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The homotopy type of the polyhedral product for shifted complexes

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Abstract

We prove a conjecture of Bahri, Bendersky, Cohen and Gitler: if K is a shifted simplicial complex on n vertices, $X_1, \ldots, X_n$ are pointed connected CW-complexes and CX_i is the cone on X_i, then the polyhedral product determined by K and the pairs (CX_i, X_i) is homotopy equivalent to a wedge of suspensions of smashes of the X_i's. Earlier work of the authors dealt with the special case where each X_i is a loop space. New techniques are introduced to prove the general case. These have the advantage of simplifying the earlier results and of being sufficiently general to show that the conjecture holds for a substantially larger class of simplicial complexes. We discuss connections between polyhedral products and toric topology, combinatorics, and classical homotopy theory.

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1. Introduction

Polyhedral products generalise the notion of a product of spaces. They are of widespread interest due to their being fundamental objects which arise in many areas of mathematics. For example, in algebraic geometry special cases of polyhedral products are toric projective varieties, in combinatorics they appear as the complements of complex coordinate subspace arrangements,

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in complex geometry they can be recognised as intersections of quadrics, in dynamical systems they arise as invariants of a system, and in robotics they are related to configuration spaces of planar linkages. Their topological properties have attracted a great deal of recent attention due to their emergence as central objects of study in toric topology. This includes the foundational work in [3,4], and treatments of their cohomology rings [4,6], rational homotopy [7,11], and homotopy types [1,2,9,10].

Let K be a simplicial complex on n vertices. For $1 \leq i \leq n$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed subspace of X_i. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^n$ be the sequence of pairs. For each simplex (face) $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be the subspace of $\prod_{i=1}^n X_i$ defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^n Y_i \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is the CW-complex

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^n X_i.$$

For example, suppose each A_i is a point. If K is a disjoint union of n points then $(\underline{X}, \underline{*})^K$ is the wedge $X_1 \vee \cdots \vee X_n$, and if K is the standard $(n-1)$-simplex then $(\underline{X}, \underline{*})^K$ is the product $X_1 \times \cdots \times X_n$.

The polyhedral product $(\underline{X}, \underline{*})^K$ is related to another case of particular interest. Observe that any polyhedral product $(\underline{X}, \underline{A})^K$ is a subset of the product $X_1 \times \cdots \times X_n$. In the special case $(\underline{X}, \underline{*})^K$, there is a homotopy fibration

$$(\underline{C\Omega X}, \underline{\Omega X})^K \longrightarrow (\underline{X}, \underline{*})^K \longrightarrow \prod_{i=1}^n X_i \tag{1}$$

where $C\Omega X$ is the cone on ΩX. Special cases of this fibration recover some classical results in homotopy theory. For example, if K is two distinct points, then $(\underline{C\Omega X}, \underline{\Omega X})^K$ is the fibre of the inclusion $X_1 \vee X_2 \longrightarrow X_1 \times X_2$. Ganea [8] identified the homotopy type of this fibre as $\Sigma \Omega X_1 \wedge \Omega X_2$. If $K = \Delta_k^{n-1}$ is the full k-skeleton of the standard n-simplex, then Porter [12] showed that for $0 \leq k \leq n-2$ there is a homotopy equivalence

$$(\underline{C\Omega X}, \underline{\Omega X})^K \simeq \bigvee_{j=k+2}^n \left(\bigvee_{1 \leq i_1 < \cdots < i_j \leq n} \binom{j-1}{k+1} \Sigma^{k+1} \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_j} \right)$$

where $j \cdot Y$ denotes the wedge sum of j copies of the space Y.

The emergence of toric topology in the 1990s brought renewed attention to these classical results, in a new context. Davis and Januszkiewicz [5] constructed a new family of manifolds with a torus action. The construction started with a simple convex polytope P with n facets, passed to the simplicial complex $K = \partial P^*$ – the boundary of the dual of P, which has n vertices – and associated to it a manifold $\mathcal{Z}_K$ with a torus action and its homotopy orbit space $DJ(K)$. As a direct consequence of these definitions, there is a homotopy fibration

$$\mathcal{Z}_K \longrightarrow DJ(K) \longrightarrow \prod_{i=1}^n \mathbb{C}P^\infty.$$

Buchstaber and Panov [3] recognised the spaces $DJ(K)$ and $\mathcal{Z}_K$ as the polyhedral products $(\underline{\mathbb{C}P^\infty}, \underline{*})^K$ and $(\underline{D}^2, \underline{S}^1)^K$ respectively, which allowed them to generalise Davis and Januszkiewicz's construction to any simplicial complex K on n vertices (not necessarily the boundary of the dual of a simple polytope) and keep the homotopy fibration

$$\mathcal{Z}_K \longrightarrow DJ(K) \longrightarrow \prod_{i=1}^{n} \mathbb{C}P^\infty$$

for any simplicial complex K on n vertices. The spaces $DJ(K)$ and $\mathcal{Z}_K$ are central objects of study in toric topology, and their thorough study in [3,4] launched toric topology into the mainstream of modern algebraic topology. The generalisation to polyhedral products soon followed in unpublished notes by Strickland and under the name K-powers in [3], appeared without fanfare in [9,10], and came to prominence in recent work of Bahri, Bendersky, Cohen and Gitler [1].

Following Ganea's and Porter's results, it is natural to ask when the homotopy type of the fibre $(\underline{C\Omega X}, \underline{\Omega X})^K$ in (1) can be recognised. It is too ambitious to hope to do this for all K, but it is reasonable to expect that it can be done for certain families of simplicial complexes. This is precisely what was done in earlier work of the authors. A simplicial complex K is *shifted* if there is an ordering on its vertices such that whenever $\sigma \in K$ and $\nu' < \nu$, then $(\sigma - \nu) \cup \nu' \in K$. This is a fairly large family of complexes, which includes Porter's case of full k-skeletons of a standard n-simplex. In [9] it was shown that for a family of complexes which contains shifted complexes (and some non-shifted complexes), there is a homotopy equivalence

$$(\underline{C\Omega X}, \underline{\Omega X})^K \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma^{\alpha(t)} \Omega X_1^{(\alpha_1)} \wedge \cdots \wedge \Omega X_n^{(\alpha_n)} \tag{2}$$

for some index set $\mathcal{I}$ (which can be made explicit), where $1 \le \alpha(t) \le n-1$, each $\alpha_i \in \{0, 1\}$ for $1 \le i \le n$, and if $\alpha_i = 0$ then the smash product is interpreted as omitting the factor X_i rather than being trivial. The homotopy equivalence (2) has implications in combinatorics. In [3], it was shown that $\mathcal{Z}_K$ is homotopy equivalent to the complement of the coordinate subspace arrangement determined by K. Such spaces have a long history of study by combinatorists. In particular, as $\mathcal{Z}_K = (\underline{D}^2, \underline{S}^1)^K$, the homotopy equivalence (2) implies that $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, which answered a major outstanding problem in combinatorics.

Bahri, Bendersky, Cohen and Gitler [1] gave a general decomposition of $\Sigma(\underline{X}, \underline{A})^K$, which in the special case of $\Sigma(\underline{CX}, \underline{X})^K$ is as follows. Regard the simplices of K as ordered sequences, $(i_1, \ldots, i_k)$ where $1 \le i_1 < \cdots < i_k \le n$. Let $\widehat{X}^I = X_{i_1} \wedge \cdots \wedge X_{i_k}$. Let $Y * Z$ be the *join* of the topological spaces X and Y, and recall that there is a homotopy equivalence $Y * Z \simeq \Sigma Y \wedge Z$. Let $K_I \subseteq K$ be the full subcomplex of K consisting of the simplices in K which have all their vertices in I, that is, $K = \{\sigma \cap I \mid \sigma \in K\}$. Let $|K|$ be the geometric realisation of the simplicial complex K. Then for any simplicial complex K, there is a homotopy equivalence

$$\Sigma(\underline{CX}, \underline{X})^K \simeq \Sigma\left(\bigvee_{I \notin K} |K_I| * \widehat{X}^I\right). \tag{3}$$

In particular, (3) agrees with the suspension of the homotopy equivalence in (2) in the case of $(\underline{C\Omega X}, \underline{\Omega X})^K$. Bahri, Bendersky, Cohen and Gitler observed that if K is shifted then each $|K_I|$ is homotopy equivalent to a wedge of spheres, and they conjectured that the decomposition (3) desuspends. Our main result is that this conjecture is true.

Theorem 1.1. *Let K be a shifted complex. Then there is a homotopy equivalence*

$$(\underline{CX}, \underline{X})^K \simeq \left(\bigvee_{I \notin K} |K_I| * \widehat{X}^I \right)$$

where each $|K_I|$ is homotopy equivalent to a wedge of spheres.

The methods used to prove the results in [9] in the case $(\underline{C\Omega X}, \underline{\Omega X})^K$ involved analysing properties of the fibration (1). In the general case of $(\underline{CX}, \underline{X})^K$, no such fibration exists, so we need to develop new methods. An added benefit is that these new methods also give a much faster proof of the results in [9]. As well, in Sections 7 and 8 we extend our methods to desuspend (3) in cases where K is not necessarily shifted.

2. A special case

Let Δ^{n-1} be the standard $(n-1)$-simplex. In this brief section we will identify $(\underline{CX}, \underline{X})^K$ when $K = \partial \Delta^{n-1}$. We begin with a general observation which holds for any $(\underline{X}, \underline{A})$.

A face $\sigma \in K$ is called *maximal* if there is no other face $\sigma' \in K$ with the property that $\sigma \subsetneq \sigma'$. In other words, a non-maximal face of K is a proper subset of another face of K. Therefore $|K| = \bigcup_{\sigma \in \mathcal{I}} |\sigma|$ where $\mathcal{I}$ runs over the list of maximal faces of K. By its definition, the polyhedral product $(\underline{X}, \underline{A})^K$ is a colimit over the faces of K, so as $\mathcal{I}$ is cofinal, we immediately obtain the following.

Lemma 2.1. *There is an equality of spaces $(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in \mathcal{I}} (\underline{X}, \underline{A})^\sigma$ where $\mathcal{I}$ runs over the list of maximal faces of K.* $\square$

For example, let $K = \partial \Delta^{n-1}$. The maximal faces of K are $\bar{\sigma}_i = (1, \ldots, \hat{i}, \ldots, n)$ for $1 \leq i \leq n$, where $\hat{i}$ means omit the ith-coordinate. Thus $|K| = \bigcup_{i=1}^n |\bar{\sigma}_i|$ and Lemma 2.1 implies that $(\underline{X}, \underline{A})^K = \bigcup_{i=1}^n \underline{X}^{\bar{\sigma}_i}$. Explicitly, we have $\underline{X}^{\bar{\sigma}_i} = X_1 \times \cdots \times A_i \times \cdots \times X_n$ so

$$(\underline{X}, \underline{A})^K = \bigcup_{i=1}^n X_1 \times \cdots \times A_i \times \cdots \times X_n.$$

As a special case, consider $(\underline{CX}, \underline{X})^K$. Then

$$(\underline{CX}, \underline{X})^K = \bigcup_{i=1}^n \underline{CX}^{\bar{\sigma}_i} = \bigcup_{i=1}^n CX_1 \times \cdots \times X_i \times \cdots \times CX_n. \quad (4)$$

Porter [12, Appendix, Theorem 3] showed that there is a homotopy equivalence

$$\Sigma^{n-1} X_1 \wedge \cdots \wedge X_n \simeq \bigcup_{i=1}^n CX_1 \times \cdots \times X_i \times \cdots \times CX_n.$$

It will be convenient later to regard $\Sigma^{n-1} X_1 \wedge \cdots \wedge X_n$ as an iterated join. Recall that the *join* of spaces A and B is $A * B = A \times I \times B / \sim$, where $(x, 0, y_1) \sim (x, 0, y_2)$ and $(x_1, 1, y) \sim (x_2, 1, y)$, and there is a homotopy equivalence $A * B \simeq \Sigma A \wedge B$. Iterating in our case, we obtain a homotopy equivalence $\Sigma^{n-1} X_1 \wedge \cdots \wedge X_n \simeq X_1 * \cdots * X_n$. Thus we obtain the following.

