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## **A novel deconvolution beamforming algorithm for virtual phased arrays**

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### **ABSTRACT**

Beamforming techniques using phased microphone arrays are one of the most common tools for localizing and quantifying noise sources. However, the use of such devices can result in a series of well-known disadvantages regarding, for instance, their very high cost or transducer mismatch. Virtual Phased Arrays (VPAs) have been proposed as an alternative solution to prevent these difficulties provided the sound field is time stationary. Several frequency domain beamforming techniques can be adapted to only use the relative phase between a fixed and a moving transducer. Therefore the results traditionally obtained using large arrays can be emulated by applying beamforming algorithms to data acquired from only two sensors. This paper presents a novel beamforming algorithm which uses a deconvolution approach to strongly reduce the presence of side lobes. A series of synthetic noise sources with negative source strength are introduced in order to maximize the dynamic range of the beamforming deconvolved map. This iterative sidelobe cleaner algorithm (ISCA) does not require the use of the covariance matrix of the array, hence it can also be applied to a VPA. The performance of ISCA is compared throughout several simulations with conventional deconvolution algorithms such as DAMAS and NNLS. The results support the robustness and accuracy of the proposed approach, providing clear localization maps in all the conditions evaluated.

Keywords: Beamforming, Virtual phased array, deconvolution algorithm.

### **1. INTRODUCTION**

There are many applications which require the utilization of microphone arrays in order to localize sound sources across a space. Traditionally, this implies the use of expensive acquisition systems. Furthermore, the resolution of the measurements would depend upon the number of transducers used and their respective positions (the geometry of the array). If the array consists of too many sensors, it becomes acoustically significant, biasing the characterization of the sound field.

A “Virtual Phased Array” (VPA) approach can be taken so as to avoid most constraints of conventional beamforming devices, assuming the sound field is time stationary. A single moving sensor is utilized to continuously acquire data across the space whilst a static reference microphone also records the event. The

acoustic signal is later split into blocks which have associated different spatial positions. Each block, or segment of the recorded signal, represents an element of the virtual phased array. The phase estimation is computed relative to the fixed reference sensor. This measurement method can potentially address many common problems due to its low cost and simple acquisition process. The use of a single moving sensor avoids array calibration issues and any other limitations derived from using a fixed array geometry.

The idea of creating synthetic or virtual arrays using a limited number of sensors had also been explored in other disciplines, most being focused upon enhancing the possibilities of conventional radars (Synthetic Aperture Radar or SAR),<sup>1,2</sup> and sonar systems (Synthetic Aperture Sonar or SAS).<sup>3,4</sup> The majority of methods developed for SAS and SAR share common ground which differs from VPA; they are active systems based on the coherent addition over many impulses across the space,<sup>5</sup> whereas the virtual array approach is based upon passive synchronization via a fixed reference sensor.

In previous works, the performance of a VPAs have been tested at low frequencies for simple outdoor scenarios.<sup>6</sup> Furthermore, VPAs have also been shown to work remarkably well in laboratory conditions to assess mid-high frequency problems.<sup>7</sup> Nonetheless, despite the accuracy and efficiency achieved with conventional beamforming algorithms, it has also been found that results presented so far have a highly constrained dynamic range. There is a large volume of studies proposing techniques and methods to overcome this problem via deconvolution algorithms, such as DAMAS or NNLS. This article explores the use of several deconvolution algorithms adapted for VPAs, but most importantly, presents the theoretical basis of a novel method to clean the beamforming map, the Iterative Sidelobe Cleaner Algorithm (ISCA). The capabilities of the proposed technique are studied in the following sections, evaluating its performance against well established deconvolution methods in different simulations scenarios.

## 2. THEORY

One common application for sensor arrays is to determine the direction of arrival (DOA) of propagating wavefronts. An array receives spatially propagating signals and processes them to estimate their direction of arrival, acting as a spatially discriminating filter.<sup>8</sup> This spatial filtering operation is known as beamforming. Conventional delay-and-sum beamforming steers a beam to a particular direction by computing a properly weighted sum of the individual sensor signals which are previously delayed by a certain amount depending on the focusing direction. As such, this procedure results in the coherent addition of signals coming from the direction of focus which maximizes the energy in the beamformer output whilst signals from other directions interfere destructively. The asynchronous time acquisition performed with a VPA implicitly constrains the range of localization techniques suitable for this technology.<sup>7</sup> Without using the cross-spectral matrix (CSM) of the data, a simple beamforming algorithm can be formulated in terms of cross-spectral densities  $S_{p_{ref}p_m}(\omega)$  between the virtual transducers ( $p_m$ ) and at the reference microphone ( $p_{ref}$ ), i.e.,

$$b(x, y) = \frac{1}{M} \sum_{m=1}^M w_m S_{p_{ref}p_m}(\omega) r_{ref} r_m e^{-jk(r_{ref}-r_m)}, \quad (1)$$

where  $M$  is the total number of transducers;  $w_m$  is a weighting factor;  $r_{ref}$  and  $r_m$  are the distances from the evaluated source point to the reference and  $m^{th}$  transducer, respectively; and  $k$  represents the wavenumber.

