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Inverse Semigroups in Coarse Geometry

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ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES
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INVERSE SEMIGROUPS IN COARSE GEOMETRY

By Martin Finn-Sell

Inverse semigroups provide a natural way to encode combinatorial data from geometric settings. Examples of this occur in both geometry and topology, where the data comes in the form of partial bijections that preserve the topology, and operator algebras, where the partial bijections encode $*$-subsemigroups of partial isometries of Hilbert space. In this thesis we explore the connections between these two pictures within the backdrop of coarse geometry.

The first collection of results is concerned primarily with inverse semigroups and their C^*-algebras. We give a construction of a six term sequence of C^*-algebras connecting the semigroup C^*-algebra to that of a naturally associated group C^*-algebra. This result is a generalisation of the ideas of Pimsner and Voiculescu, who were concerned with computing K-theory groups associated to actions of groups. We outline how to connect this picture, via groupoids, to that of a partial translation algebra of Brodzki, Niblo and Wright, and further consider applications of these sequences to computations of certain K-groups associated with group and semigroup C^*-algebras.

Secondly, we give an account of the coarse Baum-Connes conjecture associated to a uniformly discrete bounded geometry metric space and rephrase the conjecture in terms of groupoids and their C^*-algebras that can naturally be associated to a metric space. We then consider the well-known counterexamples to this conjecture, giving a unifying framework for their study in terms of groupoids and a new conjecture for metric spaces that we call the boundary coarse Baum-Connes conjecture. Generalising a result of Willett and Yu we prove this conjecture for certain classes of expanders including those of large girth by constructing a partial action of a discrete group on such spaces.

Contents

Author's Declaration.

I, Martin Finn-Sell, declare that the thesis entitled *Inverse Semigroups in Coarse Geometry* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published, as a preprint: *Spaces of Graphs, Boundary Groupoids and the Coarse Baum-Connes Conjecture*, Martin Finn-Sell and Nick Wright, 2012
  URL: http://arxiv.org/abs/1208.4237

Signed .... Date ....

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CHAPTER 1

Introduction.

Groups have played a role in geometry, topology and analysis throughout the last century. The overall theme is that to recognise objects one must focus on their symmetries; the set of structure preserving automorphisms of an object form a group that describes the ways to permute the object whilst preserving what makes it interesting *globally*. This idea works well within both geometry and topology; the fundamental group of a manifold plays an important role in both areas, as well as entering the realms of physics through representation theory. This makes group theory a natural candidate for study.

Another possible model with similar behaviour is that of a *groupoid*. An example is the fundamental groupoid that is constructed in algebraic topology similar to the path groupoid of a graph. These objects arise much more generally than groups and are able to play much more subtle roles in classifying structures such as equivalence relations (dealing with non-Hausdorff topological quotient spaces [**Con00**]), group actions and encoding coarse information about metric and topological spaces [**STY02**]. In each instance these groupoids capture local transformational data about the underlying object space.

A natural question one could ask is "What happens with the local structure?" This is where *semigroups* enter the picture; local symmetries can be captured by *partial bijections*, which form an *inverse semigroup* under composition. This point of view is less publicised than the corresponding groupoid theory, but work on these ideas enters into both geometry and topology in many places [**Mac64, Law98, KP00**], even breaking into physical questions concerning aperiodic tilings [**KP00, KL04**].

We have a dictionary between these two views; inverse semigroups admit a groupoidification [**Pat99, Exe08**]. There exists for each inverse semigroup S a universal groupoid with the same linear representation theory. These connections make inverse semigroups and groupoids very useful to study from the point of view of answering questions that require some analysis or topology. Each object has natural advantages, the combinatorial theory of semigroups is much more developed than that of groupoids, but the topological aspects of groupoids often play an important role within applications.

To illustrate this view we will consider the following very simple but general example:

EXAMPLE 1. Let Γ be a finitely generated discrete group and let X be a subset of the Cayley graph of Γ. Fix a left invariant metric on Γ and equip X with the subspace metric. The right action of Γ on itself given by multiplication by inverses gives us a set of maps:

$$t_g : \Gamma \to \Gamma, x \mapsto xg^{-1}$$

We can now restrict these maps to X, where they may not be defined everywhere. Denote the set of points in X with image in X by D_g. Then we have:

$$t_g^X : D_g \to D_{g^{-1}}, x \mapsto xg^{-1}$$

These are partial bijections of X, that is maps that are bijections between subsets of X, that move points of X bounded distances. We can then generate subsemigroup of all the partial bijections on X using this collection. This monoid belongs to the class of semigroups known as *inverse* semigroups, and it captures both the metric of X, as the group action determines the metric on Γ via the group action coarse structure. Additionally, it gives information about the local structure of X (as the D_g need not be all of X, or even connected, however the collection of partial translations does provide a partition of $X \times X$). This is an example of a *partial* action of Γ and the inverse monoid it generates belongs to a very nice combinatorial class known as *strongly 0-F-inverse* monoids that have been well-studied in the literature [**Law99, KP00, Nor12**].

The work presented in this document develops this connection between group theory and inverse semigroup theory on the one hand and topology and geometry on the other by considering the universal groupoid associated to the partial action defined above. Via this groupoid we get access to the much more well developed analytic tools of noncommutative geometry; groupoid C^*-algebras are very well studied in comparison to those of an inverse monoid. In particular, we will be interested in constructing a C^*-algebraic analogue to Example 1. With that in mind we consider a special case of Example 1, but add the operator algebra view.

EXAMPLE 2. Let $\Gamma = \mathbb{Z}$ and let $X = \mathbb{N}$. The maps defined above turn into:

$$t_n : \mathbb{N} \to \mathbb{N} \setminus \{0, 1, ..., n-1\}, x \mapsto x + n$$
$$t_{-n} : \mathbb{N} \setminus \{0, 1, ..., n-1\} \to \mathbb{N}, x \mapsto x - n$$

These partial bijections generate an inverse monoid, given by the presentation:

$$S = \langle t_1, t_{-1} | t_{-1} t_1 = 1 \rangle.$$

This is a well-known object in inverse semigroup theory: *the bicyclic monoid*. We can also consider these maps as partial isometry operators inside $\mathcal{B}(\ell^2(X))$ in a very natural way and then consider the C^*-algebra they generate. This algebra is called the translation algebra C^*T associated to the set of maps T. In this instance it coincides the the inverse semigroup C^*-algebra $C^*_r(S)$, which is defined using the natural multiplication of S as partial isometries on $\ell^2(S)$. This C^*-algebra also satisfies the presentation defined above.

This is well-known to operator algebraists: $C^*(T) \cong C^*_r(S) \cong \mathcal{T}$: the Toeplitz algebra. This fits into the short exact sequence:

$$0 \to \mathcal{K}(\ell^2(\mathbb{N})) \to \mathcal{T} \to C(S^1) \to 0.$$

This can be translated into semigroup language:

$$0 \to \mathcal{K}(\ell^2(\mathbb{N})) \to C^*_r(S) \to C^*_r(\mathbb{Z}) \to 0.$$

The last isomorphism arises from the Fourier transform, but is recorded combinatorially by the fact that $\mathbb{Z}$ is the maximal group homomorphic image of S, i.e is given by S after quotienting by a congruence.

As we saw in the construction above there is an inverse semigroup that is 0-F-inverse capturing the partial action that underlies the translation algebra. Studying representations of inverse monoids within this class one might then wonder how much of this result is true in general. A study of this is performed in Chapter 3 and this is one of the main results:

Theorem. *Let S be an F-inverse monoid and let $\mathcal{G}_{\widehat{E}}$ be its universal groupoid and let $A = C_c(\mathcal{G}_U)$. Then we have the following short exact sequence of C^*-algebras:*

$$0 \to \overline{A} \to C^*_r(\mathcal{G}) \to C^*_r(G) \to 0.$$

The above theorem also captures the work of Pimsner and Voiculescu [**PV82**] concerning the action of free groups on C^*-algebras, which was a fundamental development in noncommutative geometric techniques in operator K-theory.

We discuss the computations of the K-theory of the translation inverse monoid in Chapter 5, which are simple as there is a large machinery in the literature to compute these K-groups [**Nor12, JC12**]. We see, unlike in the example above, that the inverse semigroup reduced C^*-algebra is not the correct choice to reconstruct the calculations of [**PV82**] completely from the Theorem above; We prove a similar result about the representation connected to the translation structure:

Theorem. *Let $X \subset G$, $\mathcal{T} = \mathcal{T}_G|_X$ be a grouplike partial translation structure on X with no zero divisors and $S = \langle \mathcal{T} \rangle \hookrightarrow_\mu I(X)$ be the associated F-inverse monoid. Then we have the following short exact sequence of C^*-algebras:*

(1)
$$0 \to C^*_r(\mathcal{G}_U|_{\widehat{X}}) \to C^*_r(\mathcal{G}_{\widehat{X}}) \to C^*_r(G) \to 0.$$

Where the middle term is the translation algebra associated to X arising from $\mathcal{T}$

This does produce the correct short exact sequences for the algebras that arise from Example 1 and 2. In general the K-theory is much harder to compute but is connected to the easier computations for the inverse semigroup C^*-algebra provided by the work of [**Nor12, JC12**] via a complex of short exact sequences.

In general the inverse monoid generated by the translation structure by the construction of Example 1 will not be as well behaved as in these examples and will not satisfy the Theorems above. Through machinery generalising work of Khoshkam and Skandalis [**KS02**] captured by Milan and Steinberg [**MS11**] and suitably weak hypothesis on the group we prove:

Theorem. *Let* $X \subset \Gamma$ *where* Γ *is K-exact and let* $\mathcal{T} = \mathcal{T}_\Gamma|_X$ *be a grouplike partial translation structure on* X*. Consider* $S = \langle \mathcal{T} \rangle \hookrightarrow_\mu I(X)$ *the associated strongly 0-F-inverse monoid. Then we have the following short exact sequence of* C^**-algebras:*

$$0 \to C^*_r(\mathcal{G}_U|_{\widehat{X}}) \to C^*_r(\mathcal{G}_{\widehat{X}}) \to C^*_r(\mathcal{G}_{\widehat{E}}|_{\widehat{X} \cap \widehat{E}_{tight}}) \to 0.$$

We explore in chapter 5 the K-theory of the Cuntz algebras $\mathcal{O}_n$ from this perspective as well as applying this idea to give an alternative proof that Gromov's monster groups [**Gro03, AD08**] are not K-exact.

Another way to use the ideas of Example 1 is to see what can be said about the coarse geometry of a metric space X given that it admits a partial action by a discrete group Γ. The coarse information we are interested in is captured by the coarse Baum-Connes conjecture; recall that the coarse Baum-Connes conjecture [**HR95**] asks if a certain assembly map:

$$\mu_{X,red} : KX_*(X) \longrightarrow K_*(C^*(X))$$

is an isomorphism for X a uniformly discrete bounded geometry metric space. This conjecture is a geometric intrepretation of the well-known *Baum-Connes conjecture* [**BCH94**], and connects to it via a *descent principle* [**Roe96, HR00**]; for a finitely generated group Γ the associated Cayley graph will be a uniformly discrete space with bounded geometry and a positive result for the coarse Baum-Connes conjecture in such situations has strong implications such as the Strong Novikov conjecture concerning the homotopy invariance of the higher signatures of a smooth manifold [**Ros86**] or the existence of metrics with positive scalar curvature for manifolds M that have $\pi_1(M) \cong \Gamma$ [**HR00**].

The Baum-Connes conjecture can be developed in other directions, particularly into the realm of topological groupoids [**Tu00**]. It is a well known result from [**STY02**] that the above statement of the coarse Baum-Connes conjecture can be replaced with a conjecture with coefficients for some groupoid $G(X)$ that we can associate to any uniformly discrete bounded geometry metric space X.

In this context, the coarse Baum-Connes conjecture asks if the map:

$$\mu_r : K_*^{top}(G(X), \ell^\infty(X, \mathcal{K})) \to K_*(\ell^\infty(X, \mathcal{K}) \rtimes_r G(X))$$

is an isomorphism.

The beginning of Chapter 4 develops these ideas from considering the basics of coarse geometry through to the groupoid definition of the coarse assembly map. There are two main objectives within the chapter: first to outline the constructions of counterexamples to the conjecture [**Hig99, HLS02, WY12a, WY12b, OOY09**] and give simplifications via single unified method: the boundary coarse Baum-Connes conjecture. This conjecture, defined via groupoids, tackles the space at infinity:

CONJECTURE 3. *Let X be a uniformly discrete bounded geometry metric space, let $G(X)$ be the associated coarse groupoid on X and let $A_\partial = l^\infty(X, \mathcal{K})/C_0(X, \mathcal{K})$. Then:*

$$\mu_{bdry} : K_*^{top}(G(X)|_{\partial\beta X}, A_\partial) \to K_*(A_\partial \rtimes_r G(X)|_{\partial\beta X})$$

is an isomorphism.

This conjecture, if true, provides information about the coarse Baum-Connes conjecture via homological methods. We outline these methods in Chapter 4.

The class of spaces this conjecture is designed to study are expander graphs [**HLW06**]; these play a large role in the counterexample arguments in the literature. In particular the main result of the Chapter, generalising work of Willett and Yu [**WY12a**], is the following:

THEOREM. *The boundary coarse Baum-Connes conjecture is true for sequences of finite graphs with large girth and uniformly bounded vertex degree.*

The process to prove this associates to each such sequence a partial action of some finitely generated free group. This partial action does not generate the metric as in Example 1 but does control how the metric behaves at infinity.

Finally, in Chapter 5 we tie these ideas together. Firstly, certain examples of the short exact sequences of Chapter 3 and their K-theory are considered. Secondly, a counterexample to the boundary coarse Baum-Connes conjecture is constructed and lastly we show that Gromov monster groups, the groups that coarsely contain large girth expanders, fail to satisfy Baum-Connes with coefficients [**Gro03, AD08**], and show that there are coefficients where the conjecture holds.

In summary, in Chapter 2 we make precise the definitions and properties of inverse semigroups and groupoids that we will use within this thesis, as well as outlining the connections between them that are present in the literature. Following this we define partial actions of groups, which become the primary objects of study in later chapters. Lastly, we give the definition of a C*-algebra and develop ideas concerned with C*-algebras of groupoids and inverse semigroups as well as introducing the methods of topological K-theory.

In Chapter 3 the results concerning short exact sequences associated to F-inverse and strongly 0-F-inverse monoids that were outlined above are proved. The connections to coarse geometry are introduced; we introduce the concept of a partial translation structure and then use this to construct a short exact sequence associated to any sufficiently good subset of a finitely generated group.

In Chapter 4 the focus changes to metric spaces and coarse geometry. The coarse Baum-Connes conjecture is defined via two different approaches, one analytic and one via a groupoid construction from the literature. In this instance we focus on the groupoid version and explain how counterexamples to the conjecture are constructed. We then develop a new conjecture, the boundary coarse Baum-Connes conjecture, and prove it for certain sequences of finite graphs.

Lastly, Chapter 5 is devoted to giving examples and connections between the ideas of the previous chapters. We compute some K-theory groups associated to both translation algebras and inverse monoid C*-algebras, give a counterexample to the boundary coarse Baum-Connes conjecture and use translation structure ideas and the results of Chapter 4 to prove that Gromov monster groups are not exact.

CHAPTER 2

The Basics.

As outlined in the introduction inverse semigroups and groupoids play a large role in the development of many aspects of combinatorics, graph theory and analysis. In this section we provide some basics concerning these areas and the connections between them, developing the notion of Paterson and Exel of a *universal groupoid* associated to an inverse semigroup. We then give a brief introduction to the natural operator algebras that can be constructed from both inverse semigroups and groupoids. Lastly, we consider topological K-theory of C^*-algebras and we outline all the tools we need for the later chapters.

1. Semigroup and Groupoid Theory.

A *semigroup* is a set S, together with an associative binary operation. If additionally it has a unit element, then we say it is a *monoid*. Recall that any $s \in S$ satisfying $s^2 = s$ is said to be *idempotent*.

Definition 4. Let S be a semigroup. We say S is $\mathit{inverse}$ if there exists a unary operation $* : S \to S$ satisfying the following identities:

(1) $(s^*)^* = s$
(2) $ss^*s = s$ and $s^*ss^* = s^*$ for all $s \in S$
(3) $ef = fe$ for all idempotents $e, f \in S$

A very fundamental example of such an object is the *symmetric inverse monoid* on any set X, denoted by $I(X)$. This is defined equipping the collection of all partial bijections of X to itself equipped the composition defined on common intersections:

$$f_2 \circ f_1 : f_1^{-1}(\mathrm{im}(f_1) \cap \mathrm{dom}(f_2)) \to f_2(\mathrm{im}(f_1) \cap \mathrm{dom}(f_2)).$$

This is illustrated in Figure 1 below.

By the representation theorem of Wagner and Preston [**How95**], it is possible to realise every inverse semigroup as a semigroup of partial bijections:

Theorem 5. *Let S be an inverse semigroup. Then there exists a set X such that $S \hookrightarrow I(X)$.* □

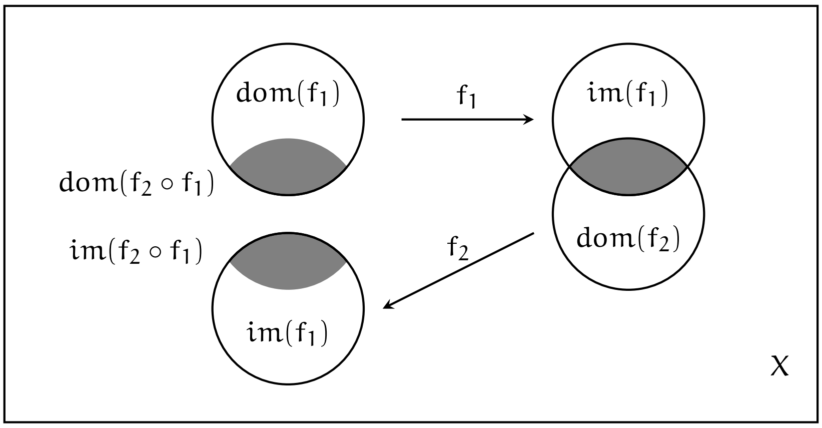


FIGURE 1. The multiplication of partial bijections

When X is a metric space we can consider the inverse submonoid of $I(X)$ consisting of those partial bijections that move elements only a bounded distance. We call these partial translations and we denote the submonoid of these by $I_b(X)$.

DEFINITION 6. Let S be an inverse monoid. We denote by $E(S)$ the set of idempotents (just by E if the context is clear). This is a meet semilattice, that is a set in which every element has a greatest lower bound. In this instance, this lower bound is given by the product of S restricted to E. In this situation we can define the following partial order:

$$e \leq f \Leftrightarrow ef = e$$

In the situation that E consists of subsets of some set X, the meet is intersection and this order corresponds to subset inclusion. This order can naturally by extended to S and should be thought of as restriction:

$$s \leq t \Leftrightarrow (\exists e \in E) \text{ such that } s = et.$$

A submonoid $T \subset S$ is said to be *full* if $E(T) = E(S)$. We remark that for a metric space X every idempotent element in $I(X)$ moves elements no distance and hence $I_b(X)$ is a full submonoid of $I(X)$.

We want to consider quotient structures of an inverse monoid. In semigroup theory, quotients correspond to applying an equivalence relation to S that is compatible with the multiplication. Such relations are called *congruences*. One possible construction arises for each *ideal* in S.

DEFINITION 7. Let I be a subset of S. I is an ideal of S if $SI \cup IS \subset I$.

From an ideal we can define a quotient at the cost of a *zero element*, that is an element $0 \in S$ such that $0s = s0 = 0$ for all $s \in S$.

Definition 8. Let S be an inverse monoid and let I be an ideal of S. Then we can define $S \backslash I$ to be the set $(S \setminus I) \cup \{0\}$, equipped with the following product:

$$s * t = \begin{cases} st \text{ if } s \text{ and } t \notin I \\ 0 \text{ if } s \text{ or } t \in I \end{cases}$$

This is an inverse monoid with 0 called the *Rees quotient* of S by I.

General quotients are given by equivalence relations and in order to get an inverse monoid structure on the equivalence classes it is enough to impose a closure condition on the relation. A relation of this type is called a *congruence* on S.

Definition 9. An equivalence relation $\sim$ on S is called a *congruence* if for every $u, s, t \in S$ such that $s \sim t$, we know that $su \sim tu$ and $us \sim ut$. This allows us to equip the quotient $S \backslash \sim$ with a product, making it into an inverse monoid.

We will be considering a specific congruence on S called the *minimum group congruence*, so called because it is the smallest congruence on S such that the quotient is a group. This congruence, denoted by σ, is given by:

$$s\sigma t \Leftrightarrow (\exists e \in E) es = et$$

A congruence is *idempotent pure* when $e \in E$ and $e \sim s$ implies $s \in E$. The equivalence class of an idempotent will contain only idempotents in this case.

Definition 10. An inverse monoid S is called *E-unitary* if for all $e \in E$ and $s \in S$ if $e \leq s$ then $s \in E$. S is *F-inverse* if the preimage of each $g \in S \backslash \sigma$ has a maximum element in the order on S. and we remark that this is equivalent to asking that for every element $s \in S$ there exists a *unique* maximal element $t \in S$ such that $s \leq t$. We denote the maximal elements of an F-inverse monoid by $Max(S)$

The minimum group congruence, on the class of E-unitary inverse monoids, is an example of an idempotent pure congruence. Additionally it is the smallest group congruence on S [**Law98**].

For an F-inverse monoid it is possible to study the minimum group congruence by considering all the maximal elements with a new product, $*$, which is defined for every $s, t \in Max(S)$ by $s * t = u$, where u is the unique maximal element in S that is above st in the partial order on S.

In general the inverse monoids we will construct will not have this property because they will have a zero element. However we can make similar definitions in this case:

DEFINITION 11. Let S be an inverse monoid. We say S is 0-E-unitary if $\forall e \in E \setminus 0, s \in S$ $e \leq s$ implies $s \in E$. We say it is 0-F-inverse if there exists a subset $T \subset S$ such that for every $s \in S$ there exists a unique $t \in T$ such that $s \leq t$ and if $s \leq u$ then $u \leq t$.

As mentioned before, the minimum group congruence on such monoids will return the trivial group. However by working in a category with a more relaxed type of morphism we can still build useful maps to groups. We develop this in Section 3 of this Chapter.

2. Groupoids.

DEFINITION 12. A *groupoid* is a set $\mathcal{G}$ equipped with the following information:

(1) A subset $\mathcal{G}^{(0)}$ consisting of the objects of $\mathcal{G}$, denote the inclusion map by $i : \mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$.
(2) Two maps, r and $s : \mathcal{G} \to \mathcal{G}^{(0)}$ such that $r \circ i = s \circ i = \mathrm{Id}$
(3) An involution map $^{-1} : \mathcal{G} \to \mathcal{G}$ such that $s(g) = r(g^{-1})$
(4) A partial product $\mathcal{G}^{(2)} \to \mathcal{G}$ denoted $(g, h) \mapsto gh$, with $\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} | s(g) = r(h)\} \subseteq \mathcal{G} \times \mathcal{G}$ being the set of composable pairs.

Moreover we ask the following:

- The product is associative where it is defined in the sense that for any pairs:

$$(g, h), (h, k) \in \mathcal{G}^{(2)} \text{ we have } (gh)k \text{ and } g(hk) \text{ defined and equal.}$$

- For all $g \in \mathcal{G}$ we have $r(g)g = gs(g) = g$.

A groupoid is *principal* if $(r, s) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is injective and *transitive* if (r, s) is surjective. A groupoid $\mathcal{G}$ is a *topological groupoid* if both $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are topological spaces, and the maps $r, s, ^{-1}$ and the composition are all continuous. A Hausdorff, locally compact topological groupoid $\mathcal{G}$ is *proper* if (r, s) is a proper map and *étale* or *r-discrete* if the map r is a local homeomorphism. When $\mathcal{G}$ is étale, s and the product are also local homeomorphisms, and $\mathcal{G}^{(0)}$ is an open subset of $\mathcal{G}$ [**Exe08**, Section 3].

DEFINITION 13. Let $\mathcal{G}$ be a groupoid and let $A, B \subset \mathcal{G}^{(0)}$. Set:

(1) $\mathcal{G}_A = s^{-1}(A)$
(2) $\mathcal{G}^A = r^{-1}(A)$
(3) $\mathcal{G}_A^B = \mathcal{G}^B \cap \mathcal{G}_A$

DEFINITION 14. A subset of $F \subseteq G^{(0)}$ is said to be *saturated* if for every element of $\gamma \in G$ with $s(\gamma) \in F$ we have $r(\gamma) \in F$.

Let A be a saturated set. Denote by $\mathcal{G}|_A$ the subgroupoid $\mathcal{G}_A^A$, called the *reduction* of $\mathcal{G}$ to A. In particular it is worth noting that the groupoids $\mathcal{G}|_{\{x\}}$ are in fact groups, and we say that for a given $x \in \mathcal{G}^{(0)}$ that the group $\mathcal{G}_x^x$ is the *isotropy* group at x.

DEFINITION 15. Let $\mathcal{G}$ be a locally compact groupoid and let Z be a locally compact space. $\mathcal{G}$ acts on Z (or Z is a $\mathcal{G}$-space) if there is a continuous, open map $r_Z : Z \longrightarrow \mathcal{G}^{(0)}$ and a continuous map $(\gamma, z) \mapsto \gamma.z$ from $\mathcal{G} * Z := \{(\gamma, z) \in \mathcal{G} \times Z | s_{\mathcal{G}}(\gamma) = r_Z(z)\}$ to Z such that $r_Z(z).z = z$ for all z and $(\eta\gamma).z = \eta.(\gamma.z)$ for all $\gamma, \eta \in \mathcal{G}^{(2)}$ with $s_{\mathcal{G}}(\gamma) = r_Z(z)$.

When it is clear we drop the subscripts on each map. Right actions are dealt with similarly, replacing each incidence of r_Z with s_Z.

DEFINITION 16. Let $\mathcal{G}$ act on Z. The action is said to be *free* if $\gamma.z = z$ implies that $\gamma = r_Z(z)$.

We end this section with some useful examples.

EXAMPLE 17. Let X be a topological Γ-space. Then the *transformation groupoid* associated to this action is given by the data $X \times G \rightrightarrows X$ with $s(x, g) = x$ and $r(x, g) = g.x$. We denote this by $X \rtimes \Gamma$. A basis $\{U_i\}$ for the topology of X lifts to a basis for the topology of $X \rtimes \Gamma$, given by sets $[U_i, g] := \{(u, g) | u \in U_i\}$.

EXAMPLE 18. The construction in the example above can be generalized to actions of étale groupoids. We are concerned with the topology here: Given an étale groupoid $\mathcal{G}$, a $\mathcal{G}$-space X and a basis $\{U_i\}$ for $\mathcal{G}^{(0)}$. We can pull this basis back to a basis for $X \rtimes \mathcal{G}$ given by $[r_z^{-1}(U_i), \gamma]$, where $U_i \subseteq s(\gamma)$.

Lastly, we consider a extended example that introduces some concepts that are highly relevant in Chapters 3 and 4.

EXAMPLE 19. Let X be a set. We will introduce the general notion of coarse structure and use this to construct a groupoid for X. This groupoid will depend on the coarse structure and will be denoted by $G(X, \mathcal{E})$. In order to do this we need to define the what we mean by a coarse structure. The details of this can be found in [**Roe03**].

DEFINITION 20. Let X be a set and let $\mathcal{E}$ be a collection of subsets of $X \times X$. If $\mathcal{E}$ has the following properties:

(1) $\mathcal{E}$ is closed under finite unions;
(2) $\mathcal{E}$ is closed under taking subsets;
(3) $\mathcal{E}$ is closed under the induced product and inverse that comes from the groupoid product on $X \times X$.
(4) $\mathcal{E}$ contains the diagonal

Then we say $\mathcal{E}$ is a *coarse structure* on X and we call the elements of $\mathcal{E}$ *entourages*. If in addition $\mathcal{E}$ contains all finite subsets then we say that $\mathcal{E}$ is *weakly connected*.

For a given family of subsets $\mathcal{S}$ of $X \times X$ we can consider the smallest coarse structure that contains $\mathcal{S}$. This is the coarse structure generated by $\mathcal{S}$. We can use this to give some examples of coarse structures.

DEFINITION 21. Let X be a coarse space with a coarse structure $\mathcal{E}$ and consider $\mathcal{S}$ a family of subsets of $\mathcal{E}$. We say that $\mathcal{E}$ is generated by $\mathcal{S}$ if every entourage $E \in \mathcal{E}$ is contained in a finite union of subsets of $\mathcal{S}$.

EXAMPLE 22. Let X be a uniformly discrete metric space with bounded geometry. We consider the collection $\mathcal{S}$ given by the R-neighbourhoods of the diagonal in $X \times X$; that is, for every $R > 0$ the set:

$$\Delta_R = \{(x, y) \in X \times X | d(x, y) \leq R\}$$

Then let $\mathcal{E}$ be the coarse structure generated by $\mathcal{S}$. This is called the *metric coarse structure* on X. It is a uniformly locally finite proper coarse structure that is weakly connected when X is a uniformly discrete bounded geometry (proper) metric space.

EXAMPLE 23. Let G be a group and let X be a right G-set. Define:

$$\Delta_g = \{(x, x.g) | x \in X\}$$

We call the coarse structure generated by the family $\mathcal{S} := \{\Delta_g | g \in G\}$ the *group action coarse structure* on X. If X is not a transitive Γ-space the group action coarse structure will not be weakly connected.

In the situation that X admits a transitive G-action by translations, the group action coarse structure generates a substructure of the metric coarse structure. If additionally, each Δ_R is contained in finitely many Δ_g, then the group action coarse structure will be the same as the metric coarse structure.

To build a groupoid from the metric coarse structure on X we consider extensions of the pair product on $X \times X$. The most natural way to do this is by making use of the entourages arising from the metric. The approach to this problem is through Corollary 10.18 of [**Roe03**], which we record as the following Lemma:

LEMMA 24. *Let X be a uniformly discrete bounded geometry metric space and let E be any entourage. Then the inclusion $E \to X \times X$ extends to an injective continuous map $\overline{E} \to \beta X \times \beta X$, where $\overline{E}$ denotes the closure of E in $\beta(X \times X)$.*

Now we can make the definition of the coarse groupoid $G(X)$:

Theorem 25. *([**Roe03**, Theorem 10.20]) Let* X *be a coarse space with uniformly locally finite, weakly connected coarse structure* $\mathcal{E}$*. Define* $G(X,\mathcal{E}) := \cup_{E\in\mathcal{E}}\overline{E}$*. Then* $G(X)$ *is a locally compact, étale groupoid with the induced product, inverse and topology from* $\beta X \times \beta X$*.*

As we are considering the metric coarse structure we can reduce this to considering only generators and we define:

$$G(X) := \bigcup_{R>0} \overline{\Delta_R}.$$

This groupoid, as well as the concept of a coarse structure, will play an important role in Chapters 3 and 4.

