Incorporating Initial Conditions into Nonlinear Input-Output Theory

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The Question

For a dynamical system represented by a set of input output pairs, how should we define initial conditions to discuss robust stability?

- Adding nonzero initial conditions to the purely I/O theory, e.g., Georgiou & Smith [1];
- Turning ISS/IOS theory of Sontag *et al.* [2] into a purely I/O theory.

Signal Spaces & System

Signal spaces. Given normed signal space $\mathcal{V}$ (e.g., $\mathcal{L}_\infty(R)$), $\tau \in \mathbb{R}$ and $\omega \in (\tau, \infty]$, we let (detailed definition not given here).

- $\mathcal{V}^+_{[\tau]} \triangleq \mathbb{P}^+_\tau \mathcal{V}$ (e.g., $\mathcal{L}_\infty[\tau,\infty)$);
- $\mathcal{V}[\tau,\omega)$: *interval space* (e.g., $\mathcal{L}_\infty[\tau,\omega)$);
- $\mathcal{V}_{[\tau,\omega)}$: contains unbounded signal (e.g., $\mathcal{L}^e_\infty[\tau,\omega)$);
- $\mathcal{V}^+_{a[\tau]} \triangleq \cup_{t>\tau} \mathcal{V}_{[\tau,t)}$: *ambient space.*

where $\mathbb{P}^+_\tau$ is an operator which maps time function $f:(t_1,t_2)\to\mathbb{R}^n$ to time function $g:[\tau,t_2)\to\mathbb{R}^n$ such that $f|_{[\tau,t_2)} = g$. ($\tau \in (t_1,t_2) \subset [-\infty,+\infty]$). We can similarly define $\mathbb{P}^-_\tau$ (i.e., if $\mathrm{dom}(f)=(t_1,t_2)$, then $\mathrm{dom}(\mathbb{P}^-_\tau f) = (t_1,\tau)$) and $\mathcal{V}^-_{a[\tau]}$. Let

$$\mathcal{V}_{a[\tau]} \triangleq \mathcal{V}^+_{a[\tau]} \oplus \mathcal{V}^-_{a[\tau]}. \tag{1}$$

System. Given normed signal spaces $\mathcal{U},\mathcal{Y}$, $\mathcal{W} \triangleq \mathcal{U}\times\mathcal{Y}$ and initial time $t_0 \in \mathbb{R}$ and $\triangle > 0$, a *system* Q is defined to be the *behaviour* denoted by $\mathfrak{B}_{Q[t_0]}$:

$$\left\{ w \in \mathcal{W}_{a[t_0]} \,\middle|\, \begin{matrix} w=(u,y)^T \text{ is input-output} \\ \text{pair of } Q, t_0 - \triangle \in \mathrm{dom}(w) \end{matrix} \right\} \tag{2}$$

The system Q is said to be *well defined* if

1. the input u is *free*, i.e., for all $u \in \mathcal{U}_{a[t_0]}$ with $t_0 - \triangle \in \mathrm{dom}(u)$, there exists a $y \in \mathcal{Y}_{a[t_0]}$ such that $(u,y)^T \in \mathcal{W}_{a[t_0]}$ is an input-output pair of Q.
2. for any $w=(u,y)^T$ and $\tilde{w}=(\tilde{u},\tilde{y})^T$ in $\mathfrak{B}_{Q[t_0]}$, if $\mathbb{P}^-_{t_0} w = \mathbb{P}^-_{t_0}\tilde{w}$ and $\mathbb{P}^+_{t_0} u = \mathbb{P}^+_{t_0}\tilde{u}$, then $\mathbb{P}^+_{t_0} y = \mathbb{P}^+_{t_0}\tilde{y}$.

