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ABSTRACT

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Numerical models of the tides usually use a finite difference form of the tidal equations. The report investigates the adjoint properties of the finite difference equations. It shows that the are not necessarily symmetrical, i.e. the finite difference form of the adjoint tidal equations may not be the same as the adjoint of the normal finite difference equations. It also shows that, with a suitable choice of the way the boundary conditions and Coriolis terms are represented, the finite difference forms can be made symmetric.

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On the Adjoint of Laplace's Tidal Equations

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Abstract.

The concept of the adjoint is regularly used in studies of both the resonances of physical systems and their response to external forcing. This report reviews the underlying theory involved in the adjoints of both differential equations and matrices and shows how the theory may be used to derive a physically meaningful adjoint to Laplace's Tidal Equations.

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1 Introduction

This note is concerned with Laplace's tidal equations and their Hermitian adjoint both as differential equations and as a set of finite difference equations.

It arose out of some work on the the tides and resonances of the Hudson Bay complex (Webb, 2013b) using a finite difference model. There were questions concerning the correct way to represent a radiational boundary condition in the model and this required some understanding of the boundary condition in the adjoint set of finite difference equations.

As Laplace's tidal equations with friction are not Hermitian, the eigenfunctions of the equations, which physically are the resonances of the system, are no longer orthogonal. Instead for a resonance expansion, one also has to make use of the eigenfunctions of the Hermitian adjoint set of equations.

Because of this connection between the resonances and the adjoint, it is helpful if the if there exists the same sort of sensible physical interpretation of the adjoint equations as there is for the normal equations. In the case of Laplace's tidal equations, although it is straightforward to generate a physically sensible Hermitian adjoint, it was found that when the tidal equations and their adjoint are converted to finite difference form there are two problems.

The first is that the two matrices formed by the complete set of finite difference equations, are not necessarily the Hermitian adjoint of each other. The second is that in such cases, the Hermitian adjoints of the two matrices do not appear to have a physically sensible interpretation.

A way of representing a radiational boundary condition which did not lead to such problems was found and used for the Hudson Bay model. However the detailed argument needed to justify the change would have detracted from the main aim of the Hudson Bay paper. It was therefore decided to publish the details in a separate technical report.

Once this decision was made it appeared sensible to include more of the background detail than would be included in a normal scientific paper. One reason for doing this is that although many textbooks cover Hermitian equations and matrices, there is less published on the properties of non-Hermitian systems.

A second reason is that the equations that make up Laplace's tidal equations form a matrix differential operator acting on a vector as opposed to the more normal set of differential operators acting on a scalar. As far as I am aware there is also nothing published on the extension to matrix differential equations or the on adjoint properties of the finite difference form of the equations.

The third reason is to publicise the use of an adjoint of the tidal equations which itself has physical meaning. The adjoint equations, whose derivation has its origins in the theory of Green's functions, have been published previously, for example by Garrett and Greenberg (1977) and Webb (1989). They are particularly useful for studying resonances where their physical basis can give insight into the processes that most efficiently excite each resonance. Their simplicity needs to be be contrasted with the results of using the Lagrange multiplier approach, widely used in data assimilation, where the physical meaning of the adjoint state variables and equations is often unclear.

The main body of the report remains the radiational boundary condition and its use in the finite difference equations representing Laplace's tidal equations. I am aware that much of the content has been previously published. Appendices are therefore used to summarise the adjoint properties matrices and the recipe given by Morse and Feshbach for determining the adjoint of a normal differential operator. The report is written primarily from the point of view of a physicist and so, while being concerned with physical interpretation of the equations, it does not cover the existence or uniqueness of solutions.

Section 2 gives the definition of an adjoint differential operator and discusses its use in the solution of differential equations. Section 3 then applies the results to the matrix form of Laplace's Tidal Equations and section 4 extends this to the finite difference form of the equations. Section 5 considers the different types of boundary condition and their impact on the adjoint symmetries of both the differential equations and their finite difference equivalents. This finally allows a informed decision to be made about the best radiational boundary condition to use for the normal and adjoint sets of finite difference equations.

2 The Hermitian Adjoint Operator

The concept of a Hermitian adjoint[1] is applicable to any linear operator $\boldsymbol{\mathcal{K}}$ which operates on functions, say $p(x)$, of position x. The functions $p(x)$ and the coordinate x may both be scalars or vectors. The theory needs an inner product to be defined which, given two such functions $p(x)$ and $q(x)$, generates a scalar S. If p and q are scalars then this is simply,

$$S = \int dx\; p^*(x)w(x)q(x)\,, \tag{1}$$

where $w(x)$ is a weighting function, often equal to one, and p^* is the complex conjugate of p.

If p and q are vectors with components represented by p_j and q_k, then

$$S = \int dx\; \sum_j \sum_k \; p_j^*(x)\mathbf{w}_{j,k}(x)q_k(x)\,, \tag{2}$$

where the weighting matrix $\mathbf{w}$ is usually a diagonal matrix.

In the rest of this analysis the summation and weighting function will be implied whenever they are missing and there is a product of the form $q^*(x)\,p(x)$ or $q^*(x)\,\boldsymbol{\mathcal{K}}(x)\,p(x)$. If the coordinate terms (x) are also missing, it implies an integral of the product over the whole of the domain of x.

For each operator $\boldsymbol{\mathcal{K}}$, a Hermitian adjoint operator, denoted by $\widetilde{\boldsymbol{\mathcal{K}}}$, exists if for each value of x, $\widetilde{\boldsymbol{\mathcal{K}}}$ and a function P, called the bilinear concomitant, can be found which satisfy the equation,

$$q^*(x)\,\boldsymbol{\mathcal{K}}(x)p(x) - (p^*(x)\,\widetilde{\boldsymbol{\mathcal{K}}}(x)q(x))^* = \nabla\cdot P(q(x),p(x)). \tag{3}$$

This is a key equation. It is the basis of all the other results reported here. A method for calculating $\widetilde{\boldsymbol{\mathcal{K}}}(x)$ and $P(q(x),p(x))$ for normal differential operators is given in Appendix B.

One reason for the importance of Eqn 3 arises because the operators $\boldsymbol{\mathcal{K}}$ and $\widetilde{\boldsymbol{\mathcal{K}}}$ have eigenvectors and eigenvalues which form complete sets and which are orthogonal. Thus operator $\boldsymbol{\mathcal{K}}$ will have eigenvalues λ_j and eigenvectors $p_j(x)$ satisfying the equation,

$$(\boldsymbol{\mathcal{K}}(x) - \lambda_j)p_j(x) = 0. \tag{4}$$

If the domain of x is bounded or it extends to infinity, then boundary conditions are required in order provide a complete set of functions[2]. In the case of a second order differential operator these would normally require the fields p to be either zero on the boundary (Dirichlet conditions), have zero normal gradient to the boundary (Neumann conditions) or a mixture of these.

[1] In systems where the solutions are all real functions, the term Hermitian adjoint is often replaced by the term adjoint. As we are concerned with wave motions where the functions are often complex and proportional to $\exp(-ikx)$ or $\exp(-i\omega t)$, the term Hermitian adjoint is more correct.

[2] If it is finite and unbounded, for example for a field defined on the surface of a sphere, then boundary conditions are not needed.

The adjoint operator also has eigenvalues and eigenvectors,

$$(\widetilde{\mathcal{K}}(x)-\mu_j)q_j(x)=0. \tag{5}$$

Integrating Eqn. 3 over the domain, when p equals p_j and q equals q_k,

$$\int dx\ [q_k^*(x)\,\mathcal{K}(x)p_j(x)-(p_j^*(x)\,\widetilde{\mathcal{K}}(x)q_k(x))^*] = \int dx\ \bigtriangledown P(q_k,p_j), \tag{6}$$

$$(\lambda_j-\mu_k^*)\int dx\ q_k^*(x)\ p_j(x) = \int ds\ \hat{\boldsymbol{n}}\cdot P(q_k,p_j). \tag{7}$$

The last term is the integral around the boundary of the normal gradient of P. If the boundary conditions for the adjoint (Eqn. 5) are such that this is zero, then, treating the integral as implicit,

$$(\lambda_j-\mu^*)\ (q^*\ p) = 0. \tag{8}$$

So either the eigenfunctions p_j and q_k are orthogonal or λ_j equals μ_k^*.

From this it is obvious that the Hermitian adjoint $\widetilde{\mathcal{K}}$ is really only of use if its boundary conditions have the right relationship to the boundary conditions of $\mathcal{K}$. Similarly, discussions of a differential operator and its adjoint need to treat each operator as consisting of both its differential form and its boundary conditions.