2.1. Groupoids from inverse monoids. In this section we outline the machine of [**Pat99, Exe08**] for producing a groupoid $\mathcal{G}_{\widehat{E}}$ from an inverse semigroup S. The way we proceed involves studying the actions of S on its semilattice E. Working with semilattices, being generalisations of Boolean algebras, we still have access to a version of Stone duality; there exists many compactifications of E, built from its order structure, that extends the natural conjugation action of S.

We outline the steps in the construction.

(1) Build an action of S on E.
(2) Build a dual space $\widehat{E}$ to E, which is locally compact and Hausdorff. Construct an action of S on $\widehat{E}$.
(3) Build the groupoid $\mathcal{G}_{\widehat{E}}$ from this data.

After the construction, we make some remarks about more general groupoids of germs built from representations of S.

Definition 26.
(1) For each $e \in E$ let $D_e = \{f \in E | f \leq e\}$. For $ss^* \in E$, we can define a map $\rho_s(ss^*) = s^*s$, extending to D_{ss^*} by $\rho_s(e) = s^*es$. This defines a partial bijection on E from D_{ss^*} to D_{s^*s}.
(2) We consider a subspace of $\mathbf{2}^E$ given by the functions ϕ such that $\phi(0) = 0$ if S has a zero and $\phi(ef) = \phi(e)\phi(f)$. We can topologise this as a subspace of $\mathbf{2}^E$, where it is closed. This makes it compact Hausdorff, with a base of topology given by $\widehat{D}_e = \{\phi \in \widehat{E} | \phi(e) = 1\}$. This admits a dual action induced from the action of S on E. This is given by the pointwise equation for every $\phi \in \widehat{D}_{s^*s}$:

$$\widehat{\rho}_s(\phi)(e) = \phi(s^*es)$$

The use of $\widehat{D}_e$ to denote these sets is not a coincidence, as we have the following map $D_e \to \widehat{D}_e$:

$$e \mapsto \phi_e, \phi_e(f) = 1 \text{ if } e \leq f \text{ and } 0 \text{ otherwise .}$$

REMARK 27. These character maps $\phi : E \to \{0, 1\}$ have an alternative interpretation, they can be considered as *filters* on E. A filter on E is given by a set $F \subset E$ with the following properties:

- for all $e, f \in F$ we have that $e = ef \in F$
- for $e \in F$ with $e \leq f$ we have that $f \in F$ and
- $0 \notin F$ if E has a zero.

the relationship between characters and filters can be summarised as: To each character ψ there is a filter:

$$F_\psi = \{e \in E | \psi(e) = 1\}.$$

And every filter F provides a character by considering χ_F, its characteristic function. This implements a 1-1 correspondence.

(3) We take the set $S \times \widehat{E}$, topologise it as a product and consider the subset $\Omega := \{(s, \phi) | \phi \in D_{s^*s}\}$ in the subspace topology. We then quotient this space by the relation:

$$(s, \phi) \sim (t, \phi') \Leftrightarrow \phi = \phi' \text{ and } (\exists e \in E) \text{ with } \phi \in \widehat{D}_e \text{ such that } es = et$$

We can give the quotient $\mathcal{G}_{\widehat{E}}$ a groupoid structure with the product set, unit space and range and source maps:

$$\mathcal{G}^{(2)}_{\widehat{E}} := \{([s, \phi], [t, \phi']) | \phi = \widehat{\rho}_t(\phi')\}$$
$$\mathcal{G}^{(0)}_{\widehat{E}} := \{[e, \phi] | e \in E\} \cong \widehat{E}$$
$$s([t, \phi]) = [t^*t, \phi], r([t, \phi]) = [tt^*, \phi],$$

and product and inverse:

$$[s, \phi].[t, \phi'] = [st, \phi'] \text{ if } ([s, \phi], [t, \phi']) \in \mathcal{G}^{(2)}_{\widehat{E}}, [s, \phi]^{-1} = [s^*, \widehat{\rho}_s(\phi)]$$

For all the details of the above, we refer to [**Exe08**, Section 4]. This groupoid is the *universal groupoid* associated to S. We collect some information about this groupoid from [**Exe08, Pat99**] in Theorem 50.

(4) Lastly we consider subspaces of $\widehat{E}$ that are closed and saturated and we outline their construction and some associated technicalities below. These subspace arise from the question: what are the ultrafilters on E?

The answer to this question and the technical obstructions that arise form a large part of the papers [**Exe08, Law10**] of Exel and Lawson. We denote the subspace of ultrafilters

$\widehat{E}_\infty$. The main technical point is that, unlike the Boolean algebra case, $\widehat{E}_\infty$ need not be a closed subset of $\widehat{E}$ when E is a semilattice. This leads Exel to consider what he calls *tight* filters, which we denote by $\widehat{E}_{tight}$. In **[Exe08]** it is shown that tight filters coincide with the closure of the ultrafilters inside $\widehat{E}$.

We will regularly make use of the following result that arises from the presence of maximal elements:

Claim 28. *Let S be 0-F-inverse. Then every element $[s, \phi] \in \mathcal{G}_{\widehat{E}}$ has a representative $[t, \phi]$ where t is a maximal element.*

Proof. Take $t = t_s$ the unique maximal element above s. Then we know

$$s = t_s s^* s \text{ and } s^* s \leq t_s^* t_s$$

The second equation tells us that $t_s^* t_s \in F_\phi$ as filters are upwardly closed, thus (t_s, ϕ) is a valid element. Now to see $[t_s, \phi] = [s, \phi]$ we need to find an $e \in E$ such that $e \in F_\phi$ and $se = t_s e$. Take $e = s^* s$ and then use the first equation to see that $s(s^* s) = t_s(s^* s)$. □

Using Claim 28 will be able to forget the non-maximal elements in the monoid S when working with $\mathcal{G}_{\widehat{E}}$. This technique will be prevalent throughout this document as it allows many natural geometric considerations to enter into what would otherwise be purely combinatorial calculations.

Lastly for this section we make remarks about the more general notion associated to a representation of $S \to I(X)$ called a *groupoid of germs.*

Remark 29. We define a topological action of S on a locally compact Hausdorff space X to be a representation of $\pi : S \to I(X)$ such that each $s \in S$ is continuous and has a open domain, where these domains satisfy $\bigcup_{s \in S} D_{s^* s} = X$.

We can construct a groupoid from the the recipe of for the universal groupoid. We do this by considering the subset of $S \times X$ given by $K := \{(s, x) | x \in D_{s^* s}\}$. We then quotient by the relation outlined in the construction of the universal groupoid and give it the same product and inverse. This turns the quotient into a groupoid; called the groupoid of germs, denoted by $X \rtimes S$.

Putting $\widehat{E}$ into this construction provides $\mathcal{G}_{\widehat{E}}$ and every other suitable representation gives us a restriction of $\mathcal{G}_{\widehat{E}}$. This follows from the work of **[Exe08]**.

3. Prehomomorphisms of inverse monoids and general partial actions of discrete groups.

In this section we outline some basic properties of partial actions of discrete groups on topological spaces, paying particular attention to the types of inverse monoid S these generate. We then use analytic information associated to the group together with properties of inverse monoid to understand analytic properties of the universal étale groupoid $\mathcal{G}_{\widehat{E}}$ that is built from the inverse monoid S. We begin with a definition.

DEFINITION 30. Let $\rho : S \to T$ be a map between inverse semigroups. This map is called a *prehomomorphism* if for every $s, t \in S$, $\rho(st) \leq \rho(s)\rho(t)$ and a *dual prehomomorphism* if for every $s, t \in S$ $\rho(s)\rho(t) \leq \rho(st)$.

We recall that a congruence is said to be *idempotent pure* if the preimage of any idempotent is an idempotent. We extend this definition to general maps in the natural way. In addition we call a map $S \to T$ *0-restricted* if the preimage of $0 \in T$ is $0 \in S$.

DEFINITION 31. Let S be a 0-E-unitary inverse monoid. We say S is *strongly 0-E-unitary* if there exists an idempotent pure, 0-restricted prehomomorphism, Φ to a group G with a zero element adjoined, that is: $\Phi : S \to G^0$. In this instance the prehomomorphism property translates into: if $s, t \in S$ with $st \neq 0$, we have $\Phi(st) = \Phi(s)\Phi(t)$, as the order structure on G^0 is simply given by $g \leq h \Leftrightarrow g = 0$ or $g = h$.

We say S is *strongly 0-F-inverse* if it is 0-F-inverse and strongly 0-E-unitary. This is equivalent to the fact that the preimage of each group element under Φ that is not 0 contains a maximum element.

This class of inverse monoids is particularly important: the idempotent pure, 0-restricted prehomomorphism onto a group (with 0) can be thought of as a generalisation of the minimum group congruence in the larger category of inverse monoids with prehomomorphisms. We will utilise this technology later to regain some of the information from a group when we cannot quotient out in any meaningful way due to the presence of a zero element.

EXAMPLE 32. In [**BR84, LMS06**] an inverse monoid was introduced that is universal for dual prehomomorphisms from a general inverse semigroup. In the context of a group G this is called the *prefix expansion*; its elements are given by pairs: (X, g) for $\{1, g\} \subset X$, where X is a finite subset of G containing 1. The set of such (X, g) is then equipped with a product and inverse:

$$(X, g)(Y, h) = (X \cup gY, gh)\ , ; (X, g)^{-1} = (g^{-1}X, g^{-1})$$

This has maximal group homomorphic image G, and it has the universal property that it is the largest such inverse monoid. We denote this by G^{Pr}. The partial order on G^{Pr} can be described by reverse inclusion, induced from reverse inclusion on finite subsets of G. It is F-inverse, with maximal elements: $\{(\{1, g\}, g) : g \in G\}$.

Definition 33. Let G be a finitely generated discrete group and let X be a (locally compact Hausdorff) topological space. A *partial action* of G on X is a dual prehomomorphism θ from G to the symmetric inverse monoid $I(X)$ that has the following properties:

(1) The domain $D_{\theta_g^*\theta_g}$ is an open set for every g.
(2) θ_g is a continuous map.
(3) The union: $\bigcup_{g\in G} D_{\theta_g^*\theta_g}$ is X.

Given this data we can generate an inverse monoid S using the set of θ_g. This would then give a representation of S into $I(X)$. If the space X admits a coarse structure, then it makes sense to ask if each θ_g is a close to the identity. In this case, we would get a representation into the bounded symmetric inverse monoid $I_b(X)$. We call such a θ a *bounded partial action* of G.

We are going to be interested in turning a partial group action into a full group action; the process of globalisation has been considered in a variety of settings [**Mac64, KP00, KL04, CL08, KS02, MS11**], each using the same central theme.

Definition 34. A *globalisation* of a partial action $\theta : G \to I(X)$ is a space Y with an injection $X \hookrightarrow Y$ and action $\tilde{\theta}$ of G such that the partial action obtained from restricting the action $\tilde{\theta}$ to X is equal to θ.

A globalisation is minimal if it injects into any other globalisation. In [**KL04**] the authors proved that for any partial action of a group G there is a unique globalisation (up to equivalence of partial actions). This is defined as follows:

Definition 35. Let X be a topological space and let G be a group acting partially on X. Then we denote by Ω the *Morita envelope* of the action of G on X, which is constructed as follows:

Consider the space $X \times G$, equipped with the product topology. Then define $\sim$ on $X \times G$ by $(x, g) \sim (y, h)$ if and only if $x(h^{-1}g) = y$. We define Ω as the quotient of $X \times G$ by $\sim$ with the quotient topology.

G acts on Ω using right multiplication by inverses on the group factor of the equivalence classes. Clearly the map that sends $x \in X$ to $[1, x] \in \Omega$ is a topological injection. The main result of [**KL04**] is that this new topological space is minimal amongst globalisations of X.

This notion will be developed further in Section 4.3 and will also play an important role in certain examples in Chapter 5.

4. C^*-algebras of groupoids and inverse semigroups.

Now we change directions slightly by introducing the analytic counterparts to topological spaces; C^*-algebras play an important role in generalising the notions of topology into a noncommutative setting. The work we initially outline below is the duality theorem of Gelfand, Naimark and Segal that connects topology with analysis. Then we develop some purely noncommutative ideas by outlining the construction of natural C^*-algebras associated to both groupoids and inverse semigroups.

4.1. Topological Spaces and Commutative C^*-algebras. We define an abstract C^*-algebra, then consider some examples.

DEFINITION 36. A Banach *-algebra is an algebra A, equipped with an involution * and a norm $\|.\|$ such that the algebra is complete in this norm.

DEFINITION 37. An abstract C^*-algebra is a Banach *-algebra A such that $\|a^*a\| = \|a\|^2$.

The fundamental example of this is bounded operators on Hilbert Space.

EXAMPLE 38. Let H be a Hilbert Space; then we consider the algebra $\mathcal{B}(H)$ of bounded linear operators on H. This has a native involution sending each $T \in \mathcal{B}(H)$ to its adjoint: $T^* \in \mathcal{B}(H)$ and a native norm arising from the inner product and this satisfies the identity above.

It is possible to connect this example to abstract C^*-algebras via the Gelfand-Naimark-Segal theorem:

THEOREM 39. *Every abstract C^*-algebra is a isometrically *-isomorphic to a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.* □

The second example links these objects to topological spaces:

EXAMPLE 40. Let X be a Hausdorff, locally compact topological space. Consider the algebra of continuous, complex valued functions that vanish at infinity $C_0(X)$ with pointwise operations:

$$(f+g)(x) = f(x) + g(x), (f \circ g)(x) = f(x)g(x)$$

We can add an involution to this algebra in the following way:

$$f^*(x) = \overline{f(x)}$$

This turns this algebra into a *-algebra. We can also define a norm in the following way:

$$\|f\| = \sup_{x \in X} |f(x)|$$

This is complete (Banach) algebra in this norm. Observe it satisfies the following identity:

$$\|f^*f\| = \|f\|^2$$

So it is a C^*-algebra. Observe also that it has a commutative product.

This example will allow us to classify *all* commutative C^*-algebras using the following result of Gelfand:

THEOREM 41. *The category of all commutative C^*-algebras and *-homomorphisms is equivalent to the opposite of the category of locally compact, Hausdorff topological spaces with proper maps.*

So the study of commutative algebras is parallel to the study of locally compact Hausdorff topological spaces. The benefit of dealing with the C^*-algebras is that we can consider *noncommutative* algebras. This concept forms the central backbone of the noncommutative geometry program of Connes [**Con00**].

4.2. Hilbert C^*-modules.

To consider groupoid C^*-algebras we want to consider representations that fiber over the unit space; in particular we need to consider *fields of Hilbert Spaces* - Hilbert Modules [**Lan95**].

DEFINITION 42. Let A be a C^*-algebra. A *Hilbert A-module* E is a right A-module equipped with an A-valued form $\langle , \rangle : \mathscr{E} \times \mathscr{E} \to A$ which satisfies the following axioms:

(1) $\langle \eta, \zeta_1 + \zeta_2 \rangle = \langle \eta, \zeta_1 \rangle + \langle \eta, \zeta_2 \rangle$;
(2) $\langle \eta, \zeta a \rangle = \langle \eta, \zeta \rangle a$;
(3) $\langle \eta, \zeta \rangle^* = \langle \zeta, \eta \rangle$;
(4) $\langle \eta, \eta \rangle \geq 0$;
(5) $\langle \eta, \eta \rangle = 0$ if and only if $\eta = 0$ and
(6) $\mathscr{E}$ is complete with respect to $\|\eta\| = \|\langle \eta, \eta \rangle\|_A^{1/2}$

REMARK 43. The axioms above imply a generalization of the Cauchy-Schwartz inequality and so Hilbert A-modules also satisfy the triangle inequality.

REMARK 44. If you put $A = \mathbb{C}$ then the above definition reduces to that of a Hilbert space. A C^*-algebra A can be thought of as a Hilbert A-module over itself using the inner product: $\langle a, a' \rangle = a^*a'$. Also we remark that not all the basic facts that apply to Hilbert spaces apply to Hilbert modules - in general the Riesz Representation Theorem fails for Hilbert Modules [**Hig90**].

4.3. Constructions of Groupoid C^*-algebras. The standard technique for constructing a C^*-algebra from a locally compact group G involves considering norm completions associated to representations of the ring of compactly supported functions on the group. We can associate a very natural representation on the space $L^2(G, \mu)$, where μ is the Haar measure on G. To extend these ideas to a locally compact groupoid we will need an analogue of this measure in a suitably fibred manner.

DEFINITION 45. A *left Haar system* for a locally compact groupoid $\mathcal{G}$ is a family $\{\lambda^u\}$, where each λ^u is a positive regular Borel measure on the locally compact Hausdorff space $\mathcal{G}^u$ such that the following hold:

(1) the support of each λ^u is the whole of $\mathcal{G}^u$;
(2) for any $g \in C_c(\mathcal{G})$, the function g^0, where:
$$g^0(u) = \int_{\mathcal{G}^u} g d\lambda^u$$
belongs to $C_c(\mathcal{G}^{(0)})$.
(3) for any $x \in \mathcal{G}$ and $f \in C_c(\mathcal{G})$,
$$\int_{\mathcal{G}^{d(x)}} f(xz) d\lambda^{d(x)}(z) = \int_{\mathcal{G}^{r(x)}} f(y) d\lambda^{r(x)}(y).$$

We now observe that when the groupoid is étale it is possible to take as a Haar system the counting measure (this fact is a consequence of Proposition 2.2.5 in [**Pat99**]). This eases the passage through calculations significantly and so we make the assumption that G is locally compact and étale from now on. We now give explicit formulae for the convolution product and adjoint on $C_c(G)$. This is taken from [**Exe08**]. For every $f, g \in C_c(\mathcal{G})$:
$$\begin{aligned} (f * g)(\gamma) &= \sum_{\substack{(\sigma,\tau)\in\mathcal{G}^{(2)} \\ \sigma\tau=\gamma}} f(\sigma)g(\tau) \\ f^*(\gamma) &= \overline{f(\gamma^{-1})} \end{aligned}$$

We outline two methods generalising the standard views from the theory of locally compact groups. First is the technique of inducing a representation of the group from a representation of the functions defined on the identity and the second involves Hilbert modules, which is outlined at the end of this section.

What follows is given in full generality and taken from [**Pat99**, Appendix D]

DEFINITION 46. A dense *-subalgebra of a C^*-algebra is called a *pre*-C^**-algebra*.

Let A and B be pre-C*-algebras such that B acts an algebra of right multipliers on A. The action of B will be denoted by: $(a, b) \to a.b$. This is assumed to be continuous.

DEFINITION 47. Let P be a linear, self-adjoint positive map from $A \to B$. We say that P is a *generalised conditional expection* if:

(1) $P(a.b) = P(a)b$
(2) for all $c \in A$ the linear map $a \mapsto P(c^*ac)$ from A to B is bounded
(3) for every $a \in A$ and every $\epsilon > 0$ there exists a c in the span A^2 of elements $a_1a_2, a_i \in A$ such that:
$$\|P((a-c)^*(a-c))\| < \epsilon$$
(4) $P(A)$ generates a dense subalgebra of B.

This map projects functions from one algebra onto another in a way that compliments the multiplier action. Take A and B to be pre-C*-algebras with a conditional expectation $P : A \to B$.

Let H be a Hilbert space and define $\pi : B \to H$ by treating H as a left Hilbert B-module by defining:

$$b\xi = \pi(b)\xi$$

A is a right Hilbert B-module as B acts on A by right multipliers. Now we can form the tensor product $A \otimes_B H$, and this becomes a (pre)-Hilbert space using the inner product:

$$\langle a \otimes \xi, a' \otimes \eta \rangle = \langle P((a')^*a)\xi, \eta \rangle_H. \tag{2}$$

Quotenting by the zero vectors in this gives a Hilbert space, and we denote this by K. We can now represent A on K as follows:

$$\mathrm{Ind}(\pi) : A \to \mathcal{B}(K), \mathrm{Ind}(\pi)(a)(a' \otimes \xi) = aa' \otimes \xi$$

Then the map $\mathrm{Ind}(\pi)$ is called the *induced representation of A* associated with π.

So consider the application of this process to the following pre-C*-algebras:

EXAMPLE 48. Let $\mathcal{G}$ be a r-discrete topological groupoid. Let $A := C_c(\mathcal{G})$ and $B := C_0(\mathcal{G}^{(0)})$. P in this case is the restriction map. We can represent B on $L(\mathcal{G}^{(0)}, \mu)$ where μ is a measure on the unit space. So for a given unit $v \in \mathcal{G}^{(0)}$ we can construct a Hilbert space and a representation as follows:

Let $\pi : B \to B(H)$ given by diagonal multiplication as above. Then we have a map $P_v : f \in A \to P(f) = f(v)$ where $P : A \to B$ is given as a sum of these maps in the following way:

$$P(f) = (f(v_1), f(v_2), ...) = \oplus_{i \in \mathbb{N}} P_{v_i}(f)$$

We can define a Hilbert space: $K_\nu = A \otimes_{\{\nu\}} H$ with the inner product: $\langle a \otimes \xi, a' \otimes \eta \rangle = \langle \pi(P_\nu(a'^* a))\xi, \eta \rangle_H$ and a Hilbert $C_0(\mathcal{G}^{(0)})$-module $K = \bigoplus_{\nu \in \mathcal{G}^{(0)}} K_\nu$. Now the induced representation is the sequence of multiplier operators:

$$\mathrm{Ind}(\nu)(f)(a \otimes \xi) = P_\nu(f) a \otimes \xi$$
$$\mathrm{Ind}(\pi) = \bigoplus_{\nu \in \mathcal{G}^{(0)}} \mathrm{Ind}(\nu)$$

We can define a norm on $C_c(\mathcal{G})$ coming from this representation:

Definition 49. Let $f \in C_c(\mathcal{G})$ Then $\|f\|_r = \sup\{\|\mathrm{Ind}(\nu)\|_{K_\nu} : \nu \in \mathcal{G}^{(0)}\}$. We call this norm *the reduced groupoid norm*

Completing $C_c(\mathcal{G})$ in this norm on $\mathcal{B}(K)$ gives the reduced groupoid C^*-algebra $C^*_r(\mathcal{G})$.

We observe that this completion arises by considering a field of Hilbert spaces over $C_0(\mathcal{G}^{(0)})$. We can also come up with an identification between this module structure and the natural field of Hilbert spaces structure outlined above. We begin by putting a natural pre-Hilbert $C_0(\mathcal{G}^{(0)})$-module structure on this function algebra by defining the inner product:

$$\langle \zeta, \eta \rangle = (\zeta^* * \eta)|_{\mathcal{G}^{(0)}}.$$

We observe that for any function $f \in C_0(\mathcal{G}^{(0)})$ we can define a right action on $C_c(\mathcal{G})$ by: $(\eta.f)(\gamma) = \eta(\gamma) f(s(\gamma))$. We can then complete this as a Hilbert module, and we denote this by $L^2(\mathcal{G})$. The algebra $C_c(\mathcal{G})$ represents naturally on this algebra using the representation: $\lambda(f)(\eta) = f * \eta$.

It is well known that any Hilbert $C_0(\mathcal{G}^{(0)})$-module $\mathcal{M}$ is the space of sections of a continuous field of Hilbert spaces $\{\mathcal{M}_x\}_{x \in \mathcal{G}^{(0)}}$, with any bounded adjointable operator T on $\mathcal{M}$ decomposing as a strongly $*$-continuous field $(T_x)_{x \in \mathcal{G}^{(0)}}$ with, $\|T\| = \sup_{x \in \mathcal{G}^{(0)}} \|T_x\|$ [**KS02**]. We use this to get easier access to the norm by explicitly constructing each $\mathcal{M}_x$. To do this, we construct an inner product for each $x \in \mathcal{G}^{(0)}$:

$$\langle \zeta, \eta \rangle_x = (\zeta^* * \eta)(x).$$

This defines an inner product on $C_c(\mathcal{G}_x)$, which we can use to complete into a Hilbert space which we denote by $L^2(\mathcal{G}_x)$. This gives us the natural field of Hilbert spaces we were looking for, namely: $\{L^2(\mathcal{G}_x)\}_{x \in \mathcal{G}^{(0)}}$. It also gives us a natural representation of $C_c(\mathcal{G})$ given by $\lambda_x(f)\eta = (f * \eta)(x)$. Hence we can conclude that $\|f\| = \sup_{x \in \mathcal{G}^{(0)}} \|\lambda(f)\| = \|\lambda(f)\|$. From this we can complete $C_c(\mathcal{G})$ in either the norm on $L^2(\mathcal{G})$ or the family of norms $\{L^2(\mathcal{G}_x)\}_{x \in \mathcal{G}^{(0)}}$, getting the same completion, denoted by $C^*_r(\mathcal{G})$.

These ideas clearly agree with the more formal construction given at the beginning of this section as the K_v are isometrically isomorphic to $L^2(\mathcal{G}|_v)$.

The last point of this section is to collect some information concerning the universal groupoid $\mathcal{G}_{\widehat{E}}$ of a inverse monoid S.

THEOREM 50. *Let* S *be a countable 0-E-unitary inverse monoid,* E *its semilattice of idempotents and* $\mathcal{G}_{\widehat{E}}$ *its universal groupoid. Then the following hold for* $\mathcal{G}_{\widehat{E}}$*:*

- $\widehat{E}$ *is a compact, Hausdorff and second countable space.*
- $\mathcal{G}_{\widehat{E}}$ *is a Hausdorff groupoid.*
- *Every representation of* S *on Hilbert space gives rise to a covariant representation of* $\mathcal{G}_{\widehat{E}}$ *and vice versa.*
- *We have* $C^*_r(S) \cong C^*_r(\mathcal{G}_{\widehat{E}})$.

PROOF. The first point is a consequence of the fact that E is countable, in this situation we know precisely that $\mathbf{2}^E$ is metrizable, hence as a closed subset we know that $\widehat{E}$ is second countable. It is compact and Hausdorff as it is a closed subset of a compact, Hausdorff space.

The second point follows from Corollary 10.9 [**Exe08**], the third point is Corollary 10.16 [**Exe08**] and the fourth point follows from [**Pat99**], but a more elementary proof is given in [**KS02**]. □

5. Semigroup Valued Cocycles and a Theorem of Milan and Steinberg.

In this section we consider the question of when a groupoid admits a transformation groupoid decompostion up to Morita equivalence. This question connects to the idea of globalising partial actions discussed in Section 3 and this has been well studied for the class of groupoids constructed from suitable inverse semigroups [**KS02, MS11**].

We follow the notation of [**KS02, MS11**]:

DEFINITION 51. Let $\mathcal{G}$ be a locally compact groupoid. Then we call a continuous homomorphism from $\mathcal{G}$ to an inverse semigroup S an *inverse semigroup valued cocycle* (or just cocycle).

DEFINITION 52. Let $\rho : \mathcal{G} \to S$ be a cocycle. We say it is:

(1) *transverse* if the map $S \times \mathcal{G} \to S \times X$, $(s, \gamma) \mapsto (s\rho(\gamma), s(\gamma))$ is open,
(2) *closed* if the map $\gamma \mapsto ((r(\gamma), \rho(\gamma), s(\gamma))$ is closed,
(3) *faithful* if the map $\gamma \mapsto ((r(\gamma), \rho(\gamma), s(\gamma))$ is injective.

We call a cocycle ρ with all these properties a *(T,C,F)-cocycle.*

Below is the main result of [**MS11**], a generalisation of the main results of [**KS02**]:

THEOREM 53. *Let* $\rho : \mathcal{G} \to S$ *be a continuous, faithful, closed, transverse cocycle where* $\mathcal{G}$ *is a locally compact groupoid and* S *is a countable inverse semigroup. Then there is a locally compact Hausdorff space* X *equipped with an action of* S *so that* $\mathcal{G}$ *is Morita equivalent to the groupoid of germs* $X \rtimes S$*. Consequently* $C^*_{max}\mathcal{G}$ *is strongly Morita equivalent to* $C_0(X) \rtimes S$*. If* S *is a group, then the analogous result holds for reduced* C^**-algebras.*

From an F-inverse monoid S it is possible to construct a (T,C,F)-cocycle onto the maximal group homomorphic image of G [**KS02**]. To prove Theorem 53 in the case that the monoid is F-inverse then makes use of the Morita envelope of the partial action that the maximal group homomorphic image G has on the unit space of the universal groupoid $\mathcal{G}_{\widehat{E}}$. By Lemma 1.7 of [**KS02**] the (T,C,F) condition on the cocycle is enough to prove that the Morita envelope is Hausdorff.

What follows from here can be found as a Corollary to Theorem 53 from [**MS11**]. We provide a direct proof of a special case using the original methods of [**KS02**]. This is possible by considering the construction of the groupoid $\mathcal{G}_{\widehat{E}}$ for a strongly 0-E-unitary inverse monoid S. It is clear that the only danger is mapping elements to 0 in Γ^0; this is overcome by the observation that the element $[0, f]$ would be defined if and only if $f \in D_0$. However, $f \in D_0$ implies that $f(0) = 1$ and hence $f \notin \widehat{E}$, so the 0 element of S contributes nothing to the groupoid $\mathcal{G}_{\widehat{E}}$.

We are interested in proving that if S is a strongly 0-F-inverse monoid then we can apply some analogue of Theorem 53. This is Corollary 6.17 from a [**MS11**], however we give a direct proof here for completeness just in the special case in which we are interested, by adapting the original techniques of [**KS02**].

DEFINITION 54. We say that S satisfies the finite cover property with respect to ϕ,, if for every $p, q \in S$, $p, q \neq 0$ there exists a finite set $U \subset S_g$ such that:

$$pS_g q = \{s \in S | \exists u \in U | s \leq u\}.$$

Where S_g is the preimage $\phi^{-1}(g)$.

