Discussion Papers in
Accounting and Management Science

The Duration Derby
A Comparison of Duration Based Strategies in Asset Liability Management

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April 2001

Number 01-176
ISSN 1356-3548

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1 Introduction

Duration has proved a successful approach to making a rough assessment of the effect of interest rate changes on single bonds and portfolios of bonds (Bierwag 1987, Bierwag et al 1990). If one only could use two numbers to describe the characteristics of a bond the obvious ones are its price and its duration. Duration has also proved effective in matching asset portfolios and liability portfolios by matching their durations, though recent developments in decomposition and sampling aspects of stochastic programming means that this more precise approach is becoming more viable for realistic problems (Birge and Louveaux, 1998)

However there are difficulties with the original Macaulay duration approach. It requires that the yield curve for the bond is flat even though the gilt market is usually suggesting something different and it does not deal with default risk explicitly. This paper reviews both these issues. In the case if default risk it shows how the idea of using the credit ratings of bonds produced by the credit rating agencies to be a measure of the default risk leads to an easy modification of the definition of duration which allows for default risk. In the case of non-flat yield curves we give a general description of the process undertaken which includes almost all of the specific forms of partial duration that have been introduced into the literature. We also introduce the idea of an approximate duration which is closer to the Macaulay duration idea of one number that describes the properties of an asset, liability or portfolio of such. Unlike the Macaulay duration though this can be thought of as the median of the cash flow of the bond rather than the mean and hence cannot be obtained for a portfolio of bonds directly form the durations of the separate bonds. However we are to describes an integer programming method of calculating this duration measure.

The effectiveness of these duration measures is investigated by describing a simulation experiment using real data to see how well these duration measures choose a portfolio of assets to match a given cash flow of liabilities. We compare five duration measures in this experiment. The first is the Macaulay duration. Two are partial durations, - one applied to a continuously compounded model with a given form of the yield curve and the other to a forward rate based discretely compounded model. The other two durations are the approximate duration applied in these two cases.

2.Risky-debt duration

One way of thinking about the Macaulay duration of a bond is to ask what is the maturity of a zero-coupon risk free bond which has the same value and the same response to a small change in interest rates. Thus if a bond has an income stream $c(t)$, $t=1,\ldots,T$, to its maturity at T, and r

is the implied interest rate or yield to maturity of the bond, the value of the bond V(r) satisfies

$$V(r) = V = \sum_t c(t) /(1+r)^t = \sum_t c(t) (1+r)^{-t} \qquad (1)$$

If one matches this by a zero-coupon bond which pays out R at time D so that its value $V_0(r) = R/(1+r)^D$ one would require

$$V(r) = V_0(r) \text{ and } dV(r)/dr = dV_0(r)/dr = -DR/(1+r)^{D+1} = -DV(r)/(1+r) \qquad (2)$$

This leads to the standard definition of Macaulay duration that

$$D = \sum_t tc(t) (1+r)^{-t} /(\sum_t c(t) (1+r)^{-t}) = \sum_t tc(t) (1+r)^{-t}/ V(r). \qquad (3)$$

A modified form of Macaulay's duration measure, the modified duration D*, provides an exact expression for the percentage in bond values resulting from a small change in the yield r

$$D^* = -(dV_0(r)/dr)/V_0(r) = D/(1+r) \qquad (4)$$

One of the difficulties with this derivation is that the same interest rate r is being used for the two types of bonds being matched and it is not clear that they have the same credit risk. The difficulty arises because there is nothing explicit in this definition which allows for the default risk that the bond might default before it is due to be paid.

The models of default risk of bonds divide into three types. see Duffee (1996). The first type of model views the firm's liabilities as contingent claims against the underlying assets and assumes that bankruptcy and bond non-payment occurs when the firm's assets are exhausted. This was the model introduced by Merton (1974), but it leads to smaller credit spreads than those that actually occur. Black and Cox (1976) adjusted the model by defining bankruptcy to occur where liabilities are some fixed proportion of the assets and this leads to more realistic credit spreads. Leland (1994) and Leland and Toft (1996) used this type of model, but with endogoneous conditions to define when bankruptcy is declared, to examine how important is the maturity of the debt as well as the amount. The difficulty with this approach is that it depends on knowledge of the firm's assets, which are not tradable and are only partially observable. Also it has to deal with the often complex priority structure of a firm's liabilities. The second type of model first developed by Hull and White (1991) assumes that on bankruptcy, the firm will pay off a pre-specified fraction of the risk-free value of the instrument where bankruptcy is again triggered when the firm's assets first reach some specified limit. This enables one to ignore the debt priority problem but still assumes knowledge of the asset value stochastic process.