Proposition 2.2. *Let $K = \partial \Delta^{n-1}$. Then there is a homotopy equivalence*

$$(\underline{CX}, \underline{X})^K \simeq X_1 * \cdots * X_n. \quad \square$$

3. Some general properties of polyhedral products

In this section we establish some general properties of polyhedral products which will be used later. First, we consider how the polyhedral product functor behaves with respect to a union of simplicial complexes. Let K be a simplicial complex on n vertices and suppose that $K = K_1 \cup_L K_2$. Relabelling the vertices if necessary, we may assume that K_1 is defined on the vertices $\{1, \ldots, m\}$, K_2 is defined on the vertices $\{m - l + 1, \ldots, n\}$ and L is defined on the vertices $\{m-l+1, \ldots, m\}$. By including the vertex set $\{1, \ldots, m\}$ into the vertex set $\{1, \ldots, n\}$, we may regard K_1 as a simplicial complex on n vertices. Call the resulting simplicial complex on n vertices $\overline{K}_1$. Note that the vertices $\{m + 1, \ldots, n\}$ are not simplices of $\overline{K}_1$. Similarly, we may define simplicial complexes $\overline{K}_2$ and $\overline{L}$ on n vertices. Then we have $K = \overline{K}_1 \cup_{\overline{L}} \overline{K}_2$. The point in doing this is that we can now construct polyhedral products for all four objects K, $\overline{K}_1$, $\overline{K}_2$ and $\overline{L}$ using the same pairs of spaces $(X_i, A_i)_{i=1}^n$.

Proposition 3.1. *Let K be a simplicial complex on n vertices. Suppose there is a pushout*

$$\begin{array}{ccc} L & \longrightarrow & K_2 \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & K \end{array}$$

where $L = K_1 \cap K_2$. Then there is a pushout

$$\begin{array}{ccc} (\underline{X}, \underline{A})^{\overline{L}} & \longrightarrow & (\underline{X}, \underline{A})^{\overline{K}_2} \\ \downarrow & & \downarrow \\ (\underline{X}, \underline{A})^{\overline{K}_1} & \longrightarrow & (\underline{X}, \underline{A})^{K} \end{array}$$

where each of the maps is an inclusion. Consequently, the polyhedral product commutes with pushouts.

Proof. Since $K = K_1 \cup_L K_2$ and K is finite, the simplices in K can be put into three finite collections: ($\mathcal{A}$) the simplices in L, ($\mathcal{B}$) the simplices in K_1 that are not simplices of L and ($\mathcal{C}$) the simplices of K_2 that are not simplices of L. Thus we have

$$L = \bigcup_{\sigma\in\mathcal{A}} \sigma$$

$$K_1 = \left(\bigcup_{\sigma\in\mathcal{A}} \sigma\right) \cup \left(\bigcup_{\sigma'\in\mathcal{B}} \sigma'\right)$$

$$K_2 = \left(\bigcup_{\sigma\in\mathcal{A}} \sigma\right) \cup \left(\bigcup_{\sigma''\in\mathcal{C}} \sigma''\right)$$

$$K = \left(\bigcup_{\sigma\in\mathcal{A}} \sigma\right) \cup \left(\bigcup_{\sigma'\in\mathcal{B}} \sigma'\right) \cup \left(\bigcup_{\sigma''\in\mathcal{C}} \sigma''\right).$$

By definition, for any simplicial complex M on n vertices, $(\underline{X}, \underline{A})^M = \bigcup_{\sigma\in M} (\underline{X}, \underline{A})^\sigma$. So in our case, we have

$$(\underline{X}, \underline{A})^{\overline{L}} = \bigcup_{\sigma \in \mathcal{A}} (\underline{X}, \underline{A})^{\sigma}$$

$$(\underline{X}, \underline{A})^{\overline{K}_1} = \left(\bigcup_{\sigma \in \mathcal{A}} (\underline{X}, \underline{A})^{\sigma} \right) \cup \left(\bigcup_{\sigma' \in \mathcal{B}} (\underline{X}, \underline{A})^{\sigma'} \right)$$

$$(\underline{X}, \underline{A})^{\overline{K}_2} = \left(\bigcup_{\sigma \in \mathcal{A}} (\underline{X}, \underline{A})^{\sigma} \right) \cup \left(\bigcup_{\sigma'' \in \mathcal{C}} (\underline{X}, \underline{A})^{\sigma''} \right)$$

$$(\underline{X}, \underline{A})^{K} = \left(\bigcup_{\sigma \in \mathcal{A}} (\underline{X}, \underline{A})^{\sigma} \right) \cup \left(\bigcup_{\sigma' \in \mathcal{B}} (\underline{X}, \underline{A})^{\sigma'} \right) \cup \left(\bigcup_{\sigma'' \in \mathcal{C}} (\underline{X}, \underline{A})^{\sigma''} \right).$$

In particular, since $(\underline{X}, \underline{A})^{\overline{L}} = (\underline{X}, \underline{A})^{\overline{K}_1} \cap (\underline{X}, \underline{A})^{\overline{K}_2}$ we have

$$(\underline{X}, \underline{A})^{K} = (\underline{X}, \underline{A})^{\overline{K}_1} \cup_{(\underline{X}, \underline{A})^{\overline{L}}} (\underline{X}, \underline{A})^{\overline{K}_2}$$

which implies the existence of the asserted pushout. □

Next, suppose K is a simplicial complex on n vertices. Let L be a subcomplex of K. Reordering the indices if necessary, assume that the vertices of L are $\{1, \ldots, m\}$ for $m \leq n$. For the application we have in mind, specialise to $(\underline{CX}, \underline{X})^K$. Let $\overline{X} = \prod_{i=m+1}^{n} X_i$. Since the indices of the factors in $\overline{X}$ are complementary to the vertex set $\{1, \ldots, m\}$ of L, the inclusion $L \longrightarrow K$ induces an inclusion $I : (\underline{CX}, \underline{X})^L \times \overline{X} \longrightarrow (\underline{CX}, \underline{X})^K$. In Proposition 3.4 we show that the restriction of I to $\overline{X}$ is null homotopic. We first need a preparatory lemma.

Lemma 3.2. *The inclusion*

$$J : X_1 \times \cdots \times X_n \longrightarrow \bigcup_{i=1}^{n} X_1 \times \cdots \times CX_i \times \cdots \times X_n$$

is null homotopic.

Proof. For $1 \leq k \leq n$, let $F_k = \bigcup_{i=1}^{k} X_1 \times \cdots \times CX_i \times \cdots \times X_n$. Then $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n$, and $\{F_k\}_{k=1}^{n}$ is a filtration of $\bigcup_{i=1}^{n} X_1 \times \cdots \times CX_i \times \cdots \times X_n$. Observe that J factors as a composite of inclusions $X_1 \times \cdots \times X_n \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_n$.

Consider first the inclusion $X_1 \times \cdots \times X_n \longrightarrow F_1 = CX_1 \times X_2 \times \cdots \times X_n$. The cone in the first coordinate of F_1 implies that this inclusion is homotopic to the composite $X_1 \times \cdots \times X_n \xrightarrow{\pi_1} X_2 \times \cdots \times X_n \xrightarrow{\varphi_1} CX_1 \times X_2 \times \cdots \times X_n$, where π_1 is the projection and φ_1 is the inclusion. (Note that in $CX_1 = [0, 1] \times X_1 / \sim$ where $(1, x) \sim (1, x')$ for all $x, x' \in X$, we assume the basepoint is $[0, *]$ so φ_1 is a pointed map.) Composing into $F_2 = CX_1 \times X_2 \times \cdots \times X_n \cup X_1 \times CX_2 \times X_3 \times \cdots \times X_n$, we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} & & X_2 \times \cdots \times X_n & \longrightarrow & CX_2 \times X_3 \times \cdots \times X_n \\ & \overset{\pi_1}{\nearrow} & \downarrow{\scriptstyle \varphi_1} & & \downarrow \\ X_1 \times \cdots \times X_n & \longrightarrow & F_1 & \longrightarrow & F_2 \end{array}$$

where the square strictly commutes and each map in the square is an inclusion. As before, the map $X_2 \times \cdots \times X_n \longrightarrow CX_2 \times X_3 \times \cdots \times X_n$ in the top row is homotopic to the composite $X_2 \times \cdots \times X_n \longrightarrow X_3 \times \cdots \times X_n \longrightarrow CX_2 \times X_3 \times \cdots \times X_n$ where the left map is the projection and the right map is the inclusion. Thus the inclusion $X_1 \times \cdots \times X_n \longrightarrow F_2$ is homotopic to the composite $X_1 \times \cdots \times X_n \xrightarrow{\pi_2} X_3 \times \cdots \times X_n \xrightarrow{\varphi_2} F_2$, where π_2 is the projection and φ_2 is an inclusion. Iterating, we obtain that the inclusion $X_1 \times \cdots \times X_n \xrightarrow{J} F_n$ is homotopic to the composite $X_1 \times \cdots \times X_n \xrightarrow{\pi_n} * \xrightarrow{\varphi_n} F_n$ where π_n is the projection and φ_n is the inclusion. Hence J is null homotopic. □

A useful consequence of Lemma 3.2 is the following.

Corollary 3.3. *Let K be a simplicial complex on n vertices and suppose that each vertex is in K. Then the inclusion $X_1 \times \cdots \times X_n \longrightarrow (\underline{CX}, \underline{X})^K$ is null homotopic.* □

Proof. Let P be the disjoint union of all the vertices in K. Then the inclusion $X_1 \times \cdots \times X_n \longrightarrow (\underline{CX}, \underline{X})^K$ factors as the composite $X_1 \times \cdots \times X_n \longrightarrow (\underline{CX}, \underline{X})^P \longrightarrow (\underline{CX}, \underline{X})^K$. Notice that $(\underline{CX}, \underline{X})^P$ is exactly the target of the map J in Lemma 3.2. Therefore the left map in the preceding composite is null homotopic, and the corollary follows. □

Proposition 3.4. *Let K be a simplicial complex on the index set $[n]$ and let L be a subcomplex of K on $[m]$, where $m \le n$. Suppose that each vertex $\{i\} \in K$ for $m+1 \le i \le n$. Let $\overline{X} = \prod_{i=m+1}^{n} X_i$. Then the restriction of $(\underline{CX}, \underline{X})^L \times \overline{X} \xrightarrow{I} (\underline{CX}, \underline{X})^K$ to $\overline{X}$ is null homotopic.*

Proof. By definition of $(\underline{CX}, \underline{X})^\sigma$, we have $(\underline{CX}, \underline{X})^{\{i\}} = X_1 \times \cdots \times CX_i \times \cdots X_n$. Since each vertex $\{i\} \in K$ for $m+1 \le i \le n$, we obtain an inclusion

$$I'' \colon \bigcup_{i=m+1}^{n} X_1 \times \cdots \times CX_i \times \cdots X_n \longrightarrow (\underline{CX}, \underline{X})^K.$$

Now there is a commutative diagram of inclusions

$$\begin{array}{ccc} X_1 \times \cdots \times X_n & \longrightarrow & \bigcup_{i=m+1}^{n} X_1 \times \cdots \times CX_i \times \cdots X_n \\ \downarrow & & \downarrow \\ (\underline{CX}, \underline{X})^L \times X_{m+1} \times \cdots \times X_n & \xrightarrow{I} & (\underline{CX}, \underline{X})^K \end{array}$$

where, by Lemma 3.2, the restriction of the top horizontal map to $\overline{X}$ is null homotopic. Therefore the restriction of I to $\overline{X}$ is null homotopic. □

4. The homotopy type of $(\underline{CX}, \underline{X})^K$ for $K = L \cup_{\partial\sigma} \sigma$

The goal of this section is to prove Theorem 4.6, which specifies properties of $K = L \cup_{\partial\sigma} \sigma$ that allow us to determine the homotopy type of $(\underline{CX}, \underline{X})^K$ from that of $(\underline{CX}, \underline{X})^L$. This will be a key tool in an inductive procedure for proving Theorem 1.1, which identifies the homotopy type of $(\underline{CX}, \underline{X})^K$ for a shifted complex K.

We begin with a standard definition from combinatorics. Given simplicial complexes K_1 and K_2 on sets $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, the *join* $K_1 * K_2$ is the simplicial complex

$$K_1 * K_2 := \{\sigma \subset \mathcal{S}_1 \cup \mathcal{S}_2 \mid \sigma = \sigma_1 \cup \sigma_2, \sigma_1 \in K_2, \sigma_2 \in K_1\}$$

on the set $\mathcal{S}_1 \cup \mathcal{S}_2$. The definition of the polyhedral product immediately implies the following.