THEOREM 55. *Let* S *be an inverse monoid. If* S *is strongly 0-E-unitary with non-trivial universal group* $U(S) = \Gamma$ *such that the prehomomorphism has the finite cover property. Then the groupoid* $\mathcal{G}_{\widehat{E}}$ *admits a transverse and faithful cocycle to a group* Γ*.*

PROOF. Let Φ be the 0-restricted, idempotent pure prehomomorphism onto Γ^0. We build an induced map on the groupoid $\mathcal{G}_{\widehat{E}}$ by considering a new map Ψ :

$$\Psi([s, x]) = \Phi(s)$$

This map is well-defined as any non-zero idempotent in S is mapped to the identity in Γ, and so for any pair $(s, f) \sim (t, f)$ there is an $e \in E \cap D_f$, in particular not 0, such that $es = et$ and hence

$\Phi(s) = \Phi(es) = \Phi(et) = \Phi(t)$. This is clearly a groupoid homomorphism to Γ. To check it is continuous observe that as Γ is a discrete group so all subsets are open. The preimage of a singleton is given by the union:

$$\Psi^{-1}(\{g\}) = \bigcup_{\Phi(u)=g} [u, D_{u^*u}]$$

which is certainly open in $\mathcal{G}_{\widehat{E}}$. The map is proper, because the preimage of any finite set in Γ is equal to a finite union of $[u, D_{u^*u}]$ by the finite cover property and these are compact by construction.

It remains to check it is a (T,C,F)-cocycle, and from the remarks prior to the Theorem the proof of this follows exactly from the proof [**KS02**, Proposition 3.6] modified suitably.

To prove the map Ψ is transverse, it is enough to prove that $\{(\Psi(\gamma), s(\gamma)) : \gamma \in \mathcal{G}_{\widehat{E}}\}$ is open in $\Gamma \times \widehat{E}$, and this in turn reduces to studying this problem for all $g \in G$, that is if $\{s(\gamma) : \Psi(\gamma) = g\}$ is open in $\widehat{E}$. This set is equal to $\bigcup_{\Psi(\gamma)=g} D_{s(\gamma)}$, which is certainly open in $\widehat{E}$ as each piece is.

To see that this is faithful, let $[u, f], [v, f'] \in \mathcal{G}_{\widehat{E}}$ such that $(f, \Phi(u), \theta_u(f)) = (f', \Phi(v), \theta_v(f'))$. Then it is clear that $f = s([u, f]) = s([v, f']) = f'$, so it is enough to prove now that $\Phi(v) = \Phi(u)$ implies $[u, f] = [v, f]$. Observe that $\Phi(u)\Phi(v)^{-1} = 1$ in Γ and $\Phi(v)^{-1} = \Phi(v^*)$, so $\Phi(uv^*) = 1$. This map is idempotent pure, so $uv^* \in E(S)$. So $[u, f][v, f]^{-1} = [uv^*, \theta_v(f)]$ is a unit in $\mathcal{G}_{\widehat{E}}$. From here it is clear that $[u, f]$ is an inverse to $[v^*, \theta_v(f)]$ and so $[u, f] = [v, f]$. □

We still need to check the fact that the cocycles are closed. Again this follows from the work of [**MS11**] or [**KS02**], but we give the proof in this setting:

LEMMA 56. *If S is an inverse monoid and has the finite cover property with respect to ϕ, then the induced cocycle ρ is closed.*

PROOF. As Γ is discrete, it is enough to prove that the graph $Gr(g)$ over g in $\widehat{E} \times \widehat{E}$ is closed. We remark also that this product space is covered by the set of $D_e \times D_f$, where e, f run though the idempotents $E(S)$, and is compact; thus only finitely many pairs $D_e \times D_f$ are necessary. The intersection $Gr(g) \cap D_e \times D_f$ is covered by $\bigcup_{u \in eS_g f}[u, \widehat{D}_{u^*u}]$ and so is a compact set if and only if:

$$Gr(g) \cap D_e \times D_f = \bigcup_{u \in U} [u, \widehat{D}_{u^*u}]$$

for some finite $U \subset S_g$. However, this is precisely implied by the finite cover property. □

COROLLARY 57. *If S is a 0-E-unitary monoid with the finite cover property and non-trivial universal group then the groupoid $\mathcal{G}_{\widehat{E}}$ is Morita equivalent a transformation groupoid $Y \rtimes \Gamma$.* □

PROOF. This follows from Theorem 1.8 from [**KS02**]. □

REMARK 58. If, in addition the inverse monoid S is 0-F-inverse, then it satisfies the finite cover property with $|U| = 1$ as each non-empty S_g will contain a unique maximal element.

6. K-theory of C^*-algebras.

In this section we give the basic definitions of operator K-theory, following the exposition of [**WO93**]. There are many alternative texts that could be followed instead, such as: [**CMR07, Bla98**] but the explicit calculations make [**WO93**] exceptionally clear. In this section we will consider only unital C^*-algebras. It is possible to perform the calculations required for the proofs of these facts for a non-unital algbera A via the *unitisation*; that is the C^*-algebra constructed from A, with underlying Banach space $A \oplus \mathbb{C}$ and the following multiplication and adjoint:

$$\begin{aligned}(a+\lambda)(b+\mu) &= (ab+\lambda b+\mu a)+\lambda\mu \\ (a+\lambda)^* &= a^*+\overline{\lambda}\end{aligned}$$

This is similar to working with locally compact spaces in topological K-theory via a one point compactification.

DEFINITION 59. A *projection* in a C^*-algebra is a self-adjoint idempotent operator. That is $p = p^* = p^2$ (i.e $p^*p = p$). A pair of projections p and q are orthogonal if $pq = 0$. A operator $v \in A$ is said to be a *partial isometry* if v^*v or vv^* is a projection.

LEMMA 60. *The sum $p + q$ of two projections p and q is a projection if and only if p and q are orthogonal.* □

PROOF. Consider $(p + q)^2 = p^2 + 2pq + q^2$. This equals $p + q$ if and only if $pq = 0$. □

DEFINITION 61. Let $p, q \in A$ be projections.

(1) p is said to be *equivalent* to q, written $p \sim q$, if there exists a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$.
(2) p is said to be *unitarily equivalent* to q, written $p \sim_u q$, if there exists a unitary $u \in A$ such that $p = u^*qu$.
(3) p is said to be *homotopic* to q, written $p \sim_h q$, is there exists a norm continuous path of projections p_t such that $p_0 = p$ and $p_1 = q$.

The following is Proposition 5.2.10 from [**WO93**]:

PROPOSITION 62. *If p and q are projections in A, then:* $p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim q$. □

These relations are not in general reversible. If one is willing to work with matrix algebras $M_n(A)$ over A, then they are however. We denote by $\mathrm{diag}(a, b)$ the matrix with diagonal entries a and b, with every other entry 0.

LEMMA 63. [**WO93**, Proposition 5.2.12] *Let* p, q *be projections in* A. *Then* $p \sim q \Rightarrow \mathrm{diag}(p, 0) \sim_u \mathrm{diag}(q, 0)$ *and* $p \sim_u q \Rightarrow \mathrm{diag}(p, 0) \sim_h \mathrm{diag}(q, 0)$.

In defining K-theory, similar to both topological K-theory and algebraic K-theory, we will actually be considering a *stabilisation* of A, i.e working with $M_\infty(A)$. Lastly, we need a result that allows a "sum" of projections to be well defined up to the equivalences defined above. The trick, again, is to rely on passing to a matrix algebra: given $p, q \in A$ projections, we observe that $\mathrm{diag}(0, p)$ is unitarily equivalent to $\mathrm{diag}(p, 0)$, and orthogonal to $\mathrm{diag}(q, 0)$. We define: $[p] + [q] = [\mathrm{diag}(p, q)]$.

DEFINITION 64. Let A be a C^*-algebra. We denote by $\mathrm{Proj}(A)$ the set of homotopy equivalence classes of projections in $M_\infty(A)$. With the sum, $+$ defined above, this is a commutative monoid.

Finally, K_0 is constructed using a process that associates to every commutative monoid a commutative group. We do this as follows. Let M be a commutative monoid with operation $+$. Now consider the following equivalence relation on $M \times M$:

$$(s, t) \sim (u, v) \Leftrightarrow (\exists p \in M) \text{ such that } s + v + p = u + t + p$$

The quotient of $M \times M$ by this relation has a group structure extending the $+$ on M [**WO93**]. This also has a natural injective map from M given by sending $m \in M$ to $[m, 1]$.

This process generalises the construction of the integers from the natural numbers and is called taking the *Grothendieck group* of M.

DEFINITION 65. Define $K_0(A)$ to be the Grothendieck group of $\mathrm{Proj}(A)$.

A morphism $A \to B$ naturally extends entrywise to $M_\infty(A) \to M_\infty(B)$; These morphisms induce maps $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$ which then pass to the Grothendieck group $K_0(A) \to K_0(B)$. In this manner, K_0 is a functor on the category of C^*-algebras.

LEMMA 66. *To any short exact sequence* $0 \to C \to B \to \frac{B}{C} \to 0$ *we get an induced half-exact sequence, using the entrywise induced maps discussed above:*

$$K_0(C) \to K_0(B) \to K_0(\frac{B}{C})$$

The definition of K_1 is more technical and is constructed from unitaries instead of projections. We present it tersely here. We denote by $GL(A)_0$ the connected component of the identity.

DEFINITION 67. $K_1(A) := \frac{GL(A)}{GL(A)_0}$.

In particular, for a finite invertible matrix $u \in GL_n(A)$, the class $[u] \in K_1(A)$ is the connected component containing $diag(u, 1_\infty) \in GL(A)$.

LEMMA 68. *To any short exact sequence* $0 \to C \to B \to \frac{B}{C} \to 0$ *we get an induced half-exact sequence:*

$$K_1(C) \to K_1(B) \to K_1(\frac{B}{C})$$

DEFINITION 69. (Boundary map)[**WO93**, Def. 8.1.1] Let $J \triangleleft A$ and let $x \in K_1(\frac{A}{J})$. Then we can find a $u \in \mathcal{U}_n^+(\frac{A}{J})$ such that $x = [u]$ and $v \in \mathcal{U}_k^+(\frac{A}{J})$ such that $diag(u, v)$ is homotopic to 1_{n+k} in $\mathcal{U}_{n+k}^+(\frac{A}{J})$. Let $wzin\mathcal{U}_{n+k}^+(A)$ be a lift of $diag(u, v)$. Then the *boundary map* $\delta : K_1(\frac{A}{J}) \to K_0(J)$ is defined by:

$$\delta(x) := [wp_n w^*] - [p_n].$$

This is a well-defined group homomorphism.

LEMMA 70. *This gives us a long exact sequence:*

$$K_1(J) \to K_1(A) \to K_1(\frac{A}{J}) \to^{\delta} K_0(J) \to K_0(A) \to K_0(\frac{A}{J})$$

Alternatively, we could have proceeded as we would have in topological K-theory, that is via *cones* and *suspensions.*

DEFINITION 71. Let A be a C^*-algebra. The Cone of A, denoted CA, is the set of functions: $\{f \in C([0,1], A) | f(0) = 0\}$. The suspension of A, denoted SA, is a subalgebra of the cone given by functions: $\{f \in CA | f(1) = 0\}$. We define the higher K-groups via suspensions: $K_n(A) := K_0(S^n A)$.

We remark that these definitions are equivalent for K_1.

THEOREM 72. *(Bott Periodicity) There is an isomorphism* $K_0(A) \cong K_0(S^2 A)$.

The proof of this result relies on constructing the *Bott map* β, which converts the long exact sequence into a *cyclic* exact sequence of length 6:

THEOREM 73. *Given a short exact sequence* $0 \to J \to A \to \frac{A}{J} \to 0$ *there is a cyclic long exact sequence:*

$$\begin{array}{ccccc} K_0(J) & \to & K_0(A) & \to & K_0(\frac{A}{J}) \\ \mu\uparrow & & & & \downarrow \\ K_1(\frac{A}{J}) & \leftarrow & K_1(A) & \leftarrow & K_1(J) \end{array}$$

This six term sequence is a key tool in computations concerning K-theory of C^*-algebras related in an extension.

The last ideas that are outlined in this section are those of Pimsner and Voiculescu on actions of groups on C^*-algebras via automorphisms.

Definition 74. Let A be a C^*-algebra represented on a Hilbert space $\mathcal{H}$ and let $\rho : G \to \mathrm{Aut}(A)$ be a representation of a group G. Then we can naturally form the C^*-algebra $A \rtimes_r G$; it is the completion of $C_c(G, A)$ equipped with a twisted convolution and completed in the norm that arises from the representation in $\mathcal{B}(\mathcal{H} \oplus \ell^2(G))$.

The following is original result of Pimsner-Voiculescu concerning the structure of infinite cyclic group actions [**PV82**]:

Theorem 75. [**WO93**, Theorem 10.2.1] *Let A be a C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. Then there is a cyclic 6-term exact sequence:*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{i_*} & K_0(A \rtimes \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes \mathbb{Z}) & \xleftarrow[i_*]{} & K_1(A) & \xleftarrow[1-\alpha_*]{} & K_1(A) \end{array}$$

The main idea of this result is that it gives an understanding of a crossed product structure by looking at only the induced action on the K-theory groups of the coefficient algebra. The main issue is that in general these can be as bad as one wants, so computations of the action could be particularly difficult.

These ideas naturally extend to free group actions by automorphisms [**PV82**]:

Theorem 76. [**WO93**, Theorem 10.8.1] *Let α_i, $i \in \{1, ..., k\}$ be elements of $\mathrm{Aut}(A)$ that give a representation of F_k. Then there is a cyclic 6-term sequence:*

$$\begin{array}{ccccc} \bigoplus_{i=1}^k K_0(A) & \xrightarrow{\rho} & K_0(A) & \xrightarrow{i_*} & K_0(A \rtimes_r F_k) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_r F_k) & \xleftarrow[i_*]{} & K_1(A) & \xleftarrow[\rho]{} & \bigoplus_{i=1}^k K_1(A) \end{array}$$

with $\rho := \sum_{i=1}^k (1 - \alpha_{i,*})$.

The major Corollary of this result gives a computation of the K-theory of a Free group C^*-algebra inductively by considering the action on $\mathbb{C}$.

Generalisations of this situation to F-inverse monoids play the main role in the next chapter.

CHAPTER 3

Partial Translations and Inverse Semigroups.

In this chapter we outline the construction of a short exact sequence of C^*-algebras associated to an F-inverse monoid S. This relates the reduced C^*-algebra of S to the reduced C^*-algebra of its maximal group homomorphic image, generalising some of the ideas present in the proof of the Pimsner-Voiculescu short exact sequence from [**PV82**]. We then make use of this result in a metric context; to any subspace X of a given finitely generated discrete group G it is possible to associate an object called a *partial translation structure* to X [**BNW07**]. This has a naturally associated inverse monoid and we investigate precisely when this inverse monoid is F-inverse. In this case, we construct an analogue of the short exact sequence for these inverse monoids. This has applications within K-theory, which we discuss at the beginning of the Chapter 5.

1. A Pimsner-Voiculescu short exact sequence for an F-inverse monoid.

In this section we construct a sequence of C^*-algebras that can naturally be associated to an F-inverse monoid and prove that it is exact. The process of doing this requires careful analysis of the representation theory of the universal groupoid associated to the inverse monoid in question. Later in Section 1.2 of this chapter we utilise this machinery to prove the following result, which is Theorem 83 in the text.

Let G be the maximal group homomorphic image of S.

Theorem. *Let S be an F-inverse monoid and let $\mathcal{G}_{\widehat{E}}$ be its universal groupoid. Then there is a distinguished element, denoted by ∞, of $\widehat{E}$. We denote the compliment of ∞ by U, which is both open and saturated and let $A = C_c(\mathcal{G}_U)$. Then we have the following short exact sequence of C^*-algebras:*

$$0 \to \overline{A} \to C^*_r(\mathcal{G}_{\widehat{E}}) \to C^*_r(G) \to 0.$$

The first step in this is to understand the representation theory of the universal groupoid.

Lemma 77. *Let S be a 0-F-inverse monoid, let $\mathcal{G} = \mathcal{G}_{\widehat{E}}$ be the universal groupoid and let $\{L^2(\mathcal{G}_x)\}_{x\in\widehat{E}}$ be the field of Hilbert spaces associated with $\mathcal{G}$. Let $x, y \in \widehat{E}$ such that $x \subset y$ Then there exists a projection $Q_{y,x} : L^2(\mathcal{G}_y) \to L^2(\mathcal{G}_x)$ such that $\lambda_x(1_{tt^*}\delta_t) = Q_{y,x}\lambda_y(1_{tt^*}\delta_t)Q^*_{y,x}$.*

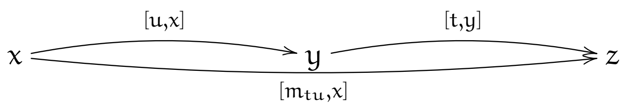


FIGURE 1. The action of $\lambda_x(1_{tt^*}\delta_t)$

PROOF. A basis for $L^2(\mathcal{G}_x)$ is given by Dirac functions of its elements, i.e $\{\delta_{[s,x]} : [s,x] \in \mathcal{G}_x\}$. Claim 28improves this by considering the maximal element in each equivalence class, $\{\delta_{[t_s,x]} : [s,x] \in \mathcal{G}_x\}$. Let $L_x = \{t \in \text{Max(S)} : [t,x] \in \mathcal{G}_x\}$. As $x \subset y$ we know that $L_x \subset L_y$ and this allows us to construct the projection from $L^2(\mathcal{G}_y)$ on the basis in the following way:

$$Q_{y,x}(\delta_{[t,y]}) = \begin{cases} \delta_{[t,x]} \text{ if } t \in L_x \\ 0 \text{ else} \end{cases} \tag{3}$$

This function is clearly surjective; we extended this linearly. To see that this is a bounded operator we observe that truncation of a Hilbert space element to a subset is norm decreasing.

Now to see the last part of the lemma we appeal to the definition of the convolution. Let $\nu = \sum_{u \in L_x} a_u \delta_{[u,x]} \in L^2(\mathcal{G}_x)$. Then

$$\lambda_x(1_{tt^*}\delta_t)(\nu)([m,x]) = \sum_{\substack{[n,y][u,x] \\ =[m,x]}} 1_{tt^*}([n,y])\nu([u,x]) = \sum_{\substack{[t,y][u,x] \\ =[m,x]}} \nu([u,x]) = \nu([u,x])$$

Where $[m,x] = [m_{tu},x]$ is the maximal representative of the element $[tu,x]$ using Claim 28. We see that:

$$\lambda_x(1_{tt^*}\delta_t)(\delta_{[u,x]}) = \delta_{[m_{tu},x]} \text{ if } u \in L_x \text{ and } m_{tu} \in L_x$$

So $\lambda_x(1_{tt^*}\delta_t)$ acts on those elements $[u,x]$ for which there exists a maximal element m and a $y \in \widehat{E}$ such that $[m,x] = [tu,x]$.

Now consider what happens for a general element $\nu = \sum_{u \in L_x} a_u \delta_{[u,x]} \in L^2(\mathcal{G}_x)$. We get the following:

$$Q_{y,x}(\lambda_y(1_{tt^*}\delta_t))Q_{y,x}^*(\nu) = Q_{y,x}(\lambda_y(1_{tt^*}\delta_t))(\nu') \tag{4}$$

where $\nu' = \sum_{u \in L_y} a_u \delta_{[u,y]} \in L^2(\mathcal{G}_y)$ with $a_u = 0$ if $u \notin L_x$. Then

$$(4) = Q_{y,x}(\sum_{\substack{m_{tu} \\ u \in L_x}} a_u \delta_{[m_{tu},y]}) = \sum_{\substack{m_{tu} \\ u \in L_x, m_{tu} \in L_x}} a_u \delta_{[m_{tu},x]} = \lambda_x(1_{tt^*}\delta_t)(\nu)$$

□

We specialise this result in the case that S is F-inverse. As such a monoid has no zero element the function $1_E : E \to \{0,1\}$ that assigns 1 to every idempotent is a valid character. We denote the ultrafilter that corresponds to that character by ∞.

COROLLARY 78. *Let S be an F-inverse monoid and let $\mathcal{G} = \mathcal{G}_{\widehat{E}}$ be its universal groupoid and let $\{L^2(\mathcal{G}_x)\}_{x\in\widehat{E}}$ be the field of Hilbert spaces associated with $\mathcal{G}$. Then for each $x \in \widehat{E} \setminus \{1_E\}$ there exists a projection map $Q_x : L^2(\mathcal{G}_\infty) \to L^2(\mathcal{G}_x)$ such that $\lambda_x(1_{tt^*}\delta_t) = Q_x\lambda_\infty(1_{tt^*}\delta_t)Q_x^*$.*

PROOF. The ultrafilter ∞ contains all filters of $E(S)$. We apply Lemma 77 to construct maps $Q_x = Q_{\infty,x}$ for each $x \in \widehat{E} \setminus \{1_E\}$. □

1.1. Representations of $C_c(\mathcal{G}_{\widehat{E}})$ for an F-inverse monoid. We discuss representations of an F-inverse monoid. We make use of the following result from [**KS02**]:

PROPOSITION 79. [**KS02**, Cor 2.4] *For a dense subset $D \subset \widehat{E}$ we have $\|f\|_r = \|\lambda(f)\| = \sup\{\|\lambda_x(f)\| : x \in D\}$.*

This is useful as the idempotents E are dense in $\widehat{E}$ as $\widehat{E}$ is a compactification of E. Additionally, recall that $\widehat{E}_{tight}$ is the closure of $\widehat{E}_\infty$ in $\widehat{E}$.

DEFINITION 80. An idempotent $e \in E$ is *primitive* if e is minimal amongst the elements of $E(S)\setminus\{0\}$.

We denote by $\mathcal{G}_{tight}$ the restriction of $\mathcal{G}_{\widehat{E}}$ to $\widehat{E}_{tight}$. We can truncate to build a quotient from $C_r^*(\mathcal{G})$ onto $C_r^*(\mathcal{G}_{tight})$:

PROPOSITION 81. *Let S be an 0-F-inverse monoid with no primitive idempotents and let $\mathcal{G} = \mathcal{G}_{\widehat{E}}$ be its universal groupoid. Then we have a surjective *-homomorphism from $C_r^*(\mathcal{G})$ onto $C_r^*(\mathcal{G}_{tight})$.*

PROOF. We construct the map q using truncation of functions:

$$q : \sum_{t\in \mathrm{Max}(S)} f_t\delta_t \mapsto \sum_{t\in \mathrm{Max}(S)} f_t|_{\widehat{E}_{tight}}\delta_t$$

We need to show that

(1) The map q is contractive
(2) The map q is a *-homomorphism.

To tackle (1) we consider the regular representations of $f = \sum_{t\in \mathrm{Max}(S)} f_t\delta_t$ and qf respectively. Using the following (commuting) diagram:

$$\begin{array}{ccc} C_c(\mathcal{G}_{\widehat{E}}) & \xrightarrow{\quad q \quad} & C_c(\mathcal{G}_{\text{tight}}) \\ \Big\downarrow \lambda & \searrow^{\lambda_{\text{tight}}} & \Big\downarrow \lambda_R \\ \mathcal{B}(L^2(\mathcal{G}_{\widehat{E}})) & \xrightarrow{\quad p \quad} & \mathcal{B}(L^2(\mathcal{G}_{\widehat{E}_{\text{tight}}})) \end{array}$$

where λ_R is the left regular representation of $C_c(\mathcal{G}_{\text{tight}})$. It follows from the definition of p that the bottom triangle commutes and the top triangle commutes as:

$$\lambda_x(f) = \sum_{t \in \text{Max(S)}} f_t(x)\lambda_x(1_{tt^*}\delta_t) = \lambda_x(qf)$$

For each $x \in \widehat{E}_{\text{tight}}$. Hence:

$$\|qf\|_r = \sup_{x \in \widehat{E}_{\text{tight}}} \{\|\lambda_x(qf)\|\} = \sup_{x \in \widehat{E}_{\text{tight}}} \{\|\lambda_x(f)\|\} \leq \|\lambda(f)\| = \|f\|_r$$

and so q is contractive (and therefore continuous).

Now to consider (2). It is enough to compute the result of products of elements of the form $f_s\delta_s$ for some $s \in$ Max(S). We then check the following two identities:

I $q(f_s\delta_s f_t\delta_t) = q(f_s\delta_s)q(f_t\delta_t)$

II $(q(f_s\delta_s))^* = q((f_s\delta_s)^*)$

To see (I) compute on a single element:

$$(f_s\delta_s f_t\delta_t)([st, \phi]) = f_s(\widehat{\rho}_t(\phi))f_t(\phi)$$

Apply q:

$$q(f_s\delta_s f_t\delta_t)([st, \phi]) = (f_s\delta_s f_t\delta_t)|_{\widehat{E}_{\text{tight}}}([st, \phi]) = f_s(\widehat{\rho}_t(\phi))f_t(\phi)$$

For all $[st, \phi] \in \mathcal{G}_{\widehat{E}_{\text{tight}}}$. Then compute the right hand side:

$$(q(f_s\delta_s)q(f_t\delta_t))([st, \phi]) = f_s|_{\widehat{E}_{\text{tight}}}(\widehat{\rho}_t(\phi))f_t|_{\widehat{E}_{\text{tight}}}(\phi)$$

Which matches for each $[st, \phi] \in \mathcal{G}_{\widehat{E}_{\text{tight}}}$.

To prove (II) we need to compute on a single element, where $(f_s\delta_s)^* = \alpha_{s^*}(\overline{f_s})\delta_{s^*}$:

$$\begin{aligned} q((f_s\delta_s)^*)([s^*,x]) &= \alpha_{s^*}(\overline{f_s})_{\widehat{E}_{tight}}(x) \\ &= \overline{f}(\widehat{\rho}_s(x)) \\ &= \overline{f(\widehat{\rho}_s(x))} \\ &= \overline{q(f)}(\widehat{\rho}_s(x)) \\ &= \alpha_{s^*}(\overline{q(f)})(x) \end{aligned}$$

Where the above holds for all $x \in \widehat{E}_{tight}$ where the function f_s is defined at $\widehat{\rho}_s(x)$ as required.

As q is a continuous *-homomorphism, it extends to the completions. □

1.2. Applying the machinery. We encode the norm estimations required for the proof of Theorem 83 in the following Lemma:

LEMMA 82. *Let S be F-inverse and let $K \subset$ Max(S) such that K is finite and $T = \sum_{t\in K} a_t\lambda(1_{tt^*}\delta_t)$ such that a_t is the constant function that has value a_t on D_{tt^*}. Then $\|T\| = \|qT\|$*

PROOF. It is immediate that $\|T\| \geq \|qT\|$ as q is contractive. We arrive at the other inequality by applying Corollary 78.

$$\|T\|_{L^2(\mathcal{G}_x)} = \|\sum_{t\in K} a_t\lambda_x(1_{tt^*}\delta_t)\|_{L^2(\mathcal{G}_x)} = \|\sum_{t\in K} a_t Q_x\lambda_\infty(1_{tt^*}\delta_t)Q_x^*\|_{L^2(\mathcal{G}_x)}$$
$$= \|Q_x(\sum_{t\in K} a_t\lambda_\infty(1_{tt^*}\delta_t))Q_x^*\|_{L^2(\mathcal{G}_x)} = \|Q_x(qT)Q_x^*\|_{L^2(\mathcal{G}_x)} \leq \|qT\|_{L^2(\mathcal{G}_\infty)}.$$

This holds for every $x \in E$ and so by $\|T\| = \|\lambda(T)\| = \sup\{\|\lambda_x(T)\| : x \in E\} \leq \|qT\|$. This gives the desired equality. □

We remark here that if S is F-inverse then minimal elements do not play a role in the ultrafilters, which was the reason for removing them when S had a zero. Additionally, in this instance the groupoid $\mathcal{G}_{tight}$ is just the maximal group homomorphic image G.

THEOREM 83. *Let S be an F-inverse monoid, let $\mathcal{G}_{\widehat{E}}$ be its universal groupoid, with $U \subset \widehat{E}$ the compliment of $\widehat{E}_{tight}$. Let G be its maximal group homomorphic image. Then we have the following short exact sequence of C^*-algebras:*

$$0 \to C_r^*(\mathcal{G}_U) \to C_r^*(\mathcal{G}_{\widehat{E}}) \to C_r^*(G) \to 0$$

PROOF. Denote by A the algebra $C_c(\mathcal{G}_U)$. We know that we have a surjective *-homomorphism q from $C_r^*(\mathcal{G}_{\widehat{E}})$ to $C_r^*(G)$, we just need to see that the kernel of this map is $\overline{A}$. The set $\overline{A}$ is contained in the kernel as elements in A are sums of functions with value at $1_E = \infty \in \widehat{E}$ of zero and projection onto this value (i.e applying q) will send the entire element to $0 \in C^*_r(G)$. So it is enough to show that A is dense in the kernel.

First consider finite sums. Let $f \in C_c(\mathcal{G}_{\widehat{\mathcal{E}}})$. We need to show that if $qf = 0$ then $f \in A$.

f has the form:

$$f = \sum_{s \in S} f_s \delta_s \text{ where } f_s \in C(D_{ss^*})$$

With only finitely many non-zero terms. This can be viewed concretely on $L^2(\mathcal{G}_{\widehat{E}})$ using

$$\lambda(f) = \sum_{s \in S} f_s \lambda(1_{ss^*}\delta_s)$$

As S is F-inverse we can reduce this sum using the observation that for each $s \in S$ we can write the term $f_s\delta_s$ as $f_s\chi_s\delta_{t_s}$ where t_s is the maximal element above s. So for each $t \in$ Max(S) we can define $f'_t = \sum_{s \leq t} f(s)\chi_s$ and then

$$\lambda(f) = \sum_{t \in \text{Max(S)}} f'_t \lambda(1_{tt^*}\delta_t) \tag{5}$$

(5) is in the kernel of q if and only if each $f'_t(\infty) = 0$ for every $t \in$ Max(S) that is if and only if $\lambda(f) \in A$.