The third approach ignores the asset value completely and again overcomes the debt liability structure by assuming that on bankruptcy a given fraction of each promised dollar is paid off. This approach assumes the bankruptcy process is specified exogenously and does not depend on the firm's underlying assets. (e.g. Jarrow and Turnbull (1995), Litterman and Iben (1991)). Lando's thesis (1994) and the paper by Jarrow et al (1997) were the first to use a Markov chain evolution of the firm's credit rating as a model for the bankruptcy process. This is the approach that motivates this generalisation of duration.

The bond has a credit rating of between 0 (default free) to M (defaulted). These correspond to the Moody or Standard and Poor's ratings. There is a transition matrix P(t) whose i,j th entry $p(t)_{ij}$ is the probability that the bond's rating will change from i to j during period t. Define

$$Q(t) = \prod_{s=1}^{t} P(s)$$

and let $q(t)_{iM}$ be the corresponding i,M th entry. Then let $d_i(t) = 1 - q(t)_{iM}$, is the probability that the bond has not defaulted by time t. The choice of the matrices P(t) could be done using historic information or including various adjustments (see Jarrow et al (1997), Thomas et al (1998), Kijima and Komoribayashi (1998)) to the historic transitions to reflect possible views of the future. If this credit risk factor is introduced into the formula for the value of the bond then the interest rate r now considered should be the risk free interest rate.

It is possible that even if the bond defaults some proportion of the future income stream may be returned to the bond holder. Fooladi, Roberts and Skinner [ ] investigated how this can be introduced into the standard form of the bond price. We assume that if the default occurs at time t then only a fraction $\alpha(t)$ of the value of the remaining income stream of the bond will be returned to the bond holders. This would give the value of a risky bond with credit rating i where the risk free interest rate is r as

$$V(r,i) = \sum_{t=1}^{T} c(t) d_i(t) / (1+r)^t + \sum_{t=1}^{T} (d_i(t-1) - d_i(t)) \alpha(t) \sum_{s=t}^{T} c(s) \Big/ \; (1+r)^s$$
$$= \sum_{t=1}^{\mathrm{T}} \frac{c(t)}{(1+r)^t} (d_i(t) + \sum_{s=1}^{T} \alpha(s)(d_i(s-1) - d_i(s)) \qquad (5)$$

Using the same argument that led to (3), the duration D(i) for such a bond would be the maturity D of a risk free bond which pays out R at its maturity so that its initial value and the effect of small changes in the risk free rate are the same for both bonds. This requires that we define $V_0(r) = R/(1+r)^D$ and require

$$V(r,i) = V_0(r) \text{ and } dV(r,i)/dr = dV_0(r)/dr = -DV_0(r)/(1+r) = -DV(r,i)/(1+r) \qquad (6)$$

So

$$D(i) = \left(\sum_{t=1}^{\mathrm{T}} \frac{c(t)}{(1+r)^t} (d_i(t) + \sum_{s=1}^{T} \alpha(s)(d_i(s-1) - d_i(s)) \right) \Big/ \mathrm{V(r,i)} \qquad (7)$$

In this case when $\alpha(t) = 0$ for all t and $d_i(t)$ is non-increasing in i, the definition puts more weight on the earlier cost terms as i increases and hence the duration will be lower.
lower. This confirms Fons (1990) empirical work that the effective duration of corporate bonds is always less than their Macaulay duration.

If this analysis is extended to the duration of a portfolio of bonds, the duration of the portfolio is the weighted average of the individual durations as follows. Let the portfolio consist of x_j units of bond j, j=1,….M where bond j has income stream $c_j(t)$ and rating i(j). Then the value of the portfolio is

$$V^P(r) = \sum_j x_j V_j(r, i(j)) = \sum_j \mathrm{x}_j \left(\sum_{t=1}^{\mathrm{T}} \frac{c_j(t)}{(1+r)^t} (d_{i(j)}(t) + \sum_{s=1}^{T} \alpha(s)(d_{i(j)}(s-1) - d_{i(j)}(s)) \right) \qquad (8)$$

Using the definition of duration given in (7) this gives that the duration of the portfolio is

$$D^P(r) = \sum_j x_j \left(\sum_{t=1}^{T} \frac{tc_j(t)}{(1+r)^t}(d_{i(j)}(t) + \sum_{s=1}^{T} \alpha(s)(d_{i(j)}(s-1) - d_{i(j)}(s)) \right) \Big/ V^P(r)$$

$$= \sum_j x_j (V_j(r,i(j)) \ /V^P(r)) \left(\left(\sum_{t=1}^{T} \frac{tc_j(t)}{(1+r)^t}(d_{i(j)}(t) + \sum_{s=1}^{T} \alpha(s)(d_{i(j)}(s-1) - d_{i(j)}(s)) \right) \Big/ V(r,i(j)) \right)$$

$$= \sum_j w_j D_j(i(j)) \qquad (9)$$

where $D_j(i(j))$ is the duration of bond j given by (7) and w_j is the current price weighted proportion of bond j in the portfolio.