2.1 Solving forced problems using eigenvectors

Once they have been determined, the eigenvalues and eigenvectors of $\mathcal{K}$ and its Hermitian adjoint can be used to solve forced problems of the type represented by the equation,

$$(\mathcal{K}(x)-\lambda)p(x)=z(x). \tag{9}$$

Here $z(x)$ usually represents the forces acting on a system and $p(x)$ the system's response.

Expanding the response in terms of the complete set of functions p_j,

$$p(x)=\sum_j \alpha_j\ p_j(x). \tag{10}$$

Then substituting Eqn. 10 in Eqn. 9,

$$\sum_j (\lambda_j-\lambda)\ \alpha_j\ p(x)=z(x). \tag{11}$$

Multiplying both sides by q_k^* and integrating over x,

$$(\lambda_k-\lambda)\ \alpha_k\ = q_k^*\ z, \tag{12}$$

$$\alpha_k\ = (q_k^*\ z)/(\lambda_k-\lambda). \tag{13}$$

Thus the solution, the response of the system, is given by,

$$p(x)\ = \sum_k p_k(x)\ (q_k^*\ z)/(\lambda_k-\lambda). \tag{14}$$

Thus the response depends on the eigenvectors of both the matrix $\mathcal{K}$ and its adjoint.

3 Laplace's tidal equations

In vector notation and with a linear friction term Laplace's tidal equations are,

$$\partial\boldsymbol{u}/\partial t+\boldsymbol{f}\times\boldsymbol{u}+(\kappa/H)\boldsymbol{u}+g\nabla\zeta = g\nabla\zeta_e\,, \qquad (15)$$
$$\partial\zeta/\partial t+\nabla\cdot(H\boldsymbol{u}) = 0\,.$$

$\boldsymbol{u}$ is the depth averaged horizontal velocity, t is time, ζ sea level, ζ_e the height of the equilibrium tide (corrected for Earth tides), κ the bottom friction coefficient, g the acceleration due to gravity, H the depth and '$\times$' indicates a vector cross product.

If the velocity $\boldsymbol{u}$ has components u and v, then the equations can be expressed in matrix form as,

$$\begin{bmatrix} \partial/\partial t+\kappa/H & -f & g\,\partial/\partial x \\ +f & \partial/\partial t+\kappa/H & g\,\partial/\partial y \\ (\partial/\partial x)H & (\partial/\partial y)H & \partial/\partial t \end{bmatrix} \begin{bmatrix} u \\ v \\ \zeta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \zeta_0 \end{bmatrix} \qquad (16)$$

If the forcing is periodic and has a time dependence of the form $\exp(-i\omega t)$, then the solution vector (u,v,ζ) will have the same time dependence. Removing the time dependent terms from the equation, it becomes,

$$\left(\begin{bmatrix} \kappa/H & -f & g\,\partial/\partial x \\ +f & \kappa/H & g\,\partial/\partial y \\ (\partial/\partial x)H & (\partial/\partial y)H & 0 \end{bmatrix} - i\omega\mathbf{1} \right) \begin{bmatrix} u \\ v \\ \zeta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \zeta_0 \end{bmatrix} \qquad (17)$$

where $\mathbf{1}$ is the unit matrix.

It is convenient to define matrix $\boldsymbol{\mathcal{K}}$ as the operator corresponding to the time dependent form of Laplace's tidal equations (Eqn. 16) and matrix $\boldsymbol{\mathcal{L}}$ as the operator corresponding to the time independent form (Eqn. 17).

$$\boldsymbol{\mathcal{K}} = \begin{bmatrix} \partial/\partial t+\kappa/H & -f & g\,\partial/\partial x \\ +f & \partial/\partial t+\kappa/H & g\,\partial/\partial y \\ (\partial/\partial x)H & (\partial/\partial y)H & \partial/\partial t \end{bmatrix}, \qquad (18)$$

$$\boldsymbol{\mathcal{L}} = \begin{bmatrix} \kappa/H & -f & g\,\partial/\partial x \\ +f & \kappa/H & g\,\partial/\partial y \\ (\partial/\partial x)H & (\partial/\partial y)H & 0 \end{bmatrix}. \qquad (19)$$

3.1 The Inner Product

The state vector $(u(\boldsymbol{x}),v(\boldsymbol{x}),\zeta(\boldsymbol{x}))$ used in Eqns. 16 and 17, includes two velocities variables and one length variable. If the state vector of the adjoint equation is represented by $(c(\boldsymbol{x}),d(\boldsymbol{x}),e(\boldsymbol{x}))$, and if the inner product is defined as,

$$S(x) = \int \boldsymbol{dx}\ (c^*(\boldsymbol{x})u(\boldsymbol{x})+d^*(\boldsymbol{x})v(\boldsymbol{x})+e^*(\boldsymbol{x})\zeta(\boldsymbol{x})), \qquad (20)$$

then because each of the products in this equation need to have the same dimensional form, the adjoint variables are forced to have different dimensional forms (i.e. if c and d are velocities, e cannot be a length).

However if the inner product is defined, using an analogy with the addition of kinetic and potential energy,

$$S(\boldsymbol{x}) = \int \boldsymbol{dx}\ (\rho/2)(c^*(\boldsymbol{x})H(\boldsymbol{x})u(\boldsymbol{x}) + d^*(\boldsymbol{x})H(\boldsymbol{x})v(\boldsymbol{x}) + e^*(\boldsymbol{x})g\zeta(\boldsymbol{x})), \tag{21}$$

then if c and d are velocities, d is a length and $S(\boldsymbol{x})$ has the dimensions of energy per unit area. In this case all the adjoint variables have a simple physical interpretation and, as shown in the next section, the adjoint operator has a simple physical form.

If $((u(\boldsymbol{x}), v(\boldsymbol{x}), \zeta(\boldsymbol{x})))$ is represented by the vector $\boldsymbol{z}(\boldsymbol{x})$, then the dot product of two such vectors, $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$, becomes,

$$S = \boldsymbol{z}_2^* \boldsymbol{\mathcal{W}} \boldsymbol{z}_1, \tag{22}$$

where there is an implied integral over x and $\boldsymbol{\mathcal{W}}$, the diagonal weighting matrix, is,

$$\boldsymbol{\mathcal{W}} = \begin{bmatrix} (\rho H)/2 & 0 & 0 \\ 0 & (\rho H)/2 & 0 \\ 0 & 0 & (\rho g)/2 \end{bmatrix}. \tag{23}$$

3.2 The Hermitian Adjoint Operator

A method for determining the adjoint of a ordinary differential equation is given in Appendix B. To see how this can be extended to operators involving matrices, let $\boldsymbol{\mathcal{K}}_t$ be the part of operator $\boldsymbol{\mathcal{K}}$ involving $\partial/\partial t$.

$$\boldsymbol{\mathcal{K}}_t = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{bmatrix} \partial/\partial t \tag{24}$$

Then for two vectors $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$

$$\begin{aligned} \boldsymbol{z}_2^* \boldsymbol{\mathcal{W}} \boldsymbol{\mathcal{K}}_t \boldsymbol{z}_1 &= (\rho/2)(Hu_2^*(\partial/\partial t)u_1 + Hv_2^*(\partial/\partial t)v_1 + g\zeta_2^*(\partial/\partial t)\zeta_1) \\ &= \partial/\partial t((\rho/2)(Hu_2^* u_1 + Hv_2^* v_1 + g\zeta_2^*\zeta_1)) \\ &\quad -(\rho/2)(Hu_1(\partial/\partial t)u_2^* + Hv_1(\partial/\partial t)v_2^* + g\zeta_1(\partial/\partial t)\zeta_2^*) \\ &= \partial/\partial t((\rho/2)(Hu_2^* u_1 + Hv_2^* v_1 + g\zeta_2^*\zeta_1)) - \boldsymbol{z}_1 \boldsymbol{\mathcal{W}} \boldsymbol{\mathcal{K}}_t \boldsymbol{z}_2^*. \end{aligned} \tag{25}$$

Thus,

$$\boldsymbol{z}_2^* \boldsymbol{\mathcal{W}} \boldsymbol{\mathcal{K}}_t \boldsymbol{z}_1 + (\boldsymbol{z}_1^* \boldsymbol{\mathcal{W}} \boldsymbol{\mathcal{K}}_t \boldsymbol{z}_2)^* = \partial/\partial t((\rho/2)(Hu_2^* u_1 + Hv_2^* v_1 + g\zeta_2^*\zeta_1)). \tag{26}$$

Thus from eqn. 3, the adjoint of $\mathcal{K}_t$ and the corresponding term of $\bigtriangledown \cdot P$ are given by,

$$\widetilde{\mathcal{K}}_t = -\mathcal{K}_t, \tag{27}$$

$$\bigtriangledown \cdot P_t(\boldsymbol{z}_2,\boldsymbol{z}_1) = \partial/\partial t((\rho/2)(Hu_2^*u_1 + Hv_2^*v_1 + g\zeta_2^*\zeta_1)).$$