An operator $\Phi : \mathcal{U}^+_{a[t_0]} \to \mathcal{Y}^+_{a[t_0]}$ is said to be *causal* if

$$\begin{cases} \forall\, u,v \in \mathcal{U}^+_{a[t_0]}, \\ \forall t \in \mathrm{dom}(u,v,\Phi u,\Phi v) \end{cases} : \begin{bmatrix} \mathbb{P}^-_t u = \mathbb{P}^-_t v \implies \\ \mathbb{P}^-_t(\Phi u) = \mathbb{P}^-_t(\Phi v) \end{bmatrix}$$

A well defined system Q is said to be *causal* if the following natural defined operators in (3) are all causal.

Initial Conditions

For a *well defined system* Q, we know that $\forall w_- \in \mathfrak{B}^-_{Q[t_0]} \triangleq \mathbb{P}^-_{t_0}\mathfrak{B}_{Q[t_0]}$, $\forall u_+ \in \mathcal{U}^+_{a[t_0]}$, there exists one and only one $y_+ \in \mathcal{Y}^+_{a[t_0]}$ such that $w_- \oplus w_+ \in \mathfrak{B}_{Q[t_0]}$ with $w_+ = (u_+,y_+)^T$. Thus for any fixed $w_- \in \mathfrak{B}^-_{Q[t_0]}$, we can naturally define an operator Q_{w_-} by:

$$Q_{w_-} : \mathcal{U}^+_{a[t_0]} \to \mathcal{Y}^+_{a[t_0]}, \quad u_+ \mapsto y_+. \tag{3}$$

Next we define an equivalence relation $\sim$ on $\mathfrak{B}^-_{Q[t_0]}$ as follows: for any $w_-, \tilde{w}_- \in \mathfrak{B}^-_{Q[t_0]}$, we say

$$w_- \sim \tilde{w}_- \text{ if and only if } Q_{w_-} = Q_{\tilde{w}_-} \tag{4}$$

The set of all equivalence classes in $\mathfrak{B}^-_{Q[t_0]}$ related to $\sim$ is denoted as $\mathfrak{S}_{Q[t_0]}$ called the *initial state space* of Q. Note that for any $x_{t_0} \in \mathfrak{S}_{Q[t_0]}$, we can easily define an operator $Q_{x_{t_0}} : \mathcal{U}^+_{a[t_0]} \to \mathcal{Y}^+_{a[t_0]}$ from (3).
The *size* of any initial state in $\mathfrak{S}_{Q[t_0]}$ is characterized by the following defined real valued function $d : \mathfrak{S}_{Q[t_0]} \to \mathbb{R}_{\geq 0}$, for any $x_{t_0} \in \mathfrak{S}_{Q[t_0]}$, define

$$d(x_{t_0}) \triangleq \inf_{w_- \in x_{t_0}} \|w_-\|_{[t_0-\triangle,\ t_0]} \tag{5}$$

Closed-loop System

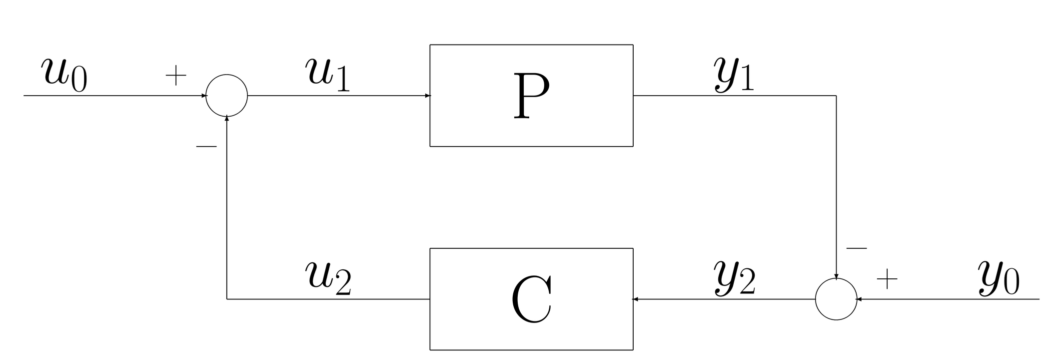

Fig. 1: Closed-loop system $[P,C]$. $w_i = (u_i,y_i)^T$, $(i=0,1,2)$, $w_0 = w_1 + w_2$, $w_1 \in \mathfrak{B}_{P[t_0]}$, $w_2 \in \mathfrak{B}_{C[t_0]}$

