Waves in Loudspeaker Cones

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Abstract

Several analytic and numerical models have been proposed to study the dynamic behaviour of loudspeaker cones, generally based on thin shell elements. The purpose of this paper is to link together some of these models by decomposing the results of a finite element analysis in terms of wave components. This is achieved using the wave finite element, WFE, method. A full finite element model is constructed first and the forced responses of the loudspeaker at different frequencies are calculated. Then, a wave finite element analysis is used to calculate the wave properties of the loudspeaker cone, in terms of wave mode shape and wavenumber, as a function of position. By decomposing the overall results from the finite element analysis, the responses of the loudspeaker cone can be interpreted in terms of wave propagation. It is shown how the loudspeaker cone moves as a rigid body at low frequencies. However, as frequency increases the transverse velocity of the cone surface becomes non-uniform, since the amplitude of the vibration increases towards the base of the cone, with a larger radius. At still higher frequencies, the motion of the base is reduced and the peak transverse displacement occurs at a lower point along the cone, which moves towards the apex as the frequency increases.

1 Introduction

The development of numerical modelling methods, such as the finite element (FE) method or the dynamic stiffness matrix (DSM) method, allows the efficient and flexible calculation of the dynamic behaviour of loudspeaker cones, the physical insight of the cone vibration patterns transition may be lost, however. The wave finite element (WFE) method is a numerical approach to investigate wave motion in waveguides and express the dynamics of the waveguide in terms of wavenumbers and wave modes^[1]. In this paper, the wave finite element analysis was used to predict the wavenumbers and wave modes as a function of position and the overall results from the finite element analysis was decomposed and interpreted in terms of wave propagation.

2 Loudspeaker cone dynamics

At low frequencies, the loudspeaker is usually assumed to be a rigid piston and the cone will move as a rigid body. However, the cone itself is not rigid at high frequencies and should be modelled as a flexible system. In this paper, the dynamic behaviour of non-rigid cone was initially calculated using thin shell theory. The cone is taken as an axisymmetric conical thin shell which is excited axially on the inner edge as shown in Figure 1.



Figure 1 Side view of the conical cone in local coordinates, x denotes position along the cone meridian, X and Y are the axial and radial positions in the global coordinates, α is the angle between the axis and meridian.

2.1 Equation of motion

Following thin shell theory^[2], the displacements and forces of the cone, which are shown in Figure 1, can be expressed by six first-order differential equations as ^[3]

$$\begin{cases} u' = -\upsilon \frac{u}{x} + \upsilon \frac{w}{x} \cot \alpha + \frac{(1-\upsilon^2)}{Eh} N_x \\ w' = \theta \\\\ \theta' = -\upsilon \frac{\theta}{x} + \frac{12(1-\upsilon^2)}{Eh^3} M_x \\\\ N'_x = Eh\left(\frac{1}{x^2} - \frac{\rho\omega^2}{E}\right) u - Eh\frac{w}{x^2} \cot \alpha + \left(\frac{\upsilon - 1}{x}\right) N_x \\\\ Q'_x = -Eh\frac{\cot \alpha}{x^2} u - h\left(\frac{\cot^2 \alpha}{x^2} E - \rho\omega^2\right) w - \upsilon \frac{\cot \alpha}{x} N_x - \frac{1}{x} Q_x \\\\ M'_x = -\frac{Eh^3}{12x^2} \theta - Q_x + \frac{(\upsilon - 1)}{x} M_x \end{cases}$$
(1)

where x is the position along the cone meridian, α is the half angle of the cone, u is the displacement along the meridian (positive in the direction of positive x), w is the displacement normal to the meridian (positive inward) and θ is the rotation of the cone's normal in the direction shown in Figure 1. N_x is the membrane force in the direction of the meridian (positive in the direction of positive x), Q_x is the normal shearing force (positive outwards normal to the meridian) and M_x is the bending moment in the circumferential direction (all per unit length of mid-plane in the circumferential direction, and is positive in counter-clockwise direction along the positive x). E is Young's modulus of the cone material, h is the cone thickness, v is Poisson's ratio, ρ is the density of cone material and ω is angular frequency.

