthermore, since each $\omega \cdot O_1$ has dimension less than $\dim(G \cdot \mathbf{n})$, none of them contain $\mathbf{n}$, hence O does not contain $\mathbf{n}$. We have that G^{rs}, O and $\mathbf{n}$ satisfy the criteria of Theorem 6.25. Therefore, there is an optimal indivisible one-parameter subgroup $\lambda \in Y(G)$ such that $P_\lambda = P_{O,\mathbf{n}}$.

Since $\lambda \in |G^{sr}, \mathbf{n}|$, the limit $\lim_{x \to 0} \lambda(x) \cdot \mathbf{n}$ exists. This means that $\lim_{x \to 0} \lambda(x) \cdot n_i$ exists for all $i \in \{1, \dots, r\}$, therefore each $n_i \in P_{O,\mathbf{n}}$. As the elements $n_1, \dots, n_r$ topologically generate N, we have that $N \subseteq P_{O,\mathbf{n}}$.

By [36, Lemma 6.8], we have $H^0 = N_H(N)^0 = (NC_H(N))^0$. If $c \in C_H(N)$, then $P_{O,c \cdot \mathbf{n}} = P_{O,\mathbf{n}}$, and by Lemma 6.27, $c \cdot P_{O,\mathbf{n}} = P_{O,c \cdot \mathbf{n}}$. Hence,

$c \cdot P_{O,\mathbf{n}} = P_{O,\mathbf{n}}$. Since $P_{O,\mathbf{n}}$ is its own normaliser, $(NC_H(N))^0 = H^0 \subseteq P_{O,\mathbf{n}}$. For each $f \in F$, we have $f \cdot P_{O,\mathbf{n}} = P_{O,f\cdot\mathbf{n}}$. Since each f acts in the same way as some $\omega \in S_{rs}$, we have $f \cdot P_{O,\mathbf{n}} = P_{O,\omega\cdot\mathbf{n}} = P_{O,\mathbf{n}}$, and so $F \subset P_{O,\mathbf{n}}$. Therefore $H = H^0 F \subseteq P_{O,\mathbf{n}}$.

This means that $\lim_{x\to 0} \lambda(x) \cdot \mathbf{h}$ exists in G^{rs+m}. Since $\lim_{x\to 0} \lambda(x) \cdot \mathbf{n}$ does not belong to $G \cdot \mathbf{n}$, and the orbit $G \cdot \mathbf{n}$ is the projection of the first rs entries of the orbit $G \cdot \mathbf{h}$, we have that the limit $\lim_{x\to 0} \lambda(x) \cdot \mathbf{h}$ does not belong to $G \cdot \mathbf{h}$. Therefore, the orbit $G \cdot \mathbf{h}$ is not closed, and by Theorem 6.19, this implies that H is not strongly reductive in G, which is a contradiction, as required. □

Remark 6.29. When $G = \mathrm{GL}(V)$, Theorem 6.28 is just a special case of Clifford's Theorem, see [14, Theorem 1.11(i)]. For N normal in G, Clifford's Theorem asserts that if V is a semisimple kG-module, then V is a semisimple kN-module, and Theorem 6.28 follows from this since semisimplicity of the module V and complete reducibility are equivalent for subgroups of $\mathrm{GL}(V)$, as noted in Lemma 6.11. An account of this observation is given in [35, §3].

7 Frobenius Morphisms

7.1 Basic Facts About Rationality

Let $k \subseteq K$ be fields of positive characteristic with K algebraically closed. In this section we introduce the notion of an algebraic group G over K being defined over the subfield k. We discuss this first in a general setting, then focus on the fields $\mathbb{F}_q \subset \overline{\mathbb{F}}_q$, where $\mathbb{F}_q$ is the finite field of characteristic p with $q = p^a$ elements for some prime p and positive integer a. For further details regarding the following refer to [6, AG.11].

Let V be a vector space over K (not necessarily finite dimensional). A k**-structure** on V is a k-module $V_k \subseteq V$ such that the homomorphism:

$$K \otimes_k V_k \longrightarrow V$$

given by $(x, v) \mapsto xv$, for all $x \in K, v \in V_k$, is an isomorphism of vector spaces. The elements of V_k are said to be **rational over k**. For a subspace U of V, we define the set $U_k := U \cap V_k$, and say that U is **rational over** k if U_k is a k-structure on U.

A K-linear map $f : V \to W$ of K-vector spaces V, W with k-structures V_k, W_k on V and W, respectively, is called a k**-morphism** if $f(V_k) \subseteq W_k$.

For a K-algebra A, we define a k**-structure** on A to be k-structure A_k that is a k-subalgebra of A. An ideal of A is k**-ideal** if it is generated by its restriction to A_k.

It is shown in [6, AG. 11.3] how to define a k**-structure** on the K-ringed space $(V, \mathcal{O}_V)$. It consists of a topology on V in which the open sets are defined over k, and are also open in V in the standard sense, such that when $\mathcal{O}_V$ is restricted to this topology it is a sheaf of K-algebras with k-structures.

A morphism $\phi : X \to Y$ of k-ringed spaces X and Y is a k**-morphism** (or is said to be **defined over** k) if ϕ is continuous when restricted to the k-topologies, and if $V \subset Y$ and $\phi(U) \subseteq V$ are k-open subspaces, then $\phi^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a k-morphism.

If an affine variety over K has a k-structure, we call it a **k-variety**. Let X be an affine k-variety and let A be its coordinate ring with a k-structure A_k. Then the **k-rational points** $X(k)$ of X consists of the points corresponding to the maximal k-ideals of A. We say that X is **defined over** k. This notion extends the correspondence between the points of an affine variety and the maximal ideals of its affine algebra. For further details see [6, §AG.13], for instance.

Example 7.1. Let $V = \mathbb{A}^1$. Its coordinate algebra satisfies

$$\overline{\mathbb{F}}_q[T] \cong \mathbb{F}_q[T] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q.$$

We have that V is defined over $\mathbb{F}_q$, and $V(\mathbb{F}_q)$ is the set of points corresponding to the maximal ideals generated by the polynomials $T - a$, for $a \in \mathbb{F}_q$. That is, $V(\mathbb{F}_q) = \mathbb{F}_q$.

An algebraic group G is called a **k-group**, or is said to be **defined over** k, if its coordinate algebra has a k-structure and the product and inverse maps on G are defined over k. As noted in [23, §34.2], if G is a **k-group**, then the k-rational points $G(k)$ is a subgroup of G.

Example 7.2. Let $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$. Then its coordinate algebra satisfies,

$$\overline{\mathbb{F}}_q[T_{i,j}, \det(T_{i,j})^{-1}] \cong \mathbb{F}_q[T_{i,j}, \det(T_{i,j})^{-1}] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q.$$

We have that G is defined over $\mathbb{F}_q$, and its group of rational points $G(\mathbb{F}_q)$ is equal to $\mathrm{GL}_n(\mathbb{F}_q)$. This is because, the maximal k-ideals are generated by the polynomials $T_{i,j} - a_{i,j}$ where $a_{i,j} \in \mathbb{F}_q$ and $\det(a_{i,j}) \neq 0$.

7.2 Frobenius Morphisms

From now on set $k = \mathbb{F}_q$, and $K = \overline{\mathbb{F}}_q$. We restrict our attention to these fields because it enables us to examine the so-called Frobenius morphisms.

This is the name given to a class of morphisms that are prominent in the field of algebraic groups. They are used to construct the finite groups of Lie type in the classification of finite simple groups. For a more extensive account of the following, see [30, §1.2].

Definition 7.3. *We define $Aut(G)$ to be the group of all automorphisms of G as an abstract group, and $Aut_{alg}(G)$ to be the automorphism group of G when G is viewed as an algebraic group.*

For the abstract automorphism $\phi : G \to G$ to belong to $\mathrm{Aut}_{\mathrm{alg}}(G)$, both ϕ and its inverse need to be morphisms of the underlying variety of G.

Example 7.4. An example of a non-algebraic morphism is the inverse of the map $\sigma_q : \mathrm{GL}_n(\overline{\mathbb{F}}_q) \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ given by $(x_{ij}) \mapsto (x_{ij}^q)$. It is easily checked that σ_q is a homomorphism given by polynomial conditions, however its inverse involves taking the q-th root, which is not an operation defined by polynomial conditions.

The comorphism of σ_q is the map $\sigma_q^* : k[\mathrm{GL}_n(\overline{\mathbb{F}}_q)] \to k[\mathrm{GL}_n(\overline{\mathbb{F}}_q)]$ given by $\sigma_q^*(f) = f \circ \sigma_q$, for $f \in k[\mathrm{GL}_n(\overline{\mathbb{F}}_q)]$. Therefore, σ_q^* is not invertible since σ_q is not invertible.

If we let G be a simple algebraic group over the algebraically closed field $\overline{\mathbb{F}}_q$ of characteristic p, then we have the following description of $\mathrm{Aut}_{\mathrm{alg}}(G)$. By [51, Theorem 30], $\mathrm{Aut}_{\mathrm{alg}}(G)$ is generated by inner automorphisms and graph automorphisms of type A_n, D_n, D_4, or E_6 (which have order $2, 2, 3$, or 2 respectively), so called as they arise from symmetries of the Dynkin diagrams of these types. The group $\mathrm{Aut}(G)$ is generated by the elements of $\mathrm{Aut}_{\mathrm{alg}}(G)$, together with non-trivial field automorphisms which are of the form $u_\alpha(x) \mapsto u_\alpha(x^q)$ for q a p-power, where u_α is as defined in §6.1, as well as automorphisms τ' of order 2 of type B_2 $(p = 2), F_4$ $(p = 2)$, or G_2 $(p = 3)$ arising from symmetries in the Dynkin diagrams of these types, and corresponding to the formula:

$$u_\alpha(x) \mapsto u_{\rho'(\alpha)}(\epsilon_\alpha x^{p(\alpha)}). \tag{3}$$

In the above $p(\alpha) = 1$ if α is a long root, $p(\alpha) = p$ if α is a short root, $\epsilon_\alpha = \pm 1$ is defined in [51, p156], and where ρ' is a permutation of the root system of G which interchanges long and short roots giving rise to an order 2 symmetry of the Dynkin diagram.

The following theorem is [50, 10.13].

Theorem 7.5. *Let G be a simple algebraic group, and let $\sigma \in Aut(G)$ be an automorphism of G as an abstract group. Then one of the following holds:*

(i) σ is in $Aut_{alg}(G)$, or

(ii) G^σ (the group of fixed points) is finite.

Definition 7.6. *In the setting of Theorem 7.5, if we are in the latter case then we call σ a **Frobenius morphism** of G. We say that a subgroup H of G is σ-**stable** if $\sigma(H) = H$.*

By [50, §11], any Frobenius morphism σ of G is G-conjugate to either σ_q or $\tau\sigma_q$ where σ_q is a non-trivial field automorphism of the form $\sigma_q : u_\alpha(x) \mapsto u_\alpha(x^q)$ for q a p-power (i.e. $q \neq 1$) and τ is a graph automorphism of type A_n, D_n, D_4, E_6 (of order $2, 2, 3$ and 2 respectively) or B_2 $(p = 2), F_4$ $(p = 2), G_2$ $(p = 3)$. In addition, in types B_2 $(p = 2), F_4$ $(p = 2), G_2$ $(p = 3)$ there are additional Frobenius morphisms τ' as described above.

Now let G be an arbitrary linear algebraic group over $\overline{\mathbb{F}}_q$. We define a Frobenius morphism of G as follows. Consider the map $\sigma_q : \mathrm{GL}_n(\overline{\mathbb{F}}_q) \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$, given by

$$\sigma_q : (x_{ij}) \mapsto (x_{ij}^q).$$

A homomorphism $\sigma : G \to G$ is called a **standard Frobenius morphism** if there exists an injective homomorphism $\iota : G \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ for some n and some $q = p^a$ such that

$$\iota(\sigma(g)) = \sigma_q(\iota(g)) \text{ for all } g \in G.$$

By [15, Proposition 3.3 (ii),(iii)], we see that G is defined over $\mathbb{F}_q$ if and only if G is σ_q-stable. Recall Example 7.2 from Section 7.1. This example shows that $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ is defined over $\mathbb{F}_q$ and, from the above, this means that $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ is σ_q-stable. The same is true for other classical algebraic groups.

A homomorphism $\sigma : G \to G$ is called a **Frobenius morphism** if some power of σ is a standard Frobenius morphism.

Definition 7.7. *Let G and H be algebraic groups, and let $f : G \to H$ be a group homomorphism and a morphism of algebraic groups. Then f is called an* ***isogeny*** *if it has finite kernel.*

The following proposition is a compilation of results that can be found in [10, §1.17].

Proposition 7.8. *Let σ be a Frobenius morphism of G.*

(1) *If H is a σ-stable closed subgroup of G, then the restriction of σ to H is a Frobenius morphism of H.*

(2) *If H is a σ-stable closed normal subgroup of G, then σ induces a homomorphism from G/H to itself which is a Frobenius morphism of G/H.*

(3) *σ is bijective.*

(4) *G^σ is finite.*

(5) *If G is semisimple and $\phi : G \to G$ is any surjective homomorphism for which G^ϕ is finite, then ϕ is a Frobenius morphism of G.*

(6) *As σ is a bijection, it is an isogeny.*

Remark 7.9. We have introduced two notions of Frobenius morphism.

(1) When G is simple, any surjective morphism $\sigma : G \to G$ which has a finite fixed-point group is called a Frobenius morphism.

(2) When G an arbitrary algebraic group any homomorphism $\sigma : G \to G$, such that there exists an embedding ι of G in some $\mathrm{GL}(V)$, and some power n of σ for which $\iota(\sigma^n(g)) = \sigma_q(\iota(g))$ for all $g \in G$, is called a Frobenius morphism.

When G is simple, these two definitions coincide, and this can be seen as follows. If σ is a Frobenius morphism of the simple algebraic group G in the second sense, by Proposition 7.8 (4), the fixed point set $G^\sigma := \{g \in G \mid \sigma(g) = g\}$ is a finite group.

Conversely, if G is a simple algebraic group and σ is a Frobenius morphism of G in the first sense, that is a surjective morphism $\sigma : G \to G$ for which G^σ is finite, then by Proposition 7.8 (5), σ is a Frobenius morphism in the second sense.

As noted in Proposition 7.8 (6), if $\sigma : G \to G$ is a Frobenius morphism, then it is an isogeny. By Proposition 7.8 (4) G^σ is finite. For example, set $G = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ and σ is the Standard Frobenius morphism, raising each element in the matrices $(x_{ij}) \in \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ to the q-th power (x_{ij}^q). Then $G^\sigma = \mathrm{SL}_n(\mathbb{F}_q)$ – its group of $\mathbb{F}_q$-points is finite. We give some more examples of groups of the form G^σ in §7.3.

Remark 7.10. In the case G is simple, [50, §11] provides a way of decomposing automorphisms of G into *elementary* automorphisms, and shows that a Frobenius morphism is the product of certain inner, graph and field automorphisms of G. However, in general we can have Frobenius morphisms arising in more exotic ways. For example, suppose that $\mathrm{char}(k) = 2$, and let B and C be simple groups of types B_n and C_n, respectively.

Chevalley describes in [12, 24-05] how to construct isogenies $\mathrm{ch} : B \to C$ and $\mathrm{ch}^* : C \to B$, which are called special isogenies, such that the composition $\mathrm{ch}^* \mathrm{ch} : B \to B$ is the 2-power map σ_2 (see [13, Lemma 7.3.2] for proof). This composition is clearly a Frobenius morphism on B_n, but factors through the special isogenies ch and ch^*, neither of which are Frobenius morphisms, in particular they are not automorphisms.

The morphism $(\mathrm{ch}, \mathrm{ch}^*) : B \times C \to B \times C$ given by $(\mathrm{ch}, \mathrm{ch}^*) : (b, c) \mapsto (\mathrm{ch}^*(c), \mathrm{ch}(b))$ is a Frobenius morphism of $B \times C$ because its square is the standard Frobenius morphism. However, $(\mathrm{ch}, \mathrm{ch}^*)$ does not decompose into the elementary automorphism types as listed in [50, §11].

The following is part of the Lang–Steinberg Theorem, see for example [48, Theorem 4.4.17]. It is an important tool in much of what follows.

Theorem 7.11 (Lang–Steinberg)**.** *Let G be a connected algebraic group, and $\sigma : G \to G$ a surjective endomorphism of G. Then the map $g \mapsto \sigma(g)g^{-1}$ from G to G is surjective.*

We now state an important consequence of Theorem 7.11. This result can be found in, for example [49, I, 2.7].

Corollary 7.12. *Let G be a connected algebraic group acting transitively on a set Δ, and let σ be a Frobenius morphism of G which acts on Δ such that $\sigma(gx) = \sigma(g)\sigma(x)$ for all $g \in G, x \in \Delta$. Then Δ contains an element fixed by σ.*

Proof. Let $x \in \Delta$. By the transitivity of the action of G on Δ, there exists some $g \in G$ with $\sigma(x) = gx$. Now by Theorem 7.11 we can write $g = \sigma(h)h^{-1}$ for some h in G. Therefore, $\sigma(h^{-1}x) = \sigma(h^{-1})\sigma(x) = \sigma(h^{-1})gx = \sigma(h^{-1})\sigma(h)h^{-1}x = h^{-1}x$, so $h^{-1}x \in \Delta$ has the desired property. $\square$

Corollary 7.13. *A connected algebraic group, with a Frobenius morphism σ, contains a σ-stable Borel subgroup.*

Proof. Let G be a connected algebraic group, and let σ be a Frobenius morphism of G. Set Δ to be the set of Borel subgroups in G. Since the Borel subgroups of G are conjugate, by Proposition 6.2 Δ forms one G-conjugacy class, and so G acts transitively on Δ. As the homomorphic image of a Borel subgroup is a Borel subgroup, Δ is σ-stable. Thus, we are in the setting of Corollary 7.12. Hence, Δ contains an element fixed by σ. $\square$

Corollary 7.14. *An algebraic group, with a Frobenius morphism σ, contains a σ-stable maximal torus.*

Proof. Let G be an algebraic group, and let σ be a Frobenius morphism of G. A maximal torus of G is also a maximal torus of G^0 due to the fact that tori are connected. Set Δ to be the set of maximal tori in G^0. By Proposition 6.2 the maximal tori in G^0 form one G-conjugacy class. Thus, G^0 acts transitively on Δ. As G^0 is a characteristic subgroup of G, it is σ-stable. As the homomorphic image of a maximal torus is a maximal torus, Δ is σ-stable. Thus, we are in the setting of Corollary 7.12. Hence, Δ contains an element fixed by σ. $\square$

Corollary 7.15. *Let G be a reductive algebraic group, with a Frobenius morphism σ. Then, each σ-stable parabolic subgroup of G contains a σ-stable Levi subgroup.*

Proof. Let P be a parabolic subgroup of G, and set Δ to be the set of Levi subgroups in P. The set of Levi subgroups of P forms one P-orbit. Thus, P acts transitively on Δ. As P is σ-stable, the homomorphic image of a Levi subgroup of P is a Levi subgroup of P, hence Δ is σ-stable. As P is connected, we are in the setting of Corollary 7.12. Hence, Δ contains an element fixed by σ. □

The converse of Corollary 7.15 does not hold in general, as we shall see in Example 10.7.

We conclude this section with a general lemma about Frobenius morphisms.

Lemma 7.16. *If H is a σ-stable subgroup of G, then so is $\overline{H}$.*

Proof. Since H is σ-stable, $H \subseteq \sigma^{-1}(\overline{H})$. Furthermore, as $\sigma : G \to G$ is a morphism, $\sigma^{-1}(\overline{H})$ is closed and thus, $\overline{H} \subseteq \sigma^{-1}(\overline{H})$. Applying σ gives $\sigma(\overline{H}) \subseteq \overline{H}$. Since σ is bijective, we must have equality. □

7.3 The Finite Groups of Lie Type

Let G be a simple algebraic group over $\overline{\mathbb{F}}_q$ and let σ be a Frobenius morphism of G. Consider the finite fixed point group $G^\sigma := \{g \in G \mid \sigma(g) = g\}$.

These finite groups G^σ are called the **finite groups of Lie type** and are classified in [30, Corollary 1.5]. In particular, for each type of simple algebraic group there exists a family of finite groups of Lie type, depending on the choice of field and Frobenius morphism. Further details on the following can be found in, for example, [10, §1.19].

Let B be a σ-stable Borel subgroup of G, which exists by Corollary 7.13. Let T be a σ-stable maximal torus of G, which exists by Corollary 7.14. Then, $R_u(B)$ is also σ-stable, and is generated by the root subgroups U_α for $\alpha \in \Phi^+(G,T)$, the positive roots in G relative to T. Therefore, σ determines a permutation ρ of these positive root subgroups such that $\sigma(U_\alpha) = U_{\rho(\alpha)}$. By extension, σ determines a permutation ρ of the root system $\Phi = \Phi(G,T)$ of G.

The groups G^σ for which ρ acts trivially on the Dynkin diagram of G are called **Chevalley groups**. The groups G^σ for which the Dynkin diagram has only single bonds and ρ acts non-trivially are called **twisted groups**.

In types B_2 $(p = 2), F_4$ $(p = 2)$ and G_2 $(p = 3)$ we have Frobenius morphisms which arise as graph automorphisms of the corresponding Dynkin diagram, each of which has a symmetry of order 2, giving a permutation of the root system of G. In each case the graph automorphism corresponds to the permutation ρ of the root system given by

$$u_\alpha(x) \mapsto u_{\rho(\alpha)}(\epsilon_\alpha x^{p(\alpha)}), \tag{4}$$

where $p(\alpha) = 1$ if α is a long root, $p(\alpha) = p$ if α is a short root, and $\epsilon_\alpha = \pm 1$, see [51, p156] for more details. Note that it is sufficient to describe the action of a graph automorphism of G by its action on the root subgroups U_α since G is generated by the U_α and a maximal torus T of G, where T can be chosen to be σ-stable, by Corollary 7.14.

The fixed points of such Frobenius morphisms give rise to the **Suzuki and Ree groups**, which we will briefly describe here. Let σ be such a graph automorphism in type B_2 $(p = 2), F_4$ $(p = 2)$ or G_2 $(p = 3)$, then the Suzuki and Ree groups arise as G^σ where G is of one of these types. We follow the convention of [24] when denoting these groups, in that we define the Suzuki and Ree groups over a field of q^2 elements where q^2 is an odd power of 2 or 3. In particular, this means that q is not an integer, and we use this convention to indicate that the square of σ is a standard Frobenius morphism. As Humphreys notes, this convention is also convenient as it resembles the group order formulas given in [24, §20.1 Table 1].

In type B_2 a group G^σ only occurs when $p = 2$ and $q^2 = 2^{2n+1}$ for some $n \geq 0$. The groups arising in this way are **Suzuki groups** denoted ${}^2B_2(q^2)$.

In type F_4 a group G^σ only occurs when $p = 2$ and $q^2 = 2^{2n+1}$ for some $n \geq 0$. The groups arising in this way are **Ree groups of type** F_4 denoted

${}^2F_4(q^2)$.

In type G_2 a group G^σ only occurs when $p = 3$ and $q^2 = 3^{2n+1}$ for some $n \geq 0$. The groups arising in this way are **Ree groups of type** G_2 denoted ${}^2G_2(q^2)$.

For each of these types, there exists one isomorphism class of groups for each q^2. The smallest ones, ${}^2B_2(2), {}^2F_4(2)$ and ${}^2G_2(3)$ are not simple, however for all other q^2, these groups are simple.

We see that the group G^σ is frequently a simple finite group, but it is not always. If we take the quotient group $G^\sigma/Z(G^\sigma)$ for G simple and simply connected we obtain a finite simple group, and these groups are called the **finite simple groups of Lie type**.

