ONE-RELATOR GROUPS WITH TORSION ARE CONJUGACY SEPARABLE

ASHOT MINASYAN AND PAVEL ZALESSKII

Abstract. We prove that one-relator groups with torsion are hereditarily conjugacy separable. Our argument is based on a combination of recent results of Dani Wise and the first author. As a corollary we obtain that any quasiconvex subgroup of a one-relator group with torsion is also conjugacy separable.

1. Introduction

Recall that a group G is said to be *conjugacy separable* if for any two non-conjugate elements $x, y \in G$ there is a homomorphism from G to a finite group M such that the images of x and y are not conjugate in M. Conjugacy separability can be restated by saying that each conjugacy class $x^G = \{gxg^{-1} \mid g \in G\}$ is closed in the profinite topology on G. The group G is said to be *hereditarily conjugacy separable* if every finite index subgroup of G is conjugacy separable. Conjugacy separability is a natural algebraic analogue of solvability of the conjugacy problem in a group and has a number of applications (see, for example, [11]). Any conjugacy separable group is residually finite, but the converse is false. Generally, it may be quite hard to show that a residually finite group is conjugacy separable.

In the recent breakthrough work [16] Dani Wise proved that one-relator groups with torsion possess so-called quasiconvex hierarchy, and groups with such hierarchy are virtually compact special. The class of *special* (or *A-special*, in the terminology of [7]) cube complexes was originally introduced by Fredéric Haglund and Dani Wise in [7], as cube complexes in which hyperplanes enjoy certain combinatorial properties. They also showed that a cube complex is special if and only if it admits a combinatorial local isometry to the Salvetti cube complex (see [3]) of some right angled Artin group. It follows that the fundamental group of every special complex $\mathcal{X}$ embeds into some right angled Artin group.

A group G is said to be *virtually compact special* if G contains a finite index subgroup P such that $P = \pi_1(\mathcal{X})$ for some compact special cube complex $\mathcal{X}$. Thus Wise's result implies that any one-relator group G, with torsion, is (virtually) a subgroup of a right

Date: November 2, 2012.
2010 *Mathematics Subject Classification.* 20E26, 20F67.
Key words and phrases. One-relator groups with torsion, conjugacy separable.
The first author was partially supported by the EPSRC grant EP/H032428/1.

angled Artin group. In particular, G is residually finite, which answers an old question of G. Baumslag.

An important fact, established by Haglund and Wise in [7], states that the fundamental group P of a compact special complex is a virtual retract of some finitely generated right angled Artin group. From the work of the first author [11] it follows that P is hereditarily conjugacy separable. This shows that any one-relator group with torsion possesses a hereditarily conjugacy separable subgroup of finite index. Unfortunately, in general conjugacy separability is not stable under passing to finite index overgroups (see [6]). The aim of this note is to prove the following:

Theorem 1.1. *If G is a one-relator group with torsion then G is hereditarily conjugacy separable.*

This theorem answers positively Question 8.69 in Kourovka Notebook [10], posed by C.Y. Tang. This question was also raised in [15] in 1982; its special cases have been considered in [15] and [1].

As a consequence of Theorem 1.1 we also derive

Corollary 1.2. *If G is a one-relator group with torsion then every quasiconvex subgroup of G is conjugacy separable.*

Our proof of Theorem 1.1 uses the above mentioned results of Wise, Haglund-Wise and the first author, and employs the quasiconvex hierarchy for one-relator groups with torsion, that was investigated by Wise in [16].

2. Background on one-relator groups with torsion

Let

$$G = \langle S \,\|\, W^n \rangle \tag{1}$$

be a *one-relator group with torsion*, where S is a finite alphabet, $n \geq 2$ and W is a cyclically reduced word, which is not a proper power in the free group $F(S)$.

Newman's spelling theorem [12, Thm. 3] (see also [9, IV.5.5]) implies that every freely reduced word over $S^{\pm 1}$, representing the identity element of G, contains a subword of W^n of length strictly greater than $(n-1)/n$ times the length of W^n. Since $(n-1)/n \geq 1/2$ it follows that the presentation (1) satisfies Dehn's algorithm ([9, IV.4]); in particular G has a linear Dehn function, and hence it is word hyperbolic. For the background on hyperbolic groups and quasiconvex subgroups the reader is referred to [2].

Another important fact, proved by Newman in [12, Thm. 2] (see also [8, p. 956]), states that centralizers of non-trivial elements in one-relator groups with torsion are cyclic.

