On positive-realness and stability of switched linear differential systems

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***Abstract*— We present some results regarding the stability of switched linear differential systems (SLDS) in the behavioral framework. Positive-realness is studied as a sufficient condition for stability and some implications derived from the use of positive-real completions are discussed.**

***Index Terms*— switched systems; behaviors; quadratic differential forms; positive-realness.**

I. Introduction

A switched system is a set of dynamical systems with a rule that orchestrates the switching among them [2]. They are usually studied in the state space framework: all the dynamical regimes share the same state space, i.e. in the linear case each system is described by $\frac{d}{dt}x = Ax + Bu$; or in descriptor form $E\dot{x} = Ax + Bu$, where E is a singular matrix, [11]. In [6],[5], a new approach has been put forward in which the dynamical regimes do not necessarily share the same state space, and they are described by sets of higher-order differential equations. We call these *switched linear differential systems* (SLDS).

Switching between stable systems may give rise to unstable responses (see [2], pp.19-20); consequently, it is important to find conditions that guarantee asymptotic stability (see e.g. [2],[3],[8]). In the state space setting, the notion of positive realness has been employed for the analysis and derivation of sufficient conditions of stability for switched linear systems (see e.g. [7],[14]). In the linear differential systems case, some results have been presented in [6],[5] using positive-realness as a sufficient condition for stability. In this contribution we present several new results using the the concept of *positive-real completion*.

II. Background

A. Notation

The space of real vectors with $\mathtt{n}$ components is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{n}\times\mathtt{m}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$ is the space of $\mathtt{n}\times\mathtt{m}$ polynomial matrices in ξ, and the space of $\mathtt{n}\times\mathtt{m}$ polynomial matrices in ζ and η is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$. A polynomial $p\in\mathbb{R}[\xi]$ is *Hurwitz* if its roots are all in the open left half-plane.

We now introduce the concept of *R-canonical representative* of a polynomial differential operator. Given $R\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ nonsingular, and $f\in\mathbb{R}^{1\times\mathtt{w}}[\xi]$; f can be uniquely written as $fR^{-1} = s+n$, where s is a vector of strictly proper rational functions, and $n\in\mathbb{R}^{1\times\mathtt{w}}[\xi]$. We define the (polynomial) R-canonical representative of f as $(f \mod R)(\xi) := s(\xi)R(\xi)$. The definition of R-canonical representative is extended in a natural way to polynomial matrices.

The set of infinitely-differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Given $f:\mathbb{R}\rightarrow\mathbb{R}$, we define $f(t^-) := \lim_{\tau\nearrow t} f(\tau)$ and $f(t^+) := \lim_{\tau\searrow t} f(\tau)$, provided that these limits exist.

B. Linear differential behaviors

We call $\mathfrak{B}\subseteq\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ a *linear time-invariant differential behavior* if $\mathfrak{B}$ is the set of solutions of a finite system of constant-coefficient linear differential equations, i.e. if there exists a polynomial matrix $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} = \{w\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\} =: \ker\ R(\frac{d}{dt})$. If $\mathfrak{B} = \ker\ R(\frac{d}{dt})$, then we call R a *kernel representation* of $\mathfrak{B}$. We denote with $\mathfrak{L}^{\mathtt{w}}$ the set of all linear time-invariant differential behaviors with $\mathtt{w}$ variables.

Autonomous behaviors are defined as follows (see Ch. 3 of [4]).

Definition 1: $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$ is *autonomous* if for all $w_1, w_2\in\mathfrak{B}$, $\{w_1(t) = w_2(t) \text{ for } t<0\} \implies \{w_1 = w_2\}$.

It can be shown that if $\mathfrak{B}$ is autonomous, it admits a kernel representation with R square and nonsingular. Moreover, it is finite-dimensional as a subspace of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, and its dimension equals $\deg(\det(R))$.

In this paper we use the notion of positive-realness [1].

Definition 2: A square matrix $B(\lambda)$ of rational functions is said to be *positive-real* if: all its entries are analytic in $\mathrm{Re}(\lambda)>0$; $B(\lambda)$ is real if λ is real; and $B(-\lambda)^\top + B(\lambda)\geq 0$ for all $\mathrm{Re}(\lambda)\geq 0$.

The third condition of Definition 2 implies that

$$B(-j\omega)^\top + B(j\omega)\geq 0 \ \forall\,\omega\in\mathbb{R}\ . \tag{1}$$

If the inequality is strict, we call B *strictly positive-real*.[1]

C. Quadratic differential forms

Let $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be a two-variable polynomial matrix. Without loss of generality we assume that $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$, i.e. that $\Phi(\zeta,\eta)$ is *symmetric*. We say that $\Phi(\zeta,\eta)$ has order L if it can be written as $\Phi(\zeta,\eta) = \sum_{k,\ell=0}^{L}\Phi_{k,\ell}\zeta^k\eta^\ell$, where $\Phi_{k,L} = \Phi_{L,k}$ is a nonzero matrix for some $k\in\mathbb{N}$. The *quadratic differential form* (QDF) Q_Φ

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[1]The definition of strictly positive real functions is not uniform in the literature; we refer to [10], Th. 2.1.

associated with $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is defined by

$$\begin{aligned} Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) &\longrightarrow \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \\ w &\mapsto Q_\Phi(w) = \sum_{k,\ell} (\frac{d^k}{dt^k}w)^\top \Phi_{k,\ell}(\frac{d^\ell}{dt^\ell}w)\ . \end{aligned}$$

We define the order of the quadratic differential form Q_Φ as the order of $\Phi(\zeta,\eta)$. Note that $\Phi(\zeta,\eta) = S_L^{\mathtt{w}}(\zeta)^\top \widetilde{\Phi}\, S_L^{\mathtt{w}}(\eta)$, where L is the order of $\Phi(\zeta,\eta)$, $S_L^{\mathtt{w}}(\xi)^\top := \begin{bmatrix} I_{\mathtt{w}} & \zeta I_{\mathtt{w}} & \cdots \xi^L I_{\mathtt{w}}\end{bmatrix}$, and $\widetilde{\Phi} \in \mathbb{R}^{L\mathtt{w}\times L\mathtt{w}}$ is the *coefficient matrix* of Φ.