In spite of the robustness of this simple method, the spatial resolution, signal to noise ratio and dynamic range can be enhanced by using deconvolution algorithms. In this section, two conventional deconvolution methods are reviewed along with the introduction of a novel approach, the Iterative Sidelobe Cleaner Algorithm (ISCA).

### 2.1. Fundamentals of iterative algorithms

The beamforming output can be related to the sources present in the sound field by means of the beamformer's point-spread function (PSF). The PSF, defined as the beamformer response to a point source with unit strength at an arbitrary position of a grid, determines the characteristics of the beamformer in terms of shape of the main beam and side lobes.<sup>9,10</sup> Assuming incoherent sources, the beamforming output can also be written as

$$b(x, y) = \sum_{(x', y')} q(x', y') A(x, y | x', y'), \quad (2)$$

where  $q$  is the source distribution, and  $A(x, y | x', y')$  is the PSF at position  $(x, y)$  in the observation grid due to a point source located at position  $(x', y')$  in the same grid. This relationship makes it possible to recover the source distribution from the measured beamformer map and the beamformer's PSF, by means of

a deconvolution procedure, imposing that the distribution of sound sources must be non-negative ( $q(x', y') \geq 0$ ). This is an inverse problem, which in matrix notation can be rewritten as

$$\mathbf{b} = \mathbf{A}\mathbf{q}, \quad (3)$$

where the components of the vectors  $\mathbf{b}$  and  $\mathbf{q}$  are  $b(x, y)$  and  $q(x', y')$ . Each vector has a length  $L$ , corresponding to the total number of grid points. On the other hand  $\mathbf{A}$  is a matrix  $L \times L$  that in each column contains the PSF for one source located at a particular position  $(x', y')$  of the grid.

## 2.2. DAMAS

In 2004 by Brooks and Humphreys in Ref. 9 suggested a method to solve the inverse problem presented in Eqs. (2) and (3) to recover the position and the strength of acoustic sources. The method, called Deconvolution Approach for the Mapping of Acoustic Sources (DAMAS), is based on an iterative algorithm.

The first step of the algorithm is the initialization of the iteration index,  $n = 0$ , and the choice of an estimate of  $q$ ,  $q_i^{(0)}$ . Typically this value is set to zero for the entire region of interest,

$$q_i^{(0)} = 0, \quad i = 1, \dots, L. \quad (4)$$

Then the estimate of the source distribution is computed as

$$q_i^{(n+1)} = \frac{1}{A_{ii}} \left[ b_i - \left[ \sum_{j=1}^{i-1} A_{ij} q_j^{(n+1)} + \sum_{j=i+1}^L A_{ij} q_j^{(n)} \right] \right], \quad (5)$$

where  $A_{ij}$ ,  $b_i$  and  $q_i^{(n)}$  are the components of the matrix  $\mathbf{A}$ , and the vectors  $\mathbf{b}$  and  $\mathbf{q}^{(n)}$  respectively. Before incrementing the iteration index  $n$ , it is ensured that the recovered source distribution is non-negative. Therefore a non-negativity constraint is applied to update the value of  $q_i$ ,

$$q_i^{(n+1)} = \max \left( q_i^{(n+1)}, 0 \right). \quad (6)$$

Once the iteration index is incremented, the steps given by Eqs. (5) and (6) are repeated at each iteration until the source distribution  $q_i$  converges. Note that although the result converges to a certain value, this method does not guarantee that this corresponds to the exact solution.<sup>11</sup>

It is common to make use of the residual to analyze the convergence. This is defined as

$$r_i^{(n+1)} = b_i - \left[ \sum_{j=1}^{i-1} A_{ij} q_j^{(n+1)} + \sum_{j=i+1}^L A_{ij} q_j^{(n)} \right]. \quad (7)$$

If  $q_i^{(n)}$  converges to the exact solution then the residual  $r_i^{(n)}$  tends to zero with increasing the number of iterations  $n$ .

This procedure allows not only to localize sound sources, but also to determine their strength. This is done by summing the mean-squared values of the recovered sound distribution over the region of interest.