Now let T be an element of $C^*_r(\mathcal{G}_{\widehat{E}})$ such that $qT = 0$. Then we need to show T can be approximated by finite sums that lie in A. Let T_n be finite sums in $C_c(\mathcal{G}_{\widehat{E}})$ with $T_n \to T$. Without loss of generality, these T_n have the following form for some finite $K_n \subset$ Max(S):

$$T_n = \sum_{t \in K_n} f^n_t \lambda(1_{tt^*}\delta_t)$$

then $qT_n = \sum_{t \in K_n} f^n_t(\infty)\lambda_\infty(1_{tt*}\delta_t)$. Define a pullback of qT_n:

$$S_n = \sum_{t \in K_n} a^n_t \lambda(1_{tt*}\delta_t) \in C_c(\mathcal{G}) \tag{6}$$

Where a^n_t is the constant function on D_{tt^*} with value $f^n_t(\infty)$. It is clear that $qS_n = qT_n$ and using Lemma 82 we have that $\|S_n\| = \|qS_n\| = \|qT_n\|$ so $\|S_n\| \to 0$

Let $U_n = (T_n - S_n)$. Then $U_n \in A$ and $U_n = (T_n - S_n) \to (T - 0) = T$, whence A is dense in $ker(q)$. □

2. A similar sequence for 0-F-inverse monoids

In this section we consider a generalisation of Theorem 83 to strongly 0-F-inverse monoids under some light conditions, and we proceed by considering saturated subsets of the unit space as defined in Chapter 2. Clearly, subsets that are invariant under the action of S are also saturated. The following Lemma outlines the connections between saturation and Morita enveloping actions.

LEMMA 84. *Let $\mathcal{G}$ be a étale locally compact Hausdorff groupoid with a (T,C,F)-cocycle ρ to Γ. Then the relation $\sim$ used in constructing the Morita envelope of $\mathcal{G}^{(0)}$ on $\mathcal{G}^{(0)} \times \Gamma$ preserves saturated subsets of $\mathcal{G}^{(0)}$*

PROOF. Let U be a saturated subset of $\mathcal{G}^{(0)}$ and let $x \in U$, $y \in U^c$. Assume for a contradiction that $(x, g) \sim (y, h)$ in $\mathcal{G}^{(0)} \times \Gamma$. Then there exists a $\gamma \in \mathcal{G}$ such that $s(\gamma) = x$, $r(\gamma) = y$ and $\rho(\gamma) = g^{-1}h$, but as U is saturated no such γ exists. □

Additionally, as we have no obvious group to consider, we introduce a universal one:

DEFINITION 85. Let S be an inverse semigroup. Then the universal group of S, denoted by $U(S)$, is the group generated by the elements of S with relations $s \cdot t = st$ if $st \neq 0$.

The map that sends $s \in S$ to $s \in U(S)$ is a partial homomorphism that is universal for partial homomorphism onto groups. It also induces a prehomomorphism $S \to U(S)^0$. An inverse monoid S is strongly 0-E-unitary if and only if this prehomomorphism is idempotent pure.

Recall that a group G is said to be *K-exact* if for every short exact sequence of G-C^*-algebras

$$0 \to A \to B \to C \to 0$$

the corresponding sequence:

$$0 \to A \rtimes_r G \to B \rtimes_r G \to C \rtimes_r G \to 0$$

is exact at the level of K-theory groups (i.e gives rise to a long exact sequence in K-theory).

THEOREM 86. *Let S be a strongly 0-F-inverse monoid with universal group $G := U(S)$. If G is infinite and K-exact then the sequence:*

$$0 \to C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}_{\widehat{E}}) \to C^*_r(\mathcal{G}_{\widehat{E}_{tight}}) \to 0$$

is exact at the level of K-theory.

PROOF. We begin by using either Theorem 55 or 53 to construct a transformation groupoid $Y_{\widehat{E}} \rtimes G$ and a Morita equivalence between $\mathcal{G}_{\widehat{E}}$ and $Y_{\widehat{E}} \rtimes G$. We can repeat this process for both $\widehat{E}_{tight}$ and $U := \widehat{E}^c_{tight}$, and by Lemma 84 and the fact that $\widehat{E}_{tight}$ is closed in $\widehat{E}$ we can conclude that we have a natural sequence of commutative C^*-algebras:

$$0 \to C_0(Y_U) \to C_0(Y_{\widehat{E}}) \to C_0(Y_{\widehat{E}_{tight}}) \to 0$$

each of which is a G-algebra. We now act by the reduced cross product, which produces a sequence of C^*-algebras:

$$0 \to C_0(Y_U) \rtimes_r G \to C_0(Y_{\widehat{E}}) \rtimes_r G \to C_0(Y_{\widehat{E}_{tight}}) \rtimes_r G \to 0.$$

This may not be exact in the middle term. However by K-exactness of G it is exact at the level of K-theory, so consider exact sequence:

$$\begin{array}{ccccccccc}
\dots & \to & K_0(C_0(Y_U) \rtimes_r G) & \to & K_0(C_0(Y_{\widehat{E}}) \rtimes_r G) & \to & K_0(C_0(Y_{\widehat{E}_{tight}}) \rtimes_r G) & \to & \dots \\
 & & \cong\uparrow & & \cong\uparrow & & \cong\uparrow & & \\
\dots & \longrightarrow & K_0(C_r^*(\mathcal{G})) & \longrightarrow & K_0(C_r^*(\mathcal{G}_{\widehat{E}})) & \longrightarrow & K_0(C_r^*(\mathcal{G}_{\widehat{E}_{tight}})) & \longrightarrow & \dots
\end{array}$$

where the isomorphisms are induced by the Morita equivalences given by Theorems 55 and 53. This concludes the proof. □

3. Translation Structures to Inverse Monoids

In this section we outline the definition of a partial translation structure and describe some of the results concerning them from the literature. Focusing on a special case, which we call *grouplike partial translation structures*, we connect uniform embeddability into groups for metric spaces to translation structures. We then outline an inverse monoid approach to understanding the translation algebra that can be naturally associated to a partial translation structure.

Definition 87. Let X be a metric space and let t be a partial bijection of X. The map t is a *partial translation* if there exists $R > 0$ such that for every $x \in \mathrm{Dom}(t)$ we have $d(x, t(x)) \leq R$. Let $\mathcal{T}$ be a collection of disjoint partial bijections of X. Then a partial bijection u of X is a *cotranslation of* $\mathcal{T}$ if for every $t \in \mathcal{T}$ ($x \in \mathrm{Dom}(t)$ and $x, t(x) \in \mathrm{Dom}(u)$) implies $u(x) \in \mathrm{Dom}(t)$ and $t(u(x)) = u(t(x))$.

The concept of a partial translation structure was first introduced in [**BNW07**]. By associating to a metric space this additional information, namely a collection of partial bijections that form entourages in the metric coarse structure, it is possible to use the local symmetries of the space to control the metric.

These definitions allow for a rephrasing to the language of inverse semigroups of the definitions in [**BNW07**].

Definition 88. A partial translation structure on X is a collection $\mathcal{T}$ of partial translations of X such that for all $R > 0$ there exists a finite subset $\mathcal{T}_R$ of disjoint partial translations in $\mathcal{T}$ and a collection Σ_R of partial cotranslations of $\mathcal{T}_R$ satisfying the following axioms:

(1) The union of the partial translations $t \in \mathcal{T}_R$ contains the R-neighbourhood of the diagonal.
(2) There exists k such that for each $x, x' \in X$ there are at most k elements $\sigma \in \Sigma_R$ such that $\sigma x = x'$.
(3) For each $t \in \mathcal{T}_R$ and for all $(x, y), (x', y') \in t$ there exists $\sigma \in \Sigma_R$ such that $\sigma x = x'$ and $\sigma y = y'$.

Definition 89. (Freeness, Global control) A partial translation structure on X is said to be *free* if in Definition 88 ii) $k = 1$; i.e there is a unique cotranslation such that for each pair $(x, y) \in X \times X$ we have $\sigma x = y$.
A partial translation structure on X is said to be *globally controlled* if the partial cotranslation orbits are partial translations.

The following is Theorem 19 from [**BNW07**].

Lemma 90. *Let* G *be a group equipped with a proper left invariant metric and let* $X \subseteq G$ *equipped with the induced metric. Then the restriction of the action of* G *on itself by right multiplication to* X *is a partial translation structure on* X *that is free and globally controlled, with cotranslations given by the left action by isometries.*

The intuition for the Definition 88 is a metric version of a group action for spaces, with freeness and global control giving conditions that are similar to a free and transitive action of group.

Definition 91. Let $\mathcal{T}$ be a partial translation structure. Then we say $\mathcal{T}$ has *zero divisors* if there exists a product of disjoint translations $t_1, t_2, ..., t_n \in \mathcal{T}$ such that $t_1 t_2 t_3 ... t_n$ is empty (i.e has empty domain). We say $\mathcal{T}$ has no zero divisors if no such product is empty.

We specialize our definition slightly in light of the following proposition, the proof of which can be found in [**Put10**, Proposition 8.1]

Recall that a subspace Y of a metric space X is said to be *coarsely dense* if there exists $c > 0$ such that for every $x \in X$ there is a $y \in Y$ satisfying $d(x, y) < c$.

Proposition 92. *Let* G *be a countable discrete group with a proper left invariant metric and let* $X \subseteq G$ *be a metric subspace. The following are equivalent:*

(1) X^c *is not coarsely dense in* G.
(2) For every $R > 0$ *there exists* $g \in G$ *such that* $B_R(g) \subseteq X$.
(3) The monoid generated by the translations obtained by restriction of the right action of G *on itself has no zero element.*

The definition provided below is stronger than the definition provided in [**BNW07**], however this better emulates the situation that arises when you consider a space that is uniformly embedded into a group.

DEFINITION 93. Let X be a countable set. A collection of partial bijections $\mathcal{T}$ is a called an *abstract partial translation structure* for X if:

(1) $\mathcal{T}$ partitions $X \times X$.
(2) $\forall t_i, t_j \in \mathcal{T}$ $\exists t_k \in \mathcal{T}$ such that $t_i t_j \subseteq t_k$ (i.e $\mathcal{T}$ is subclosed).
(3) $\forall t \in \mathcal{T}$ we have $t^* \in \mathcal{T}$.
(4) $\mathcal{T}$ has a global identity, denote this t_0.

REMARK 94. Additionally if X is equipped with a partial translation structure from Definition 93 it is possible to construct a coarse structure on X using the entourages generated by $\mathcal{T}$. However, this may not be the metric coarse structure if X is a metric space. Even if we assume that all the partial bijections in $\mathcal{T}$ are partial translations we may also not capture the metric entirely. So we make the following definition for metric spaces:

DEFINITION 95. A (grouplike) partial translation structure on a metric space X is an abstract partial translation structure $\mathcal{T}$ where:

(1) each $t \in \mathcal{T}$ is a partial translation;
(2) for each $R > 0$ the R-neighbourhood of the diagonal is contained within finitely many distinct elements of $\mathcal{T}$.

Condition (2) ensures that such a structure is compatible with the metric.

As a consequence of the Wagner-Preston Theorem [**How95**] partial bijections move us toward inverse semigroup theory.

PROPOSITION 96. *Let X be a metric space equipped with a group-like partial translation structure $\mathcal{T}$. Then the subsemigroup of $I_b(X)$ generated by $\mathcal{T}$ is an inverse monoid.*

PROOF. The axioms given in Definition 93 include the existence of inverse elements, which proves the subsemigroup is inverse. From these axioms we additionally know that $\mathcal{T}$ includes a global identity, which proves that the subsemigroup is a monoid. □

LEMMA 97. *The inverse monoid S generated by a partial translation structure $\mathcal{T}$ is 0-F-inverse, with maximal element set $\mathcal{T}$.*

PROOF. First we prove maximality of the translations. We prove that for any $s \in S \setminus \{0\}$ there exists a unique $t \in \mathcal{T}$ such that $s \leq t$. Property (2) from the Definition 93 implies that any product of elements of $\mathcal{T}$ is less than a unique $t \in \mathcal{T}$. As $\mathcal{T}$ partitions $X \times X$ we have that for any pair $t_i, t_j \in \mathcal{T}$ $et_i = et_j \Leftrightarrow t_i = t_j$.

Now we prove that S is 0-E-unitary. Let $e \in E(S)\backslash\{0\}$ and $s \in S\backslash\{0\}$. As any product of translations is included in a unique translation, it is enough to consider the case that s is maximal. It follows that $e \leq s$ implies that s fixes some elements of X. However, $\mathcal{T}$ partitions $X \times X$ and so $s \leq id_X$. We assumed that s was maximal however, so $s = id_X$. Now any general word in $\mathcal{T}$ satisfies: $e \leq s \implies s \leq id_X$, hence s is idempotent. □

3.1. An Embeddability Theorem for Metric Spaces with Grouplike Partial Translation Structures. The precise nature of the relationship between partial translation structures in the sense of Definition 93 and uniform embeddings is partially understood from the work of Brodzki, Niblo and Wright [**BNW07**]. It follows from Theorem 19 of [**BNW07**] that given any space that admits a uniform embedding into a group, we can equip it with a translation structure satisfying Definition 93. The inverse monoid generated by this translation structure can also be described using Lemma 97.

In this section we provide a partial converse to Theorem 19 of [**BNW07**]:

THEOREM 98. *Let* X *be a countable discrete metric space equipped with a grouplike partial translation structure* $\mathcal{T}$*, where* $\mathcal{T}$ *has no zero divisors. Then there exists a countable discrete group* G *and an embedding* $X \hookrightarrow G$ *such that the translation structure provided by* G *restricted to* X *denoted* $\mathcal{T}_G|_X$ *is equal to* $\mathcal{T}$*.*

PROOF. Consider the inverse monoid $S = \langle \mathcal{T} \rangle$. By Lemma 97 S is 0-F-inverse, and as $\mathcal{T}$ has no zero divisors we know that in fact it must be F-inverse. As every $t \in \mathcal{T}$, $t \neq 1$ is not an idempotent, we can conclude that that S/σ is a non-trivial group. Denote that group by G. The aim now is to embed X into G. The maximal elements in $\mathcal{T}$ generate this group, and σ induces an inverse semigroup homomorphism from S into G, which is a bijection between the maximal elements and G. Denote by T_{x_0} the following:

$$T_{x_0} := \{t \in \mathcal{T} : \exists x \in X \text{ with } tx = x_0\} \tag{7}$$

where x_0 is a base point in X. Observe that because $\mathcal{T}$ partitions $X \times X$ we can construct a bijection between X and T_{x_0}, specifically $x \mapsto t_x$, where t_x is the element of T_{x_0} satisfying $t_x(x) = x_0$. Restricting to the image of T_{x_0} under σ we get a subspace of the group that is in bijection with X, i.e we can view X as a subset of the group G. To finish the proof, we need the translation structure $\mathcal{T}$ to come from the group.

Take a translation $t_j \in \mathcal{T}$. For every $x \in \mathrm{Dom}(t_j)$ there exists a unique $t_x \in T_{x_0}$ such that $t_x x = x_0$. For each $x \in \mathrm{Dom}(t_j)$ there exists a unique $y \in X$ such that $t_j x = y$ Taking inverses: $t_j^* y = x$. This gives a map: $t_x t_j^* y = x_0$ and y corresponds to some element in T_{x_0}, denote this t_y. This gives the following situation:

$$t_x t_j^* \subseteq t_y. \tag{8}$$

Under σ we have:

$$g_x g_j^{-1} = g_y \tag{9}$$

This action on the right by inverses agrees with the typical translation structure of a group restricted to X, as we can define, using the map σ and the above information:

$$t_{g_j} : g_x \mapsto g_y \tag{10}$$

where this construction holds for all $x \in \mathrm{Dom}(t_j)$. This tells us that $\mathrm{Dom}(t_j) \subseteq \mathrm{Dom}(t_{g_j})$. All that remains is to show the reverse inclusion. Let $h \in \mathrm{Dom}(t_{g_j})$ Then $h \in X \cap X g_j$ so $h = h^{'} g_j$ and:

$$t_{g_j} : h \mapsto h^{'} \tag{11}$$

Pulling h and $h^{'}$ back into X using the original bijection, we get a pair $(x, y) \in X \times X$. As $\mathcal{T}$ partitions $X \times X$ we have a unique $t_p \in \mathcal{T}$ such that $t_p x = y$. Via σ we get the following situation:

$$h = h^{'} g_p = h^{'} g_j \Leftrightarrow g_p = g_j \tag{12}$$

And pulling back this gives us $t_p = t_j$. So for every point $x \in \mathrm{Dom}(t_{g_j})$ we have that $x \in \mathrm{Dom}(t_j)$.

Hence for each map in $\mathcal{T}$ we have a corresponding map in $\mathcal{T}_G|_X$ which is defined in the same places and is equal where it is defined. This implies $\mathcal{T} = \mathcal{T}_G|_X$ s required. □

The following is a direct corollary of Theorem 98 and Proposition 92:

COROLLARY 99. *The compliment of* $\sigma(T_{x_0})$ *is not coarsely dense in* G. □

In summary, Theorem 98 provides us a wealth of examples of F-inverse monoids with the added information of a concrete representation on an interesting metric space. It turns out that this provides a simplification to Theorem 83 when dealing with such representations.

3.2. Translation Algebras. Let X be a uniformly discrete bounded geometry metric space.

DEFINITION 100. The translation algebra associated with a partial translation structure $\mathcal{T}$ on X, denoted by $C^*\mathcal{T}$, is the C^*-algebrea generated by $\mathcal{T}$, when each translation is viewed as bounded operator on $\ell^2(X)$

The aim of this section is to give a description of the partial translation algebra associated to a grouplike partial translation structure $\mathcal{T}$ with no zero divisors as the C^*-algebra of a groupoid, where the groupoid is related to the inverse monoid generated by the partial translations. We then recast Theorem 8.3 of Brodzki, Niblo, Putwain and Wright [**Put10**] outlining a short exact sequence of C^*-algebras arising from such translation structures. We later consider some examples in Chapter 5.

Given the information of Lemma 97 we have an inverse monoid that we can associate to a grouplike translation structure. This has a natural C^*-algebra, as outlined in Chapter 2. However, we have not used the geometric representation of this inverse monoid on $I(X)$, which determines a representation on $\ell^2(X)$ in the standard way. This representation will be the focus of this section. The following is Proposition 10.6 [**Exe08**]

Proposition 101. *Let μ be a representation of S on a Hilbert Space H. Then there exists a unique *-representation π_μ of $C_0(\widehat{E})$ on H such that $\pi_\mu(1_e) = \mu(e)$ for every $e \in E$ In addition $(\pi_\mu \times \mu)$ is a covariant representation for $\mathcal{G}_{\widehat{E}}$.*

The proof of the above result relies on the spectrum of the commutative C^*-subalgebra $A = C^*_{\pi_\mu}(E)$ of $C^*_{\pi_\mu}(S)$. We denote the spectrum of A by $\widehat{X}$. The key aspect of the proof of Proposition 101 is the continuous map j defined by:

$$\begin{aligned} j \;:\; & \widehat{X} \to \widehat{E} \\ & \psi \mapsto \phi = \psi \circ \mu \end{aligned}$$

This map is well defined as the elements of A are self adjoint projections on $\ell^2(X)$ and certainly injective as two elements ψ, ϕ satisfy $j(\phi) = j(\psi)$ if and only if for every $e \in E$ $\psi(\mu(e)) = \phi(\mu(e))$, and the $\mu(e)$ are dense in the algebra A.

Additionally, we will be interested in describing the elements of $\widehat{X}$ in terms of filters and ultrafilters of X. We restrict now to the case that μ is a bounded representation and S is a monoid. In this instance $A \subset \ell^\infty(X)$, and by considering spectra we get a natural map: $\beta X \twoheadrightarrow \widehat{X}$.

We remark here also that $\widehat{X}$ contains a dense image of X by pushing through the map from βX. This gives a description of the image of X by considering elements: $\psi_x : \mu(E) \to \{0, 1\}$ that have value 1 at $\mu(e)$ if $x \in \mu(e)$ and 0 otherwise.

Returning to the situation that we are interested in: Let X be a metric space equipped with a grouplike partial translation structure $\mathcal{T}$ we get an inverse monoid $S = \langle \mathcal{T} \rangle$ and a representation $\mu : S \hookrightarrow I_b(X)$ from Proposition 96. So applying Proposition 101 we arrive at a representation π_μ of $C(\widehat{E})$ on $\ell^2(X)$.

PROPOSITION 102. *Let S be a 0-F-inverse monoid and let $\mu : S \hookrightarrow I_b(X)$ be a geometric representation. Then the following hold for $\widehat{X}$:*

(1) $\widehat{X} \hookrightarrow \widehat{E}$ is a topological embedding

(2) $\beta X \twoheadrightarrow \widehat{X}$ is a quotient map.

Moreover the topologies are all compatible with the topology endowed as the spectrum of $A = C^*_{\pi_\mu}(E)$.

PROOF. We give a concrete proof when S has no 0: First we show (1) using the map j defined above. $j(\widehat{X})$ is compact as j is continuous and closed because $\widehat{E}$ is Hausdorff.

For (2) we observe that the quotient map is given by the equivalence relation

$$\phi \sim \phi' \leftrightarrow \phi \cap E(S) = \phi' \cap E(S)$$

This map is surjective as given any $\psi \in \widehat{X}$ we can view this as a filter on X be considering the set:

$$F_\psi = \{e \in E(S) | \psi(\sigma(e)) = 1\}$$

We can complete this to an ultrafilter in βX in many ways using Zorn's Lemma, however it is enough to show we can do it such that $F_{\psi,UF} \cap E(S) = \psi$. So it is enough to pick subsets according to the following rules. Let $M, M^c \in \{0,1\}^X$ and

- If $M \in E(S)$ then add M^c to F_ψ
- If $M \notin E(S)$ then add M to F_ψ
- If $M, M^c \notin E(S)$ add either to F_ψ

It is impossible for both M and M^c to be contained in F_ψ as $E(S)$ has no zero element.

Now F_ψ has the correct property and is an ultrafilter of βX that maps onto ψ. Observe that the image of βX is again compact, and thus closed, hence the map is a quotient.

In the case that S has a zero element, we appeal to universal properties and another result of Exel [**Exe08**]. First we need a definition, making use of the groupoid $G(X)$ defined in Chapter 2.

DEFINITION 103. Let $G(X)$ be the coarse groupoid of X as defined in Chapter 2. Then the *uniform Roe algebra*, denoted by $C^*_u(X)$, is the reduced groupoid C^*-algebra $C^*_r(G(X))$.

We now finish the proof of Proposition 102. By Proposition 10.10 [**Exe08**] the space $\widehat{X}$ is closed and invariant.

Recall that since $t \in \mathcal{T}$ is an element of $I_b(X)$ the algebra $C^*_\pi(S)$ is a subalgebra of the uniform Roe algebra $C^*_u X$. We now remark that the representation π_X, when restricted to $C^* E$ assigns each idempotent a projection in $C^*_u X$, that is $C^*_{\pi_\mu}(E) = \pi(C^*E) \subset \ell^\infty(X)$. Taking the spectra associated to this inclusion then gives us a map:

$$r_{\beta X} : \beta X \twoheadrightarrow \widehat{X}$$

which is continuous. In particular as both βX and $\widehat{X}$ are compact Hausdorff spaces, this map is closed (and open) and hence a quotient. □

REMARK 104. Two elements of the image of X in $\widehat{X}$, say ψ_x and ψ_y, are equal if and only if every idempotent $\mu(e)$ that contains x also contains y.

Recall that associated to an inverse monoid there is a universal groupoid $\mathcal{G}_{\widehat{E}}$ with unit space $\widehat{E}$. We consider the restriction of $\mathcal{G}_{\widehat{E}}$ to the subset $\widehat{X}$ and denote this by $\mathcal{G}_{\widehat{X}}$. It is immediate (using [**Exe08**, Prop 10.10]) that the set $\widehat{X}$ is invariant under the action of S. To compute the groupoid and groupoid C^*-algebras associated to $\widehat{X}$ explicitly in terms of X we need to know more about the Hilbert spaces fibers associated to $\mathcal{G}_{\widehat{X}}$ and as well as the action of $\mathcal{G}_{\widehat{X}}$ on the set $B := \{\psi_x | x \in X\}$, which is the image of X in $\widehat{X}$.

PROPOSITION 105. *Let* $t \in \mathcal{T}$. *Then* $\widehat{\rho}_t(\psi_x) = \psi_{t(x)}$ *for all* $x \in \mathrm{Dom}(t)$.

PROOF. First some observations:

(1) $\widehat{\rho}_t(\psi_x)$ is defined as $\widehat{\rho}_t(\psi_x) \in D_{tt^*} \Leftrightarrow \psi_x \in D_{t^*t} \Leftrightarrow t^*t \in \psi_x \Leftrightarrow x \in t^*t = \mathrm{Dom}(t)$.

(2) $(\widehat{\rho}_t(\psi_x))(tet^*) = \psi_x(t^*(tet^*)t) = \psi_x(e)$. Hence $e \in \psi_x \Leftrightarrow tet^* \in \widehat{\rho}_t(\psi_x)$.

(3) $\psi_x = \psi_y \Leftrightarrow \psi_{t(x)} = \psi_{t(y)}$, in fact more is true as: $\psi_{t(x)} = \psi_{t'(y)} \Leftrightarrow \mathrm{Dom}(t') = \mathrm{Dom}(t)$.

We prove inclusions. First $\widehat{\rho}_t(\psi_x) \subset \psi_{t(x)}$. Without loss of generality, we can take tet^* to be the general form of an element of $\widehat{\rho}_t(\psi_x)$ and then: $tet^* \in \psi_{t(x)} \Leftrightarrow t(x) \in tet^* \Leftrightarrow tet^*(t(x)) = t(x)$, which is the case if and only if $e \in \psi_x$.

To see the reverse inclusion let $f \in \psi_{t(x)}$. Then $f \in \widehat{\rho}_t(\psi_x) \Leftrightarrow t^*ft \in \psi_x \Leftrightarrow t(x) \in f \Leftrightarrow f \in \psi_{t(x)}$. □

PROPOSITION 106. B *is invariant and* $\mathcal{G}_B$ *acts transitively on* B.

PROOF. B is invariant as a consequence of Proposition 105 and $\mathcal{G}_B$ acts transitively by the first property of grouplike partial translation structures - $\mathcal{T}$ partitions $X \times X$. □

Recall that for each point $x \in X$ there exists a unique translation t_y to each other point $y \in X$. This then defines a bijective map $[t_y, \psi_x] \mapsto t_y(x) = y$. This bijection provides a unitary isomorphism between these spaces, denote this map at the level of Hilbert spaces by U_x for each $x \in X$.

Proposition 107. U_x *implements a spacial equivalence of norms, that is:* $\|\lambda(1_{tt^*}\delta_t)\| = \|\mu(t)\|_{\ell^2(X)}$ *for all* $t \in \mathcal{T}$.

Proof. The proof of this fact follows from a computation on the basis of $\ell^2(X)$ using the unitary isomorphism U_x. We compute $U_x\lambda_{\psi_x}(1_{tt^*}\delta_t)U_x^{-1}$ evaluated on a basis element $\delta_y \in \ell^2(X)$.

(1) $U_x^{-1}(\delta_y) = \delta_{[t_y,\psi_x]}$

(2) $\lambda_{\psi_x}(1_{tt^*}\delta_t)(\delta_{[t_y,\psi_x]})(\delta_{[s,\psi_x]}) = \sum_{\substack{[n,\psi_z][u,\psi_x] \\ =[s,\psi_x]}} 1_{tt^*}([n,\psi_z]\delta_{[t_y,\psi_x]}([u,\psi_x]) = \delta_{[tt_y,\psi_x]}$.

Now there is a unique s satisfying $tt_y \leq s$ by the definition of $\mathcal{T}$, whence $\lambda_{\psi_x}(1_{tt^*}\delta_t)$ moves the basis element $\delta_{[t_y,\psi_x]}$ to the basis element $\delta_{[s,\psi_x]}$.

(3) $U_x(\delta_{[tt_y,\psi_x]}) = \delta_{t(t_y(hx))} = \delta_{t(y)} = \mu(t)(\delta_y)$.

This holds for all y in the domain of t, as the multiplication in the groupoid is defined for only that situation.

As we have this equality for each $x \in X$; we get that $\|\lambda(1_{tt^*}\delta_t)\|_r = \sup\{\|\lambda_{\psi_x}(1_{tt^*}\delta_t)\| : x \in X\} = \|\mu(t)\|_{\ell^2(X)}$. □

This extends to finite sums:

Lemma 108. *Let* $K \subset \widehat{X}$ *be a finite subset and let* a_t *be the constant function valued* a_t *on* D_{tt^*}. *Then* $\|\sum_{t\in K} a_t\delta_t\|_r = \|\sum_{t\in K} a_t\mu(t)\|_{\ell^2(X)}$

Proof. First we show that $\sum_{t\in K} a_t\delta_t$ represents as $\sum_{t\in K} a_t\mu(t)$ on the basis of $\ell^2(X)$. We proceed as in Proposition 105.

First compute $U_x^{-1}(\delta_y)$:

$$U_x^{-1}(\delta_y) = \delta_{[t_y,\psi_x]}$$

Then compute:

$$\begin{aligned}
(\sum_{t\in K}\lambda_{\psi_x}(a_t\delta_t))(\delta_{[t_y,\psi_x]}) &= \sum_{\left\{t\in K: \substack{[t,\psi_y][t_y,\psi_x] \\ =[tt_y,\psi_x]}\right\}} a_t([t,\psi_y])\delta_{[t_y,\psi_x]}([t_y,\psi_x]) \\
&= \sum_{\{t\in K, y\in \mathrm{Dom}(t)\}} a_t([t,t_y(\psi_x)])\delta_{[tt_y,\psi_x]} \\
&= \sum_{t\in K, y\in \mathrm{Dom}(t)} a_t\delta_{[tt_y,\psi_x]}
\end{aligned}$$

Lastly move back to $\ell^2(X)$ via U_x:

$$\begin{aligned} U_x(\sum_{t\in K, y\in Dom(t)} a_t\delta_{[tt_y,\psi_x]}) &= \sum_{t\in K, y\in Dom(t)} a_t\delta_{t(y)} \\ &= (\sum_{t\in K} a_t\mu(t))(\delta_y) \end{aligned}$$

So both finite sums transform the basis in the same way. This equality holds for each ψ_x in B, so we can conclude that $\| \sum_{t\in K} a_t\delta_t\|_r = \sup\{\|\lambda_{\psi_x}(\sum_{t\in K} a_t\delta_t)\| : \psi_x \in B\} = \| \sum_{t\in K} a_t\mu(t)\|_{\ell^2(X)}$. □

This lets us define a map on the basis of $C_c(\mathcal{G}_{\widehat{X}})$, which when extended linearly has image $\mu(\mathbb{C}S)$. This map is given by:

$$\mathcal{Q} : \lambda(1_{tt^*}\delta_t) \mapsto \mu(t)$$

Now we can state and prove the main result of this section:

Theorem 109. *Let X be a subset of a countable discrete group G equipped with the natural partial translation structure $\mathcal{T}_G|_X v$ arising from a proper left invariant metric. Let μ denote the representation of $S = \langle \mathcal{T}_G|_X\rangle$ on X. If $\mathcal{T}_G|_X$ has no zero divisors then we have an isomorphism $C^*_r(\mathcal{G}_{\widehat{X}}) \cong C^*_\mu(S) = C^*\mathcal{T}$.*

Proof. The map $\mathcal{Q}$ is surjective onto $\mathbb{C}S$ (mapping to the generators of S), so it remains to see that it passes to the completion and is injective. To show this, we appeal to Lemma 108 to show that the norms match under the map $\mathcal{Q}$ up to finite sums - making the map on the incomplete algebras uniformly continuous. Ideally, we would now complete - however we need to be careful as the incomplete *-algebra M generated by finite sums of $1_e\delta_t$ may not be dense (and is the source of the map $\mathcal{Q}$).