Many results about one-relator groups are proved using induction on some complexity depending on the word W. In this paper we will use the *repetition complexity* $RC(W)$ of W employed by Wise in [16]. This is defined as the difference between the length of W, and the number of distinct letters from S that occur in W. For example, if $S = \{a, b, c\}$ then $RC(ab^2a^{-1}c^{-3}) = 7 - 3 = 4$.

Start with a one relator-group G given by presentation (1). Recall that a *Magnus subgroup* M of G is a subgroup generated by a subset $U \subset S$ such that U omits at least one generator appearing in W. By the famous Magnus's Freiheitssatz, M is free and U is its free generating set.

Observe that if $RC(W) = 0$ then every letter appears in W exactly once. In this case, using Tietze transformations, it is easy to see that G is isomorphic to the free product of a free group of rank $|S| - 1$ with the cyclic group of order n.

Assume, now, that $RC(W) > 0$. Then, following [16, 18.2], one can let $H = G * \langle t \rangle$, and represent H as an HNN-extension of another one-relator group $K = \langle \overline{S} \,\|\, \overline{W}^n \rangle$, where $|\overline{S}| < \infty$, $\overline{W}$ is some cyclically reduced word in the free group $F(\overline{S})$, and the associated subgroups are Magnus subgroups M_1, M_2 of K. In other words, there are subsets $U_1, U_2 \subset \overline{S}$, each of which omits some letter of $\overline{W}$, and a bijection $\alpha : U_1 \to U_2$ such that $M_i = \langle U_i \rangle$, $i = 1, 2$, and H has the presentation

$$H = \langle \overline{S}, t \,\|\, \overline{W}^n, tut^{-1} = \alpha(u) \text{ for all } u \in U_1 \rangle. \tag{2}$$

Moreover, in [16, 18.3] Wise shows that one can do this in such a way that $RC(\overline{W}) < RC(W)$.

Lemma 2.1. *The group H defined above contains a finite index normal subgroup $L \lhd H$ such that L is hereditarily conjugacy separable.*

Proof. In [16, Ch. 18] Wise shows that H is virtually compact special. By the work of Haglund and Wise from [7, Ch. 6], H contains a finite index subgroup L such that L is a virtual retract of some finitely generated right angled Artin group A. Now, a result of the first author [11, Cor. 2.1] implies that L is hereditarily conjugacy separable. □

The next statement follows from a combination of results of Wise [16] and Haglund-Wise [7]:

Lemma 2.2. *Let P be a finite index subgroup of K or M_1, or M_2. Then P is closed in the profinite topology of H.*

Proof. The group H is hyperbolic as a free product of two hyperbolic groups, and by [16, Lemma 18.8] K, M_1 and M_2 are all quasiconvex subgroups of H. Since a finite index subgroup of a quasiconvex subgroup is itself quasiconvex, it follows that P is quasiconvex in H.

As we already mentioned above, [16, Cor. 18.3] states that H is virtually compact special. Now we can use [7, Thm. 7.3, Lemma 7.5], which imply that any quasiconvex subgroup of H is separable in H. Thus the lemma is proved. □

3. Some auxiliary facts

First let us specify some notation. If A is a group and $C, D \subseteq A$, then C^D will denote the subset defined by $C^D = \{dcd^{-1} \mid c \in C, d \in D\}$. If $x \in A$ and $E \leqslant A$ then $C_E(x) = \{g \in E \mid gx = xg\}$ will denote the centralizer of x in E.

Recall that a subset C of a group A is said to be *separable* if C is closed in the profinite topology of A. This is equivalent to the following property: for every $y \in A \setminus C$ there exist a finite group Q and an epimorphism $\psi : H \to Q$ such that $\psi(y) \notin \psi(C)$ in Q.

The following notion is helpful for proving hereditary conjugacy separability of groups. It is similar to [11, Def. 3.1].

Definition 3.1. Let H be a group and $x \in H$. We will say that the element x satisfies the *Centralizer Condition in H* (briefly, CC_H), if for every finite index normal subgroup $P \lhd H$ there is a finite index normal subgroup $N \lhd H$ such that $N \leqslant P$ and $C_{H/N}(\psi(x)) \subseteq \psi\left(C_H(x)P\right)$ in H/N, where $\psi : H \to H/N$ is the natural homomorphism.