3. Non-flat yield curves: partial durations and approximate durations.

The definitions of Macaulay duration and the extension proposed in section two are based on the idea that the yield curve is flat. By explicitly estimating the default risk, the yield parameter in section two can be thought of as the risk-free interest rate, which the market rarely considers to be flat. There are two ways the market describes non-constant yields over time – often called the term structure of interest rates. The first is to assume that the yield or spot rate y(t) is continuously compounded, so the value of a bond with income stream c(t) and rating i is

$$V(y,i) = \sum_t e^{-ty(t)} c(t) d_i(t) \qquad (10)$$

[For ease of notation we will assume from now on that $\alpha(t) = 0$ so there is no partial payment on default. The results still hold in the general case when $\alpha(t) \neq 0$.]

The second way is to have discretely compounded rates at appropriate intervals (annual compounding is used in the Eurodollar market and semi-annual compounding in the US Treasury market). In that case if the spot rates are given by a vector $\mathbf{r} = (r_1, r_2, \ldots, r_T)$ then the value of a bond with income stream c(t) and rating i is

$$V(\mathbf{r}, i) = \sum_t c(t) d_i(t)/(1+r_t)^t \qquad (11)$$

Instead of using risk free spot rates one could use risk-free forward rates. If $\mathbf{f} = (f_1, f_2, \ldots, f_T)$ is the vector of forward rates then the value of the bond with income stream c(t) and rating i is

$$V(\mathbf{f}, i) = \sum_t c(t) d_i(t)/\prod_{s \le t}(1+f_s) \qquad (12)$$

Whichever formulation is used, one has to model the term structure or equivalently the discount factor b(t) where

$$V(\mathbf{b}, i) = \sum_t b(t) c(t) d_i(t) \qquad (13)$$

and $b(t) = e^{-ty(t)}$ in (10); $b(t) = (1+r_t)^{-t}$ in (11); and $b(t) = 1/\prod_{s \le t}(1+f_s)$ in (12). There are two main approaches to modelling the term structure. The first is to choose a specific form of the yield curve and use the market data to estimate its parameters. Thus [Harry who??] suggested a yield curve of the form

$$y(t) = \left((a_1 + a_3 t)e^{-a_4 t} + a_2\right) \qquad (14)$$

so $b(t) = e^{-y(t)t}$. This has the advantage that the parameters have an obvious interpretation. $a_1 = y(0)$ is the short rate level; $a_2 = y(\infty)$ is the long rate level; $a_3 = y'(0)$ is the slope at the short rate and $a_4 = -y''(\infty) / y'(\infty)$ describes the ratio of curvature to slope in the long run but is also the rate of convergence to the steady state long run rate. (Note that a_4 must be positive). Other forms of the yield curve that have been used include $y(t) = a_1+a_2 t+a_3 t^2$ [britten-jones]

The second approach is to describe the movements in the term structure by a set of factors. In this case it is assumed that

$$y(t) = \sum_i a_i F_i(t) + w(t) \qquad (15)$$

where w(t) is a stochastic process with zero mean. The factors F(.) are determined empirically (see Dahl(1993) , Litterman and Scheinkman (1988)) using factor analysis on the historical returns of pure discount bonds or the historical estimated term structures. In both cases one ends up with a discount function b(t) which is a function of a few critical parameters , i.e. $b(t) = b(t, a_1, a_2, \ldots, a_n)$

If one looks at the discretely compounded models, it is clear that the discount factor is again a function of a discrete number of parameters, i.e.