We say that a QDF Q_Φ is *nonnegative along* $\mathfrak{B}$, denoted $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$, if $(Q_\Phi(w))(t) \geq 0$ for all $w \in \mathfrak{B}$ and $t \in \mathbb{R}$. If a QDF Q_Φ is nonnegative for every trajectory in $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ we write $Q_\Phi \geq 0$ and say that Q_Φ is *nonnegative definite*. Note that Φ is nonnegative definite if and only if $\widetilde{\Phi} \geq 0$. We say that Q_Φ is *positive* along $\mathfrak{B}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{>} 0$, if $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $Q_\Phi(w) = 0$, $w \in \mathfrak{B}$, implies that $w = 0$. A QDF is *positive definite* if it is positive along $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$; this happens if and only if $\widetilde{\Phi} > 0$. We define $Q_\Phi \overset{\mathfrak{B}}{<} 0$, etc. in an analogous manner.

The derivative $\frac{d}{dt}Q_\Phi =: Q_{\overset{\bullet}{\Phi}}$ of a QDF Q_Φ is also a QDF, and the associated two-variable polynomial matrix is $\overset{\bullet}{\Phi}(\zeta,\eta) := (\zeta+\eta)\Phi(\zeta,\eta)$ (see [12], section 3).

A *Lyapunov function* for a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is defined as a quadratic differential form Q_Φ whose values $Q_\Phi(w)$ are nonnegative and decrease with the time for all $w \in \mathfrak{B}$, i.e. $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $\frac{d}{dt}Q_\Phi \overset{\mathfrak{B}}{<} 0$.

The concept of R-canonical representative is employed for two-variable polynomial matrices. Let $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be nonsingular and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Factorize $\Phi(\zeta,\eta) = M(\zeta)^\top N(\eta)$ and compute the R-canonical representatives $M' = M \mod R$; and $N' = N \mod R$. Then the R-canonical representative of $\Phi(\zeta,\eta)$ is defined as $\Phi(\zeta,\eta) \mod R := M'(\zeta)^\top N'(\eta)$. In this sense, the QDFs Q_Φ, $Q_{\Phi'}$ are *equivalent along* $\ker\ R\left(\frac{d}{dt}\right)$, which means that $Q_{\Phi'}(w) = Q_\Phi(w)$ for all $w \in \ker\ R\left(\frac{d}{dt}\right)$.

III. Switched linear differential systems

We recall the basic definitions of [6], [5].

Definition 3: A *switched linear differential system* (SLDS) Σ is a quadruple $\Sigma = \{\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G}\}$ where: $\mathcal{P} = \{1,\dots,N\} \subset \mathbb{N}$ is the *set of indices*; $\mathcal{F} = (\mathfrak{B}_1,\dots,\mathfrak{B}_N)$, with $\mathfrak{B}_j \in \mathfrak{L}^{\mathtt{w}}$ for $j \in \mathcal{P}$, is the *bank of behaviors*; $\mathcal{S} = \{s : \mathbb{R} \to \mathcal{P}\}$ with s piecewise constant and right-continuous, is the set of admissible *switching signals*; and $\mathcal{G} = \{(k,\ell), G^+_{k\to\ell}(\xi), G^-_{k\to\ell}(\xi)\}$, where $(G^+_{k\to\ell}(\xi), G^-_{k\to\ell}(\xi)) \in (\mathbb{R}[\xi]^{\bullet\times\mathtt{w}})^2$ and $(k,\ell) \in \mathcal{P}\times\mathcal{P}$, $k \neq \ell$, is the set of *gluing conditions*. For a given $s \in \mathcal{S}$, the set of *switching instants* with respect to s is $\mathbb{T}_s := \{t \in \mathbb{R} \mid \lim_{\tau\nearrow t} s(\tau) \neq s(t)\} = \{t_1,t_2,\dots\}$ where $t_i < t_{i+1}$.

We make the standard assumption that the switching signal is arbitrary and *well-defined*, i.e. every finite interval of $\mathbb{R}$ contains only a finite number of switching instants (see [9]).

Definition 4: Let Σ be a SLDS and $s \in \mathcal{S}$. The *s-switched behavior* $\mathfrak{B}^s$ *with respect to* Σ is the set of trajectories satisfying the following conditions: 1) for all $t_i, t_{i+1} \in \mathbb{T}_s$, there exists $k \in \mathcal{P}$ such that $w_{|[t_i,t_{i+1})} \in \mathfrak{B}_{k|[t_i,t_{i+1})}$; 2) w satisfies the gluing conditions $\mathcal{G}$ at the switching instants:

$$(G^+_{s(t_{i-1})\to s(t_i)}(\frac{d}{dt}))w(t_i^+) = (G^-_{s(t_{i-1})\to s(t_i)}(\frac{d}{dt}))w(t_i^-)\ ,$$

for each $t_i \in \mathbb{T}_s$.

The *switched behavior* $\mathfrak{B}^\Sigma$ *of* Σ is defined by $\mathfrak{B}^\Sigma := \bigcup_{s\in\mathcal{S}} \mathfrak{B}^s$.

In the rest of this paper we consider scalar ($\mathtt{w} = 1$) behaviors, and "standard" gluing conditions which are defined as follows. Let Σ be a SLDS and let $\mathfrak{B}_k := \ker\ p_k\left(\frac{d}{dt}\right)$, $\mathfrak{B}_\ell := \ker\ p_\ell\left(\frac{d}{dt}\right)$ be a pair behaviors in $\mathcal{F}$, where $(p_k,p_\ell) \in \mathbb{R}[\xi]$ and $n_k := \deg(p_k)$, $n_\ell := \deg(p_\ell)$. We define the standard gluing conditions when we switch from the behavior $\mathfrak{B}_k$ to $\mathfrak{B}_\ell$ for all $t_i \in \mathbb{T}_s$ as