In this way it is straightforward to show that, if the above weighting is used, the adjoint of the Laplace tidal operator $\widetilde{\mathcal{K}}$ and the corresponding bilinear concomitant are,

$$\widetilde{\mathcal{K}} = -\begin{bmatrix} \partial/\partial t - \kappa/H & -f & g\,\partial/\partial x \\ +f & \partial/\partial t - \kappa/H & g\,\partial/\partial y \\ (\partial/\partial x)H & (\partial/\partial y)H & \partial/\partial t \end{bmatrix}, \tag{28}$$

$$\bigtriangledown \cdot P(\boldsymbol{z}_2,\boldsymbol{z}_1) = \partial/\partial t[(\rho/2)(g\zeta_2^*\zeta_1 + Hu_2^*u_1 + Hv_2^*v_1)]$$
$$+\partial/\partial x[(\rho g H/2)(\zeta_2^*u_1 + u_2^*\zeta_1)] + \partial/\partial y[(\rho g H/2)(\zeta_2^*v_1 + v_2^*\zeta_1)].$$

The equations show that the matrix operator $-\widetilde{\mathcal{K}}$ is the same as $\mathcal{K}$ except for the sign of the friction term. This is useful, because instead of being just an abstract mathematical concept, the solutions of the adjoint equation can be thought of as long tidal waves going backward in time. As an example, as when the adjoint is used in data assimilation, it is straight forward to understand how errors at one time can be propagated back in time to identify where the errors may have arisen.

The same scheme can be used for the time independent operator $\mathcal{L}$. Thus,

$$\widetilde{\mathcal{L}} = -\begin{bmatrix} -\kappa/H & -f & g\,\partial/\partial x \\ +f & -\kappa/H & g\,\partial/\partial y \\ (\partial/\partial x)H & (\partial/\partial y)H & 0 \end{bmatrix}, \tag{29}$$

$$\bigtriangledown \cdot P(\boldsymbol{z}_2,\boldsymbol{z}_1) = \partial/\partial x[(\rho g H/2)(\zeta_2^*u_1 + u_2^*\zeta_1)] + \partial/\partial y[(\rho g H/2)(\zeta_2^*v_1 + v_2^*\zeta_1)]$$

The last equation is used to determine the adjoint boundary conditions for the normal modes. The integral of the r.h.s. of Eqn. 6 will only be zero if P is zero at each point on the boundary. Thus on the boundary,

$$\zeta_2^*(\boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}}) + (\boldsymbol{u}_2 \cdot \hat{\boldsymbol{n}})^*\zeta_1 = 0. \tag{30}$$

or,

$$(\boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}})/\zeta_1 = -(\boldsymbol{u}_2 \cdot \hat{\boldsymbol{n}})^*/\zeta_2^*, \tag{31}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are vectors and $\hat{\boldsymbol{n}}$ is the outward normal vector on the boundary.

If the physical system is such that sea level is specified on the boundary (and the eigenfunctions are zero there), then the adjoint eigenfunctions must have zero sea level there. If instead the normal velocity is zero, the adjoint eigenfunctions must have zero normal velocity there. For other cases where the ratio between the normal velocity and sea level is specified, the adjoint eigenfunctions have the same boundary condition but with opposite sign.

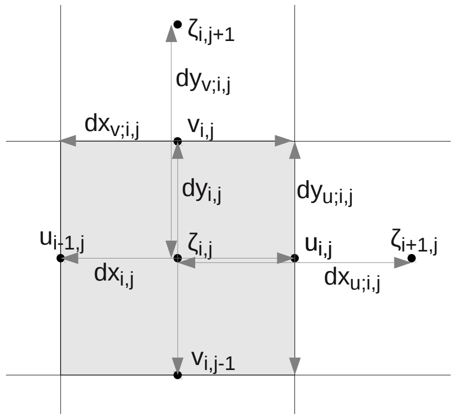

Fig. 1. The Arakawa C-Grid, showing the relative position of sea level grid points (ζ) and the two components of velocity u and v. The shaded area is the 'grid cell' surrounding the point $\zeta_{i.j}$ with width $dx_{i,j}$ and height $dy_{i,j}$. Similar cells surround the points $u_{i,j}$ and $v_{i,j}$ the corresponding dimensions denoted using an extra subscript, i.e. $dx_{u;i,j}$. When the grid is curvilinear, for example in a latitude-longitude grid, the dimensions vary from one grid square to the next.

4 Laplace's Tidal Equations in Finite-Difference Form

Computer models of the tides usually convert Laplace's tidal equations into finite difference form. In such a model, the fields of sea surface height (ssh) and velocity are defined at the vertices of a regular grid. The underlying differential equations are then approximated in the vicinity of each vertex making use of the model variables at neighbouring vertices.

If the system of equations is linear, the resulting set of finite difference equations can be represented as the matrix equation,

$$\mathbf{L}\boldsymbol{z} = \boldsymbol{z}_0. \tag{32}$$

in which the matrix $\mathbf{L}$, representing the finite difference operators, acts on the vector $\boldsymbol{z}$ consisting of all the grid point model variables. Vector $\boldsymbol{z}_0$ represents the forcing.

Arakawa (1966) investigated the properties of a number of finite difference grids when used to solve a form of Laplace's tidal equations on a flat surface with constant depth and constant Coriolis term. On the basis of his results, most tidal models use the Arakawa-C grid shown in fig 1. This was the scheme used by Webb (2013a,b) and is the scheme analysed in the rest of this paper.

Let x and y represent the model's horizontal axes, usually east and north. Let ζ be sea surface height and u and v the velocity in the x and y directions. Individual vertices are denoted by the indices i and j, where i increases in the x direction and j in the y direction. Thus $\zeta_{i,j}$ represents the sea surface height at vertex (i,j). All the model variables together form the single vector z of Eqn.

32.

For simplicity, first consider the case where where the Coriolis term is zero. To ensure conservation of volume, the equation including $\partial\zeta/\partial t$ has to be written in volume conserving form. The finite difference equations for points $u_{i,j}$, $v_{i,j}$ and $\zeta_{i,j}$ lying within the model region are then,

$$\begin{aligned}
\frac{\partial}{\partial t}u_{i,j}+(\kappa/H_{u,i,j})u_{i,j}+(g/dx_{u;i,j})(\zeta_{i+1,j}-\zeta_{i,j}) &= 0, \\
\frac{\partial}{\partial t}v_{i,j}+(\kappa/H_{v,i,j})v_{i,j}+(g/dy_{v;i,j})(\zeta_{i,j+1}-\zeta_{i,j}) &= 0. \\
\frac{\partial}{\partial t}\zeta_{i,j}+(dy_{u;i,j}H_{u,i,j}u_{i,j}-dy_{u;i-1,j}H_{u,i-1,j}u_{i-1,j})/(dx_{i,j}dy_{i,j}) & \\
+(dy_{v;i,j}H_{v,i,j}v_{i,j}-dy_{v;i,j-1}H_{v,i,j-1}v_{i,j-1})/(dx_{i,j}dy_{i,j}) &= 0,
\end{aligned} \tag{33}$$

Here $H_{u,i,j}$ and $H_{v,i,j}$ are the depths and $dx_{u;i,j}$ etc. are the grid spacings at the $u_{i,j}$ and $v_{i,j}$ grid vertices. For later use it is useful to define the areas A and volumes V of the cells,

$$\begin{aligned}
A_{i,j} &= dx_{i,j}dy_{i,j}, & V_{i,j} &= dx_{i,j}dy_{i,j}H_{u,i,j}, \\
A_{u;i,j} &= dx_{u;i,j}dy_{u;i,j}, & V_{u;i,j} &= dx_{u;i,j}dy_{u;i,j}H_{u,i,j}, \\
A_{v;i,j} &= dx_{v;i,j}dy_{v;i,j}, & V_{v;i,j} &= dx_{v;i,j}dy_{v;i,j}H_{v,i,j}.
\end{aligned} \tag{34}$$