$b(t, r_1, r_2, \ldots, r_T) = (1+r_t)^{-t}$ or $b(t, f_1, f_2, \ldots, f_T) = \prod_{s\leq t}(1+f_s)^{-1}$

One need not require the discount be a function of all the rates. It is usual to pick a few key rates and make the discount factor a function of these. Suppose for example one chose the second, fifth and tenth period rates as key rates, then one can define the discount factor by

$$b(t,r_2,r_5,r_{10}) = \begin{cases} (1+r_2)^{-t} & t \leq 2 \\ \left(1+\left(\frac{5 - t}{3}\right)r_2 + \left(\frac{t-2}{3}\right)r_5\right)^{-t} & 2 \leq t \leq 5 \\ \left(1+\left(\frac{10 - t}{5}\right)r_5 + \left(\frac{t-5}{5}\right)r_{10}\right)^{-t} & 5 \leq t \leq 10 \\ (1+r_{10})^{-t} & 10 \leq t \end{cases} \qquad (16)$$

Thus whichever model of compounding of the “yield curve one chooses, one arrives at a model for the value of a risky bond which depend on its rating i and a vector of parameter $a = (a_1, a_2, \ldots, a_n)$ which describe the yield curve, spot rates or forward rates, so that

$$V(\mathbf{a}, i) = \sum_t b(t, \mathbf{a})c(t)d_i(t) \qquad (17)$$

Following the analogy with the derivation of duration in (1)-(3), one would ask what is the maturity of a zero coupon risk-free bond (paying out R so its value is $V_0(r) = Rb(D,\mathbf{a})$) that has the same value as the previous bond and the same response to small changes in the risk-free rates.

The problem is that there are now a number of ways the risk free rate can change, not just the parallel shifts in the term structure that is implicit in the Macaulay duration. What is normally suggested in the literature is to calculate the duration for each of the ways that this rate can change and seek to match asset and liability portfolios in each of these durations. One assumes that each change in the risk free rate corresponds to a change in one of the parameters that make up the risk free interest rate term structure and hence the discount factors $b(t,\mathbf{a})$. Thus approach was first suggested by {Cooper 1997}}and subsequently these durations became called partial durations as they are extensions of the basic modified duration. Given the bond price model of (17), then the i^{th} partial duration is

$$D_j^* = -(dV(\mathbf{a},i)/da_j)/V(\mathbf{a}, i) \qquad (18)$$

As examples, consider using the yield curve formulation of (14) in the continuously compounded case. This leads to partial durations of the form.

$$D_1^* = -\left(dV(\mathbf{a},i)da_1\right)/V(\mathbf{a},i) = \sum_t te^{-a_4 t}c(t)d_i(t)e^{-ty(t)} \Big/ V(\mathbf{a},i)$$
$$D_2^* = -\left(dV(\mathbf{a},i)da_2\right)/V(\mathbf{a},i) = \sum_t tc(t)d_i(t)e^{-ty(t)} \Big/ V(\mathbf{a},i)$$
$$D_3^* = -\left(dV(\mathbf{a},i)da_3\right)/V(\mathbf{a},i) = \sum_t t^2 e^{-a_4 t}c(t)d_i(t)e^{-ty(t)} \Big/ V(\mathbf{a},i)$$
$$D_4^* = -\left(dV(\mathbf{a},i)da_4\right)/V(\mathbf{a},i) = \sum_t t^2(a_1 + a_2 t)e^{-a_4 t}c(t)d_i(t)e^{-ty(t)} \Big/ V(\mathbf{a},i)$$

Similarly in the discretely compounded case using all the forward rates as the parameters of the interest rate term structure, the partial durations are of the form

$$D_j^* = -\left(dV(f)df_j\right)/V(\mathbf{a},i) = \sum_{t\geq j} \frac{c(t)d_i(t)}{\prod_{j\leq t}(1+f_t)(1+f_j)} c(t)d_i(t) \Big/ V(f,i)$$

Given one is seeking to allow for all the possible changes in the term structure that one has identified one would expect fitting portfolios by matching all their partial durations would be much more successful than just matching on the one Macaulay duration. This is what Chambers et al [] investigated and they did find an improvement in immunizing the terminal values of the portfolio, while transaction costs are ignored.

There is though an alternative one duration measure that may be even more robust than the Macaulay duration and which has the advantage that the user can specify which types of change to the interest rate term structure are of most concern to him. This approach is to find what duration measure minimises the weighted sum of the errors in the sensitivity of the bond to changes to each of the possible parameter changes in the discount factor. This duration which we call the approximate duration is obtained as follows for a bond whose price V(**b,** i) is given by (17). As is the case with the Macaulay duration, we wish to find the maturity, D, of the zero-coupon risk free bond which most closely matches the weighted sum of the individual changes in the yield curve. The price of such a bond is Vo(**a**) = Rb(D,**a).** The weighting put on the individual changes is given by the weights vector **w**. So the approximate duration is defined by finding the D(**w**) where

$$\mathrm{V}(\mathbf{a},\mathrm{i}) = \mathrm{V}_0(\mathbf{a}) \text{ and } \sum_{s=1}^{n} w(s)\left| \partial\mathrm{V}(\mathbf{a},\mathrm{i}) \Big/ \partial a_s - \partial\mathrm{V}_0(\mathbf{a}) \Big/ \partial a_s \right| \text{ is minimised} \qquad (19)$$

If $w(s)_t = \delta_{st}$, D(**w**) is the partial duration corresponding to a_t.

