# A Proof of the Four Colour Theorem

Jing Tang Xing
*Faculty of Engineering & the Environments, University of Southampton,*
*Southampton SO17 1BJ, UK. jtxing@soton.ac.uk*

**ABSTRACT**

The four-colour problem remained unsolved for more than a hundred years has played a role of the utmost importance in the development of graph theory. The four-colour theorem was confirmed in 1976, which is not completely satisfied due to: i) part of the proof using computers cannot be verified by hand; ii) even the part, supposedly hand-checkable, is extraordinarily complicated and tedious, and as far as we know, no one has entirely verified it. Seeking a hand-checkable proof of the four-colour theorem is one of world-interested problems, which is addressed in this paper. A necessary and sufficient condition for *n*-colour theorem in a space is: there exists a largest *n*-complete graph base in the same space. Examples are given to illustrate applications.

**KEY WORDS:** Four colour theorem, Hand-checkable proof, Graph theory.

**INTRODUCTION**

The history of the four colour problem may be read in many references, for example, the references [1]-[3] in which the originality of the problem, the historic demonstrations, comments and further developments including the original publications are given. Therefore, it is not necessary to repeat this information in this paper. The four colour conjecture was confirmed by Appel and Haken in 1976, when they published their proof of the four colour theorem [4, 5]. A discussion of errors, their correction, and other potential problems were reported in [6]. The Appel-Haken proof is not completely satisfactory due to the following two main reasons: i) part of the Appel-Haken proof uses a computer, and cannot be verified by hand, and ii) even

**Copyright** © By Jing Tang Xing, Univ. of Southampton, September 18, 2013. 

the part that is supposedly hand-checkable is extraordinarily complicated and tedious, and as far as we know, no one has verified it in its entirety [2,3,7,8,9]. Seeking a simple hand-checkable proof of the four colour theorem is still one of very interested problems in the world. This paper provides a simple demonstration on this theorem. This demonstration can be fully verified by readers, even young school students. A necessary and sufficient condition for *n*-colour theorem in a space is given by a theorem based on which the four coloured theorem can be confirmed. Examples are given to illustrate applications of the proved theorem in the paper.

**DEFINITIONS**

**Spaces.** The definition on a space follows the geometrical concepts commonly known in the Euclid's geometry. An *m*-dimensional (*m*-D) space, ($m$ = 0, 1, 2), defines a continuous domain in which each point requires $m$ independent coordinates to determine its position. For example, a point is a 0-D space and a plane is a 2-D space, etc. If a closed curve in a space can be contracted to a point in the same space, this space is called as a *simply-connected* space, otherwise a *multiply-connected* one. A continuous line that starts and ends at two different points, respectively, is a simply-connected 1-D space, but a closed curve of which its starting and ending points are at a same point is a multiply-connected 1-D space.

**Graph.** A graph $g_m^n$ in the *m*-D space is defined by a configuration consisting of $n$ vertices connected by lines (edges) without any crossing each other except at the vertex points.

***k*-Complete Graph**. A graph $K_m^k$ in the *m*-D space is *k-complete* if it consists of $k$ vertices of which each pair of vertices is connected by an edge. Therefore, each vertex of the graph $K_m^k$ is connected to the other $k$-1 vertices by $k$-1 edges.

**Copyright** © By Jing Tang Xing, Univ. of Southampton, September 18, 2013. 

**Largest *k*-Complete Graph.** A graph $G_m^k$ in the *m*-D space is *the largest k-complete* if it is impossible to add a **new** vertex with the **new** *k* edges connecting to the *k* vertices in $G_m^k$ to generate a (*k*+1)-complete graph $G_m^{k+1}$ in the same space.

***n*-Coloured Graph.** A graph is said to be *n-coloured* if it is possible to assign one of **minimum** *n* colours to each vertex in such a way that no two connected vertices have a same colour. Obviously, a *k*-complete graph is *k*-coloured graph.

**Graph sets**. A graph set $S_m^k$ is defined as the set of all possible graphs consisting of *k* vertices in the *m*-D space. Obviously, adding a vertex into a given graph $g_m^k \in S_m^k$ produces a corresponding graph $g_m^{k+1} \in S_m^{k+1}$. The graph set of all possible graphs in the *m*-D space is given by $S_m = \bigcup_{k=0}^{\infty} S_m^k$.