If the solution z' of the adjoint operator is represented by u', v' and ζ', the finite difference equations for the Hermitian adjoint operator $\widetilde{\mathcal{K}}$ are,

$$\begin{aligned}
\frac{\partial}{\partial t}u'_{i,j}-(\kappa/H_{u,i,j})u'_{i,j}+(g/dx_{u;i,j})(\zeta'_{i+1,j}-\zeta'_{i,j}) &= 0, \\
\frac{\partial}{\partial t}v'_{i,j}-(\kappa/H_{v,i,j})v'_{i,j}+(g/dy_{v;i,j})(\zeta'_{i,j+1}-\zeta'_{i,j}) &= 0. \\
\frac{\partial}{\partial t}\zeta'_{i,j}+(dy_{u;i,j}H_{u,i,j}u'_{i,j}-dy_{u;i-1,j}H_{u,i-1,j}u'_{i-1,j})/(dx_{i,j}dy_{i,j}) & \\
+(dy_{v;i,j}H_{v,i,j}v'_{i,j}-dy_{v;i,j-1}H_{v,i,j-1}v'_{i,j-1})/(dx_{i,j}dy_{i,j}) &= 0,
\end{aligned} \tag{35}$$

As the grid spacing may not be constant, the finite difference form of the weighting function also needs to include the area of each cell. Thus, within the weighting matrix $\mathbf{W}$, the rows and columns corresponding the variables $u_{i,j}$, $v_{i,j}$ and $\zeta_{i,j}$, are

$$\mathbf{W}=\begin{pmatrix} V_{u;i,j} & 0. & 0. \\ 0. & V_{v;i,j} & 0. \\ 0. & 0. & g\,A_{i,j} \end{pmatrix}, \tag{36}$$

Let $\mathbf{K}$ represent the matrix containing the complete set of finite difference equations corresponding to $\mathcal{K}$. This includes the above equations plus those representing boundary conditions. Let $\widetilde{\mathbf{K}}$ represent the similar matrix corresponding to $\widetilde{\mathcal{K}}$. Although $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are adjoints of each other, $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ may not be matrix adjoints (discussed in Appendix 1).

The equivalent of Eqn. 3 is then,

$$z'^{*}\mathbf{W}\mathbf{K}z\;-\;(z^{*}\mathbf{W}\widetilde{\mathbf{K}}z')^{*}=\mathbf{M}_{\zeta}+\mathbf{M}_{u}+\mathbf{M}_{v}, \tag{37}$$

where,

$$\mathbf{M}_\zeta = \sum_{i,j} \frac{g}{2}\Big[A_{i,j} \frac{\partial}{\partial t}(\zeta'^{*}_{i,j}\zeta_{i,j}) \tag{38}$$

$$+\zeta'^{*}_{i,j}(dy_{u;i,j}H_{u,i,j}u_{i,j} - dy_{u;i-1,j}H_{u,i-1,j}u_{i-1,j})$$

$$+\zeta'^{*}_{i,j}(dx_{v;i,j}H_{v,i,j}v_{i,j} - dx_{v;i,j-1}H_{v,i,j-1}v_{i,j-1})$$

$$+\zeta_{i,j}(dy_{u;i,j}H_{u,i,j}u'^{*}_{i,j} - dy_{u;i-1,j}H_{u,i-1,j}u'^{*}_{i-1,j})$$

$$+\zeta_{i,j}(dx_{v;i,j}H_{v,i,j}v'^{*}_{i,j} - dx_{v;i,j-1}H_{v,i,j-1}v'^{*}_{i,j-1})\Big],$$

$$\mathbf{M}_u = \sum_{i,j} \frac{H_{u,i,j}}{2}\Big[A_{u;i,j}\frac{\partial}{\partial t}(u'^{*}_{i,j}u_{i,j}) \tag{39}$$

$$+g\,dy_{u;i,j}u'^{*}_{i,j}(\zeta_{i+1,j} - \zeta_{i,j}) + g\,dy_{u;i,j}u_{i,j}(\zeta'_{i+1,j} - \zeta'_{i,j})\Big],$$

$$\mathbf{M}_v = \sum_{i,j} \frac{H_{v,i,j}}{2}\Big[A_{v;i,j}\frac{\partial}{\partial t}(v'^{*}_{i,j}v_{i,j}) \tag{40}$$

$$+g\,dx_{v;i,j}v'^{*}_{i,j}(\zeta_{i,j+1} - \zeta_{i,j}) + g\,dx_{v;i,j}v_{i,j}(\zeta'_{i,j+1} - \zeta'_{i,j})\Big],$$

and the summations are over all the corresponding ζ, u or v vertices within the model region.

If a sea surface height point and an adjacent velocity point both lie with the model region then the corresponding coloured terms in the above equations cancel and all that is left is the change of the system's energy with time. This has the right form for the bilinear concomitant. However if one of the two points lies on the boundary, then there is no cancellation and the result depends on the boundary condition and the way it is implemented.

5 Boundaries

When setting up the finite-difference equations there are two ways of handling the boundary conditions. In the first, variables on the boundary are included in the model vector and for each of these a finite difference equations is added which represents the boundary condition at the boundary point.

In the second method, the boundary condition is treated implicitly. Boundary values are not included in the model vector and instead the boundary condition is used to modify the standard finite difference equation for points adjacent to the boundary.

5.1 Coastlines

At coastlines, unless special provision is made for wetting and drying, it is usual to assume that the velocity normal to the boundary is zero. The simplest way to implement this is to approximate coastlines by lines running along the edges of a set of ζ grid squares.

If the grid box surrounding $\zeta_{i,j}$ has a coastline running through the point $u_{i,j}$, then in the first method, $u_{i,j}$ is treated as one of the model variables and the following equation is added to the model,

$$u_{i,j} = 0. \tag{41}$$

For the adjoint system it is tempting to use the same approach, i.e.

$$u'_{i,j} = 0. \tag{42}$$

To see if this is valid, let the variable $u_{i,j}$ be the last variable of the array z, i.e. $u_{i,j} = z_n$, and let the neighbouring height point be the $(n-1)$th term, so $\zeta_{i,j} = z_{n-1}$. The term $\mathbf{K}_{n,n}$ then equals one and $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ have the form,

$$\mathbf{K} = \begin{pmatrix} \dots & \dots & \dots & 0. \\ \dots & \dots & \dots & 0. \\ \dots & \dots & \partial/\partial t & H_{u;i,j}/dx' \\ 0. & 0. & 0. & 1.0 \end{pmatrix}, \tag{43}$$

$$\widetilde{\mathbf{K}} = \begin{pmatrix} \dots & \dots & \dots & 0. \\ \dots & \dots & \dots & 0. \\ \dots & \dots & \partial/\partial t & H_{u;i,j}/dx' \\ 0. & 0. & 0. & 1.0 \end{pmatrix},$$

where H is the depth of cell $u_{i,j}$ and dx' represents the term $dx_{i,j}dy_{i,j}/dy_{u;i,j}$ of Eqn. 33. Substituting into Eqn. 37, the terms involving z_n and z'_n are,

$$\begin{aligned} z'^*_{n-1}(H_{u;i,j}/dx')z_n + z'^*_n z_n - (z^*_{n-1}(\ H_{u;i,j}\ /dx')z'_n + z^*_n z'_n)^* \\ = \quad (\ H_{u;i,j}\ /dx')(z'^*_{n-1}z_n - z'^*_n z_{n-1}). \end{aligned} \tag{44}$$

Because of Eqns.41 and 42, the right hand side is zero for all solutions $\boldsymbol{z}$ and $\boldsymbol{z}'$. As a result the adjoint operator condition appears satisfied. However the matrices $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ are not the simple matrix adjoints of each other.

For the time independent form of the equations (i.e. Eqn. 17) there has to be a diagonal term proportional to $(-i\omega)$. This can be done by writing the boundary condition as

$$-i\omega u_{i,j} = 0. \tag{45}$$

Again the matrix $\mathbf{L}$ and its adjoint satisfy Eqn. 37 but the terms of row n of matrix $\mathbf{L}$ are now all zero. As a result both matrices now have zero determinant.

5.1.1 Implicit Method

The second, implicit, method drops the point $u_{i,j}$ from the set of model variables. As u is zero on the boundary, the term $-g/dx$ in Eqn. 33 and its adjoint make no contribution. The form of the matrices $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ are then the same as in Eqn. 36 except that the row and column corresponding to $u_{i,j}$ are missing.

The new $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ satisfy the operator adjoint equation but this time they are also the simple matrix adjoints of each other. This is also true for finite-difference matrices $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ representing the time independent operators. Computationally the implicit method is also preferred because it reduces both the storage required for the model matrix and the number of computer operations required to solve the equations.

5.2 Open Boundaries

If the model is being forced by sea level variations on the open boundary, then the system of equations is simplest if the open boundary is defined to follow a line of sea level vertices.

In the first, explicit, method of defining the boundary conditions, the boundary values are retained and for each variable $\zeta_{i.j}$ the following equation added,

$$\zeta_{i,j} = \zeta_b, \tag{46}$$

or for the time independent case

$$-i\omega\zeta_{i,j} = -i\omega\zeta_b, \tag{47}$$

where ζ_b is the value imposed at the boundary.