The condition CC_H defined above is actually quite natural from the viewpoint of the profinite completion $\widehat{H}$ of H. Indeed, in [11, Prop. 12.1] it is shown that if H is residually finite then $x \in H$ has CC_H if and only if $C_{\widehat{H}}(x) = \overline{C_H(x)}$, where the right-hand side is the closure of $C_H(x)$ in the profinite completion $\widehat{H}$.

The next two lemmas were proved by the first author in [11, Lemmas 3.4 and 3.7]. The first one shows why the Centralizer Condition is useful, and the second lemma provides a partial converse to the first one.

Lemma 3.2. *Suppose that H is a group, $H_1 \leqslant H$ and $x \in H$. Assume that the element x satisfies CC_H and the conjugacy class x^H is separable in H. If the double coset $C_H(x)H_1$ is separable in H, then the H_1-conjugacy class x^{H_1} is also separable in H.*

Lemma 3.3. *Let H be a group. Suppose that $x \in H$, $P \lhd H$ and $|H : P| < \infty$. If the subset x^P is separable in H, then there is a finite index normal subgroup $N \lhd H$ such that $N \leqslant P$ and $C_{H/N}(\psi(x)) \subseteq \psi\left(C_H(x)P\right)$ in H/N (where $\psi : H \to H/N$ denotes the natural homomorphism).*

The proof of Theorem 1.1 will also use the following two auxiliary statements.

Lemma 3.4. *Let A be a group and let $C_1, C_2 \leqslant A$ be isomorphic subgroups with a fixed isomorphism $\varphi : C_1 \to C_2$. Let $B = \langle A, t \,\|\, tgt^{-1} = \varphi(g) \text{ for all } g \in C_1\rangle$ be the corresponding HNN-extension of A. Suppose that $x, y \in A$ are elements such that $y \notin x^A$ and $x \notin C_i^A$ for $i = 1, 2$. Then $y \notin x^B$ and $C_B(x) = C_A(x)$ in B.*

Proof. Let $\mathcal{T}$ be the Bass-Serre tree associated to the splitting of B as an HNN-extension of A. Then x fixes a particular vertex v of $\mathcal{T}$, where the stabilizer $St_B(v)$ of v in B is equal to A. The stabilizer of any edge e, adjacent to v, is C_i^a for some $i \in \{1, 2\}$ and some $a \in A$ (see [14]). Therefore, the assumptions imply that x does not fix any edge of $\mathcal{T}$ adjacent to v. Since the fixed point set of an isometry of a tree is connected, it follows that v is the only vertex of $\mathcal{T}$ fixed by x.

Arguing by contradiction, suppose that $y \in x^B$, thus there is $b \in B$ such that $y = bxb^{-1}$ in B. Then $b \circ v$ is the only vertex of $\mathcal{T}$ fixed by y. Since $A = St_B(v)$ and $y \in A$ the latter implies that $b \circ v = v$. Hence $b \in St_B(v) = A$, i.e., $y \in x^A$, contradicting one of the assumptions. Thus $y \notin x^B$, as claimed.

For the final assertion, suppose that $b \in C_B(x)$, i.e., $x = bxb^{-1}$. The same argument as above shows that $b \in A$, hence $b \in C_A(x)$. □

Lemma 3.5. *Let A be a group with a free subgroup $F \leqslant A$ and let $g \in A \setminus \{1\}$ be an element of finite order. Suppose that every finite index subgroup of F is separable in A. Then there exists a finite index normal subgroup $N \lhd A$ such that $\psi(g) \notin \psi(F)^{A/N}$, where $\psi : A \to A/N$ denotes the natural epimorphism.*

Proof. Since every finite index subgroup of F is separable in A and F is residually finite, the assumptions imply that A is residually finite and the profinite topology of A induces the full profinite topology on F. Therefore by Lemma 3.2.6 in [13] the closure $\overline{F}$, of F, in the profinite completion $\widehat{A}$, of A, is naturally isomorphic to the profinite completion $\widehat{F}$ of F. Then in the profinite completion $\widehat{A}$, of A, the claim of the lemma reads as follows: g is not conjugate to $\overline{F} \cong \widehat{F}$ in $\widehat{A}$. Indeed, $\overline{F}^{\widehat{A}} = \varprojlim \psi_N(F)^{A/N}$, where $\psi_N : F \to F/N$ denotes the natural epimorphism and the inverse limit is taken over the directed set of all finite index normal subgroups $N \lhd_f A$. Therefore $\psi_N(g) \notin \psi_N(F)^{A/N}$ for some $N \lhd_f A$ if and only if $g \notin \overline{F}^{\widehat{A}}$. But $\overline{F} \cong \widehat{F}$ is torsion-free by Proposition 22.4.7 in [5], hence the result follows. □