One can define the approximate duration for any of the types of interest rate model described above. We will describe its calulcation in the case of the discretely compounded model where the parameters are all the forward interest rates. In this case D(**w**) is the value where

$$\mathrm{V}(\mathbf{f},\mathrm{i}) = \mathrm{V}_0(\mathbf{r}) \text{ and } \sum_{s=1}^{T} w(s)\left| \partial\mathrm{V}(\mathbf{f},\mathrm{i}) \Big/ \partial f_s - \partial\mathrm{V}_0(\mathbf{f}) \Big/ \partial f_s \right| \text{ is minimised} \qquad (20)$$

If $w(s)_t = \delta_{st}$, D(**w**) is the partial duration when there are only changes in the t-period forward rate. If $w(s)_t = 1$ for all t (**w= 1**) one is seeking to cover changes in each forward rate equally. The difference between the s-th partial derivatives in (20) is

$$\sum_{t\geq s} -c_t d_i(t) \prod_{u\leq t} (1+f_u)^{-1} (1+f_s)^{-1} - RV_0(\mathbf{f})(1+f_s)^{-1} \qquad (21)$$

if s ≤ D* while the second term disappears if s > D*.

One can find the integer D* that minimises this by solving the following integer programming problem

$$\text{Minimise } \sum_t w(s)(a(s)+b(s)) \qquad (22)$$

subject to the following for all s=1,2,..,T

$$\sum_{t\geq s} c(t) d_i(t) \prod_{u\leq t} (1+f_u)^{-1} (1+f_s)^{-1} + a(s) - b(s) = V(r)(1+f_s)^{-1} \left(\sum_{u=1}^{s} y(u)\right)$$

$$\sum_{u=1}^{T} y(u) = 1$$

$$a(s), b(s), y(s), \geq 0 \text{ for all } s = 1,....,T \text{ and } y(s) \text{ are integer}$$

Since the bonds are usually for an integer number of periods and this is a measure which is minimising errors rather than matching exactly the effect of one particular change in the forward rates it may be more appropriate to leave D* as an integer.

D* is a different measure of stability than the Macaulay duration family of measures which concentrate on one dimensional changes in the forward rates. To see this consider the case when the yield curve is again flat, i,e **f** = (r,r,r,….., r).Then the error term given by (21) become

$(1+r)^{-1}(\sum_{t<s} c(t)d_i(t)(1+r)^{-t})$ for s ≤ D* and $(1+r)^{-1}(\sum_{t\geq s} c(t)d_i(t)(1+r)^{-t})$ for s > D*. (23)

If we define $g(t) = c(t)\, d_i(t)(1+r)^{-t}$ for t=1,…,T this duration D* is the D that minimises

$E(D)=\sum_{s\leq D} w(s)\sum_{u<s} g(u) + \sum_{s>D} w(s)\sum_{u\geq s} g(u) =$

$\sum_{s<D}(\sum_{u:s+1\leq u\leq D} w(u))g(s) + \sum_{s>D}(\sum_{u:D+1\leq u\leq s} w(u))g(s)$

Hence

$$E(D+1) - E(D) = w(D+1)(\sum_{s\leq D} g(s) - \sum_{s\geq D+1} g(s))$$

and it follows that the minimum occurs at

$$D^* = \max\{ D \mid \sum_{s\leq D} g(s) < \sum_{s\geq D+1} g(s) \} \qquad (24)$$

This is the median of the discounted expected cost whereas the Macaulay duration is the mean ($\sum sg(s)$). Note that this result holds no matter what the weighting **w** of the importance of the different periods sensitivities, provided they are non-zero. So if one assumes the forward rates are able to move independently rather than together in a one dimensional family, medians of the cashflow may be more appropriate than the means as measures of duration.

Unlike the mean, the median of a linear combination of measures need not be the linear combination of the individual medians and hence the value D* of a portfolio of bonds has to be calculated by considering the total cash flow from the portfolio rather than by combining the durations of the individual bonds.

4. Data for the simulation experiment between the various duration types

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