The analysis of the adjoint properties proceeds as in the previous section but this time the right hand side of Eqn. 37 can be non-zero for realistic solutions of the equations. In principal the eigenfunctions of the time independent equation are always zero on the boundary, in these cases saving Eqn. 37, but there is still the problem that the determinant of matrix $\mathbf{K}$ is zero.

In the second method of handling the boundary conditions, the boundary value is treated implicitly by moving the terms involving the boundary value of sea surface height to the right hand side of the equations. Thus if velocity point $u_{i,j}$ has an open boundary at the adjacent point at $\zeta_{i+1.j}$ with value ζ_0, the correspond part of eqn. 33 becomes,

$$\frac{\partial}{\partial t} u_{i,j} + (\kappa/H_{u,i,j})\, u_{i,j} - (g/dx_{u;i,j})\, \zeta_{i-1,j} = -(g/dx_{u;i,j})\, \zeta_0. \tag{48}$$

As before this essentially solves all the problems. The operators $\mathbf{K}$, $\widetilde{\mathbf{K}}$, $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ all satisfy the adjoint equation and, as well as having normal determinants, they are all normal matrix adjoints of each other.

5.3 Open Boundary with Radiation

Another common situation is a system forced at the open boundary by a specified incoming wave but which also allows the model to loose energy via an outgoing wave. For time dependent tidal models this is usually implemented by using the scheme proposed by Flather (1976). In this the normal velocity at the boundary is given by,

$$u = u_{\mathcal{I}} + (\zeta - \zeta_{\mathcal{I}})c/H, \tag{49}$$

where ζ is the sea level on the boundary, u is the velocity normal to the boundary and $\zeta_{\mathcal{I}}$ and $u_{\mathcal{I}}$ are the corresponding values for the incoming forced wave. c is the gravity wave speed, equal to $(gH)^{1/2}$, where H is depth.

We consider here the case of a regular grid where all variables have a time dependence of the form $\exp(-i\omega t)$. At the open boundary the solution is represented as the sum of incoming and outgoing waves propagating normal to the boundary,

$$\zeta = (A\exp(ikx) + B\exp(-ikx))exp(-i\omega t), \tag{50}$$
$$u = (A\exp(ikx) - B\exp(-ikx))(c/h)exp(-i\omega t).$$

where A and B are the amplitudes of the incident and reflected waves, x is the coordinate normal to the boundary and k, the wavenumber in deep water, equals ω/c.

In terms of the Arakawa-C grid (see Fig. 1), consider the case where $\zeta_{i,j}$ lies on the open boundary at the point where x equals zero. Assume also that $u_{i,j}$ lies in the positive x direction and that this time it is within the model domain. If the grid spacing in the x direction is δx, then,

$$\zeta_{i,j} = (A+B)exp(-i\omega t), \tag{51}$$
$$u_{i,j} = (A\Delta - B\Delta^{-1})(c/h)exp(-i\omega t).$$

where Δ equals $\exp(i\omega\delta x/(2c))$. After eliminating B, the outgoing wave, and dropping the time dependent terms,

$$\zeta_{i,j} + (H/c)u_{i,j} = A(1+\Delta^2). \tag{52}$$

This equation can be used as before to explicitly relate the value of ζ on the boundary to the neighbouring velocity u or, implicitly, to replace the value on ζ on the boundary in the finite difference equation for the adjacent velocity cell. However because Δ is a function of ω, it means that in the time independent form of the tidal equations, the matrix $\mathbf{L}$ itself becomes a function of ω and this complicates the analysis.

The alternative is to ignore the term δx in the definition of Δ. This gives,

$$\zeta_{i,j} + (H/c)u_{i,j} = 2A. \tag{53}$$

The operator $\mathbf{L}$ is now independent of ω. The change was investigated by Webb (2013b) for the Hudson Bay region. In that case the effect was found to be small.

5.4 Adjoint Properties of the Radiational Boundary Condition

This time let $\zeta_{i,j}$ be the last element of the array z, i.e. $\zeta_{i,j} = z_n$, and let the neighbouring velocity point be the $(n-1)$th element, so $u_{i-1,j} = z_{n-1}$. Also let the depth be H at both points.

Using the explicit form of the boundary condition, the matrices $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ are,

$$\mathbf{K} = \begin{pmatrix} \dots & \dots & \dots & 0. \\ \dots & \dots & \dots & 0. \\ \dots & \dots & \partial/\partial t + \kappa/H & (g/dx) \\ 0. & 0. & H/c & 1.0 \end{pmatrix}, \tag{54}$$

$$\widetilde{\mathbf{K}} = \begin{pmatrix} \dots & \dots & \dots & 0. \\ \dots & \dots & \dots & 0. \\ \dots & \dots & \partial/\partial t - \kappa/H & (g/dx) \\ 0. & 0. & H/c & 1.0 \end{pmatrix},$$

where here dx refers to $dx_{u;i-1,j}$.

Substituting into Eqn. 37 the terms involving z_n and z'_n are now,

$$\begin{aligned} &z'^*_{n-1}(g/dx)z_n + z'^*_n(H/c)z_{n-1} + z'^*_n z_n - (z^*_{n-1}(g/dx)z'_n + z^*_n(H/c)z'^*_{n-1} + z^*_n z'_n)^* \\ &\qquad = (g/dx)(z'^*_{n-1}z_n - z'^*_n z_{n-1}) + (H/c)(z'^*_n z_{n-1} - z^*_n z'_{n-1}). \end{aligned} \tag{55}$$

With a radiational boundary condition, ζ and ζ' will always be non zero somewhere on the boundary. As a result in this case the right hand side is never zero and so the adjoint operator condition is not satisfied. Both $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ have simple matrix adjoints satisfy the matrix adjoint equation but the physics behind these adjoint matrices is not obvious.

With the implicit form of the boundary condition the row and columns corresponding the values of ζ on the boundary are missing. If the neighbouring velocity point is now z_n, then

$$\begin{aligned} \mathbf{K}_{n,n} &= \partial/\partial t + (\kappa/H) + (g/dx)(H/c), \\ \widetilde{\mathbf{K}}_{n,n} &= -\partial/\partial t + (\kappa/H) + (g/dx)(H/c). \end{aligned} \tag{56}$$

The bilinear concomitant, the r.h.s of Eqn. 37 is then zero, so the matrices are the true adjoints of each other.

For the case, where the time dependence is of the form $\exp(-i\omega t)$, then the corresponding terms of the matrices $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ are,

$$\begin{aligned} \mathbf{L}_{n,n} &= +(\kappa/H) + (g/dx)(H/c), \\ \widetilde{\mathbf{L}}_{n,n} &= +(\kappa/H) + (g/dx)(H/c). \end{aligned} \tag{57}$$

The bilinear concomitant is zero so again the matrices of the adjoint operators are themselves the adjoint of each other.

6 The Coriolis Term

When using the Arakawa-C grid, the Coriolis term at 'u' velocity point $u_{i,j}$ depends on the 'v' velocities at points $v_{i,j}$, $v_{i,j-1}$, $v_{i+1,j-1}$ and $v_{i+1,j}$. The Coriolis term at 'v' points similarly depends on four adjacent 'u' points.

If $(fv)_{u;i,j}$ is the Coriolis term at point $u_{i,j}$, then the simplest symmetric approximation is,

$$(fv)_{u;i,j} = f(v_{i,j} + v_{i,j-1} + v_{i+1,j-1} + v_{i+1,j})/4, \tag{58}$$

with a corresponding term for $v_{i,j}$.