Part II

Complete Reducibility and Frobenius Morphisms

8 Introduction to (G,σ)-Complete Reducibility

In this chapter we let G be a connected reductive algebraic group over the algebraically closed field $k = \overline{\mathbb{F}}_q$, where $\mathbb{F}_q$ is the finite field with q elements of characteristic p, where q is some positive power of the prime p, unless otherwise stated, and let $\sigma : G \to G$ be a Frobenius morphism of G.

8.1 (G,σ)-Complete Reducibility

In this section we define analogues of Serre's notions of G-complete reducibility and G-irreducibility, from [45], which we introduced in Definition 6.10. Recall that a subgroup H of G is called σ-stable if $\sigma(H) = H$.

Definition 8.1. *Let H be a σ-stable subgroup of G.*

(1) *We say H is (G,σ)-**completely reducible** (or (G,σ)-**cr**) if whenever H is contained in a σ-stable parabolic subgroup P of G, then H is contained in a σ-stable Levi subgroup of P.*

(2) *We say H is (G,σ)-**irreducible** (or (G,σ)-**ir**) if H is not contained in any proper σ-stable parabolic subgroup of G.*

Recall that, according to Serre [45], a subgroup H of G is called G-**completely reducible** if whenever H is contained in a proper parabolic subgroup P of G it is contained in a Levi subgroup of P. Also H is said to be G-**irreducible** if it is not contained in any proper parabolic subgroup of G, see [47, Part II, Lecture 1].

Clearly, a σ-stable G-irreducible subgroup is trivially (G,σ)-irreducible, and a (G,σ)-irreducible subgroup is trivially (G,σ)-completely reducible. However, a (G,σ)-irreducible subgroup need not be G-irreducible, as the following example shows.

Example 8.2. Consider the case when G is the group GL_{m+n} with the Frobenius morphism σ where $\sigma : g \mapsto (\sigma_q(g^{-1}))^T$, for all $g \in G$ and where T denotes the transpose map.

The parabolic subgroup $P := \begin{pmatrix} \mathrm{GL}_n & * \\ 0 & \mathrm{GL}_m \end{pmatrix}$ of G is sent by the Frobenius morphism σ to the opposite parabolic subgroup $P^- = \begin{pmatrix} \mathrm{GL}_n & 0 \\ * & \mathrm{GL}_m \end{pmatrix}$ of G, and we have that these are not conjugate if $m \neq n$. In this case the Levi subgroup $L := \begin{pmatrix} \mathrm{GL}_n & 0 \\ 0 & \mathrm{GL}_m \end{pmatrix}$ is a σ-stable Levi subgroup of G. Furthermore, L is a Levi subgroup of a maximal parabolic subgroup of G. Thus any of the parabolic subgroups of G containing L actually contain L as a Levi subgroup and are therefore maximal themselves. But neither of these is σ-stable.

Our ultimate aim in this chapter is to investigate when the notions of G-complete reducibility and (G, σ)-complete reducibility are equivalent for a σ-stable subgroup of G.

In Theorem 8.6 we show that a σ-stable G-completely reducible subgroup H of G is (G, σ)-completely reducible. This is a generalisation of [33, Theorem 9] in that we remove restrictions that were placed on H and G, namely that $H \subseteq G^\sigma$ and G is of exceptional type.

We proceed in §9 to investigate the converse of Theorem 8.6 for finite σ-stable subgroups of G. We first state Proposition 9.1 and Proposition 9.3, which are due to Liebeck, Martin and Shalev, see [31], which show that if F is a finite σ-stable subgroup of G that is not strongly reductive in G, then F is contained in a proper σ-stable parabolic subgroup of G. Furthermore, if G^σ is not a Ree or Suzuki group, then F is not contained in any Levi subgroup of P. Therefore, this is a partial converse to Theorem 8.6. In Lemma 9.9 we partially extend these results to the case G is reductive where we show that a finite σ-stable subgroup F of G that is not strongly reductive in G is contained in a proper σ-stable parabolic subgroup P of G.

In §9.2 we extend Proposition 9.1 to include the Ree and Suzuki cases. We examine the symmetries of the Dynkin diagrams that occur in these types to identify the conjugacy classes of σ-stable parabolic subgroups that exist in these cases. For G whose fixed point group under the action of σ is a large Ree group, we require Lemma 9.9 because we restrict our attention to Levi subgroups of G, and use the fact that Levi subgroups are reductive. The main result of §9.2 is Theorem 9.12, which shows that when G is simple σ-stable group, a finite σ-stable subgroup F of G is G-completely reducible if, and only if, it is (G, σ)-completely reducible. Furthemore, if F is not G-completely reducible then we use Lemma 8.15 to show the stronger result in one direction that F is contained in a σ-stable parabolic subgroup P of G and in no Levi subgroup of P.

In §9.3 we extend these results to the reductive case in Proposition 9.15, whose proof follows a similar methodoloy to Lemma 9.9, however now that our results about Ree and Suzuki groups are established we can restrict to any simple factor of G and employ Theorem 9.12. This gives that when G is a reductive σ-stable group, a finite σ-stable subgroup of G is G-completely reducible, if and only if, F is (G, σ)-completely reducible.

In §10 we develop the notion of a finite σ-structure which allows us to pass from an infinite σ-stable subgroup of G to a finite σ-stable subgroup of G that shares the infinite group's (G, σ)-complete reducibility properties. This enables us to extend Proposition 9.15 to the case F is infinite. We summarise our main results in our study of (G, σ)-complete reducibility in Theorem 10.6 which gives an equivalence between the notions of G-complete reducibility and (G, σ)-complete reducibility for σ-stable subgroups of G.

We also have the stronger result in one direction: if H is a σ-stable subgroup of G that is not G-completely reducible, then H is contained in a proper σ-stable parabolic subgroup P of G, and not in any Levi subgroup of P.

We begin by establishing the following basic facts about σ-stable subgroups of G.

Lemma 8.3. *If H is a σ-stable subgroup of G, then so are $C_G(H)$ and $N_G(H)$.*

Proof. Let $h \in H$, and let $\cdot$ denote the action of conjugation of elements of G on G, given by $g \cdot h = ghg^{-1}$. Since the map $\sigma : H \to H$ is surjective, there exists some $h' \in H$ such that $h = \sigma(h')$. Let $c \in C_G(H)$. Then $\sigma(c) \cdot h = \sigma(c) \cdot \sigma(h') = \sigma(c \cdot h') = \sigma(h') = h$ and so $\sigma(c) \in C_G(H)$. This shows that $\sigma(C_G(H)) \subseteq C_G(H)$. Since σ is bijective we have $\sigma(C_G(H)) = C_G(H)$.

Now let $n \in N_G(H)$. Then $\sigma(n) \cdot H = \sigma(n \cdot H) = \sigma(H) = H$ and so $\sigma(n) \in N_G(H)$ and hence $\sigma(N_G(H)) \subseteq N_G(H)$. Again, since σ is bijective $\sigma(N_G(H)) = N_G(H)$. □

The next proposition states that (G, σ)-completely reducible subgroups of G are reductive. This is an analogue of [47, Property 4] by Serre, showing that a closed G-completely reducible subgroup of G is reductive. The proof uses a construction given in [23, 30.3] and shows that we need H to be σ-stable in Definition 8.1, for if not we would not be able to construct a σ-stable parabolic subgroup of G containing H.

Proposition 8.4. *If H is (G, σ)-completely reducible, then H is reductive.*

Proof. Suppose that H is (G, σ)-cr and $U := R_u(H) \neq e$. Since H is σ-stable, so is U, being a characteristic subgroup of H. By Lemma 8.3, $N_1 := N_G(U)$ is also σ-stable and so is $U_1 := R_u(N_1)$. Inductively define the σ-stable subgroups $N_i := N_G(U_{i-1})$ and $U_i := R_u(N_i)$ of G.

Since U is a connected normal unipotent subgroup of N_1 we have $U \subseteq R_u(N_1)$, and similarly $U_i \subseteq R_u(N_{i+1})$. Hence, $U_{i+1} \supseteq U_i \supset \cdots \supset U$, and clearly $\dim U_{i+1} > \dim U_i$ unless $U_{i+1} = U_i$. In particular the two sequences (N_i) and (U_i) must stabilise, say $U_{j-1} = U_j = \cdots, N_j = N_{j+1} = \cdots$. Set

$P := N_j, V := U_j$. We have that $N_G(U)$ normalises P. Since U is connected and solvable it must lie in some Borel subgroup of G and so [23, Corollary 30.3A] gives that P is a σ-stable parabolic subgroup of G with $N_G(U) \subseteq P$ and $U \subseteq R_u(P)$ and all these groups are σ-stable.

Since H is (G, σ)-cr and P is σ-stable, H is contained in a σ-stable Levi subgroup, L say, of P. So $U \subseteq H \cap R_u(P) \subseteq L \cap R_u(P) = e$ which is a contradiction. $\square$

Remark 8.5. Recall that Theorem 6.8 shows that for a non-trivial unipotent subgroup U of G, we have a proper parabolic subgroup P of G for which $U \subseteq R_u(P)$ and $N_G(U) \subseteq P$. This result is due to A. Borel and J. Tits, see [7]. By [7, Theorem 2.5], if U is σ-stable, then this construction leads to a σ-stable parabolic subgroup of G that satisfies the same conditions as P does. Note that we can draw the same conclusion by following the argument given in the proof of Proposition 8.4. This method for constructing such a parabolic subgroup of G is used in several places throughout this thesis, and we refer to this construction as the *construction of Borel-Tits*.

For the proof of the following theorem we adapt the argument used at the end of the proof of [33, Theorem 9] to the context of σ-stability.

Theorem 8.6. *A σ-stable G-completely reducible subgroup of G is (G, σ)-completely reducible.*

Proof. Let H be a closed σ-stable G-cr subgroup of G, and suppose that $H \subseteq P$, for some proper σ-stable parabolic subgroup P of G. Since H is G-cr, $H \subseteq L$ for a Levi subgroup L of P, and $P = R_u(P)L$.

If $H \subseteq L^u$ for some $u \in R_u(P)$, then $H^{u^{-1}} \subseteq L \cap (R_u(P)H) = H$, so $u \in N_{R_u(P)}(H)$. Define the non-empty set

$$\Delta := \{L^u \mid u \in R_u(P),\ H \subseteq L^u\}.$$

Now let u be any element of $N_{R_u(P)}(H)$ and $h \in H$. Then $huh^{-1}u^{-1} \in R_u(P) \cap H = e$, since $H \subseteq L$. Therefore, $N_{R_u(P)}(H) = C_{R_u(P)}(H)$. Thus $C := C_{R_u(P)}(H)$ acts transitively on Δ by conjugation.

Next we show that C is connected. Let $S = Z(L)^0$, then by Proposition 6.4 $C_G(S) = L$ and so $H \subseteq C_G(S)$. We claim that $S \subseteq N_G(C)$. This can be seen since $S \subseteq C_G(H)$ and S normalises $R_u(P)$, therefore we have that S normalises $C_G(H) \cap R_u(P) = C$. Thus S acts on C.

The torus S acts on C^0, because C^0 is characteristic in C, and hence S acts trivially on the finite group C/C^0, by [23, Proposition 8.2].

Since S centralizes C/C^0, we have $scs^{-1} \in cC^0$ for some $s \in S, c \in C$. Therefore,

$$c(C^0S)c^{-1} = C^0(cSc^{-1}) = C^0(C^0S) = C^0S.$$

Where the first equality holds because C^0 is normal in C, and the second equality holds because $sc^{-1}s^{-1} \in c^{-1}C^0$ and hence $csc^{-1} \in C^0s$, for all $s \in S$. Thus, C acts on C^0S. There is just one class of maximal tori in C^0S. We have that C^0S is normal in CS. Therefore, if $x \in CS$, then S^x is a maximal torus in C^0S, so $S^x = S^a$ for some $a \in C^0$. Hence, $xa^{-1} \in N_{CS}(S)$ and $x \in N_{CS}(S)C^0 = C^0N_{CS}(S)$.

As x was arbitrary, we now have $CS = C^0N_{CS}(S)$, and as $N_{CS}(S) = SN_C(S)$ this gives $CS = C^0SN_C(S)$. However, $[N_C(S), S] \subseteq C \cap S = e$, so that $N_C(S) = C_C(S)$.

Suppose that C/C^0 is non-trivial, then it follows that $C_C(S)$ is not trivial. Hence, this argument produces elements in C fixed by S.

However $C_C(S) \subseteq C_{R_u(P)}(S) = C_G(S) \cap R_u(P) = L \cap R_u(P) = e$, this is a contradiction. Therefore, $C = C^0$.

Next we claim that Δ is σ-stable. Let $H \subseteq L^u \in \Delta$. Note that $R_u(P)$ is σ-stable since $R_u(P)$ is characteristic in P. Then $H = \sigma(H) \subseteq \sigma(L^u) = \sigma(L)^{\sigma(u)}$. But $P = \sigma(P) = \sigma(R_u(P)L) = R_u(P)\sigma(L)$. Thus $\sigma(L) = L^v$ for some $v \in R_u(P)$. Hence $\sigma(L)^{\sigma(u)} = L^{v \cdot v'} \in \Delta$ for $v' = \sigma(u) \in R_u(P)$ and so Δ is σ-stable. It now follows from Corollary 7.12, that Δ contains an element fixed by σ. $\square$

Remark 8.7. Theorem 8.6 provides one direction of the desired equivalence between the notions of G-complete reducibility and (G, σ)-complete reducibility for a σ-stable subgroup of the reductive group G. The converse is given in Proposition 9.14 for finite σ-stable subgroups of G, and is extended in Proposition 10.5 to include infinite σ-stable subgroups of G.

Remark 8.8. For $H \subseteq G^\sigma$, with G simple and of exceptional type, Theorem 8.6 is obtained from the proof of the last part of [33, Theorem 9].

Example 8.9. A map $\phi : H \longrightarrow \mathrm{GL}(V)$, for an algebraic group H, is called a **rational representation of** H if ϕ is a homomorphism of algebraic groups, see for example [6, Examples 1.6].

A subgroup H of G is called **linearly reductive** if all of its rational representations are semisimple, see for example [42, §1.2]. By [1, Lemma 2.6], if H is linearly reductive, it is G-completely reducible. Therefore, a σ-stable linearly reductive subgroup of G is (G, σ)-completely reducible.

Example 8.10. Let S be any torus in G. Since S is linearly reductive, by [1, Lemma 2.6], S is G-completely reducible. Therefore, [1, Theorem 3.14] gives that $C_G(S)^0$ is G-completely reducible. By [6, Corollary 11.12], $C_G(S)^0 = C_G(S)$, and by Proposition 6.4 $C_G(S)$ is a Levi subgroup of G. Moreover, every Levi subgroup of G is of this form. Hence, Theorem 8.6 shows that any σ-stable Levi subgroup of G is (G, σ)-completely reducible.

Example 8.11. A subgroup H of G is called **regular** if it is normalised by a maximal torus of G. By [1, Proposition 3.20], if H is regular, it is G-completely reducible. Therefore, a σ-stable regular subgroup of G is (G, σ)-completely reducible.

Corollary 8.12. *If H is σ-stable and G-completely reducible, then H^0 is (G, σ)-completely reducible.*

Proof. As H is σ-stable then so is H^0 since it is characteristic in H. Since H is G-cr, by [1, Theorem 3.10] so is H^0. Therefore, the result follows from Theorem 8.6. $\square$

Next we establish a generalisation of [1, Proposition 3.40]. This argument was provided by Michael Bate, Tim Burness and Martin Liebeck.

Proposition 8.13. *Let H be a σ-stable G-irreducible subgroup of G such that H^0 is not G-irreducible. Then $C_G(H^0)$ contains a σ-stable non-central maximal torus.*

Proof. Since H is G-ir, it is G-cr and therefore, by [1, Theorem 3.10], H^0 is G-cr. Since H^0 is not G-ir, it is contained in a proper parabolic subgroup P of G, and hence a proper Levi subgroup L of P. By Proposition 6.4 $L = C_G(S)$ for some non-central torus S of G. In particular, since $H^0 \subseteq C_G(S)$ we have $S \subseteq C_G(H^0)$. In particular, every maximal torus of $C_G(H^0)$ is non-central (because they are all conjugate in $C_G(H^0)$).

Consider the set $\Delta = \{S^g \mid g \in C_G(H^0)^0\}$ to be the conjugacy class in $C_G(H^0)^0$ containing its maximal tori. We have that $C_G(H^0)^0$ acts transitively on Δ and Δ is σ-stable. Hence, by Corollary 7.12, $C_G(H^0)^0$ (and hence $C_G(H^0)$) contains a σ-stable maximal torus. $\square$

We conclude this section with the following lemma and its subsequent corollary, which were provided by Michael Bate, Tim Burness and Martin Liebeck.

Lemma 8.14. *Let G be a reductive algebraic group with a Frobenius morphism σ. Let H be a σ-stable subgroup of G that is contained in a proper σ-stable parabolic subgroup P of G. Then, H is contained in a Levi subgroup of P if, and only if, H is contained in a σ-stable Levi subgroup of P.*

Proof. Suppose that H is contained in a Levi subgroup L of P, and $P = R_u(P)L$. Define the non-empty set

$$\Delta := \{L^u \mid u \in R_u(P),\ H \subseteq L^u\}.$$

As in the proof of Theorem 8.6, whenever $H \subseteq L^u$ for some $u \in R_u(P)$ we can conclude that $u \in N_{R_u(P)}(H)$. The arguments of Theorem 8.6 show that $N_{R_u(P)}(H) = C_{R_u(P)}(H)$. Since any Levi subgroup of P is $R_u(P)$-conjugate to L, the group $C_{R_u(P)}(H)$ acts transitively on Δ. The same proof shows that this group is connected.

Let $H \subseteq L^u \in \Delta$. Note that $R_u(P)$ is σ-stable since $R_u(P)$ is characteristic in P. Then $H = \sigma(H) \subseteq \sigma(L^u) = \sigma(L)^{\sigma(u)}$. But $P = \sigma(P) = \sigma(R_u(P)L) = R_u(P)\sigma(L)$. Thus $\sigma(L) = L^v$ for some $v \in R_u(P)$. Hence $\sigma(L)^{\sigma(u)} = L^{v \cdot v'} \in \Delta$ for $v' = \sigma(u) \in R_u(P)$ and so Δ is σ-stable.

Therefore, by applying Corollary 7.12, we see that Δ contains an element fixed by σ. The converse is immediate, and this gives the lemma. □

Lemma 8.15. *Let G be a reductive algebraic group with a Frobenius morphism σ. Let H be a σ-stable subgroup of G. Then if H is not (G, σ)-completely reducible, it is contained in a σ-stable parabolic subgroup P of G, and in no Levi subgroup of P.*

Proof. Since H is not (G, σ)-completely reducible, it is contained in a proper σ-stable parabolic subgroup P of G, and not in any σ-stable Levi subgroup of P. Suppose, by way of contradiction, that H is contained in a Levi subgroup L of P that is not σ-stable. Then Lemma 8.14 implies that H is contained in a σ-stable Levi subgroup of P. However, this contradicts our hypothesis and therefore H is not contained in any Levi subgroup of P. □

9 Finite Subgroups

Let G be a reductive algebraic group over the algebraically closed field $k = \overline{\mathbb{F}}_q$, where $\mathbb{F}_q$ is the finite field with q elements of characteristic p, where q is some positive power of the prime p.

In this section we recall some geometric invariant theory and state an equivalence result between the notions of G-complete reducibility and (G, σ)-complete reducibility under certain conditions, which follows from work of Liebeck, Martin and Shalev, [31]. When σ is a Frobenius morphism of G and G is simple, we obtain an equivalence between the notions of G-complete reducibility and (G, σ)-complete reducibility in Theorem 9.12 for finite σ-stable subgroups of G. We generalise this equivalence further in Theorem 9.15 to the case when G is a reductive group.

9.1 A Result of Liebeck, Martin, Shalev

We introduce the same setup as that of §6.6. Much of the following uses the argument given in [31, Proposition 2.2 and Remark 2.4]. As a brief reminder, for an arbitrary affine G-variety V, let $v \in V$ and $\lambda \in Y(G)$, and let S be a closed G-stable subvariety of V which does not contain v. If the limit $\lim_{x\to 0} \lambda(x)v\lambda(x)^{-1}$ exists and is equal to u, we define the morphism $M_v(\lambda) : k \to V$ as in §6.2, that is $M_v(\lambda)(x) = \lambda(x)v\lambda(x)^{-1}$ for every $x \in k^*$, and $M_v(\lambda)(0) = u$. Then, because of the G-invariance of S, we have $M_v(\lambda)(k^*) \cap S = \emptyset$. Hence $M_v(\lambda)(k) \cap S \neq \emptyset$ if and only if $M_v(\lambda)(0) \in S$. Therefore, $M_v(\lambda)^{-1}(U) = \{0\}$ and we can conclude there is some integer m such that $M_v(\lambda)^{-1}(U)$ is determined by $\mathrm{Spec}(k[T]/(T^m))$, for an indeterminate T. Thus, m depends on S, v and λ, so we denote the degree m by $\alpha_{S,v}(\lambda)$, and note this is a non-negative integer in general, and positive if $M_v(\lambda)(0) \in S$.

We use this setup in the following way. Let G be a simple algebraic group, then we consider the affine G-variety G^n, where " $\cdot$ " denotes the action of G on G^n by simultaneous conjugation, that is:

$$g \cdot (x_1, \ldots, x_n) = (gx_1g^{-1}, \ldots, gx_ng^{-1}),$$

for $g \in G, (x_1, \ldots, x_n) \in G^n$.

Define a subset $\mathrm{Aut}^+(G)$ of $\mathrm{Aut}(G)$ to be the group of those abstract automorphisms of G generated by inner automorphisms, field automorphisms and graph automorphisms of G which are of type A_n, D_n or E_6, see §7.2. Note that these graph automorphisms are automorphisms of G as an algebraic group, however the remaining ones, those of type B_2 $(p = 2), F_4$ $(p = 2)$ or G_2 $(p = 3)$, are only automorphisms of G as an abstract group and their inverses are not morphisms, and they are excluded from $\mathrm{Aut}^+(G)$.

The group $\mathrm{Aut}^+(G)$ contains the Frobenius morphisms which are G-conjugate to either σ_q, a non-trivial field automorphism, or $\tau\sigma_q$, for τ a graph automorphism of G if G is of type A_n, D_n or E_6. The Frobenius morphisms which are not contained in $\mathrm{Aut}^+(G)$ are those Frobenius morphisms of G which are G-conjugate to either τ', or $\tau'\sigma_q$, where τ' is an automorphism of G of the form defined in equation (3) for G of type B_2 $(p = 2), F_4$ $(p = 2)$ or G_2 $(p = 3)$. For details see [30, Theorem 1.4]. Hence, $\mathrm{Aut}^+(G)$ contains every Frobenius morphism of G if G is of type $A_n(n \geq 1), B_n(n \geq 3), C_n(n \geq 3), D_n(n \geq 4), E_6, E_7$ or E_8, and only those that are G-conjugate to σ_q when G is of type B_2 $(p = 2), F_4$ $(p = 2)$ or G_2 $(p = 3)$.

We have a component-wise action of $\mathrm{Aut}^+(G)$ on G^n given by $\beta \cdot (x_1, \ldots, x_n) \mapsto (\beta(x_1), \ldots, \beta(x_n))$ for $\beta \in \mathrm{Aut}^+(G)$ and $(x_1, \ldots, x_n) \in G^n$. The action of $\mathrm{Aut}^+(G)$ on G^n permutes the G-orbits in G^n. The obvious action of the symmetric group S_n on G^n commutes with these two actions.

In [31, §2] it is shown how to construct a length function $|| - ||_1$ on $Y(G)$ which is invariant under $\mathrm{Aut}^+(G)$.