## 2.3. Non-Negative Least Squares (NNLS)

Non-Negative Least Squares is a well established deconvolution technique, originally proposed by Lawson and Hanson in 1974.<sup>12</sup> The method solves the deconvolution problem in Eq. (3) algebraically, by minimizing the square sum of the residuals, subject to the condition that there are no sources of negative strength ( $\mathbf{q} \geq 0$ ), which makes it robust. The method is well defined in the sense that if a solution exists, convergence is guaranteed.

For this study a gradient type version is used, where no matrix factorizations are required, as described in Ref. 11. The NNLS algorithm aims at minimizing the squared sum

$$\|\mathbf{A}\mathbf{q} - \mathbf{b}\|_2. \quad (8)$$

A brief overview of the steps of the gradient type implementation is summarized here. The reader is referred to Ref. 11 for a more detailed description of the steps. First, a residual vector is estimated from a given solution  $\mathbf{q}^{(n)}$ , based upon which the gradient of the residual vector  $\mathbf{w}^{(n)}$  is calculated, and the projected

gradient  $\tilde{\mathbf{w}}^{(n)}$  is used to define a path through the solution space given by  $\mathbf{q}^{(n)}$ . Then the auxiliary vector is obtained as

$$\mathbf{s}^{(n)} = \mathbf{q}^{(n)} + \lambda \tilde{\mathbf{w}}^{(n)}, \quad (9)$$

where  $\lambda$  is the optimal step factor, to find the optimal location along the search path that yields the minimum value of Eq. (8), the squared sum of the residuals. Given this minimum value, the iteration step is finalized by applying the non-negative constraint

$$q_i^{(n+1)} = \max(s_i^{(n)}, 0), \quad (10)$$

that yields the values of the solution vector used in the subsequent iterations. A detailed description of the method can be found in Refs. 11 and 12.

## 2.4. Iterative Sidelobe Cleaner Algorithm (ISCA)

The combination of a robust localization algorithm with an iterative approach to enhance the dynamic range of the beamformer output has lead to develop the Iterative Sidelobe Cleaner Algorithm (ISCA). This deconvolution method is based upon the reconstruction of the measurement data from a set of synthetic sound sources via an adaptive approach.

Similarly to other deconvolution algorithms, the first step is the initialization of the iteration index,  $n = 0$ , and the choice of an estimate of  $q$ ,  $\mathbf{q}^{(0)}$ . Typically this value is set to zero for the entire region of interest,

$$\mathbf{q}^{(0)} = 0. \quad (11)$$

In addition, an estimate of the cross-spectra perceived by the virtual array is initialized with the measurement data

$$\hat{S}_{p_{ref}p_m}^{(0)} = S_{p_{ref}p_m}. \quad (12)$$

In a first stage, a conventional delay-and-sum beamformer is applied to the acquired data following Eq. (1). Next, the maximum of the beamforming output is used to extract information about the main excitation point

$$\psi(x, y) = \max(\mathbf{b}). \quad (13)$$

It is then introduced a negative synthetic source with a fraction of the power found at the maximum of the beamformer output

$$q^{(n+1)}(x, y) = q^{(n)}(x, y) - \beta \psi(x, y), \quad (14)$$

where  $\beta$  is an adaptive parameter which control the amplitude assigned to the synthetic sources in each iteration. This factor is defined by the gradient of the error, i.e.,

$$\beta = \begin{cases} \epsilon_0 & n = 0 \\ e_{RMS}^{(n-1)} - e_{RMS}^{(n)} & n \geq 1 \end{cases} \quad (15)$$

where  $\epsilon_0 \in (0, 1)$  is a constant value used when there is not enough data to calculate the gradient and  $e_{RMS}$  represents the squared 2-norm between the original beamforming map and the synthesized solution, thus

$$e_{RMS} = \|\mathbf{A}\mathbf{q} - \mathbf{b}\|_2. \quad (16)$$

As the error between the the original and synthetic beamforming output decreases, also the step parameter  $\beta$  is proportionally reduced. Once the clean source map  $\mathbf{q}^{(n)}$  has been updated, then we can propagate the data from the source location to the array plane, hence

$$\xi_m = \frac{-\beta \psi(x, y)}{r_{ref} r_m} e^{jk(r_{ref} - r_m)}. \quad (17)$$

Since the excitation sources have been assumed to be uncorrelated, it is then possible to directly add the original measured data with the synthetic sound field created

$$\hat{S}_{p_{ref}p_m}^{(n+1)} = \hat{S}_{p_{ref}p_m}^{(n)} + \xi_m^{(n)}. \quad (18)$$

The application of a delay-and-sum beamforming algorithm to the superposition of both original and synthetic signals leads to a reduction of the energy maximum previously found. This process has a direct

impact on the beamforming dynamic range, since the energy of the real source is directly decreased as well as the residual sidelobes. Ghost sources and any masking effect produced by the presence of strong sidelobes will be progressively reduced. The processing limits of ISCA are determined by a constraint imposed in terms of dynamic range. The algorithm will iteratively remove energy within a dynamic range established by the user.