$$\begin{aligned}
&\begin{bmatrix} 1 \\ \frac{d}{dt} \\ \vdots \\ \frac{d^{n_\ell-1}}{dt^{n_\ell-1}} \end{bmatrix} w(t_i^+) = \begin{bmatrix} 1 \\ \frac{d}{dt} \\ \vdots \\ \frac{d^{n_k-1}}{dt^{n_k-1}} \end{bmatrix} w(t_i^-) \ \text{ if } n_k = n_\ell; \\
&\begin{bmatrix} 1 \\ \frac{d}{dt} \\ \vdots \\ \frac{d^{n_\ell-1}}{dt^{n_\ell-1}} \end{bmatrix} w(t_i^+) = \begin{bmatrix} 1 \\ \frac{d}{dt} \\ \vdots \\ \frac{d^{n_\ell-1}}{dt^{n_\ell-1}} \end{bmatrix} w(t_i^-) \ \text{ if } n_k > n_\ell; \\
&\begin{bmatrix} 1 \\ \frac{d}{dt} \\ \vdots \\ \frac{d^{n_\ell-1}}{dt^{n_\ell-1}} \end{bmatrix} w(t_i^+) = \begin{bmatrix} 1 \\ \frac{d}{dt} \\ \vdots \\ \frac{d^{n_k-1}}{dt^{n_k-1}} \\ \Pi \begin{bmatrix} \frac{d^{n_\ell}}{dt^{n_\ell}} \\ \vdots \\ \frac{d^{n_k-1}}{dt^{n_k-1}} \end{bmatrix} \end{bmatrix} w(t_i^-) \ \text{ if } n_k < n_\ell,
\end{aligned} \tag{2}$$

where $\Pi \in \mathbb{R}^{(n_\ell-n_k)\times n_k}$ is such that

$$\begin{bmatrix} \xi^{n_k} \\ \vdots \\ \xi^{n_\ell-1} \end{bmatrix} \mod p_k = \Pi \begin{bmatrix} 1 \\ \vdots \\ \xi^{n_k-1} \end{bmatrix}\ .$$

In words, when switching from a dynamical regime $\mathfrak{B}_k$ to $\mathfrak{B}_\ell$, we rewrite if necessary every derivative of w of order higher than $n_k - 1$ as a linear combination of derivatives of order at most $n_k - 1$, according to the canonical representative of ξ^j modulo p_k, $j = 0,...,n_\ell - 1$, (see section II-A). Thus at every switching instant, the state of the active behavior is uniquely specified as a linear function of the state of the behavior before the switch, allowing the continuation of the trajectories of the switched behavior by providing a full set of "initial conditions" after the switch. We call a SLDS with such gluing conditions a *standard switched linear differential system.*

IV. STABILITY AND POSITIVE-REALNESS

Asymptotically stable SLDS are defined as follows.

Definition 5: A SLDS Σ is *asymptotically stable* if $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}^{\Sigma}$.

We prove the stability of a SLDS showing the existence of a *Lyapunov function* Q_Ψ, i.e. a QDF such that: $Q_\Psi \overset{\mathfrak{B}_k}{\geq} 0$ and $\frac{d}{dt}Q_\Psi \overset{\mathfrak{B}_k}{<} 0$ for all $k \in \mathcal{P}$; and the value of Q_Ψ does not increase at the switching instants, i.e. $Q_\Psi(w)(t_i^-) \geq Q_\Psi(w)(t_i^+)$ for all $t_i \in \mathbb{T}_s$.

We summarize previous results (see [6], [5]) on the stability of SLDS with two behaviors in the following theorem.

Theorem 1: Let $p_j \in \mathbb{R}[\xi]$, $j = 1, 2$, be Hurwitz polynomials, and define $n_j := \deg(p_j)$, $j = 1, 2$. Let $\mathcal{F} = \{\mathfrak{B}_1, \mathfrak{B}_2\}$ with $\mathfrak{B}_j := \ker\ p_j\left(\frac{d}{dt}\right)$, $j = 1, 2$. Assume that $\frac{p_2}{p_1}$ is strictly positive-real with $n_1 \geq n_2$. Define $x_1(\xi) := \begin{bmatrix} 1 & \cdots & \xi^{n_1-1} \end{bmatrix}^\top$, $x_2(\xi) := \begin{bmatrix} 1 & \cdots & \xi^{n_2-1} \end{bmatrix}^\top$, and the set of gluing conditions $\mathcal{G}$ with $G^-_{2\to1}(\xi) = x_1(\xi) \bmod p_2$; $G^+_{2\to1}(\xi) = x_1(\xi)$; and $G^-_{1\to2}(\xi) = x_2(\xi) = G^+_{1\to2}(\xi)$. Define $\Phi(\zeta,\eta) := p_1(\zeta)p_2(\eta) + p_2(\zeta)p_1(\eta)$. Then, there exists a polynomial vector $d \in \mathbb{R}^{\bullet\times1}[\xi]$ such that

1. $p_1(-\xi)p_2(\xi) + p_2(-\xi)p_1(\xi) = d(-\xi)^\top d(\xi)$.
2. $\Psi(\zeta,\eta) := \frac{\Phi(\zeta,\eta) - d(\zeta)^\top d(\eta)}{\zeta+\eta} \in \mathbb{R}[\zeta,\eta]$.
3. Q_Ψ is a Lyapunov function for $\mathcal{F}$.

Proof: See [6] Theorem 10, and [5] Theorem 2.3. ■

As shown in [13] Th. 5.10, if we assume that $\frac{p_2}{p_1}$ is strictly positive-real, then the degree of p_1 and p_2 cannot differ by more than one, consequently, Theorem 1 only covers the situation where $n_1 - n_2 = 0$ or $n_1 - n_2 = 1$. To study the stability of behaviors whose state space dimension differs arbitrarily, we introduce the concept of *positive-real completion*.

Definition 6: Let Σ be a standard SLDS. The polynomial $m \in \mathbb{R}[\xi]$ is a *strictly positive-real completion* of $\frac{p_2}{p_1}$ if $\frac{mp_2}{p_1}$ is strictly proper and strictly positive-real.

Remark 1: Not every pair of Hurwitz polynomials has a strictly- positive-real completion, for example the polynomials $p_1(\xi) := 2523677 + 435616\xi + 81559\xi^2 + 7000\xi^3 + 603\xi^4 + 24\xi^5 + \xi^6$ and $p_2(\xi) := 65 + 46\xi + 26\xi^2 + 6\xi^3 + \xi^4$.