The velocity components of eqn. 33 then become

$$\begin{aligned}\frac{\partial}{\partial t}u_{i,j} + (\kappa/H_{u,i,j})u_{i,j} + (f/4)(v_{i,j} + v_{i,j-1} + v_{i+1,j-1} + v_{i+1,j})& \\ + (g/dx)(\zeta_{i+1,j} - \zeta_{i,j}) &= 0, \quad (59)\\ \frac{\partial}{\partial t}v_{i,j} + (\kappa/H_{v,i,j})v_{i,j} + (f/4)(u_{i,j} + u_{i,j+1} + u_{i-1,j+1} + u_{i-1,j})& \\ + (g/dy)(\zeta_{i,j+1} - \zeta_{i,j}) &= 0, \quad (60)\end{aligned}$$

and the corresponding adjoint equations are,

$$\begin{aligned}\frac{\partial}{\partial t}u'_{i,j} - (\kappa/H_{u,i,j})u'_{i,j} + (f/4)(v'_{i,j} + v'_{i,j-1} + v'_{i+1,j-1} + v'_{i+1,j})& \\ + (g/dx)(\zeta'_{i+1,j} - \zeta'_{i,j}) &= 0, \quad (61)\\ \frac{\partial}{\partial t}v'_{i,j} - (\kappa/H_{v,i,j})v'_{i,j} + (f/4)(u'_{i,j} + u'_{i,j+1} + u'_{i-1,j+1} + u'_{i-1,j})& \\ + (g/dy)(\zeta'_{i,j+1} - \zeta'_{i,j}) &= 0. \quad (62)\end{aligned}$$

Eqn. 37 becomes,

$$z'^{*}\mathbf{W}\mathbf{K}z \;-\; (z^{*}\mathbf{W}\widetilde{\mathbf{K}}z')^{*} = \mathbf{M}_\zeta + \mathbf{M}_u + \mathbf{M}_v + \mathbf{M}_f, \tag{63}$$

where, if $f_{u,i,j}$ is the Coriolis factor for cell $u_{i,j}$,

$$\begin{aligned}\mathbf{M}_f = &-\sum_{i,j} f_{u;i,j} V_{u;i,j}\, u'^{*}_{i,j}(v_{i,j} + v_{i,j-1} + v_{i+1,j-1} + v_{i+1,j})/4 \quad (64)\\ &+\sum_{i,j} f_{u;i,j} V_{v;i,j}\, v'^{*}_{i,j}(u_{i,j} + u_{i,j+1} + u_{i-1,j+1} + u_{i-1,j})/4\\ &-\sum_{i,j} f_{u;i,j} V_{u;i,j}\, u_{i,j}(v'^{*}_{i,j} + v'^{*}_{i,j-1} + v'^{*}_{i+1,j-1} + v'^{*}_{i+1,j})/4\\ &+\sum_{i,j} f_{u;i,j} V_{v;i,j}\, v_{i,j}(u'^{*}_{i,j} + u'^{*}_{i,j+1} + u'^{*}_{i-1,j+1} + u'^{*}_{i-1,j})/4.\end{aligned}$$

If the Coriolis terms all have the value f_0 and the cell volumes are all equal to V_0 then,

$$\begin{aligned}\mathbf{M}_f = &-\sum_{i,j} f_0 V_0\, u'^{*}_{i,j}(v_{i,j} + v_{i,j-1} + v_{i+1,j-1} + v_{i+1,j})/4 \quad (65)\\ &+\sum_{i,j} f_0 V_0\, v'^{*}_{i,j}(u_{i,j} + u_{i,j+1} + u_{i-1,j+1} + u_{i-1,j})/4\\ &-\sum_{i,j} f_0 V_0\, u_{i,j}(v'^{*}_{i,j} + v'^{*}_{i,j-1} + v'^{*}_{i+1,j-1} + v'^{*}_{i+1,j})/4\\ &+\sum_{i,j} f_0 V_0\, v_{i,j}(u'^{*}_{i,j} + u'^{*}_{i,j+1} + u'^{*}_{i-1,j+1} + u'^{*}_{i-1,j})/4.\end{aligned}$$

and, as before, when points $u_{i,j}$ and $v_{i,j}$ both lie with the model domain, the red terms cancel. Similarly for the other colours.

If points $v_{i,j}$ and $v_{i,j-1}$ lie within the model region but $u_{i,j}$ lies on the boundary there is a residual term,

$$(f_0 H_0/4)(-v'^*_{i,j} u_{i,j} + v_{i,j} u'^*_{i,j} - v'^*_{i,j-1} u_{i,j} + v_{i,j-1} u'^*_{i,j}). \tag{66}$$

For the system to be self adjoint, the residual needs to be zero. At a coastline defined by velocity points, $u_{i,j}$ will be zero and so the residual vanishes.

At an open boundary defined by sea surface height points, $u_{i,j}$ is the velocity along the boundary. If it is set to zero then the residue $\mathbf{M}_f$ vanishes and the finite difference equations corresponding the Laplace's tidal equations and its adjoint are themselves the adjoint of each other.

6.1 Varying Depth and Coriolis Parameter

Espelid et al. (2000) showed that the approximation,

$$(fv)_{u;i,j} = f(v_{i,j} + v_{i,j-1} + v_{i+1,j-1} + v_{i+1,j})/4, \tag{67}$$

only conserves energy when the Coriolis parameter, f, and all the grid box volumes are equal. For the more realistic case where depths and/or grid box areas vary, a weighting has to be introduced which ensures that the work done by velocity $u_{i,j}$ in its own grid box against the Coriolis term involving $v_{i,j}$ exactly balances the work done by velocity $v_{i,j}$ in its own grid box against the Coriolis term involving $u_{i,j}$.

Espelid et al. (2000) and Webb (2013a) propose different methods of ensuring balance. They are both of the form,

$$(fv)_{u;i,j} = \sum_{i',j'} f^u_{i,j;i',j'} v_{i',j'}/4, \tag{68}$$

$$f^u_{i,j;i',j'} = \frac{2 f_{u;i,j;i'j'} f_{v;i',j';i,j} V_{v;i',j'}}{f_{u;i,j;i'j'} V_{u;i,j} + f_{v;i',j';i,j} V_{v;i'j'}}, \tag{69}$$

where the sum is over the v points (i',j') surrounding $u_{i,j}$. Here $f_{u;i,j;i'j'}$, the approximation to the value of f at point $u_{i,j}$ used for the term involving $v_{i'.j'}$. Rewriting Eqn. 64 in this form,

$$\begin{aligned}
\mathbf{M}_f = & -\sum_{i,j} V_{u;i,j}\, u'^*_{i,j} (f^u_{i,j;i,j} v_{i,j} + f^u_{i,j;i,j-1} v_{i,j-1} + f^u_{i,j;i+1,j-1} v_{i+1,j-1} + f^u_{i,j;i+1,j} v_{i+1,j})/4 \\
& + \sum_{i,j} V_{v;i,j}\, v'^*_{i,j} (f^v_{i,j;i,j} u_{i,j} + f^v_{i,j;i,j+1} u_{i,j+1} + f^v_{i,j;i-1,j+1} u_{i-1,j+1} + f^v_{i,j;i-1,j} u_{i-1,j})/4 \\
& - \sum_{i,j} V_{u;i,j}\, u_{i,j} (f^u_{i,j;i,j} v'^*_{i,j} + f^u_{i,j;i,j-1} v'^*_{i,j-1} + f^u_{i,j;i+1,j-1} v'^*_{i+1,j-1} + f^u_{i,j;i+1,j} v'^*_{i+1,j})/4 \\
& + \sum_{i,j} V_{v;i,j}\, v_{i,j} (f^v_{i,j;i,j} u'^*_{i,j} + f^v_{i,j;i,j+1} u'^*_{i,j+1} + f^v_{i,j-1;i,j+1} u'^*_{i-1,j+1} + f^v_{i,j;i-1,j} u'^*_{i-1,j})/4\,.
\end{aligned} \tag{70}$$

The terms of $\mathbf{M}_f$ cancel as before.

7 Summary and Discussion

This note has investigated the adjoint properties of Laplace's tidal equations. It has shown, as reported previously, that if a suitable weighting function is used, the adjoint set of equations have a form similar to the original equations. Physically they represent a tidal wave propagating backwards in time.

For practical solution in a numerical model, Laplace's tidal equations are usually converted into finite difference form on a regular grid. This note investigated the adjoint properties when the finite difference model uses an Arakawa-C grid. This is the grid most often used for tidal studies.

The resulting finite difference equations can be represented by a simple matrix equation of the form,

$$\mathbf{K}\boldsymbol{z} = \boldsymbol{z_0}, \tag{71}$$

where $\mathbf{K}$ is a matrix containing the finite difference coefficients, z a vector containing all the model variables and z_0 the forcing terms. The finite difference form of the adjoint to the tidal equaitons can be represented by a similar matrix $\widetilde{\mathbf{K}}$.

If the variables all have a time dependence $\exp(-i\omega t)$, the system of equations can also be written,

$$(\mathbf{L} - i\omega\mathbf{1})\boldsymbol{z} = \boldsymbol{z_0}, \tag{72}$$

where $\mathbf{1}$ is the unit diagonal matrix. Again the finite difference form of the adjoint set of equasions can be represnted by a matrix $\widetilde{\mathbf{L}}$.

The analysis has shown that, away from the boundaries of the model, the matrices $\mathbf{K}$ and $\widetilde{\mathbf{K}}$, are symmetric such that all but the $\partial/\partial t$ term in the bilinear concomitant cancel. As a result, in a model with cyclic boundary conditions, $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ are the simple matrix adjoints of each other. For the time independent probelem, the bilinear concomitant is zero and the two matrices, $\mathbf{L}$ and $\widetilde{\mathbf{L}}$, are again the simple matrix adjoints of each other.

Boundary conditions can either be included explicitly or implicit. In the explicit form, the model variable on the boundary is included in the model vector z and an additional row added to the matrices $\mathbf{K}$ or $\mathbf{L}$ representing the boundary condition.