Let R be a subgroup of $\mathrm{Aut}^+(G)$, and let $F = \{f_1, \ldots, f_n\}$ be a finite R-invariant subgroup of G which is not strongly reductive in G. Set $\mathbf{f} :=$

$(f_1, \ldots, f_n) \in G^n$. Since F is not strongly reductive in G, Theorem 6.19 gives that the orbit $G \cdot \mathbf{f}$ is not closed in G^n. By [39, No. 8], the closure of $G \cdot \mathbf{f}$ contains a unique closed orbit, $D'(\mathbf{f})$ say. By [6, Proposition 1.8], $D'(\mathbf{f})$ has strictly lower dimension that $\overline{G \cdot \mathbf{f}}$. Set $D(\mathbf{f}) := \cup_{\pi \in S_n} \pi \cdot D'(\mathbf{f})$. We have $\mathbf{f} \notin D'(\mathbf{f})$, because $D'(\mathbf{f})$ is an orbit of lower dimension than $G \cdot \mathbf{f}$. Furthermore, $D(\mathbf{f})$ is closed, since it is a finite union of closed G-orbits, each of which does not contain $\mathbf{f}$ and so $\mathbf{f} \notin D(\mathbf{f})$.

Hence $G^n, D(\mathbf{f})$, and $\mathbf{f}$ satisfy the hypotheses of Theorem 6.25, thus there exists a one-parameter subgroup $\lambda \in \Lambda_{D(\mathbf{f}),\mathbf{f}}$ such that $\lim_{x \to 0} \lambda(x) \cdot \mathbf{f}$ exists and belongs to $D(\mathbf{f})$. To the one-parameter subgroup λ we have an associated optimal destabilising parabolic subgroup $P_{D(\mathbf{f}),\mathbf{f}}$ of G (see Notation 6.26) such that $P_{D(\mathbf{f}),\mathbf{f}} = P_\lambda$ where, as in Lemma 6.6, $P_\lambda = \{g \in G \mid \lim_{x \to 0} \lambda(x) \cdot g \text{ exists}\}$. Hence we can conclude that $F \subseteq P_{D(\mathbf{f}),\mathbf{f}}$.

First, we show $P_{\beta \cdot D(\mathbf{f}), \beta \cdot \mathbf{f}} = P_{D(\mathbf{f}),\mathbf{f}}$ for any $\beta \in R$. By construction $D(\mathbf{f})$ is S_n-invariant, hence an argument of Martin (see [35, p672]), gives that $P_{D(\mathbf{f}),\pi \cdot \mathbf{f}} = P_{D(\mathbf{f}),\mathbf{f}}$.

As F is finite, $\beta \cdot \mathbf{f} = \pi \cdot \mathbf{f}$ for some $\pi \in S_n$. Furthermore, since $\mathrm{Aut}^+(G)$ and S_n act by homeomorphisms on G^n, and these two actions commute, $\beta \cdot D(\mathbf{f}) = D(\beta \cdot \mathbf{f}) = D(\pi(\mathbf{f})) = D(\mathbf{f})$ for all $\beta \in R$. Hence,

$$P_{\beta \cdot D(\mathbf{f}), \beta \cdot \mathbf{f}} = P_{D(\mathbf{f}),\pi \cdot \mathbf{f}} = P_{D(\mathbf{f}),\mathbf{f}}. \tag{5}$$

This equality holds, in fact, even if we take R to be an arbitrary subset of $\mathrm{Aut}(G)$. This is because in the argument used, we only require β to stabilise F, act by homeomorphisms on G^n and to commute with the S_n action, which all the automorphisms of G that stabilise F satisfy. The following discussion, however, is only proved for subsets of $\mathrm{Aut}^+(G)$.

In [31, §2], for all $\beta \in \mathrm{Aut}^+(G)$, it is shown that

$$\beta \cdot P_{D(\mathbf{f}),\mathbf{f}} = P_{\beta \cdot D(\mathbf{f}), \beta \cdot \mathbf{f}}. \tag{6}$$

The way the argument works for any graph automorphism requires that the inverse morphism is applied. However, in types B_2 $(p = 2), F_4$ $(p = 2)$ or G_2 $(p = 3)$ a graph automorphism does not have an inverse that is a morphism, and so we cannot apply this argument in these cases.

For inner automorphisms equation (6) is given in [26, Corollary 3.5(a)], for field automorphisms in [26, Lemma 4.1] and for graph automorphisms β which are algebraic automorphisms of G we have the following argument, further details of which can be found in [31, p.547]. Recall that $|G^n, \mathbf{f}|_{D(\mathbf{f})}$ is the set of all optimal indivisible one-parameter subgroups λ of G whose limit $\lim_{x\to 0} \lambda(x) \cdot \mathbf{f}$ belongs to the set $D(\mathbf{f})$.

As β is an invertible morphism, the limit $\lim_{x\to 0} \lambda(x)\cdot\mathbf{f}$ exists if, and only if, the limit $\beta\cdot(\lim_{x\to 0} \lambda(x)\cdot\mathbf{f})$ exists. Furthermore, the limit $\beta\cdot(\lim_{x\to 0} \lambda(x)\cdot \mathbf{f})$ exists if, and only if, the limit $\lim_{x\to 0} \beta \cdot (\lambda(x) \cdot \mathbf{f})$ exists. Hence, we have:

$$\begin{aligned}
|G^n, \beta \cdot \mathbf{f}|_{\beta\cdot D(\mathbf{f})} &= \{\lambda \in Y(G) \mid \lim_{x\to 0} \lambda(x) \cdot (\beta \cdot \mathbf{f}) \text{ exists and lies in } \beta \cdot D(\mathbf{f})\} \\
&= \{\lambda \in Y(G) \mid \beta\beta^{-1} \cdot (\lim_{x\to 0} \lambda(x) \cdot (\beta \cdot \mathbf{f})) \text{ exists and lies in } \beta \cdot D(\mathbf{f})\} \\
&= \{\lambda \in Y(G) \mid \lim_{x\to 0} \beta \cdot (\beta^{-1} \circ \lambda(x) \cdot (\beta^{-1}\beta \cdot \mathbf{f})) \text{ exists and lies in } \beta \cdot D(\mathbf{f})\} \\
&= \{\beta \circ \lambda' \in Y(G) \mid \lim_{x\to 0} \beta \cdot (\lambda'(x) \cdot \mathbf{f}) \text{ exists and lies in } \beta \cdot D(\mathbf{f})\} \\
&= \{\beta \circ \lambda' \in Y(G) \mid \lim_{x\to 0} \lambda'(x) \cdot \mathbf{f} \text{ exists and lies in } D(\mathbf{f})\} \\
&= \beta \cdot |G^n, \mathbf{f}|_{D(\mathbf{f})}.
\end{aligned}$$

Because the inverse of β features, this argument cannot be applied to those graph automorphisms in $\mathrm{Aut}(G)$ which are not in $\mathrm{Aut}^+(G)$, since they are precisely the automorphisms whose inverse is not a morphism.

We also have that $\beta \circ M_v(\lambda) = M_v(\beta \circ \lambda)$ and that $\alpha_{\beta\cdot D(\mathbf{f}),\beta\cdot\mathbf{f}}(\beta \circ \lambda) = \alpha_{D(\mathbf{f}),\mathbf{f}}(\lambda)$. The function $|G^n, \mathbf{f}|_{D(\mathbf{f})} \to \mathbb{R}$ given in Theorem 6.25 by $\lambda \mapsto \alpha_{D(\mathbf{f}),\mathbf{f}}(\lambda)/||\lambda||_1$ reaches an upper bound for some $\lambda \in Y(G)$, and since $||-||_1$ is $\mathrm{Aut}^+(G)$-invariant, reaches the same upper bound at all $\beta(\lambda)$ for all $\beta \in \mathrm{Aut}^+(G)$. This gives that $\beta \cdot P_{D(\mathbf{f}),\mathbf{f}} = P_{\beta\cdot D(\mathbf{f}),\beta\cdot\mathbf{f}}$, as required.

Combining (5) and (6) gives that $\beta \cdot P_{D(\mathbf{f}),\mathbf{f}} = P_{D(\mathbf{f}),\mathbf{f}}$ for any $\beta \in R \subseteq \mathrm{Aut}^+(G)$.

Set $P_{D(\mathbf{f}),\mathbf{f}} = P_\lambda$ for λ in the optimal class $\Lambda_{D(\mathbf{f}),\mathbf{f}}$. By the discussion following Lemma 6.6, every Levi subgroup of P_λ is of the form L_λ for $\lambda \in \Lambda_{D(\mathbf{f}),\mathbf{f}}$. Suppose that F is contained in a Levi subgroup L_λ of P_λ. Then $u \cdot \mathbf{f} \in L_\lambda^n$ for some $u \in R_u(P_\lambda)$. Since $u \cdot f_i \in L_\lambda$ for each i, we have $\lambda(k^*)$ centralises $u \cdot f_i$. Hence

$$\begin{aligned}
u \cdot f_i &= \lim_{x\to 0} \lambda(x) u f_i u^{-1} \lambda(x)^{-1} \\
&= \lim_{x\to 0} \lambda(x) u \lambda(x)^{-1} \lambda(x) f_i \lambda(x)^{-1} \lambda(x) u^{-1} \lambda(x)^{-1} \\
&= \lim_{x\to 0} (\lambda(x) u \lambda(x)^{-1}) \lim_{x\to 0} (\lambda(x) f_i \lambda(x)^{-1}) \lim_{x\to 0} (\lambda(x) u^{-1} \lambda(x)^{-1}) \\
&= \lim_{x\to 0} \lambda(x) f_i \lambda(x)^{-1}.
\end{aligned}$$

Therefore, $\lim_{x\to 0} \lambda(x) \cdot \mathbf{f} = u \cdot \mathbf{f} \in G$ which lies inside the orbit $G \cdot \mathbf{f}$. However, by hypothesis $\lim_{x\to 0} \lambda(x) \cdot \mathbf{f}$ does not lie within the orbit $G \cdot \mathbf{f}$, hence we have reached a contradiction. Therefore, we conclude that F is not contained in any Levi subgroup of $P_{D(\mathbf{f}),\mathbf{f}}$.

The following proposition is [31, Proposition 2.2] combined with the argument given after its proof, see [31, p. 547], and it follows from the discussion above.

Proposition 9.1. *Let F be a finite subgroup of the simple group G, and let R be a subgroup of $\mathrm{Aut}^+(G)$ such that F is R-invariant. Then one of the following holds:*

(1) *F is strongly reductive in G, or*

(2) *F is contained in a proper R-invariant parabolic subgroup P of G, but not in any Levi subgroup of P.*

Remark 9.2. In [31, Remark 2.4], a partial extension to Proposition 9.1 is given that takes into account the situation where G is of type B_2 ($p =$

$2), F_4$ ($p = 2$) or G_2 ($p = 3$) where $\mathrm{Aut}^+(G)$ is replaced by $\langle \mathrm{Aut}^+(G), \phi\rangle$, and ϕ is a graph automorphism of G as introduced in equation (4). In these cases ϕ^2 is a field automorphism, and ϕ normalises $\mathrm{Aut}^+(G)$.

Suppose that F is a finite subgroup of G, that is invariant under a subgroup S of $\langle \mathrm{Aut}^+(G), \phi\rangle$. The argument in [31, Remark 2.4] states that if we set $S_0 = S \cap \mathrm{Aut}^+(G)$, then $S = \langle S_0, \sigma\rangle$, where $\sigma^2 \in S_0$.

Suppose that F is not strongly reductive in G. By Proposition 9.1, we have that F is contained in the S_0-invariant parabolic subgroup P' of G, and hence F is also contained in $\sigma(P')$. If $P' \cap \sigma(P')$ is reductive, then by Lemma 6.9, this intersection is a Levi subgroup of P'. However this is a contradiction because we showed in Proposition 9.1 that F is not contained in any Levi subgroup of P'. Therefore, $U := R_u(P' \cap \sigma(P')) \neq e$. We construct the parabolic subgroup P of G from this non-trivial unipotent radical using the construction of Borel-Tits. Since U is S-invariant, so is P, and since U is non-trivial we have that P is proper in G. Although it is known that F is not contained in any Levi subgroup of P', the relationship between the Levi subgroups of P and those of P' is not well understood. It is therefore not trivial to infer from the arguments given in [31, Remark 2.4] whether F is contained in a Levi subgroup of P, or not.

We present an argument in §9.2 which shows that in the case $S = \sigma$ there is a σ-stable parabolic subgroup P'' of G containing F, and F is not contained in any Levi subgroup of P''.

The next result follows from Proposition 9.1 and Remark 9.2.

Proposition 9.3. *Let G be a simple algebraic group and let σ be a Frobenius morphism of G. Let F be a finite σ-stable subgroup of G. Then one of the following holds:*

(1) *F is strongly reductive in G, or*

(2) *F is contained in a proper σ-stable parabolic subgroup P of G.*

In the remainder of this section we will show that Proposition 9.3 can be extended to the case where G is a reductive algebraic group. This result is achieved in Lemma 9.9.

Lemma 9.4. *Let G_1, G_2 and G_3 be algebraic groups. Given isogenies $f : G_1 \to G_2$ and $g : G_2 \to G_3$, the map $h : G_1 \to G_3$ where $h(x) = g(f(x))$ for all $x \in G_1$, is also an isogeny.*

Proof. The composition of the map f and g is indeed a morphism from G_1 to G_3. We have to show that it has a finite kernel.

Set $K := \ker(h)$, and let $x \in K$. Then, $g(f(x)) = e$, and so $f(x) \in \ker(g)$. That is, $f(K) \subseteq \ker(g)$. Therefore, $f(K)$ is finite. Since, $K \cap \ker(f) = \ker(f|_K)$ is finite, we have that K is finite. $\square$

Notation 9.5. Let $G = G_1 \times \cdots \times G_n$ be a product of n groups, and let σ be a Frobenius morphism of G. We say that σ permutes the G_i transitively if for all i we have $\sigma(G_i) \subseteq G_{i+1 \bmod n}$. That is, for any $(g_1, \ldots, g_n) \in G$ and each i we have morphisms $\sigma_i : G_i \to G_{i+1 \bmod n}$ such that σ is given by the map $(g_1, \ldots, g_n) \mapsto (\sigma_n(g_n), \sigma_1(g_1) \ldots, \sigma_{n-1}(g_{n-1}))$.

Lemma 9.6. *Let $H = G_1 \times \cdots \times G_n$ be a direct product of the simple algebraic groups $G_1, \ldots, G_n$, and let σ be a Frobenius morphism of H that permutes the G_i transitively. Suppose that F is a finite σ-stable subgroup of H. Then one of the following holds:*

(1) *F is strongly reductive in H, or*

(2) *F is contained in a proper σ-stable parabolic subgroup P of H.*

Proof. Label the G_i so that $\sigma(G_i) = G_{i+1 \bmod n}$. Let $\pi_i : H \to G_i$, be the projection of H onto the i-th component G_i, and let $F_i := \pi_i(F)$. As F is σ-stable, for $f_i \in F_i$, we have $\pi_{i+1}(\sigma(f_i)) \in F_{i+1 \bmod n}$. Therefore, we have

$\sigma(F_i) \subseteq F_{i+1 \bmod n}$. Furthermore, $\sigma^{n-1}(F_{i+1}) \subseteq F_i$. Since σ is bijective, all the F_is are finite and of the same order. Hence, $\sigma(F_i) = F_{i+1 \bmod n}$, for each i.

For each i, let $\mathbf{f}_i := (f_{i_1}, \ldots, f_{i_m})$ be a tuple of the m elements of F_i. We write $\sigma(\mathbf{f}_i)$ for $(\sigma(f_{i_1}), \ldots, \sigma(f_{i_m}))$. Then, $\sigma(\mathbf{f}_i) = \omega \cdot \mathbf{f}_{i+1 \bmod n}$, for some $\omega \in S_m$. The tuple $\omega \cdot \mathbf{f}_i$ generates F_i for all i, and all $\omega \in S_m$.

Suppose that F is not strongly reductive in H. Then, by [1, Theorem 3.1], F is not H-cr. By [1, Lemma 2.12(i)], there is some $j \in \{1, \ldots, n\}$ such that F_j is not G_j-cr.

As $\sigma : H \to H$ is a bijection, it has trivial kernel, hence σ is an isogeny. By Lemma 9.4, $\sigma^a : G_i \to G_{i+a}$ is an isogeny for any positive integer a. Therefore, [1, Lemma 2.12(ii)(b)] gives that $\sigma^a(F_j)$ is not G_{j+a}-cr. By construction, $\sigma^a(F_j) = F_{j+a \bmod n}$. Hence, F_i is not G_i-cr for all i.

In particular F_1 is not G_1-cr and both are σ^n-stable. We may apply Proposition 9.3. Therefore we can construct a parabolic subgroup P_1 of G_1 that is σ^n-stable and contains F_1.

Since P_1 is σ^n-stable, the parabolic subgroup $P := P_1 \times \sigma(P_1) \times \cdots \times \sigma^{n-1}(P_1)$ of H is σ-stable. Furthermore, F is contained in P because $\sigma^a(F_1) = F_{a+1 \bmod n}$ is contained in $\sigma^a(P_1) = P_{a+1 \bmod n}$, for all n. This gives the result. $\square$

The following lemma shows that in the context of our study of (G, σ)-complete reducibility we can reduce from the case where G is reductive to the case where G is semisimple.

Lemma 9.7. *Let G be a reductive algebraic group and let σ be a Frobenius morphism of G. Suppose F is a finite σ-stable subgroup of G. Then, there exists a Frobenius morphism σ' of $[G, G]$ and a finite σ'-stable subgroup F' of $[G, G]$ such that the following hold:*

(1) *F is G-completely reducible if and only if F' is $[G,G]$-completely reducible, and*

(2) *F is (G,σ)-completely reducible if and only if F' is $([G,G],\sigma')$-completely reducible.*

Proof. Since G is reductive, we have $G = [G,G]Z$, where $Z = Z(G)^0$. Let $\alpha : [G,G] \times Z \to G$ be the product map. Then α is an isogeny.

There exists a Frobenius morphism σ' on $[G,G] \times Z$ which when composed with the multiplication map gives the Frobenius morphism σ on G. Set σ' to be the map defined by $(g,z) \mapsto (\sigma(g), \sigma(z))$ for $g \in [G,G], z \in Z$. Then, $\ker(\alpha)$ is a σ'-stable normal subgroup of $[G,G] \times Z$. Therefore, by Proposition 7.8 (2), σ' induces the Frobenius morphism σ on $[G,G] \times Z/\ker(\alpha) \cong G$. Hence, we have the commutative diagram:

$$\begin{array}{ccc} [G,G] \times Z & \xrightarrow{\alpha} & G \\ \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ [G,G] \times Z & \xrightarrow{\alpha} & G. \end{array}$$

Let $F^{-1} := \alpha^{-1}(F)$, for a finite σ-stable subgroup F of G. We wish to show that F^{-1} is finite and σ'-stable. We have that $\alpha(\sigma'(F^{-1})) = \sigma(\alpha(F^{-1})) = \sigma(F) = F$. Therefore, $\alpha^{-1}(\alpha(\sigma'(F^{-1}))) = \alpha^{-1}(F) = F^{-1}$. Since $\ker(\alpha)$ is σ'-stable and $\ker(\alpha) \subseteq F^{-1}$, we have that $F^{-1} = \alpha^{-1}(\alpha(\sigma'(F^{-1}))) \supseteq \sigma'(F^{-1})\ker(\alpha) = \sigma'(F^{-1}\ker(\alpha)) = \sigma'(F^{-1})$. Note that, as α is an isogeny, F^{-1} is finite. Hence, as σ' is a bijection we must have equality. This shows F^{-1} is finite and σ'-stable.

By [1, Lemma 2.12(ii)], F is G-cr if and only if F^{-1} is $([G,G] \times Z)$-cr. Let $\pi_{[G,G]} : [G,G] \times Z \to [G,G]$ be the projection onto $[G,G]$, and $\pi_Z : [G,G] \times Z \to Z$ be the projection onto Z.

By [1, Lemma 2.12(i)], if F^{-1} is $([G,G] \times Z)$-cr, then $F' := \pi_{[G,G]}(F^{-1})$ is $[G,G]$-cr. Again, by [1, Lemma 2.12(i)], if F^{-1} is not $([G,G] \times Z)$-cr, then either F' is not $[G,G]$-cr, or $\pi_Z(F^{-1})$ is not Z-cr, or both of these statements hold. Since $\pi_Z(F^{-1}) \subseteq Z$, it is linearly reductive, and hence, by [25, Lemma 11.24], is G-cr. Hence if F^{-1} is not $([G,G] \times Z)$-cr, then F' is not $[G,G]$-cr. Therefore, combining the above gives that F is G-cr if and only if F' is $[G,G]$-cr.

Since $[G,G]$ and F^{-1} are σ'-stable, F' is also σ'-stable, and since F^{-1} is finite, so is F'.

Given a parabolic subgroup P of G, we have $P' := \pi_{[G,G]}(\alpha^{-1}(P))$ is a parabolic subgroup of $[G,G]$, and every parabolic subgroup of $[G,G]$ arises in this way. Also, if F is contained in P, then F' is contained in P', and vice-versa. Furthermore, P is σ-stable if, and only if, P' is σ'-stable. A corresponding argument holds for Levi subgroups of P, giving the result. $\square$

Notation 9.8. Let G be a semisimple algebraic group, and let σ be a Frobenius morphism of G. Then G is the almost direct product of n simple factors (see Definition 5.21), and the image under σ of each simple factor is another simple factor. Therefore, σ naturally partitions G into a fixed number, k say, of σ-orbits denoted H_j for $j \in \{1, \dots, k\}, k \leq n$. If the j-th such σ-orbit H_j is a product of l_j simple groups, we say that l_j is the **length** of the σ-orbit H_j.

We may assume without loss of generality that the simple factors G_i of G, for $i = 1, \dots, n$, are labeled such that within the j-th σ-orbit σ sends G_i to $G_{i+1 \bmod l_j}$. Thus, for each $j \in \{1, \dots, k\}$ we may choose a corresponding number $a_j \in \{1, \dots, n\}$ to denote the index of the first simple factor in the σ-orbit H_j, and $\sigma(G_{a_j}) = G_{a_j+1 \bmod l_j}$.Therefore, we can write the $H_j = G_{a_j} \cdots G_{a_j+l_j-1}$, where $a_1 = 1$ and $a_k + l_k - 1 = n$.

Therefore, we have

$$\begin{aligned} G &= H_1 \cdots H_k \\ &= (G_1 \cdots G_{1+l_1-1}) \cdots (G_{a_k} \cdots G_n) \\ &= G_1 \cdots G_n. \end{aligned}$$

Necessarily, we have $\Sigma_{j=1}^{k} l_j = n$ and the a_j are numbers in the set $\{1, \ldots, n\}$ such that $a_j < a_{j+1}$ for all j, but are not necessarily consecutive (the two indices a_j, a_{j+1} are consecutive if, and only if, $l_j = 1$).

We will refer to this labeling of the simple factors G_i of G, and of its σ-orbits H_j as a **compatible decomposition of G with respect to σ.**

No restrictions are placed on G and σ in defining this decomposition, and it is clear that every semisimple algebraic group with a Frobenius morphism σ has a compatible decomposition with respect to σ. The objective of defining this decomposition is to simplify subsequent arguments by considering a concrete decomposition of G into its σ-orbits.

Clearly this decomposition is not unique, for we may begin each σ-orbit at any of the simple factors occurring within that orbit, however for our purposes it is sufficient to pick any compatible decomposition. It should be noted that the decomposition into σ-orbits is unique up to isomorphism, and so the number of orbits, their lengths, and the types of groups that occur as simple factors are all uniquely determined.