4. Proofs

Proof of Theorem 1.1. Let G be a one-relator group given by the presentation (1). The result will be proved by induction on $RC(W)$. If $RC(W) = 0$ then G is isomorphic to the free product $F_m * \mathbb{Z}/n\mathbb{Z}$, where $m = |S| - 1$ and F_m is the free group of rank m. Therefore G is virtually free and so it is hereditarily conjugacy separable by Dyer's theorem [4].

Thus we can further assume that $RC(W) > 0$. Let $H \cong G * \mathbb{Z}$, K, M_1, M_2, U_1, U_2 and $\alpha : M_1 \to M_2$ be as described in Section 2. Then $K = \langle \overline{S} \,\|\, \overline{W}^n \rangle$, where $RC(\overline{W}) < RC(W)$, and so K is hereditarily conjugacy separable by induction. Since G is a retract of H, to prove the theorem it is enough to show that H is hereditarily conjugacy separable (cf. [11, Lemma 9.5]).

Observe that H is itself a one-relator group with torsion. Therefore, by Newman's theorem [12, Thm. 2], centralizers of non-trivial elements in H are cyclic. We also recall that, according to Lemma 2.1, H contains a finite index normal subgroup L which is hereditarily conjugacy separable.

Let $H_1 \leqslant H$ be an arbitrary finite index subgroup and let $x \in H$ be an arbitrary element. We will show that the subset x^{H_1} is separable in H by considering two different cases.

Case 1: x has infinite order in H. Since L is hereditarily conjugacy separable, $L_1 = H_1 \cap L$ is a normal conjugacy separable subgroup of finite index in H. Set $l = |H : L_1|$. Then $x^l \in L_1 \setminus \{1\}$ and $C_H(x^l)$ is infinite cyclic. It follows that for any $y \in H \setminus x^{H_1}$, $y^l \notin (x^l)^{H_1}$. Indeed, if $x^l = hy^lh^{-1}$ for some $h \in H_1$, then both x and hyh^{-1} belong to the infinite cyclic subgroup $C_H(x^l)$. But in the infinite cyclic group any element can have at most one l-th root, thus $x = hyh^{-1}$, contradicting the assumption that $y \notin x^{H_1}$.

Since L_1 is conjugacy separable, $(x^l)^{L_1}$ is closed in the profinite topology of L_1, and since $|H : L_1| < \infty$ this implies that $(x^l)^{L_1}$ is separable in H. Moreover, we can also deduce that the subset $(x^l)^{H_1}$ is separable in H, because it equals to a finite union of conjugates of $(x^l)^{L_1}$, as L_1 has finite index in H_1. Since $y^l \notin (x^l)^{H_1}$, there are a finite group Q and an epimorphism $\psi : H \to Q$ such that $\psi(y^l) \notin \psi\left((x^l)^{H_1}\right) = \left(\psi(x)^l\right)^{\psi(H_1)}$. Therefore $\psi(y) \notin \psi(x^{H_1})$ in Q, as required. Thus x^{H_1} is separable in H.

Case 2: x has finite order in H. Note that we can assume that $x \neq 1$ in H because otherwise $x^{H_1} = \{1\}$ is separable in H as H is residually finite (by Wise's work [16] H possesses a finite index subgroup that embeds into a right angled Artin group, and right angled Artin groups are well-known to be residually finite). Now we are going to verify that all the assumptions of Lemma 3.2 are satisfied.

Claim I: the conjugacy class x^H is separable in H.

By the torsion theorem for HNN-extensions ([9, IV.2.4]), $x \in K^H$. Thus, without loss of generality, we can assume that $x \in K$.

Consider any element $y \in H \setminus x^H$. If y has infinite order then, since H is residually finite, there is a finite group Q and an epimorphism $\psi : H \to Q$, such that the order of $\psi(y)$ in Q is greater than the order of x in H (and, hence, of $\psi(x)$ in Q). It follows that $\psi(x)$ is not conjugate to $\psi(y)$ in Q.