In this case $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ or $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ are not simple matrix adjoints of each other. However for closed boundaries and for open boundaries where the tidal height is imposed, the bilinear concomitant is zero for all eigenvalues of the operator $\mathbf{K}$.

The analysis has also shown that, when the implicit form of the boundary conditions is used, such problems do not arise. The matrix $\widetilde{\mathbf{K}}$ is the simple matrix adjoint of $\mathbf{K}$ and the same is true for $\mathbf{L}$ and $\widetilde{\mathbf{L}}$. The result is true for coastlines where the normal velocity is zero and also for open boundaries, where the tidal height is forced and there may or may not be an additional radiation condition.

A final result concerns the radiation condition used for the time independent problem. For this boundary condition, there is an advantage in ignoring the small separation between the open boundary and the first row of velocity points. The approximation introduces a small error, but it means that both the matrix operator $\mathbf{L}$ and its adjoint remain independent of angular velocity.

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Appendix A The Adjoint of a Matrix

Let $\mathbf{M}$ represent a N by N matrix with complex elements $\mathbf{M}_{n,m}$. The adjoint of $\mathbf{M}$, also known as the conjugate transpose, Hermitian transpose and Hermitian conjugate is the matrix $\mathbf{M}^*$ whose elements are defined by the equation,

$$(\mathbf{M}^*)_{n,m} = (\mathbf{M})^*_{m,n}, \tag{A1}$$

where $(\mathbf{M})_{m,n}$ represents element (m,n) of matrix $\mathbf{M}$ and $'*'$ attached to a scalar represents its complex conjugate. Thus for two vectors ψ and ϕ,

$$(\boldsymbol{\phi}^*\mathbf{M}^*\boldsymbol{\psi})^* = \boldsymbol{\psi}^*\mathbf{M}\boldsymbol{\phi} \tag{A2}$$

Many physical systems can be represented by an equation of the form,

$$(\mathbf{M} - \lambda\mathbf{1})\boldsymbol{\phi} = \boldsymbol{z}, \tag{A3}$$

where $\boldsymbol{\phi}$ is a vector representing the state of the system (the state vector) and $\boldsymbol{z}$ is a vector representing the forcing. $\mathbf{1}$ represents the unit diagonal matrix and λ is a scalar. In the time independent form of Laplace's tidal equations, λ is equal to $(-i\omega)$ where i is the complex root of -1 and ω is the angular velocity of the forcing.

One way of solving this equation is in terms of the eigenvalues and eigenvectors of the matrix $\mathbf{M}$. These are given by the solutions of the equation,

$$(\mathbf{M} - \lambda_j\mathbf{1})\boldsymbol{\phi_j} = 0, \tag{A4}$$

where $\boldsymbol{\phi_j}$ is the j'th eigenvector and λ_j the corresponding eigenvalue.

In most cases of interest, a matrix $\mathbf{M}$ has N independent eigenvectors that can be combined to represent any general vector $\boldsymbol{\phi}$. Thus there exist a set of scalars a_j such that,

$$\boldsymbol{\phi} = \sum_{j=1}^{N} a_j\boldsymbol{\phi_j}. \tag{A5}$$

The coefficients a_j are determined using the adjoint matrix $\mathbf{M}^*$. Let $\boldsymbol{\psi}_j$ and μ_j be the eigenvectors and eigenvalues of the equation,

$$(\mathbf{M}^* - \mu_j\mathbf{1})\boldsymbol{\psi_j} = 0. \tag{A6}$$

Taking the dot product of $\boldsymbol{\phi_j}$ with eqn A6 and of $\boldsymbol{\psi_k}$ with eqn A3,

$$\boldsymbol{\phi_j}^*(\mathbf{M}^* - \mu_k \mathbf{1})\boldsymbol{\psi_k} = 0, \tag{A7}$$
$$\boldsymbol{\psi_k}^*(\mathbf{M} - \lambda_j \mathbf{1})\boldsymbol{\phi_j} = 0.$$

Taking the complex conjugate of the first and subtracting,

$$(\boldsymbol{\phi_j}^* \mathbf{M}^* \boldsymbol{\psi_k})^* - \boldsymbol{\psi_k}^* \mathbf{M} \boldsymbol{\phi_j} + \lambda_j \boldsymbol{\psi_k}^* \boldsymbol{\phi_j} - \mu_k^* (\boldsymbol{\phi_j}^* \boldsymbol{\psi_k})^* = 0. \tag{A8}$$

But $(\boldsymbol{\phi_j}^* \boldsymbol{\psi_k})^*$ equals $\boldsymbol{\psi_k}^* \boldsymbol{\phi_j}$ and, from eqn A1, $(\boldsymbol{\phi_j}^* \mathbf{M}^* \boldsymbol{\psi_k})^*$ equals $\boldsymbol{\psi_k}^* \mathbf{M} \boldsymbol{\phi_j}$, so,

$$(\lambda_j - \mu_k^*)(\boldsymbol{\psi_k}^* \boldsymbol{\phi_j}) = 0. \tag{A9}$$

Thus either the eigenvectors $\boldsymbol{\psi_k}$ and $\boldsymbol{\phi_j}$ are orthogonal to each other or λ_j equals μ_k^*.

If the adjoint vector is normalised so that the dot product $(\boldsymbol{\psi_j}^* \boldsymbol{\phi_j})$ equals one, then the sum over terms $(\boldsymbol{\phi_j} \boldsymbol{\psi_j}^*)$ (where $\boldsymbol{\phi_j}$ is a vertical vector and $\boldsymbol{\psi_j}^*$ a horizontal vector) behaves as the unit diagonal matrix. For example, acting on the eigenvector $\boldsymbol{\phi_j}$,

$$\begin{aligned} \sum_{k=1}^{N} (\boldsymbol{\phi_k} \boldsymbol{\psi_k}^*) \boldsymbol{\phi_j} &= \sum_{k=1}^{N} \boldsymbol{\phi_k} \delta_{k,j}, \\ &= \boldsymbol{\phi_j}. \end{aligned} \tag{A10}$$

Here $\delta_{j,k}$ is a delta function, equal to one when j equals k and zero otherwise.

The eigenfunctions can now be used to solve equation A3. Let,

$$\boldsymbol{\phi} = \sum_{j=1}^{N} a_j \boldsymbol{\phi_j}. \tag{A11}$$

Substitute in eqn A3,

$$\begin{aligned} (\mathbf{M} - \lambda \mathbf{1}) \sum_{j=1}^{N} a_j \boldsymbol{\phi_j} &= \boldsymbol{z}, \\ \sum_{j=1}^{N} (\mathbf{M} - \lambda \mathbf{1}) a_j \boldsymbol{\phi_j} &= \boldsymbol{z}, \\ \sum_{j=1}^{N} (\lambda_j - \lambda \mathbf{1}) a_j \boldsymbol{\phi_j} &= \boldsymbol{z}. \end{aligned} \tag{A12}$$

Taking the dot product of both side with the adjoint vector $\boldsymbol{\psi_k}^*$,

$$(\lambda_k - \lambda) a_k = \boldsymbol{\psi_k^*} \boldsymbol{z}. \tag{A13}$$

Solving for a_k and substituting in eqn. A11

$$\boldsymbol{\phi} = \sum_{j=1}^{N} \boldsymbol{\phi_j} (\boldsymbol{\psi_j^*} \boldsymbol{z}) / (\lambda_j - \lambda). \tag{A14}$$

Note that the contribution of each mode to the final solution depends, first, on the dot produce between its adjoint and the forcing vector and, secondly, on the separation between λ and the mode's eigenvalue. The contribution will be largest when the adjoint is a good match to the forcing and the eigenvalue λ_j lies close to the value of λ.

A1 Modified Dot Product

If the components of a vector $\boldsymbol{\phi}$ have different dimensions, for example if some represent sea level and some velocity, it is normal to redefine the dot product between vectors $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ to have the form $\boldsymbol{\psi}^* \mathbf{W} \boldsymbol{\phi}$, where $\mathbf{W}$ is a diagonal matrix and, when the dimensions of components of $\boldsymbol{\psi}$ are the same as those of ϕ, the products $\psi_i^* \mathbf{W}_{i,i} \phi_i$ all have the same dimensions.

In the case where the vector $\boldsymbol{\phi}$ represents the finite difference solution of Laplace's tidal equations this can be done by defining,

$$\begin{aligned} \mathbf{W}_{j,j} &= (\rho g/2) A_j \text{ when } \phi_j \text{ represents sea surface height,} \\ &= (\rho/2) V_j \text{ when } \phi_j \text{ represents velocity.} \end{aligned} \tag{A15}$$

where A_j is the area and V_j the volume of the cell. In this case the dot product has the dimensions of energy.