***n*-Coloured Graph Base.** For an *n*-coloured graph $g_m^p, \ (p > n)$ in *m*-D space, minimum *n* colours are needed. We can do:

i) retain its *n* vertices with *n* different colours;

ii) delete the other *p* - *n* vertices with repeating colours but allow the edges at each deleted vertex to be connected;

iii) move each deleted vertex point onto a retained colour vertex by allowing the related edges to be extended or compressed;

iv) merge the repeating edges between two retained colour vertices.

The above process does not reduce or increase the number of minimum colours required in the graph $g_m^p, \ (p > n)$. The resultant graph is one with *n* vertices of *n* colours. **If this resultant graph is still an *n*-coloured graph, i.e. requiring the minimum *n* colours, it is called the *n*-coloured graph base**.

**Copyright** © By Jing Tang Xing, Univ. of Southampton, September 18, 2013. 

Since the above works in i)-iv) can be done in the reverse process from iv) to i), so that from the obtained *n*-coloured graph base we can recover the original graph structure.

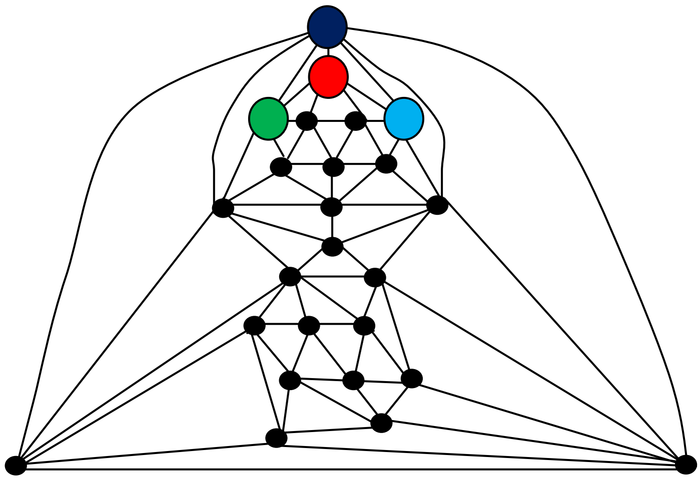


Figure 1 The Heawood's planar graph with 25 vertices.

To illustrate the above process, we consider the Heawood's planar graph [1] as shown in Figure 1. This graph needs at least 4 colours to be coloured. We can generate its 4-coloured graph base as follows. For example, we retain the 4 large coloured vertices and delete all the other smaller black vertices, and then follow the steps iii)-iv) to generate the 4-coloured graph base, as shown in Figure 3 a), which is a 4-complete graph.

**LEMMAS AND THEOREM**

**Lemma I.** *An existed n-complete graph* $K_m^n$ *in m-D space can always be generated by using one or both of the following methods:* i) *adding a* ***new*** *vertex and* $(n-1)$ ***new*** *edges connecting to the* $n-1$ *vertices of a* $(n-1)$*-complete graph* $K_m^{n-1}$*;* ii) *deleting a vertex and its n edges of an existing* $(n+1)$*-complete graph* $K_m^{n+1}$.

**PROOF**: Based on the definition, a ( $n-1$ )-complete graph $K_m^{n-1}$ consists of $n-1$ vertices of which each vertex is connected by $n-2$ edges. Therefore, the

**Copyright** © By Jing Tang Xing, Univ. of Southampton, September 18, 2013.

resultant graph generated by method i) is a graph consisting of *n* vertices of which each vertex is connected by $n-1$ edges, and hence an *n*-complete graph $K_m^n$. Similarly, a graph generated by method ii) is also a graph consisting of *n* vertices of which each vertex is connected by $n-1$ edges, i.e. an *n*-complete graph.

It is necessary to note that the $(n-1)$ ***new edges*** are required while using the method i) in the Lemma. A new vertex on one of the existing $(n-1)$ edges of a $(n-1)$-complete graph can be connected to their $n-1$ vertices using only $(n-3)$ ***new edges***, so that the resultant graph is not an *n*-complete graph.

Based on this Lemma, if there exists an *n*-complete graph in a space, there must exist the *m*-complete graphs ($m < n$) in the same space. Also, if there exists no an *n*-complete graph in a space, there must exist no *m*-complete graphs ($m > n$) in the same space.

**Lemma II**. The *n*-coloured graph base of an *n*-coloured graph $g_m^p$, $(p > n)$, in *m*-D space, must be an *n* complete graph $K_m^n$.