Remark 2: Strictly- positive-real completions are not unique; for instance the rational function $\frac{mp_2}{p_1}$ with $p_1(\xi) := (\xi+1)(\xi+3)(\xi+6)$ and $p_2 := \xi + 2$ is positive-real with m equal to $\xi + 4$, $\xi + 5$ and many other options.

A. Computation of a positive-real completion

To compute a strictly-proper positive-real completion m we can use the positive-real lemma [1]. Define $p_3 := mp_2$ and $n_3 := \deg(p_3)$; in the following we assume that $n_1 = n_3 + 1$. A realization $(A, B, C, 0)$ of $\frac{p_3(\xi)}{p_1(\xi)}$ can be written in controllable canonical form, i.e. $Ax(\xi) := \xi x(\xi) \bmod p_1 = \xi x(\xi) - Bp_1(\xi)$, and $p_3(\xi) = Cx(\xi)$, where $x(\xi) = \begin{bmatrix} 1 & \cdots & \xi^{n_1-1} \end{bmatrix}^\top$. The coefficients of m are parameters to be determined, so we write

$$C^\top := \underbrace{\begin{bmatrix} p_{2,0} & 0 & 0 & \cdots & 0 \\ p_{2,1} & p_{2,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \end{bmatrix}}_{=:\tilde{T}} \underbrace{\begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_{n_1-n_2-1} \end{bmatrix}}_{=:\tilde{m}} \tag{3}$$

where $\tilde{T} \in \mathbb{R}^{n_1\times(n_1-n_2)}$ is a Töplitz matrix containing the coefficients $p_{2,j}$ of $p_2(\xi)$; and $\tilde{m} \in R^{(n_1-n_2)\times1}$ contains the unknown coefficients of $m(\xi)$.

Now if for some $\varepsilon \geq 0$ and for some m_i, $i = 0, \ldots, n_1 - n_2 - 1$, the inequality

$$\begin{bmatrix} A^\top\tilde{\Psi} + \tilde{\Psi}A + 2\varepsilon\tilde{\Psi} & \tilde{\Psi}B - C^\top \\ B^\top\tilde{\Psi} - C & 0 \end{bmatrix} \leq 0\,, \tag{4}$$

has a positive-definite solution $\tilde{\Psi} = \tilde{\Psi}^\top \in \mathbb{R}^{n_1\times n_1}$, then $G(\xi) = \frac{p_3(\xi)}{p_1(\xi)} = C(\xi I - A)^{-1}B$ is strictly positive-real, and m is a completion. The LMI (4) can be solved using standard computational methods. On the other hand, if (4) has no solution, we conclude that the pair p_1, p_2 does not have a positive-real completion, see Remark 1.

B. Stability of SLDS using positive-real completions

In the following section we analyse some further consequences of the existence of positive-real completions.

V. MAIN RESULTS

To discuss the main results of this paper we need to illustrate first an important structural property of a Lyapunov function Q_Ψ for a SLDS Σ with $\mathcal{F} := \{\mathfrak{B}_i := \ker\ p_i\left(\frac{d}{dt}\right)\}_{i=1,2}$ with $p_i \in \mathbb{R}[\xi]$, $i = 1, 2$, and gluing conditions as in (2). Let $\Psi(\zeta,\eta)$ induce a Lyapunov function for a standard SLDS as in def. 4, and write

$$\Psi(\zeta,\eta) = \begin{bmatrix} 1 & \cdots & \zeta^{n_1-1} \end{bmatrix} \underbrace{\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^\top & \Psi_{22} \end{bmatrix}}_{=:\tilde{\Psi}} \begin{bmatrix} 1 \\ \vdots \\ \eta^{n_1-1} \end{bmatrix},$$

for suitable matrices $\Psi_{11} \in \mathbb{R}^{n_2\times n_2}$, $\Psi_{12} \in \mathbb{R}^{n_2\times(n_1-n_2)}$ and $\Psi_{22} \in \mathbb{R}^{(n_1-n_2)\times(n_1-n_2)}$. Note that since Q_Ψ is positive along $\mathfrak{B}_1$, the coefficient matrix

$$\tilde{\Psi} := \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^\top & \Psi_{22} \end{bmatrix} \tag{5}$$

is positive definite. Now consider the following Lemma.

Lemma 1: Let Σ be SLDS with $\mathcal{F} := \{\mathfrak{B}_i := \ker\ p_i\left(\frac{d}{dt}\right)\}_{i=1,2}$ with $p_i \in \mathbb{R}[\xi]$, $i = 1, 2$, and gluing conditions as in (2). Define $n_i := \deg(p_i)$, $i = 1, 2$ and assume that $n_1 > n_2$. Assume that there exists a Lyapunov function Q_Ψ for Σ and let its coefficient matrix $\tilde{\Psi}$ be partitioned as in (5), then $\Psi_{12} = -\Pi^\top\Psi_{22}$.

Proof: In order to prove the claim, define $z := \begin{bmatrix} w & \cdots & \frac{d^{n_2-1}}{dt^{n_2-1}}w \end{bmatrix}^\top$ and $v := \begin{bmatrix} \frac{d^{n_2}}{dt^{n_2}}w & \cdots & \frac{d^{n_1-1}}{dt^{n_1-1}}w \end{bmatrix}^\top$, then taking the standard gluing conditions (2) into account, when switching from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ at t_k, the inequality $Q_\Psi(w)(t_k^-) - Q_\Psi(w)(t_k^+) \geq 0$ holds true if and only if

$$\begin{bmatrix} z(t_k^-) \\ v(t_k^-) \end{bmatrix}^\top \left(\tilde{\Psi} - \begin{bmatrix} I_{n_2} & \Pi^\top \\ 0 & 0 \end{bmatrix} \tilde{\Psi} \begin{bmatrix} I_{n_2} & 0 \\ \Pi & 0 \end{bmatrix} \right) \begin{bmatrix} z(t_k^-) \\ v(t_k^-) \end{bmatrix} \geq 0 \,.$$

Since $[z^\top(t_k^-) \;\; v^\top(t_k^-)]$ is arbitrary in $\mathbb{R}^{n_1}$ for the trajectories of Σ, the last equality implies that

$$\tilde{\Psi} - \begin{bmatrix} I_{n_2} & \Pi^\top \\ 0 & 0 \end{bmatrix} \tilde{\Psi} \begin{bmatrix} I_{n_2} & 0 \\ \Pi & 0 \end{bmatrix} \geq 0 \,. \tag{6}$$

After standard linear algebra manipulations we find that (6) is equivalent to

$$\begin{bmatrix} -(\Psi_{12} + \Pi^\top \Psi_{22})\Psi_{22}^{-1}(\Psi_{12}^\top + \Psi_{22}\Pi) & 0 \\ 0 & \Psi_{22} \end{bmatrix} \geq 0 \,. \tag{7}$$

Now consider that the $(1,1)$ block in (7) is negative semidefinite; consequently, (7) holds if and only if the $(1,1)$ block is zero, i.e. if and only if $\Psi_{12} = -\Pi^\top \Psi_{22}$. The claim is proved. ■

A. Positive-realness and stability of SLDS with three behaviors

We now prove a sufficient condition for the asymptotic stability of a SLDS with *three* behaviors.