Thus we can further suppose that y also has finite order in H; as before this allows us to assume that $y \in K$. Consequently $y \in K \setminus x^K$, and by conjugacy separability of K, we can find a finite index normal subgroup $K_0 \lhd K$ such that the images of x and y, under the natural epimorphism $K \to K/K_0$, are not conjugate in K/K_0.

According to Lemmas 2.2 and 3.5, H contains finite index normal subgroups $N_1, N_2 \lhd H$ such that the image of x in H/N_i is not conjugate to the image of M_i for $i = 1, 2$. By Lemma 2.2, K_0 is separable in H, hence there exists a finite index normal subgroup $N_0 \lhd H$ such that $N_0 \cap K \subseteq K_0$. Let $N' \lhd H$ and $K_1 \lhd K$ denote the finite index normal subgroups of H and K respectively, defined by $N' = N_0 \cap N_1 \cap N_2$ and $K_1 = K \cap N'$.

Let $\xi : K \to K/K_1$ denote the natural epimorphism. Note that the isomorphism $\alpha : M_1 \to M_2$ gives rise to the isomorphism $\bar{\alpha} : \xi(M_1) \to \xi(M_2)$, defined by $\bar{\alpha}(\xi(g)) = \xi(\alpha(g))$ for all $g \in M_1$. Indeed, the fact that $\bar{\alpha}$ is well-defined is essentially due to the construction of K_1 as the intersection of K with a normal subgroup N' of H, and so ξ is a restriction to K of $\tilde{\xi} : H \to H/N'$. Thus for any $g, h \in M_1$ with $\xi(g) = \xi(h)$ we have

$$\bar{\alpha}(\xi(g)) = \tilde{\xi}(\alpha(g)) = \tilde{\xi}(tgt^{-1}) = \tilde{\xi}(t)\tilde{\xi}(g)\tilde{\xi}(t^{-1}) = \tilde{\xi}(tht^{-1}) = \tilde{\xi}(\alpha(h)) = \bar{\alpha}(\xi(h)).$$

Let $\bar{H}$ be the HNN-extension of K/K_1 with associated subgroups $\xi(M_1)$ and $\xi(M_2)$, defined by

$$\bar{H} = \langle K/K_1, \bar{t} \,\|\, \bar{t}\xi(u)\bar{t}^{-1} = \bar{\alpha}(\xi(u)) \text{ for all } u \in U_1\rangle.$$

Note that $\bar{H}$ is virtually free since $|K/K_1| < \infty$ (see, for example, [14, II.2.6, Prop. 11]). Clearly ξ extends to a homomorphism $\eta : H \to \bar{H}$, given by $\eta(t) = \bar{t}$ and $\eta(g) = \xi(g)$ for all $g \in K$.

Let us show that $\eta(x) = \xi(x)$ is not conjugate to $\eta(y) = \xi(y)$ in $\bar{H}$. Indeed, $\xi(y) \notin \xi(x)^{K/K_1}$ because the homomorphism $K \to K/K_0$ factors through ξ by construction (as $K_1 = K \cap N' \subseteq K \cap N_0 \subseteq K_0$) and the images of x and y are not conjugate in K/K_0. On the other hand, since $K_1 \subseteq N_1 \cap N_2$, we have $\xi(x) \notin \xi(M_i)^{K/K_1}$ for $i = 1, 2$. Therefore, $\xi(y) \notin \xi(x)^{\bar{H}}$ by Lemma 3.4.

It remains to recall that $\bar{H}$ is conjugacy separable by Dyer's theorem [4], and so there exist a finite group Q and a homomorphism $\zeta : \bar{H} \to Q$ such that $\zeta(\eta(y)) \notin \zeta(\eta(x))^Q$ in Q. Hence the homomorphism $\psi = \zeta \circ \eta : H \to Q$ distinguishes the conjugacy classes of x and y, as required. Thus we have shown that x^H is separable in H.

Claim II: x satisfies the Centralizer Condition CC_H from Definition 3.1.

This will be proved similarly to Claim I. As above, without loss of generality, we can assume that $x \in K$. Consider any finite index normal subgroup $P \lhd H$ and let $R = K \cap P$.

Since K is hereditarily conjugacy separable by induction, the finite index subgroup $E = R\langle x \rangle \leqslant K$ is conjugacy separable. Hence the subset $x^E = x^R$ is separable in E. And since $|K : E| < \infty$ we see that x^R is separable in K. Therefore we can apply Lemma 3.3 to find a finite index normal subgroup $K_0 \lhd K$ such that $K_0 \leqslant R$ and the centralizer of the image of x in K/K_0 is contained in the image of $C_K(x)R$ in K/K_0.