Let behaviours $\mathfrak{B}_{P[t_0]}$ with initial state space $\mathfrak{S}_{P[t_0]}$ and $\mathfrak{B}_{C[t_0]}$ with initial state space $\mathfrak{S}_{C[t_0]}$ represent two well defined subsystems called the *plant* P and *controller* C, respectively.
The *closed-loop system* $[P,C]$ represented by the behaviour $\mathfrak{B}_{P//C[t_0]}$ is defined as the interconnection of the *plant* P and *controller* C shown in Fig. 1:

$$\left\{ \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \in \mathcal{W}^2_{a[t_0]} \,\middle|\, \begin{matrix} w_0 \text{ is input, } w_1 \in \mathfrak{B}_{P[t_0]}, \\ w_0 - w_1 \in \mathfrak{B}_{C[t_0]} \end{matrix} \right\} \tag{6}$$

It is natural to define the initial state of the closed loop system $[P,C]$ by $\mathfrak{S}_{P[t_0]} \times \mathfrak{S}_{C[t_0]}$, here we do not require $[P,C]$ to be well defined. The size of any element in the product space $\mathfrak{S}_{P[t_0]} \times \mathfrak{S}_{C[t_0]}$ is defined in the usual way.

Well-posedness of $[P,C]$

Fixed any $x_{t_0} \in \mathfrak{S}_{P[t_0]} \times \mathfrak{S}_{C[t_0]}$. For any $w_{0+} \in \mathcal{W}^+_{[t_0]}$, a pair $(w_{1+},w_{2+}) \in \mathfrak{B}^+_{P[t_0]} \times \mathfrak{B}^+_{C[t_0]}$ is a *solution* if it satisfies $w_{0+} = w_{1+} + w_{2+}$ (see Fig. 1) on time domain of $\mathrm{dom}(w_{1+},w_{2+})$. The closed-loop system $[P,C]$ is said to have the *existence property* if $\forall x_{t_0}, \forall w_{0+}$, the solution exists. $[P,C]$ is said to have the *uniqueness property* if $\forall x_{t_0}, \forall w_{0+}$, any two solutions equal at any same time domain. To define the following closed-loop operator in (7) for any fixed x_{t_0}, we let (w_{1+},w_{2+}) be the solution with maximal time domain of existence:

$$\Pi^{x_{t_0}}_{P//C} : \mathcal{W}^+_{[t_0]} \to \mathcal{W}^+_{a[t_0]}, \qquad w_{0+} \mapsto w_{1+}. \tag{7}$$

$[P,C]$ is said to be *well-posed* if it has the existence and uniqueness properties and $\Pi^{x_{t_0}}_{P//C}$ is causal for any x_{t_0}. We next define the *graph* $\mathcal{G}_P$ of the plant P and the *graph* $\mathcal{G}_C$ of the controller C respectively by

$$\mathcal{G}_P \triangleq \mathcal{W}^+_{[t_0-\triangle]} \cap \mathfrak{B}_{P[t_0]}, \qquad \mathcal{G}_C \triangleq \mathcal{W}^+_{[t_0-\triangle]} \cap \mathfrak{B}_{C[t_0]}.$$

The Idea

The initial state space is defined by a set of equivalent classes of past input-output signals up to initial time. The size of any initial state is defined by the smallest input-output pairs in terms of norm in that corresponding equivalent class. A robust stability theorem with initial conditions is then given.