Theorem 2: Let $p_i \in \mathbb{R}[\xi]$, $i = 1, 2$, be Hurwitz polynomials such that $\deg(p_1) > \deg(p_2)$. Assume that there exists $m \in \mathbb{R}[\xi]$, with $\deg(m) = \deg(p_1) + 1$, and a Lyapunov function Q_Ψ for ker $p_i\left(\frac{d}{dt}\right)$, $i = 1, 2$, as in Lemma 1, such that the coefficient matrices $\tilde{m}$ and $\tilde{\Psi}$ satisfy the LMI (4) with C as in (3). Define $p_3(\xi) := m(\xi)p_2(\xi)$, $\mathfrak{B}_j :=$ ker $p_j\left(\frac{d}{dt}\right)$, $j = 1, 2, 3$, and denote $n_j := \deg(p_j)$, $j = 1, 2, 3$. Moreover, define $x_2(\xi) := \begin{bmatrix} 1 & \cdots & \xi^{n_2 - 1} \end{bmatrix}^\top$; $x_3'(\xi) := \begin{bmatrix} \xi^{n_2} & \cdots & \xi^{n_3 - 1} \end{bmatrix}^\top$, $x_3 := \begin{bmatrix} x_2(\xi) & x_3'(\xi) \end{bmatrix}^\top$ and $x_1'(\xi) := \xi^{n_1 - 1}$.

Consider the SLDS Σ' with $\mathcal{F}' = (\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$ and gluing conditions

$$\left(G^-_{2\to1}(\xi), G^+_{2\to1}(\xi)\right) := \left(\begin{bmatrix} x_2(\xi) \\ \Pi_1 x_2(\xi) \end{bmatrix}, \begin{bmatrix} x_2(\xi) \\ x_3'(\xi) \\ x_1'(\xi) \end{bmatrix} \right) \,,$$

$$\left(G^-_{1\to2}(\xi), G^+_{1\to2}(\xi)\right) := (x_2(\xi), x_2(\xi)) \,,$$

$$\left(G^-_{3\to1}(\xi), G^+_{3\to1}(\xi)\right) := \left(\begin{bmatrix} x_3(\xi) \\ \Pi_3 x_3(\xi) \end{bmatrix}, \begin{bmatrix} x_3(\xi) \\ x_1'(\xi) \end{bmatrix} \right) \,,$$

$$\left(G^-_{1\to3}(\xi), G^+_{1\to3}(\xi)\right) := (x_3(\xi), x_3(\xi)) \,,$$

$$\left(G^-_{2\to3}(\xi), G^+_{2\to3}(\xi)\right) := \left(\begin{bmatrix} x_2(\xi) \\ \Pi_2 x_2(\xi) \end{bmatrix}, \begin{bmatrix} x_2(\xi) \\ x_3'(\xi) \end{bmatrix} \right) \,,$$

$$\left(G^-_{3\to2}(\xi), G^+_{3\to2}(\xi)\right) := (x_2(\xi), x_2(\xi)) \,,$$

where $\Pi_1 \in \mathbb{R}^{(n_1 - n_2)\times n_2}$, $\Pi_2 \in \mathbb{R}^{(n_3 - n_2)\times n_2}$, $\Pi_3 \in \mathbb{R}^{(n_1 - n_3)\times n_3}$ are such that $\begin{bmatrix} x_3'(\xi) \\ x_1'(\xi) \end{bmatrix} \bmod p_2 = \Pi_1 x_2(\xi)$; $x_3'(\xi) \bmod p_2 = \Pi_2 x_2(\xi)$; and $x_1'(\xi) \bmod p_3 = \Pi_3 x_3(\xi)$. Then there exists a Lyapunov function Q_Ψ for $\mathcal{F}'$.

Proof: In order to show that Q_Ψ is a Lyapunov function for $\mathcal{F}'$, we prove the following statements:

S1. $Q_\Psi \overset{\mathfrak{B}_1}{\geq} 0$ and $\frac{d}{dt} Q_\Psi \overset{\mathfrak{B}_1}{<} 0$.
S2. $Q_\Psi \overset{\mathfrak{B}_2}{\geq} 0$ and $\frac{d}{dt} Q_\Psi \overset{\mathfrak{B}_2}{<} 0$.
S3. $Q_\Psi \overset{\mathfrak{B}_3}{\geq} 0$ and $\frac{d}{dt} Q_\Psi \overset{\mathfrak{B}_3}{<} 0$;

and moreover, we show that the value of Q_Ψ does not increase at the switching instants, i.e.

S4. when we switch from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ and viceversa.
S5. when we switch from $\mathfrak{B}_1$ to $\mathfrak{B}_3$ and viceversa.
S6. when we switch from $\mathfrak{B}_3$ to $\mathfrak{B}_2$ and viceversa.

Note that statements **S1** and **S2** and **S4** hold, since Q_Ψ is a Lyapunov function for $\{\mathfrak{B}_1, \mathfrak{B}_2\}$.