Arguing as in Claim I, we can choose finite index normal subgroups $N_0, N_1, N_2 \lhd H$ such that $K \cap N_0 \subseteq K_0$, and the image of x is not conjugate to the image of M_i in H/N_i for $i = 1, 2$. Set $N' = N_0 \cap N_1 \cap N_2$ and $K_1 = K \cap N'$. Similarly to Claim I, the homomorphism $\xi : K \to K/K_1$ extends to a homomorphism $\eta : H \to \bar{H}$, where $\bar{H}$ is an HNN-extension of K/K_1 with associated subgroups $\xi(M_1)$ and $\xi(M_2)$.

Denote $\bar{x} = \eta(x) = \xi(x) \in K/K_1 \leqslant \bar{H}$. As before, since $K_1 \leqslant N_i$, we have that $\bar{x} \notin \xi(M_i)^{K/K_1}$, $i = 1, 2$, and so we can use Lemma 3.4 to conclude that $C_{\bar{H}}(\bar{x}) = C_{K/K_1}(\bar{x})$. Recall that $K_1 \leqslant K_0$, hence the epimorphism from K to K/K_0 factors through ξ. Therefore in $\bar{H}$ we have

$$C_{\bar{H}}(\bar{x}) = C_{K/K_1}(\bar{x}) \subseteq \xi(C_K(x)RK_0) = \xi(C_K(x)R) \subseteq \eta(C_H(x)P), \tag{3}$$

because $K_0 \leqslant R \leqslant P$ by construction.

Once again, $\bar{H}$ is virtually free and so is any subgroup of it. Therefore $\bar{P}\langle \bar{x} \rangle \leqslant \bar{H}$ is conjugacy separable by Dyer's theorem [4], where $\bar{P} = \eta(P)$ is a finite index normal subgroup of $\bar{H}$. As above this yields that the subset $\bar{x}^{\bar{P}\langle \bar{x} \rangle} = \bar{x}^{\bar{P}}$ is separable in $\bar{H}$. By Lemma 3.3 there exists a finite index normal subgroup $\bar{N} \lhd \bar{H}$ such that $\bar{N} \leqslant \bar{P}$ and

$$C_{\bar{H}/\bar{N}}(\zeta(\bar{x})) \subseteq \zeta\left(C_{\bar{H}}(\bar{x})\bar{P}\right), \tag{4}$$

where $\zeta : \bar{H} \to \bar{H}/\bar{N}$ is the natural epimorphism.

Let $N = \eta^{-1}(\bar{N})$ be the full preimage of $\bar{N}$ in H, and let $\psi : H \to H/N$ be the natural homomorphism. Then $\psi = \zeta \circ \eta$ and $\bar{H}/\bar{N}$ can be identified with H/N. A combination of (4) and (3) gives the following inclusion in H/N:

$$C_{H/N}(\psi(x)) \subseteq \zeta\left(C_{\bar{H}}(\bar{x})\bar{P}\right) \subseteq \zeta\left(\eta(C_H(x)P)\bar{P}\right) = \psi(C_H(x)P). \tag{5}$$

To finish the proof of Claim II it remains to show that $N \leqslant P$. Since $\eta(N) = \bar{N} \leqslant \bar{P} = \eta(P)$, it is enough to prove that $\ker\eta \leqslant P$. To this end, observe that $\ker\eta$ is the normal closure of $K_1 = \ker\xi$ in H (this easily follows from the universal property of HNN-extensions and is left as an exercise for the reader). Since $K_1 \leqslant K_0 \leqslant R \leqslant P$ and $P \lhd H$, we see that the normal closure of K_1 in H must also be contained in P. Thus $\ker\eta \leqslant P$, implying that $N \leqslant P$, which finishes the proof of Claim II.

In order to apply Lemma 3.2 we should also note that the subset $C_H(x)H_1$ splits in a finite union of left cosets modulo H_1 in H because $|H : H_1| < \infty$, and hence this subset is separable in H. In view of Claims I, II we see that all of the assumptions of Lemma 3.2 are satisfied. Therefore x^{H_1} is separable in H, and the consideration of Case 2 is finished.