**PROOF**: A proof by mathematical induction is used to confirm this lemma as follows.

(i) Consider an *n*-coloured graph $g_m^n \in S_m^n$ in *m*-D space. Since this *n*-coloured graph $g_m^n \in S_m^n$ has *n* vertices and needs minimum *n* colours to be coloured, it must be an complete graph $K_m^n$.

(ii) Consider an *n*-coloured graph $g_m^{n+1} \in S_m^{n+1}$ in *m*-D space. Since this *n*-coloured graph $g_m^{n+1} \in S_m^{n+1}$ has *n* vertices coloured by the *n* different colours and one vertex coloured by a repeating colour. Therefore between the two vertices coloured by the same colour there is no edge, so that one of two same colour vertices could be deleted to generate its base graph, so that the resultant graph must be a complete graph $K_m^n$.

(iii) Assume that the lemma is valid for $n$-coloured graph $g_m^p \in S_m^p,\ (p > n),$ which implies it has an $n$ complete graph base $K_m^n$ with $p$ - $n$ vertices of which each is coloured by a repeating one of $n$ colours. The completion of the process for its $n$ complete graph base by deleting these p - $n$ vertices will lead to an $n$ complete graph base $K_m^n$ . Now, we need to prove for $n$-coloured graph $g_m^{p+1} \in S_m^{p+1},\ (p > n)$ to have the $n$ complete graph base $K_m^n$. Actually, the $n$-coloured graph $g_m^{p+1}$ is generated by adding a vertex into $g_m^p$. This added vertex cannot be connected to that original $n$ vertices coloured by the $n$ different colours and constructed the $n$ complete graph base $K_m^n$ of $g_m^p \in S_m^p,\ (p > n)$, otherwise this added vertex must be coloured by a new colour and the graph $g_m^{p+1}$ would be an ($n$+1)-coloured graph. Therefore, the new added vertex must be coloured by one repeating colour of n colours. Following its base generation process, this added extra vertex with one repeating colour can be deleted from the $n$-coloured graph $g_m^{p+1}$, which will generate the corresponding $n$-coloured graph $g_m^p$ with the $n$ complete graph base $K_m^n$, so that the $n$-coloured graph $g_m^{p+1}$ has the $n$ complete graph base $K_m^n$. Therefore the lemma is valid.

**Lemma III.** A graph $g_m^p(G_m^k)$, ($p$ > $k$), generated by adding $p$-$k$ new vertices and some allowed edges to the largest $k$-complete graph $G_m^k$ in $m$-D space, can be coloured by no more than $k$ colours.

**PROOF:** Since $G_m^k$ is the largest *k*-complete graph in *m*-D space, according to its definition, there is no $G_m^{k+1}$ in *m*-D space. Adding a new vertex with allowed edges, we generate a graph $g_m^{k+1}(G_m^k) = [G_m^k + 1 \ \text{vertex}] \in S_m^{k+1}$ with no an extra colour being introduced. Actually, if the new vertex is added on one edge of $G_m^k$, this new added vertex and the two vertices of this edge can be coloured by the two colours used by the two vertices of this edge. If the new vertex is added at a point not on any edges, it cannot be connected to at least one vertex of $G_m^k$ due to $G_m^k$ is largest, so that this new vertex can be coloured using the colour at the non-connected vertex. Therefore, the $g_m^{k+1}(G_m^k)$ can be coloured by no more than *k* colours. This demonstration concludes that adding a vertex to the largest *k*-complete graph $G_m^k$ in *m*-D space does not change the number *k* of required minimum colours.

Assume that the graph $g_m^{k+p}(G_m^k)$ in *m*-D space can be coloured by minimum *k+1* colours, from Lemma II, its base graph is $K_m^{k+1}$ that implies the largest complete graph in *m*-D space, which contravenes the prescribed condition of $K_m^k$ being the largest *k*-complete graph $G_m^k$ in *m*-D space.

**Theorem.** *All graphs in m-D space can be colourable using minimum n colours if and only if there exists a largest n-coloured graph base* $G_m^n$ *in m-D space* .

PROOF: Since graphs $G_m^p$ , (*p* < *n*), can be definitely coloured using no more than *n* colours, we need to prove the case of $p \geq n$. The necessity of the theorem is obvious. According to lemma II, an *n*-coloured graph base must be an *n* complete graph $K_m^n$. Since minimum *n* colours are required for all graphs in *m*-D space, the $K_m^n$ is the largest *n*-coloured graph base $G_m^n$ in m-D space.

Based on Lemma III, the sufficiency condition is also valid. The existing largest *n*-coloured graph base $G_m^n$ requires minimum *n* colours, and also all graphs $g_m^p(G_m^n)$ are *n*-coloured.