The dynamic behaviour at a given position along the cone depends on whether the excitation is above or below the ring frequency at this location^[4]. The ring frequency is given by

$$f_R = \frac{c}{2\pi R},\tag{2}$$

where c is the speed of longitudinal waves given by $\sqrt{E/\rho}$, R is the distance between the cone and cone axis measured perpendicular to the cone meridian, as shown in Figure 1, and this is plotted for the cone used below in Figure 2. Below the ring frequency the dynamics are dominated by the membrane stiffness, resulting in mostly in-plane motion. Above the ring frequency the dynamics are dominated by the bending stiffness, resulting in flexural motion. Since the ring frequency varies along the length of the cone, however, the dynamic behaviour of the cone depends on whether the frequency is in one of three regions, as shown in Figure 2. In region I, the excitation frequency is below the ring frequency at every position along the cone, so the cone moves entirely in phase as a quasi-rigid body. In region II, the excitation frequency is below the ring frequency at the apex of the cone but above it at the base of the cone, so that part of the cone is still moving as a quasi-rigid body and part of the cone has "broken up" into flexural motion. In region III, the excitation frequency is above the ring frequency all along the cone, so the dynamics are dominated by bending waves at all positions.



Figure 2 The ring frequency as a function of position along the cone axis indicating the transition between mostly in-plane (///) and mostly bending (\\\) behaviour.

The dynamic stiffness matrix, **D**, of a conical cone, whose geometry and material are listed in Table 1, was calculated using the Runge-Kutta approximation ^[5] with a step size of 0.1 mm. A finite element model of the dynamics of the cone thus can be expressed as

$$\mathbf{Dq} = \mathbf{f},\tag{3}$$

where **q** is the vector of displacement, u, w, and θ at each element along the cone and **f** is the corresponding vector of forces N_x , M_x and Q_x at each element, which defined the excitation applied on the cone at the apex.

A [rad]	Cone length [m]	Initial value on the X axis [m]	E [Pa]	ρ [kg/m ³]	Loss factor η	Poisson's ratio v	Thickness <i>h</i> [m]
2π/9	0.108	0.0209	1.5×10^{9}	900	0.02	0.33	0.5×10 ⁻³

 Table 1 Assumed properties of the loudspeaker cone.

2.2 Forced responses

The forced response of the cone due to a harmonic force f, where $e^{i\omega t}$ is suppressed here for brevity, which is applied uniformly over the circumference of the narrow neck of the cone in the axial direction, is calculated using the finite element method at different frequencies and instantaneous snap shots of the resulting vibration patterns are shown in Figure 3. The boundary condition at the apex is fixed axially and free at the base. Different forces are applied at different frequencies, as shown in Figure 3, in order to show clearly the difference between the deformation displacement and the un-deformed surface. The dashed lines in each subfigure are the un-deformed middle surfaces and the solid lines denote the deformed middle surfaces.



Figure 3 Forced responses of the cone at (a) 100 Hz, (b) 2000 Hz, (c) 3000 Hz and (d) 10000 Hz, the displacements are plotted in the global coordinates denoted by solid lines and the un-deformed middle surface of the cone is represented by dashed lines.

It can be seen that at low frequency, 100 Hz for example, the cone moves as a rigid body, since the frequency is in region I, and the amplitude of motion in the axial direction, X, is much greater than that in the radial direction, Y. Also, the real part of the displacement in the axial direction is much greater than its imaginary part and is constant along the cone. The amplitude of this displacement is about 15.7 mm which is close to the theoretical result, which is calculated using Newton's Second Law of Motion assuming the cone acts as a mass.

At higher frequencies, 2 kHz and 3 kHz when the excitation frequency is in region II, cone "break-up" occurs, in which the cone no longer vibrates as a rigid body, but some sections of the cone still move in phase. In this frequency range, a bending wave cannot propagate in the region close to the apical edge of the cone, due to the high stiffness, but can propagate on the outer part of the cone and then build up into a standing wave. The position, the "transition point", where bending wave starts to cut-on, moves towards the apical edge as driving frequency increases. At 10 kHz the excitation frequency is in region III and the whole cone moves with a bending motion. The more detailed behaviours of bending and longitudinal motions are studied and discussed below using the wave finite element method (WFE).

3 Waves in the loudspeaker cone

3.1 Transfer matrix

Wave motion on the loudspeaker cone can be analysed using the wave finite element method^[1]. This approach starts by modelling a short section of the cone using conventional finite elements. Then all the complex forces, at a given excitation frequency, can be expressed as a function of all the complex displacements multiplied by the corresponding dynamic stiffness. The vector of these forces and displacements is then partitioned into those on the left and right hand side of the *n*-th section of the cone, so that

$$\begin{bmatrix} \mathbf{D}_{\mathrm{LL}}(n) & \mathbf{D}_{\mathrm{LR}}(n) \\ \mathbf{D}_{\mathrm{RL}}(n) & \mathbf{D}_{\mathrm{RR}}(n) \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\mathrm{L}}(n) \\ \mathbf{q}_{\mathrm{R}}(n) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathrm{L}}(n) \\ \mathbf{f}_{\mathrm{R}}(n) \end{bmatrix},$$
(4)