Lemma 4.1. *Let K_1 and K_2 be simplicial complexes on the index sets $\{1, \ldots, n\}$ and $\{n+1, \ldots, m\}$, respectively. Then $(\underline{X}, \underline{A})^{K_1 * K_2} = (\underline{X}, \underline{A})^{K_1} \times (\underline{X}, \underline{A})^{K_2}$.* □

If K is a simplicial complex on the index set $[n]$ then the *cone* on K is $K * \{n+1\}$. Applying Lemma 4.1, we obtain the following, which will be of use later.

Corollary 4.2. *Let K be a simplicial complex on the index set $[n]$. Then $(\underline{X}, \underline{A})^{K*\{n+1\}} = (\underline{X}, \underline{A})^K \times X_{n+1}$. Consequently, $(\underline{CX}, \underline{X})^{K*\{n+1\}} = (\underline{CX}, \underline{X})^K \times CX_{n+1}$.* □

As another useful observation, consider the inclusions of $\partial\Delta^{n-1}$ into $\partial\Delta^{n-1} * \{n+1\}$ and Δ^{n-1}.

Lemma 4.3. *There is a pushout*

$$\begin{array}{ccc} \partial\Delta^{n-1} & \longrightarrow & \Delta^{n-1} \\ \downarrow & & \downarrow \\ \partial\Delta^{n-1} * \{n+1\} & \longrightarrow & \partial\Delta^n. \end{array} \quad \square$$

Applying Proposition 3.1 in the case of $(\underline{CX}, \underline{X})$ to the pushout in Lemma 4.3, we obtain a homotopy equivalence

$$(\underline{CX}, \underline{X})^{\partial\Delta^n} \simeq (\underline{CX}, \underline{X})^{\partial\Delta^{n-1}*\{n+1\}} \cup_{(\underline{CX}, \underline{X})^{\overline{\partial\Delta}^{n-1}}} (\underline{CX}, \underline{X})^{\overline{\Delta}^{n-1}}. \tag{5}$$

It will be useful to state this homotopy equivalence more explicitly. By (4),

$$(\underline{CX}, \underline{X})^{\partial\Delta^{n-1}} = \bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n.$$

So by the definition of $\overline{\partial\Delta}^{n-1}$ we have

$$(\underline{CX}, \underline{X})^{\overline{\partial\Delta}^{n-1}} = \left(\bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n\right) \times X_{n+1}.$$

As well, by the definition of the polyhedral product, we have

$$(\underline{CX}, \underline{X})^{\Delta^{n-1}} = CX_1 \times \cdots \times CX_n.$$

So by the definition of $\overline{\Delta}^{n-1}$ we have

$$(\underline{CX}, \underline{X})^{\overline{\Delta}^{n-1}} = CX_1 \times \cdots \times CX_n \times X_{n+1}.$$

By Corollary 4.2, we have

$$(\underline{CX}, \underline{X})^{\partial\Delta^{n-1}*\{n+1\}} = (\underline{CX}, \underline{X})^{\partial\Delta^{n-1}} \times CX_{n+1}.$$

Thus we obtain

$$(\underline{CX}, \underline{X})^{\partial\Delta^{n-1}*\{n+1\}} = \left(\bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n\right) \times CX_{n+1}.$$

Therefore (5) states the following.

Lemma 4.4. *There is a pushout*

$$\begin{array}{ccc} \left(\bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n\right) \times X_{n+1} & \xrightarrow{b} & CX_1 \times \cdots \times CX_n \times X_{n+1} \\ \big\downarrow a & & \big\downarrow \\ \left(\bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n\right) \times CX_{n+1} & \longrightarrow & (\underline{CX}, \underline{X})^{\partial\Delta^n} \end{array}$$

where the maps a and b are coordinate-wise inclusions. □

Note that this pushout identifies $(\underline{CX}, \underline{X})^{\partial\Delta^n}$ as $\bigcup_{i=1}^{n+1} CX_1 \times \cdots \times X_i \times \cdots \times CX_{n+1}$, which matches the description in (4). Since a is a coordinate-wise inclusion and CX_{n+1} is contractible, a is homotopic to the composite

$$\overline{\pi}_1\colon \left(\bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n\right) \times X_{n+1} \xrightarrow{\pi_1} \bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n$$
$$\xrightarrow{i_1} \left(\bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n\right) \times CX_{n+1}$$

where π_1 is the projection and i_1 is the inclusion. Similarly, since b is a coordinate-wise inclusion and $CX_1 \times \cdots \times CX_n$ is contractible, b is homotopic to the composite

$$\overline{\pi}_2\colon \left(\bigcup_{i=1}^{n} CX_1 \times \cdots \times X_i \times \cdots \times CX_n\right) \times X_{n+1} \xrightarrow{\pi_2} X_{n+1} \xrightarrow{i_2} CX_1 \times \cdots \times CX_n \times X_{n+1}$$

where π_2 is the projection and i_2 is the inclusion.

The pushout in Lemma 4.4 and the description of the maps a and b play a key role in helping to identify the homotopy types of certain $(\underline{CX}, \underline{X})^K$'s in Theorem 4.6. Before stating this, we need another preliminary lemma which identifies the homotopy type of a certain pushout. Let $\pi_j\colon \prod_{i=1}^{n} X_i \longrightarrow X_j$ be the projection onto the jth-factor. For spaces A and B, the *left half-smash* of A and B is $A \ltimes B = A \times B/\sim$ where $(a, *) \sim *$, and the *right half-smash* of A and B is $A \rtimes B = A \times B/\sim$ where $(*, b) \sim *$.

Lemma 4.5. *Suppose there is a homotopy pushout*

$$\begin{array}{ccc} A \times B \times C & \xrightarrow{\pi_2 \times \pi_3} & B \times C \\ \big\downarrow f & & \big\downarrow \\ P & \longrightarrow & Q \end{array}$$

where f factors as the composite $A \times B \times C \xrightarrow{\pi_1 \times \pi_3} A \times C \longrightarrow A \rtimes C \xrightarrow{f'} P$. Then there is a homotopy equivalence

$$Q \simeq D \vee [(A * B) \rtimes C]$$

where D is the cofibre of f'.

Proof. We start by recalling two general facts. First, the pushout of the projections $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$ is homotopy equivalent $X * Y$, and the map from each of X and Y into $X * Y$

is null homotopic. Second, if Q is the pushout of maps $X \xrightarrow{a} Y$ and $X \xrightarrow{b} Z$ then, for any space T, the pushout of $X \times T \xrightarrow{a\times 1} Y \times T$ and $X \times T \xrightarrow{b\times 1} Z \times T$ is $Q \times T$.

In our case, since f factors through the projection onto $A \times C$, there is a diagram of iterated homotopy pushouts

$$\begin{array}{ccc}
A \times B \times C & \xrightarrow{\pi_2\times\pi_3} & B \times C \\
\downarrow{\scriptstyle \pi_1\times\pi_3} & & \downarrow \\
A \times C & \xrightarrow{g} & R \\
\downarrow{\scriptstyle \bar{f}} & & \downarrow \\
P & \longrightarrow & Q
\end{array}$$

which defines the space R and the map g. Observe that the top square is the product of C with the pushout of the projections $A \times B \xrightarrow{\pi_1} A$ and $A \times B \xrightarrow{\pi_2} B$. Thus $R \simeq (A * B) \times C$ and $g \simeq * \times 1$. The identification of R and g lets us write the bottom pushout above as a diagram of iterated homotopy pushouts

$$\begin{array}{ccccc}
A \times C & \xrightarrow{\pi_2} & C & \xrightarrow{i} & (A * B) \times C \\
\downarrow & & \downarrow & & \downarrow \\
P & \longrightarrow & Q' & \longrightarrow & Q
\end{array}$$

where i is the inclusion of the second factor. By hypothesis, the restriction of $A \times C \longrightarrow P$ to C is null homotopic. Thus we can pinch out C in the previous diagram to obtain a diagram of iterated homotopy pushouts

$$\begin{array}{ccccc}
A \rtimes C & \longrightarrow & * & \longrightarrow & (A * B) \rtimes C \\
\downarrow{\scriptstyle f'} & & \downarrow & & \downarrow \\
P & \longrightarrow & D & \longrightarrow & Q.
\end{array}$$

The left pushout implies that D is the homotopy cofibre of f', and the right pushout immediately implies that $Q \simeq D \vee [(A * B) \rtimes C]$. $\square$

Let K be a simplicial complex on n vertices and suppose that $K = L \cup_{\partial\sigma} \sigma$ for some simplex σ and a simplicial complex L containing $\partial\sigma$. We consider cases where L contains a simplicial cone on $\partial\sigma$, and use this to help identify the homotopy type of $(\underline{CX}, \underline{X})^K$. This requires some notation.

For a sequence $(i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq n$, let $\Delta^{i_1,\ldots,i_k}$ be the $(k-1)$-dimensional simplex on the vertices $\{i_1, \ldots, i_k\}$. To match a later application of Theorem 4.6 in Section 5, we will assume that in $K = L \cup_{\partial\sigma} \sigma$ we have $\sigma = (i_1, \ldots, i_k)$ and $i_1 \geq 2$. Let $(j_1, \ldots, j_{n-k-1})$ be the complement of $(i_1, \ldots, i_k)$ in $(2, \ldots, n)$, and assume that $j_1 < \cdots < j_{n-k-1}$. Let $\overline{X} = \prod_{t=1}^{n-k-1} X_{j_t}$. Let $\overline{\partial\sigma}$ be $\partial\sigma$ regarded as a simplicial complex on the vertices $\{1, \ldots, n\}$. Note that the vertices $1, j_1, \ldots, j_{n-k-1}$ are not vertices of $\overline{\partial\sigma}$. The inclusion $\overline{\partial\sigma} \longrightarrow L$ induces a map of polyhedral products $f \colon (\underline{CX}, \underline{X})^{\overline{\partial\sigma}} \longrightarrow (\underline{CX}, \underline{X})^L$. By the definition of the polyhedral product, $(\underline{CX}, \underline{X})^{\overline{\partial\sigma}} = (\underline{CX}, \underline{X})^{\partial\sigma} \times X_1 \times \overline{X}$. Assuming that each of the vertices $\{i\}$ is in L for $1 \leq i \leq n$, by Proposition 3.4, the map f is null homotopic when restricted to $X_1 \times \overline{X}$. Thus f

factors through $(\underline{CX}, \underline{X})^{\partial\sigma} \rtimes (X_1 \times \overline{X})$. Restricting f to $(\underline{CX}, \underline{X})^{\partial\sigma} \times \overline{X}$, we obtain a map

$$f' : (\underline{CX}, \underline{X})^{\partial\sigma} \rtimes \overline{X} \longrightarrow (\underline{CX}, \underline{X})^{L}.$$

Theorem 4.6. *Let K be a simplicial complex on n vertices. Suppose that $K = L \cup_{\partial\sigma} \sigma$ where:*
(a) *for $1 \le i \le n$, the vertex $\{i\} \in L$;*
(b) *$\sigma = (i_1, \dots, i_k)$ for $2 \le i_1 < \cdots < i_k \le n$;*
(c) *$\sigma \notin L$;*
(d) *$(1) * \partial\sigma \subseteq L$.*
Then there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^{K} \simeq D \vee [((X_{i_1} * \cdots * X_{i_k}) * X_1) \rtimes \overline{X}]$$

where D is the cofibre of the map $(X_{i_1} * \cdots * X_{i_k}) \rtimes \overline{X} \simeq (\underline{CX}, \underline{X})^{\partial\sigma} \rtimes \overline{X} \xrightarrow{f'} (\underline{CX}, \underline{X})^{L}$.

Proof. Since the inclusion $\partial\sigma \longrightarrow L$ factors as the composite $\partial\sigma \longrightarrow (1) * \partial\sigma \longrightarrow L$, we obtain an iterated pushout diagram

$$\begin{array}{ccc} \partial\sigma & \longrightarrow & \sigma \\ \downarrow & & \downarrow \\ (1) * \partial\sigma & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ L & \longrightarrow & K \end{array}$$

which defines the simplicial complex K_1. Proposition 3.1 therefore implies that there is an iterated pushout diagram

$$\begin{array}{ccc} (\underline{CX}, \underline{X})^{\overline{\partial\sigma}} & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{\sigma}} \\ \downarrow & & \downarrow \\ (\underline{CX}, \underline{X})^{\overline{(1)*\partial\sigma}} & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K_1}} \\ \downarrow & & \downarrow \\ (\underline{CX}, \underline{X})^{L} & \longrightarrow & (\underline{CX}, \underline{X})^{K} \end{array} \tag{6}$$

where the bar over each of $\partial\sigma, \sigma, (1) * \sigma$ and K_1 means they are to be regarded as simplicial complexes on the index set $[n]$.