Lemma 9.9. *Let G be a reductive algebraic group and let σ be a Frobenius morphism of G. Let F be a finite σ-stable subgroup of G. Then one of the following holds:*

(1) *F is strongly reductive in G, or*

(2) *F is contained in a proper σ-stable parabolic subgroup P of G.*

Proof. By Lemma 9.7 we may assume without loss that $G = [G, G]$. According to Notation 9.8, let $G = G_1 \cdots G_n = H_1 \cdots H_k$ be a compatible decomposition of G with respect to σ. That is, the G_i, for $i \in \{1, \ldots, n\}$, are the simple factors of G, and the H_j, for $j \in \{1, \ldots, k\}$, are the σ-orbits each of length l_j of G, and within each σ-orbit H_j we have $\sigma(G_i) = G_{i+1 \bmod l_j}$.

Set $\tilde{H}_j = G_{a_j} \times \cdots \times G_{a_j+l_j-1}$, and $\tilde{G} = \tilde{H}_1 \times \cdots \times \tilde{H}_k$. Then, $\tilde{G}$ is equal to the product $G_1 \times \cdots \times G_n$ of n simple groups. Let $\iota_i : G_i \to G$ be the inclusion map for each i, and let $\epsilon : \tilde{G} \to G$ be the product map, defined by $\epsilon : (g_1, \ldots, g_n) \mapsto \iota_1(g_1) \cdots \iota_n(g_n)$, for $g_i \in G_i$.

There exists a Frobenius morphism σ' on $\tilde{G}$ which when composed with the product map, ϵ, gives the Frobenius morphism σ on G. Define the Frobenius morphism $\sigma_j : \tilde{H}_j \to \tilde{H}_j$ by

$$\sigma_j(g_{a_j}, \ldots, g_{a_j+l_j-1}) = (\sigma(g_{a_j+l_j-1}), \sigma(g_{a_j}), \ldots, \sigma(g_{a_j+l_j-2})).$$

Then define $\sigma' : \tilde{G} \to \tilde{G}$ by $\sigma'(\mathbf{h}_1, \ldots, \mathbf{h}_k) = (\sigma_1(\mathbf{h}_1), \ldots, \sigma_k(\mathbf{h}_k))$ where each $\mathbf{h}_j \in \tilde{H}_j$. This defines a Frobenius morphism on $\tilde{G}$ because σ_j is a homomorphism and for each j there is some power p_j of σ_j such that $\sigma_j^{p_j}$ acts like the standard Frobenius morphism on each simple factor of $\tilde{H}_j$, for each j.

We have $\ker(\epsilon)$ is a σ'-stable normal subgroup of $\tilde{G}$. Therefore, by Proposition 7.8 (2), σ' induces the Frobenius morphism σ on $\tilde{G}/\ker(\epsilon) \cong G$. Hence, we have the commutative diagram:

$$\begin{tikzcd} \tilde{G} \arrow[r, "\epsilon"] \arrow[d, "\sigma'"] & G \arrow[d, "\sigma"] \\ \tilde{G} \arrow[r, "\epsilon"] & G. \end{tikzcd}$$

Note, the product map, ϵ, is an isogeny because G is an almost direct product of the simple groups G_i, so there are only finitely many elements in its kernel.

Let F be a finite σ-stable subgroup of G. Then, by the argument given in the proof of Lemma 9.7, we have $\tilde{F} := \epsilon^{-1}(F)$ is also finite and σ'-stable.

Suppose F is not strongly reductive in G, then by [1, Theorem 3.1], F is not G-cr. By [1, Lemma 2.12 (ii)] we have $\tilde{F}$ is not $\tilde{G}$-cr. Let $\pi_j : \tilde{G} \to \tilde{H}_j$ be the projection of $\tilde{G}$ onto the j-th σ-orbit $\tilde{H}_j$. Then, by [1, Lemma 2.12 (i)], there exists some j such that $F_j := \pi_j(\tilde{F})$ is not $\tilde{H}_j$-cr. Furthermore, we have $\sigma_j(\pi_j(\tilde{F})) \subseteq \pi_j(\tilde{F})$, so F_j is σ_j-stable.

By Lemma 9.6, we have F_j is contained in a proper σ_j-stable parabolic subgroup $\tilde{P}_j$ of $\tilde{H}_j$. Thus, $\tilde{F}$ is contained in the proper σ'-stable parabolic subgroup $\tilde{P} := \tilde{H}_1 \times \cdots \times \tilde{H}_{j-1} \times \tilde{P}_j \times \tilde{H}_{j+1} \times \cdots \times \tilde{H}_k$ of $\tilde{G}$. By [1, Lemma 2.11], we can conclude that F is contained in the proper parabolic subgroup $P := \epsilon(\tilde{P})$ of G. Finally, because $\tilde{P}$ is σ'-stable, P is σ-stable. $\square$

9.2 Extension to the Ree and Suzuki Case

We now analyse the cases where G is a simple algebraic group of type B_2 ($p = 2$), F_4 ($p = 2$) or G_2 ($p = 3$) with root system Φ. Let σ be a Frobenius morphism of G that gives rise to an order two permutation of the roots in Φ. Let F be a finite σ-stable subgroup of G that is not strongly reductive in G. Note that in Remark 9.2 we obtained a partial extension of Proposition 9.1, in that we were able to construct a σ-stable parabolic subgroup of G containing F. Below we show that F is contained in a proper σ-stable parabolic subgroup P of G, but not in any Levi subgroup of P.

Case 1 *G is of type $B_2, p = 2$ or $G_2, p = 3$*

Since F is σ-stable, it is σ^2-stable and $\sigma^2 \in \mathrm{Aut}^+(G)$, so by Proposition 9.1, we have $F \subset P \subsetneq G$ for P a proper σ^2-stable parabolic subgroup of G and F is in no Levi subgroup of P. Furthermore, $F \subset P \cap \sigma(P)$ and this intersection is σ-stable. We cannot have $U := R_u(P \cap \sigma(P)) = e$ for then the intersection $P \cap \sigma(P)$ would be a Levi subgroup of P containing F, which contradicts Proposition 9.1. Therefore, $U \neq e$, and is σ-stable. Hence, by the construction of Borel-Tits, U gives rise to a σ-stable parabolic subgroup P_1 of G, which contains F. Since U is non-trivial, the parabolic subgroup P_1 is proper in G.

By Corollary 7.13 P_1 contains a σ-stable Borel subgroup B of G, and by Corollary 7.14 this Borel subgroup contains a σ-stable maximal torus T of G. We choose a base Δ of the root system Φ of G with respect to B and T. Suppose that $\Delta = \{\alpha, \beta\}$, then $\sigma(\alpha) = \beta$ and vice-versa.

From §6.2, we see that for each subset I of Δ, there is a corresponding conjugacy class P_I of parabolic subgroups of G, and $U_{-\gamma} \subset P_I$ if, and only if, $\gamma \in I$. We have Δ corresponds to G and $\emptyset$ corresponds to the Borel subgroups of G.

By checking the Dynkin diagrams in these types, we see that there are only two conjugacy classes of proper (non-Borel) parabolic subgroups in G, and due to our choice of base for Φ these correspond to the simple roots α and $\sigma(\alpha) = \beta$, where σ swaps these simple roots. The maximal proper parabolic subgroup $P_\alpha = \langle B, U_{-\alpha} \rangle$ corresponding to α is sent to the maximal proper parabolic subgroup $P_{\sigma(\alpha)} = \langle B, U_{\sigma(-\alpha)} \rangle$. Thus, the only σ-stable parabolic subgroups in this case are the σ-stable Borel subgroups.

Therefore P_1 must be a Borel subgroup of G. By Theorem 6.18, F is not G-cr, hence it is not contained in a torus of P_1, giving the result. That is, F is not (G, σ)-completely reducible. In particular, F is contained in the σ-stable parabolic subgroup P_1 of G, and in no Levi subgroup of P_1.

Case 2 *G is of type $F_4, p = 2$*

Let F be a finite σ-stable subgroup of G, and suppose that F is not G-completely reducible, but is (G, σ)-completely reducible. By Lemma 9.9 we have that $F \subset P \subset G$ for P a proper σ-stable parabolic subgroup of G.

Since F is (G, σ)-completely reducible we have $F \subset L \subset P$ for L a σ-stable Levi subgroup of P.

By [45, Proposition 3.2], since L is a Levi subgroup, F is G-completely reducible if and only if F is L-completely reducible. Hence F is not L-completely reducible.

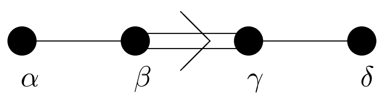
Therefore, by Lemma 9.9, we can conclude $F \subseteq Q \subseteq L$, for Q a proper σ-stable parabolic subgroup of L.

By [8, Proposition 4.4 (c)], the subgroup $V = QR_u(P)$ is also a parabolic subgroup of G and is contained in P. It is σ-stable by construction.

Lemma 9.10. *Let G be a simple algebraic group of type F_4, and let σ be a Frobenius morphism of G that induces a non-trivial permutation of its simple roots. Then, there are no proper inclusions amongst the set of proper, non-Borel, σ-stable parabolic subgroups of G.*

Proof. In this case we note that, by [30, Corollary 1.5], $G^\sigma = {}^2F_4$. By [34, Main Theorem], we have that every proper (non-Borel) parabolic subgroup of ${}^2F_4$ is maximal.

By [32, Theorem 8] the maximal parabolic subgroups of ${}^2F_4$ are the fixed point groups of maximal σ-stable parabolic subgroups of G. Pick a σ-stable Borel subgroup B of G, containing a σ-stable maximal torus T of G. Then, with respect to B and T we can form a base $\Delta = \{\alpha, \beta, \gamma, \delta\}$ of the root system of G, giving the following Dynkin diagram.



Dynkin diagram of type F_4

The Frobenius morphism σ acts on these simple roots by $\sigma(\alpha) = \delta$ and vice-versa, and also $\sigma(\beta) = \gamma$ and vice-versa. The conjugacy classes of σ-stable parabolic subgroups of G correspond to σ-stable subsets of these simple roots, which are the sets $\{\alpha, \beta, \gamma, \delta\}$, $\{\alpha, \delta\}$, $\{\beta, \gamma\}$ and $\emptyset$. As before, $P_{\{\alpha,\beta,\gamma,\delta\}} = G$ and $P_{\emptyset} = B$.

Thus, the only conjugacy classes of proper non-Borel parabolic subgroups of G are represented by $P_{\{\alpha,\delta\}}$ and $P_{\{\beta,\gamma\}}$, and these classes give rise to the two conjugacy classes of maximal parabolic subgroups in ${}^2F_4$ listed in [34, Main Theorem]. Therefore, the proper, non-Borel, σ-stable parabolic subgroups of G are maximal, thus we conclude that there are no proper inclusions among this set of subgroups of G. $\square$

We have three cases to consider:

(1) $V = G$. This is not possible since $V \subseteq P \neq G$.

(2) If $V = B$, a Borel subgroup of G. Then we have $F \subseteq B \subseteq G$ and is not G-completely reducible. Therefore, F is not contained in a torus of B, and is therefore not (G, σ)-completely reducible. This contradicts our hypothesis.

(3) If V is another parabolic subgroup of G that is contained in P then, by Lemma 9.10, we must have $V = P$. That is $QR_u(P) = P$. We have, $\dim(P) = \dim(L) + \dim(R_u(P))$. However, $\dim(V) \leq \dim(Q) + \dim(R_u(P))$. Since $\dim(Q) \lneq \dim(L)$, we cannot have the equality $V = P$. Thus, we have a contradiction.

None of these cases are possible, and so we have reached a contradiction to our hypothesis. Therefore, we conclude that F is not (G, σ)-completely reducible. We complete the argument by applying Lemma 8.15, to obtain that H is contained in a proper σ-stable parabolic subgroup P' of G, and not in any Levi subgroup of P'.

Combining the above results and Proposition 9.1 gives the following, which provides a partial converse to Theorem 8.6.

Proposition 9.11. *Let G be a simple algebraic group and let σ be a Frobenius morphism of G. Let F be a finite σ-stable subgroup of G. Then one of the following holds:*

(1) *F is strongly reductive in G, or*

(2) *F is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

We can now present our main result about finite subgroups of simple groups, which follows immediately from Theorem 8.6 and Proposition 9.11

Theorem 9.12. *Let G be a simple algebraic group, and let σ be a Frobenius morphism of G. Suppose that F is a finite σ-stable subgroup of G, then*

(1) *F is G-completely reducible if and only if it is (G, σ)-completely reducible, and*

(2) *if F is not G-completely reducible, then F is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

9.3 Extension to Reductive Groups

The aim of this section is to generalise Theorem 9.12 to the case where G is a reductive algebraic group. In the following lemma we extend Proposition 9.11 to the case where G is a direct product of simple groups. This will be used later on in the proof of our main result in this section, which is Proposition 9.14.

Lemma 9.13. *Let $H = G_1 \times \cdots \times G_n$ be a direct product of the simple algebraic groups $G_1, \ldots, G_n$, and let σ be a Frobenius morphism of H that permutes the G_i transitively. Suppose that F is a finite σ-stable subgroup of H. Then one of the following holds:*

(1) *F is strongly reductive in H, or*

(2) *F is contained in a proper σ-stable parabolic subgroup P of H and not in any Levi subgroup of P.*

Proof. Label the G_i such that $\sigma(G_i) = G_{i+1 \bmod n}$. Let $\pi_i : H \to G_i$, be the projection of H onto the i-th component G_i, and let $F_i := \pi_i(F)$.

Suppose that F is not strongly reductive in H. As in the proof of Lemma 9.6, we can conclude that $\sigma(F_i) = F_{i+1 \bmod n}$, and that F_i is not G_i-cr for each i.

For all i we have F_i and G_i are σ^n-stable. By Proposition 9.11, we can construct a proper parabolic subgroup P_i of G_i that is σ^n-stable that contains F_i, such that F_i is not contained in any Levi subgroup of P_i.

Without loss, set $i = 1$, then F_1 is not G_1-cr, and F_1 is contained in the proper σ^n-stable parabolic subgroup P_1 of G_1, and in no Levi subgroup of P_1. The parabolic subgroup $P := P_1 \times \sigma(P_1) \times \cdots \times \sigma^{n-1}(P_1)$ of H is σ-stable, and contains F.

Suppose that F is contained in a Levi subgroup $L := L_1 \times \cdots \times L_n$ of P. We have L_1 is a Levi subgroup of P_1 containing F_1, and L_2 is a Levi subgroup of P_2 containing F_2, etc. However, this contradicts the previous assertion. Hence F is not contained in any Levi subgroup of P. $\square$

We now extend Proposition 9.11 to the case where G is reductive, and observe that the following result provides the converse to Theorem 8.6 for finite σ-stable subgroups of G.

Proposition 9.14. *Let G be a reductive algebraic group and let σ be a Frobenius morphism of G. Let F be a finite σ-stable subgroup of G. Then one of the following holds:*

(1) *F is strongly reductive in G, or*

(2) *F is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

Proof. By Lemma 9.7 we may assume without loss that $G = [G, G]$. According to Notation 9.8, let $G = G_1 \cdots G_n = H_1 \cdots H_k$ be a compatible decomposition of G with respect to σ. That is the G_i, for $i \in \{1, \ldots, n\}$, are the simple factors of G, and the H_j, for $j \in \{1, \ldots, k\}$, are the σ-orbits each of length l_j of G, and within each σ-orbit H_j we have $\sigma(G_i) = G_{i+1 \bmod l_j}$.

Set $\tilde{H}_j = G_{a_j} \times \cdots \times G_{a_j+l_j-1}$, and $\tilde{G} = \tilde{H}_1 \times \cdots \times \tilde{H}_k$. Then, $\tilde{G}$ is equal to the product $G_1 \times \cdots \times G_n$ of n simple groups. Let $\iota_i : G_i \to G$ be the inclusion map for each i, and let $\epsilon : \tilde{G} \to G$ be the product map, defined by $\epsilon : (g_1, \ldots, g_n) \mapsto \iota_1(g_1) \cdots \iota_n(g_n)$, for $g_i \in G_i$.

As in the proof of Lemma 9.9, there exists a Frobenius morphism σ' on $\tilde{G}$ which when composed with the product map, ϵ, gives the Frobenius morphism σ on G. We define the Frobenius morphism $\sigma' : \tilde{G} \to \tilde{G}$ by its action on each $\tilde{H}_j$. For $(g_{a_j}, \ldots, g_{a_j+l_j-1}) \in \tilde{H}_j$ we set:

$$\sigma_j(g_{a_j}, \ldots, g_{a_j+l_j-1}) = (\sigma(g_{a_j+l_j-1}), \sigma(g_{a_j}), \ldots, \sigma(g_{a_j+l_j-2})).$$

Then we set $\sigma' : \tilde{G} \to \tilde{G}$ by $\sigma'(\mathbf{h}_1, \ldots, \mathbf{h}_k) = (\sigma_1(\mathbf{h}_1), \ldots, \sigma_k(\mathbf{h}_k))$ where each $\mathbf{h}_j \in \tilde{H}_j$.

Let F be a finite σ-stable subgroup of G. Then, by the argument given in the proof of Lemma 9.7, we have $\tilde{F} := \epsilon^{-1}(F)$ is also finite and σ'-stable.

Suppose F is not strongly reductive in G, then by [1, Theorem 3.1], F is not G-cr. By [1, Lemma 2.12 (ii)] we have $\tilde{F}$ is not $\tilde{G}$-cr. Let $\pi_j : \tilde{G} \to \tilde{H}_j$ be the projection map. Then, by [1, Lemma 2.12 (i)], there exists some j such that $F_j := \pi_j(\tilde{F})$ is not $\tilde{H}_j$-cr. Since $\sigma_j(\pi_j(\tilde{F})) \subseteq \pi_i(\tilde{F})$, we have that F_j is σ_j-stable.

Therefore, by Lemma 9.13, we have F_j is contained in a proper σ_j-stable parabolic subgroup $\tilde{P}_j$ of $\tilde{H}_j$, and in no Levi subgroup of $\tilde{P}_j$.

Thus, $\tilde{F}$ is contained in the σ'-stable parabolic subgroup $\tilde{P} := \tilde{H}_1 \times \cdots \times \tilde{H}_{j-1} \times \tilde{P}_j \times \tilde{H}_{j+1} \times \cdots \times \tilde{H}_k$ of $\tilde{G}$. The Levi subgroups of $\tilde{P}$ are of the form $\tilde{H}_1 \times \cdots \times \tilde{H}_{j-1} \times \tilde{L}_j \times \tilde{H}_{j+1} \times \cdots \times \tilde{H}_k$, where $\tilde{L}_j$ a Levi subgroup of $\tilde{P}_j$. Therefore, $\tilde{F}$ is not contained in any Levi subgroup of $\tilde{P}$.

By [1, Lemma 2.11], we can conclude that $F = \epsilon(\tilde{F})$ is contained in the proper parabolic subgroup $P := \epsilon(\tilde{P})$ of G. Because $\tilde{P}$ is σ'-stable, we have that P is σ-stable. Similarly, by [1, Lemma 2.11], F is not contained in any Levi subgroup of P. $\square$

It is interesting to note that Proposition 9.14 takes into account Frobenius morphisms of G that are composed of elementary morphisms which are not necessarily Frobenius morphisms. For example, suppose $\mathrm{char}(k) = 2$ and $G = B_n \times C_n$. Let $\sigma : G \to G$ be the homomorphism which acts by $\sigma : (x, y) \mapsto (\mathrm{ch}^*(y), \mathrm{ch}(x))$ where $\mathrm{ch}, \mathrm{ch}^*$ are the isogenies from B_n to C_n and vice-versa, respectively, as introduced in Remark 7.10 and $x \in B_n, y \in C_n$. Then $\sigma^2 = \sigma_2$ is a standard Frobenius morphism, and hence σ is a Frobenius morphism of G and thus we can apply our results to this case. Another important feature of the proof of Proposition 9.14, is that it is not necessary to consider the specific decomposition of σ into its elementary components.

We have now arrived at the main result of this section, which follows immediately from Theorem 8.6 and Proposition 9.14.

Theorem 9.15. *Let G be a reductive algebraic group, and let σ be a Frobenius morphism of G. Suppose that F is a finite σ-stable subgroup of G, then*

(1) *F is G-completely reducible if and only if it is (G, σ)-completely reducible, and*

(2) *if F is not G-completely reducible, then F is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

Remark 9.16. Alternative proofs of Theorem 9.15, or special cases of it, are available. For example, part (1) of Theorem 9.15 is proved in [18].

If we take $\sigma = \sigma_q$ to be a standard Frobenius morphism, then Theorem 9.12 (1) and Theorem 9.15 (1) are special cases of [1, Theorem 5.8], and this can be seen as follows. Consider the extension $\overline{\mathbb{F}}_q/\mathbb{F}_q$ of perfect fields. A group is defined over $\mathbb{F}_q$ if and only if it is σ_q-stable. A σ-stable subgroup H of G is said to be **G-completely reducible over $\mathbb{F}_q$** if it is contained in a parabolic subgroup P of G defined over $\mathbb{F}_q$ implies that it is contained in a Levi subgroup L of P defined over $\mathbb{F}_q$. In these terms, [1, Theorem 5.8] states that a σ_q-stable subgroup H of G is G-completely reducible if and only if it is (G, σ_q)-completely reducible, and this is because being G-completely reducible over $\overline{\mathbb{F}}_q$ is the same as being (G, σ_q)-completely reducible.

Example 9.17. In characteristic p, a finite subgroup F of G is G-completely reducible provided p does not divide $|F|$. This is the well known Maschke's Theorem of representation theory. This gives plenty of examples of G-completely reducible subgroups. Therefore, by Theorem 8.6, a finite σ-stable subgroup of G is (G, σ)-completely reducible if its order is not divisible by p.

The following example provides numerous examples of finite (G, σ)-completely reducible subgroups of a reductive group G, with Frobenius morphism σ.

Example 9.18. Let G be a reductive group, and σ an arbitrary Frobenius morphism of G. The finite group G^σ of G is (G, σ)-irreducible.

We can see this as follows. Suppose that $G^\sigma \subseteq P$, for a proper σ-stable parabolic subgroup P of G. Let L be a σ-stable Levi subgroup of P, which exists by Corollary 7.15. Then, the opposite parabolic subgroup $-P$ relative to L is also σ-stable. Its unipotent radical $R_u(-P)$ intersects trivially with P, yet contains fixed points under the action of σ (see [10, p.

76] for details). This is a contradiction. Therefore, G^σ is not contained in any proper σ-stable parabolic subgroup of G.

Remark 9.19. Let L be a σ-stable Levi subgroup of some parabolic subgroup P of G. One consequence of Example 9.18 is that since L^σ is (L, σ)-irreducible, P is in fact a minimal parabolic subgroup of G containing L^σ.

10 Infinite Subgroups

In this section, unless otherwise stated, G will denote a reductive algebraic group over the field $k = \overline{\mathbb{F}}_q$ of characteristic p, and where q is some positive power of p. Recall that by reductive, we mean a connected reductive group. Let σ be a Frobenius morphism of G. The aim of this section is to generalise Theorem 9.15 to a result about arbitrary closed subgroups of G. We can do this once we know Lemma 10.4, which is an analogue of Lemma 6.24 in the σ-stability setting. This is an important step in the generalisation because within the context of G-complete reducibility it enables us to model any σ-stable subgroup of G as a finite σ-stable subgroup of G.

10.1 Extension to Infinite Groups

It is shown in [36, Lemma 3.2] that for a reductive group G there exists an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite subgroups of G whose union is dense in G. If G has such a sequence with each G_i σ-stable then we say that G has a **finite σ-structure** given by the chain $\{G_i\}$.

Proposition 10.1. *Let G be a reductive algebraic group and σ a Frobenius morphism of G. Then G has a finite σ-structure.*

Proof. Since σ is a Frobenius morphism of G there is an injective homomorphism $\iota : G \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ such that $\iota(\sigma^a(g)) = \sigma_q(\iota(g))$ for some positive power a of σ. Therefore, there is a Frobenius morphism $\sigma' : \iota(G) \to \iota(G)$, given by $\sigma'(x) = \iota(\sigma(\iota^{-1}(x)))$ for all $x \in \iota(G)$. Thus, we have $\iota(\sigma(g)) = \sigma'(\iota(g))$ for all $g \in G$, and $\sigma'^a(\iota(g)) = \sigma_q(\iota(g))$.