where the subscripts L and R represent the displacement and forcing vector on the left and right hand side of the *n*-th section. The terms in equation (4) can be re-arranged to express the forces and displacements on one side of the section in terms of those on the other side, as

$$\begin{bmatrix} \mathbf{q}_{\mathrm{R}}(n) \\ -\mathbf{f}_{\mathrm{R}}(n) \end{bmatrix} = \mathbf{T}(n) \begin{bmatrix} \mathbf{q}_{\mathrm{L}}(n) \\ \mathbf{f}_{\mathrm{L}}(n) \end{bmatrix},$$
(5)

where $\mathbf{T}(n)$ is the transfer matrix of the *n*-th section and can be expressed in terms of the dynamic stiffness as ^[1]

$$\mathbf{T}(n) = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}(n)\mathbf{D}_{LL}(n) & \mathbf{D}_{LR}^{-1}(n) \\ -\mathbf{D}_{RL}(n) + \mathbf{D}_{RR}(n)\mathbf{D}_{LR}^{-1}(n)\mathbf{D}_{LL}(n) & -\mathbf{D}_{RR}(n)\mathbf{D}_{LR}^{-1}(n) \end{bmatrix}.$$
 (6)

Assuming the dynamics only change a little from one element to the next, a particular vibrating pattern of displacements and forces, due to the *m*-th wave, on the right hand side of the section is equal to the same pattern on the left hand side of the section, multiplied by a complex constant of proportionality λ_m , so that

$$\begin{bmatrix} \mathbf{q}_{\mathrm{R}}(n,m) \\ -\mathbf{f}_{\mathrm{R}}(n,m) \end{bmatrix} = \lambda_{m} \begin{bmatrix} \mathbf{q}_{\mathrm{L}}(n,m) \\ \mathbf{f}_{\mathrm{L}}(n,m) \end{bmatrix}.$$
(7)

This vibration pattern then would travel as a propagating, evanescent or oscillating wave along the cone with a complex wavenumber, k_m , determined by

$$\lambda_m = e^{-ik_m\Delta},\tag{8}$$

where Δ is the length of the cone section and the type of wave depends on the relative magnitude of the real and imaginary parts of the wavenumber. The right hand side of equation (5) must now be equal to the right hand side of equation (7), and so λ_m , and the corresponding distribution of displacements and forces, must be an eigenvalue, and the corresponding eigenvector, of the transfer matrix for this section.

3.2 Wavenumber distributions

The wavenumber can be obtained directly from solving the eigenvalue problem for the transfer matrix T(n). Figure 4 shows the distribution of the real and imaginary parts of the wavenumber along the cone

axis at 3000 Hz calculated using this method. Since the finite element model for each of the 512 segments of the cone has 3 degrees of freedom on each face of the cone section, there are thus 3 pairs of eigenvalues for the transfer matrix $\mathbf{T}(n)$. Half of these, whose imaginary part is negative, are forward going waves. We can divide the cone into 2 regions along the cone axis, region A includes apex to about 0.06 m corresponding to the transition point at 3000 Hz shown in Figure 2 and region B corresponds to from 0.06 m to the base of the cone. It can be seen that not all waves can propagate along the cone in region A, since they have non-zero imaginary wavenumber indicating they are oscillating and decaying. Specifically, wave 1 propagates with a gradually decreasing speed and decays less than the other two waves, since wave 1 has a small non-zero imaginary part of wavenumber. This shows that in the region A, the longitudinal motion dominates the vibration pattern of the cone. Beyond the position 0.06 m, wave 2 starts to propagate towards the base. The wave 1 becomes an evanescent wave beyond the position 0.06 m. Wave 3 has a large non-zero imaginary part of wavenumber along the whole range of the cone, which indicates that this wave does not play a significant role in cone vibration.



Figure 4 Wavenumber distribution along the cone axis at 3000 Hz using the WFE, with only wavenumbers corresponding to forward going waves plotted, and solid lines being the real component of the wavenumber *k*, and dashed lines being the imaginary component.

Although we can find the properties of each wave from the wavenumber variations, the transition between different vibration patterns and the contribution of each wave to the cone vibration cannot be directly seen from this analysis. Next, the forced response from the full finite element analysis will be decomposed into wave components, which can show the contribution of each wave to the cone vibration.