By hypothesis, $\sigma = (i_1, \dots, i_k)$, so $\sigma = \Delta^{i_1,\dots,i_k}$. The pushout defining K_1 therefore implies that $K_1 = \partial\Delta^{1,i_1,\dots,i_k}$. Now, arguing in the same way that produced the diagram in Lemma 4.4, an explicit description of the upper pushout in (6) is as follows

$$\begin{array}{ccc} \left(\bigcup_{j=1}^{k} CX_{i_1} \times \cdots \times X_{i_j} \times \cdots \times CX_{i_k}\right) \times X_1 \times \overline{X} & \xrightarrow{\ b\ } & CX_{i_1} \times \cdots \times CX_{i_k} \times X_1 \times \overline{X} \\ \downarrow{\scriptstyle a} & & \downarrow \\ \left(\bigcup_{j=1}^{k} CX_{i_1} \times \cdots \times X_{i_j} \times \cdots \times CX_{i_k}\right) \times CX_1 \times \overline{X} & \longrightarrow & (\underline{CX}, \underline{X})^{\partial\Delta^{1,i_1,\dots,i_k}} \times \overline{X} \end{array} \tag{7}$$

where a and b are the inclusions. Observe that, rearranging the indices, (7) is just the product of a pushout as in Lemma 4.4 with $\overline{X}$. As well, as noted after Lemma 4.4, up to homotopy, a factors through the projection onto $\left(\bigcup_{j=1}^{k} CX_{i_1} \times \cdots \times X_{i_j} \times \cdots \times CX_{i_k}\right) \times \overline{X}$ and b factors through the projection onto $X_1 \times \overline{X}$. By Proposition 2.2, there are homotopy equivalences

$$\left(\bigcup_{j=1}^{k} CX_{i_1} \times \cdots \times X_{i_j} \times \cdots \times CX_{i_k}\right) \simeq X_{i_1} * \cdots * X_{i_k}$$

and

$$(\underline{CX}, \underline{X})^{\partial\Delta^{1,i_1,\ldots,i_k}} \simeq X_1 * X_{i_1} * \cdots * X_{i_k}.$$

Thus, up to homotopy equivalences, (7) is equivalent to the homotopy pushout

$$\begin{array}{ccc} (X_{i_1} * \cdots * X_{i_k}) \times X_1 \times \overline{X} & \xrightarrow{\text{proj}} & X_1 \times \overline{X} \\ \downarrow{\scriptstyle \text{proj}} & & \downarrow \\ (X_{i_1} * \cdots * X_{i_k}) \times \overline{X} & \longrightarrow & (X_1 * X_{i_1} * \cdots * X_{i_k}) \times \overline{X}. \end{array}$$

Therefore, up to homotopy equivalences, (6) is equivalent to the iterated homotopy pushout diagram

$$\begin{array}{ccc} (X_{i_1} * \cdots * X_{i_k}) \times X_1 \times \overline{X} & \xrightarrow{\text{proj}} & X_1 \times \overline{X} \\ \downarrow{\scriptstyle \text{proj}} & & \downarrow \\ (X_{i_1} * \cdots * X_{i_k}) \times \overline{X} & \longrightarrow & (X_1 * X_{i_1} * \cdots * X_{i_k}) \times \overline{X} \\ \downarrow & & \downarrow \\ (\underline{CX}, \underline{X})^L & \longrightarrow & (\underline{CX}, \underline{X})^K. \end{array}$$

By hypothesis, each vertex $\{i\} \in L$ for $1 \leq i \leq n$, so Proposition 3.4 implies that the restriction of $(X_{i_1} * \cdots * X_{i_k}) \times \overline{X} \longrightarrow (\underline{CX}, \underline{X})^L$ to $\overline{X}$ is null homotopic. Thus the outer perimeter of the previous diagram is a homotopy pushout

$$\begin{array}{ccc} (X_{i_1} * \cdots * X_{i_k}) \times X_1 \times \overline{X} & \xrightarrow{\text{proj}} & X_1 \times \overline{X} \\ \downarrow{\scriptstyle f} & & \downarrow \\ (\underline{CX}, \underline{X})^L & \longrightarrow & (\underline{CX}, \underline{X})^K \end{array}$$

where f factors as the composite $(X_{i_1} * \cdots * X_{i_k}) \times X_1 \times \overline{X} \xrightarrow{\pi_1 \times \pi_3} (X_{i_1} * \cdots * X_{i_k}) \times \overline{X} \longrightarrow (X_{i_1} * \cdots * X_{i_k}) \rtimes \overline{X} \xrightarrow{f'} (\underline{CX}, \underline{X})^L$. Lemma 4.5 therefore implies that

$$(\underline{CX}, \underline{X})^K \simeq D \vee [((X_{i_1} * \cdots * X_{i_k}) * X_1) \rtimes \overline{X}]$$

where D is the cofiber of f'. $\square$

5. Polyhedral products for shifted complexes

In this section we prove Theorem 1.1. To begin, we introduce some definitions from combinatorics.

Definition 5.1. Let K be a simplicial complex on n vertices. The complex K is *shifted* if there is an ordering on its vertices such that whenever $\sigma \in K$ and $\nu' < \nu$, then $(\sigma - \nu) \cup \nu' \in K$.

It may be helpful to interpret this definition in terms of ordered sequences. Let K be a simplicial complex on $[n]$ and order the vertices by their integer labels. If $\sigma \in K$ with vertices $\{i_1, \ldots, i_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$, then regard σ as the ordered sequence $(i_1, \ldots, i_k)$. The shifted condition states that if $\sigma = (i_1, \ldots, i_k) \in K$ then K contains every simplex $(t_1, \ldots, t_l)$ with $l \leq k$ and $t_1 \leq i_1, \ldots, t_l \leq i_l$.

Examples 5.2. We give three examples.

(1) Let K be the 1-dimensional simplicial complex with vertices $\{1, 2, 3, 4\}$ and edges $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$. So $|K|$ is two copies of $|\partial \Delta^2|$ glued along the common edge $(1, 2)$. In the given ordering of the vertices, K is shifted. Notice that the ordering of the vertices is important, for if the vertices of K were reordered so the common face was $(3, 4)$ then the shifted condition fails.
(2) Let K be the 1-dimensional simplicial complex with vertices $\{1, 2, 3, 4\}$ and edges $\{(1, 2), (1, 4), (2, 3), (3, 4)\}$. So $|K|$ is the boundary of a square. Then K is not shifted.
(3) For $0 \leq k \leq n - 1$, the full k-skeleton of Δ^n is shifted.

Definition 5.3. Let K be a simplicial complex on the index set $[n]$. The *star*, *restriction* (or *deletion*) and *link* of a simplex $\sigma \in K$ are the subcomplexes

$$\begin{aligned}
&\text{star}_K\sigma = \{\tau \in K \mid \sigma \cup \tau \in K\};\\
&\text{rest}_K[n] \setminus \sigma = \{\tau \in K \mid \sigma \cap \tau = \emptyset\};\\
&\text{link}_K\sigma = \text{star}_K\sigma \cap \text{rest}_K[n] \setminus \sigma.
\end{aligned}$$

There are three standard facts that follow straight from the definitions. First, there is a pushout

$$\begin{array}{ccc}
\text{link}_K\sigma & \longrightarrow & \text{rest}_K[n]\backslash\sigma \\
\downarrow & & \downarrow \\
\text{star}_K\sigma & \longrightarrow & K.
\end{array}$$

Second, if K is shifted then so are $\text{star}_K\sigma$, $\text{rest}_K[n] \setminus \sigma$ and $\text{link}_K\sigma$ for each $\sigma \in K$. Third, $\text{star}_K\sigma$ is a join: $\text{star}_K\sigma = \sigma * \text{link}_K\sigma$.

For K a simplicial complex on $[n]$ and σ being a vertex (i), we write $\text{star}(i)$, $\text{rest}\{1, \ldots, \hat{i}, \ldots, n\}$ and $\text{link}(i)$ for $\text{star}_K\sigma$, $\text{rest}_K[n] \setminus \sigma$ and $\text{link}_K\sigma$. To illustrate, take $i_1 = 1$. Then $\text{star}(1)$ consists of those simplices $(i_1, \ldots, i_k) \in K$ where $1 \leq i_1 < \cdots < i_k \leq n$ and $i_1 = 1$; $\text{rest}\{2, \ldots, n\}$ consists of those simplices $(j_1, \ldots, j_k) \in K$ where $1 < j_1 < \cdots < j_k \leq n$, and $\text{link}(1) = \text{star}(1) \cap \text{rest}\{2, \ldots, n\}$. The three useful facts mentioned above become the following. First, there is a pushout

$$\begin{array}{ccc}
\text{link}(1) & \longrightarrow & \text{rest}\{2, \ldots, n\} \\
\downarrow & & \downarrow \\
\text{star}(1) & \longrightarrow & K.
\end{array}$$

Second, if K is shifted then so are star(1), rest$\{2, \ldots, n\}$ and link(1). Third, star(1) is a join: star(1) $= (1) * $ link(1).

Next, we require four lemmas to prepare for the proof of Theorem 1.1. The first two are about shifted complexes, and the next two are about decompositions.

Lemma 5.4. *Let K be a shifted complex on the index set $[n]$. If $\sigma \in$ rest$\{2, \ldots, n\}$, then $\partial\sigma \in$ link(1).*

Proof. Suppose the ordered sequence corresponding to σ is $(i_1, \ldots, i_k)$. Then $\partial\sigma = \bigcup_{j=1}^{k} \sigma_j$ for $\sigma_j = (i_1, \ldots, \hat{i}_j, \ldots, i_k)$, where $\hat{i}_j$ means omit the jth-coordinate. So to prove the lemma it is equivalent to show that $\sigma_j = (i_1, \ldots, \hat{i}_j, \ldots, i_k) \in$ link(1) for each $1 \leq j \leq k$.

Fix j. Observe that $\sigma_j = (i_1, \ldots, \hat{i}_j, \ldots, i_k)$ is a sequence of length $k - 1$ and $2 \leq i_1 < \cdots < i_k \leq n$. We claim that the sequence $(1, i_1, \ldots, \hat{i}_j, \ldots, i_k)$ of length k represents a face of K. This holds because, as ordered sequences, we have $(1, i_1, \ldots, \hat{i}_j, \ldots, i_k) < (i_1, \ldots, i_k)$, and the shifted property for K implies that as $(i_1, \ldots, i_k) \in K$, any ordered sequence less than $(i_1, \ldots, i_k)$ also represents a face of K. Now, as $(1, i_1, \ldots, \hat{i}_j, \ldots, i_k) \in K$, we clearly have $(1, i_1, \ldots, \hat{i}_j, \ldots, i_k) \in$ star(1). Thus the sub-simplex $(i_1, \ldots, \hat{i}_j, \ldots, i_k)$ is also in star(1). That is, $\sigma_j \in$ star(1). Hence $\sigma_j \in$ star(1) $\cap$ rest$\{2, \ldots, n\} =$ link(1), as required. $\square$

Remark 5.5. In Lemma 5.4, it may be that σ itself is in link(1), but this need not be the case. For by the definition of link(1), we have $\sigma \in$ link(1) if and only if $(1, i_1, \ldots, i_k) \in K$.

Remark 5.6. It is also worth noting that as $\partial\sigma \in$ link(1) and star(1) $= (1) *$ link(1), we have $(1) * \partial\sigma \subseteq$ star(1). That is, the cone on $\partial\sigma$ is in star(1).

We say that a face τ of a simplicial complex K is *maximal* if there is no other face $\tau' \in K$ with τ a proper subset of τ'.