A subgroup H of G is σ-stable if, and only if, $\iota(H)$ is σ'-stable, and clearly H is finite if, and only if $\iota(H)$ is finite. Therefore, G has a finite σ-structure if, and only if, $\iota(G)$ has a finite σ'-structure. We set:

$$G(i) := (\iota(G)\cap\mathrm{GL}_n(\mathbb{F}_{q^{i!}}))\cap\sigma'(\iota(G)\cap\mathrm{GL}_n(\mathbb{F}_{q^{i!}}))\cap\cdots\cap\sigma'^{a-1}(\iota(G)\cap\mathrm{GL}_n(\mathbb{F}_{q^{i!}})).$$

Each $G(i)$ is σ'-stable because $\sigma'^a = \sigma_q$ and $\iota(G)$ and $\mathrm{GL}_n(\mathbb{F}_{q^{i!}})$ are σ_q-stable.

Let $x \in \iota(G)$. Since $\overline{\mathbb{F}}_q = \cup_{i\in\mathbb{N}}\mathbb{F}_{q^{i!}}$, we have that $\iota(G) = \cup_{i\in\mathbb{N}}(\iota(G) \cap \mathrm{GL}_n(\mathbb{F}_{q^{i!}}))$, see [36, Notation 3.3]. Therefore, we have $x \in \iota(G) \cap \mathrm{GL}_n(\mathbb{F}_{q^{i!}})$ for some i. Furthermore, we have that $\iota(G) = \cup_i \sigma'^b(\iota(G) \cap \mathrm{GL}_n(\mathbb{F}_{q^{i!}}))$ for each positive integer b. Therefore, for each such b there exists a corresponding i such that $x \in \sigma'^b(\iota(G) \cap \mathrm{GL}_n(\mathbb{F}_{q^{i!}}))$. By picking i to be sufficiently large, we have $x \in G(i) \neq \{1\}$, and for all i we have $G(i) \subseteq G(i+1)$. Hence we obtain an ascending sequence of finite subgroups $G(i)$ of G, such that each element of $\iota(G)$ is contained in a finite group $G(i)$, for sufficiently large i.

The union over i of each of the chains $\sigma'^b(\iota(G) \cap \mathrm{GL}_n(\mathbb{F}_{q^{i!}}))$, for each $b \in \mathbb{N}$, is dense in $\iota(G)$. Therefore, the union $\cup_{i\in\mathbb{N}} G(i)$ is also dense in $\iota(G)$ giving the result. □

Note that in the proof of Proposition 10.1 we require $G(i)$ to be a subgroup of $\mathrm{GL}_n(\mathbb{F}_{q^{i!}})$. Since $i!$ divides $(i+1)!$, we can embed $G(i)$ as a subgroup in $G(i+1)$ by canonically embedding $\mathbb{F}_{q^{i!}}$ as a subfield in $\mathbb{F}_{q^{(i+1)!}}$.

Example 10.2. We present an example of a finite σ-structure of a reductive group. Consider the reductive group $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ which has a non-standard Frobenius morphism σ given by $\sigma(g) = (\sigma_q(g^{-1}))^T$, where T denotes the transpose map.

With respect to a suitable basis, σ sends a parabolic subgroup P of G which is of block upper triangular form to the parabolic subgroup P^- of G which is of block lower triangular form, and leaves $P \cap P^-$ stable. Furthermore, $\sigma^2 = \sigma_{q^2}$. Thus, the fixed point group $\mathrm{GL}_n(\overline{\mathbb{F}}_q)^\sigma$ is the group of all matrices $g \in \mathrm{GL}_n(\mathbb{F}_{q^2})$ for which $g = (\sigma_q(g^{-1}))^T$, that is, the group $\mathrm{U}_n(\mathbb{F}_{q^2})$ of all **unitary transformations** of $\mathbb{F}_{q^2}^n$. Let us define the finite groups $G(i)$, as in Proposition 10.1, by

$$G(i) := \mathrm{GL}_n(\mathbb{F}_{q^{2i!}}) \cap \sigma(\mathrm{GL}_n(\mathbb{F}_{q^{2i!}})).$$

The $G(i)$ are finite, σ-stable, and form a chain $\{G(i)\}$ with the inclusions $G(i) \subseteq G(i+1)$. All that needs to be checked is that the union of the $G(i)$ is dense in G. Consider the chains given by $H(i) := \mathrm{GL}_n(\mathbb{F}_{q^{2i!}})$ and $H'(i) := \sigma(\mathrm{GL}_n(\mathbb{F}_{q^{2i!}}))$. These chains are σ_{q^2}-structures for G. Therefore, any element of G must simultaneously be in $H(i)$ and $H'(i)$ for sufficiently large i. Thus, the chain $\{G(i)\}$ endows G with a finite σ-structure.

In Proposition 10.5, a finite σ-structure of an arbitrary closed subgroup of G is required. Therefore, as part of our generalisation of Theorem 9.15 to a statement about arbitrary closed subgroups of G, we extend Proposition 10.1 to arbitrary algebraic groups in the following result, and for the proof we appeal to [4, Lemma 2.3].

Proposition 10.3. *Let G be an algebraic group defined over $\overline{\mathbb{F}}_q$, and let σ be a Frobenius morphism of G. Then G has a finite σ-structure.*

Proof. We proceed by induction on $\dim G$. If G is reductive, then Proposition 10.1 gives the result. Suppose that G is not reductive, then $Z := Z(R_u(G))^0$ is a non-trivial closed connected unipotent normal subgroup of G, and Z is a characteristic subgroup of $Z(R_u(G))$ and so is σ-stable. By [6, III.10.6(2)], Z contains a subgroup isomorphic to the additive group $\mathbb{G}_a$. Let C be the subgroup of Z generated by the subgroups of Z that are isomorphic to $\mathbb{G}_a$. Since σ is a morphism, the image of each of these subgroups under σ is another subgroup of Z that is isomorphic to $\mathbb{G}_a$. By [20, Theorem 5.4], C is a vector space. Therefore, C is a σ-stable, finite-dimensional vector space over k which is normal in G.

As in the proof of Proposition 10.1, there is an injective homomorphism $\iota : G \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$, such that $\iota(\sigma^a(g)) = \sigma_q(\iota(g))$ for some positive power a of σ. Therefore, there is a Frobenius morphism $\sigma'' : \iota(G) \to \iota(G)$, given by $\sigma''(x) = \iota(\sigma(\iota^{-1}(x)))$ for all $x \in \iota(G)$. Thus, we have $\iota(\sigma(g)) = \sigma''(\iota(g))$ for all $g \in G$, and $\sigma''^a(\iota(g)) = \sigma_q(\iota(g))$.

A subgroup H of G is σ-stable if, and only if, $\iota(H)$ is σ''-stable, and clearly H is finite if, and only if $\iota(H)$ is finite. Hence, G has a finite σ-structure if, and only if, $\iota(G)$ has a finite σ''-structure. We may therefore assume that $G = \iota(G)$ and $\sigma = \sigma''$.

To the group C we may associate a chain $C_1 \subseteq C_2 \subseteq \cdots$ of finite σ_q-stable subgroups of C whose union is dense in C by setting $C_i := C \cap \mathrm{GL}_n(\mathbb{F}_{q^{i!}})$. We have $C_i' := C_i \cap \sigma(C_i) \cap \cdots \cap \sigma^{a-1}(C_i)$ is a finite σ-stable subgroup of C. As in the proof of Proposition 10.1, there is some i such that $C_i' \neq e$, and for all i we have that $C_i' \subseteq C_{i+1}'$. We have that $C = \cup_i C_i'$. Therefore C has a finite σ-structure given by the set $\{C_i'\}$.

Since C is a σ-stable subgroup of G, by Proposition 7.8, the morphism σ' on $M := G/C$ defined by $\sigma' : gC \mapsto \sigma(g)C$ for $g \in G$ is a Frobenius morphism. Furthermore, we have the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\sigma"] & M \arrow[d, "\sigma'"] \\ G \arrow[r, "\pi"] & M. \end{tikzcd}$$

By induction M has an ascending sequence of finite σ'-stable groups $M_1 \subseteq M_2 \subseteq \cdots$ whose union is dense in M. Let $\pi : G \to M$ be the canonical projection. Suppose that $M_i = \{g_{i_1}C, \ldots, g_{i_{k_i}}C\}$ for each i.

Denote by G_i the subgroup of G generated by the finite set $C_i' \cup \{g_{i_1}, \ldots, g_{i_{k_i}}\}$. We have, $\pi(G_i) \supseteq M_i$, and the G_i form an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of subgroups where G_i contains C_i'. Since G_i is finitely generated and $G_i \cap C$ is of finite index in G_i, by [44, Theorem 11.54] $G_i \cap C$ is finitely generated. Since C is a vector space, $G_i \cap C$ is finite. Therefore G_i is finite. We wish to show that the G_i are σ-stable. If there is no $g_{i_a} \in G_i - C$, then the G_i coincide with the C_i' and hence are σ-stable. Therefore, suppose that there is some $g_{i_a} \in G_i - C$. As M_i is σ'-stable, we have $\sigma'(g_{i_a}C_i') \in M_i$. Since the C_i' are σ-stable, we have $\sigma(g_{i_a}) \in G_i - C$.

Thus, G_i is σ-stable. Therefore we have that the G_i form an ascending sequence of finite σ-stable subgroups.

Let G' be the closure of the union of the G_i. Then G' is a closed σ-stable subgroup of G containing C. Its image $\pi(G')$ is a closed subgroup of M containing the M_i and is therefore equal to M. Therefore $G' = G$. $\square$

By Proposition 10.3 we can associate a finite σ-structure to a closed σ-stable subgroup of G, enabling us to prove the following lemma, which is an adaptation of Lemma 6.24.

Lemma 10.4. *Let G be a σ-stable reductive group. Suppose that H is a closed σ-stable subgroup of G with a finite σ-structure given by the chain of subgroups $\{H_i\}$. Then, there is some $a \in \mathbb{N}$ such that for all $b \geq a$ we have H is G-completely reducible if and only if H_b is G-completely reducible.*

Proof. List representatives of the G-conjugacy classes of parabolic subgroups of G as $P_1, \ldots, P_m$ and representatives of the G-conjugacy classes of Levi subgroups as $L_1, \ldots, L_n$. Let $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, n\}$ and $l \in \mathbb{N}$.

Since $\cup_i H_i$ is dense in H we have that $\cup_i H_i$ is contained in a parabolic subgroup P of G (resp. a Levi subgroup of P) if and only if H is contained in a parabolic subgroup P of G (resp. a Levi subgroup of P). We may therefore assume, without loss of generality, that $\cup_i H_i = H$. For any subgroup $H' \subseteq H$ define the sets $C_j(H') := \{g \in G \mid H' \subseteq gP_jg^{-1}\}$ and $D_k(H') := \{g \in G \mid H' \subseteq gL_kg^{-1}\}$ for all j, k, which are closed by [23, Proposition 8.2 (a)]. For any $H'' \subseteq H'$ we have the inclusions $C_j(H') \subseteq C_j(H'')$ and $D_k(H') \subseteq D_k(H'')$.

For each j define $\mathcal{C}_j$ to be the set of all $C_j(H')$ where H' ranges over the set of all finite subgroups of H in the chain $\{H_i\}$ giving H its finite σ-structure, and for each k define $\mathcal{D}_k$ to be the set of all $D_k(H')$ correspondingly. By the descending chain condition on closed sets, these two chains give rise to a minimal element in each of $\mathcal{C}_j$ and $\mathcal{D}_k$. Therefore, there exists some finite group H^*, say, in $\{H_i\}$ such that for any other H^{**} in $\{H_i\}$ with $H^* \subseteq H^{**}$ we have $C_j(H^*) \subseteq C_j(H^{**})$ and $D_k(H^*) \subseteq D_k(H^{**})$ for each j and k. The reverse inclusions are noted to hold above, so in fact we have that $C_j(H^*) = C_j(H^{**})$ and $D_k(H^*) = D_k(H^{**})$.

It follows that there are some $a', a'' \in \mathbb{N}$ such that $C_j(H_{a'}) = C_j(H)$ and $D_k(H_{a''}) = D_k(H)$ for all j, k. This can be seen as follows. Suppose, by way of contradiction, that $C_j(H) \subsetneq C_j(H_{a'})$ for some j, then there exists some $g \in C_j(H_{a'})$ which is not in $C_j(H)$. Hence, there exists some $g \in C_j(H_{a'})$, and some $h \in H$ with $ghg^{-1} \notin P_j$. Since $\cup_i H_i = H$, we have that $h \in H_{a'+l}$ for some $l \in \mathbb{N}$ and so $C_j(H_{a'+l}) \subsetneq C_j(H_{a'})$ since $g \notin C_j(H_{a'+l})$, which is a contradiction. A similar argument shows that $D_k(H_{a''}) = D_k(H)$.

It also follows, from the minimality of $C_j(H_{a'})$ and $D_k(H_{a''})$, and the fact that $C_j(H_{a'}) \subseteq C_j(H_{a'+l})$ and $D_k(H_{a''}) \subseteq D_k(H_{a''+l})$ for all $l \in \mathbb{N}$, that $C_j(H_{a'+l}) = C_j(H)$ and $D_k(H_{a''+l}) = D_k(H)$ for all $l \in \mathbb{N}$.

Suppose that $H \subseteq P$ for some parabolic subgroup P of G. Then P is G-conjugate to P_j for some j, so $gPg^{-1} = P_j$, say. Hence $gHg^{-1} \subseteq P_j$. Hence, $g \in C_j(H) = C_j(H_{a'+l})$ and so $gH_{a'+l}g^{-1} \subseteq P_j$, so $H_{a'+l} \subseteq P$ for all l. This argument is reversible and a corresponding argument works for Levi subgroups of G. Let $a = \max(a', a'')$. The group H_a satisfies the conditions in the statement of the Lemma, giving the result. $\square$

We can now show the converse to Theorem 8.6.

Proposition 10.5. *Let G be a reductive algebraic group with Frobenius morphism σ. Let H be a σ-stable subgroup of G. Then one of the following holds:*

(1) *H is strongly reductive in G, or*

(2) *H is contained in a proper σ-stable parabolic subgroup P of G, and not in any Levi subgroup of P.*

Proof. Suppose that H is σ-stable and not strongly reductive in G. By [1, Theorem 3.1], H is not G-cr. By Proposition 10.3, H admits a finite σ-structure, $\{H_i\}$ say. By Lemma 10.4, there exists some j such that H_{j+l} is not G-cr, for all $l \geq 0$, and hence by Theorem 6.18 H_{j+l} is not strongly reductive in G. Therefore, by Proposition 9.14, for each $l \geq 0$ we have that each H_{j+l} is contained in some proper σ-stable parabolic subgroup of G. Suppose P is such a σ-stable parabolic subgroup of G containing H_{j+l} for some $l \geq 0$.

As in the proof of Lemma 10.4, list representatives of the conjugacy classes of parabolic subgroups of G by $P_1, \ldots, P_m$. Suppose that P is G-conjugate to P_k for some $k \in \{1, \ldots, m\}$, say $P = gP_kg^{-1}$ for some $g \in G$. Then, as in the proof of Lemma 10.4, $C_k(H_j) = C_k(H_{j+l})$, hence $H_{j+l} \subseteq gP_kg^{-1} = P$ for all $l > 0$. We have that $H_{j+l} \subset P$ for all $l > 0$ and so $H \subseteq P$. Furthermore, by Proposition 9.14, each H_{j+l} is not contained in any Levi subgroup of P, and so H is also not contained in any Levi subgroup of P, as required. □

We have arrived at our main result, which is an immediate consequence of Theorem 8.6 and Proposition 10.5.

Theorem 10.6. *Let G be a reductive algebraic group, and let σ be a Frobenius morphism of G. Suppose that H is a σ-stable subgroup of G, then*

(1) *H is G-completely reducible if and only if it is (G, σ)-completely reducible, and*

(2) *if H is not G-completely reducible, then H is contained in a proper σ-stable parabolic subgroup P of G and not in any Levi subgroup of P.*

Example 10.7. Care must be taken in applying Theorem 10.6, for the fact that a (G, σ)-irreducible subgroup of G may not be G-irreducible.

With reference to the notation of Example 8.2, we note that although L is (G, σ)-irreducible, it is not G-irreducible. However, as L is a Levi subgroup of G it is G-completely reducible, and since it is (G, σ)-irreducible, it is (G, σ)-completely reducible.

We now present a collection of corollaries to Theorem 10.6. In each we assume that G is a reductive algebraic group with Frobenius morphism σ. The first of the corollaries gives an understanding of the geometric nature of (G, σ)-completely reducible subgroups of G. Each is the result of a combination of Theorem 10.6 and a corresponding result from [1] or [2]. We have chosen this collection of results to provide an indication of the properties of a (G, σ)-completely reducible subgroup of G, and because they are of general interest.

Corollary 10.8. *Let H be a σ-stable subgroup of G topologically generated by $\{x_1, \ldots, x_n\}$. Then H is (G, σ)-completely reducible if and only if the orbit $G \cdot (x_1, \ldots, x_n)$ is closed in G^n.*

Proof. Follows from Theorem 10.6 and [1, Corollary 3.7]. □

Corollary 10.9. *Assume that p is good for G or $p > 3$. Let A and B be σ-stable commuting connected (G, σ)-completely reducible subgroups of G. Then AB is (G, σ)-completely reducible.*

Proof. Follows from Theorem 10.6 and [2, Corollary 4.19]. The bound on p is a result of the case by case analysis in [2]. □

Corollary 10.10. *Let H be a closed σ-stable subgroup of G and let N be a σ-stable normal subgroup of H. If H is (G, σ)-completely reducible, then so is N. In particular, H^0 is (G, σ)-completely reducible.*

Proof. Follows from Theorem 10.6 and [1, Theorem 3.10]. □

Corollary 10.11. *Let H be a (G,σ)-completely reducible subgroup of G and let K be a σ-stable closed subgroup of G satisfying $HC_G(H)^0 \subseteq K \subseteq N_G(H)$. Then, K is (G,σ)-completely reducible.*

Proof. Follows from Theorem 10.6 and [1, Corollary 3.14]. □

The following two results are immediate consequences of Corollaries 10.10 and 10.11.

Corollary 10.12. *Let H be a closed σ-stable subgroup of G. Then H is (G,σ)-completely reducible if and only if $N_G(H)$ is.*

Corollary 10.13. *Let H be a closed σ-stable subgroup of G. If H is (G,σ)-completely reducible then so is $C_G(H)$.*

The following result is an analogue of [1, Corollary 3.22] in the setting of σ-stability.

Corollary 10.14. *Let K be a closed σ-stable subgroup of a σ-stable Levi subgroup L of G. Then K is (L,σ)-completely reducible if and only if K is (G,σ)-completely reducible.*

Proof. Suppose K is (L,σ)-cr. Then, by Theorem 10.6, K is L-cr. Therefore, by [1, Corollary 3.22], K is G-cr. Thus K is (G,σ)-cr, again by Theorem 10.6.

Conversely, suppose K is (G,σ)-cr. Then, by Theorem 10.6, K is G-cr. Therefore, by [1, Corollary 3.22], K is L-cr. Thus K is (L,σ)-cr, by Theorem 8.6. □

Recall that a subgroup H of G is called **regular** if it is normalised by a maximal torus of G. The following result is an analogue of [1, Corollary 3.26] in the setting of σ-stability, and in which the restriction that p is good for G is required.

Corollary 10.15. *Suppose that p is good for G. Let K be a closed σ-stable subgroup of a σ-stable regular reductive subgroup H of G. Then K is (H,σ)-completely reducible if and only if K is (G,σ)-completely reducible.*

Proof. Suppose K is (H,σ)-cr. Then, by Theorem 10.6, K is H-cr. Therefore, by [1, Theorem 3.26], K is G-cr. Thus K is (G,σ)-cr, again by Theorem 10.6.

Conversely, suppose K is (G,σ)-cr. Then, by Theorem 10.6, K is G-cr. Therefore, by [1, Theorem 3.26], K is H-cr. Thus K is (H,σ)-cr, by Theorem 10.6. □

Example 10.16. Let $\mathrm{char}(k) = 2$. We embed the group $\mathrm{Sp}_m(k)$ diagonally in the maximal rank subgroup $\mathrm{Sp}_m(k) \times \mathrm{Sp}_m(k)$ of $\mathrm{Sp}_{2m}(k)$. Then, by [1, Example 3.45], $\mathrm{Sp}_m(k)$ is not $\mathrm{Sp}_{2m}(k)$-cr, even though $\mathrm{Sp}_m(k)$ is reductive. Let $\sigma = \sigma_{2^a}$ be a standard Frobenius morphism of $\mathrm{Sp}_{2m}(k)$, where $a \in \mathbb{N}$. Clearly, the diagonally embedded copy of $\mathrm{Sp}_m(k)$ is also σ-stable, and by Theorem 10.6, is not $(\mathrm{Sp}_{2m}(k),\sigma)$-completely reducible.

This provides an example of a reductive subgroup of $\mathrm{Sp}_{2m}(k)$ which is not $(\mathrm{Sp}_{2m}(k),\sigma)$-completely reducible.

10.2 Groups of Fixed Points

Let G be a connected reductive algebraic group over a field of characteristic p, with a Frobenius morphism σ. In this section we deal with subgroups of G^σ. For such groups, we present the following definition, which is an analogue of Definition 8.1 for subgroups of G^σ.

Definition 10.17. *Let G be a reductive algebraic group, and let σ be a Frobenius morphism of G. Let H be a subgroup of G^σ.*

(1) *We say that H is G^σ**-completely reducible** (or G^σ**-cr**) if whenever H is contained in P^σ for a σ-stable parabolic subgroup P of G, then H is contained in L^σ, for a σ-stable Levi subgroup L of P.*

(2) *We say that H is G^σ-**irreducible** (or G^σ-**ir**) if H is not contained in P^σ for any proper σ-stable parabolic subgroup P of G.*

Example 10.18. The observation made in Example 9.18 gives that G^σ is G^σ-irreducible, and is therefore trivially G^σ-completely reducible.

Proposition 10.19. *Let G be a reductive algebraic group and let H be a subgroup of G^σ. Then H is (G, σ)-completely reducible if and only if it is G^σ-completely reducible.*

Proof. Let H be (G, σ)-cr. Suppose that $H \subseteq P^\sigma$, where P is a σ-stable parabolic subgroup of G. Then $H \subseteq P$. Because H is (G, σ)-cr, $H \subseteq L$ for some σ-stable Levi subgroup L of P. Therefore $H \subseteq L^\sigma$.

Conversely, suppose that H is G^σ-cr and that $H \subseteq P$ for some σ-stable parabolic subgroup P of G. Then $H \subseteq P^\sigma$. Because H is G^σ-cr, $H \subseteq L^\sigma$ for some σ-stable Levi subgroup L of P. Hence, $H \subseteq L$, giving the result. ☐

Corollary 10.20. *Let G be a reductive algebraic group and σ a Frobenius morphism of G. The following are equivalent for a subgroup H of G^σ:*

(1) *H is G^σ-completely reducible,*

(2) *H is (G, σ)-completely reducible, and*

(3) *H is G-completely reducible.*

Proof. This follows from Theorem 10.6 and Proposition 10.19. ☐

Using the notion of G^σ-complete reducibility, we can find more examples of (G, σ)-completely reducible subgroups of G, as shown in the following.

Example 10.21. Let G be a reductive algebraic group with a Frobenius morphism σ, and let L be a σ-stable Levi subgroup of G. As in Example 10.18, L^σ is L^σ-completely reducible. By Proposition 10.19, L^σ is (L, σ)-completely reducible. Therefore, by Corollary 10.14, L^σ is (G, σ)-completely reducible.