Thus we have shown that x^{H_1} is separable in H for all $x \in H$ and any finite index subgroup $H_1 \leqslant H$. Since the profinite topology of a subgroup is finer than the topology induced from the ambient group, we can conclude that x^{H_1} is separable in H_1 whenever $x \in H_1$. Consequently H_1 is conjugacy separable. Since H_1 was chosen as an arbitrary finite index subgroup of H, we see that H is hereditarily conjugacy separable. □

Proof of Corollary 1.2. Let $H \leqslant G$ be a quasiconvex subgroup. By Newman's theorem [12, Thm. 2], for any $x \in H\setminus\{1\}$ there is $g \in G$ such that $C_G(x) = \langle g\rangle$. Hence $x = g^k \in H$ for some $k \in \mathbb{N}$ and so the subset $C_G(x)H$ splits in a finite union of left cosets modulo H. Now, since G is virtually compact special by [16, Cor. 18.3], quasiconvex subgroups are separable in G by [7, Thm. 7.3, Lemma 7.5]. It follows that H and, hence, $C_G(x)H$ are separable in G, for an arbitrary $x \in H$ (if $x = 1$ then $C_G(x)H = G$).

By Theorem 1.1, G is hereditarily conjugacy separable and so every element $x \in G$ satisfies CC_G (see [11, Prop. 3.2]). Therefore we can apply Lemma 3.2 to conclude that x^H is separable in G (and, hence, in H). Thus H is conjugacy separable, as claimed. □

References

[1] R.B.J.T Allenby, C.Y. Tang, Conjugacy separability of certain 1-relator groups with torsion. *J. Algebra* **103** (1986), no. 2, 619-637.

[2] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, H. Short, Notes on word hyperbolic groups. In: Short, H.B., ed. Group Theory from a Geometrical Viewpoint, Proc. ICTP Trieste, *World Scientific Publishing Co., Inc., River Edge, NJ*, pp. 3-63.

[3] R. Charney, An introduction to right-angled Artin groups. *Geom. Dedicata* **125** (2007), 141-158.

[4] J.L. Dyer, Separating conjugates in free-by-finite groups.*J. London Math. Soc. (2)* **20** (1979), no. 2, 215-221.

[5] M.D. Fried, M. Jarden, Field Arithmetic. Third edition. Revised by Jarden. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 11. *Springer-Verlag, Berlin*, 2008.

[6] A.V. Goryaga, Example of a finite extension of an FAC-group that is not an FAC-group (Russian). *Sibirsk. Mat. Zh.* **27** (1986), no. 3, 203-205, 225.

[7] F. Haglund, D.T. Wise, Special cube complexes. *Geom. Funct. Anal.* **17** (2008), no. 5, 1551-1620.

[8] A. Karrass, D. Solitar, The free product of two groups with a malnormal amalgamated subgroup. *Canad. J. Math.* **23** (1971), 933-959.
[9] R.C. Lyndon, P.E. Schupp,Combinatorial group theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. *Springer-Verlag, Berlin-New York*, 1977. xiv+339 pp.
[10] V.D. Mazurov, E.I. Khukhro (eds.), The Kourovka notebook. Unsolved problems in group theory. Sixteenth edition. Including archive of solved problems. *Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk*, 2006. 178 pp.
[11] A. Minasyan, Hereditary conjugacy separability of right angled Artin groups and its applications. *Groups Geom. Dyn.* **6** (2012), no. 2, 335-388.
[12] B.B. Newman, Some results on one-relator groups. *Bull. Amer. Math. Soc.* **74** (1968), 568-571.
[13] L. Ribes, P.A. Zalesskii, Profinite groups. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 40. *Springer-Verlag, Berlin,* 2010. xvi+464 pp.
[14] J.-P. Serre, Arbres, amalgames, SL_2. *Astérisque*, no. **46**. Société Mathématique de France, Paris, 1977. 189 pp.
[15] C.Y. Tang, Conjugacy separability of certain 1-relator groups. *Proc. Amer. Math. Soc.* **86** (1982), no. 3, 379-384.
[16] D.T. Wise, The structure of groups with a quasiconvex hierarchy. Preprint (2011). Available from `http://www.math.mcgill.ca/wise/papers.html`.

(Ashot Minasyan) School of Mathematics, University of Southampton, Highfield, Southampton, SO17 1BJ, United Kingdom.

E-mail address: aminasyan@gmail.com

(Pavel Zalesskii) Departamento de Matemática, Universidade de Brasília, 70910-900 Brasília-DF, Brazil.

E-mail address: pz@mat.unb.br