However we observe that by the Stone-Weierstrass Theorem every element in $C(\widehat{X})$ can be approximated by elements of the form 1_e, i.e for all $f_t \in C(\widehat{X})$:

$$f_t = \lim_n(\sum_{e\in E} a^n_e 1_e)$$

So for a particular element $f = \sum_{t\in K} f_t\delta_t \in C_c(\mathcal{G}_{\widehat{X}})$ we can approximate each f_t in turn by limits of $\sum_{e\in E} a_e 1_e$ giving us an approximation by finite sums of elements in M. Hence $\overline{M} = C^*_r(\mathcal{G}_{\widehat{X}})$, allowing $\mathcal{Q}$ to pass to completions (by uniform continuity).

After passing to the completion, the map is isometric; hence injective. This gives us the first isomorphism. To see the equality, observe that by definition the translation algebra is the algebra generated in $\mathcal{B}(\ell^2(X))$ by the set of operators $\{\mu(t)|t \in \mathcal{T}\}$. This is precisely $C^*_\mu(S)$. □

3.3. A Short Exact Sequence of Translation Algebras. In this situation we still have access to all the tools available in the general inverse monoid case, as well as all the geometric properties arising from the representation μ.

THEOREM 110. *Let* $X \subset G$, $\mathcal{T} = \mathcal{T}_G|_X$ *be a grouplike partial translation structure on* X *with no zero divisors and* $S = \langle \mathcal{T} \rangle \hookrightarrow_\mu I(X)$ *be the associated F-inverse monoid. Then we have the following short exact sequence of* C^**-algebras:*

$$0 \to C^*_r(\mathcal{G}_U|_{\widehat{X}}) \to C^*_r(\mathcal{G}_{\widehat{X}}) \to C^*_r(G) \to 0 \tag{13}$$

Where the middle term is the translation algebra associated to X *arising from* $\mathcal{T}$

PROOF. The proof follows the same lines as the proof of Theorem 83.

(1) The map defined on finite sums still has the desired properties, i.e a finite sum is in the kernel if and only if all its components are 0.
(2) We still have the same pullbacks of constant functions to the entire space; This enables the same construction of approximating elements; who each have the same norm control property provided by Corollary 78.
(3) We can then conclude that the kernel is the desired algebra with a density argument.

□

CHAPTER 4

Counterexamples to Baum-Connes and Large Girth Expanders.

In this chapter we develop the basic concepts of coarse geometry for metric spaces and outline some coarse properties that can be associated to a metric space. We then outline certain analytic group and groupoid properties and connect these to their coarse counterparts. Secondly, we introduce the coarse Baum-Connes conjecture for proper metric spaces and outline machinery to construct counterexamples to this conjecture by converting the coarse assembly map into a groupoid assembly map.

We then present a unifying approach to all known counterexamples to this conjecture by developing the groupoid centric viewpoint of [**HLS02**] further, introducing a new conjecture known as the *boundary coarse Baum-Connes conjecture*. First however we outline the construction of the algebras involved in defining the coarse assembly map μ, then make some connections to groupoid equivariant KK-theory. These ideas then allow us to formulate the boundary coarse Baum-Connes conjecture and apply homological algebra techniques to connect it to the coarse Baum-Connes conjecture.

As an example of the flexibility of this new conjecture we verify it for certain large classes of expander graph, generalising work of Willett and Yu [**WY12a**].

1. Coarse geometry, Groupoids and Assembly.

In this section we outline the coarse geometry and group theoretic properties that are going to be considered throughout this chapter. The overall scheme of this section is first to consider some general coarse ideas associated to metric spaces and then move onto discussions of certain analytic properties held by discrete groups. We will develop these coarse ideas further by introducing abstract coarse structures and their relationship to metric spaces.

1.1. Coarse geometry. The notions of coarse geometry are similar in spirit to those of topology, however the focus is on the large as opposed to the small. Recall that a function f between topological spaces X and Y is continuous if the preimage of every open set in Y is open in X. Suppose additionally that X and Y are metric spaces equipped with the metric topology. The key idea in coarse geometry is somehow to supplant this notion of continuous by replacing the occurrences of open in the definition with *bounded*. We call such maps *metrically proper*. If additionally suppose that the map f sends sets of diameter R to sets of diameter at most S for some $S > 0$. Then we call such a map *bornologous*. Combining these two, we arrive at a definition:

Definition 111. Let $f : X \to Y$ be a map of metric spaces. f is *coarse* if it is both metrically proper and bornologous.

A coarse map, intuitively, preserves the structure of a metric space on large scales. We call a pair of maps $f : X \to Y$, $g : Y \to X$ *close* if there is a uniform bound R such that $d(g(f(x), x) < R$ and $d(y, f(g(y)) < R$. Two metric spaces X and Y are *coarsely equivalent* if we can find maps $f : X \to Y$, $g : Y \to X$ that are both coarse such that the pair are close. Classifying spaces by their coarse type is one of the basic goals of coarse geometry. An example where such coarse equivalences turn out to be useful is in analysing the topology of manifolds via their fundamental group; this idea also motivates the more technical coarse geometric ideas that permeate throughout this chapter.

Lemma 112. *(Švarc - Milnor) Let Γ be a countable discrete group and let X be an proper, free, cocompact Γ-metric space. Additionally suppose that the action is isometric. Then X is coarsely equivalent to Γ.* □

Example 113. Let M be a closed compact manifold and let $\pi_1(M) = \Gamma$ be its fundamental group. Then Γ is coarsely equivalent to $\tilde{M}$, the universal cover of M by the Lemma above.

We introduce now another concept, similar to the previous notion of coarse map, that allows us to talk about uniformly controlled embeddings:

Definition 114. A map is called *effectively proper* if additionally, for each $R > 0$ there exists an $S > 0$ such that the preimage of each ball of radius R in Y is contained in a ball of radius S in X.

This notion seems a little less natural than a metrically proper mapping, however it plays an important role in describing an embedding in this category. In particular, focusing on coarse embeddings, it is enough to consider pairs of maps that are effectively proper and bornlogous. The following is proved in [**Gue**].

Lemma 115. *Let X and Y be coarsely equivalent metric spaces. Then there exists f and g that are effectively proper and bornologous that implement this coarse equivalence.*

The notion of a coarse embedding is fundamental to the applications of this theory to topology and analysis, so we make it precise here:

Definition 116. A metric space X is said to admit a coarse embedding into Hilbert space $\mathcal{H}$ if there exist maps $f : X \to \mathcal{H}$, and non-decreasing $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}$ such that:

(1) for every $x, y \in X$, $\rho_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x, y))$;
(2) for each i, we have $\lim_{r\to\infty} \rho_i(r) = +\infty$.

This connects to the notion of an effectively proper, bornologous map by observing that our controls (i.e the S's that appear within the definitions) will arise as the value of the control functions $\rho_\pm$ at R.

This property is exceptionally flexible; many constructions of metric spaces preserve coarse embeddability [**Gue**]. This property will be important in later in the chapter where it plays an important role in results concerning groupoids and the coarse Baum-Connes conjecture.

1.2. Properties of finitely generated discrete groups and étale groupoids. Via the construction of a Cayley graph, it is possible to introduce a metric to a finitely generated group that is compatible with the group structure. Many of the ideas in geometric group theory rely on computing coarse properties, such as coarse embeddings, for this metric. Additionally, coarse properties of this metric often connect to analytic properties of groups that are equivariant with respect to the group action. In the global context of the Baum-Connes conjecture for groups and groupoids the following property plays an important role. As we primarily consider transformation groupoids we focus the definition on them.

Definition 117. Let H be a continuous field of Hilbert spaces over X. we say that a transformation groupoid $G := X \rtimes \Gamma$ acts on H by affine isometries if for every $(x, g) \in G$ there exists an affine isometry $A_{(x,g)} : H_{xg} \to H_x$ such that:

(1) $A_{(x,e)}$ is the identity map.
(2) Whenever the pair $((x, g), (y, g^{'})) \in G^{(2)}$ the maps $A_{(x,g)}$, $A_{(y,g')}$ and $A_{(x,gg')}$ satisfy the composition law $A_{(x,g)} \circ A_{(y,g')} = A_{(x,gg')}$.
(3) For any continuous vector field $h(x)$ in H and every $g \in \Gamma$ we have that $A_{(x,g)}(h(xg))$ is a continuous vector field in H.

This definition can be connected to cohomology and analysis via positive and negative type kernels [**BdlHV08, AD05**]. The role of positive and conditionally negative type kernels within group theory is well known and plays an important role in studying both analytic and representation theoretic properties of groups [**BdlHV08, HK97**]. These ideas were extended to groupoids by Tu [**Tu99**], and we define and consider them in that generality. Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid.

Definition 118. A continuous function $F : \mathcal{G} \to \mathbb{R}$ is said to be of *conditionally negative type* if

(1) $F|_{\mathcal{G}^{(0)}} = 0$;

(2) $\forall x \in \mathcal{G}, F(x) = F(x^{-1})$;
(3) Given $x_1, ..., x_n \in \mathcal{G}$ all having the same range and $\sigma_1, ..., \sigma_n \in \mathbb{R}$ such that $\sum_i \sigma_i = 0$ we have $\sum_{j,k} \sigma_j \sigma_k F(x_j^{-1} x_k) \leq 0$.

The important feature of functions of this type is their connection to affine actions of locally compact, σ-compact groupoids.

THEOREM 119. *Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid. Then the following are equivalent* [**Tu99**]*:*

(1) There exists a proper conditionally negative type function on $\mathcal{G}$
(2) There exists a continuous field of Hilbert spaces over $\mathcal{G}^{(0)}$ with a proper affine action of $\mathcal{G}$.

This result sparks the following definition:

DEFINITION 120. A locally compact Hausdorff groupoid $\mathcal{G}$ is said to have the *Haagerup property* if it satisfies either condition (1) or (2) of Theorem 119.

This property has many connections to the Baum-Connes conjecture for locally compact groupoids via the work of Tu [**Tu99, Tu00, Tu12**] that we will discuss in detail later in the chapter. Here however we would like to remark on a connection with coarse embeddings that directly highlights why we are interested in this property [**CCJ⁺01**].

PROPOSITION 121. *Let Γ be a discrete finitely generated group. If Γ has the Haagerup property then Γ coarsely embeds into Hilbert space.* □

2. The coarse Baum-Connes conjecture.

In this section we outline the construction of the coarse assembly map using only elementary K-theoretic methods. The exposition for this follows Chapters 5 and 8 from [**Roe96**], however we could equally have developed this using the latter half of [**HR00, HR95**]. We then connect this construction to a KK-theoretic formulation of the Baum-Connes conjecture introduced by Tu [**Tu99, Tu00**] via a groupoid constructed by Skandalis, Tu and Yu [**STY02**].

2.1. The algebras of locally compact and pseudolocal operators. Let X be a locally compact Hausdorff topological space and let $\mathcal{H}$ be a Hilbert space. We say that $\mathcal{H}$ is an X-module if it admits a representation, by bounded linear operators, of $C_0(X)$. Where appropriate we denote this representation by ϕ.

DEFINITION 122. An X-module is *adequate* if $\overline{\phi(C_0(X))\mathcal{H}} = \mathcal{H}$ and $\phi(C_0(X)) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

REMARK 123. Such an X-module is a suitably faithful representation of X into Hilbert space: it leaves the Hilbert space invariant and acts essentially non-trivially from the perspective of index theory. By results of Voiculescu [**Voi76**] and Brown-Douglas-Fillmore [**BDF77**] it is sufficient to consider only adequate X-modules from the perspective of defining K-homology. In fact, it can be shown that the choice of adequate X-module is irrelevant for this task.

Fix an adequate X-module $\mathcal{H}_X$.

DEFINITION 124. Let $T \in \mathcal{B}(\mathcal{H}_X)$. We say that T is *pseudolocal* if $[T, f] \in \mathcal{K}(\mathcal{H}_X)$ for all $f \in C_0(X)$. If Tf and fT are compact for all $f \in C_0(X)$ we say that T is *locally compact.*

These operators are important from the perspective of defining K-homology; the set of pseudolocal operators forms a C^*-algebra which we denote by $\mathcal{D}^*(X)$ and the locally compact operators form an ideal in $\mathcal{D}^*(X)$ that we denote by $\mathcal{C}^*(X)$. This gives us a short exact sequence of C^*-algebras:

$$0 \to \mathcal{C}^*(X) \to \mathcal{D}^*(X) \to \frac{\mathcal{D}^*(X)}{\mathcal{C}^*(X)} \to 0$$

DEFINITION 125. We define the analytic K-homology $K_*(X)$ to be the topological K-theory group $K_{*+1}(\frac{\mathcal{D}^*(X)}{\mathcal{C}^*(X)})$.

Using the properties of K-theory it is possible to prove that the functor $X \mapsto K_*(X)$ is covariant functor to abelian groups that satisfies the properties of a generalised homology theory that is dual to K-theory. For a detailed proof of these facts we refer the reader to [**Roe96, HR00**].

2.2. The Roe algebra and the assembly map. Let X be a proper metric space. We fix once and for all a countable dense subset Z and a separable Hilbert space $\mathcal{H}_0$ and we consider bounded operators T on the Hilbert space $\ell^2(Z, \mathcal{H}_0)$. We write $T = (T_{xy})_{x,y\in Z}$ for the "matrix" decomposition of T as an operator with entries $T_{xy} \in \mathcal{B}(\mathcal{H}_0)$. We then call T *locally compact* if:

(1) $T_{xy} \in \mathcal{K}(\mathcal{H}_0)$;
(2) For every bounded subset $B \subset X$ the set:

$$\{(x, y) \in (B \times B) \cap (Z \times Z) | T_{xy} \neq 0\}$$

is finite;

The propagation of an operator T is defined to be:

$$\mathrm{prop}(T) := \inf\{S > 0 | T_{xy} = 0 \text{ for all } x, y \in Z \text{ such that } d(x, y) > S\}$$

This preamble leads us to:

DEFINITION 126. The algebraic Roe algebra, denoted $\mathbb{C}[X]$, is the $*$-subalgebra of $\mathcal{B}(\mathcal{H}_0)$ consisting of all finite propagation operators T that are also locally compact. The Roe algebra, denoted C^*X, is the norm closure of $\mathbb{C}[X]$ in the operator norm associated to the Hilbert space $\ell^2(Z, \mathcal{H}_0)$.

REMARK 127. The Hilbert space $\ell^2(Z, \mathcal{H}_0))$ is an adequate X-module [**WY12a**] and $C^*X = \mathcal{C}^*X \cap \{\text{finite propagation operators on } \ell^2(Z, \mathcal{H}_0))\}$.

We now consider a finite propagation version of the K-homology defined in the previous section:

DEFINITION 128. Define $D^*(X)$ to be the intersection of the finite propagation operators on $\ell^2(Z, \mathcal{H}_0)$ with $\mathcal{D}^*(X)$.

Similarly the finite propagation operators we defined above in $C^*(X)$ form an ideal in $D^*(X)$. So again we get a long exact sequence:

$$0 \to C^*(X) \to D^*(X) \to \frac{D^*(X)}{C^*(X)} \to 0$$

Remarkably, this short exact sequence connects to K-homology [**Roe96**, Corollary 5.9], [**HR00**, Lemma 12.3.2]:

THEOREM 129. *K-homology is equally well defined with finite propagation operators. That is* $K_*(X) \cong K_{*+1}(\frac{D^*(X)}{C^*(X)})$. □

Essentially, this Theorem reflects that K-homology elements can be chosen to have arbitrarily small propagation.

Consider now long exact sequence in K-theory associated to the short exact sequence defined above:

$$\begin{array}{ccccc} K_0(C^*(X)) & \to & K_0(D^*(X)) & \to & K_0(\frac{D^*(X)}{C^*(X)}) \\ \mu\uparrow & & & & \downarrow\mu \\ K_1(\frac{D^*(X)}{C^*(X)}) & \leftarrow & K_1(D^*(X)) & \leftarrow & K_1(C^*(X)) \end{array}$$

Where the maps μ are the boundary map in K-theory. Using the identification provided by Theorem 129 we get the following long exact sequence:

$$\begin{array}{ccccc} K_0(C^*(X)) & \to & K_0(D^*(X)) & \longrightarrow & K_1(X) \\ \mu\uparrow & & & & \downarrow\mu \\ K_0(X) & \longleftarrow & K_1(D^*(X)) & \leftarrow & K_1(C^*(X)) \end{array}$$

So we have constructed a map, using only the K-theory long exact sequence, that connects the K-homology of X to the K-theory of the Roe algebra $C^*(X)$; this is the coarse assembly map.

CONJECTURE 130. *(Coarse Baum-Connes conjecture I) Let* X *be a proper metric space and let* $\mu : K_*(X) \to K_*(C^*(X))$ *be boundary map defined above. Then* μ *is an isomorphism.*

This conjecture is slightly naive, as it is not in fact functorial under coarse maps. The left hand side, $K_*(X)$ is a topological homology theory, functorial under proper continuous maps (see Chapter 5 of [**Roe96**]), whilst the right hand side is a coarse invariant. Thus, the class of spaces that naturally fit this conjecture are those whose topology is uniformly controlled, that is *uniformly contractible spaces*

CONJECTURE 131. *Let* X *be a uniformly contractible simplicial complex of bounded geometry. Then the coarse Baum-Connes conjecture I is true for* X.

There are striking positive results to this conjecture in the case that the space is *scalable* [**HR95**]:

DEFINITION 132. A proper complete metric space X is *scalable* if there is a uniform map $r : X \to X$ that scales the metric, that is $d(r(x), r(y)) \leq \frac{1}{2}d(x, y)$ for all pairs $(x, y) \in X \times X$, and is coarsely homotopic to the identity.

THEOREM 133. *Let* X *be a scalable metric space, such as a nonpositively curved complete Riemannian manifold. Then the coarse Baum-Connes conjecture I is true for* X.

The proof of this result relies on showing that $K_*(D^*X)$, the analytic structure set of X, is trivial, then the result follows from the long exact sequence above.

In order to improve the conjecture we rely on making the left hand side term $K_*(X)$ functorial under coarse maps. To do this we *coarsen* X.

DEFINITION 134. A *coarsening* of X is uniformly contractible space $\underline{E}X$ that is equipped with a coarse equivalence $X \to \underline{E}X$.

Given such a space $\underline{E}X$ we could then apply Conjecture 130 directly and use coarse invariance of the right side. The issue is that these spaces need not exist in general and so we must work around this problem by constructing a weaker notion that does always exist.

The correct weaker notion is a scale dependent sequence of metric simplicial spaces equipped with suitable coarse maps, that is a directed system $\{X_R\}_{R>0}$ of metric simplicial complexes and proper maps that have, for every $R > 0$, a coarse map $X_R \to X$. such a system constructs a system of assembly maps and natural composites:

$$K_*(X_R) \to K_*(C^*(X_R)) \to K_*(C^*(X))$$

By taking direct limits we get the following:

$$\lim_{R>0} K_*(X_R) \to \lim_{R>0} K_*(C^*(X_R)) \to K_*(C^*(X)).$$

We can now consider the natural "coarsened" version of Conjecture 130 by considering the direct limit of the maps $\mu_R : K_*(P_R(X)) \to K_*(C^*(P_R(X)) \to K_*(C^*(X))$. (In general, the maps $K_*(C^*(P_R(X)) \to K_*(C^*(X))$ need not be isomorphisms however in the direct limit they will be. See for example [**Wri05**, Theorem 2.17] in the case of nerves of covers). We define the coarse K-homology, denoted by $KX_*(X)$ to be the direct limit $\lim_{R>0} K_*(X_R)$ for any such $R > 0$.

CONJECTURE 135. *(The coarse Baum-Connes conjecture II). Let* X *be a proper metric space. Then the coarse assembly map:* $\mu_\infty : KX_*(X) = \lim_R K_*(P_R(X)) \to K_*(C^*(X))$ *is an isomorphism.*

An example of such a sequence of complexes is the family of nerves of uniformly bounded covers of X or the family of Rips complexes on X.

EXAMPLE 136. (Rips complex) Fix $R > 0$. Then the *Rips complex on scale* R is a simplicial complex $P_R(X)$, where a set of points $\{x_1, ..., x_n\}$ spans a simplex if and only if $d(x_i, x_j) \leq R$ for every i, j. If $S > R$, then every R simplex will certainly be an S simplex - so we have inclusions $P_R(X) \hookrightarrow P_S(X)$. Additionally these inclusions are proper. For each $R > 0$, there is a natural coarse map that sends any point in the Rips complex to the nearest vertex in the complex, which are given by the points in X. Additionally, the union $X_\infty := \bigcup_{R>0} P_R(X)$ is a uniformly contractible space but will not be coarsely equivalent to X and so is not a coarsening. The K-homology groups $KX_*(X)$ in this case are computing only the compactly supported part of the K-homology of this infinite simplex X_∞.

REMARK 137. This simplifies in the case of a finitely generated discrete group Γ. The inclusion, $\Gamma \hookrightarrow P_R(\Gamma)$ is a coarse map in this case and coupled with the projection map defined above gives a coarse equivalence between Γ and $P_R(\Gamma)$ for each $R > 0$. This may fail in the case of a general metric space X however.

As before there are broadly positive results to this conjecture. The following is a striking result of Guoliang Yu concerning coarsely embeddable spaces [**Yu00**]:

THEOREM 138. *Let* X *be a proper metric space that admits a coarse embedding into Hilbert space. Then the coarse Baum-Connes conjecture II is true for* X

Later in the Chapter we will explore counterexamples to this conjecture by rephrasing it using groupoids. Lastly, consider a discrete group Γ acting on a space X properly, freely and cocompactly. Then we can consider the following algebras:

Definition 139. Let T be an operator on $\ell^2(X, \mathcal{H}_0)$. Then T is Γ-invariant if for all $g \in \Gamma$ the matrix entries $T_{x,y}$ and $T_{gx,gy}$ are equal. Denote by $\mathbb{C}[X]^\Gamma$ the collections of operators that are finite propagation, locally compact and Γ-invariant. Similarly, denote $\mathbb{D}(X)^\Gamma$ the operators that are finite propagation, pseudolocal and Γ-invariant. Lastly, denote their completions by $C^*(X)^\Gamma$ and $D^*(X)^\Gamma$.

These algebras also fit into the same short exact sequence, as $C^*(X)^\Gamma$ is an ideal in $D^*(X)^\Gamma$, just as above. This allows us to compute the K-theory long exact sequence:

$$\begin{array}{ccccc} K_0(C^*(X)^\Gamma) & \to & K_0(D^*(X)^\Gamma) & \longrightarrow & K_1^\Gamma(X) \\ \mu\uparrow & & & & \downarrow\mu \\ K_0^\Gamma(X) & \longleftarrow & K_1(D^*(X)^\Gamma) & \leftarrow & K_1(C^*(X)^\Gamma) \end{array}$$

We define the equivariant K-homology of X, denoted by $K_*^\Gamma(X)$ to be the K-theory $K_{*+1}(\frac{D^*(X)^\Gamma}{C^*(X)^\Gamma})$. Now the boundary maps become the equivariant coarse assembly map for the action of Γ on X. By virtue of the following lemma (for a proof see [**WY12a**]), we can simplify this further.

Lemma 140. *If the action of Γ on X is cocompact in addition to being free and proper then there is a Morita Equivalence between C^*X^Γ and $C_r^*(\Gamma)$.*

This gives us, using the boundary maps, assembly maps $K_*^\Gamma(P_R(X)) \to K_*(C^*(P_R(X))^\Gamma) \cong K_*(C^*(X)^\Gamma)$.

By applying the ideas of coarsening introduced above using a Rips complex over X we get the following system: $\{K_*^\Gamma(P_R(X)) \to K_*(C^*(P_R(X))^\Gamma) \cong K_*(C^*(X)^\Gamma)\}$. By taking direct limits we get an assembly map: $\lim_{R>0} K_*^\Gamma(P_R(X)) \to K_*(C_r^*(\Gamma))$. As remarked in Example 136, the left hand side in this instance is compactly supported.

Conjecture 141. *(Baum-Connes conjecture) Let Γ be a finitely generated discrete group and let $\underline{E}\Gamma$ be its classifying space for proper actions obtained by taking direct limits over Rips complexes of Γ. Then the map μ obtained by taking direct limits of the maps defined above: $\mu : K_*^{\Gamma,c}(\underline{E}\Gamma) \to K_*(C_r^*(\Gamma))$ is an isomorphism.*

There are also positive results in this setting [**HK97**]:

Theorem 142. *Let Γ be a discrete group. If Γ has the Haagerup property then the Baum-Connes conjecture defined above holds for Γ.*

This provides some evidence of the connection between analysis on groups and coarse geometry.

3. The Coarse Groupoid of a Metric Space.

Let X be a uniformly discrete bounded geometry (sometimes denoted uniformly locally finite) metric space. In this section we construct a groupoid $G(X)$ associated to X that captures coarse properties of X. Explicitly, for each coarse property of X that we are interested in there is a corresponding analytic property of $G(X)$, this includes an encoding of the coarse Baum-Connes conjecture as a groupoid Baum-Connes conjecture.

We briefly recall how to build a groupoid from the metric coarse structure on X by considering extensions of the pair product on $X \times X$. The most natural way to do this is by making use of the entourages arising from the metric. The approach to this problem is through Lemma 24 introduced in Chapter 2, which we recall below.

LEMMA 143. *Let X be a uniformly discrete bounded geometry metric space and let E be any entourage. Then the inclusion $E \to X \times X$ extends to an injective continuous map $\overline{E} \to \beta X \times \beta X$, where $\overline{E}$ denotes the closure of E in $\beta(X \times X)$.*

Now we can recall the coarse groupoid is defined as:

$$G(X) := \bigcup_{R>0} \overline{\Delta_R}.$$

This groupoid plays an important role in coarse geometry through the work of [**STY02**]. We summarise these results in the following theorem:

THEOREM 144. *Let X be a uniformly discrete bounded geometry metric space. Then following hold:*

(1) $G(X)$ *is an étale locally compact Hausdorff principal topological groupoid with unit space* $G(X)^{(0)} = \beta X$. [**Roe03**, Theorem 10.20][**STY02**, Proposition 3.2]*;*
(2) $C^*_r(G(X))$ *is isomorphic to the uniform Roe algebra* $C^*_u(X)$. [**Roe03**, Proposition 10.29]*;*
(3) The coarse Baum-Connes conjecture for X *is equivalent to the Baum-Connes conjecture for* $G(X)$ *with coefficients in* $\ell^\infty(X, \mathcal{K})$. [**STY02**, Lemma 4.7].

So this lets us appeal to the theory of groupoids to conclude coarse information about a given metric space X. In fact, this is precisely the strategy of [**HLS02**] when it comes to dealing with counterexamples to the coarse Baum-Connes conjecture.

4. Equivariant KK-theory for groupoids and assembly.

We recall the definitions of groupoid equivariant KK-theory. For this section let G be a locally compact, σ-compact, Hausdorff groupoid with Haar system. The basic notion here is that of a G-C^*-algebra:

DEFINITION 145. A C^*-algebra A is called a G-C^*-algebra if it is a $C_0(G^{(0)})$-algebra and admits a G-action, that is:

(1) there is a $*$-homomorphism to the centre of the multiplier algebra of A, $\theta : C_0(G^{(0)}) \to Z(M(A))$, such that $\overline{\theta(C_0(G^{(0)}))A} = A$
(2) there is an isomorphism from $\alpha : s^*A \to r^*A$ such that for each $(g, h) \in G^{(2)}$ the morphisms $\alpha(g) : A_{s(g)} \to A_{r(g)}$ satisfy $\alpha(g) \circ \alpha(h) = \alpha(gh)$

We are going to be concerned with *proper* G-algebras. Let Z be a G-space, then under the previous definition a $Z \rtimes G$-algebra is both a G-algebra and a $C_0(Z)$-algebra, with compatibility between the two structures. We then say a G-algebra A is proper if there exists a proper G-space Z such that A is a $Z \times G$-algebra.

In this context we can also extend the action of the groupoid G from a G-algebra A to any Hilbert module E over A. See [**Tu00**] for more details.

Definition 146. (KK_G-cycles) [**LG99, Tu99, Tu00**]. Let A and B be G-algebras. Then a Kasparov (A, B) G-equivariant bimodule consists of a triple (E, ϕ, F) where E is a G-equivariant, $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert B-module, $\phi : A \to \mathcal{L}(E)$ is a G-equivariant $*$-homomorphism and $F \in \mathcal{L}(E)$ is of degree 1 and satisfies, for all $a \in A$ and $a_1 \in r^*A$:

(1) $a(F - F^*) \in \mathcal{K}(E)$
(2) $a(F^2 - 1) \in \mathcal{K}(E)$
(3) $[a, F] \in \mathcal{K}(E)$
(4) $a_1(V(s^*F)V^* - r^*F) \in r^*\mathcal{K}(E)$, where V denotes the unitary operator that implements the action of G on E.

Then we denote by $KK_G(A, B)$ the group of homotopy classes of Kasparov (A, B) G-equivariant bimodules.

This theory has many of the same features as the more traditional non-equivariant KK groups, namely:

- $KK_G(A, B)$ is covariant in second variable and contravariant in the first;
- Bott peroidicity holds for this theory; define: $KK^n_G(A, B) = KK_G(A, B \otimes C_0(\mathbb{R}^n))$, then $KK^n_G(A, B) = KK^{n+2}_G(A, B)$;
- For any G-algebra D there is a natural transformation:

$$\sigma_{G^{(0)},D} : KK_G(A, B) \to KK_G(A \otimes_{C_0(G^{(0)})} D, B \otimes_{C_0(G^{(0)})} D).$$

- There is a natural associative product, which is comparable with $\sigma_{G^{(0)},\_}$ in the obvious way.
- There are descent morphisms compatible with the product:

$$j_{G,(red)} : KK_G(A, B) \to KK(A \rtimes_{(red)} G, B \rtimes_{(red)} G)$$

The interest in understanding proper G-spaces is motivated by the construction of topological K-theory, a homology theory on groupoids, in this context.