Lemma 5.7. *Let K be a shifted complex on the index set $[n]$. Then the inclusion link$(1) \longrightarrow$ rest$\{2, \ldots, n\}$ is filtered by a sequence of simplicial complexes*

$$\textit{link}(1) = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = \textit{rest}\{2, \ldots, n\}$$

where $L_i = L_{i-1} \cup \tau_i$ and τ_i satisfies:

(a) *τ_i is maximal in rest$\{2, \ldots, n\}$;*
(b) *$\tau_i \notin$ link(1);*
(c) *$\partial\tau_i \in$ link(1).*

Proof. In general, if L is a connected simplicial complex and $L_0 \subseteq L$ is a subcomplex (not necessarily connected), it is possible to start with L_0 and sequentially adjoin faces one at a time to get L. That is, there is a sequence of simplicial complexes $L_0 =\subseteq L_1 \subseteq \cdots \subseteq L_m = L$ where $L_i = L_{i-1} \cup \tau_i$ for some simplex $\tau_i \in L$, $\tau_i \notin L_{i-1}$ and the union is taken over the boundary $\partial\tau_i$ of τ_i. In addition, it may be assumed that the adjoined faces τ_i are maximal in L. Thus parts (a) and (b) of the lemma follow. For part (c), since K is shifted and each $\tau_i \in$ rest$\{2, \ldots, n\}$, Lemma 5.4 implies that $\partial\tau_i \in$ link(1). $\square$

Next, we turn to the two decomposition lemmas.

Lemma 5.8. *For any spaces $M, N_1, \ldots, N_m$, there is a homotopy equivalence*

$$\Sigma M \rtimes (N_1 \times \cdots \times N_m) \simeq \Sigma M \vee \left(\bigvee_{1 \leq t_1 < \cdots < t_k \leq m} \Sigma M \wedge N_{i_1} \wedge \cdots \wedge N_{i_k} \right).$$

Proof. In general, $\Sigma X \rtimes Y \simeq \Sigma X \vee (\Sigma X \wedge Y)$, so it suffices to decompose $\Sigma M \wedge (N_1 \times \cdots \times N_m)$. Iterating the basic fact that $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y)$, we obtain a homotopy equivalence $\Sigma(N_1 \times \cdots \times N_m) \simeq \bigvee_{1 \leq t_1 < \cdots < t_k \leq m} (\Sigma N_{i_1} \wedge \cdots \wedge N_{i_k})$. Thus

$$\begin{aligned} \Sigma M \wedge (N_1 \times \cdots \times N_m) &\simeq M \wedge \Sigma(N_1 \times \cdots \times N_m) \\ &\simeq M \wedge \left(\bigvee_{1 \leq t_1 < \cdots < t_k \leq m} \Sigma N_{i_1} \wedge \cdots \wedge N_{i_k} \right) \\ &\simeq \bigvee_{1 \leq t_1 < \cdots < t_k \leq m} M \wedge \Sigma N_{i_1} \wedge \cdots \wedge N_{i_k}. \quad \Box \end{aligned}$$

Recall from Section 4 that if K is a simplicial complex on the index set $[n]$ then $\Delta^{i_1, \ldots, i_k}$ is the full $(k-1)$-dimensional simplex on the vertex set $\{i_1, \ldots, i_k\}$ for $1 \leq i_1 < \cdots < i_k \leq n$.

Lemma 5.9. *Let K be a simplicial complex on the index set $[n]$. Suppose for some sequence $1 \leq i_1 < \cdots < i_k \leq n$ that $\partial\Delta^{i_1, \ldots, i_k}$ is a full subcomplex of K. Then the map $(\underline{CX}, \underline{X})^{\partial\Delta^{i_1, \ldots, i_k}} \longrightarrow (\underline{CX}, \underline{X})^K$ induced by the inclusion $\partial\Delta^{i_1, \ldots, i_k} \longrightarrow K$ has a left inverse. Consequently, $X_{i_1} * \cdots * X_{i_k}$ is a retract of $(\underline{CX}, \underline{X})^K$.*

Proof. This is a consequence of a result in [6] which states that if K_I is a full subcomplex of K then $(\underline{X}, \underline{A})^{K_I}$ is a retract of $(\underline{X}, \underline{A})^K$. $\Box$

We expand on Lemma 5.9. To simplify notation, let $\sigma = \Delta^{i_1, \ldots, i_k}$. We again assume that $\partial\sigma$ is a full subcomplex of K, so $\sigma \notin K$. Let $\{j_1, \ldots, j_{n-k}\}$ be the vertices in $[n]$ which are complementary to $\{i_1, \ldots, i_k\}$. Let $\overline{X} = \prod_{t=1}^{n-k} X_{j_t}$ and let $\overline{\partial\sigma}$ be $\partial\sigma$ regarded as a simplicial complex on the vertex set $[n]$. Note that $\{j_1, \ldots, j_{n-k}\}$ are not vertices of $\overline{\partial\sigma}$. By definition of the polyhedral product,

$$(\underline{CX}, \underline{X})^{\overline{\partial\sigma}} = (\underline{CX}, \underline{X})^{\partial\sigma} \times \overline{X}.$$

The inclusion $\overline{\partial\sigma} \longrightarrow K$ induces a map of polyhedral products $f : (\underline{CX}, \underline{X})^{\partial\sigma} \times \overline{X} \longrightarrow (\underline{CX}, \underline{X})^K$. If each vertex of $[n]$ is in K, Proposition 3.4 implies that the restriction of f to $\overline{X}$ is null homotopic. Thus f factors through a map

$$f' : (\underline{CX}, \underline{X})^{\partial\sigma} \rtimes \overline{X} \longrightarrow (\underline{CX}, \underline{X})^K.$$

We now apply the natural decomposition of (3) to f. By (3), there are homotopy equivalences

$$\Sigma(\underline{CX}, \underline{X})^{\overline{\partial\sigma}} \simeq \Sigma \left(\bigvee_{J \notin \overline{\partial\sigma}} |(\overline{\partial\sigma})_J| * \widehat{X}^J \right) \tag{8}$$

$$\Sigma(\underline{CX}, \underline{X})^K \simeq \Sigma \left(\bigvee_{I \notin K} |K_I| * \widehat{X}^I \right). \tag{9}$$

In the first case, observe that $\sigma \notin \overline{\partial\sigma}$ so $\overline{\partial\sigma}_\sigma = \overline{\partial\sigma}$, implying that $|(\overline{\partial\sigma})_\sigma| \simeq S^{k-2}$. Therefore

$$|(\overline{\partial\sigma})_\sigma| * \widehat{X}^\sigma \simeq \Sigma^{k-1} X_{i_1} \wedge \cdots \wedge X_{i_k}.$$

In other words, this term is $(\underline{CX}, \underline{X})^{\partial\sigma}$. More generally, suppose $J \notin \overline{\partial\sigma}$. There are three cases: (i) if J contains a proper subset J' of $\{i_1, \ldots, i_k\}$ then $\overline{\partial\sigma}_J = \overline{\partial\sigma}_{J'}$ and the right term is contractible, (ii) if J is disjoint from $i_1, \ldots, i_k$ then $\overline{\partial\sigma}_J = \emptyset$ since $\overline{\partial\sigma}$ is a simplicial complex on the vertices $i_1, \ldots, i_k$; this case corresponds to a summand of $\Sigma\overline{X}$, which we factor out in f'; (iii) if J contains $i_1, \ldots, i_k$ and some subset of $\{j_1, \ldots, j_{n-k}\}$ then, as $\overline{\partial\sigma}$ is a simplicial complex on the vertices $i_1, \ldots, i_k$, we have $(\overline{\partial\sigma})_J = \overline{\partial\sigma}_\sigma = \overline{\partial\sigma}$. Therefore, if $\mathcal{J}$ consists of the sequences J which contain J in case (iii), then from (8) we obtain

$$\Sigma(\underline{CX}, \underline{X})^{\partial\sigma} \rtimes \widehat{X} \simeq \Sigma\left(\bigvee_{J\in\mathcal{J}} |(\overline{\partial\sigma})_J| * \widehat{X}^J\right)$$
$$\simeq \Sigma\left(\bigvee_{J\in\mathcal{J}} S^{k-2} * \widehat{X}\right) \simeq \Sigma\left(\bigvee_{J\in\mathcal{J}} \Sigma^{k-1}\widehat{X}\right). \tag{10}$$

Since f is induced by the map $\overline{\partial\sigma} \longrightarrow K$ of simplicial complexes, the naturality of the decompositions in (8) and (9) implies that Σf decomposes as the wedge sum of the maps $\Sigma|(\overline{\partial\sigma})_J| * \widehat{X} \longrightarrow \Sigma|K_J| * \widehat{X}$, indexed by the sequences $J \notin \overline{\partial\sigma}$. In general, the quotient map $Y \times Z \longrightarrow Y \rtimes Z$ has a right homotopy inverse after suspending, so the wedge decomposition of Σf implies a wedge decomposition of $\Sigma f'$, where now – as in the previous paragraph – the wedge decomposition is indexed by the sequences $J \in \mathcal{J}$.

Since $\partial\sigma$ is a full subcomplex of K and $\overline{\partial\sigma}$ is simply $\partial\sigma$ regarded as a simplicial complex on the n vertices of K, we obtain that $\overline{\partial\sigma}$ is a full subcomplex of K. Therefore, for any sequence J containing σ, $(\overline{\partial\sigma})_J$ is a full subcomplex of K_J. In particular, $(\overline{\partial\sigma})_J$ is retract of K_J, so $|(\overline{\partial\sigma})_J|$ is a retract of $|K_J|$. Therefore the map $\Sigma^{k-1}\widehat{X} \simeq \Sigma|(\overline{\partial\sigma})_J| * \widehat{X} \longrightarrow \Sigma|K_J| * \widehat{X}$ has a left homotopy inverse. This is true for every $J \in \mathcal{J}$, so $\Sigma f'$ has a left homotopy inverse. This proves part (a) of the following.

Proposition 5.10. *Let K be a simplicial complex on the index set $[n]$ for which each vertex is in K. Suppose for some sequence $1 \le i_1 < \cdots < i_k \le n$ that $\partial\Delta^{i_1,\ldots,i_k}$ is a full subcomplex of K. Then there is a homotopy equivalence*

$$\Sigma((\underline{CX}, \underline{X})^{\partial\Delta^{i_1,\ldots,i_k}} \rtimes \overline{X}) \simeq \Sigma\left(\bigvee_{J\in\mathcal{J}} \Sigma^{k-1}\widehat{X}\right)$$

such that the map $\Sigma((\underline{CX}, \underline{X})^{\partial\Delta^{i_1,\ldots,i_k}} \rtimes \overline{X}) \xrightarrow{\Sigma f'} \Sigma(\underline{CX}, \underline{X})^K$ satisfies:

(a) *$\Sigma f'$ has a left homotopy inverse;*
(b) *if the decomposition for $\Sigma(\underline{CX}, \underline{X})^K$ in (9) desuspends, then part (a) also desuspends.*

Proof. It remains to show part (b). This follows by the same argument as for part (a) provided the decomposition in (10) for $\Sigma((\underline{CX}, \underline{X})^{\partial\Delta^{i_1,\ldots,i_k}} \rtimes \overline{X})$ also desuspends. As before, to simplify notation, let $\sigma = \Delta^{i_1,\ldots,i_k}$. The functorial decomposition in (3) was proved in [1] as a generalisation of the fact that $\Sigma(Y \times Z) \simeq \Sigma Y \vee \Sigma Z \vee (\Sigma Y \wedge Z)$. In our case we consider the restriction to $Y \rtimes Z$. If Y is a suspension then $Y \rtimes Z \simeq Y \vee (Y \wedge Z)$. So in our case, as

$(\underline{CX}, \underline{X})^{\partial\sigma} \simeq \Sigma^{k-1} X_{i_1} \wedge \cdots \wedge X_{i_k}$, we obtain

$$(\underline{CX}, \underline{X})^{\partial\sigma} \rtimes \overline{X} \simeq (\underline{CX}, \underline{X})^{\partial\sigma} \vee ((\underline{CX}, \underline{X})^{\partial\sigma} \wedge \overline{X}).$$

Since $\overline{X} = \prod_{t=1}^{n-k} X_{j_t}$ and $(\underline{CX}, \underline{X})^{\partial\sigma}$ is a suspension, iterating the general decomposition $\Sigma(Y \times Z) \simeq \Sigma Y \vee \Sigma Z \vee (\Sigma Y \wedge Z)$ shows that there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^{\partial\sigma} \rtimes \widehat{X} \simeq \left(\bigvee_{J \in \mathcal{J}} |(\overline{\partial\sigma})_J| * \widehat{X}^J \right).$$

As before, $|(\overline{\partial\sigma})_J| \simeq S^{k-2}$, so this homotopy equivalence desuspends (10). □

We are now ready to prove the main result in the paper. For convenience, let $\mathcal{W}_n$ be the collection of spaces which are either contractible or homotopy equivalent to a wedge of spaces of the form $\Sigma^j X_{i_1} \wedge \cdots \wedge X_{i_k}$ for $j \geq 1$ and $1 \leq i_1 < \cdots < i_k \leq n$. Note that for each $n > 1$, $\mathcal{W}_{n-1} \subseteq \mathcal{W}_n$.