In order to prove **S3**, define $\Psi_3(\zeta, \eta) := \Psi(\zeta, \eta) \bmod p_3$. Note that since $Q_\Psi \geq 0$ and $Q_\Psi \overset{\mathfrak{B}_3}{=} Q_{\Psi_3}$, it follows that $Q_{\Psi_3} \geq 0$. To prove the second part of the statement, since $\frac{p_3}{p_1}$ is strictly positive-real, then

$$(\zeta + \eta)\Psi(\zeta, \eta) = p_1(\zeta)p_3(\eta) + p_3(\zeta)p_1(\eta) - d(\zeta)^\top d(\eta) \tag{8}$$

for some polynomial vector $d \in \mathbb{R}^{\bullet \times 1}[\xi]$ (see Theorem 1, section IV). From standard results in the theory of quadratic differential forms (see [12], p.1716), we know that the derivative of Q_{Ψ_3} is induced by the two variable polynomial $(\zeta + \eta)\Psi(\zeta, \eta) \bmod p_3 = -d'(\zeta)^\top d'(\eta)$, where $d' := d \bmod p_3$. Therefore, to prove that the derivative of Q_{Ψ_3} decreases along $\mathfrak{B}_3$ it is enough to check that $\mathrm{col}(d'(\lambda), p_3(\lambda))$ is full column rank for all $\lambda \in \mathbb{C}$, which guarantees that $\frac{d}{dt}(Q_{\Psi_3}(w))$ is non zero for the trajectories of $\mathfrak{B}_3$. By contradiction, assume that there exists $\lambda \in \mathbb{C}$ such that $p_1(\lambda) = 0$ and $d(\lambda) = 0$. Note that since p_1 is Hurwitz necessarily $\lambda \in \mathbb{C}_-$, the open left half-plane. Substitute $\zeta = \overline{\lambda}$ and $\eta = \lambda$ in the expression in (8), obtaining $(\overline{\lambda} + \lambda)\Psi(\overline{\lambda}, \lambda) = 0$. Since $\lambda \in \mathbb{C}_-$, this is equivalent with $\Psi(\overline{\lambda}, \lambda) = 0$, which implies that $\widetilde{\Psi}$ is not positive-definite, a contradiction.

The validity of statement **S5** follows from Th. 1, since $\frac{p_3}{p_1}$ is strictly positive-real and $\deg(p_3) = \deg(p_1) - 1$.

It remains to prove **S6**. When we switch from $\mathfrak{B}_3$ to $\mathfrak{B}_2$, the condition $Q_\Psi(w)(t_i^-) - Q_\Psi(w)(t_i^+) \geq 0$ must be satisfied. Since

$$\left(\begin{bmatrix} x_2(\xi) \\ x_3'(\xi) \\ x_1'(\xi) \end{bmatrix} \bmod p_3 \right) \bmod p_2 = \begin{bmatrix} x_2(\xi) \\ x_3'(\xi) \\ x_1'(\xi) \end{bmatrix} \bmod p_2 \,,$$

the condtion can be written as

$$Q_{\Psi \bmod p_3}(w) - Q_{(\Psi \bmod p_3) \bmod p_2}(w) \geq 0 \,. \tag{9}$$

In the following, we aim to express condition (9) in terms of a matrix inequality. We proceed by expressing the relation between Π_1, Π_2 and Π_3, and we first compute

$$\begin{bmatrix} x_2(\xi) \\ x_3'(\xi) \\ x_1'(\xi) \end{bmatrix} \bmod p_2 = \begin{bmatrix} x_2(\xi) \\ \Pi_1 x_2(\xi) \end{bmatrix} \,. \tag{10}$$

Partition $\Pi_3 := \begin{bmatrix}\Pi_3' & \Pi_3''\end{bmatrix}$ with $\Pi_3' \in \mathbb{R}^{(n_1-n_3)\times n_2}$ and $\Pi_3'' \in \mathbb{R}^{(n_1-n_3)\times(n_3-n_2)}$, then

$$\begin{bmatrix} x_2(\xi) \\ x_3'(\xi) \\ x_1'(\xi)\end{bmatrix} \bmod p_3 = \begin{bmatrix} x_2(\xi) \\ \Pi_2 x_2(\xi) \\ \Pi_3' x_2(\xi) + \Pi_3'' x_3'(\xi)\end{bmatrix} .$$

Consequently

$$\left(\begin{bmatrix} x_2(\xi) \\ x_3'(\xi) \\ x_1'(\xi)\end{bmatrix} \bmod p_3\right) \bmod p_2 = \begin{bmatrix} x_2(\xi) \\ \begin{bmatrix}\Pi_2 \\ \Pi_3' + \Pi_3''\Pi_2\end{bmatrix} x_2(\xi)\end{bmatrix} . \tag{11}$$

By comparing equations (10) and (11) we have that $\Pi_1 = \begin{bmatrix}\Pi_2 \\ \Pi_3' + \Pi_3''\Pi_2\end{bmatrix}$. Now consider the coefficient matrix of the Lyapunov function Q_Ψ and partition it as

$$\tilde{\Psi} := \begin{bmatrix}\Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{12}^\top & \Psi_{22} & \Psi_{23} \\ \Psi_{13}^\top & \Psi_{23}^\top & \Psi_{33}\end{bmatrix} , \tag{12}$$

with $\Psi_{11} \in \mathbb{R}^{n_2\times n_2}$, $\Psi_{12} \in \mathbb{R}^{n_2\times(n_3-n_2)}$, $\Psi_{13} \in \mathbb{R}^{n_2\times(n_1-n_3)}$, $\Psi_{22} \in \mathbb{R}^{(n_3-n_2)\times(n_3-n_2)}$, $\Psi_{23} \in \mathbb{R}^{(n_3-n_2)\times(n_1-n_3)}$ and $\Psi_{33} \in \mathbb{R}^{(n_1-n_3)\times(n_1-n_3)}$. From the results of Lemma 1, since the Lyapunov function Q_Ψ does not increase when switching from $\mathfrak{B}_1$ to $\mathfrak{B}_2$, this implies that

$$\begin{aligned}\begin{bmatrix}\Psi_{12}^\top \\ \Psi_{13}^\top\end{bmatrix} &= -\begin{bmatrix}\Psi_{22} & \Psi_{23} \\ \Psi_{23}^\top & \Psi_{33}\end{bmatrix}\Pi_1 \\ &= -\begin{bmatrix}\Psi_{22} & \Psi_{23} \\ \Psi_{23}^\top & \Psi_{33}\end{bmatrix}\begin{bmatrix}\Pi_2 \\ \Pi_3' + \Pi_3''\Pi_2\end{bmatrix} ,\end{aligned}$$

and consequently

$$\Psi_{12}^\top = -(\Psi_{22}\Pi_2 + \Psi_{23}\Pi_3' + \Psi_{23}\Pi_3''\Pi_2) , \tag{13}$$

and

$$\Psi_{13}^\top = -(\Psi_{23}\Pi_2 + \Psi_{33}\Pi_3' + \Psi_{33}\Pi_3''\Pi_2) . \tag{14}$$

The following lemma provides important structural properties of $Q_{\Psi \bmod p_3}$ that will be essential for the rest of the proof.