Although the method was designed for enhancing source localization maps obtained with a VPA, the approach presented in this section could be generalized for multichannel phased array systems. In that case, the cross-spectral matrix (CSM) would be the term updated in each iteration instead of the cross-spectral estimates. Further research on this topic will be undertaken in future work.

### 3. SIMULATIONS

The validation and characterization of the proposed method has been studied by evaluating several simulated scenarios. Results have also been compared with alternative methods in order to understand the capabilities of ISCA with respect to current deconvolution solutions.

Let us begin by defining the geometry of the simulation scenario. A square array of 441 regularly distributed elements has been located across an area of 2 by 2 meters. Even though the total number of transducers could seem excessively large, this choice was based upon the amount of virtual transducers which is normally achieved using a VPA.<sup>6,7</sup> A set of six uncorrelated point sources radiating at a frequency of 1 kHz were located 30 meters away from the plane. The individual positions are stated in Table 1.

	$Q_1$	$Q_2$	$Q_3$	$Q_4$	$Q_5$	$Q_6$
Azimuth ( $^\circ$ )	30	10	0	-10	-20	15
Elevation ( $^\circ$ )	-20	10	0	10	0	0

Table 1 – Location of the sound sources used in the simulation environment.

An area of 40 by 30 meters of potential source positions was evaluated using a regular grid of 1225 points. Figure 1 presents a sketch of the simulated scenario.

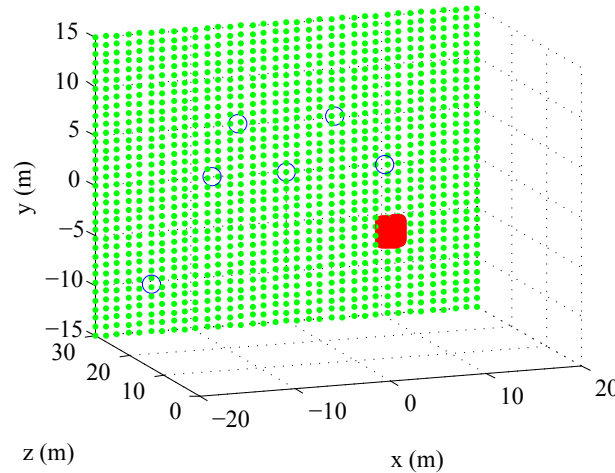


Fig. 1 – Geometry of the simulated scenario. Evaluated source positions are plotted in green; array sensor positions are represented with red dots; the location of the noise sources are shown with blue circular markers.

All results presented along this section with DAMAS and NNLS were computed after a 10000 iterations process. The localization maps obtained with ISCA were computed till the maximum level of the beamforming output was reduced by 10 dB, which corresponds to 138 iterations for the first experiment; 90 for the second and 109 for the last test.

#### 3.1. Convergence of ISCA

One of the key features of iterative algorithms is the the ratio of convergence towards an optimal solution. It is therefore required to study the behavior of the method when is used under different sound field

characteristics. A first order gradient algorithm (also known as steepest descent) has been implemented in order to control the effort applied in each iteration. The cost function used to extract the gradient towards the optimal solution was based upon the difference between the initial data and the result of applying sound propagation models to a clean map of synthetic sources. The root mean squared value (RMS) is reduced as the clean map improves in each step, hence reducing the synthetic source strength added every iteration. The left hand side of Figure 2 shows how the cost function, the error between the “clean” map and the original output, is iteratively reduced under several source conditions. The power of multiple point sources was varied between 0, 3, 6 and 9 dB range. As it can be seen, the highest reductions are achieved during first steps of the process. Moreover, the number of iterations required to reduce the energy of a beamforming map by 10 dB depends directly upon the nature of the sound field. In the presence of sources with equal source strength (0 dB variance between source strengths) the algorithm needs long time to converge. In contrast, when the source strength between them differs within a range of 9 dB, it is possible to minimize the energy more rapidly. Therefore, it can be inferred that the convergence of the algorithm depends upon the strength range of the excitation sources.