The definition of the adjoint matrix also changes. For the first two terms in equation A8 to cancel, the revised definition of dot product requires,

$$(\boldsymbol{\phi_j}^* \mathbf{W} \mathbf{M}^* \boldsymbol{\psi_k})^* = \boldsymbol{\psi_k}^* \mathbf{W} \mathbf{M} \boldsymbol{\phi_j}. \tag{A16}$$

Equating the terms on each side proportional to $(\boldsymbol{\psi_k})^*_m$ and $(\boldsymbol{\phi}_j)_n$,

$$((\mathbf{W})_{n,n} ((\mathbf{M}^*)_{n,m})^* = (\mathbf{W})_{m,m} (\mathbf{M})_{m,n}. \tag{A17}$$

Then if the elements of D are real, the elements of the adjoint matrix are given by,

$$(\mathbf{M}^*)_{n,m} = ((\mathbf{M})_{m,n})^* (\mathbf{W})_{m,m} / (\mathbf{W})_{n,n}, \tag{A18}$$

Appendix B The Adjoint of a Linear Differential Operator

Physical systems that involve fields are usually described in terms of linear differential operators acting on continuous functions. Laplace's tidal equations is an example where the partial differential operators $\partial/\partial t$, $\partial/\partial x$ and $\partial/\partial y$ act on continuous fields representing sea surface height ζ and water velocity $\boldsymbol{u}$. Such systems have many properties which are similar to those represented by finite sized matrix equations. However the differences are important.

First the fields have an infinite number of degrees of freedom and this means that they have an infinite number of eigenvalues and eigenfunctions. In systems of finite size, the eigenvalues are usually separate unless some symmetry is involved. [3] If the system is infinite in size then many of the eigenvalues will not be separate but will form a continuum. An example is the case of the modes of an infinitely long canal.

[3] An example of the latter is a square drumhead, where there might be one mode with two nodes in one direction and one in the second direction, and a second mode with the directions reversed, with both modes having the same resonant frequency.

A second important difference is that partial differential equations need boundary conditions. The form of the boundary condition depends on whether the differential equations are hyperbolic, parabolic or elliptic.

In the full time dependent form of the tidal equations, the solution depends on the initial model state but does require the state at the final time to be known beforehand. In this case the differential equations are said to be parabolic.

When studying tidal waves of fixed angular velocity, such as the M2 tide, studies it is normal to assume a fixed time dependence of the form $\exp(-\imath\omega t)$. This removes the time dependent term in the equations but leaves a set of equations whose solution depends on the boundary conditions all around the model region. In this case the differential equations are said to be elliptic.

These changes affect the definition of adjoint. The transfer from a finite to an infinite number of degrees of freedom is relatively straightforward but the boundary conditions introduce an extra level of complication.

B1 The Basic Equation

A detailed derivation of the theory behind adjoint differential operators and their relation to Green's functions is given by Morse and Feshbach (1953). Let $\boldsymbol{\mathcal{A}}$ be a partial differential operator. Then if ψ and ϕ are two function, the equivalent of the adjoint condition (eqn. A2) is to find an operator $\widetilde{\boldsymbol{\mathcal{A}}}$ and a vector $\boldsymbol{P}$ such that,

$$(\psi^* \boldsymbol{\mathcal{A}}\phi)^* - \psi^* \widetilde{\boldsymbol{\mathcal{A}}}^* \phi = \nabla \cdot \boldsymbol{P}(\psi^*, \phi). \tag{B1}$$

$\boldsymbol{P}$ is called the bilinear concomitant, is a generalised vector function of ψ^* and ϕ. The operator $\widetilde{\boldsymbol{\mathcal{A}}}$ is then the adjoint of $\boldsymbol{\mathcal{A}}$.

The adjoint boundary condition is determined by integrating this equation over the physical domain. The terms on the left of the equation cancel. Then, after applying the divergence theorem, the term on the right gives an integral around the boundary between the vector $\boldsymbol{P}$ and and the unit vector normal to the boundary. This then determines the adjoint boundary conditions acting on ϕ corresponding to the normal boundary conditions acting on ψ.

B1.1 The Del-squared Operator in 1-D

As a simple example, consider the del-squared operator in one dimension. If,

$$\boldsymbol{\mathcal{A}}\phi = p\ d^2\phi/dx^2, \tag{B2}$$

then

$$\psi^* \boldsymbol{\mathcal{A}}\phi = \psi^* p\ \frac{d^2}{dx^2}\phi. \tag{B3}$$

Express this as the derivative of the r.h.s. plus a correction,

$$\psi^* \boldsymbol{\mathcal{A}}\phi = \frac{d}{dx}(\psi^* p \frac{d}{dx}\phi) - \frac{d}{dx}(\psi^* p)\frac{d}{dx}\phi \tag{B4}$$

Repeat, taking the derivative of the rightmost term,

$$\psi^* \boldsymbol{\mathcal{A}}\phi = \frac{d}{dx}(\psi^* p \frac{d}{dx}\phi) - \frac{d}{dx}(\frac{d}{dx}(\psi^* p)\phi) + \frac{d^2}{dx^2}(\psi^* p)\phi. \tag{B5}$$

Thus,

$$\psi^* p \ \frac{d^2}{dx^2}\phi - \frac{d^2}{dx^2}(\psi^* p)\phi = \frac{d}{dx}(\psi^* p \frac{d}{dx}\phi - \frac{d}{dx}(\psi^* p)\phi). \tag{B6}$$

In terms of equation B1, this means that the adjoint operator and bilinear concomitant are given by,

$$\widetilde{\boldsymbol{\mathcal{A}}}\psi = \frac{d^2}{dx^2}(p^*\psi), \tag{B7}$$

$$P(\psi,\phi) = \psi^* p \frac{d}{dx}\phi - \frac{d}{dx}(\psi^* p)\phi \tag{B8}$$

and if $P(\phi,\psi)$ is zero on the boundary then on the boundary,

$$\frac{1}{\phi}\frac{d}{dx}(\phi) = \frac{1}{p\psi^*}\frac{d}{dx}(p\psi^*). \tag{B9}$$

Thus if the main problem uses Drichlet conditions for which ϕ is zero on the boundary, then eqn. B9 requires $p^*\psi$ to have the same boundary conditions. Similarly if ϕ has Neumann or mixed boundary conditions, then $p^*\psi$ will again have the same boundary conditions.

B1.2 Results for 1-D and 2-D

The above derivation is readily extended to higher order derivatives. In general if $\boldsymbol{\mathcal{A}}$ is the operator,

$$\boldsymbol{\mathcal{A}}\phi = p \ d^n\phi/dx^n, \tag{B10}$$

then,

$$\widetilde{\boldsymbol{\mathcal{A}}}\psi = (-1)^n d^n(p\psi)/dx^n, \tag{B11}$$

and,

$$P(\psi,\phi) = p\psi^* d^{n-1}\psi/dx^{n-1} - d(p\psi^*)/dx \ d^{n-2}\psi^*/dx^{n-2} + \cdots \ (-1)^{n-1} d^{n-1}(p\psi^*)/dx^{n-1}\phi. \tag{B12}$$

Any other differential operator in one dimension can be made up of a sum of such terms.

In two dimensions, the most general differential operator has the form

$$\boldsymbol{\mathcal{A}}\phi = p(x_1,x_2)\frac{d^{n+m}}{dx_1^n dx_2^m}\phi. \tag{B13}$$

This has the adjoint and bilinear concomitant,

$$\widetilde{\boldsymbol{\mathcal{A}}}\psi = (-1)^{n+m}\frac{d^{n+m}}{dx_1^n dx_2^m} p(x_1,x_2)\psi. \tag{B14}$$

$$P(\psi,\phi) = \boldsymbol{a_1} \ [p\psi^* \frac{d^{n+m-1}}{dx_1^{n-1}dx_2^m}\phi - \frac{d}{dx_1}(p\psi^*)\frac{d^{n+m-2}}{dx_1^{n-2}dx_2^m}\phi + ... + (-1)^{n-1}\frac{d^{n-1}}{dx_1^{n-1}}(p\psi^*)\frac{d^m}{dx_2^m}\phi]$$
$$+(-1)^n \ \boldsymbol{a_2} \ [\frac{d^n}{dx_1^n}(p\psi^*)\frac{d^{m-1}}{dx_2^{m-1}}\phi - \frac{d^{n+1}}{dx_1^n dx_2}(p\psi^*)\frac{d^{m-2}}{dx_2^{m-2}}\phi + ... + (-1)^{m-1}\frac{d^{n+m-1}}{dx_1^n dx_2^{m-1}}(p\psi^*) \ \phi].$$

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