Proof of Theorem 1.1. The proof is by induction on the number of vertices. If $n = 1$ then $K = \{1\}$, which is shifted, and the definition of the polyhedral product implies that $(\underline{CX}, \underline{X})^K = CX$, which is contractible. Thus $K \in \mathcal{W}_1$.

Assume the theorem holds for all shifted complexes on k vertices, with $k < n$. Let K be a shifted complex on the index set $[n]$. Consider the pushout

$$\begin{array}{ccc} \text{link}(1) & \longrightarrow & \text{rest}\{2, \ldots, n\} \\ \downarrow & & \downarrow \\ \text{star}(1) & \longrightarrow & K \end{array}$$

and recall that $\text{star}(1) = (1) * \text{link}(1)$. Since K is shifted, so are star(1), $\text{rest}\{2, \ldots, n\}$ and link(1). Note that $\text{rest}\{2, \ldots, n\}$ is a shifted complex on $n - 1$ vertices, and as link(1) is a subcomplex of $\text{rest}\{2, \ldots, n\}$, it too is a shifted complex on $n-1$ vertices. Therefore, by inductive hypothesis, $(\underline{CX}, \underline{X})^{\text{link}(1)} \in \mathcal{W}_{n-1}$.

By Lemma 5.7, the map $\text{link}(1) \longrightarrow \text{rest}\{2, \ldots, n\}$ is filtered by a sequence of simplicial complexes

$$\text{link}(1) = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = \text{rest}\{2, \ldots, n\}$$

where $L_i = L_{i-1} \cup \tau_i$ and τ_i satisfies: (i) τ_i is maximal in $\text{rest}\{2, \ldots, n\}$; (ii) $\tau_i \notin \text{link}(1)$; and (iii) $\partial\tau_i \in \text{link}(1)$. In particular, for each $1 \leq i \leq m$, there is a pushout

$$\begin{array}{ccc} \partial\tau_i & \longrightarrow & \tau_i \\ \downarrow & & \downarrow \\ L_{i-1} & \longrightarrow & L_i. \end{array} \tag{11}$$

Let $K_0 = \text{star}(1)$, and for $1 \leq i \leq m$, define K_i as the simplicial complex obtained from the pushout

$$\begin{array}{ccc} L_{i-1} & \longrightarrow & L_i \\ \downarrow & & \downarrow \\ K_{i-1} & \longrightarrow & K_i. \end{array} \tag{12}$$

Observe that we obtain a filtration of the map $\text{star}(1) \longrightarrow K$ as a sequence $\text{star}(1) = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K$. Juxtaposing the pushouts in (11) and (12) we obtain a pushout

$$\begin{array}{ccc} \partial\tau_i & \longrightarrow & \tau_i \\ \downarrow & & \downarrow \\ K_{i-1} & \longrightarrow & K_i. \end{array} \tag{13}$$

Since $\partial\tau_i \in \text{link}(1)$, Remark 5.6 implies that $(1) * \partial\tau_i \in \text{star}(1)$. Thus as $\text{star}(1) = K_0$, the map $\partial\tau_i \longrightarrow K_{i-1}$ factors as the composite $\partial\tau_i \longrightarrow (1) * \partial\tau_i \longrightarrow \text{star}(1) = K_0 \longrightarrow K_{i-1}$. That is, the inclusion of $\partial\tau_i$ into K_{i-1} factors through the cone on $\partial\tau_i$.

We now argue that each $(\underline{CX}, \underline{X})^{K_j} \in \mathcal{W}_n$. First consider $(\underline{CX}, \underline{X})^{K_0}$. Since $K_0 = \text{star}(1) = (1) * \text{link}(1)$, by Lemma 4.1 we have $(\underline{CX}, \underline{X})^{K_0} = (\underline{CX}, \underline{X})^{(1)} \times (\underline{CX}, \underline{X})^{\text{link}(1)}$. By the definition of the polyhedral product, $(\underline{CX}, \underline{X})^{(1)} = CX_1$, so $(\underline{CX}, \underline{X})^{K_0}$ is homotopy equivalent to $(\underline{CX}, \underline{X})^{\text{link}(1)}$. By inductive hypothesis, $(\underline{CX}, \underline{X})^{\text{link}(1)} \in \mathcal{W}_{n-1}$. Thus $(\underline{CX}, \underline{X})^{K_0} \in \mathcal{W}_n$.

Next, fix an integer j such that $1 \le j \le m$, and assume that $(\underline{CX}, \underline{X})^{K_{j-1}} \in \mathcal{W}_n$. We have $K_j = K_{j-1} \cup_{\partial\tau_j} \tau_j$. Let $[n]_j$ be the subset of $[n]$ labelling the vertices of K_j and suppose $[n]_j$ has m_j elements. Note that $1 \in [n]_j$. Since $\tau_j = \Delta^{i_1,\ldots,i_k}$ for some sequence $(i_1, \ldots, i_k)$, we have $\partial\tau_j = \partial\Delta^{i_1,\ldots,i_k}$. Let $(j_1, \ldots, j_{m_j-k-1})$ be the complement of $(1, i_1, \ldots, i_k)$ in $[n]_j$, and let $\overline{X} = \prod_{t=1}^{m_j-k-1} X_{j_t}$. Let $\overline{\partial\tau_j}$ be $\partial\tau_j$ regarded as a simplicial complex on the vertex set $[n]_j$. Note that the vertices $1, j_1, \ldots, j_{m_j-k-1}$ are not vertices of $\overline{\partial\tau_j}$. The inclusion $\overline{\partial\tau_j} \longrightarrow K_j$ induces a map of polyhedral products $f\colon (\underline{CX}, \underline{X})^{\overline{\partial\tau_j}} \longrightarrow (\underline{CX}, \underline{X})^{K_j}$. By the definition of the polyhedral product, $(\underline{CX}, \underline{X})^{\overline{\partial\tau_j}} = (\underline{CX}, \underline{X})^{\partial\tau_j} \times X_1 \times \overline{X}$. As each of the vertices in $[n]_j$ is in K_j, by Proposition 3.4, the map f is null homotopic when restricted to $X_1 \times \overline{X}$. Thus f factors through $(\underline{CX}, \underline{X})^{\partial\tau_j} \rtimes (X_1 \times \overline{X})$. Restricting f to $(\underline{CX}, \underline{X})^{\partial\tau_j} \times \overline{X}$, we obtain a map

$$f'\colon (\underline{CX}, \underline{X})^{\partial\tau_j} \rtimes \overline{X} \longrightarrow (\underline{CX}, \underline{X})^{K_j}.$$

Since $\partial\tau_j \longrightarrow K_{j-1}$ factors through $(1) * \partial\tau_j$, by Theorem 4.6 there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^{K_j} \simeq D_j \vee \big(((X_{i_1} * \cdots * X_{i_k}) * X_1) \rtimes \overline{X}\big)$$

where D_j is the cofiber of f'. Since $\overline{X}$ is a product, if we take $M = X_{i_1} * \cdots * X_{i_k} * X_1$ then Lemma 5.8 implies that $((X_{i_1} * \cdots * X_{i_k}) * X_1) \rtimes \overline{X} \in \mathcal{W}_n$. If $D_j \in \mathcal{W}_n$ as well, then $(\underline{CX}, \underline{X})^{K_j} \in \mathcal{W}_n$. Therefore, by induction, $(\underline{CX}, \underline{X})^{K_m} \in \mathcal{W}_n$. But $(\underline{CX}, \underline{X})^{K} = (\underline{CX}, \underline{X})^{K_m}$, so $(\underline{CX}, \underline{X})^{K} \in \mathcal{W}_n$, which completes the inductive step on the number of vertices and therefore proves the theorem.

It remains to show that $D_j \in \mathcal{W}_n$. Consider the cofibration

$$(X_{i_1} * \cdots * X_{i_k}) \rtimes \overline{X} \simeq (\underline{CX}, \underline{X})^{\partial\tau_j} \rtimes \overline{X} \xrightarrow{f'} (\underline{CX}, \underline{X})^{K_{j-1}} \longrightarrow D_j. \tag{14}$$

Notice that the definition of the map f' coincides with that which appears preceding Proposition 5.10. Since $\partial\tau_j$ is a full subcomplex of K_{j-1} and $(\underline{CX}, \underline{X})^{\partial\tau_j} \rtimes \overline{X} \simeq (X_{i_1} * \cdots * X_{i_k}) \rtimes \overline{X} \in \mathcal{W}_n$, by Proposition 5.10 (b) the map f' has a left homotopy inverse. Since $(\underline{CX}, \underline{X})^{K_{j-1}} \in W_n$, it is a suspension, so the existence of a left homotopy inverse for

f' implies that the cofibration (14) splits to give a homotopy decomposition $(\underline{CX}, \underline{X})^{K_{j-1}} \simeq ((\underline{CX}, \underline{X})^{\partial\tau_j} \rtimes \overline{X}) \vee D_j$. Thus D_j is a retract of a space in $\mathcal{W}_n$ so $D_j \in \mathcal{W}_n$ as well.

Hence by induction, $(\underline{CX}, \underline{X})^K \in \mathcal{W}_n$. Finally, as the inductive step produces decompositions by Proposition 5.10, which is based on desuspending Bahri, Bendersky, Cohen and Gitler's decomposition in (3), we obtain a homotopy decomposition

$$(\underline{CX}, \underline{X})^K \simeq \left(\bigvee_{I \notin K} |K_I| * \widehat{X}^I\right)$$

which desuspends (3). □

6. Examples

We consider the two shifted cases from Examples 5.2. First, let $K = \Delta^{n-1}_k$ be the full k-skeleton of Δ^{n-1}. Phrased in terms of polyhedral products, Porter [13] showed that for any simply-connected spaces $X_1, \ldots, X_n$, there is a homotopy equivalence

$$(\underline{C\Omega X}, \underline{\Omega X})^K \simeq \bigvee_{j=k+2}^{n} \left(\bigvee_{1\le i_1<\cdots<i_j\le n} \binom{j-1}{k+1} \Sigma^{k+1}\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_j}\right).$$

Theorem 1.1 now generalises this. If $X_1, \ldots, X_n$ are any path-connected spaces, there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^K \simeq \bigvee_{j=k+2}^{n} \left(\bigvee_{1\le i_1<\cdots<i_j\le n} \binom{j-1}{k+1} \Sigma^{k+1} X_{i_1} \wedge \cdots \wedge X_{i_j}\right).$$

For example, this decomposition holds not just for $X_i = \Omega S^{n_i}$ as in Porter's case, but also for the spheres themselves, $X_i = S^{n_i}$.

Second, let K be the simplicial complex in Examples 5.2(1), whose geometric realisation is two copies of $|\partial\Delta^2|$ glued along a common edge. Specifically, K is the simplicial complex with vertices $\{1, 2, 3, 4\}$ and edges $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$. To illustrate the algorithmic nature of the proof of Theorem 1.1, we will carry out the iterative procedure for identifying the homotopy type of $(\underline{CX}, \underline{X})^K$. Starting with $K_0 = \text{star}(1)$, we adjoin one edge at a time: let $K_1 = K_0 \cup_{\{2,3\}} (2, 3)$ and $K_2 = K_1 \cup_{\{2,4\}} (2, 4)$. Note that $K_2 = K$. We begin to identify homotopy types.