Example 10.22. Let G be a reductive algebraic group, with a Frobenius morphism σ and suppose that p is good for G. Suppose that H is a closed σ-stable regular reductive subgroup of G. As in Example 10.18, H^σ is H^σ-completely reducible. By Proposition 10.19, H^σ is (H,σ)-completely reducible. Therefore, by Corollary 10.15, H^σ is (G,σ)-completely reducible.

10.3 Strong σ-Reductivity in G

We define an analogue in the setting of σ-stability to Richardson's notion of strong reductivity, see [42, §16], which we discussed in §6.4.

Definition 10.23. *A σ-stable subgroup H of G is **strongly σ-reductive in** G if H is not contained in any proper σ-stable parabolic subgroup of $C_G(S)$, where S is a σ-stable maximal torus of $C_G(H)$.*

Remark 10.24. Note that in Definition 10.23 it makes sense to require S to be a σ-stable maximal torus of $C_G(H)$ since such an S exists, by Corollary 7.14.

Theorem 6.18 shows that the notions of G-complete reducibility and strong reductivity are equivalent. In the following we generalise this result to the σ-stability setting. The proof of the forward direction was provided by Michael Bate, Tim Burness and Martin Liebeck.

Theorem 10.25. *Let H be a σ-stable subgroup of G. Then, H is strongly σ-reductive in G if, and only if, it is (G,σ)-completely reducible.*

Proof. Suppose that H is strongly σ-reductive in G; so H is not contained in any proper σ-stable parabolic subgroup of $C_G(S)$ where $S \subseteq C_G(H)$ is a σ-stable maximal torus. Then, H is $(C_G(S),\sigma)$-ir, and thus is $(C_G(S),\sigma)$-cr. Therefore, by Theorem 10.6, H is $C_G(S)$-cr, and by [1, Corollary 3.5] is G-cr. As H is σ-stable, by Theorem 8.6, it is (G,σ)-cr.

Suppose that H is (G, σ)-cr. By Theorem 10.6, H is G-cr. Pick any σ-stable maximal torus S_1 of $C_G(H)$, then by [1, Corollary 3.5] H is $C_G(S_1)$-ir. Therefore, H is $(C_G(S_1), \sigma)$-ir. This gives that H is not contained in any proper σ-stable parabolic subgroup of $C_G(S_1)$ where S_1 is any σ-stable maximal torus of $C_G(H)$. Thus, H is strongly σ-reductive in G. $\square$

Remark 10.26. We note that the proof of Theorem 10.25 shows that Definition 10.23 is independent of the choice of σ-stable maximal torus of $C_G(H)$.

We have the following analogue of [42, Lemma 16.3], which justifies the use of the terminology strongly σ-reductive.

Lemma 10.27. *Suppose that H is strongly σ-reductive in G. Then H is reductive.*

Proof. Let S be a σ-stable maximal torus of $C_G(H)$. We have that $C_G(S)$ is reductive. Suppose that $R_u(H) \neq e$. Since H is σ-stable, so is $R_u(H)$. Then, by the construction of Borel-Tits, there is a proper σ-stable parabolic subgroup P of $C_G(S)$ such that $R_u(H) \subseteq R_u(P)$, and $H \subseteq P$. However, this contradicts the hypothesis. Hence H is reductive. $\square$

Part III

Complete Reducibility for Lie Algebras

11 Complete Reducibility for Lie Algebras

Let G be a reductive algebraic group. In this section we analyse the notion of G-complete reducibility for a Lie subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$, which is due to McNinch, see [37]. In Theorem 11.38, we obtain an analogue in the Lie algebra setting of Theorem 6.28. As a consequence of this result, we obtain Corollary 11.40, which demonstrates that if G is a simple algebraic group then any $\mathrm{Ad}(G)$-invariant ideal in $\mathfrak{g}$ is G-completely reducible.

Notation 11.1. In this section algebraic groups will be represented with capital Roman letters, $G, H, K, \ldots$, and to each group the corresponding Lie algebra will be denoted by the same letter in Gothic $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}, \ldots$.

From §5.8 we have that the adjoint representation of G gives an action of G on $\mathfrak{g}$ given by $\mathrm{Ad}(g) : X \mapsto \mathrm{Ad}(g)(X)$ for all $g \in G, X \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$, and let "0" denote the identity element of the Lie algebra $\mathfrak{g}$. A Lie subalgebra of $\mathfrak{g}$ is called a parabolic (resp. Levi) subalgebra if it is the Lie algebra of a parabolic (resp. Levi) subgroup of G.

11.1 G-Complete Reducibility for Lie Algebras

Let G be a reductive algebraic group over an algebraically closed field k, and let H be a closed subgroup of G. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$, as defined in §5.7. If H is reductive then $\mathfrak{h}$ is called **reductive**.

We combine some definitions and results of [42, §2], and [1, Lemma 2.4]. We note that we originally introduced the definitions of $P_\lambda, L_\lambda, R_u(P_\lambda)$, and the map $c_\lambda : P_\lambda \to L_\lambda$ in §6.2.

Definition 11.2. *Let $\lambda \in Y(G)$, and $x \in k^*$.*

(1) $P_\lambda := \{g \in G \mid \lim_{x\to 0} \lambda(x) \cdot g$ *exists*$\}$ *is a parabolic subgroup of G.*

(2) $L_\lambda := \{g \in G \mid \lim_{x\to 0} \lambda(x) \cdot g = g\}$ *is a Levi subgroup of P_λ.*

(3) $U_\lambda := \{g \in G \mid \lim_{x\to 0} \lambda(x) \cdot g = e\} = R_u(P_\lambda)$.

(4) $\mathfrak{p}_\lambda := \{X \in \mathfrak{g} \mid \lim_{x\to 0} \mathrm{Ad}(\lambda(x))X \text{ exists}\} = \mathrm{Lie}(P_\lambda)$ *is a parabolic subalgebra of* $\mathfrak{g}$.

(5) $\mathfrak{l}_\lambda := \{X \in \mathfrak{g} \mid \lim_{x\to 0} \mathrm{Ad}(\lambda(x))X = X\} = \mathrm{Lie}(L_\lambda)$ *is a Levi subalgebra of* $\mathfrak{g}$.

(6) $\mathfrak{u}_\lambda := \{X \in \mathfrak{g} \mid \lim_{x\to 0} \mathrm{Ad}(\lambda(x))X = 0\} = \mathrm{Lie}(U_\lambda)$.

For $g \in P_\lambda$ the map $c_\lambda : P_\lambda \to L_\lambda$ given by

$$c_\lambda(g) = \lim_{x\to 0} \lambda(x) g \lambda(x)^{-1}$$

is a surjective homomorphism of algebraic groups. Clearly, $R_u(P_\lambda)$ is the kernel of the map c_λ. Corresponding to the group case, for $X \in \mathfrak{p}_\lambda$ define the projection $c_\lambda : \mathfrak{p}_\lambda \to \mathfrak{l}_\lambda$ given by

$$c_\lambda(X) = \lim_{x\to 0} \mathrm{Ad}(\lambda(x))X.$$

Since the limit $\lim_{x\to 0} \mathrm{Ad}(\lambda(x))X$ exists we have that $\mathrm{Ad}(\lambda(x)) \in \mathrm{GL}(\mathfrak{g})$, for all $x \in k$, and so $\mathrm{Ad}(\lambda(x))$ is an automorphism of $\mathfrak{g}$.

Thus, c_λ preserves the Lie bracket and as such c_λ is a homomorphism of Lie algebras. As in the group case, the kernel of the map c_λ is $\mathfrak{u}_\lambda$ and we have the decomposition $\mathfrak{p}_\lambda = \mathfrak{l}_\lambda \oplus \mathfrak{u}_\lambda$, for details see, for instance, [27, Equation 5.93(b)].

Suppose that G and H are reductive groups with $H \subseteq G$. Recall that each one-parameter subgroup $\lambda \in Y(H)$ may be considered as a cocharacter of G. Therefore, corresponding to λ is one parabolic subgroup $P_\lambda(H)$ of H and one parabolic subgroup $P_\lambda(G)$ of G with $P_\lambda(H) = P_\lambda(G) \cap H$. For details see [1, Corollary 2.5]. We will write P_λ for $P_\lambda(G)$, and will only write $P_\lambda(H)$ when H is a proper subgroup of G.

We denote $\mathfrak{p}_\lambda(H) = \mathrm{Lie}(P_\lambda(H))$ and $\mathfrak{p}_\lambda(G) = \mathrm{Lie}(P_\lambda(G))$. We have $\mathfrak{p}_\lambda(H)$ is the Lie algebra of a parabolic subgroup of H and $\mathfrak{p}_\lambda(G)$ is the Lie algebra of a parabolic subgroup of G. As in the group case, we will write $\mathfrak{p}_\lambda$ for $\mathfrak{p}_\lambda(G)$, and will only write $\mathfrak{p}_\lambda(H)$ when H is a proper subgroup of G.

Following [37], we give the following definition.

Definition 11.3. *Let G be a reductive algebraic group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$.*

(1) *We say that $\mathfrak{h}$ is G-**completely reducible** (or G-**cr**) if whenever $\mathfrak{h} \subseteq \mathrm{Lie}(P)$ for some parabolic subgroup P of G, then $\mathfrak{h} \subseteq \mathrm{Lie}(L)$ for some Levi subgroup L of P.*

(2) *We say that $\mathfrak{h}$ is G-**irreducible** (or G-**ir**) if $\mathfrak{h}$ is not contained in the Lie algebra of any proper parabolic subgroup of G.*

(3) *We say that $\mathfrak{h}$ is G-**indecomposable** (or G-**ind**) if $\mathfrak{h}$ is not contained in the Lie algebra of any proper Levi subgroup of G.*

Remark 11.4. Suppose that G is a non-connected algebraic group, with subgroup H. Then, the Lie subalgebras of G coincide with those of G^0. Therefore, $\mathrm{Lie}(H)$ is G-completely reducible if and only if it is G^0-completely reducible.

Remark 11.5. Let H be a closed subgroup of $G = \mathrm{GL}(V)$ where V is a finite dimensional vector space. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. We claim that $\mathfrak{h}$ is $\mathrm{GL}(V)$-completely reducible if and only if V is a semisimple $\mathfrak{h}$-module. This is an analogous result in the Lie algebra case to Lemma 6.11, and can be seen as follows.

A parabolic subgroup P of $\mathrm{GL}(V)$ is the stabiliser of a flag $\mathcal{F} := (V_1, \ldots, V_m)$ of subspaces $\{0\} \neq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m$ of V, where $m \leq \dim(V)$. By [6, Theorem 5.1], the Lie algebra $\mathfrak{p} = \mathrm{Lie}(P)$ also stabilises the flag $\mathcal{F}$ of V.

As in Lemma 6.11, we may choose a complement W_i to V_{i-1} in V_i such that $V_i = V_{i-1} \oplus W_i$. Then a Levi subgroup L of P is isomorphic to $\mathrm{GL}_{n_1}(k) \times$

$\cdots \times \mathrm{GL}_{n_m}(k)$, where each $n_i = \dim(W_i)$. To each subspace U of V that is stabilised by L, there is a complement to U in V that is also stabilised by L. Thus V is a semisimple L-module.

We have $\mathfrak{l} = \mathrm{Lie}(L)$ is isomorphic to the direct sum $\bigoplus_i \mathfrak{gl}_{n_i}(k)$ in $\mathfrak{p}$, and V is a semisimple $\mathfrak{l}$-module.

Suppose that $\mathfrak{h}$ is G-completely reducible, and stabilises a subspace U of V. Then $\mathfrak{h}$ is contained in the Lie algebra $\mathfrak{p}$ of a parabolic subgroup P of G that also stabilises U. As $\mathfrak{h}$ is G-completely reducible, $\mathfrak{h}$ is contained in the Lie algebra $\mathfrak{l}$ of some Levi subgroup L of P. Each Levi subalgebra of $\mathfrak{p}$ stabilises U, and a complement to U. As U has an $\mathfrak{l}$-stable complement and $\mathfrak{h} \subseteq \mathfrak{l}$, it has the same $\mathfrak{h}$-stable complement. Thus, V is a semisimple $\mathfrak{h}$-module.

Conversely, suppose that V is a semisimple $\mathfrak{h}$-module, and that $\mathfrak{h}$ is contained in the Lie algebra $\mathfrak{p}$ of a parabolic subgroup P of G. Since $\mathfrak{p}$ acts on V by stabilising a flag $(V_1, \ldots, V_n)$, we have that $\mathfrak{h}$ also stabilises $(V_1, \ldots, V_n)$. Since V is a semisimple $\mathfrak{h}$-module $\mathfrak{h}$ is of block diagonal form, and so is contained in a Levi subalgebra of $\mathfrak{p}$.

We discuss the corresponding situation when $G = \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ in Remark 11.29.

The following is an analogue of [1, Corollary 2.7].

Lemma 11.6. *Let H be a reductive subgroup of G. Suppose that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and is contained in $\mathfrak{h} = \mathrm{Lie}(H)$. Then:*

(1) *if $\mathfrak{k}$ is G-irreducible, it is H-irreducible, and*

(2) *if $\mathfrak{k}$ is G-indecomposable, it is H-indecomposable.*

Proof. (1) Suppose that $\mathfrak{k}$ is contained in the Lie algebra $\mathrm{Lie}(Q)$ of a proper parabolic subgroup Q of H. Then, since H is reductive in G, by [1, Corollary

2.5], there exists a proper parabolic subgroup P of G such that $Q \subseteq P$, and $\mathfrak{k} \subseteq \mathrm{Lie}(P)$, a contradiction. Hence, $\mathfrak{k}$ is in no such Lie algebra of $\mathfrak{h}$, i.e. $\mathfrak{k}$ is H-ir.

(2) Suppose that $\mathfrak{k}$ is contained in the Lie algebra $\mathrm{Lie}(M)$ of a Levi subgroup M of a proper parabolic subgroup Q of H. Then, since H is reductive in G, by [1, Corollary 2.5], there exists a proper parabolic subgroup P of G such that $Q \subseteq P$, and $M \subseteq L$ for L a Levi subgroup of P. Thus $\mathfrak{k} \subseteq \mathrm{Lie}(L)$, a contradiction. Hence, $\mathfrak{k}$ is in no such Lie algebra of $\mathfrak{h}$, i.e. $\mathfrak{k}$ is H-ind. □

Remark 11.7. Let H be a closed subgroup of G. If $\mathfrak{h}$ is G-irreducible, then so is H. This is easy to see. Indeed if H is not G-irreducible then $H \subseteq P$ for some proper parabolic subgroup P of G. Then $\mathfrak{h} \subseteq \mathrm{Lie}(P)$, a contradiction.

Notation 11.8. We consider the simultaneous adjoint action of G on $\mathfrak{g}^n$ for $g \in G$ and $\mathbf{X} := (X_1, \ldots, X_n) \in \mathfrak{g}^n$ by:

$$\mathrm{Ad}(g)(X_1, \ldots, X_n) = (\mathrm{Ad}(g)X_1, \ldots, \mathrm{Ad}(g)X_n).$$

Following Richardson [42], we denote the Lie subalgebra of $\mathfrak{g}$ generated by the X_i by $\mathfrak{a}(\mathbf{X})$.

Let $\mathbf{X} \in \mathfrak{g}^n$ and set $\mathfrak{h} = \mathfrak{a}(\mathbf{X})$. Recall the centraliser in G of $\mathfrak{h}$ is the set $C_G(\mathfrak{h}) = \{g \in G \mid \mathrm{Ad}(g)X = X \text{ for all } X \in \mathfrak{h}\}$. For $X \in \mathfrak{g}$, we denote the stabiliser in G of X by G_X. Note that $C_G(\mathfrak{h}) = G_{\mathbf{X}}$.

Let H be a closed subgroup of G. Set $\mathfrak{c}_{\mathfrak{g}^n}(H) := \{\mathbf{X} \in \mathfrak{g}^n \mid \mathrm{Ad}(h)\mathbf{X} = \mathbf{X} \text{ for all } h \in H\}$. Finally, we set $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) := \{X \in \mathfrak{g} \mid [Y, X] = 0 \text{ for all } Y \in \mathfrak{h}\}$.

The next result is an adaptation of [37, Theorem 1.1], due to McNinch. In [37], the tuple $\mathbf{X}$ is taken to be a basis of $\mathfrak{h}$, however we note that the result holds also when $\mathbf{X}$ is a generating tuple of $\mathfrak{h}$ (that is for $\mathbf{X}$ such that $\mathfrak{a}(\mathbf{X}) = \mathfrak{h}$). This is because the tuple $\mathbf{X}$ is used in the proof to generate the Lie algebra $\mathfrak{h}$, see [37, Proof of Theorem 1], much like we have used a generating tuple below in Theorem 11.38. Therefore, this result is stated in this more general form here. See also [3, Theorem 5.30] for a different approach to part (2) of this theorem.

Theorem 11.9. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{a}(\mathbf{X})$ for some $\mathbf{X} \in \mathfrak{g}^n$. Then:*

(1) *The Lie algebra $\mathfrak{h}$ is G-completely reducible if and only if the G-orbit of $\mathbf{X}$ is closed in $\mathfrak{g}^n$.*

(2) *Let H be a closed subgroup of G, and let $\mathfrak{h} = \mathrm{Lie}(H)$. If H is G-completely reducible, then $\mathfrak{h}$ is G-completely reducible.*

Remark 11.10. Part (1) is proved in [37] using techniques similar to those of Richardson, see [42].

Once Part (1) is known, Part (2) is proved in the following way. Let S be a maximal torus of $C_G(H)$. Then, $H \subseteq L := C_G(S)$, and so $\mathfrak{h} \subseteq \mathrm{Lie}(L)$. By [37, Lemma 2] it is sufficient to show that $\mathfrak{h}$ is L-completely reducible. Since S was chosen to be maximal, we have that H is not contained in any proper Levi subgroup of L. Since H is G-completely reducible, H is not contained in any proper parabolic subgroup of L.

By Theorem 11.9 (1), to show that $\mathfrak{h}$ is L-completely reducible, it is sufficient to show that $\mathrm{Ad}(L)(\mathbf{X})$ is closed in $\mathrm{Lie}(L)^n$, where $\mathbf{X} = (X_1, \ldots, X_n)$ is a generating tuple of $\mathfrak{h}$. If $\mathrm{Ad}(L)(\mathbf{X})$ is not closed, then the boundary $S = \overline{\mathrm{Ad}(L)(\mathbf{X})} - \mathrm{Ad}(L)(\mathbf{X})$ is non empty, and we can refer to Theorem 6.25 to obtain that the destabilising parabolic subgroup $P_{S,\mathbf{X}}$ is a proper parabolic subgroup of L. Since $\mathrm{Ad}(H)$ leaves $\mathfrak{h}$ invariant, by [37, Corollary 7], we obtain that $h \cdot P_{S,\mathbf{X}} = P_{S,\mathrm{Ad}(h)\cdot\mathbf{X}} = P_{S,\mathbf{X}}$, for all $h \in H$. As a parabolic subgroup is its own normaliser, we conclude that $H \subseteq P_{S,\mathbf{X}}$. This contradicts our assumption, and so we conclude that $\mathfrak{h}$ is L-completely reducible, and therefore G-completely reducible.

Example 11.11. By [1, Corollary 3.22], any Levi subgroup L of G is G-completely reducible. Therefore, by Theorem 11.9, any Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ is G-completely reducible.

Example 11.12. Let H be a G-completely reducible subgroup. Then, by [1, Corollaries 3.16, 3.17], $C_G(H)$ and $N_G(H)$ are both G-completely reducible. Hence, by Theorem 11.9 (2), $\mathrm{Lie}(C_G(H))$ and $\mathrm{Lie}(N_G(H))$ are both G-completely reducible.

We present the following examples which show that the converse to Theorem 11.9 (2) does not always hold even in the case when H is connected.

Example 11.13. Any finite unipotent subgroup U of G is not G-completely reducible since, by [23, §30.3], $U \subseteq R_u(P)$ for some parabolic subgroup P of G, and U is not contained in any Levi subgroup of P. However, as U is finite, its Lie algebra is trivial and is therefore trivially G-completely reducible.

The following example is taken from [37] and is attributed to Ben Martin.

Example 11.14. Let H be a semisimple group. Let $\rho_i : H \to \mathrm{SL}(V_i)$ for $i = 1, 2$, be two representations of H with ρ_1 semisimple and ρ_2 not. Consider the representation $\rho : H \to \mathrm{SL}(V_1 \oplus V_2)$ given by $\rho(h) \mapsto \rho_1(h) \oplus \rho_2(\sigma(h))$ where σ is a standard Frobenius morphism, and set $G := \mathrm{SL}(V_1 \oplus V_2)$. Then $\rho(H)$ is not G-completely reducible, since $V_1 \oplus V_2$ is not a semisimple $\rho(H)$-module. Recall that the differential map $\partial_e(\rho)$ of ρ at e introduced in §5.5 maps $\mathfrak{h}$ to $\mathrm{Lie}(\mathrm{SL}(V_1 \oplus V_2))$. The Lie algebra $\mathrm{Lie}(\rho(H)) = \mathrm{im}(\partial_e \rho)$ lies in the Lie algebra of $M = \mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$ (which is the semisimple part of a Levi subgroup of G), and $\mathrm{im}(\partial_e \rho) = \mathrm{im}(\partial_e \rho_1) \oplus \mathrm{im}(\partial_e \rho_2 \circ \sigma) = \mathrm{im}(\partial_e \rho_1) \oplus 0$ lies in $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$. We justify the last equality as follows. By [23, §5.4] ∂_e acts on the functions that define $\sigma(\mathrm{SL}(V_2))$ by taking all partial derivatives of these functions, all of which involve a p-power. Hence, this factor vanishes in the above. By [1, Lemma 2.12 (i)], the image of $\rho_1 \times 1 : H \to M$ is M-completely reducible and, by Theorem 11.9 (i) $\mathrm{Lie}(\rho(H))$ is M-completely reducible. Hence, by [37, Lemma 2] $\mathrm{Lie}(\rho(H)) = \mathrm{im}(\partial_e\rho_1 \times 1)(H)$ is G-completely reducible.

The following is an analogue in the Lie algebra setting of the notion of a strongly reductive subgroup of G due to Richardson, [42].

Definition 11.15. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We say $\mathfrak{h}$ is **strongly reductive in** G if $\mathfrak{h}$ is not contained in the Lie algebra of any proper parabolic subgroup of $C_G(S)$, where S is a maximal torus of $C_G(\mathfrak{h})$.*

We follow the argument of Richardson [42, §16] to obtain the following result.

Theorem 11.16. *Let $\mathbf{X} \in \mathfrak{g}^n$. Then $\mathfrak{a}(\mathbf{X})$ is strongly reductive in G if and only if the orbit $\mathrm{Ad}(G)\mathbf{X}$ is closed in $\mathfrak{g}^n$.*

Recall from Definition 6.21, that for a G-variety X, and $Z = \cap_{x\in X} C_G(x)$, we say $x \in X$ is a **stable point** for the action of G if the orbit $G \cdot x$ is closed in X and $C_G(x)/Z$ is finite.

Proposition 11.17. *Let $\mathbf{X} \in \mathfrak{g}^n$. Then $\mathbf{X}$ is a stable point of $\mathfrak{g}^n$ if and only if $\mathfrak{a}(\mathbf{X})$ is not contained in the Lie algebra of any proper parabolic subgroup of G.*

Proof. We sketch the proof of this result. Suppose $\mathfrak{a}(\mathbf{X})$ is not contained in the Lie algebra of any proper parabolic subgroup of G. We can conclude, as in the proof of Proposition 6.23, that $\mathrm{Ad}(G)\mathbf{X}$ is closed and affine. Therefore, by [41, Theorem A], $G^0_{\mathbf{X}}$ is reductive. Let S be a maximal torus of $G_{\mathbf{X}}$, then by following the argument in 6.23 we can conclude that $G^0_{\mathbf{X}} = S$.