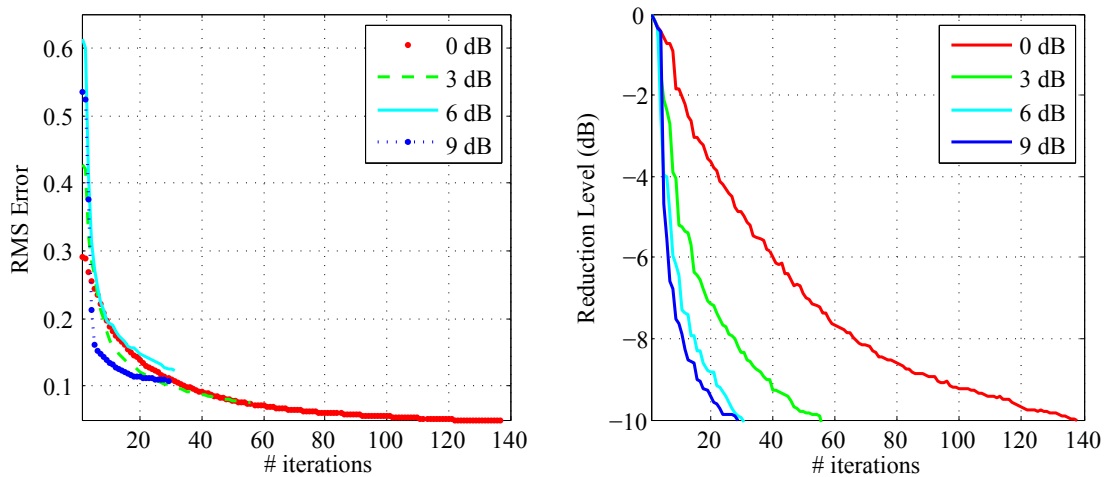


Fig. 2 – Convergence of the method for different dynamic ranges of source strengths.

### 3.2. Localization of uncorrelated sources with equal source strength

A first simulation case has been used to test the different deconvolution algorithms. The six sources described at the beginning of this section have been set with an equal source strength level of 90 dB. Measurement noise has been avoided in this first simulation and, consequently, even the source grid and real source location have been matched to minimize the localization error. Figure 3 presents a comparison of conventional beamforming CBF (top left), DAMAS (top right), NNLS (bottom left), and the proposed algorithm ISCA (bottom right).

As expected, the three iterative deconvolution methods are able to achieve a spatial resolution far higher than the conventional delay-and-sum beamforming. It is apparent from Figure 3 that all methods resolve most of the source positions accurately. Only DAMAS presents a significant drift in the localization of source 1. This algorithm also presents several weak ghost sources distributed across the output map. Both ISCA and NNLS lead to similar localization maps, although the accuracy of ISCA seems to be slightly better in terms of spatial resolution and energy spread.

### 3.3. Localization of uncorrelated excitations with multiple source strengths

A second simulation scenario has been studied for sources that have different excitation levels. Consequently, the source strengths have been distributed along a 6 dB dynamic range. The aim of this test is to evaluate the ability to detect sound sources which were masked in the conventional beamforming map by the sidelobes of the most powerful excitations. Table 2 provides the strengths assigned.

Similarly to the previous section, measurement noise has been disregarded. Figure 4 illustrates a comparison of conventional beamforming CBF, DAMAS, NNLS, and ISCA for multiple source excitations with different source strengths. The representation range has been expanded with respect to previous results since the dynamic range of the excitation sources is larger in this experiment.