4 Decomposition into wave components

In general, the $N \times N$ transfer matrix **T** has a linearly independent set of N eigenvectors and we can express the eigenvalue, eigenvector decomposition of the transfer matrix for the *n*-th section in the form

$$\mathbf{T}(n) = \mathbf{Q}(n) \mathbf{\Lambda}(n) \mathbf{Q}^{-1}(n), \qquad (9)$$

so that

$$\mathbf{T}(n)\mathbf{Q}(n) = \mathbf{Q}(n)\mathbf{\Lambda}(n), \tag{10}$$

and

$$\mathbf{Q}^{-1}(n)\mathbf{T}(n) = \mathbf{\Lambda}(n)\mathbf{Q}^{-1}(n).$$
(11)

The right eigenvectors of $\mathbf{T}(n)$ thus correspond to the columns of $\mathbf{Q}(n)$ and the left eigenvectors of $\mathbf{T}(n)$ correspond to the rows of $\mathbf{Q}^{-1}(n)$. The wavenumbers are given by the elements of the diagonal eigenvalue matrix $\mathbf{\Lambda}(n)$.

Using equation (9), equation (5) can also be written as

$$\mathbf{Q}^{-1}(n)\begin{bmatrix}\mathbf{q}_{\mathrm{R}}(n)\\-\mathbf{f}_{\mathrm{R}}(n)\end{bmatrix} = \mathbf{\Lambda}(n)\mathbf{Q}^{-1}(n)\begin{bmatrix}\mathbf{q}_{\mathrm{L}}(n)\\\mathbf{f}_{\mathrm{L}}(n)\end{bmatrix}.$$
(12)

Since $\Lambda(n)$ is diagonal, the inner product of each row of $\mathbf{Q}^{-1}(n)$, which is a left eigenvector of $\mathbf{T}(n)$, with the "state vectors" on the right and left hand side gives an equation of the form

$$a_{\rm Rm}(n) = \lambda_m(n) a_{\rm Lm}(n), \qquad (13)$$

where $a_{Rm}(n)$ and $a_{Lm}(n)$ can be interpreted as the complex amplitudes of the *m*-th wave on the right and left hand side of the *n*-th section ^[6]. The vector of all such wave amplitudes, on the right hand side of this segment, for example, can be written as

$$\mathbf{a}_{\mathrm{R}}(n) = \mathbf{Q}^{-1}(n) \begin{bmatrix} \mathbf{q}_{\mathrm{R}}(n) \\ -\mathbf{f}_{\mathrm{R}}(n) \end{bmatrix}.$$
(14)

The axial displacement, d(n,m), due to the *m*-th wave at the *n*-th position is given by

$$d(n,m) = \mathbf{sq}(n,m)a(n,m), \tag{15}$$

where s is the weighting vector, $\mathbf{s} = [\cos \alpha \sin \alpha]$, $\mathbf{q}(n,m)$ is the displacement vector for the *m*-th wave at the *n*-th position and a(n,m) is the wave amplitude for the *m*-th wave at the *n*-th position. The contributions to the axial displacement distributions, calculated using the full finite element model, due to each of the forward going waves shown in Figure 4, are plotted in Figure 5.



Figure 5 Decomposition of the overall axial displacement calculated from the full finite element model into components due to forward going waves in Figure 4, calculated from the wave finite element model.

The contribution of wave 1 is seen to be in reasonable agreement with the overall result from the full finite element method for positions apical to the peak response at this frequency, at about 0.06 m along the cone axis. The contribution of wave 1 is significantly less than the overall result of the full finite element for positions beyond the peak response, region B, however, where the contribution of wave 2 dominates the overall response. There is also a negative going component of wave 2 in this region, due to the reflection from the free basal end of the cone and the interference between this and the positive going wave 2 gives rise to the interference pattern seen in the full finite element results. The contribution of wave 3 decays away on either side of this peak, and the amplitude is too small to significantly affect the overall response. It can be concluded that, inside region A the wavelength is long due to the predominance of longitudinal motion, and inside region B the wavelength is much shorter due to the predominance of bending motion. The energy is converted from longitudinal motion to bending motion at the transition point.

5 Conclusion

The dynamics of the loudspeaker cone have previously been studied using different methods such as the Dynamic Stiffness Matrix method, Finite Element method etc., but these methods have not previously been interpreted in terms of wave propagations. This research combines the conventional Finite Element method and the Wave Finite Element to study waves in the loudspeaker and the decomposition of the results from the full FE model into wave components, in order to illustrate the contribution of each wave to the overall response. The overall response of the cone considered results from a coupling between longitudinal and bending motion at an excitation frequency of 3000 Hz. The longitudinal motion plays a major part from the apex to the transition point but the bending motion dominates the overall response beyond this region.

The wave finite element method provides a method of studying the waves that travel in the loudspeaker cone and, more importantly, decomposing the response of the full finite element model into the components due to each of these waves, in order to explore how they interact. In this way the insight provided by the wave approach can be brought to bear on the numerical results from more detailed finite element models.

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