To conclude that $\mathbf{X}$ is a stable point we need that $G_{\mathbf{X}}/Z$ is finite where $Z = \cap_{\mathbf{Y}\in\mathfrak{g}^n} G_{\mathbf{Y}}$. We have that $G_{\mathbf{X}}/S$ is finite. Because $S \subseteq Z(G)^0$, and $Z(G)^0$ is the kernel of the adjoint representation, we have $S \subseteq Z$. Since $G_{\mathbf{X}}/Z$ is a quotient of $G_{\mathbf{X}}/S$, it follows that $G_{\mathbf{X}}/Z$ is also finite.

Conversely, suppose there is some non-central $\lambda \in Y(G)$ with $\mathfrak{a}(\mathbf{X}) \subseteq \mathfrak{p}_\lambda = \mathrm{Lie}(P_\lambda)$, for P_λ a proper parabolic subgroup of G. Let $\mathbf{X} = (X_1, \ldots, X_n)$, and set $\mathbf{Y} = (\lim_{x\to 0} \mathrm{Ad}(\lambda(x))X_1, \ldots, \lim_{x\to 0} \mathrm{Ad}(\lambda(x))X_n)$. Then, we have $\lambda(k^*) \subseteq G^0_{\mathbf{Y}}$, and hence $\lambda \in Y(G_{\mathbf{Y}})$.

Since P_λ is proper in G, so L_λ is also proper in G, and hence $\mathfrak{l}_\lambda = \mathrm{Lie}(L_\lambda)$ is proper in $\mathfrak{g}$. We have that $\mathfrak{l}_\lambda$ consists of the elements of $\mathfrak{g}$ which are fixed by $\mathrm{Ad}(\lambda(x))$ for all $x \in k^*$. As $\mathrm{Ad}(\lambda(k^*))$ does not fix all of $\mathfrak{g}$, $\lambda(k^*)$ is not contained in Z.

Hence, $Z^0 \subsetneq G^0_{\mathbf{Y}}$. These two groups must therefore have different dimensions, because they are connected, and so $G^0_{\mathbf{Y}}/Z^0$ is infinite. Thus $\mathbf{Y}$ is not a stable point.

If $\mathbf{Y} \in \mathrm{Ad}(G)\mathbf{X}$, then the quotient $G_{\mathbf{X}}/Z$ is also infinite since $G_{\mathbf{X}}$ and $G_{\mathbf{Y}}$ are conjugate in G and, as before, $\mathbf{X}$ is not a stable point. Now, suppose $\mathbf{Y} \notin \mathrm{Ad}(G)\mathbf{X}$, then the orbit of $\mathbf{X}$ is not closed because it does not contain the limit $\mathbf{Y} = \lim_{x\to 0} \mathrm{Ad}(\lambda(x))\mathbf{X}$. Again, we conclude that $\mathbf{X}$ is not a stable point of $\mathfrak{g}^n$. □

We state two preliminary results of Richardson before giving the proof of Theorem 11.16. The first is a special case of [43, Theorem C], and the second is [42, Lemma 16.6].

Lemma 11.18. *Let S be a linearly reductive subgroup of G and let $\mathbf{X} \in \mathfrak{c}_{\mathfrak{g}^n}(S)$. Then $\mathrm{Ad}(G)\mathbf{X}$ is closed in $\mathfrak{g}^n$ if and only if $\mathrm{Ad}(C_G(S))\mathbf{X}$ is closed in $\mathfrak{c}_{\mathfrak{g}^n}(S)$.*

Lemma 11.19. *Let $\mathbf{X} \in \mathfrak{g}^n$ and S be a maximal torus of $G_{\mathbf{X}}$. Then $\mathrm{Ad}(G)\mathbf{X}$ is closed in $\mathfrak{g}^n$ if and only if $\mathbf{X}$ is a stable point for the action of $C_G(S)$ on $\mathfrak{c}_{\mathfrak{g}^n}(S)$.*

We now prove Theorem 11.16.

Proof. Let S be a maximal torus of $G_{\mathbf{X}}$. Suppose that $\mathfrak{a}(\mathbf{X})$ is strongly reductive in G. Since $\mathbf{X} \in \mathfrak{c}_{\mathfrak{g}}(S)^n$ we have $\mathfrak{a}(\mathbf{X}) \subseteq \mathfrak{c}_{\mathfrak{g}}(S)$ and by [43, Lemma 4.1] $\mathfrak{c}_{\mathfrak{g}}(S) = \mathrm{Lie}(C_G(S))$. Hence, $\mathfrak{a}(\mathbf{X}) \subseteq \mathrm{Lie}(C_G(S))$. As $\mathfrak{a}(\mathbf{X})$ is strongly reductive in G, it is not contained in the Lie algebra of any proper parabolic subgroup of $C_G(S)$. By Proposition 11.17 applied to $C_G(S)$, we have $\mathbf{X}$ is a stable point for the action of $C_G(S)$ on $\mathfrak{c}_{\mathfrak{g}}(S)^n$. Therefore, the orbit $\mathrm{Ad}(C_G(S))\mathbf{X}$ is closed in $\mathfrak{c}_{\mathfrak{g}}(S)^n = \mathfrak{c}_{\mathfrak{g}^n}(S)$, and hence by Lemma 11.18, $\mathrm{Ad}(G)\mathbf{X}$ is closed in $\mathfrak{g}^n$.

For the converse, suppose that $\mathrm{Ad}(G)\mathbf{X}$ is closed in $\mathfrak{g}^n$. Then, by Lemma 11.19, $\mathbf{X}$ is a stable point for the action of $C_G(S)$ on $\mathfrak{c}_{\mathfrak{g}^n}(S) = \mathfrak{c}_{\mathfrak{g}}(S)^n$. Hence, by Proposition 11.17, $\mathfrak{a}(\mathbf{X})$ is not contained in the Lie algebra of any proper parabolic subgroup of $C_G(S)$, in other words, $\mathfrak{a}(\mathbf{X})$ is strongly reductive in G. $\square$

The three results that follow are corollaries of the preceding results, and constitute analogues in the Lie algebra setting of [1, Theorem 3.1], [1, Corollary 3.5] and [1, Corollary 3.21], respectively.

Corollary 11.20. *A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is G-completely reducible if and only if it is strongly reductive in G.*

Remark 11.21. In the algebraic group setting, the equivalence between G-complete reducibility and strong reductivity in G follows from group theoretic methods. The equivalence leads to the 'geometric approach' to G-complete reducibility developed by Bate, Martin and Röhrle, described in [1]. This constitutes a new method that is available to tackle problems in G-complete reducibility and gives rise to a number of results, such as the fact in [1, Corollary 3.7] which links G-completely reducible subgroups of G to closed G-orbits in G^n.

In the Lie algebra setting, we require Theorem 11.16 in order to show that the equivalence between G-complete reducibility and strong reductivity holds. We note that Theorem 11.16 follows from the geometric methods of Richardson [42].

Corollary 11.22. *A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is G-completely reducible if and only if it is $C_G(S)$-irreducible where S is a maximal torus of $C_G(\mathfrak{h})$.*

Recall that a subgroup H of G is called **linearly reductive** if all of its representations are semisimple.

Corollary 11.23. *Let S be a linearly reductive subgroup of G and let $\mathbf{X} \in \mathfrak{c}_{\mathfrak{g}^n}(S)$. Then, $\mathfrak{a}(\mathbf{X})$ is G-completely reducible if and only if it is $C_G(S)$-completely reducible.*

Proof. By Lemma 11.18, $\mathrm{Ad}(G)\mathbf{X}$ is closed in $\mathfrak{g}^n$ if and only if $\mathrm{Ad}(C_G(S))\mathbf{X}$ is closed in $\mathfrak{c}_{\mathfrak{g}^n}(S)$. Thus, since $\mathfrak{c}_{\mathfrak{g}^n}(S) = \mathfrak{c}_{\mathfrak{g}}(S)^n$ we see that $\mathfrak{a}(\mathbf{X})$ is G-cr if and only if it is $C_G(S)$-cr. $\square$

The reverse direction of the following proposition provides a condition for the converse of Theorem 11.9 (2) to hold.

Proposition 11.24. *Let H be a closed subgroup of G such that H is contained in $C_G(S)$ where S is a maximal torus of $C_G(\mathfrak{h})$. Then H is G-completely reducible if and only if $\mathfrak{h}$ is G-completely reducible.*

Proof. Suppose that H is G-cr. Then, by Theorem 11.9, $\mathfrak{h}$ is G-cr.

For the converse, suppose $\mathfrak{h}$ is G-cr. By Corollary 11.22, this is true if and only if $\mathfrak{h}$ is $C_G(S)$-ir, where S is a maximal torus of $C_G(\mathfrak{h})$. Since $H \subseteq C_G(S)$ by hypothesis, Remark 11.7 gives that H is $C_G(S)$-ir. Since $S \subseteq C_G(H)$ we have that S is a maximal torus of $C_G(H)$, and so H is G-cr, giving the result. $\square$

Remark 11.25. Let G be a reductive group, and let $\mathfrak{g} = \mathrm{Lie}(G)$. Suppose that H is a subgroup of G, and let S be a torus in $C_G(H)$. Denote $\mathrm{Lie}(H) =$

$\mathfrak{h}$. Then by [6, §8.17] we have that $\mathfrak{h} \subseteq \mathfrak{c}_{\mathfrak{g}}(S) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(s)X = X \text{ for all } s \in S\}$. That is $S \subseteq C_G(H)$ implies $S \subseteq C_G(\mathfrak{h})$.

Remark 11.26. We can replace the condition that H is contained in $C_G(S)$ where S is a maximal torus of $C_G(\mathfrak{h})$ with the stronger condition that $C_G(H) = C_G(\mathfrak{h})$ in Proposition 11.24. This is because if S is a maximal torus of $C_G(\mathfrak{h})$, and $C_G(H) = C_G(\mathfrak{h})$, then $H \subseteq C_G(S)$.

Example 11.27. Let H be a connected subgroup of G that is not G-completely reducible, such that $\mathrm{Lie}(H)$ is G-completely reducible. For instance we may take for H and G the setup described in Example 11.14. Let S be a maximal torus of $C_G(H)$ contained in a maximal torus T of $C_G(\mathfrak{h})$. Then we may conclude that:

(1) S is properly contained in T, and

(2) H is not contained in the normaliser of T.

The following result follows immediately from [42, 16.8] together with Theorem 11.9.

Lemma 11.28. *Let G be a reductive algebraic group, and let S be a linearly reductive group that acts on G. Let $\mathfrak{g} = \mathrm{Lie}\, G$, and let $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathfrak{c}_{\mathfrak{g}^n}(S)$ generate the Lie algebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h}$ is G-completely reducible if, and only if, $\mathfrak{h}$ is $C_G(S)$-completely reducible.*

Remark 11.29. Suppose that $\mathrm{char}(k) \neq 2$, and let V be an n-dimensional vector space over k. Let $\tau : \mathrm{GL}(V) \to \mathrm{GL}(V)$ be a non-trivial graph automorphism. Then τ is of the form $\tau(g) = A(g^T)^{-1}A^{-1}$, where by a suitable choice of basis for V, A has the anti-diagonal form $A = \begin{pmatrix} 0 & \cdots & \epsilon_n \\ \vdots & \diagup & \vdots \\ \epsilon_1 & \cdots & 0 \end{pmatrix}$, $\epsilon_i = \pm 1$, and A is either skew symmetric or symmetric. In either case $\tau^2 = e$.

By [51, §11], if A is skew symmetric, then n is even, and $\mathrm{GL}(V)^\tau = \mathrm{Sp}(V)$, and if A is symmetric, then $(\mathrm{GL}(V)^\tau)^0 = \mathrm{SO}(V)$.

Let $S = \langle \tau \rangle$ be the subgroup of $\mathrm{Aut}(\mathrm{GL}(V))$ generated by the automorphism τ. Then $C_{\mathrm{GL}(V)}(S) = \{g \in \mathrm{GL}(V) \mid \tau(g) = g\} = \mathrm{GL}(V)^\tau$.

Since $|S|$ does not divide $\mathrm{char}(k)$, Maschke's Theorem gives that all of its rational representations are semisimple. Therefore S is linearly reductive. We apply Lemma 11.28 with $G = \mathrm{GL}(V)$, and S as above. This gives that a subalgebra $\mathfrak{h}$ of $\mathrm{Lie}(\mathrm{GL}(V)^\tau)$ is $\mathrm{GL}(V)^\tau$-completely reducible if and only if $\mathfrak{h}$ is $\mathrm{GL}(V)$-completely reducible, and, by Remark 11.4, $\mathfrak{h}$ is $\mathrm{GL}(V)^\tau$-completely reducible if and only $\mathfrak{h}$ is $(\mathrm{GL}(V)^\tau)^0$-completely reducible.

Therefore, by Remark 11.5, when $\mathrm{char}(k) \neq 2$ we have $\mathfrak{h}$ is $\mathrm{Sp}(V)$ (resp. $\mathrm{SO}(V)$)-completely reducible if and only if V is a semisimple $\mathfrak{h}$-module.

The next proposition shows that for a subgroup K of G, if $C_G(\mathfrak{k})$ and $C_G(K)$ share a common maximal torus, then there is a certain class of Levi subgroups of G for which the two implications of [37, Lemma 2] are equivalent. The proof below was provided by Michael Bate and Tim Burness.

Proposition 11.30. *Let K be a subgroup of G such that $C_G(\mathfrak{k})$ and $C_G(K)$ contain a common maximal torus T. Let $H = C_G(S)$, for $S \subseteq T$. Then the following are equivalent:*

(1) *K is G-completely reducible,*

(2) *K is H-completely reducible,*

(3) *$\mathfrak{k}$ is G-completely reducible, and*

(4) *$\mathfrak{k}$ is H-completely reducible.*

Proof. Let T be a common maximal torus of both $C_G(\mathfrak{k})$ and $C_G(K)$, and let $S \subseteq T$ be a torus of $C_G(K)$.

We have that (1) and (2) are equivalent by [1, Corollary 3.22]. The equivalence between (1) and (3) is given by Proposition 11.24 applied to G. Finally, since $T \subseteq H$, we have that T is a common maximal torus of

$C_H(K)$ and $C_H(\mathfrak{k})$, and so the equivalence between (2) and (4) is given by Proposition 11.24. $\square$

11.2 Separability

In this section we introduce the notion of separability for Lie algebras.

An extension E of a field F, denoted $E \supseteq F$, is said to be **separable** if for each $x \in E$, the minimal polynomial of x over F is a separable polynomial (i.e. has distinct roots in E).

Consider the morphism $\phi : X \to Y$ of irreducible varieties. As discussed in §5.2, the comorphism $\phi^* : k[Y] \to k[X]$ induces an embedding of $\phi^*(k(Y))$ in $k(X)$. If $k(X) \supseteq \phi^*(k(Y))$ is a separable extension of fields, then ϕ is said to be **separable**.

Suppose that H is topologically generated by the elements $x_1, \ldots, x_n$ in G, and let $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$. It is shown in [6, Proposition 6.7] that the orbit map $\mu : G \to G \cdot \mathbf{x}$ is separable if and only if $(\partial_e \mu) : \mathfrak{g} \to T_x(G \cdot \mathbf{x})$, the differential map of μ at e as introduced in §5.5, is a surjective map, and if this occurs $G \cdot \mathbf{x}$ is isomorphic to $G/C_G(\mathbf{x})$.

By [6, Proposition 6.7], for any subgroup H of G we have $\mathrm{Lie}(C_G(H)) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$. The map $(\partial_e \mu)$ is surjective if and only if we have equality.

In the next definition we follow [1, §3.5], where the various centralisers are defined in Notation 11.8.

Definition 11.31. *Let H be a closed subgroup of G.*

(1) *If $\mathrm{Lie}(C_G(H)) = \mathfrak{c}_{\mathfrak{g}}(H)$, then H is said to be **separable in** G.*

(2) *If $\mathrm{Lie}(C_G(\mathfrak{h})) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$, then $\mathfrak{h}$ is said to be **separable in** $\mathfrak{g}$.*

Lemma 11.32. *Let H be a closed separable subgroup of G such that $\mathfrak{g}$ is semisimple as an H-module. Then $\mathfrak{c}_{\mathfrak{g}}(H)$ is G-completely reducible. In particular, if H is a linearly reductive subgroup of G, then $\mathfrak{c}_{\mathfrak{g}}(H)$ is G-completely reducible.*

Proof. By [1, Theorem 3.46], H is G-cr and so, by [1, Corollary 3.17], we have $C_G(H)$ is G-cr. By Theorem 11.9, and since H is separable in G, we get that $\mathrm{Lie}(C_G(H)) = \mathfrak{c}_{\mathfrak{g}}(H)$ is G-cr. $\square$

The following is an analogue of [1, Lemma 2.17]. The proof translates over from the group case without major changes.

Lemma 11.33. *Let* $\mathbf{X} = (X_1, \ldots, X_n) \in \mathfrak{g}^n$*. Then* $\mathfrak{a}(\mathbf{X})$ *is strongly reductive in* G *if and only if for every cocharacter* λ *of* G *with* $\mathfrak{a}(\mathbf{X}) \subseteq \mathfrak{p}_\lambda (= \mathrm{Lie}(P_\lambda))$*, there exists some* $g \in G$ *such that* $c_\lambda(Y) = \mathrm{Ad}(g)Y$ *for all* $Y \in \mathfrak{a}(\mathbf{X})$*.*

The following lemma is standard, and can be found, for example, in [10, §1.5] or [48, Corollary 5.3.3.].

Lemma 11.34. *Let* G *act transitively on an algebraic variety* X*, and let* $x \in X$*. Then* $\dim(G \cdot x) = \dim(G) - \dim(\mathrm{Stab}_G(x))$*. In particular, for any* $\mathbf{X} \in \mathfrak{g}^n$*, we have that* $\dim(\mathrm{Ad}(G)\mathbf{X}) = \dim(G) - \dim(C_G(\mathfrak{a}(\mathbf{X})))$*.*

The following is an analogue of [1, Theorem 3.46], and provides a criterion for $\mathfrak{h}$ to be G-completely reducible. We sketch the proof here, and note that is closely resembles the argument in the group case.

Theorem 11.35. *Let* $\mathfrak{h}$ *be separable in* $\mathfrak{g}$*. If* $\mathfrak{g}$ *is semisimple as an* $\mathfrak{h}$*-module, then* $\mathfrak{h}$ *is* G*-completely reducible.*

Proof. We sketch the proof of this result. Suppose that $\mathfrak{h}$ is not G-cr. Choose $\mathbf{X} = (X_1, \ldots, X_n)$ to be a generating tuple of $\mathfrak{h}$ in $\mathfrak{g}^n$ in that $\mathfrak{h} = \mathfrak{a}(\mathbf{X})$. Then, by Theorem 11.9, the orbit $\mathrm{Ad}(G)\mathbf{X}$ is not closed in $\mathfrak{g}^n$. By Theorem 6.20, there exists a cocharacter λ of G such that $\lim_{x\to 0} \mathrm{Ad}(\lambda(x))\mathbf{X} =: \mathbf{X}'$ exists, and the orbit $\mathrm{Ad}(G)\mathbf{X}'$ is closed.

Let $\mathfrak{h}' = \mathfrak{a}(\mathbf{X}')$. As in the proof of [1, Theorem 3.46], we can show that $\dim(\mathrm{Ad}(G)\mathbf{X}') < \dim(\mathrm{Ad}(G)\mathbf{X})$, and $\dim \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}') > \dim \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$.

Let $\mathfrak{m} = \mathrm{ad}(\mathfrak{h})$, and $\mathfrak{m}' = \mathrm{ad}(\mathfrak{h}')$. Then $\mathfrak{m}' = c_{\mathrm{ad}\circ\lambda}(\mathfrak{m})$. Since $\mathfrak{g}$ is $\mathfrak{h}$-semisimple, $\mathfrak{m}$ is $\mathrm{GL}(\mathfrak{g})$-cr. Therefore, by Theorem 11.16, $\mathfrak{m}$ is strongly reductive in $\mathrm{GL}(\mathfrak{g})$, and hence, by Lemma 11.33, and since $\mathfrak{h} \subseteq \mathfrak{p}_\lambda$, we have $\mathfrak{m} \subseteq \mathrm{ad}(\mathfrak{p}_\lambda)$. Hence, $\mathfrak{m}' = c_{\mathrm{ad}\circ\lambda}(\mathfrak{m}) = \mathrm{Ad}(g)\mathfrak{m}$ for some $g \in \mathrm{GL}(\mathfrak{g})$. That is, $\mathfrak{m}'$ is $\mathrm{GL}(\mathfrak{g})$-conjugate to $\mathfrak{m}$.

We have $\mathfrak{c}_\mathfrak{g}(\mathfrak{h})$ (resp. $\mathfrak{c}_\mathfrak{g}(\mathfrak{h}')$) is the set of fixed points of $\mathfrak{m}$ (resp. $\mathfrak{m}'$) in $\mathfrak{g}$, and so $\mathfrak{c}_\mathfrak{g}(\mathfrak{h})$ is $\mathrm{GL}(\mathfrak{g})$-conjugate to $\mathfrak{c}_\mathfrak{g}(\mathfrak{h}')$. Therefore, $\dim \mathfrak{c}_\mathfrak{g}(\mathfrak{h}) = \dim \mathfrak{c}_\mathfrak{g}(\mathfrak{h}')$ which is a contradiction, and hence $\mathfrak{h}$ is G-cr. □

Recall that the prime p is said to be good for G is p does not divide any of the coefficients in the expressions obtained when each root in the root system of G is written as a sum of simple roots. Then, p is said to be very good for G if p is good for G and p does not divide $n+1$ for any of the simple components of type A_n that occur in the decomposition of G into its simple factors.

The following is an analogue of [4, Theorem 1.7], and follows immediately from [4, Theorem 1.2] and Theorem 11.35.

Theorem 11.36. *Let G be a connected reductive group, and suppose that char(k) is very good for G. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}$ is semisimple as an $\mathfrak{h}$-module. Then $\mathfrak{h}$ is G-completely reducible.*

11.3 Ad-Invariant Lie subalgebras

We recall some results from geometric invariant theory. Suppose V is an affine G-variety, and suppose that $v \in V$. Let S be a closed G-stable subvariety of V that does not contain v, but such that S meets the closure of $G \cdot v$. We let $|V, v|_S$ denote the set of one-parameter subgroups $\lambda \in Y(G)$ for which $\lim_{x\to 0} \lambda(x) \cdot v$ exists and lies in S. We call a one-parameter subgroup $\lambda \in Y(G)$ indivisible if $\lambda = n\mu$ for some one-parameter subgroup $\mu \in Y(G)$ if and only if $n = 1$.

Let W be the Weyl group of G, and $||-||_W$ the W-invariant length function on $Y(G)$, as described in §6.6. We are in the setting of Theorem 6.25. Let $\Delta_{S,v}$ denote the set of indivisible optimal cocharacters in $|V, v|_S$. The parabolic subgroup P_λ arising in this theorem is associated to an indivisible optimal cocharacter λ in $\Delta_{S,v}$, and is determined uniquely by V, v and S. We denote P_λ by $P_{S,v}$, and we call $P_{S,v}$ optimal for v and S. Recall also that, due to Lemma 6.27, we have $gP_{S,v}g^{-1} = P_{S,g\cdot v}$ for all $g \in G$.

Notation 11.37. For V, S, v and $P_{S,v}$ as above, we denote $\mathrm{Lie}(P_{S,v})$ by $\mathfrak{p}_{S,v}$.