DEFINITION 147. A *classifying space for proper actions* of G is a proper G-space $\underline{E}G$ such that for any proper G-space Z there exists a G-equivariant map $Z \to \underline{E}G$ that is unique up to G-homotopy.

Such a space always exists [**Tu99**, Section 11] and one example of a model for this is given by a collection of compactly supported positive measures on G [**Tu99**]. This construction can also be given the structure of a G-simiplical complex [**Tu12**].

DEFINITION 148. (K^{top}) Let G be a locally compact, σ-compact Hausdorff groupoid with Haar system and let $\underline{E}G$ be its classifying space for proper actions. Then we define:

$$K^{top}(G, B) = \varinjlim_{Y \subseteq \underline{E}G} KK_G(C_0(Y), B)$$

where the limit runs through all possible G-compact subspaces $Y \subset \underline{E}G$. If one takes $B = C_0(G^{(0)})$ then we denote by $K^{top}(G)$ *the topological K-theory of* G.

REMARK 149. For a G-compact, proper G-space Y we can define an assembly map by composing the descent morphism $j_{G,red}$ by a suitable partition associated to a proper G-space, that is:

$$KK_G(C_0(Y), B) \to KK(C_0(Y) \rtimes_r G, B \rtimes_r G) \to K_*(B \rtimes_r G)$$

By taking limits through G-compact subspaces, one arrives at a map:

$$\mu_* : K_*^{top}(G, B) \to K_*(B \rtimes_r G).$$

This is the Baum-Connes assembly map for G with coefficients in B. The Baum-Connes conjecture asserts that this map is an isomorphism for all possible G-algebras B, and is known in this context to have counterexamples [**HLS02**], some of which arise from coarse geometry.

We now return to the coarse groupoid $G(X)$ of a uniformly discrete bounded geometry metric space X. We recall that this groupoid is étale, locally compact, σ-compact. Hence we can define $KK_{G(X)}(A, B)$ for any pair of $G(X)$-algebras A, B.

DEFINITION 150. Let X be a coarse space with a uniformly locally finite coarse structure $\mathcal{E}$ and $G(\mathcal{E})$ the coarse groupoid. For each $E \in \mathcal{E}$ we define $P_E(X)$ to be the simplical complex in which each finite subset $F \subset E$ spans a simplex. We denote by $P_E(G(\mathcal{E}))$ the closure of $P_E(X) \times X$ in the weak $*$-topology in the dual of $C_c(G(\mathcal{E}))$ (viewing each element of $P_E(X)$ as a positive measure in the obvious way).

Definition 151. Let X be a coarse space. The coarse K-homology of X relative to $\mathcal{E}$, denoted $KX_*(X, \mathcal{E})$ is defined to be the directed limit:

$$\varinjlim_{E \in \mathcal{E}} KK(C_0(P_E(X)), \mathbb{C})$$

Remark 152.
- If X is a uniformly discrete bounded geometry metric space and $\mathcal{E}$ is the metric coarse structure, then the $P_{\Delta_R}(X)$ are equal to the standard Rips complex $P_R(X)$.
- The limit through the directed set of entourages $E \in \mathcal{E}$ of the $P_E(X)$ gives us a directed system we can use to coarsen the K-homology of X. The limit of the $P_E(G(X))$ is a model of $\underline{E}G(X)$.
- Lemma 4.7 in [**STY02**] proves that the inclusion of a point $\{x\}$ viewed as a subgroupoid of $G(X)$ gives rise to a restriction map in KK-theory and this map induces an isomorphism:

  $$KK_{G(X)}(C_0(P_E(G(X)), \ell^\infty(X, \mathcal{K})) \cong KK(C_0(P_E(X)), \mathbb{C}).$$

  Taking limits, gives us:

  $$K^{top}(G(X), \ell^\infty(X, \mathcal{K})) \cong KX_*(X).$$

- The content of Lemma 4.4 from [**STY02**] is that $\ell^\infty(X, \mathcal{K}) \rtimes_r G(X) \cong C^*X$. Hence, we can use the assembly map defined above for $G(X)$ with coefficients in $\ell^\infty(X, \mathcal{K})$ to define:

  $$\mu_* : K_*^{top}(G(X), \ell^\infty(X, \mathcal{K})) \to K_*(\ell^\infty(X, \mathcal{K}) \rtimes_r G(X)).$$

  This map is equivalent to the traditional coarse assembly map:

  $$\mu_* KX_*(X) \to K_*(C^*X).$$

- These maps are again both considering the compactly supported part of the corresponding K-homology group.

One flexibility that the groupoid picture provides is the ability to consider natural maps associated to saturated subsets. We outline a technical Lemma of Tu concerning equivariant KK-theory associated to saturated subsets required to build a long exact sequence in the topological K-theory associated to a groupoid decomposition.

Lemma 153. *Let G be a locally compact, second countable, proper groupoid with a Haar system and let Z be a second countable G-space. If F is a closed saturated subset and U is its open complement then for any G-C^*-algebra A there is a long exact sequence:*

$$\begin{array}{ccccc} KK_G(C_0(U), A) & \to & KK_G(C_0(Z), A) & \to & KK_G(C_0(F), A) \\ \uparrow & & & & \downarrow \\ KK^1_G(C_0(F), A) & \leftarrow & KK^1_G(C_0(Z), A)) & \leftarrow & KK^1_G(C_0(U), A) \end{array}$$

Choosing Z to be a model of $\underline{E}G$ gives us a long exact sequence in K-homology associated to any closed saturated subset F and coefficient algbera A that naturally connects to the assembly maps μ, μ_F, μ_U.

5. Counterexamples to the Coarse Baum-Connes Conjecture and Boundary Groupoids.

The new material that begins in this section and continues until the end of the chapter is joint work with Nick Wright and appears in the paper [**FSW12**].

Throughout this section let G be an étale, Hausdorff locally compact topological groupoid. Let F be a closed saturated subset of $G^{(0)}$. We remark also that the complement F^c is an open saturated set. This gives rise to the algebraic decomposition:

$$G = G_{F^c} \sqcup G_F$$

This lets us construct maps on the $*$—algebras of compactly supported functions associated with G,G_F and G_{F^c}:

$$0 \to C_c(G_{F^c}) \to C_c(G) \to C_c(G_F) \to 0.$$

By the properties of the maximal C^*-norm this extends to the maximal groupoid C^*-algebras:

$$0 \to C^*_{max}(G_{F^c}) \to C^*_{max}(G) \to C^*_{max}(G_F) \to 0.$$

On the other hand this may fail to be an exact sequence when we complete in the norm that arises from any specific representation, for example the left regular representation λ_G; this can be detected at the level of K-theory, as discussed in [**HLS02**], by considering the sequence:

$$K_0(C^*_r(G_U)) \to K_0(C^*_r(G)) \to K_0(C^*_r(G_F)) \tag{14}$$

This was used in [**HLS02**] to construct multiple different types of counterexample to the Baum-Connes conjecture for groupoids - each of which invokes the following Lemma:

LEMMA 154. *([**HLS02**, Lemma 1]) Assume the sequence (14) is not exact at its middle term.*

(1) If the Baum-Connes map $K_0^{top}(G_F) \to K_0(C^*_r(G_F))$ is injective then the Baum-Connes map $K_0^{top}(G) \to K_0(C^*_r(G))$ fails to be surjective.
(2) If the map $K_0(C^*_{max}(G_F)) \to K_0(C^*_r(G_F))$ is injective then the map $K_0(C^*_{max}(G)) \to K_0(C^*_r(G))$ fails to be surjective and a fortiori the Baum-Connes map $K_0^{top}(G) \to K_0(C^*_r(G))$ fails to be surjective.

We observe that whilst the sequence:

$$0 \longrightarrow C^*_r(G_{F^c}) \xrightarrow{\alpha} C^*_r(G) \xrightarrow{q} C^*_r(G_F) \longrightarrow 0$$

may fail to be exact in the middle term the maps α and q both exist and we can see that the map q is also surjective by considering the following diagram.

$$\begin{array}{ccc} C^*_{max}(G) & \twoheadrightarrow & C^*_{max}(G_F) \\ \downarrow & & \downarrow \\ C^*_r(G) & \longrightarrow & C^*_r(G_F) \end{array}$$

It is also clear that the image of α is contained in the kernel of q, whence we can make the sequence exact artificially by replacing $C^*_r(G_{F^c})$ by the ideal $I := \ker(q)$. We can then define a new assembly map in the first term to be the composition of the original assembly map μ_{F^c} and the K-theory map induced by inclusion $i_* : K_*(C^*_r(G_{F^c})) \to K_*(I)$. Then in terms of assembly maps this gives us a new commutative diagram:

$$\begin{array}{cccccccccccc} \to & K_1(C^*_r(G_F) & \to & K_0(I) & \to & K_0(C^*_r(G)) & \to & K_0(C^*_r(G_F) & \to & K_1(I) & \to \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \to & K_1^{top}(G_F) & \to & K_0^{top}(G_{F^c}) & \to & K_0^{top}(G) & \to & K_0^{top}(G_F) & \to & K_1^{top}(G_{F^c}) & \to \end{array}$$

where the rows here are exact.

As in [**HLS02**] we would now choose suitable groupoids G and subsets F of the unit space $G^{(0)}$ that allow us to use the above sequence to analyse the Baum-Connes conjecture for the groupoid G. We have in mind the situation that $G = G(X)$, the coarse groupoid associated to some uniformly discrete bounded geometry metric space X.

5.1. The Coarse Groupoid Conjecture. Let X be a uniformly discrete bounded geometry metric space. From what was described above we can associate to each closed saturated subset F of the unit space space βX a long exact sequence in K-theory. We consider the obvious closed saturated subset: $\partial\beta X \subset G(X)^{(0)}$. This gives us the following commutative diagram (omitting coefficients):

$$\begin{array}{cccccccccccc} \to & K_1(C^*_r(G(X)|_{\partial\beta X}) & \to & K_0(I) & \to & K_0(C^*_r(G(X))) & \to & K_0(C^*_r(G(X)|_{\partial\beta X}) & \to & K_1(I) & \to \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow \mu_{bdry} & & \uparrow & \\ \to & K_1^{top}(G(X)|_{\partial\beta X}) & \to & K_0^{top}(X \times X) & \to & K_0^{top}(G(X)) & \to & K_0^{top}(G(X)|_{\partial\beta X}) & \to & K_1^{top}(X \times X) & \to \end{array}$$

We can now properly formulate the boundary conjecture (replacing the coefficients):

CONJECTURE 1. *[Boundary Coarse Baum-Connes Conjecture] Let X be a uniformly discrete bounded geometry metric space. Then the assembly map:*

$$\mu_{bdry} : K_*^{top}(G(X)|_{\partial\beta X}, l^\infty(X, \mathcal{K})/C_0(X, \mathcal{K})) \to K_*((l^\infty(X, \mathcal{K})/C_0(X, \mathcal{K})) \rtimes_r G(X)|_{\partial\beta X})$$

is an isomorphism.

5.2. Expanders, Asymptotic Coverings and Ghost operators. We depart from assembly maps to outline the type of metric space that give rise to counterexamples to the coarse Baum-Connes conjecture.

DEFINITION 155. Let $\{X_i\}$ be a sequence of finite graphs. Then $X = \sqcup_{i\in\mathbb{N}} X_i$ equipped with any metric such that $d_X(X_i, X_j) \to \infty$ as $i + j \to \infty$ and $d_X|_{X_i} = d_{X_i}$ is called the *space of graphs* associated with the sequence $\{X_i\}$.

We will be considering only sequences that grow in size, that is $|X_i| \to \infty$ as $i \to \infty$.

We denote the girth of a finite graph X by $\mathtt{girth}(X)$, by which we mean the length of the shortest simple cycle of the graph. We say a sequence of graphs has *large girth* if $\mathtt{girth}(X_i) \to \infty$ as $i \to \infty$. Another way to think of large girth sequences is that they are the only sequences for which the universal covering sequence $\{\tilde{X}_i, p_i\}$ is *asymptotically faithful*, that is that for any $R > 0$ there exists an $n \in \mathbb{N}$ such that for all $i > n$ we have $x_i \in \tilde{X}_i$ such that $B_R(x_i) \cong B_R(p_i(x_i))$.

As we are going to be concerning ourselves with counterexamples to the coarse Baum-Connes conjecture, we need to consider a class of sequences known as *expanders*. Being an expander is described by measuring the connectedness of each of the finite graphs X_i in our sequence, and for this we use the (finite, weighted) graph Laplacian, a bounded linear operator on $\ell^2(V(X_i))$:

$$(\Delta_i f)(x) = f(x) - \sum_{d(x,y)=1} \frac{f(y)}{\sqrt{\deg(x)\deg(y)}}$$

If each X_i were a regular graph, then this would reduce to the traditional graph laplacian, as in the above equation we are weighting by the degree of each vertex.

DEFINITION 156. Let $\{X_i\}$ be a sequence of finite graphs and let X be the associated space of graphs. Then the space X (or the sequence $\{X_i\}$) is an *expander* if:

(1) There exists $k \in \mathbb{N}$ such that all the vertices of each X_i have degree at most k.
(2) $|X_i| \to \infty$ as $i \to \infty$.
(3) There exists $c > 0$ such that $\mathtt{spectrum}(\Delta_i) \subseteq \{0\} \cup [c, 1]$ for all i.

REMARK 157. Each Laplacian Δ_i has propagation 1, so we can form the product in the (algebraic) Roe algebra:

$$\Delta := \prod_i \Delta_i \in C^*_u(X)$$

Now we can consider projection p onto the kernel of Δ. For an expander X we have $\mathtt{spectrum}(\Delta) \subseteq \{0\} \cup [c, 1]$ for some $c > 0$, and so by an application of the functional calculus we can conclude that $p \in C^*_u X$. As Δ breaks up as a product we observe that its $\mathtt{Ker}(\Delta) = \oplus_i \mathtt{Ker}(\Delta_i)$ and so the projection p decomposes as a product:

$$p = \prod_i p^{(i)}.$$

Additionally, it is easy to see that a function in $\ell^2(X_i)$ is an element of the kernel of Δ_i if and only if it is a constant function, and so each p^i has matrix entries $p^i_{x,y} = \frac{1}{|X_i|}$.

The following notion is due to Guoliang Yu (unpublished):

Definition 158. An operator $T \in C^*X$ is a *ghost operator* if $\forall \epsilon > 0$ there exists a bounded subset $B \subset X \times X$ such that the norm: $\|T_{xy}\| \leq \epsilon$ for all $(x, y) \in (X \times X) \setminus B$.

It is clear, from the definition as a product given above, that the kernel p of the Laplacian Δ is a ghost operator in the uniform Roe algebra. However by considering $q := \prod_i p^i \otimes q$ for some rank one projection q we have a ghost operator $q \in C^*_u(X) \otimes \mathcal{K} \subset C^*X$.

6. Homological Connections between Coarse Assembly and the Boundary Conjecture.

The intuitive view of the boundary conjecture in the context of an expander is supposed to be "quotient by the ghost ideal and consider the K-theory". We confirm the technical approach meets the intuitive one by proving that the kernel I is precisely the ghost ideal I_G. In order to prove this we need the technology of Lemma 9 from [**HLS02**]:

Lemma 159. *If an étale topological groupoid $\mathcal{G}$ acts on a C^*-algebra A, then the map $C_c(\mathcal{G}, A) \to C_0(\mathcal{G}, A)$ extends to an injection (functorial in A) from $A \rtimes_r \mathcal{G}$ to $C_0(\mathcal{G}, A)$.* □

Remark 160. The phrase "functorial in A" allows us, given a map: $A \to B$ of $\mathcal{G} - C^*$-algebras, to build the following square:

$$\begin{array}{ccc} A \rtimes_r \mathcal{G} & \longrightarrow & B \rtimes_r \mathcal{G} \\ \downarrow & & \downarrow \\ C_0(\mathcal{G}, A) & \longrightarrow & C_0(\mathcal{G}, B) \end{array}$$

Remark 161. The map provided is not a $*$—homomorphism as it takes convolution in $A \rtimes_r \mathcal{G}$ to pointwise multiplication in $C_0(\mathcal{G}, A)$. It suffices for applications however as it is continuous.

Proposition 162. *Let X be a uniformly discrete bounded geometry metric space. Then the kernel of the map*

$$l^\infty(X, \mathcal{K}) \rtimes_r G(X) \to (l^\infty(X, \mathcal{K})/C_0(X, \mathcal{K})) \rtimes_r G(X)|_{\partial\beta X}$$

is the ghost ideal I_G.

PROOF. Lemma 159 implies that the following diagram commutes:

$$\begin{array}{ccc} \ell^\infty(X,\mathcal{K}) \rtimes G(X) & \xrightarrow{q} & \frac{\ell^\infty(X,\mathcal{K})}{C_0(X,\mathcal{K})} \rtimes G(X) \\ \Big\downarrow i' & & \Big\downarrow i \\ C_0(G(X),\ell^\infty(X,\mathcal{K})) & \xrightarrow{q'} & C_0(G(X),\frac{\ell^\infty(X,\mathcal{K})}{C_0(X,\mathcal{K})}) \end{array}$$

The downward maps being injective implies that the kernel is precisely the kernel of induced map:

$$q : C^*X = \ell^\infty(X,\mathcal{K}) \rtimes_r G(X) \to C_0(G(X), l^\infty(X,\mathcal{K})/C_0(X,\mathcal{K})).$$

We can compute this kernel:

$$\begin{aligned} I &= \{f \in C^*X | i(q(f)) = 0\} \\ &= \{f \in C^*X | q'(i'(f)) = 0\} \\ &= \{f | i'(f) \in C_0(G(X), C_0(X,\mathcal{K}))\} \\ &= \{f | \forall \epsilon > 0 \exists K \subset X \times X \text{ compact } : |f_{xy}| \leq \epsilon \forall (x,y) \in X \times X \setminus K\}. \end{aligned}$$

As X is uniformly discrete with bounded geometry and $X \times X$ is equipped with the product topology we can replace compact by bounded. So:

$$I = \{f | \forall \epsilon > 0 \exists K \subset X \times X \text{ bounded } : |f_{xy}| \leq \epsilon \ \forall (x,y) \in X \times X \setminus K\} = I_G$$

□

Recall that the assembly map μ_{I_G} associated to the open saturated subset X is given by the composition of $\mu_X : K_*^{top}(X) \to K_*(\mathcal{K}(\ell^2(X))$ with the inclusion $i_* K_*(\mathcal{K}(\ell^2(X)) \to K_*(I_G)$. So to understand how the assembly map μ_{I_G} behaves, it is enough to consider the behaviour of the inclusion i_* as the map μ_X is an isomorphism.

PROPOSITION 163. *Let* $\{X_i\}_{i\in\mathbb{N}}$ *be an expander sequence. Then:*

$$K_*^{top}(X) \cong K_*(\mathcal{K}(\ell^2(X)) \to K_*(I_G)$$

is not surjective but is injective.

PROOF. The proof is an adaptation of a well known argument and relies on considering the algebra C^*X_∞ which is the Roe algebra of the space of graphs with the disjoint union 'metric' and the following exact sequences.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i\in\mathbb{N}} \mathcal{K}(\ell^2(X_i)) & = & \bigoplus_{i\in\mathbb{N}} C^*X_i & \hookrightarrow & C^*X_\infty & \overset{q}{\twoheadrightarrow} & \frac{C^*X_\infty}{\bigoplus_{i\in\mathbb{N}} C^*X_i} \\
 & & & & & \searrow & \downarrow & & \downarrow i \quad \searrow \\
 & & & & & & \prod_{i\in\mathbb{N}} C^*X_i & \twoheadrightarrow & \frac{\prod_{i\in\mathbb{N}} C^*(X_i)}{\bigoplus_{i\in\mathbb{N}} C^*X_i} \longrightarrow 0
\end{array}
$$

We remark that we can conclude $C^*X = C^*X_\infty + \mathcal{K}$. Let $I_{G,\infty} = I_G \cap C^*X_\infty$. Then by the second isomorphism theorem:

$$\frac{I_G}{\mathcal{K}(\ell^2(X)} \cong \frac{I_{G,\infty}}{\bigoplus_{i\in\mathbb{N}} \mathcal{K}(\ell^2(X_i))}$$

We define the map d to be the compositions of q and the inclusion i. The map d induces a map on K-theory that is used to detect non-triviality of certain classes of projections - namely those associated to expander sequences. In particular we observe that d restricts to a map:

$$d|_G : I_{G,\infty} \to \frac{\prod_{i\in\mathbb{N}} C^*(X_i)}{\bigoplus_{i\in\mathbb{N}} C^*X_i}$$

In particular: if $d|_{G,*}([p]) \neq 0$ then $q_*([p]) \neq 0$. Let p be the ghost projection associated to Laplacians of the X_is, defined from the Laplacians in remark 157. This lies in C^*X as X is an expander, and is clearly an element of $C^*(X)_\infty$ as it is defined piecewise. p evaluates to a non-trivial class under $d_{G,*}$ so we know that $K_0(\frac{I_{G,\infty}}{\bigoplus_{i\in\mathbb{N}} C^*X_i}) \neq 0$. From here we see that $K_0(\mathcal{K}) \to K_0(I_G)$ is not surjective and so μ_{I_G} is not surjective either.

To see injectivity it suffices to show that $K_1(I_G) \to K_0(\mathcal{K})$ is the zero map. Consider the following diagram:

$$
\begin{array}{ccccccc}
\bigoplus_{i\in\mathbb{N}} M_i & \longrightarrow & I_{G,\infty} & \hookrightarrow & C^*X_\infty & \longrightarrow & \prod_{i\in\mathbb{N}} M_i \\
\| & & & & & & \| \\
\bigoplus_{i\in\mathbb{N}} M_i & & & \hookrightarrow & & & \prod_{i\in\mathbb{N}} M_i
\end{array}
$$

Given that the bottom long arrow defines an injection on K-theory we can deduce that the first map is also an injection on K-theory. Now we ask this in the Roe algebra by considering the following diagram:

$$
\begin{array}{ccccccccc}
K_1(\frac{I_{G,\infty}}{\bigoplus_{i\in\mathbb{N}} M_i}) & \overset{0}{\longrightarrow} & K_0(\bigoplus_{i\in\mathbb{N}} M_i) & \hookrightarrow & K_0(I_{G,\infty}) & \twoheadrightarrow & K_0(\frac{I_{G,\infty}}{\bigoplus_{i\in\mathbb{N}} M_i}) & \longrightarrow & \\
\| & & \downarrow & & \downarrow & & \| & & \\
K_1(\frac{I_G}{\mathcal{K}}) & \longrightarrow & K_0(\mathcal{K}) & \longrightarrow & K_0(I_G) & \twoheadrightarrow & K_0(\frac{I_G}{\mathcal{K}}) & \longrightarrow &
\end{array}
$$

To finish the proof take an $x \in K_0(\mathcal{K})$ that goes to $0 \in K_0(I_G)$. Then it comes from a $y \in K_1(\frac{I_G}{\mathcal{K}})$ as the rows are exact. Hence there is a $y' \in K_1(\frac{I_{G,\infty}}{\bigoplus_{i\in\mathbb{N}} M_i})$ that by the commutativity of the diagram maps, via the zero map, to x. □

Conjecture 1 has applications to the coarse Baum-Connes conjecture for spaces of graphs:

Proposition 164. *If X satisfies the boundary coarse Baum-Connes conjecture then the following hold:*

(1) The coarse Baum-Connes assembly map for X is injective.
(2) If X is an expander then the coarse Baum-Connes assembly map for X fails to be surjective.

Proof. Consider the long exact sequences:

$$\begin{array}{ccccccccccc} \to K_1(C^*_r(G(X)|_{\partial\beta X}) & \to & K_0(I_G) & \to & K_0(C^*_r(G(X))) & \to & K_0(C^*_r(G(X)|_{\partial\beta X}) & \to & K_1(I_G) & \to \\ \wr\| \uparrow & & f_1 \uparrow & & f_2 \uparrow & & \wr\| \uparrow & & f_3 \uparrow & \\ \to K_1^{top}(G(X)|_{\partial\beta X}) & \to & K_0^{top}(X\times X) & \to & K_0^{top}(G(X)) & \to & K_0^{top}(G(X)|_{\partial\beta X}) & \to & K_1^{top}(X\times X) & \to \end{array}$$

We remark here that f_1 is not surjective and that f_1 and f_3 are injective by Proposition 163. The assumptions and these remarks, coupled with the five lemma, conclude the proof. □

In fact, the previous result can be improved by considering how the K-theory of the compact operators sits inside the K-theory of the Roe algebra.

Proposition 165. *Let X be a space of graphs. Then $K_0(\mathcal{K}) \hookrightarrow K_0(C^*X)$.*

Proof. As in the proof of Proposition 163 we can just work with the subalgebras $\bigoplus_i M_{n_i}$ and C^*X_∞. We observe that we have a similar diagram:

$$\begin{array}{ccccc} \bigoplus_{i\in\mathbb{N}} M_i & \longrightarrow & C^*X_\infty & \longrightarrow & \prod_{i\in\mathbb{N}} M_i \\ \| & & & & \| \\ \bigoplus_{i\in\mathbb{N}} M_i & & \hookrightarrow & & \prod_{i\in\mathbb{N}} M_i \end{array}$$

As the long bottom arrow is certainly injective on K-theory we can see that the arrow into C^*X_∞ is also injective on K-theory. From this we can deduce, using a similar argument to Proposition 163, that this map actually induces an injection on K-theory between the compact operators $\mathcal{K}$ and C^*X. □

Combining this proposition with the fact that the assembly map μ_{I_G} factors through the standard assembly map $\mu_{X\times X}$ we can conclude:

Proposition 166. *Let X be a space of graphs for some sequence $\{X_i\}$ and assume μ_{bdry} be injective. Then:*

(1) *The coarse Baum-Connes assembly map* μ *is injective.*
(2) *If* X *is an expander then the coarse Baum-Connes assembly map fails to be surjective.*

Proof. Consider the diagram:

$$\begin{array}{ccccccccccc}
 & & & \nearrow & \dfrac{\prod_{i\in\mathbb{N}} K_0(C^*X_i)}{\bigoplus_{i\in\mathbb{N}} K_0(C^*X_i)} & & & & & & \\
 & & & & d_* \uparrow & & & & & & \\
\to K_1(C_r^*(G(X)|_{\partial\beta X}) & \to & K_0(I_G) & \to & K_0(C_r^*(G(X))) & \to & K_0(C_r^*(G(X)|_{\partial\beta X}) & \to & K_1(I_G) & \to & \\
\uparrow & & \uparrow & \nearrow & \mu\uparrow & & \uparrow & & 0\uparrow & & \\
\to K_1^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0^{top}(G(X)) & \longrightarrow & K_0^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & 0 & \longrightarrow &
\end{array}$$

We prove (1) by considering an element $x \in K_0^{top}(G(X))$ such that $\mu(x) = 0$. Then x maps to $0 \in K_0^{top}(G(X)|_{\partial\beta X})$ and so comes from an element $y \in \mathbb{Z}$. Each square commutes hence y maps to $0 \in K_0(C_r^*(G(X))$. As the composition up and left (as indicated in the diagram) is injective by Proposition 163, we know that $y \in \mathbb{Z}$ is in fact $0 \in \mathbb{Z}$. Hence $x = 0$.

To see (2): take any non-compact ghost projection $p \in K_0(I_G)$. This does not lie in the image of $\mathbb{Z}$ as it does not vanish under the trace d_*. Push this element to $q \in K_0(C_r^*(G(X))$. Assume for a contradiction that μ is surjective, so $q = \mu(x)$ for some $x \in K_0^{top}(G(X))$. As q maps to $0 \in K_0(C_r^*(G(X)|_{\partial\beta X}))$ and μ_{bdry} is injective we can deduce that x maps to $0 \in K^{top}(G(X)|_{\partial\beta X})$. Hence there is an element $y \in \mathbb{Z}$ that maps to q via x, and so $d_*(q) = 0$, as the image of compact operators lies in the kernel of the map d_*. This gives the desired contradiction. ☐

6.1. Tools to Prove the Boundary Conjecture.

In general it is going to be very hard to compute what might happen in the boundary as our geometric intuition breaks down when considering non-principal ultrafilters. However we can salvage something in the situation that we have more information about the global geometry, by requiring the space of graphs X admits a group action for example.

Proposition 167. *Let* $\{X_i\}$ *be a sequence of finite graphs, let* X *be the corresponding space of graphs and let* Γ *be a finitely generated discrete group. If* Γ *acts on* X *such that the induced action on* βX *is free on* $\partial\beta X$ *and the action generates the metric coarse structure on* X *then* $G(X)|_{\partial\beta X} \cong \partial\beta X \rtimes \Gamma$

Proof. From earlier; we know that $G(X)|_{\partial\beta X}$ is the groupoid constructed by considering the coronas of each entourage in the metric coarse structure $\mathcal{E}$:

$$G(X)|_{\partial\beta X} = \bigcup_{E\in\mathcal{E}} \overline{E} \setminus E = \bigcup_{R>0} \overline{\Delta_R} \setminus \Delta_R$$

We can also form the groupoid $\partial\beta X \rtimes \Gamma$. In order to check that this groupoid is really the coarse boundary groupoid $G(X)|_{\partial\beta X}$ we need to build maps between them, and then check they are continuous. In order to do this, we must first worry about isotropy groups for the action; under the assumption that the action is free on the boundary, we observe that these are all trivial. Now we construct a map locally as follows; making heavy use of the fact that it is always possible, for any subset U in the basis for $\partial\beta X$, to build an entourage such that:

$$[g, U] \leftrightarrow U \times U.g \leftrightarrow \overline{E} \setminus E \subset \overline{\Delta_g} \setminus \Delta_g$$

The first homeomorphism is due to the principality of the transformation groupoid $\partial\beta X \rtimes \Gamma$ and the second requires the fact that all the basis elements are the boundaries of closures of subsets of X. This implies that locally we have a topological equivalence of groupoids - so taking a direct limit in $g \in G$ and closed basis elements U, we find that we have isomorphisms of $\partial\beta X \rtimes \Gamma = \bigcup_{g,U}[g, U] \leftrightarrow \bigcup_g \overline{\Delta_g} \setminus \Delta_g$. So it is enough to prove that the right hand side is what we want, that is the coarse boundary groupoid $G(X)|_{\partial\beta X}$.