**EXAMPLES**

**0-D Space.** There exists only a 1-coloured graph consisting of a vertex. Hence, in 0-D space, all graphs can be coloured using only one colour.

**1-D Simply-Connected Space.** In this space, there exists the largest 2-complete graph consisting of two vertices connected by one edge. Since this largest 2-complete graph divides the total 1-D finite space into the two regions: one is the internal domain bounded by the two vertices and another is the external domain including two non-connected parts, as shown in Figure 2 a). Therefore, any new vertex in each region cannot be connected to the two vertices without any edges crossing each other, so that there exist no larger *m*-coloured graphs ($m > 2$) in the finite 1-D space. Any graphs in this space can be coloured using no more than 2 colours.

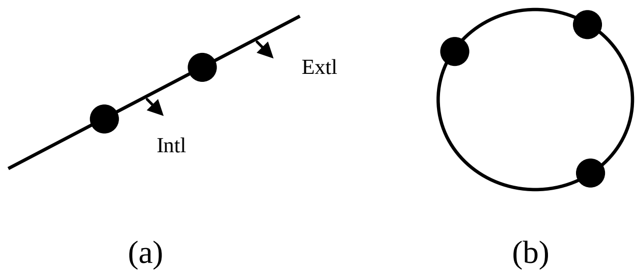
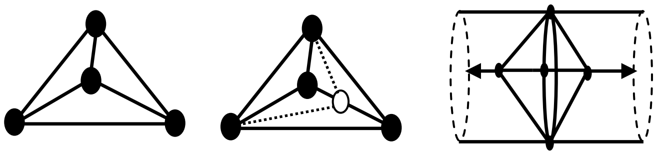



Figure 2 a) A largest 2-complete graph divides a 1-D simply-connected space; b) A largest 3-complete graph in a 1-D multiply-connected space (a circle).

**1-D Multiply-Connected Space**. Assuming the two ends of the line shown by Figure 2 a) are connected at a point, we construct a 1-D multiply-connected space shown by Figure 2 b). In this space, a largest 3-complete graph consisting of three vertices of which each connected by two edges divides the total circle into three regions of which each is bounded by two vertices. We cannot add any new vertex to be connected to all the three vertices with no edges crossing each other on this circle. This demonstrates that no largest *m*-complete graphs ($m > 3$) exist in this space. Therefore, all possible graphs in a 1-D multiply-connected space can be coloured with no more than 3 colours.

**2-D Simply-Connected Space.** If the 1-D space in Figure 2 b) is extended to a 2-D space, a fourth vertex can be added inside the circle to construct a largest 4-complete graph as shown in Figure 3 a).



(a) (b) (c)