Recall that G acts on $\mathfrak{g}$ via the adjoint representation Ad, and we call a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ **G-invariant** if it is $\mathrm{Ad}(G)$-invariant, that is if $\mathrm{Ad}(g)\mathfrak{h} \subseteq \mathfrak{h}$ for all $g \in G$.

The following result uses the techniques of B. Martin [35] and McNinch [37]. Note that the proof below bears many similarities to the proof of [35, Theorem 2].

Theorem 11.38. *Let $H \subseteq G$ and suppose that $\mathfrak{h} = \mathrm{Lie}(H)$ is a G-completely reducible Lie subalgebra of $\mathfrak{g}$. Then, any H-invariant Lie subalgebra of $\mathfrak{h}$ is G-completely reducible.*

Proof. Let $\mathfrak{k}$ be an H-invariant Lie subalgebra of $\mathfrak{h}$, and let $\mathbf{K} := (K_1, \ldots, K_m)$ be a generating tuple for $\mathfrak{k}$ as a subalgebra of $\mathfrak{h}$. Suppose that $\mathfrak{k}$ is not G-cr. Then, by Theorem 11.9, $\mathrm{Ad}(G)\mathbf{K}$ is not closed in $\mathfrak{g}^m$. By [39, No.8], the closure of this orbit contains a unique closed orbit, O say, and $\mathbf{K} \notin O$. Therefore, $\mathfrak{g}^m, O$ and $\mathbf{K}$ satisfy the hypothesis of Theorem 6.25. Therefore, there exists a cocharacter λ of G contained in $|\mathfrak{g}^m, \mathbf{K}|$. Furthermore, the limit $\lim_{x\to 0} \mathrm{Ad}(\lambda(x))\mathbf{K}$ exists, and therefore the limit $\lim_{x\to 0} \mathrm{Ad}(\lambda(x))K_i$ exists for each i, and so $\mathfrak{k} \subseteq \mathfrak{p}_\lambda = \mathfrak{p}_{O,\mathbf{K}} \subsetneq \mathfrak{g}$, where the last containment is proper, because $\mathfrak{p}_{O,\mathbf{K}} = \mathrm{Lie}(P_{O,\mathbf{K}})$, and $P_{O,\mathbf{K}} \neq G$.

Consider the proper parabolic subgroup $P_{O,\mathbf{K}}$ of G. We apply Lemma 6.27, where the action of G on $\mathfrak{g}^n$ is the adjoint action Ad, to get $gP_{O,\mathbf{K}}g^{-1} =$

$P_{O,\mathrm{Ad}(g)\mathbf{K}}$ for all $g \in G$. By hypothesis we have that $\mathrm{Ad}(h)\mathfrak{k} = \mathfrak{k}$ for any $h \in H$, hence $\mathbf{K}' := \mathrm{Ad}(h)\mathbf{K}$ is another generating tuple of $\mathfrak{k}$. Hence, we may apply [37, Corollary 7] to $\mathbf{K}'$, for any $h \in H$, to get that $|\mathfrak{g}^n, \mathbf{K}| = |\mathfrak{g}^n, \mathbf{K}'|$, and $\alpha_{O,\mathbf{K}}(\lambda) = \alpha_{O,\mathbf{K}'}(\lambda)$ (where $\alpha_{O,\mathbf{K}}(\lambda)$ is defined in §6.6), for all $\lambda \in |O, \mathbf{K}|$. Therefore $\Delta_{O,\mathbf{K}} = \Delta_{O,\mathbf{K}'}$ and hence $P_{O,\mathbf{K}} = P_{O,\mathbf{K}'}$. Thus, $hP_{O,\mathbf{K}}h^{-1} = P_{O,\mathrm{Ad}(h)\mathbf{K}} = P_{O,\mathbf{K}'} = P_{O,\mathbf{K}}$. Since $P_{O,\mathbf{K}}$ is its own normaliser, $H \subseteq P_{O,\mathbf{K}}$. We conclude that $\mathfrak{h} \subseteq \mathfrak{p}_{O,\mathbf{K}}$.

Let $(H_1, \ldots, H_l)$ be a generating tuple for $\mathfrak{h}$. Define the tuple $\mathbf{H} := (H_1, \ldots, H_l, K_1, \ldots, K_m)$. The limit $\lim_{x\to 0} \mathrm{Ad}(\lambda(x))\mathbf{H}$ exists in $\mathfrak{g}^{m+l}$ since $\mathfrak{h} \subseteq \mathfrak{p}_{O,\mathbf{k}}$. However, this limit is not in $\mathrm{Ad}(G)\mathbf{H}$, since $\lim_{x\to 0} \mathrm{Ad}(\lambda(x))\mathbf{K} \notin \mathrm{Ad}(G)\mathbf{K}$. Therefore, the orbit $\mathrm{Ad}(G)\mathbf{H}$ is not closed, so Theorem 11.9 implies that $\mathfrak{h}$ is not G-cr, which is a contradiction. □

Corollary 11.39. *Suppose that H is a subgroup of G such that $\mathfrak{h}$ is G-invariant and H is contained in $C_G(S)$ for S a maximal torus of $C_G(\mathfrak{h})$. Then H is G-completely reducible.*

Proof. If $\mathfrak{h}$ is G-invariant, then by Theorem 11.38 with $H = G$, $\mathfrak{h}$ is G-cr. Hence, by Corollary 11.24, H is G-cr. □

11.4 Ideals in $\mathfrak{g}$

Let G be a simple algebraic group, with Lie algebra $\mathfrak{g}$. In our discussion of ideals in $\mathfrak{g}$ we chiefly follow the notation of [21]. The reader should be aware that many authors use the symbols $\mathfrak{e}, \mathfrak{f}, \mathfrak{h}$ as generating elements of certain Lie algebras, however the notation of [21] is unrelated. A complete list of G-invariant ideals of $\mathfrak{g}$ is given in [21, Table 1].

In characteristic 2, if G is of type A_1, B_2 or C_n, for any n, then there are ideals in $\mathfrak{g}$ that are not G-invariant, which we describe in the following discussion.

Let T be a maximal torus of G, and Φ be the root system of G with respect to the lattice $X(T)$, see §5.11. The Lie algebra $\mathfrak{g}$ of G can be realised as $\mathfrak{g} = \mathfrak{t} \oplus \coprod_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where the $\mathfrak{g}_\alpha$ are the root spaces and $\mathfrak{t}$ is the 0-weight space. Let $\mathfrak{e}$ be the subspace of $\mathfrak{g}$ generated by the $\mathfrak{g}_\alpha$. Let Φ_S be the subset of Φ consisting of all the short roots and let $\mathfrak{e}_S$ be the subspace of $\mathfrak{g}$ generated by the $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_S$. Similarly let Φ_L be the subset of Φ consisting of all the long roots and let $\mathfrak{e}_L$ be the subspace of $\mathfrak{g}$ generated by the $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_L$.

For G of type A_1 where $p = 2$, the ideals $\{\mathfrak{t} + kX\}$ and $\{kX\}$ for each $0 \neq X \in \mathfrak{e}$ of $\mathfrak{g}$, are not G-invariant. For G of type B_2 or C_n where $p = 2$, we have that any ideal $\mathfrak{h}$ of $\mathfrak{g}$ for which $[\mathfrak{g},\mathfrak{g}] \subsetneq \mathfrak{h} \neq \mathfrak{g}$ is not G-invariant. For G of type C_n where $p = 2$, we have that the ideals $\mathfrak{e}_S + \mathfrak{f} + \mathfrak{t}_\Phi$ with $0 \neq \mathfrak{f} \subsetneq \mathfrak{e}_L$ are not G-invariant, where $\mathfrak{t}_\Phi$ is a certain subalgebra of $\mathfrak{t}$, as defined in [21, §1].

In characteristic p, where $p \geq 3$, every ideal of $\mathfrak{g}$ is G-invariant. Without restriction on p, if G is a simple group of exceptional type, then any ideal in $\mathfrak{g}$ is G-invariant.

We may now present the following corollary, which is an analogue in the Lie algebra setting of [35, Theorem 2].

Corollary 11.40. *Let G be a simple algebraic group over k. Let $\mathfrak{m}$ be an ideal in $\mathfrak{g}$. If $\mathfrak{m}$ is G-invariant, then $\mathfrak{m}$ is G-completely reducible. In particular, if char$(k) \geq 3$, then any ideal in $\mathfrak{g}$ is G-completely reducible.*

Proof. Apply Theorem 11.38 with $H = G$, noting that, when $\mathrm{char}(k) \geq 3$, any ideal in $\mathfrak{g}$ is G-invariant, see [21, Table 1]. $\square$

Example 11.41. Let G be a simple algebraic group over a field k of characteristic 2. Further, suppose that G is not of type A_1, B_2 or C_n, for any n. Then, any ideal of $\mathfrak{g}$ is G-invariant, by [21, Table 1]. Hence, in this case, any ideal of $\mathfrak{g}$ is G-completely reducible.

Example 11.42. Let G be a simple algebraic group G over k. By [21, Table 1], the ideal $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is G-invariant. Therefore, $[\mathfrak{g}, \mathfrak{g}]$ is G-cr.

Corollary 11.43. *Let G be a simple algebraic group over k. Let M be a subgroup of G for which $\mathfrak{m}$ is a G-invariant ideal in $\mathfrak{g}$, and suppose that M is contained in $C_G(S)$ for S a maximal torus of $C_G(\mathfrak{m})$. Then M is G-completely reducible.*

Proof. Since $\mathfrak{m}$ is a G-invariant ideal in $\mathfrak{g}$, Corollary 11.40 gives that $\mathfrak{m}$ is G-cr. By Proposition 11.24, M is G-cr. $\square$

Example 11.44. According to [21, Table 1], for G of type B_2, C_n for n even and for $\mathrm{char}(k) = 2$, there exists an ideal $\mathfrak{i} = \mathfrak{e}_S + \mathfrak{h}_S$ which is G-invariant, where $\mathfrak{h}_S$ is a certain subalgebra generated by semisimple elements. Therefore, by Corollary 11.40 $\mathfrak{i}$ is G-completely reducible.

Part IV

Conclusion

12 Conclusion and Topics for Further Study

Let G be a reductive algebraic group over a field $k = \overline{\mathbb{F}}_q$ where $\mathbb{F}_q$ is the finite field of $q = p^a$ elements, for a prime p and positive integer a. Let σ be a Frobenius morphism of G, and let H be a closed σ-stable subgroup of G.

In Part I of this thesis we described the structure of reductive algebraic groups and provided some background on the theory of G-complete reducibility, that was introduced by Serre in [46], and later developed by Bate, Martin and Röhrle in [1]. Suppose that H is topologically generated by the elements $h_1, \ldots, h_n$. We described how the group-theoretic notion of G-complete reducibility is equivalent to Richardson's notion of strong reductivity in G, which is a geometric notion in that it classifies the closed G-orbits in G^n. This link between geometry and group theory enables the geometric theory of Richardson to be used to study G-complete reducibility.

In Part II we introduced the notion of (G, σ)-complete reducibility as an analogue in the setting of σ-stability of the notion of G-complete reducibility. Our first important result, Theorem 8.6, is to extend part of [33, Theorem 9] to show that a σ-stable G-completely reducible subgroup of G is (G, σ)-completely reducible.

The main result in Part II of this thesis is Theorem 9.15, in which we demonstrated that a finite σ-stable subgroup F of G is G-completely reducible if and only if it is (G, σ)-completely reducible. This is an attractive result since neither of the implications in this equivalence are obvious. It is also a significant generalisation of [31, Proposition 2.2] because we do not impose any restrictions on the type of Frobenius morphism, and we allow G to be a reductive group rather than simple. In addition, we show that if F is not G-completely reducible then it is contained in a σ-stable parabolic subgroup P of G, and in no Levi subgroup of P. This, in turn, is a significant improvement of the result of [18], and of [1, Theorem 5.8] in the case when σ is a standard Frobenius morphism. Finally, in Theorem 10.6 we see that Theorem 9.15 can be extended to the case F is not necessarily finite.

In [31], Liebeck, Martin and Shalev used what is reproduced in this thesis as Proposition 9.1 to investigate the number of conjugacy classes of maximal subgroups of simple groups. Our Proposition 9.14 is an extension of this result from simple algebraic groups to reductive algebraic groups, and it may be possible to infer information about the number of conjugacy classes of maximal subgroups of the reductive groups we consider using the methods of [31].

We conclude Part II by discussing an analogue in the setting of σ-stability of the notion of strong reductivity in G. For a closed σ-stable subgroup H of G, we defined the notion of strong σ-reductivity in G. In Theorem 10.25 we proved that H is (G, σ)-completely reducible if and only if H is strongly σ-reductive in G.

In Part III we discuss the notion of G-complete reducibility for Lie subalgebras of $\mathrm{Lie}(G)$, as introduced by McNinch in [37]. In Proposition 11.24 we provided conditions under which a subgroup of G is G-completely reducible if and only if its Lie algebra is. The equivalence is non-trivial as there exist non-trivial examples of non-G-completely reducible subgroups of G whose Lie algebras are G-completely reducible. Thus, we have discovered an interesting connection between the behaviour a group and its Lie algebra.

We proceed to study ideals in $\mathfrak{g}$, and we show that any G-invariant ideal in $\mathfrak{g}$ is G-completely reducible, see Corollary 11.40. This result is therefore an analogue in the Lie algebra setting of Martin's result about normal subgroups given in [35], since normal subgroups of G and ideals in the Lie algebra $\mathfrak{g}$ are closely related, in fact in characteristic zero, they are in one-to-one correspondence. Corollary 11.40 does not, however, talk about other subalgebras of $\mathfrak{g}$, and their G-complete reducibility. The question of how other subalgebras behave, with regard to G-complete reducibility remains an open question worthy of further study. One approach to tackle this could be to investigate under what conditions a closed subgroup H of G is contained in $C_G(S)$, where S is a maximal torus of $C_G(\mathrm{Lie}(H))$. This is a sufficient condition for the equivalence in Proposition 11.24 to hold.

For a simple group G, it would be interesting to explore if a non-G-completely reducible subgroup of G gives rise to an $\mathrm{Ad}(G)$-invariant Lie subalgebra of $\mathrm{Lie}(G)$, as listed in [21]. In this case, by Corollary 11.40, we know that such a Lie subalgebra is G-completely reducible.

In order to identify non-normal subgroups of a simple algebraic group G, one place to look is for non-G-completely reducible subgroups (for any normal subgroup of G is G-completely reducible, see [35, Theorem 2]). Consider the situation described in [1, Example 3.45]. Then, for $\mathrm{Char}(k) = 2$, and $n \geq 4$ is even, we have that $\mathrm{Sp}_n(k)$ is not $\mathrm{Sp}_{2n}(k)$-completely reducible, and thus is not a normal subgroup in $\mathrm{Sp}_{2n}(k)$. However, in this case there are no non-$\mathrm{Ad}(G)$-invariant ideals in $\mathrm{Lie}(\mathrm{Sp}_{2n}(k))$. Stewart has surveyed the non-G-completely reducible subgroups of exceptional groups, see [52] for details. Again, in these cases there are no non-$\mathrm{Ad}(G)$-invariant ideals in $\mathrm{Lie}(G)$. Such subgroups may give rise to new and interesting examples of non-G-completely reducible subgroups of G, whose Lie algebras are G-completely reducible. The question is, are the Lie algebras of these non-G-completely reducible subgroups of G ideals in $\mathrm{Lie}(G)$?

It would be interesting to extend Corollary 11.40 to the case where G is semisimple. In [21, §3] Hogeweij discusses the situation of ideals in the Lie algebras of semisimple groups. This direction of study could yield a significant generalisation of the results of §11.4 from the case where G is simple, to the case of G semisimple.

Bibliography

[1] Bate, M., Martin, B., Röhrle, G.: *A Geometric Approach to Complete Reducibility*, Invent. Math., **161**, p177–218, (2005).

[2] Bate, M., Martin, B., Röhrle, G.: *Complete Reducibility and Commuting Subgroups,* J. Reine Angew. Math., **621**, p213–235, (2008).

[3] Bate, M., Martin, B., Röhrle, G., Tange, R.: *Closed Orbits and Uniform S-Instability in Geometric Invariant Theory*, to appear in Trans. AMS, (2011).

[4] Bate, M., Martin, B., Röhrle, G., Tange, R.: *Complete Reducibility and Separability*, Trans. AMS. **362** Issue 8, p4283–4311, (2010).

[5] Bate, M., Martin, B., Röhrle, G., Tange, R.: *Complete Reducibility and Conjugacy Classes of Tuples in Algebraic Groups and Lie Algebras*, Math. Zeit., **269**, p809–832, (2011).

[6] Borel, A.: *Linear Algebraic Groups*, Springer-Verlag, New York, (1991).

[7] Borel, A., Tits. J.: *Éléments Unipotents et Sous-Groupes Paraboliques de Groupes Réductifs*, I., Invent. Math., **12**, p95–104, (1971).

[8] Borel, A., Tits. J.: *Groupes Réductifs*, Inst. Hautes Études Sci. Publ. Math., **27**, p55–150, (1965).

[9] Bourbaki, N.: *Groupes et Algèbres de Lie*, Ch. IV-VI, Hermann, Paris, (1968).

[10] Carter, R. W.: *Finite Groups of Lie Type. Conjugacy Classes and Complex Characters*, Wiley, Chichester, (1993).

[11] Carter, R. W.: *Simple Groups of Lie Type*, Wiley, London-New York, (1972).

[12] Chevalley, C.: *Existence d'isogenies,I. Seminaire Claude Chevalley*, **2 No.23**, (1956-1958).

[13] Conrad, B., Gabber, O., Prasad, G.: *Pseudo-reductive Groups*, Cambridge University Press, New York, (2010).

[14] Curtis, C. W., Reiner, I.: *Methods of Representation Theory with Applications to Finite Groups and Orders*, **I**, Wiley, New York, (1981).

[15] Digne, F., Michel, J.: *Representations of Finite Groups of Lie Type*, London Math. Soc, Student Texts, **21**, Cambridge University Press, (1991).

[16] Fogarty, J., Kirwan, F., Mumford, D.: *Geometric Invariant Theory*, Springer-Verlag, New York, (1965).

[17] Fontana, M., Kabbaj, S-E., Olberding, B., Swanson, I.: *Commutative Algebra - Noetherian and Non-Noetherian Perspectives*, Springer, New York, (2010).

[18] Gold, D., Herpel, S., Röhrle, G.: *Complete Reducibility and Steinberg Endomorphisms*, Comp. Rend. Math., **349** Issues 5–6, p243–246, (2011).

[19] Gorenstein, D., Lyons, R., Solomon, R.: *The Classification of Finite Simple Groups, Number 3*, Amer. Math. Soc., **40** No. 3, Rhode Island, (1995).

[20] Hochschild, G. P.: *Basic Theory of Algebraic Groups and Lie Algebras*, Graduate Texts in Mathematics, 75, Springer-Verlag, New York-Berlin, (1981).

[21] Hogeweij, G. M. D.: *Almost-Classical Lie Algebras*, 1, Indag. Math., **44**, p441–452, (1982).

[22] Humphreys, J. E.: *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Springer-Verlag, New York, (2000).

[23] Humphreys, J. E.: *Linear Algebraic Groups*, Graduate Texts in Mathematics, Springer-Verlag, New York, (1998).

[24] Humphreys, J. E.: *Modular Representations of Finite Groups of Lie Type*, Lon. Math. Soc. Lecture Series 325, Cambridge University Press, Cambridge, (2006).

[25] Jantzen, J. C.: *Nilpotent Orbits in Representation Theory*, In: Lie Theory, Lie Algebras and Representations, Prog. Math., **228**, ed. by J.-P. Anker, B. Orsted, Boston, Birkhäuser (2004).

[26] Kempf, G. R.: *Instability in Invariant Theory*, Ann. Math., **108**, p299–316, (1978).

[27] Knapp, A. W.: *Lie Groups Beyond an Introduction Second Edition*, Birkhäuser, New York, (2005).

[28] Knus, M-A., Merkujev, A., Rost, M., Tignol, J-P.: *The Book of Involutions* Amer. Math. Soc., Colloquium Publications **40** Rhode Island (1998).

[29] Lang, S.: *Algebra*, Springer-Verlag, New York, (2005).

[30] LIEBECK, M. W.: *Subgroups of Simple Algebraic Groups and of Related Finite and Locally Finite Groups of Lie Type*, B. Hartley *et al*, (*eds*), Finite and Locally Finite Groups, p71–96, Kluwer, (1995).

[31] LIEBECK, M. W., MARTIN, B. M. S., SHALEV, A.: *On Conjugacy Classes of Maximal Subgroups of Finite Simple Groups, and a Related Zeta Function*, Duke Math., **128**, p541–557, (2005).

[32] LIEBECK, M. W., SEITZ, G. M.: *A Survey of Maximal Subgroups of Exceptional Groups of Lie Type.* `http://www2.imperial.ac.uk/ mwl/durhamliesei.pdf`

[33] LIEBECK, M. W., SEITZ, G. M.: *On the Subgroup Structure of Exceptional Groups of Lie Type*, Trans. of the AMS, **350 No.9**, p3409–3482, (1998).

[34] MALLE, G.: *The Maximal Subgroups of* $^2F_4(q^2)$, J. Alg., **139**, p52–69, (1991).

[35] MARTIN, B. M. S.: *A Normal Subgroup of a Strongly Reductive Subgroup is Strongly Reductive*, J. Alg., **265**, p669–674, (2003).

[36] MARTIN, B. M. S.: *Reductive Subgroups of Reductive Groups in Nonzero Characteristic*, J. Alg., **262**, p265–286, (2003).

[37] MCNINCH, G.: *Completely Reducible Lie Subalgebras*, Transform. Groups, **12**, **No.1**, p127–135, (2007).

[38] MUMFORD, D.: *Geometric Invariant Theory*, Springer-Verlag, Berlin, (1965).

[39] NAGATA, M.: *Invariants of a Group in an Affine Ring*, J. Math. Kyoto, **3 No.3**, p369–377, (1964).

[40] Newstead, P. E.: *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, **51**, (1978).

[41] Richardson, R. W.: *Affine Coset Spaces of Reductive Algebraic Groups*, Bull. London Math. Soc., **9 No.1**, p38–41, (1977).

[42] Richardson, R. W.: *Conjugacy Classes of n-Tuples in Lie Algebras and Algebraic Groups*, Duke Math., **57 No.1**, p1–35, (1998).

[43] Richardson, R. W.: *On Orbits of Algebraic Groups and Lie Groups*, Bull. Austral. Math. Soc., **25 No.1**, p1–29, (1982) .

[44] Rotman, J. J.: *An Introduction to the Theory of Groups, Fourth Edition*, Graduate Texts in Mathematics, 148, Springer-Verlag, New York, (1995).

[45] Serre, J. P.: *Complète Réductibilité*, Séminaire Bourbaki, 56ème année, n^o 932, (2003-2004).

[46] Serre, J. P.: *La Notion de Complète Réductibilité dans les Immeubles Sphrériques et les Groupes Réductifs*, Séminaire au Collége de France, résumé dans, **44**, p93–98, (1997).

[47] Serre, J. P.: *The Notion of Complete Reducibility in Group Theory*, Moursund Lectures, University of Oregon, (1998).

[48] Springer, T. A.: *Linear Algebraic Groups*, Birkhäuser, Boston, (1998).

[49] Springer, T. A., Steinberg, R.: *Conjugacy Classes, in: Seminar on Algebraic Groups and Related Topics (ed. Borel et al.)*, Lecture Notes in Math., **131**, Springer-Verlag, Berlin, p168–266, (1970).

[50] Steinberg, R.: *Endomorphisms of Linear Algebraic Groups*, Memoirs AMS., **80**, (1968).

[51] Steinberg, R.: *Lectures on Chevalley Groups*, Yale University Lecture Notes, (1968).

[52] Stewart, D. I.: *Non-G-Completely Reducible Subgroups of the Exceptional Algebraic Groups*, arXiv:1201.5310, (2012).