Step 1: For K_0 we have $\text{star}(1) = (1) * \text{link}(1)$ where $\text{link}(1) = \{2, 3, 4\}$. So Lemma 4.1 implies that $(\underline{CX}, \underline{X})^{\text{star}(1)} \simeq CX_1 \times (\underline{CX}, \underline{X})^{\text{link}(1)} \simeq (\underline{CX}, \underline{X})^{\text{link}(1)}$. Since $\text{link}(1) = \Delta^2_0$, we can apply the previous example to obtain a homotopy equivalence

$$(\underline{CX}, \underline{X})^{K_0} \simeq (\Sigma X_2 \wedge X_3) \vee (\Sigma X_2 \wedge X_4) \vee (\Sigma X_3 \wedge X_4) \vee 2 \cdot (\Sigma X_2 \wedge X_3 \wedge X_4).$$

Step 2: Since $K_1 = K_0 \cup_{\{2,3\}} (2, 3)$, Theorem 4.6 implies that there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^{K_1} \simeq D_1 \vee [(X_2 * X_3 * X_1) \rtimes X_4] \tag{15}$$

where there is a cofibration $(X_2 * X_3) \rtimes X_4 \longrightarrow (\underline{CX}, \underline{X})^{K_0} \longrightarrow D_1$. As $(X_2 * X_3) \rtimes X_4 \simeq (\Sigma X_2 \wedge X_3) \vee (\Sigma X_2 \wedge X_3 \wedge X_4)$, the homotopy equivalence for $(\underline{CX}, \underline{X})^{K_0}$ in Step 1 implies that the cofibration splits and there is a homotopy equivalence

$$D_1 \simeq (\Sigma X_2 \wedge X_4) \vee (\Sigma X_3 \wedge X_4) \vee (\Sigma X_2 \wedge X_3 \wedge X_4).$$

Thus (15) implies that there is a homotopy equivalence

$$\begin{aligned}(\underline{CX}, \underline{X})^{K_1} \simeq\ & (\Sigma X_2 \wedge X_4) \vee (\Sigma X_3 \wedge X_4) \vee (\Sigma X_2 \wedge X_3 \wedge X_4) \\ & \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_3) \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4).\end{aligned}$$

Step 3: Since $K_2 = K_1 \cup_{\{2,4\}} (2, 4)$, Theorem 4.6 implies that there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^{K_2} \simeq D_2 \vee [(X_2 * X_4 * X_1) \rtimes X_3] \tag{16}$$

where there is a cofibration $(X_2 * X_4) \rtimes X_3 \longrightarrow (\underline{CX}, \underline{X})^{K_1} \longrightarrow D_2$. As $(X_2 * X_4) \rtimes X_3 \simeq (\Sigma X_2 \wedge X_4) \vee (\Sigma X_2 \wedge X_3 \wedge X_4)$, the homotopy equivalence for $(\underline{CX}, \underline{X})^{K_1}$ in Step 2 implies that the cofibration splits and there is a homotopy equivalence

$$D_2 \simeq (\Sigma X_3 \wedge X_4) \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_3) \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4).$$

Thus (16) implies that there is a homotopy equivalence

$$\begin{aligned}(\underline{CX}, \underline{X})^{K} = (\underline{CX}, \underline{X})^{K_2} \simeq\ & (\Sigma X_3 \wedge X_4) \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_3) \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_4) \\ & \vee 2 \cdot (\Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4).\end{aligned}$$

7. Extensions of the method I: gluing along a common face

The basic idea behind proving Theorem 1.1 was to present $(\underline{CX}, \underline{X})^K$ as the end result of a sequence of pushouts, and then analyse the homotopy theory of the pushouts. In these terms, the key ingredient of the proof was Lemma 4.5. The idea behind the method is therefore very general. One can look for different constructions of K which translate to a sequence of homotopy pushouts constructing $(\underline{CX}, \underline{X})^K$, whose homotopy theory can be analysed. This may apply to different classes of complexes K other than the shifted class. In this section we give such a construction. As a consequence, we find that the decomposition in the statement of Theorem 1.1 holds for a class of simplicial complexes that contains more than just shifted complexes.

Let K be a simplicial complex on the index set $[n]$. Suppose $K = K_1 \cup_\tau K_2$ for τ a simplex in K. Geometrically, $|K|$ is the result of gluing $|K_1|$ and $|K_2|$ together along a common face. Relabelling the vertices if necessary, we may assume that K_1 is defined on the vertices $\{1, \ldots, m\}$, K_2 is defined on the vertices $\{m - l + 1, \ldots, n\}$ and τ is defined on the vertices $\{m - l + 1, \ldots, m\}$. Let $\overline{K}_1$, $\overline{K}_2$ and $\overline{\tau}$ be K_1, K_2 and τ regarded as simplicial complexes on $[n]$. So $K = \overline{K}_1 \cup_{\overline{\tau}} \overline{K}_2$.

Let $\sigma \in K_1$ and let $\overline{\sigma}$ be its image in $\overline{K}_1$. By the definition of $\overline{\sigma}$, we have $i \notin \overline{\sigma}$ for $i \in \{m + 1, \ldots, n\}$. Thus $(\underline{CX}, \underline{X})^{\overline{\sigma}} = (\underline{CX}, \underline{X})^{\sigma} \times X_{m+1} \times \cdots \times X_n$. Consequently, taking the union over all the faces in $\overline{K}_1$, we obtain

$$(\underline{CX}, \underline{X})^{\overline{K}_1} = (\underline{CX}, \underline{X})^{K_1} \times X_{m+1} \times \cdots \times X_n.$$

Similarly, we have

$$(\underline{CX}, \underline{X})^{\overline{K}_2} = X_1 \times \cdots \times X_{m-l} \times (\underline{CX}, \underline{X})^{K_2}.$$

Since $\tau = \Delta_{m-l-1}$, we have $(\underline{CX}, \underline{X})^{\tau} = CX_{m-l+1} \times \cdots \times CX_m$, so as above we obtain

$$(\underline{CX}, \underline{X})^{\overline{\tau}} = X_1 \times \cdots \times X_{m-l} \times CX_{m-l+1} \times \cdots \times CX_m \times X_{m+1} \times \cdots \times X_n.$$

Since $K = K_1 \cup_{\tau} K_2$, by Proposition 3.1 there is a pushout

$$\begin{array}{ccc} X_1 \times \cdots \times X_{m-l} \times (\underline{CX}, \underline{X})^{\tau} \times X_{m+1} \times \cdots \times X_n & \xrightarrow{a} & (\underline{CX}, \underline{X})^{K_1} \times X_{m+1} \times \cdots \times X_n \\ \big\downarrow{\scriptstyle b} & & \big\downarrow \\ X_1 \times \cdots \times X_{m-l} \times (\underline{CX}, \underline{X})^{K_2} & \longrightarrow & (\underline{CX}, \underline{X})^{K} \end{array} \tag{17}$$

where a and b are coordinate-wise inclusions.

We next identify the homotopy classes of a and b. We use the Milnor–Moore convention of writing the identity map $Y \longrightarrow Y$ as Y. To simplify notation, let $M = X_1 \times \cdots \times X_{m-l}$ and $N = X_{m+1} \times \cdots \times X_n$. Then the domain of a and b is $M \times (\underline{CX}, \underline{X})^{\tau} \times N$. Since a and b are coordinate-wise inclusions, their homotopy classes are determined by their restrictions to M, $(\underline{CX}, \underline{X})^{\tau}$ and N. Consider a. Since each vertex $\{i\} \in K$ for $1 \leq i \leq m-l$, Corollary 3.3 implies that the restriction of a to M is null homotopic. Since $(\underline{CX}, \underline{X})^{\tau}$ is a product of cones, it is contractible, so the restriction of a to $(\underline{CX}, \underline{X})^{\tau}$ is null homotopic. Since a is a coordinate-wise inclusion, it is the identity map on $X_{m+1} \times \cdots \times X_n$. Thus $a \simeq * \times * \times N$. Similarly, $b \simeq M \times * \times *$. Thus we can rewrite (17) as a pushout

$$\begin{array}{ccc} M \times (\underline{CX}, \underline{X})^{\tau} \times N & \xrightarrow{f \times N} & (\underline{CX}, \underline{X})^{K_1} \times N \\ \big\downarrow{\scriptstyle M \times g} & & \big\downarrow \\ M \times (\underline{CX}, \underline{X})^{K_2} & \longrightarrow & (\underline{CX}, \underline{X})^{K} \end{array} \tag{18}$$

where f and g are null homotopic.

To identify the homotopy type of $(\underline{CX}, \underline{X})^{K}$ we use a general lemma.

Lemma 7.1. *Let*

$$\begin{array}{ccc} A \times E \times B & \xrightarrow{f \times B} & C \times B \\ \big\downarrow{\scriptstyle A \times g} & & \big\downarrow \\ A \times D & \longrightarrow & Q \end{array}$$

be a homotopy pushout, where E is contractible and f and g are null homotopic. Then there is a homotopy equivalence

$$Q \simeq (A * B) \vee (A \ltimes D) \vee (C \rtimes B).$$

Proof. Let $j: A \times B \longrightarrow A \times E \times B$ be the inclusion. Observe that $(f \times B) \circ j \simeq * \times B$ and $(A \times g) \circ j \simeq A \times *$. Thus as E is contractible, j is a homotopy equivalence and the pushout in the statement of this lemma is equivalent, up to homotopy, to the pushout in the statement of Lemma 4.5. The homotopy equivalence for Q now follows. □

Applying Lemma 7.1 to the pushout in (18), we obtain the following.

Theorem 7.2. *Let K be a simplicial complex on the index set $[n]$. Suppose that $K = K_1 \cup_\tau K_2$ where τ is a common face of K_1 and K_2. Then there is a homotopy equivalence*

$$(\underline{CX}, \underline{X})^K \simeq (M * N) \vee ((\underline{CX}, \underline{X})^{K_1} \rtimes N) \vee (M \ltimes (\underline{CX}, \underline{X})^{K_2})$$

where $M = X_1 \times \cdots \times X_{m-l}$ and $N = X_{m+1} \times \cdots \times X_n$. □

In the light of Theorem 1.1, the value of Theorem 7.2 is that it can be applied to examples of simplicial complexes which are not shifted. On the other hand, Theorem 7.2 can also be applied to some examples which are shifted in order to give a quicker evaluation of the homotopy type of $(\underline{CX}, \underline{X})^K$. We illustrate the latter first, by reconsidering the simplicial complex K in Examples 5.2 (1). Geometrically, $|K|$ is obtained by gluing two copies of $|\partial\Delta^2|$ along a common edge. Specifically, $K = K_1 \cup_\tau K_2$ where K_1 is the simplicial complex on vertices $\{1, 2, 3\}$ having edges $\{(1, 2), (1, 3), (2, 3)\}$; K_2 is the simplicial complex on vertices $\{1, 2, 4\}$ having edges $\{(1, 2), (1, 4), (2, 4)\}$; and τ is the edge $(1, 2)$. Since $K_1 = \partial\Delta^2$, Proposition 2.2 implies that $(\underline{CX}, \underline{X})^{K_1} \simeq \Sigma^2 X_1 \wedge X_2 \wedge X_3$. Similarly, $(\underline{CX}, \underline{X})^{K_2} \simeq \Sigma^2 X_1 \wedge X_2 \wedge X_4$. In general, the space M is the product of the X_i's where i is not a vertex of K_2, and similarly for N and K_1. So in this case $M = X_3$ and $N = X_4$. Theorem 7.2 therefore implies that there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^K \simeq (X_3 * X_4) \vee ((\Sigma^2 X_1 \wedge X_2 \wedge X_3) \rtimes X_4) \vee (X_3 \ltimes \Sigma^2 X_1 \wedge X_2 \wedge X_4).$$

In general, there is a homotopy equivalence $\Sigma A \rtimes B \simeq \Sigma A \vee (\Sigma A \wedge B)$, and similarly for the right half-smash. Thus in our case we obtain a homotopy equivalence

$$\begin{aligned}(\underline{CX}, \underline{X})^K \simeq\ & (\Sigma X_3 \wedge X_4) \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_3) \vee (\Sigma^2 X_1 \wedge X_2 \wedge X_4) \\ & \vee 2 \cdot (\Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4).\end{aligned}$$

This matches the answer in Section 6.

Next, we apply Theorem 7.2 to a family of nonshifted complexes. Let L_2 be the previous example of two copies of $\partial\Delta^2$ glued along the common edge $(1, 2)$. Let L_3 be the simplicial complex on 5 vertices obtained by gluing another copy of $\partial\Delta^2$ to L_2 along the common edge $(1, 4)$. We will show that L_3 is not shifted. If it were, then so is the restriction rest$\{2, 3, 4, 5\}$. But this is the simplicial complex on vertices $\{2, 3, 4, 5\}$ with edges $\{(2, 3), (2, 4), (4, 5)\}$. Any connected simplicial complex which is shifted has a distinguished vertex which is connected by an edge to any other vertex. This is not the case with rest$\{2, 3, 4, 5\}$, and there is no reordering of the vertices which would make this the case. Thus rest$\{2, 3, 4, 5\}$ is not shifted, implying that L_3 is not shifted. Nevertheless, since $(\underline{CX}, \underline{X})^{L_2} \in \mathcal{W}_4$ and $(\underline{CX}, \underline{X})^{\partial\Delta^2} \simeq \Sigma^2 X_1 \wedge X_4 \wedge X_5$, Theorem 7.2 implies that $(\underline{CX}, \underline{X})^{L_3} \in \mathcal{W}_5$. In the same way, we can iteratively construct L_{n-2} by gluing a copy of $\partial\Delta^2$ to L_{n-3} along the common edge $(1, n-1)$ to obtain a simplicial complex on n vertices which is not shifted and which satisfies $(\underline{CX}, \underline{X})^{L_{n-2}} \in \mathcal{W}_n$.