Lemma 2: Let Q_Ψ, its coefficient matrix $\tilde{\Psi}$ and $\Pi_3 := \begin{bmatrix}\Pi_3' & \Pi_3''\end{bmatrix}$, be as previously defined and let $\tilde{\tilde{\Psi}}$ be the coefficient matrix of $Q_{\Psi \bmod p_3}$. Consider the partition

$$\tilde{\tilde{\Psi}} := \begin{bmatrix}\tilde{\tilde{\Psi}}_{11} & \tilde{\tilde{\Psi}}_{12} \\ \tilde{\tilde{\Psi}}_{12}^\top & \tilde{\tilde{\Psi}}_{22}\end{bmatrix} , \tag{15}$$

with $\tilde{\tilde{\Psi}}_{11} \in \mathbb{R}^{n_2\times n_2}$, $\tilde{\tilde{\Psi}}_{12} \in \mathbb{R}^{n_2\times(n_3-n_2)}$ and $\tilde{\tilde{\Psi}}_{22} \in \mathbb{R}^{(n_3-n_2)\times(n_3-n_2)}$. Then

$$\begin{aligned}\tilde{\tilde{\Psi}}_{11} &= (\Psi_{11} + \Pi_3'\Psi_{13}^\top + \Psi_{13}\Pi_3' + \Pi_3'^\top\Psi_{33}\Pi_3') , \\ \tilde{\tilde{\Psi}}_{12} &= (\Psi_{12} + \Pi_3'^\top\Psi_{23}^\top + \Psi_{13}\Pi_3'' + \Pi_3'^\top\Psi_{33}\Pi_3'') , \\ \tilde{\tilde{\Psi}}_{22} &= (\Psi_{22} + \Pi_3''^\top\Psi_{23}^\top + \Psi_{23}\Pi_3' + \Pi_3''^\top\Psi_{33}\Pi_3'') .\end{aligned}$$

Proof: Following the same procedure as in Lemma 1 and considering the partitions (12) and (15), we conclude that the coefficient matrix of $Q_{\Psi \bmod p_3}$ is

$$\begin{bmatrix}\tilde{\tilde{\Psi}}_{11} & \tilde{\tilde{\Psi}}_{12} \\ \tilde{\tilde{\Psi}}_{12}^\top & \tilde{\tilde{\Psi}}_{22}\end{bmatrix} = \begin{bmatrix}I_{n_2} & 0 \\ 0 & I_{(n_3-n_2)} \\ \Pi_3' & \Pi_3''\end{bmatrix}^\top \begin{bmatrix}\Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{12}^\top & \Psi_{22} & \Psi_{23} \\ \Psi_{13}^\top & \Psi_{23}^\top & \Psi_{33}\end{bmatrix} \begin{bmatrix}I_{n_2} & 0 \\ 0 & I_{(n_3-n_2)} \\ \Pi_3' & \Pi_3''\end{bmatrix} . \tag{16}$$

The desired equalities follow by inspection. ■

Now we return to the proof of the main Theorem. Note that from the inequality (9) we can obtain

$$\begin{bmatrix}\tilde{\tilde{\Psi}}_{11} & \tilde{\tilde{\Psi}}_{12} \\ \tilde{\tilde{\Psi}}_{12}^\top & \tilde{\tilde{\Psi}}_{22}\end{bmatrix} - \begin{bmatrix}I_{n_2} & \Pi_2^\top \\ 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{\tilde{\Psi}}_{11} & \tilde{\tilde{\Psi}}_{12} \\ \tilde{\tilde{\Psi}}_{12}^\top & \tilde{\tilde{\Psi}}_{22}\end{bmatrix}\begin{bmatrix}I_{n_2} & 0 \\ \Pi_2 & 0\end{bmatrix} \geq 0 .$$

Note that similarly to Lemma 1, this inequality holds if and only if $\tilde{\tilde{\Psi}}_{12}^\top + \tilde{\tilde{\Psi}}_{22}\Pi_2 = 0$, or equivalently from Lemma 2, the condition is satisfied if and only if

$$\begin{aligned}&\Psi_{12}^\top + \Pi_3''^\top\Psi_{13}^\top + \Psi_{23}\Pi_3' + \Pi_3''^\top\Psi_{33}\Pi_3' = \\ &-(\Psi_{22} + \Pi_3''^\top\Psi_{23}^\top + \Psi_{23}\Pi_3' + \Pi_3''^\top\Psi_{33}\Pi_3'')\Pi_2 .\end{aligned}$$

Substituting (14) in the latter equation we obtain (13) and we conclude that

$$\left\{\begin{bmatrix}\Psi_{12}^\top \\ \Psi_{13}^\top\end{bmatrix} = -\begin{bmatrix}\Psi_{22} & \Psi_{23} \\ \Psi_{23}^\top & \Psi_{33}\end{bmatrix}\Pi_1\right\} \implies \left\{\tilde{\tilde{\Psi}}_{12}^\top = -\tilde{\tilde{\Psi}}_{22}\Pi_2\right\} .$$

Consequently Q_Ψ does not increase when switching from $\mathfrak{B}_3$ to $\mathfrak{B}_2$. It is a matter of straighforward verification to check that when we switch from $\mathfrak{B}_2$ to $\mathfrak{B}_3$ the value of Q_Ψ remains the same before and after the switch. This concludes the proof of the Theorem. ■

Theorem 2 shows that the existence of a strictly positive-real completion m associated to a SLDS Σ with two behaviors $\mathfrak{B}_j := \ker\ p_j\left(\frac{d}{dt}\right)$, $j = 1, 2$, in the bank $\mathcal{F}$, implies the existence of a third behavior $\mathfrak{B}_3 := \ker\ p_3\left(\frac{d}{dt}\right)$ with $p_3 := mp_2$, in an augmented bank $\mathcal{F}'$ of a SLDS Σ'. We defined the standard gluing conditions for Σ', associated to the switching among the behaviors $\mathfrak{B}_i$, $i = 1, 2, 3$, as in (2) following that $n_1 > n_3 > n_2$. Consequently, the stability conditions derived from the analysis of the switching between the behaviors in $\mathcal{F}$ are compatible with the stability conditions for $\mathcal{F}'$ concluding that if Σ is asymptotically stable, so is Σ'.