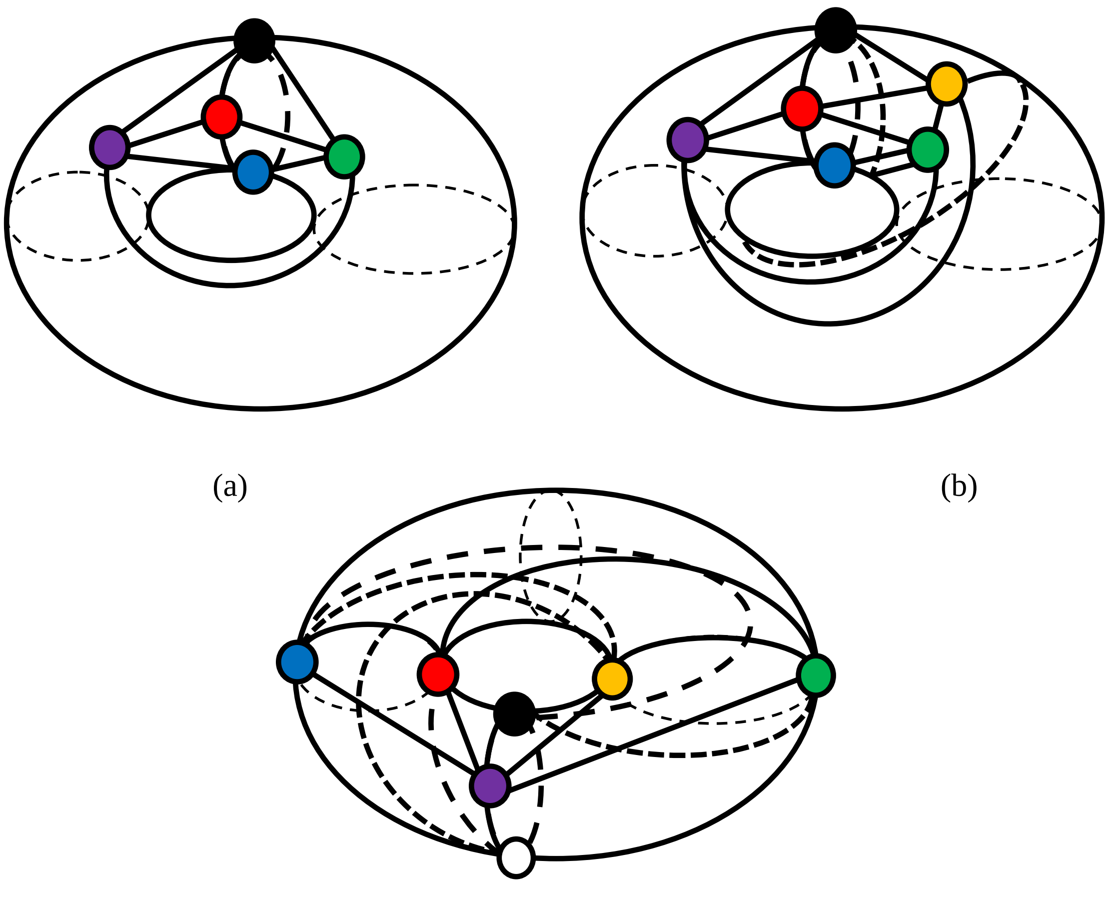
Figure 3 a) A largest 4-complete graph in a simply-connected 2-D space; b) a new vertex on one of edges is connected to the four vertices using only two new edges; c) A surface of finite long cylinder.

This largest 4-complete graph divides the total 2-D plane into four triangles including an external one extending to infinity. A new vertex on one of the six edges in Figure 3 a) can be connected to the four vertices in Figure 3 a) but using only two new edges shown by the dashed lines in Figure 3 b), therefore the resultant graph is not a 5-complete graph. Any other new vertices in this space must be closed in one of the four triangles in Figure 3 a). Because any point in a closed triangle cannot be connected to a point outside the same triangle using a line without crossing with one of its three edges, it is not possible to generate a 5-complete graph in this space by adding ***4 new edges*** without any edges crossing each other. Therefore, there exist no complete *m*-coloured graphs ($m > 4$) in this space. Based on the theorem given in this

**Copyright** © By Jing Tang Xing, Univ. of Southampton, September 18, 2013. 

paper, all the graphs in a 2-D simply-connected space can be coloured using no more than 4 colours, as described by the four colour theorem.

**2-D Multiply-Connected Space**. Figure 3 c) shows a surface of a finite long cylinder on which it is possible to draw a circle that cannot be contracted to a point on the surface, so that it is a 2-D multiply-connected space. On the surface of the cylinder, it is not possible to connect the two side vertices to construct a 5-complete graph, because the circle on the cylinder surface divides the surface into the two parts and any line to connect these two side vertices must cross with the circle. This cylinder surface is still a 4 colourable space.



(a) (b) (c)

Figure 4 A multiply-connected 2-D space (a torus): a) a 5-complete graph; b) a 6-complete graph; c) a largest 7-complete graph. The thin dashed lines are not the edges but to show the 3-D torus shape.

Now if the two ends of the cylinder are allowed to be connected to generate a torus as shown in Figure 4 a), a 5-complete graph can be produced. Furthermore, a new vertex can be added in Figure 4 a) to generate a 6-complete graph as shown in Figure 4 b) which is a base graph to generate a largest 7-complete graph shown in Figure 4 c) by introducing an extra vertex and the related new edges. There exist no *m*-complete graphs ($m > 7$) in this space, therefore all the graphs in a torus can be coloured using no more than 7 colours [2.3].

**A Discussion for 3-D Case:** Figure 5 a) shows a tetrahedron which is a 4-complete graph based on which, a 6-complete graph as shown in Figure5 b) can be generated in 3-D by adding the two new vertices and the required new edges using the lemma I. Since the colour condition does not restrict a new edge having an **intersection with the face constructed by three vertices in 3-D space**, it is not difficult to generate a 7-complete graph in 3-D space. Figure 6 a) shows a practical 3-D structure consisting of N domains of which each connects to all the others. This structure corresponds to a graph shown in Figure 6 b) in which all the vertices are located on a vertical line. The edges for one vertex to connect to the other vertices are located in a half plane bounded by the vertical line and determined by a position angle $\theta$ about the vertical line. There exist infinite numbers of this type of half planes which intersect only along the vertical line. The edges located on one of the half plane cannot cross with the edges located on all the other half planes except at the vertex points. The graph shown in Figure 6 b) is an *N*-complete graph with any possible *N* values. Therefore, in the 3-D space, the largest *n*-complete graphs is infinite large.