It is worth pointing out that we assume that any simplicial complex contains the empty set as its simplex. Thus Theorem 7.2 also treats the case when K is obtained as a disjoint union of K_1 and K_2, that is $K = K_1 \coprod K_2$. For example, let $K = K_1 \coprod K_2$ where $K_1 = \Delta^1$ is the 1-simplex on vertices $\{1, 2\}$ and $K_2 = \Delta^1$ is the 1-simplex on vertices $\{3, 4\}$. Any n-simplex is shifted so both K_1 and K_2 are shifted. However, K is not shifted since the edge $(3, 4) \in K$ would imply the edge $(1, 4) \in K$, but this is not the case and there is no reordering of the vertices which will make this the case. Since K_1 and K_2 are 1-simplices, we have $(\underline{CX}, \underline{X})^{K_1} = CX_1 \times CX_2$

and $(\underline{CX}, \underline{X})^{K_2} = CX_3 \times CX_4$. Both spaces are contractible, so Theorem 7.2 implies that $(\underline{CX}, \underline{X})^K \simeq M * N$ where $M = X_3 \times X_4$ and $N = X_1 \times X_2$. Thus $(\underline{CX}, \underline{X})^K \in \mathcal{W}_4$.

We now discuss how Theorem 7.2 allows us to generalise Theorem 1.1. Let $\mathcal{W}$ be the family of simplicial complexes K for which the polyhedral product $(\underline{CX}, \underline{X})^K$ is homotopy equivalent to a wedge of suspensions of smashes of the X_i's. By Theorem 1.1, this family is non-empty and contains shifted complexes.

Proposition 7.3. *Let K be a simplicial complex on the index set $[n]$. Suppose that $K = K_1 \cup_\tau K_2$ where τ is a common face of K_1 and K_2 and $K_1, K_2 \in \mathcal{W}$. Then $K \in \mathcal{W}$, that is, $(\underline{CX}, \underline{X})^K$ is homotopy equivalent to a wedge of suspensions of smashes of the X_i's.*

Proof. By Theorem 7.2,

$$(\underline{CX}, \underline{X})^K \simeq (M * N) \vee ((\underline{CX}, \underline{X})^{K_1} \rtimes N) \vee (M \ltimes (\underline{CX}, \underline{X})^{K_2}).$$

As K_1 and K_2 belong to $\mathcal{W}$, the corresponding polyhedral products are wedges of suspensions of smashes of the X_i's. Now the statement follows by applying the homotopy equivalences $\Sigma A \rtimes B \simeq \Sigma A \vee (\Sigma A \wedge B)$ and $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B)$. □

8. Extensions of the method II: the simplicial wedge construction

Let K be a simplicial complex on vertices $\{v_1, \ldots, v_n\}$. Fix a vertex v_i. By doubling the vertex v_i, define a new simplicial complex $K(v_i)$ on the $n+1$ vertices $\{v_1, \ldots, v_{i-1}, v_{i,1}, v_{i,2}, v_{i+1}, \ldots, v_n\}$ by

$$K(v_i) = (v_{i,1}, v_{i,2}) * \mathrm{link}_K(v_i) \cup \{v_{i,1}, v_{i,2}\} * \mathrm{rest}_K(v_i)$$

where $(v_{i,1}, v_{i,2})$ denotes the one dimensional simplex on the vertices $v_{i,1}$ and $v_{i,2}$. The simplicial complex $K(v_i)$ is called the *simplicial wedge* of K on v_i. This construction arises in combinatorics (see [14]) and has the important property that if K is the boundary of the dual of a polytope then so is $K(v_i)$.

As in [2], the construction can be iterated. To set this up, let $(1, \ldots, 1)$ be an n tuple of 1s, corresponding to the single appearance of each vertex in the vertex set $\{v_1, \ldots, v_n\}$. The vertex doubling operation of v_i in the simplicial wedge construction gives a new vertex set for $K(v_i)$ – listed above – to which we associate the n-tuple $(1, \ldots, 1, 2, 1, \ldots, 1)$ with 2 in the i-position, which records that the vertex v_i appears twice. The sequence $(1, \ldots, 1, 3, 1, \ldots, 1)$ then corresponds to either the simplicial wedge $(K(v_i))(v_{i,1})$ or to $(K(v_i))(v_{i,2})$. However, these two complexes are equivalent, so the choice of vertex $v_{i,1}, v_{i,2}$ does not matter. More generally, let $J = (j_1, \ldots, j_n)$ be an n tuple of positive integers, and let $m = \sum_{i=1}^n j_i$. Define a new simplicial complex $K(J)$ on m-vertices

$$\{v_{1,1}, \ldots, v_{1,j_1}, v_{2,1}, \ldots, v_{2,j_2}, \ldots, v_{n,1}, \ldots, v_{n,j_n}\}$$

by iteratively applying the simplicial wedge construction, starting with K.

We shall show that if K is a shifted complex on n vertices then, for any $J = (j_1, \ldots, j_n)$, the polyhedral product determined by $K(J)$ and $m = (\sum_{i=1}^n j_i)$ topological pairs (CX_i, X_i) is homotopy equivalent to a wedge of suspensions of smashes of the X_i's. This improves on Theorem 1.1 because the class of simplicial complexes obtained from shifted complexes by iterating the simplicial wedge construction is strictly larger than the class of shifted complexes. We give an example to illustrate this.

Example 8.1. Let K be the 1-dimensional simplicial complex consisting of vertices $\{1, 2, 3, 4\}$ and edges $\{(1, 2), (1, 3)\}$. Observe that K is shifted using this particular ordering of vertices. Apply the simplicial wedge product which doubles vertex 4, that is, let $J = (1, 1, 1, 2)$. Then $K(J)$ is a simplicial complex on vertices $\{1, 2, 3, 4a, 4b\}$. We have $K(J) = (4a, 4b) * \text{link}_K(4) \cup \{4a, 4b\} * \text{rest}_K(4)$. As $\text{link}_K(4) = \emptyset$ and $\text{rest}_K(4)$ consists of vertices $\{1, 2, 3\}$ and edges $\{(1, 2), (1, 3)\}$, the simplicial wedge complex $K(J)$ is the simplicial complex on $\{1, 2, 3, 4a, 4b\}$ with the maximal faces $\{(4a, 4b), (1, 2, 4a), (1, 2, 4b), (1, 3, 4a), (1, 3, 4b)\}$.

We claim that $K(J)$ is not shifted. Observe that the edge $(2, 3) \notin K(J)$, but every other possible edge is in $K(J)$. That is, $(x, y) \in K(J)$ for every $x, y \in \{1, 2, 3, 4a, 4b\}$ except $(2, 3)$. Thus with the ordering $1 < 2 < 3 < 4a < 4b$, $K(J)$ does not satisfy the shifted condition as $(2, 4a) \in K(J)$ would imply that $(2, 3) \in K(J)$. So if $K(J)$ is to be shifted, we must reorder the vertices. Let $\{1', 2', 3', 4', 5'\}$ be the new labels of the vertices. To satisfy the shifted condition we need to send the vertices $\{2, 3\}$ to $\{4', 5'\}$. The vertices $\{1, 4a, 4b\}$ are therefore sent to $\{1', 2', 3'\}$. Now observe that the face $(1, 4a, 4b) \notin K(J)$. Thus in the new ordering, the face $(1', 2', 3') \notin K(J)$. The shifted condition therefore implies that no 2-dimensional faces are in $K(J)$, a contradiction. Hence there is no reordering of the vertices of $K(J)$ for which the shifted condition holds. Hence $K(J)$ is not shifted.

In [2], polyhedral products related to the simplicial wedge constriction were studied. The authors started with a simplicial complex K on n vertices, n topological spaces $\underline{X} = (X_1, \ldots, X_n)$ and an n-tuple of integers $J = (j_1, \ldots, j_n)$. After defining a family of topological pairs by

$$(\underline{C(\circledast_J X)}, \underline{\circledast_J X}) = \{(C(\underbrace{X_i * \cdots * X_i}_{j_i}), \underbrace{X_i * \cdots * X_i}_{j_i})\}_{i=1}^n$$

it was shown that there is a homeomorphism of polyhedral products

$$(\underline{C(\circledast_J X)}, \underline{\circledast_J X})^K \longrightarrow (\underline{CX}, \underline{X})^{K(J)}$$

where, to match the vertices of $K(J)$, $(\underline{CX}, \underline{X})^{K(J)}$ was defined by requiring that

$$(\underline{CX}, \underline{X}) = (\underbrace{(CX_1, X_1), \ldots, (CX_1, X_1)}_{j_i}, \ldots, \underbrace{(CX_n, X_n), \ldots, (CX_n, X_n)}_{j_n}).$$

We generalise these results by removing the restriction on the topological pairs defining the polyhedral product $(\underline{CX}, \underline{X})^{K(J)}$. Let K be a simplicial complex on n vertices and let $J = (j_1, \ldots, j_n)$ be an n-tuple of positive integers. Let $m = \sum_{k=1}^n j_k$ and let $\underline{X} = (X_1, \ldots, X_m)$ be m topological spaces. Let $m_0 = 0$ and for $1 \le i \le n$, let $m_i = \Sigma_{k=1}^i j_k$. Note that $m_n = m$. The m topological spaces $X_1, \ldots, X_m$ are then written as

$$X_{m_0+1}, \ldots, X_{m_1}, X_{m_1+1}, \ldots, X_{m_2}, X_{m_2+1}, \ldots, X_{m_{n-1}}, X_{m_{n-1}+1}, \ldots X_{m_n}.$$

Define

$$(\underline{C(*_J X)}, \underline{*_J X}) = \left\{\left(C(X_{m_{i-1}+1} * \cdots * X_{m_i}), X_{m_{i-1}+1} * \cdots * X_{m_i}\right)\right\}_{i=1}^n.$$

Note that there are j_i terms in $X_{m_{i-1}+1} * \cdots * X_{m_i}$, and the definition of $(\underline{C(*_J X)}, \underline{*_J X})$ coincides with the definition of $(\underline{C(\circledast_J X)}, \underline{\circledast_J X})$ if, for $1 \le i \le n$, the spaces $X_{m_{i-1}+1}, \ldots, X_{m_i}$ are all equal. For example, if $n = 2$ and $J = (2, 2)$ then $m = 4$, there are 4 topological spaces

$X_1, \ldots, X_4$ which are grouped as X_1, X_2 corresponding to j_1 and X_3, X_4 corresponding to j_2, and we have

$$(\underline{C(*_J X)}, \underline{*_J X}) = \{(C(X_1 * X_2), X_1 * X_2), (C(X_3 * X_4), X_3 * X_4)\}.$$

The following lemma, which is a classical result (see for example, [12]), is the key ingredient in the generalisation.

Lemma 8.2. *For any finite CW-complexes X and Y, there is a homeomorphism of pairs*

$$(C(X * Y), X * Y) \longrightarrow (CX, X) \times (CY, Y). \quad \square$$

Proposition 8.3. *For a simplicial complex K on n-vertices, an n-tuple $J = (j_1, \ldots, j_n)$ of positive integers and $\sum_{i=1}^{n} j_i$ topological pairs (CX_i, X_i) where X_i is a finite CW-complex, there is a homeomorphism of polyhedral products*

$$(\underline{C(*_J X)}, \underline{*_J X})^K \longrightarrow (\underline{CX}, \underline{X})^{K(J)}.$$

Proof. The proof is along the lines of that in [2, Theorem 8.2], using Lemma 8.2 instead of the special case $(C(X * X), X * X) \xrightarrow{\cong} (CX, X) \times (CX, X)$ in [2, Lemma 8.1]. $\square$

Proposition 8.4. *Let K belong to $\mathcal{W}$ and J be an n-tuple of positive integers. Then there is a homotopy equivalence*

$$(\underline{CX}, \underline{X})^{K(J)} \simeq \left(\bigvee_{I \notin K} |K_I| * (\widehat{*_J X}^I) \right).$$

Proof. This is a direct consequence of Proposition 8.3 and the defining property of simplicial complexes belonging to $\mathcal{W}$. $\square$

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