Robust Stability Theorem

Theorem. Given normed signal spaces $\mathcal{U},\mathcal{Y}$, $\mathcal{W} \triangleq \mathcal{U}\times\mathcal{Y}$, initial time $t_0 \in \mathbb{R}$ and $\triangle > 0$. Let P denoted by $\mathfrak{B}_{P[t_0]}$ with initial state space $\mathfrak{S}_{P[t_0]}$ and C denoted by $\mathfrak{B}_{C[t_0]}$ with initial state space $\mathfrak{S}_{C[t_0]}$ be causal well defined systems representing the nominal plant and the controller, respectively. Suppose that $[P,C]$ is well-posed relatively continuous and time-invariant. Let $[P,C]$ be *input to output stable*, i.e., there exist functions $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}_\infty$ such that, $\forall x_{t_0} = (x_{1t_0}, x_{2t_0})^T \in \mathfrak{S}_{P[t_0]} \times \mathfrak{S}_{C[t_0]}$, $\forall w_{0+} \in \mathcal{W}^+_{[t_0]}$, $\forall t > t_0$,

$$\left|(\Pi^{x_{t_0}}_{P//C} w_{0+})(t)\right| \leq \beta\left(d(x_{t_0}), t - t_0\right) + \gamma(\|w_{0+}\|_{[t_0,t)}), \tag{8}$$

Let $\tilde{P}$ denoted by $\mathfrak{B}_{\tilde{P}[t_0]}$ with initial state space $\mathfrak{S}_{\tilde{P}[t_0]}$ be causal well defined system representing the perturbed plant of P. If $[\tilde{P},C]$ has the uniqueness property, and there exists a causal surjective operator $\Phi : \mathcal{G}_P \to \mathcal{G}_{\tilde{P}}$ and functions $\beta_0(\cdot,\cdot) \in \mathcal{KL}$, $\sigma(\cdot) \in \mathcal{K}_\infty$, $\sigma_0(\cdot) \in \mathcal{K}_\infty$, such that, $\forall t > h \geq t_0$, $\forall w \in \mathcal{G}_P$,

$$\|\Phi w\|_{[t_0-\triangle,t_0]} \geq \sigma_0^{-1}(\|w\|_{[t_0-\triangle,t_0]}), \tag{9}$$

$$|((\Phi - I)w)(t)| \leq \beta_0(\|w\|_{[t_0-\triangle,h]}, t-h) + \sigma(\|w\|_{[h,t)}). \tag{10}$$

and $\mathbb{P}^-_t(\Phi - I)$ is compact for any $t \geq t_0$, and if there exist two functions $\rho(\cdot), \varepsilon(\cdot)$ of class $\mathcal{K}_\infty$ such that

$$\sigma \circ (I+\rho) \circ \gamma(s) \leq (I+\varepsilon)^{-1}(s), \qquad \forall s \geq 0. \tag{11}$$

Then the closed loop system $[\tilde{P},C]$ is also *input to output stable*, i.e., there exist functions $\tilde{\beta}(\cdot,\cdot) \in \mathcal{KL}$ and $\tilde{\gamma}(\cdot) \in \mathcal{K}_\infty$ such that, $\forall \tilde{x}_{t_0} \in \mathfrak{S}_{\tilde{P}[t_0]} \times \mathfrak{S}_{C[t_0]}$, $\forall \tilde{w}_{0+} \in \mathcal{W}^+_{[t_0]}$, $\forall t > t_0$,

$$\left|(\Pi^{\tilde{x}_{t_0}}_{\tilde{P}//C} \tilde{w}_{0+})(t)\right| \leq \beta\left(d(\tilde{x}_{t_0}), t - t_0\right) + \gamma(\|\tilde{w}_{0+}\|_{[t_0,t)}). \tag{12}$$

References

[1] T.T. Georgiou and M.C. Smith *Robustness analysis of nonlinear feedback systems: an input-output approach*, IEEE Transactions on Automatic Control **42**, 1997.

[2] http://www.math.rutgers.edu/ sontag/

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