Fix $R > 0$. As the metric coarse structure is finitely generated by the group action, we can find some $S > 0$ and finitely many $g \in \Gamma$ with length less than S such that $\overline{\Delta_R} \setminus \Delta_R \subset \bigcup_{|g|<S} \overline{\Delta_g} \setminus \Delta_g$. Given any element $\gamma \in \overline{\Delta_R} \setminus \Delta_R$, it belongs to precisely one $\overline{\Delta_g} \setminus \Delta_g$. It follows that $\bigcup_{R>0} \overline{\Delta_R} \setminus \Delta_R = \bigcup_g \overline{\Delta_g} \setminus \Delta_g$, as desired.

A last remark about the topology here; a basis for $G(X)|_{\partial\beta X}$ is given by all the coronas of the entourages $\overline{E} \setminus E$, each of which is contained in some finite union $\bigcup_F \overline{\Delta_g} \setminus \Delta_g$. Using Lemma 143 this is now an closed subset (after identification) of $\bigcup_F U \times g.U$ for some open $U \subset \partial\beta X$. Lastly, using the fact that $\partial\beta X \rtimes \Gamma$ is principal we can pull this back to a subset of the set $\bigcup_F [g, U]$, which is certainly closed. □

This proposition provides a collection of examples of sequences that we can deal with, and will in general be the conduit we want to pass through to verify the conjecture in the presence of a group action.

Example 168. (Box spaces) Let Γ be a finitely generated residually finite group and let $\{N_i\}$ be a family of finite index normal subgroups such that $N_i \leq N_{i+1}$, $\bigcap_{i\in\mathbb{N}} N_i = 1$. Fix a generating set S. Let $\square\Gamma = \sqcup_{i\in\mathbb{N}} \frac{\Gamma}{N_i}$, equipped with a metric that restricts to the metric induced from the generating sets $\pi_i(S)$ for each i, and has the property that $d(\frac{\Gamma}{N_i}, \frac{\Gamma}{N_j}) \to \infty$ as $i + j \to \infty$. This is called a *box space* for Γ. Then the sequence of groups $\{\Gamma/N_i\}$ admit a right action of Γ via quotient maps. The Stone-Čech boundary of the box space admits a free action of the group and the metric structure is generated at infinity by action of Γ. Proposition 167 then provides a description of the boundary groupoid and converts the boundary conjecture into a case of the Baum-Connes conjecture with coefficients for Γ.

This process did not require a normal subgroup: box spaces can constructed using Schreier quotients. The conditions on the subgroups change to reflect the absence of normality. Let Γ be a residually finite group and let $\{H_i\}$ be a family of nested subgroups of finite index with trivial intersection, and additionally satisfying: each $g \in \Gamma$ belongs to only finitely many conjugates of the subgroups from the family $\{H_i\}$. Fixing a left invariant metric on Γ the box space can be constructed using the left quotients of Γ by the H_i. In this instance these spaces are graphs with no left action of Γ. However they do retain a right action of Γ that determines the metric at infinity and becomes free on the boundary (this is due to the additional constraint concerning conjugates of the H_i).

This gives us some examples of situations where we can immediately verify the conjecture. We remark that a group is said to have the *Strong Baum-Connes property* if it satisfies the Baum-Connes conjecture with arbitrary coefficients. In particular this includes all amenable, a-T-menable [**HK97**] and, by remarkable recent results of Lafforgue [**Laf12**], groups that act on weakly geodesic strongly bolic metric spaces.

Theorem 169. *The Boundary Conjecture holds for sequences of graphs that are generalised box spaces of residually finite discrete groups that have the Strong Baum-Connes property.* ☐

This covers in particular certain expanding sequences that come from property (T) groups or property (τ) with respect to the corresponding family of finite index subgroups.

Explicitly this behaviour occurs for the sequence $\{SL_2(\mathbb{Z}/p^n\mathbb{Z})\}_{n\in\mathbb{N}}$; coming from congruence quotients in $SL_2(\mathbb{Z})$. In fact, this example motivates [**OOY09**] - this sequence of finite graphs has small girth as $SL_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6$ implies that the group has cycles of length 4 and 6 in its Cayley graph. However this also acts as an upper bound on cycle length - it otherwise looks like a tree as it is a virtually free group. In particular the space of graphs for any family is coarsely equivalent to one of large girth.

Proposition 166 on the other hand tells us that the coarse Novikov conjecture holds in much more generality than this.

Theorem 170. *The Boundary assembly map μ_{bdry} is injective for the generalised box spaces associated to all uniformly embeddable groups.*

Proof. We can use Proposition 167 to decompose our groupoid $G(X)|_{\partial\beta X}$ as $\partial\beta X \rtimes \Gamma$ for any generalised box space X of Γ. As Γ is exact we can conclude that the conjecture for $G(X)|_{\partial\beta X}$ is injective as it is equivalent to a conjecture with coefficients for Γ. Proposition 166 then allows us to conclude that the coarse Novikov conjecture holds for X. ☐

Corollary 171. *Let Γ be an uniformly embeddable group and let X be a generalised box space of Γ. Then the following hold:*

(1) The coarse Novikov conjecture holds for X.
(2) If X is an expander, then the assembly map μ fails to be surjective.

Proof. Proof follows from Theorem 170 and Proposition 166. □

This includes property (T) groups such as $SL_3(\mathbb{Z})$, and hence tells us interesting things for small girth expanders. Using the recent results of Sako on the relationship between property A and the operator norm localization property for uniformly discrete bounded geometry spaces [**Sak**] we get a simpler proof of [**CTWY08**, Theorem 7.1]. Given that any countable subgroup of $GL(n, K)$ is exact for any field K [**GHW05**] we can also conclude [**GTY11**, Theorem 5.3].

6.2. Some Remarks on the Max Conjecture. In addition to the coarse assembly map defined at the beginning of the Chapter it is possible to define a *maximal* variant, very much in analogy with the maximal Baum-Connes conjecture for groups. The primary difference between the maximal and reduced cases is that the maximal algebra has much nicer functorial properties. In particular the sequence:

$$0 \to \mathcal{K}(\ell^2(X,\mathcal{K})) \to \ell^\infty(X,\mathcal{K}) \rtimes_{max} G(X) \to (\frac{\ell^\infty(X,\mathcal{K})}{C_0(X,\mathcal{K})}) \rtimes_{max} G(X) \to 0$$

is exact. So we get the following ladder diagram at the level of the K-theory groups (as usual omitting coefficients):

$$\begin{array}{ccccccccccc} \to K_1(C^*_{max}(G(X)|_{\partial\beta X}) & \longrightarrow & K_0(\mathcal{K}) & \longrightarrow & K_0(C^*_{max}(G(X))) & \to & K_0(C^*_{max}(G(X)|_{\partial\beta X}) & \longrightarrow & K_1(\mathcal{K}) & \longrightarrow \\ \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mu_{bdry,max}} & & \uparrow & \\ \longrightarrow K_1^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & K_0^{top}(X\times X) & \longrightarrow & K_0^{top}(G(X)) & \longrightarrow & K_0^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & K_1^{top}(X\times X) & \to \end{array}$$

We use this functoriality to prove:

Proposition 172. *Let X be the space of graphs arising from a sequence of finite graphs $\{X_i\}$. Then*

(1) the maximal coarse Baum-Connes assembly map is an isomorphism if and only if the maximal Boundary coarse Baum-Connes map is an isomorphism.
(2) the maximal coarse assembly map is injective if and only if the maximal boundary assembly map is injective.

Proof. As before we consider a diagram, this time of maximal algebras:

$$\begin{array}{ccccccccccc}
\to K_1(C^*(G(X)|_{\partial\beta X}) & \longrightarrow & K_0(\mathcal{K}) & \longrightarrow & K_0(C^*(G(X))) & \to & K_0(C^*(G(X)|_{\partial\beta X}) & \longrightarrow & K_1(\mathcal{K}) & \longrightarrow \\
\uparrow & & \wr\| \uparrow & & \uparrow & & \mu_{bdry} \uparrow & & \wr\| \uparrow & \\
\longrightarrow K_1^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & K_0^{top}(X \times X) & \longrightarrow & K_0^{top}(G(X)) & \longrightarrow & K_0^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & K_1^{top}(X \times X) & \to
\end{array}$$

Both parts follow from a simple diagram chase. □

Theorem 173. *The maximal coarse Baum-Connes assembly map is an isomorphism for a space of graphs whose boundary groupoid decomposes as $\partial\beta X \rtimes \Gamma$ where Γ is a finitely generated discrete group with the Haagerup property.*

Proof. The Haagerup property provides us with the Strong Baum-Connes property for the maximal group conjecture. It then follows from Proposition 172. □

This result captures completely [**OOY09**, Corollary 4.18], but the proof is very much more elementary. It is also clear that the argument above also works in the case of injectivity, which allows us to strengthen Theorem 5.2.(2) of [**GWY08**] by requirements on the classifying space for proper actions of Γ:

Corollary 174. [**GWY08**, Theorem 5.2.(2)] *Let Γ be a finitely generated residually finite group. Then Γ satisfies the strong Novikov conjecture if and only if the maximal coarse assembly map μ for the box space $\Box\Gamma$ is injective.* □

7. The Boundary Conjecture for Spaces of Graphs with Large Girth.

The aim of the remainder of the chapter is to prove the following result:

Theorem 175. *The boundary coarse Baum-Connes conjecture holds for spaces of graphs with large girth and uniformly bounded vertex degree.*

A route to this Theorem is through Proposition 167, which characterizes the boundary groupoid as a transformation groupoid in the presence of a free boundary action that generates the metric coarse structure. For this we need to cook up a group action on an arbitrary sequence of finite graphs with large girth. Initially we consider some easier cases and for these we work via the universal covering sequence. We recall Pedersens Lemma [**Kön90**, Theorem 7, Chapter XI] from finite graph theory:

Lemma 176. *Let X be a finite graph. If $2k$ edges go into any vertex then all the edges of X can be divided into k classes such that two edges from the same class go into any vertex.* □

Definition 177. Let X be a finite $2k$-regular graph. A *k-orientation* is a choice of edge orientation and labelling in letters $a_1, ..., a_k$ that are compatible in the sense that precisely one edge oriented into and out of a vertex is labelled a_i for all i.

Lemma 176 provides us an avenue to construct k-orientations. We record this in a Lemma below.

Lemma 178. *Every finite $2k$-regular graph X can be $k-$oriented. Such a graph admits an action of the free group of rank k on the right by translations.*

Proof. The fact that any finite $2k$-regular graph can be k-oriented follows from Lemma 176 that tells us we can partition our edge set into k-pieces such that every vertex is incident on exactly two edges in each of the k partitioning sets. We can then orient this by mapping to a k-leafed bouquet and this covering map induces an inclusion of finite index of the group of deck transformations $\widehat{G}(X)$ into $\pi_1(\bigvee_{i=1}^{k} S^1) = F_k$, whose action is on the left by isometries on the Cayley graph of F_k. So we allow F_k to act on itself on the right - this commutes with the left action and hence passes to the left quotient by $\widehat{G}(X)$ giving us an action by translations on $\widehat{G}(X)\backslash F_k = X$ □

Remark 179. For the $4-$regular case we can get this result by appealing to the Eulerian tour that exists for all $2k$-regular graphs; we label around any such tour using the letters a and b alternatively. This provides us almost with what we want as from this we then re-label so that every vertex has the orientation of a ball of radius 1 in the free group [**Hat02**, pg 57]. It is not easy to produce a labelling of an Euler tour that is compatible with the necessary labelling we are after for any k larger than two however.

Remark 180. Let X be a finite $2k$-regular graph. A k-orientation can be thought of as providing a recipe to understand the action provided by the free group. Every vertex has entering (and leaving) precisely one edge labelled in the $a_1, ..., a_k$ and every undirected, not necessarily simple, path in the finite graph is now labelled in the letters $a_1, ..., a_k$ and has some assigned orientations. To describe the action, take a word in the free group with reduced form $w = \prod_i a_i^{e_i}$ and let $v \in V(X)$. Then we can apply w to v simply by following a walk along the letters a_i that make up w. In addition, if we choose to vertices connected by some path, that path is now labelled and oriented and by reading the labels from this path we will attain an element of the free group that takes us from x to y.

Lemma 181. *Let $\{X_i\}$ be a sequence of finite connected graphs that have $|X_i| \to \infty$ as $i \to \infty$ and are $2k$-regular and let X be the associated space of graphs. Then the action of F_k generates the metric coarse structure on X.*

Proof. Let $(x, y) \in \Delta_R$ with $x, y \in X_i$ for some i. Then they are joined by a path that as a consequence of a k-orientation is labelled in the generators of F_k and has assigned orientations.

This provides us enough information to read the action of F_k, whence there is a $w \in F_k$ that takes x to y. This implies $(x, y) \in \Delta_w$. Let F be the finitely many pairs $(x, y) \in \Delta_R$ that come from distinct X_is. Then Δ_R decomposes as:

$$\Delta_R = F \cup \bigcup_{|w| \leq R} \Delta_w$$

□

The intuition for this action at infinity can be gathered from the *ultralimit* of the sequence in the following way: If sequences of points in each X_i, when viewed as subsets of X, are fixed then the action is not free. An asymptotically faithful covering sequence essentially tells us that no sequence is fixed.

LEMMA 182. *The action on X of F_k extends to βX and is free on $\partial\beta X$.*

PROOF. Firstly, the action is continuous as we are acting on a discrete space X, hence it extends to a continuous action on the Stone-Čech compactification βX. We now deal with the second part of the claim.

Let $g \in F_k$ and for each i fix a basepoint $x_i \in X_i$. As the graph is finite there exists an n_i such that g^{n_i} translates x_i to itself. We assume that there is only a single orbit for the purposes of the following argument as the case of multiple orbits is similar. This gives us an action of $\mathbb{Z}/n_i\mathbb{Z}$ on X_i for each i. We note that there are two cases as for any $\omega \in \partial\beta X$ as we know that the pieces:

$$X_{even} = \bigsqcup_{n_i \equiv 0 \bmod 2} X_i$$
$$X_{odd} = \bigsqcup_{n_i \equiv 1 \bmod 2} X_i$$

are mutually complimentary and union to the entire of X, hence ω picks either X_{even} or X_{odd}.

For the even case break the space into two complimentary pieces in the following way:

$$A_{i,0} := \{x_i.g^n | n \equiv 0 \text{ mod } 2\}$$
$$A_{i,1} := \{x_i.g^n | n \equiv 1 \text{ mod } 2\}$$

and let $A_j = \sqcup_{i|n_i \in \{even\}} A_{i,j}$. We assume for a contradiction that $\omega = \omega.g$ and then observe that g permutes A_0 to A_1, so if, without loss of generality, $A_0 \in \omega$ we can deduce that $A_0.g = A_1 = A_0^c \in \omega$, which is a contradiction.

The odd case is similar only we break each X_i represented into three pieces:

$$B_{i,0} := \{x_i.g^n | n \equiv 0 \bmod 2 \text{ and } n \neq n_i - 1\}$$
$$B_{i,1} := \{x_i.g^n | n \equiv 1 \bmod 2\}$$
$$B_{i,2} := \{x_i^{n_i - 1}\}$$

$B_{i,2}$ is necessary here as the action of g sends that point to $B_{i,0}$, which would otherwise have been a map from $B_{i,0}$ to itself. We build the corresponding $B_j = \sqcup_{\{i | n_i \in \{odd\}\}} B_{i,j}$. Again let $\omega.g = \omega$ and observe that $B_j \in \omega$ for some j. Acting by g gives: $B_j.g \in \omega.g$, hence $B_j.g \in \omega$. Considering j mod 3: $B_j.g \subset B_{j+1} \sqcup B_{j+2} = B_j^c$ which again gives a contradiction. □

THEOREM 183. *Conjecture 1 holds for spaces of graphs of large girth and $2k$-regularity.*

PROOF. Lemmas 181 and 182 combine with Proposition 167 to give us that $G(X)|_{\partial\beta X}$ is isomorphic to $\partial\beta X \rtimes F_k$. The proof follows using either a Pismner-Voiculescu argument or appealing to the Strong Baum-Connes property for F_k. □

COROLLARY 184. *([**WY12a**, Theorem 1.5]) For sequences of large girth and vertex degree $2k$ we have that the Coarse Baum-Connes assembly map is injective and if the sequence forms an expander then it also fails to be surjective.*

PROOF. Combine Proposition 164 and Theorem 183. □

7.1. Some Finite Graph Theory. The main idea in the previous Theorem was that we could utilise Pedersens Lemma to build an action of the free group on a large girth sequence. To adjust the results to a situation in which the vertex degree is odd everywhere we use some finite graph theory, this time we make use of 1-factors.

DEFINITION 185. Let X be a connected finite graph. A *1-factor* is a spanning subgraph M such that for every vertex $v \in V(M) = V(X)$ we have $\deg(v) = 1$.

Graph factorisation is well studied [**AK85**] and hence using some more of this theory we can arrive at an analogue of the Theorem 183. The issue with a direct analogue is that the universal cover of a $2k+1$-regular graph is a $2k+1$-regular infinite tree, which is not automatically a cayley graph of a free group. However there is still technology to deal with this. The following is [**Sum74**, Corollary 2].

PROPOSITION 186. *Let X be a finite connected graph with $|X| = 2n$ and no induced subgraphs isomorphic to $K_{1,3}$ then X has a 1-factor.* □

The removal of a 1-factor from a finite graph of uniform odd vertex degree $2k+1$ gives a new finite graph that is a disjoint union of finitely many connected components that are each $2k$ regular.

Proposition 187. *Let* X *be a* $2k+1$*-regular finite graph that has a* $1-\mathtt{factor}$*. Then there is an action of* $F_k * C_2$ *on* X *via labellings.*

Proof. Consider the $1-$factor $M \subset X$. Then consider the graph X' with the same vertex set as X but with the edges of M removed; this is a finite disjoint union of $2k$-regular induced subgraphs that we can now label and act on using Lemma 178. Now add back the edges of M but with no orientation. We observe that the edges of M can be thought of as ways to reflect in the graph. Hence we attain an action of $F_k * C_2$ by combining the obvious actions of both the factors. □

We can use Propositions 167 and 187 to prove the following:

Theorem 188. *Let* $\{X_i\}$ *be a sequence of finite graphs that are* $2k+1$*-regular and cofinitely many contain no induced* $K_{1,3}$*'s and let* X *be the associated space of graphs. Then conjecture 1 holds for* X*.*

Proof. We argue as we did in the $2k$-regular case. As the $2k+1$-regular infinite tree forms an asymptotically faithful covering sequence for the X_i's we can conclude that our boundary groupoid: $G(X)|_{\partial\beta X}$ is homeomorphic to $\partial\beta X \rtimes (F_k * C_2)$. Now we can conclude the proof using either the strong Baum-Connes property or using an elementary argument in K-theory using the results of [**Lan83**] on free products and a Pimsner-Voiculescu argument. □

However this is not very satisfying as there are many finite $2k+1$-regular graphs with edge chromatic number $2(k+1)$ which do not immediately admit 1-factors - not to mention the fact that we would like this result to hold in much more generality than sequences of regular graphs. We are interested in sequences with only a uniform upper bound on their regularity in order to reach the most general results of [**WY12a**]. To tackle this we need a more flexible way to allow the free group to affect our finite graphs. We proceed via the notion of a *partial action* introduced in Chapter 2.

7.2. The General Strategy via Graph Colourings. We begin by considering the more general result that accompanies Pedersens Lemma. The following is [**Kön90**, Theorem 6, Chapter XI]:

Lemma 189. *Let* X *be a finite graph. If at most* $2k$ *edges go into any vertex then all the edges of* X *can be divided into* k *classes such that at most two edges from the same class go into any vertex.* □

We want to use this to label any sequence of graphs that have uniformly bounded degree, which without loss of generality can be chosen to be an even uniform upper bound. We call such a labelling a *partial k-orientation* and we say such graphs are *partially or almost k-oriented.* From the point of view of building a group action Lemma 189 is completely useless, however if we are willing to work with a reasonable generalisation of a group action Lemma 189 provides us ample information. When considering the space of graphs X of a sequence of finite graphs $\{X_i\}$ the strategy is as follows:

(1) Construct from a partial k-orientation a collection of *partial bijections* of each finite graph X_i. These will have disjoint support, whence they can be "added" together when we pass to the space of graphs - giving us, for each group element, a partial bijection on X. A natural thing to do then is ask how such things can be composed; they generate a submonoid of the *symmetric inverse monoid* over X.
(2) Applying the work of Exel in [**Exe08**] (or Paterson [**Pat99**]) to this inverse monoid we can associate a groupoid over X. Combining this with an augmentation of Proposition 167 we can get a description of the boundary groupoid for the space X.
(3) We utilise properties of the inverse monoid to prove that this groupoid has the Haagerup property. This in turn provides us with the Baum-Connes conjecture being an isomorphism with any coefficients for this groupoid. We use this to conclude that the boundary coarse Baum-Connes assembly map is an isomorphism for X.

The remainder of this section is making these ideas precise.

REMARK 190. We can always assume that the $2k$ here is minimal; there is a smallest even integer that bounds above the degree of all our graphs. This in particular stops us from doing something unnatural like embedding the 4-regular tree into a 6-regular tree.

Let $X = \sqcup X_i$ be a space of graphs admitting a bounded partial action of a discrete group G. We remark that in this setting partial bijections in the group can have the following form:

$$\theta_g = \theta_g^0 \sqcup \bigsqcup_{i>i_0} \theta_g^i.$$

Where i_0 is the first i for which the distances between the X_is is greater than the upper bound of the distance moved by θ_g, and the θ_g^i are componentwise partial bijections of the X_i. We collect all the additional pieces that act only between the first i_0 terms into θ_g^0, which could be the empty translation. We remark now that it is possible that there are partial bijections θ_g that could have finite support, that is only finitely many terms that are non-empty after i_0. To avoid this, we observe the following:

PROPOSITION 191. *Let* $S = \langle \theta_g | g \in G \rangle$ *and let* $I_{fin} = \{\theta_g | \mathrm{supp}(\theta_g)$ *is finite*$\}$. *Then* I_{fin} *is an ideal and the Rees quotient* $S_{inf} = \frac{S}{I}$ *is an inverse monoid with* 0.

PROOF. To be an ideal, it is enough to show that $I_{fin}S \subset I_{fin}, SI_{fin} \subset I_{fin}$. Using the description of the multiplication of partial bijections from Chapter 2, it is clear that either combination si or is

yields an element of finite support. Now we can form the Rees quotient, getting an inverse monoid with a zero - the zero element being the equivalence class of elements with finite support. □

We want to utilise a partial action to construct a groupoid, so we apply the general construction outlined in Section 2.1 of Chapter 2 to get an improved version of Proposition 167.

As we are interested in specialising to a partial action that will somehow generate the boundary coarse groupoid we would like to know that we can get information about the metric coarse structure from the partial action when X is a metric space. However in general it is too much to expect that our partial action generates the metric coarse structure completely. To understand this we need to define the length of a partial bijection:

DEFINITION 192. The length of each θ_g is defined to be:

$$|\theta_g| = \sup\{d(x, \theta_g(x)) : x \in \mathrm{Dom}(\theta_g)\}.$$

DEFINITION 193. We say a bounded partial action generates the metric coarse structure at infinity if for all $R > 0$ there exists $S > 0$ such that $\overline{\Delta_R} \setminus \Delta_R = \bigcup_{|\theta_g|<S} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$. We say it finitely generates the metric coarse structure if the number of θ_g required for each R is finite.

REMARK 194. Recall a groupoid G is said to be *principal* if the map $(s, r) : G \to G^{(0)} \times G^{(0)}$ is injective.

PROPOSITION 195. *Let* $\{X_i\}$ *be a sequence of finite graphs and let* X *be the corresponding space of graphs. If* $\theta : G \to \mathcal{I}(X)$ *is a bounded partial action of* G *on* X *such that the induced action on* βX *is free on* $\partial\beta X$*, the inverse monoid* S_{inf} *is 0-F-inverse and the partial action finitely generates the metric coarse structure at infinity then there is a second countable, étale ample topological groupoid* $\mathcal{G}_{\widehat{X}}$ *such that* $G(X)|_{\partial\beta X} \cong \partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$.

PROOF. Observe now that the finite θ_g play no role in the action on the boundary and so we work with S_{inf}. We build the groupoid from the bottom up, by first constructing the unit space using Proposition 10.6 and Theorem 10.16 from [**Exe08**].

We consider the representation of the inverse monoid S_{inf} on $\ell^2(X)$ induced by θ to get a representation $\pi_\theta : S \to \mathcal{B}(\ell^2(X))$. We can complete the semigroup ring in this representation to get an algebra $C^*_{\pi_\theta}S$, which has a unital commutative subalgebra $C^*_{\pi_\theta}E$. Proposition 10.6 [**Exe08**] then tells us that the spectrum of this algebra, which we will denote by $\widehat{X}$, is a subspace of $\widehat{E}$ that is closed and invariant under the action of S on which the representation π_θ is supported.

As the space $\widehat{X}$ is closed and invariant we can reduce the universal groupoid $\mathcal{G}_{\widehat{E}}$ for S_{inf} to $\widehat{X}$. This we denote by $\mathcal{G}_{\widehat{X}}$.

We show this groupoid acts on βX; we make use of the assumption that θ_g is a bounded partial bijection for each $g \in G$ and again of the representation π_θ. Each θ_g bounded implies that the algebra $C^*_\theta(S)$ is a subalgebra of $C^*_u X$. We now remark that the representation π_X, when restricted to C^*E assigns each idempotent a projection in $C^*_u X$, that is $C^*_{\pi_X}(E) = \pi_X(C^*E) \subset \ell^\infty(X)$. Taking the spectra associated to this inclusion then gives us a map:

$$r_{\beta X} : \beta X \twoheadrightarrow \widehat{X}$$

which is continuous. In particular as both βX and $\widehat{X}$ are compact Hausdorff spaces; this map is closed (and open) and hence a quotient. We make use of this to define an action on βX. By Claim 28 we have that each element of our groupoid $\mathcal{G}_{\widehat{X}}$ can be represented by a pair $[\theta_g, \phi]$, for some $\phi \in \widehat{X}$. Observe also that as X is discrete so are all of it subspaces, hence the maps θ_g are continuous (open) for each $g \in G$. These then extend to βX, and so coupled with the map $r_{\beta X}$ we can act by:

$$[\theta_g, \phi].\omega = \theta_g(\omega)$$

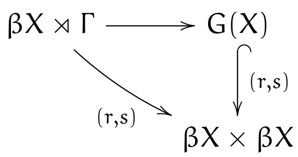
for all $\omega \in D_{\theta_g^*\theta_g}$ with $r_Z(\omega) = \phi$. We see that $r_Z(\omega).\omega = [\theta_e, r_Z(\omega)].\omega = \omega$. and for all $([\theta_g, \phi], [\theta_h, \phi']) \in \mathcal{G}^{(2)}_{\widehat{X}}$ with $\phi' = r_Z(\omega)$ we have:

$$[\theta_g\theta_h, \phi'].\omega = \theta_g\theta_h(\omega) = \theta_g([\theta_h, \phi'].\omega) = [\theta_g, \phi].([\theta_h, \phi'].\omega)$$

as $r_Z([\theta_h, \phi'].\omega) = \theta_h(\phi') = \phi$.

It remains to prove the isomorphism of topological groupoids: $G(X)|_{\partial\beta X} \cong \partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$. We follow the scheme of Proposition 167 and build a map from $\beta X \rtimes \mathcal{G}_{\widehat{X}}$ to $G(X)$. Recall that as the partial action of G generates the metric coarse structure at infinity $G(X) = (\bigcup_g \Delta_{\theta_g}) \cup (X \times X)$. We observe that each Δ_{θ_g} maps bijectively onto the domain of θ_g, a subset of X.

This map extends to the closure of the domain precisely as in Proposition 167, where here we map the pair $(\omega, [\theta_g, \phi])$ to the element $\gamma_{g,\phi}$ that is the limit $\lim_\lambda(x_\lambda, \theta_g(x_\lambda))$ for some net $\{x_\lambda\}$ that converges to ω (and also to ϕ). This map is well defined as the groupoid $G(X)$ is principal, and it fits into the following commutative diagram:



Again by principality, we can deduce that the covering map is a groupoid homomorphism.

We now restrict this map to the boundary $\partial\beta X$. As we know that the group action generates the metric coarse structure at infinity and that the partial action of the group G is free on the boundary. Using these facts we can see that:

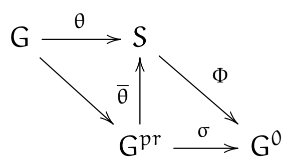
(1) $\partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$ is principal.
(2) $G(X)|_{\partial\beta X} = \bigsqcup_g \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$.

From both (1) and (2) we can further deduce that the covering map is a bijection on the boundary. Both groupoids are also étale and so each component $\overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$ is mapped homeomorphically onto its image and is therefore clopen. It follows then that we get the desired isomorphism $\partial\beta X \rtimes \mathcal{G}_{\widehat{X}} \cong G(X)|_{\partial\beta X}$ of topological groupoids. □

7.3. Relating $\mathcal{G}_{\widehat{X}}$ to G. We are interested in understanding those analytic properties the groupoid $\mathcal{G}_{\widehat{X}}$ has, in particular we are interested in showing that the groupoid has the Haagerup property, that is admits a proper affine isometric action on a field of Hilbert spaces. From results of Tu in [**Tu99**] this enough to conclude the Baum-Connes assembly map is an isomorphism for all coefficients for this groupoid. To do this we study the inverse monoid S associated to the partial action θ.

Proposition 196. *Let* $S = \langle \theta_g | g \in G \rangle$, *where* $\theta : G \to S$ *is a dual prehomomorphism. If* S *is 0-F-inverse with* $\mathrm{Max}(S) = \{\theta_g | g \in G\}$ *where each nonzero* θ_g *is not idempotent when* $g \neq e$ *then* S *is strongly 0-F-inverse.*

Proof. We build a map Φ back onto G^0. Let $m : S \setminus \{0\} \to \mathrm{Max}(S)$ be the map that sends each non-zero s to the maximal element $m(s)$ above s and consider the following diagram:



where G^{pr} is the prefix expansion of G. Define the map $\Phi : S \to G^0$ by:

$$\Phi(s) = \sigma(m(\overline{\theta}^{-1}(m(s)))), \Phi(0) = 0$$

For each maximal element the preimage under $\overline{\theta}$ is well defined as the map θ_g has the property that $\theta_g = \theta_h \Rightarrow g = h$ precisely when $\theta_g \neq 0 \in S$. Given the preimage is a subset of the F-inverse monoid G^{pr} we know that the maximal element in the preimage is the element $(\{1, g\}, g)$ for each $g \in G$, from where we can conclude that the map σ takes this onto $g \in G$.