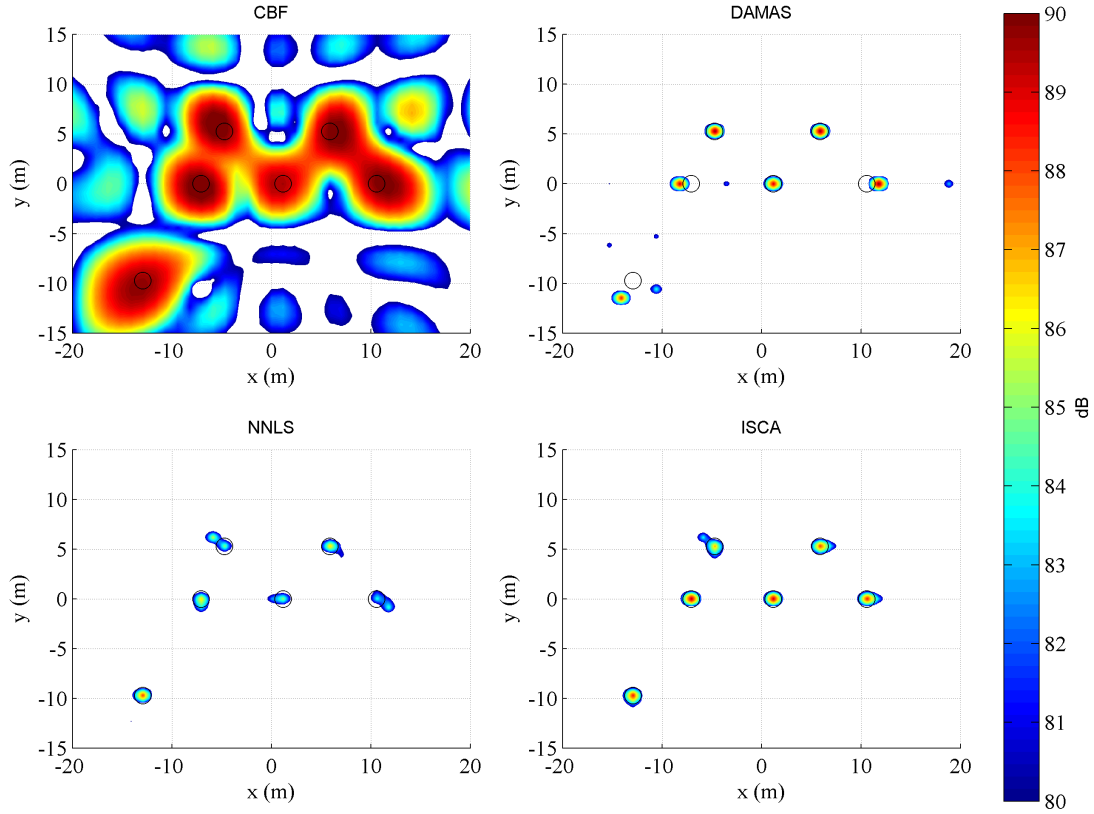


Fig. 3 – Comparison of several algorithms to localize multiple uncorrelated sources of equal source strength. The actual position of the sources is indicated with circles.

	$Q_1$	$Q_2$	$Q_3$	$Q_4$	$Q_5$	$Q_6$
Source Strength (dB)	85	86	87	88	89	90

Table 2 – Source strengths of the different excitation sources used in the simulation scenario.

It can be seen that DAMAS presents some difficulties when the dynamic range is expanded. The number and power of the ghost sources is increased respect to the previous test. In contrast, the performance of NNLS and ISCA are very similar to the results previously obtained. However, in this second experiment NNLS present some drift in the location estimation of two of the sources. On the other hand, ISCA is able to produce a clear localization map without the presence of ghost sources, although the energy seems to be slightly more spread around the true source location.

### 3.4. Localization of uncorrelated sources with multiple source strength including noise

The first two experiments previously presented evaluate the performance of all the algorithms in simple conditions. Nonetheless, this is not representative of a real case scenario. It is therefore required to include an additional step in the simulations which allow us to study the robustness of the beamforming algorithms when the data deviate from ideal conditions. For this matter, the location of the sources has been modified to avoid a matching grid between beamforming observation points and real source locations. In addition, random complex error has been added to the sound pressure perceived by each virtual array transducer. The variance of the error has been limited up to 50 percent of the original received sound pressure (signal to noise ratio of 6 dB). Figure 5 shows the assessed array output before and after adding noise.

Experimental errors derived from spectral estimation, lack of excitation stationarity or the presence of background noise can then be modeled using this procedure. Figure 6 illustrates the results of CBF, DAMAS, NNLS, and ISCA when multiple excitations sources are assessed in presence of measurement noise.

As can be seen in Figure 6, despite the severe noise added to the array signal, the deconvolution algorithms are able to provide similar solutions to the ones found in previous experiments. The appearance of