For 3-D case, it might be interested that any edges do not allow having intersects on any surfaces constructed by three vertices except at a vertex point. Following this new rule, it is not difficult to find that a graph with more than 6 vertices in Fig. 6 b) cannot be constructed, since the 7th vertex must intersects with a surface defined by three vertices in Fig. 5 b). Under this new colour condition, it would be interested to ask if the theorem in this paper is also valid for 3-D cases. This problem may be further investigated by interested readers.

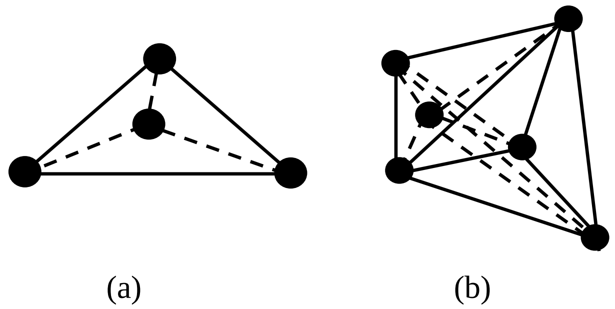



Figure 5 a) A 4-complete graph; b) a 6-coloured graph in 3-D space.

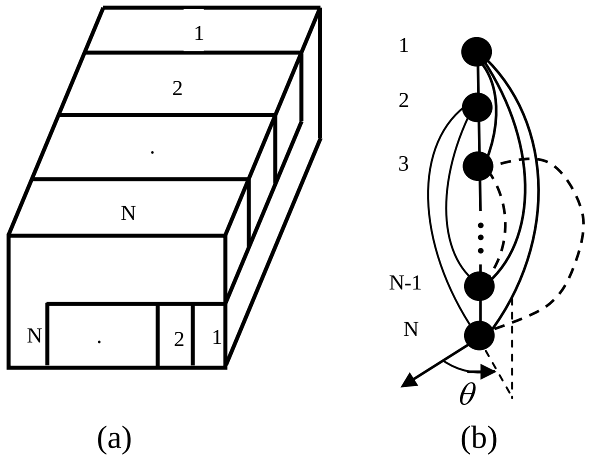



Figure 6 a) A practical structure consisting of N domains in the 3-D space; b) the corresponding graph of the structure shown in a).

**CONCLUSION**

This paper gives a simple hand-checkable demonstration of the four colour theorem. The demonstration and the theorem cover not only the four colour theorem but also more wider cases. The sufficient and necessary condition, that graphs in *m*-D space ($m \leq 2$) can be coloured using no more than *n* colours, is that there exists a largest *n*-complete graph base in the same space. The paper gives the lemma to find the largest *n*-complete graph in the space.

**REFERENCES**

[1] N. L. Biggs, E. K. Lloyd & R. J. Wilson, *Graph Theory, 1736-1936,* Oxford University Press, 1976.

[2] R. Wilson, 100 years of graph theory. In: *Jubilee Conference on Discrete Mathematics*, 11-13 January (2009), Banasthali University, India.

[3] Shouchun Xu, Four-color problem explained by figures, (in Chinese), Peking University Press, 2008.

[4] K. Appel and W. Haken, Every planar map is four colorable. Part I. Discharging, Illinois J. Math. 21 (1977), 429-490.

[5] K. Appel, W. Haken and J. Koch, Every planar map is four colorable. Part II. Reducibility, Illinois J. Math. 21 (1977), 491--567.

[6] K. Appel and W. Haken, Every planar map is four colorable, Contemporary Math. 98 (1989). MR 91m:05079.

[7] R, Thomas, An update on the four-color theorem, Notice of the AMS, 45(7) (1998), 848-859.

[8] N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The four colour theorem, J. Combin. Theory Ser. B. 70 (1997), 2-44.

[9] N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, A new proof of the four colour theorem, Electron. Res. Announc. Amer. Math. Soc. 2 (1996), 17-25 (electronic).

**Copyright** © By Jing Tang Xing, Univ. of Southampton, September 18, 2013. 