B. Positive-realness and stability of families of three-behaviors

Another consequence of the notion of positive-real completion is given in the following Theorem.

Theorem 3: Let Σ' be a SLDS as in Theorem 2. Assume that there exist two different strictly positive-real completions m_1 and m_2 for $\frac{p_2}{p_1}$, and let $\alpha \in [0, 1]$. Then $m_\alpha := \alpha m_1 + (1-\alpha)m_2$ is also a completion, i.e. $\frac{m_\alpha p_2}{p_1}$ is strictly positive-real.

Moreover, define

$$\mathfrak{F}'_\alpha := \{\ker\ p_1(\frac{d}{dt}), \ker\ p_2(\frac{d}{dt}), \ker\ p_{3,\alpha}(\frac{d}{dt})\}\ ,$$

with $p_{3,\alpha} := m_\alpha p_2$ and the standard gluing conditions as in Theorem 2. Then $\mathfrak{F}_\alpha$ is stable.

Proof: The fact that m_α for all $\alpha \in [0,1]$ is strictly positive-real follows from straightforward computations:

$$\begin{aligned}
&\frac{m_\alpha(-j\omega)p_2(-j\omega)}{p_1(-j\omega)} + \frac{m_\alpha(j\omega)p_2(j\omega)}{p_1(j\omega)} \\
&= \frac{(\alpha m_1(-j\omega) + (1-\alpha)m_2(-j\omega))\, p_2(-j\omega)}{p_1(-j\omega)} \\
&\quad + \frac{(\alpha m_1 + (1-\alpha)m_2)\, p_2(j\omega)}{p_1(j\omega)} \\
&= \alpha \underbrace{\left(\frac{m_1(-j\omega)r_2(-j\omega)}{p_1(-j\omega)} + \frac{m_1 p_2(j\omega)}{p_1(j\omega)}\right)}_{>0 \text{ for all } \omega\in\mathbb{R}} \\
&\quad + (1-\alpha)\underbrace{\left(\frac{m_2(-j\omega)p_2(-j\omega)}{p_1(-j\omega)} + \frac{m_2 p_2(j\omega)}{p_1(j\omega)}\right)}_{>0 \text{ for all } \omega\in\mathbb{R}},
\end{aligned}$$

To prove that $\mathfrak{F}_\alpha$ is stable, use Theorem 2. ■

Theorem 3 shows that the existence of two separate completions allows to establish the stability of a whole family of parameter-dependent SLDS with three behaviors $\mathfrak{F}_\alpha$. This result also shows that the asymptotic stability of a completion established in Theorem 2 is robust: perturbations of a given completion, parametrized by α as in Theorem 3, also result in a stable SLDS.

We now provide a method to compute more than one strictly- positive-real completion; the intuition behind this procedure is to consider small perturbations of a positive-real completion that result in other completions satisfying the frequency domain inequality (1).

Consider the realization $(A, B, C, 0)$ associated to a strictly positive real function $G(\xi) := C(\xi I - A)^{-1}B$ in section IV-A, and the LMI (4) with C as in (3). Consider that $G(\xi-\varepsilon)$ is strictly positive-real for some constant $\varepsilon > 0$ (see [10], Th. 3.3). We can use this fact to numerically compute different solutions $\tilde{m}$ and $\tilde{\Psi}$ for a given pair of polynomials (p_1, p_2) by defining different values of $\varepsilon \geq 0$. In order to define an upper bound for ε, define $Q := A^\top\tilde{\Psi} + \tilde{\Psi}A$. Since $\tilde{\Psi}$ is symmetric and positive definite, there exists a nonsingular matrix $N \in \mathbb{R}^{n_1\times n_1}$ such that $\tilde{\Psi} := N^\top N$. Consequently, ε is such that $Q + 2\varepsilon\tilde{\Psi} < 0$ if and only if $N^{-\top}QN^{-1} + 2\varepsilon < 0$. In order for this to hold, ε must be less than $-\frac{1}{2}\lambda_{max}$, where λ_{max} is the largest eigenvalue of $N^{-\top}QN^{-1}$. Consequently, ε must necessarily belong to the interval $[0, -\frac{1}{2}\lambda_{max})$.

Based on this discussion, we state the following algorithm.

Algorithm 1:

Input: Hurwitz polynomials p_1, p_2 with $n_1 > n_2 + 1$.

Output: If they exist, two strictly- positive-real completions.

Step 1: Define A, B as in the controllable canonical realization of $\frac{1}{p_1}$, and $C^\top := \tilde{T}\tilde{m}$ as in (3).

Step 2: Solve the LMI (4) with $\varepsilon = 0$, to obtain $\tilde{\Psi}_0$ and the coefficient vector $\tilde{m}_0$. If there is no solution, **EXIT**.

Step 3: Compute a factorization $\tilde{\Psi}_0 := N_0^\top N_0$ and define $Q_0 := A^\top\tilde{\Psi}_0 + \tilde{\Psi}_0 A$.

Step 5: Compute the largest eigenvalue $\lambda_{max,0}$ of $N_0^{-\top}Q_0N_0^{-1}$, and choose $\varepsilon_1 \in (0, -\frac{1}{2}\lambda_{max,0})$.

Step 2: Solve the LMI (4) with $\varepsilon = \varepsilon_1$, to obtain $\tilde{\Psi}_1$ and the coefficient vector $\tilde{m}_1$.

Step 6: **RETURN** $\tilde{m}_0$ and $\tilde{m}_1$.

VI. Conclusions

We studied the stability of scalar switched linear differential systems with three behaviors using the concept of positive-real completion, and we illustrated how a family of switched differential systems can be obtained the convex combination of two completions.

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