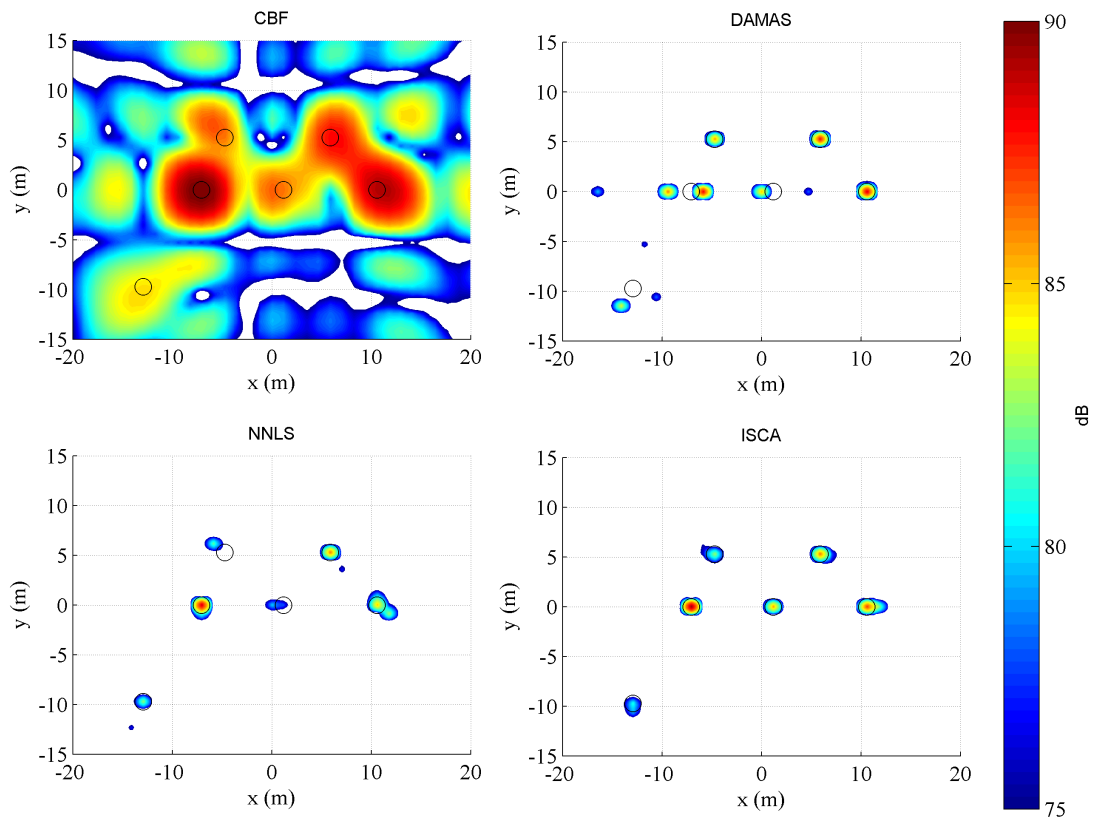


Fig. 4 – Comparison of several algorithms to localize multiple uncorrelated sources which power is distributed along a 6 dB dynamic range. The actual position of the sources is indicated with circles.

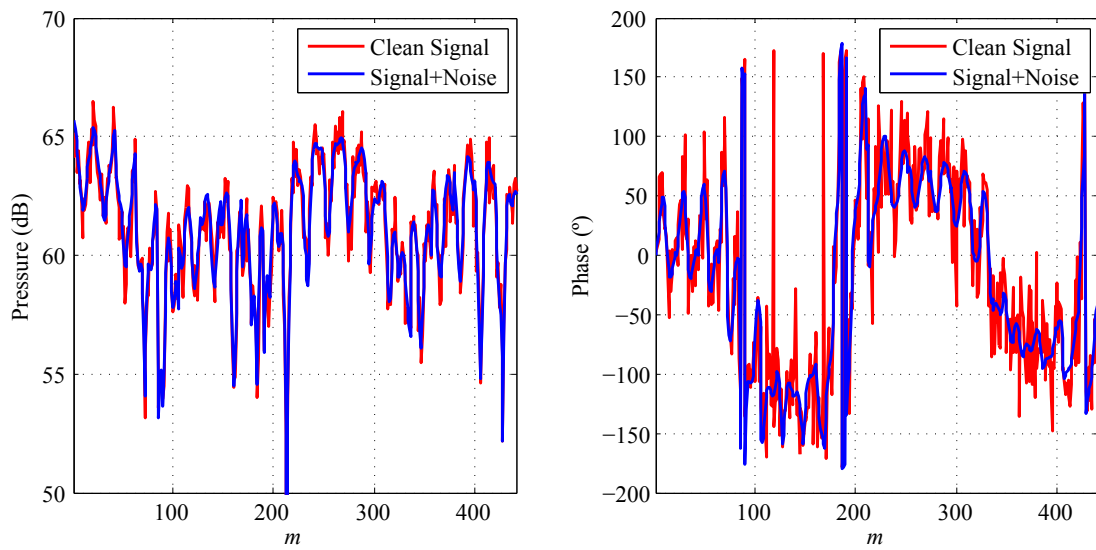


Fig. 5 – Signal received by the virtual array in terms of magnitude (left) and phase (right).



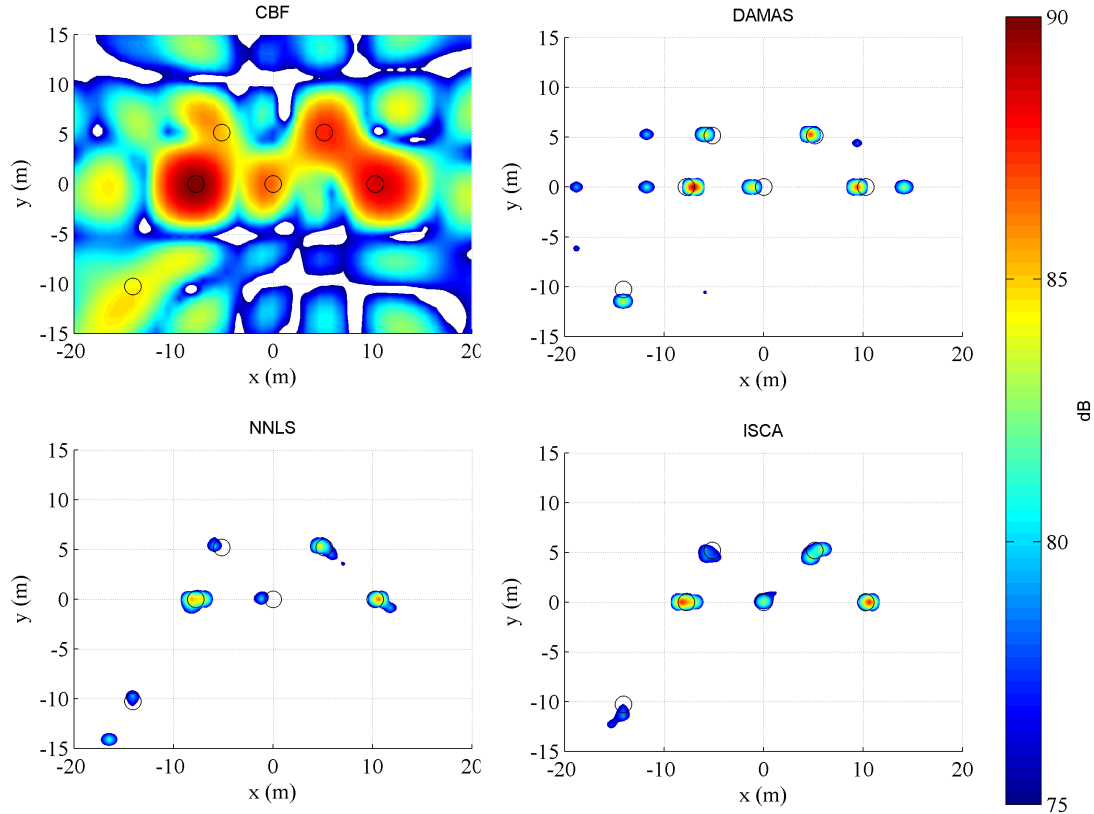


Fig. 6 – Comparison of several algorithms to localize multiple uncorrelated sources including random spectral estimation error.

additional ghost sources is noticeable in DAMAS and NNLS but not in ISCA. On the other hand, the use of a non-matching grid together with the inclusion of random noise seems to cause a slight drift in the estimated source locations which affects the three algorithms.

The robustness of ISCA can be attributed to the fact that only the range of data with the best signal to noise ratio is used for computing the clean source map. In the simulation presented above, the original energy of the beamforming output was iteratively reduced by 10 dB, but mainly acting over the excitation maxima. In contrast, NNLS and DAMAS tend to converge to a numerically optimal solution using the complete dataset available. Therefore, the accuracy of these methods is slightly more biased by the presence of noise than the proposed algorithm.

#### 4. CONCLUSIONS

Several deconvolution methods have been adapted and tested for VPAs for source localization purposes. Spatial resolution, dynamic range and accuracy improvements are achieved by applying deconvolution techniques to a conventional delay-and-sum beamforming output. A novel iterative sidelobe cancellation algorithm (ISCA) has been introduced and validated against conventional deconvolution methods such as DAMAS and NNLS. Noise localization experiments of multiple uncorrelated sources have been undertaken under different excitations conditions, with and without including noise in the virtual array data acquired. It has been shown that the performance of the novel method proposed exceeds conventional iterative deconvolution algorithms due to the nature of the investigation process: ISCA mainly interacts with the data at the points where the energy is maximized along a limited dynamic range. Contrarily, NNLS and DAMAS seek for the solution of the system algebraically, converging towards an answer which best fits the complete dataset. This argument is in agreement with the simulation results, providing clear evidence of the robustness and accuracy of ISCA, even in the presence of severe measurement noise. Further research needs to be undertaken in order to generalize ISCA for multichannel phased arrays and be able to assess partially coherent sound sources.

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