We now prove it is a prehomomorphism. Let $\theta_g, \theta_h \in S$, then:

$$\Phi(\theta_g) = \sigma(m(\overline{\theta}^{-1}(\theta_g))) = \sigma(\{1, g\}, g) = g$$
$$\Phi(\theta_h) = \sigma(m(\overline{\theta}^{-1}(\theta_h))) = \sigma(\{1, h\}, h) = h$$
$$\Phi(\theta_{gh}) = \sigma(m(\overline{\theta}^{-1}(\theta_{gh}))) = \sigma(\{1, gh\}, gh) = gh$$

Hence whenever θ_g, θ_h and θ_{gh} are defined we know that $\Phi(\theta_g\theta_h) = \Phi(\theta_g)\Phi(\theta_h)$. They fail to be defined if:

(1) If $\theta_{gh} = 0$ in S but θ_g and $\theta_h \neq 0$ in S, then $0 = \Phi(\theta_g\theta_h) \leq \Phi(\theta_g)\Phi(\theta_h)$
(2) If (without loss of generality) $\theta_g = 0$ then $0 = \Phi(0.\theta_h) = 0.\Phi(\theta_h) = 0$

So prove that the inverse monoid S is strongly 0-F-inverse it is enough to prove then that the map Φ is idempotent pure, and without loss of generality it is enough to consider maps of only the maximal elements - as the dual prehomomorphism property implies that in studying any word that is non-zero we will be less than some θ_g for some $g \in G$.

So consider the map Φ applied to a θ_g:

$$\Phi(\theta_g) = \sigma(m(\overline{\theta}^{-1}(\theta_g))) = \sigma(\{1, g\}, g) = g$$

Now assume that $\Phi(\theta_g) = e_G$. Then it follows that $\sigma(m(\overline{\theta}^{-1}(\theta_g))) = e_G$. As σ is idempotent pure, it follows then that $m(\overline{\theta}^{-1}(\theta_g)) = 1$, hence for any preimage $t \in \theta^{-1}(\theta_g)$ we know that $t \leq 1$, and by the property of being 0-E-unitary it then follows that $t \in E(G^{pr})$. Mapping this back onto θ_g we can conclude that θ_g is idempotent, but by assumption this only occurs if $g = e$. □

PROPOSITION 197. *Let* $S = \langle \theta_g | g \in G \rangle$ *be a strongly 0-F-inverse monoid with maximal elements* $Max(S) = \{\theta_g : g \in G\}$, *where* $\theta : G \to S$ *is a dual prehomomorphism. Then the groupoid* $G_{\widehat{E}}$ *admits a continuous proper groupoid homomorphism onto the group* G.

PROOF. Using the map Φ we construct a map $\rho : \mathcal{G}_{\widehat{E}} \to G$ as follows:

$$\rho([m, \phi]) = \Phi(m)$$

A simple check proves this is a groupoid homomorphism. This map sends units to units as the map Φ is idempotent pure. We prove continuity by considering preimage of an open set in G:

$$\rho^{-1}(U) = \bigcup_{g \in U} [\theta_g, D_{\theta_g^*\theta_g}]$$

This is certainly open as each $[\theta_g, D_{\theta_g^*\theta_g}]$ are elements of the basis of topology of $\mathcal{G}_{\widehat{E}}$. We check it is proper by observing that for groups G compact sets are finite, and they have preimage:

$$\rho^{-1}(F) = \bigcup_{g \in F} [\theta_g, D_{\theta_g^*\theta_g}], \ |F| < \infty$$

This is certainly compact as these are open and closed sets in the basis of topology for the groupoid $\mathcal{G}_{\widehat{E}}$. □

As $\mathcal{G}_{\widehat{X}} \subseteq \mathcal{G}_{\widehat{E}}$ we also get a continuous proper groupoid homomorphism from $\mathcal{G}_{\widehat{X}}$ onto a group. A remark that comes from considering the work of Lawson in [**Law98**] is that we can consider the category of inverse monoids with prehomomorphisms equivalent to the category of ordered groupoids with groupoid homomorphisms, so it is reasonable to consider such maps when we want to understand the structure of the universal groupoid associated to S.

We recall a special case of [**Tu99**, Lemme 3.12].

LEMMA 198. *Let* G *and* H *be locally compact, Hausdorff, étale topological groupoids and let* $\varphi : G \to H$ *be a continuous proper groupoid homomorphism. If* H *has the Haagerup property then so does* G. □

This lets us conclude the following:

COROLLARY 199. *Let* θ *be a partial action of* G *on* X *such that all the conditions of Proposition 197 are satisfied. If* G *has the Haagerup property then so does* $\mathcal{G}_{\widehat{X}}$.

PROOF. The map induced by the idempotent pure 0-restricted prehomomorphism from S_{inf} to G induces a continuous proper groupoid homomorphism from $G_{\widehat{X}}$ to G. This then follows from Lemma 198. □

7.4. Partial actions on sequences of graphs. Let $\{X_i\}$ be a sequence of finite graphs with degree $\leq 2k$ and large girth.

LEMMA 200. *Such a sequence can be almost* k*-oriented and this defines a bounded partial action of* F_k *on* X

PROOF. We work on just the X_i. Using Lemma 189, we partition the edges $E(X_i)$ into at most k sets E_j such that every vertex appears in at most 2 edges from each subset. Pick a generating set $S = \{a_j | j \in \{1, ..., k\}\}$ for F_k and assign them to the edge sets E_j, and label the edges that appear in each E_j by the corresponding generator. Pedersens Lemma ensures that no more than 2 edges at each point have the same label. This defines a map from the edges to the wedge $\bigvee_{j=i}^{k} S^1$. Choose an orientation of each circle and pull this back to the finite graph X_i - this provides the partial k-orientation. Now define for each generator the partial bijection $\theta^i_{a_j}$ that maps any vertex appearing as the source of any edge in E_j to the range of that edge. I.e:

$$\theta^i_{a_j}(v) = \begin{cases} r(e) \;\text{ if } \exists e \in E_j : s(e) = v \\ \text{undefined otherwise} \end{cases}$$

For $g = a_l^{e_l} ... a_m^{e_m}$ we define θ^i_g as the product $\theta_{a_l^{e_l}} ... \theta_{a_m^{e_m}}$; i.e θ^i_g moves vertices along any path that is labelled by the word g in the graph X_i. We observe that for $i \neq i'$ the domain $D^i_{\theta^*_g\theta_g} \cap D^{i'}_{\theta^*_g\theta_g}$ is empty hence we can add these partial bijections in $I(X)$ to form $\theta_g = \sqcup \theta^i_g$. It is a remark that as the topology of X is discrete these maps are all continuous and open. It is clear that as each X_i is connected that the partial bijections have the property that $\cup_g D_{\theta^*_g\theta_g} = X$. Lastly, this map is a dual prehomomorphism as for each $g, h \in G$ we have that $\theta_g\theta_h = \theta_{gh}$ precisely when both θ_h and θ_{gh} are defined and moreover if $\theta_g\theta_h$ is defined then so is θ_{gh}. Hence this collection forms a partial action of G on X. We also remark that as each bijection is given translation along a labelling in the free group it is clear that these move elements only a bounded distance and are therefore elements of $I_b(X)$. □

REMARK 201. If we consider the proof of the above Lemma then it would seem that for every θ_g, θ_h we have that $\theta_g\theta_h = \theta_{gh}$. However this might not be the case because of cancellation that occurs in the group but not in the partial bijections.

We would want to show that the partial action generates the metric coarse structure, we recall the length of a partial bijection:

DEFINITION 202. The length of each θ_g is defined to be:

$$|\theta_g| = \sup\{d(x, \theta_g(x)) : x \in \mathrm{Dom}(\theta_g)\}.$$

REMARK 203. As we have a concrete description of each θ_g, given on each X_i, we can see that the length on each X_i is given by:

$$|\theta_g^i| = max\{|p| : p \in \{ \text{ paths in } X_i \text{ labelled by g}\}.$$

Then $|\theta_g| = \sup_i |\theta_g^i|$.

In this situation we require that the partial action contains plenty of infinitely supported elements.

PROPOSITION 204. *Let $\theta : F_k \to I_b(X)$ be the dual prehomomorphism corresponding to the bounded partial actions on each X_i. Then for each $R > 0$ there exist finitely many infinite θ_g with $|\theta_g| = |g| < R$.*

PROOF. In the general case we know the following for each $R > 0$ and $i \in \mathbb{N}$: $|\theta_g^i| \leq |g| \leq R$. From Lemma 200 we know that the partial action is defined by moving along paths inside each individual X_i. So for each R we count the number of words in F_k with length less than R; this is finite (consider the Cayley graph, which has bounded geometry). Now we observe that on the other hand there are infinitely many simple paths of length less than R, thus we must repeat some labellings infinitely many times. These labellings will be contained in words in F_k of length less than R hence when we take the supremum we observe that $|\theta_g| = |g| < R$. □

COROLLARY 205. *The bounded partial action θ of F_k on X finitely generates the metric coarse structure at infinity, that is the set $\overline{\Delta_R} \setminus \Delta_R = \bigcup_{|g|<R} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$ where the index set is finite.*

PROOF. We proceed by decomposing Δ_R as we did in the proof of Proposition 167.

$$\Delta_R = (\bigcup_{|\theta_g|<R} \Delta_{\theta_g}) \cup F_R$$

Where F_R is the finitely many elements of Δ_R who move between components. We now consider the following decomposition of the set $A := \{\theta_g || \theta_g| < R\}$ into:

$$A_\infty = \{\theta_g || \theta_g| < R \text{ and } |supp(\theta_g)| = \infty\}$$
$$A_{fin} = \{\theta_g || \theta_g| < R \text{ and } |supp(\theta_g)| < \infty\}$$

The first of these is in bijection with the words in F_k that have $|g| < R$ and define an infinite θ_g from Proposition 204. Then:

$$\Delta_R = (\bigcup_{g\in A_\infty} \Delta_{\theta_g}) \cup (\bigcup_{g\in A_{fin}} \Delta_{\theta_g}) \cup F_R$$

We complete the proof by observing that for each $\theta_g \in A_{fin}$ the set Δ_{θ_g} is finite. Therefore:

$$\overline{\Delta_R} \setminus \Delta_R = \bigcup_{g\in A_\infty} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g} = \bigcup_{|g|<R} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$$

□

As in the case of uniform regularity we also need to see that the action is free. Ideally we would argue as if we were in the group case; for each $g \in F_k$ choose a point in each X_i and consider its orbits as was implemented in Lemma 182. However, this argument does not work; we are faced with the problem that for a partial action the concept of orbit is not well-defined. In particular it may not always be possible to apply an element $\theta_i(g)$ twice to things within its domain.

The following is a concept to replace that of an orbit:

DEFINITION 206. We say a subset $\mathcal{A}_g \subset X$ is a *stable core* for the element θ_g if $\theta_g(\mathcal{A}_g) = \mathcal{A}_g$.

Recall that for each subset A of X, $\widehat{D}_A := \{\omega \in \partial\beta X | A \in \omega\}$.

LEMMA 207. *Let* $\theta : F_k \to I_b(X)$ *be the partial action defined above. If* $\mathcal{A}_g$ *is a stable core then for any* $\omega \in \widehat{D}_{\mathcal{A}_g}$ $\omega \neq \theta_g(\omega)$.

PROOF. Take $x_i \in \mathcal{A}^i_g$. Then let n_i the the smallest integer such that $\theta_g(x_i) = x_i$. We observe also that there are two cases as for any $\omega \in D_{\mathcal{A}_g}$ as we know that the pieces:

$$\mathcal{A}_{even} = \bigsqcup_{n_i \equiv 0 \bmod 2} \mathcal{A}^i_g$$
$$\mathcal{A}_{odd} = \bigsqcup_{n_i \equiv 1 \bmod 2} \mathcal{A}^i_g$$

are mutually complimentary and union to the entire of $\mathcal{A}_g^i$, hence ω picks either $\mathcal{A}_{even}$ or $\mathcal{A}_{odd}$.

For the even case break each $\mathcal{A}_g$ into two complimentary pieces in the following way:

$$A_{i,0} := \{\theta_g^n(x_i) | n \equiv 0 \bmod 2\}$$
$$A_{i,1} := \{\theta_g^n(x_i) | n \equiv 1 \bmod 2\}$$

and let $A_j = \sqcup_{i|n_i \in \{even\}} A_{i,j}$. We assume for a contradiction that $\omega = \theta_g(\omega)$ and then observe that g permutes A_0 to A_1, so if, without loss of generality, $A_0 \in \omega$ we can deduce that $\theta_g(A_0) = A_1 = A_0^c \in \omega$, which is a contradiction.

The odd case is similar only we break each $\mathcal{A}_g^i$ represented into three pieces:

$$B_{i,0} := \{\theta_g^n(x_i) | n \equiv 0 \bmod 2 \text{ and } n \neq n_i - 1\}$$
$$B_{i,1} := \{\theta_g^n(x_i) | n \equiv 1 \bmod 2\}$$
$$B_{i,2} := \{\theta_g^{n_i - 1}(x_i)\}$$

$B_{i,2}$ is necessary here as the action of θ_g sends that point to $B_{i,0}$, which would otherwise have been a map from $B_{i,0}$ to itself. We build the corresponding $B_j = \sqcup_{\{i|n_i \in \{odd\}\}} B_{i,j}$. Again let $\theta_g(\omega) = \omega$ and observe that $B_j \in \omega$ for some j. Acting by θ_g gives: $\theta_g(B_j) \in \theta_g(\omega)$, hence $\theta_g(B_j) \in \omega$. Considering j mod 3: $B_j.g \subset B_{j+1} \sqcup B_{j+2} = B_j^c$ which again gives a contradiction. □

LEMMA 208. *The partial action of* F_k *defined above extends to* βX *and is free on the boundary* $\partial\beta X$.

PROOF. For each $g \in F_k$ the domain of θ_g breaks into three pieces. The first piece, index denoted I_0, consists of all the $D^i_{g^*g}$ such that $D^i_{\theta_g^*\theta_g} \cap D^i_{\theta_g\theta_g^*}$ is empty; the second, indexed by I_1, consists of all the $D^i_{\theta_g^*\theta_g}$ such that $D^i_{\theta_g^*\theta_g} \cap D^i_{\theta_g\theta_g^*}$ is not empty but $D^i_{\theta_g^*\theta_g}$ is not contained in $D^i_{\theta_g\theta_g^*}$; and the third, denoted by I_2 is all the $D^i_{\theta_g^*\theta_g}$ such that $D^i_{\theta_g^*\theta_g} = D^i_{\theta_g\theta_g^*}$.

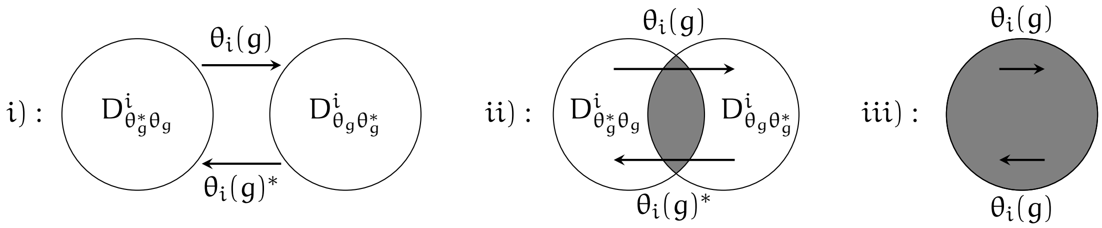


FIGURE 1. The three cases for Lemma 208

Case i) is clear. In Case iii) we observe that $A_3 := \sqcup_{i \in I_3} D^i_{\theta_g^*\theta_g}$ is a stable core. By Lemma 207, $\omega \neq \omega_g(\omega)$ if $A_3 \in \omega$. This leaves case ii).

Let $A_{0,i} := D^i_{\theta_g^*\theta_g} \setminus D^i_{\theta_g^*\theta_g} \cap D^i_{\theta_g\theta_g^*}$ and $A_1 = D^i_{\theta_g^*\theta_g} \cap D^i_{\theta_g\theta_g^*}$. Then set $A_j = \bigsqcup_{i \in I_2} A_{j,i}$ and let $\omega \in \partial\beta X$. If $A_0 \in \omega$ then $\theta(g)(A_0) \subset A_0^c$, which implies that $\theta(g)(A_0) \notin \theta(g)(\omega)$.

It remains to deal with $A_1 \in \omega$. Assume for a contradiction that $\theta(g)(\omega) = \omega$. Then $\theta(g)(A_1) \cap A_1 \in \omega$, and we can apply $\theta(g)$ again - denote by $A_1^m = A_1 \cap \theta(g)(A_1) \cap ... \cap \theta(g)^m(A_1)$. For each $i \in I_2$ there exists a power m_i of $\theta(g)$ such that this intersection stabilises in X_i, that is $A_{1,i}^{m_i} = A_{1,i}^{m_i+1}$.

Let $\mathcal{A} = \bigcap_{m \in \mathbb{N}} A_1^m$. This is a stable core. If $\mathcal{A} \in \omega$ then Lemma 207 gives $\omega \neq \theta_g(\omega)$.

So the last case to consider is that $\mathcal{A}^c \in \omega$. It suffices to work in A_1, so let $B_1 = \mathcal{A}^c \cup A_1$ and then define $B_i = \theta_g(B_{i-1}) \cap B_1$. It follows from the construction of $\mathcal{A}$ that every element $x \in B_1$ has an associated smallest natural number n_x such that $\theta_g^{n_x}(x) \notin B_{n_x}$. It is clear from the definition that $B_{i+1} \subset B_i$ for every i. Lastly, define $B_{i+1}^{-1} := \theta_g^{-1}(B_{i+1}) \subset B_i$. From this we consider the decomposition of B_i into two disjoint infinite pieces:

$$B_{i,even}^{\pm 1} := \{x \in B_i^{\pm 1} | n_x \equiv 0 \bmod 2\}$$
$$B_{i,odd}^{\pm 1} := \{x \in B_i^{\pm 1} | n_x \equiv 1 \bmod 2\}.$$

ω must choose precisely one of these two pieces for each i. Assume without loss that $B_{1,even} \in \omega$. It is clear that $B_2^{-1} \cap B_{1,even} = B_{2,even}^{-1} \in \omega$ is sent, by θ_g, to $B_{2,odd}$ and so $B_{2,odd} \in \theta_g(\omega)$. From the assumption that $\theta_g(\omega) = \omega$, we can conclude that $B_{2,odd} \in \omega$. As ultrafilters are upwardly closed, we know also that $B_{1,odd} \in \omega$, which is a contradiction. □

This freeness gives us a tool to understand the structure of S_{inf}.

LEMMA 209. *Let $\{X_i\}$ be a sequence of graphs and let G be a group which acts partially on each X_i. If G fixes any sequence in $\{X_i\}$ then the partial action is not free on $\partial\beta X$.*

PROOF. Let θ_g denote the disjoint union of the θ_g^i arising from the partial action of G on each X_i. To prove this it is enough to show that there is a single $\omega \in \partial\beta X$ that is fixed by the action of some $g \in G$. The hypothesis that G fixes a sequence gives us $\mathbf{x} := \{x_n\}_I$ with I infinite and $\theta_g(\{x_n\}) = \{\theta_g^n(x_n)\}_I = \{x_n\}_I$.

Now consider an ultrafilter $\omega \in \partial\beta X$ that picks $\mathbf{x}$. Then this ultrafilter ω is an element of D_{g^*g} as $\mathbf{x} \subset D_{g^*g}$. Now for any $A \in \omega$ and consider the intersection $A \cap \mathbf{x}$. This is fixed by the action of g, as it is a subset of $\mathbf{x}$. Hence we have: $\theta_g(A \cap \mathbf{x}) \in \theta_g(\omega)$ for every $A \in \omega$. As $\theta_g(\omega)$ is an ultrafilter $A \in \theta_g(\omega)$, so in particular $\omega \subseteq \theta_g(\omega)$, whence $\theta_g(\omega) = \omega$. □

Recall that the inverse monoid S_{inf} is represented geometrically by partial bijections on $I(X)$. This representation gives us access to the geometry of X, which we can utilise, in addition to Lemma 209, to understand the structure of S_{inf}.

LEMMA 210. *Consider the inverse monoid S_{inf} as a submonoid of $I(X)$. Then the following hold:*

(1) S_{inf} *has the property that* $g \neq e_G$ *and* $\theta_g \neq 0$ *implies* θ_g *is not an idempotent;*
(2) S_{inf} *is* 0*-E-unitary;*
(3) S_{inf} *has maximal element set* $\{\theta_g : g \in F_k\}$.

Proof.

(1) We prove that no non-zero θ_g are idempotent. To do this we pass to the induced action on βX. We observe that if θ_g is idempotent on X then it extends to an idempotent on βX, hence on the boundary $\partial\beta X$. θ_g is non-zero implies that there is a non-principal ultrafilter ω in the domain $\widehat{D}_{\theta_g}$. The result then follows from the observation that $\theta_g \circ \theta_g(\omega) = \theta_g(\omega)$ implies that θ_g must now fix the ultrafilter $\theta_g(\omega)$, which by Lemma 208 cannot happen.
(2) For 0-E-unitary is is enough to prove that $f \leq \theta_g$ implies $\theta_g \in E(S)$. Again, we extend the action to βX. We observe that if θ_g contains an idempotent, then we can build a sequence of elements of $x_i \in f \cap \widehat{D}_{\theta_g^*\theta_g} \cap X_i$ such that θ_g fixes the sequence, and hence fixes any ultrafilter ω that picks this sequence by Lemma 209. This is a contradiction, from where we deduce that the only situation for which $f \leq \theta_g$ is precisely when $g = e_G$ hence trivially $e \leq \theta_g$ implies $\theta_g \in E(S)$. For the general case, we remark that by the above statement coupled with the dual prehomomorphism property shows that $f \leq s$ implies $s \leq \theta_{e_G}$, hence is an idempotent.
(3) We construct the maximal elements. Observe that using the dual prehomomorphism it is clear that every non-zero word $s \in S$ lives below a non-zero θ_g. So it is enough to prove that for $\theta_g, \theta_h \neq 0$, $\theta_g \leq \theta_h \Rightarrow \theta_g = \theta_h$. Let $\theta_g \leq \theta_h$. This translates to $\theta_h\theta_g^*\theta_g = \theta_g$, hence for all $x \in \widehat{D}_{\theta_g^*\theta_g} : \theta_h(x) = \theta_g(x)$. Hence $\theta_g^*\theta_h \in E(S)$. From here we see that $\theta_g^*\theta_h \leq \theta_e$. From (2) we can deduce: $\theta_g^*\theta_h \leq \theta_{g^{-1}h}$ implies $\theta_{g^{-1}h} \in E(S_{inf})$. By (1) this implies $\theta_{g^{-1}h} = \theta_e$, and this happens if and only if $g^{-1}h = e$, i.e $g = h$. □

Appealing to the machinery we developed earlier in Propositions 196 and 197 we get the following corollary immediately.

Corollary 211. *The inverse monoid* S_{inf} *is strongly 0-F-inverse.*

We now have enough tools to prove the general version of Theorems 183 and 188.

Theorem 212. *Let* $\{X_i\}$ *be a sequence with large girth and vertex degree uniformly bounded above by* $2k$. *Let* X *be the corresponding space of graphs. Then the boundary Baum-Connes conjecture holds for* X.

PROOF. Using Proposition 195 we know the form of the boundary groupoid again in this case: $G(X)|_{\partial\beta X} \cong \partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$. Using results of Tu (namely Theorem 3.13 from [**Tu12**]): we know that for any $G(X)|_{\partial\beta X}$-C^*-algebra A the Baum-Connes conjecture for $G(X)|_{\partial\beta X}$ with coefficients A holds if and only if the conjecture for $\mathcal{G}_{\widehat{X}}$ with coefficients in A holds. By choosing $A = \frac{\ell^\infty(X,\mathcal{K})}{C_0(X,\mathcal{K})}$ and remarking that $\mathcal{G}_{\widehat{X}}$ has the Haagerup property by by Corollary 211 and Corollary 199 it follows that the boundary coarse Baum-Connes conjecture holds for X. $\square$

CHAPTER 5

Applications and Connections.

The focus of this chapter is developing ideas that appeared in Chapters 3 and 4 further as well as connecting these ideas together. Firstly, we consider some examples that arise from the short exact sequence of Chapter 3. Secondly, we outline a construction of a counterexample to the boundary conjecture; this space and its construction were first introduced in [**Wan07**] and its properties are developed further in this Chapter. Lastly, we connect the ideas of Chapter 3 and 4 together by outlining how the concept of a partial translation structure and associated groupoid can be used to describe why a Gromov monster group, a group that coarsely contains an expander, fails to be C^*-exact.

1. K-theory examples.

In this section we construct some examples, some well known in the literature, of inverse monoids associated to subspaces of groups. We then consider applications of the results outlined in the previous sections of the paper combined with a result of Norling [**Nor12**] concerning the K-theory of $C^*_r(S)$, when S is strongly 0-F-inverse. We first begin with a seemingly disconnected topological notion:

Definition 213. Let X be a second countable totally disconnected space. A set $\mathcal{V}$ is said to be a *regular basis* for the topology of X if:

(1) $\mathcal{V} \cup \{\emptyset\}$ is closed under finite intersections;
(2) $\mathcal{V}$ generates the compact open sets of X under finite unions, finite intersections and compliments.;
(3) $\mathcal{V}$ is independent, that is for every finite family $X, X_1, ..., X_n \in \mathcal{V}$ such that $X = \cup_{i=1}^n X_i$ there exists an $i \in \{1, .., n\}$ such that $X = X_i$.

Such a basis is countable by Lemma 2.10 of [**JC12**]. We index the basis by a countable set I. If this basis is also G-invariant, we get a natural action on the indexing set I, given by the representation of each $g \in G$ as the unique element of the symmetric group on I that induces the map: $g(V_i) = V_{gi}$ for each i.

In [**JC12**] Cuntz, Echterhoff and Li compute the K-theory for transformation groupoid C^*-algebras associated to actions of discrete groups G on totally disconnected spaces Ω that carry a regular G-invariant basis. We state a weaker version of their main result below:

Theorem 214. *Let G be a discrete group. Suppose that G acts on a second countable totally disconnected space Ω with G-invariant regular basis $\mathcal{V}$ and let I be the countable discrete index set for the basis $\mathcal{V}$. Then there is an isomorphism:* $K_*(C_0(I) \rtimes_r G) \cong K_*(C_0(\omega) \rtimes_r G)$.

Following Remark 3.13 from [**JC12**] as I is discrete computing the left hand side of the above isomorphism reduces to:

$$K_*(C_0(I) \rtimes_r G) \cong \bigoplus_{[i]\in G\backslash I} K_*(C_r^*(G_i))$$

And so in this situation we can recover the K-theory of the transformation groupoid $\Omega \rtimes G$ using only the K-theory of certain subgroups of G.

Furthermore, Norling [**Nor12**] extended these ideas to the class of strongly 0-F-inverse monoids S using the Morita envelope of the action of S on $\widehat{E}$ that was outlined in Chapter 2. First, he proved that the basis $\{\widehat{D}_e | e \in E\}$ associated to a strongly 0-F-inverse monoid S is a regular basis of $\widehat{E}$ and that this extends to a basis of the Morita envelope $Y_{\widehat{E}}$ that is still regular but also invariant under the universal group action. The final part of his paper deals with how to compute the stabiliser groups that appear within the Cuntz, Echterhoff and Li result outlined above. To do this requires an understanding of the following equivalence relation:

Definition 215. Let $e, f \in E(S) \setminus \{0\}$. Then $e \approx f$ if $\exists s \in S$ such that $e \leq s^*s$ and $ses^* = f$.

Whilst this looks asymmetric, the conditions placed on e in the above definition ensure that $e = s^*fs$; which proves the relation is symmetric. The proof of transitivity relies on a similar observation.

Remarks before Theorem 3.5 from [**Nor12**] outlines that the G_e whilst the stabiliser group for the element $[1, e]$ in $Y_{\widehat{E}}$ it is possible to calculate it using only the strongly 0-F-inverse monoid S. We denote by Φ the 0-restricted prehomomorphism from S to G^0.

$$G_e = \{\Phi(s) | s \in Max(S), e \leq s^*s, ses^* = e\}.$$

It is possible to utilise this relation combined with the fact that the idempotent set $E(S)$ indexes a G-invariant basis for $Y_{\widehat{E}}$ and Theorem 214 to prove the main result of [**Nor12**]:

THEOREM 216. *Let* S *be a strongly 0-F-inverse monoid with universal group* G*, where* G *has the Haagerup property. Then there is an isomorphism:*

$$K_*(C_r^*(S)) \cong \bigoplus_{[e] \in \frac{E^\times}{\approx}} K_*(C_r^*(G_e))$$

Where G_e *is the stabiliser of the action of* G *on the globalisation* $Y_{\widehat{E}}$ *the class* $[1, e]$.

This Theorem gives a method for computing the K-theory for certain reduced semigroup C^*-algebras, in particular those constructed from partial translation structures arising from groups. We consider some general natural inverse monoids and compute both the K-theory groups and the associated long exact sequence that arises from the corresponding Pimnser-Voiculescu type sequence.

1.1. The Examples. We consider the examples outlined in the introduction as well as other inverse monoids that we have introduced throughout the document.

EXAMPLE 217. (Toeplitz extension) We begin by considering the first C^*-algebraic construction outlined in the introduction. Let $X = \mathbb{N} \subset \mathbb{Z} = G$. We arrive at a partial translation structure for X by considering the maps:

$$t_n : \mathbb{N} \to \mathbb{N} \setminus \{0, 1, ..., n-1\}, x \mapsto x + n$$
$$t_{-n} : \mathbb{N} \setminus \{0, 1, ..., n-1\} \to \mathbb{N}, x \mapsto x - n$$

These partial bijections generate an inverse monoid, given by the presentation:

$$S = \langle t_n, t_n^* = t_{-n} | (\forall n \in \mathbb{N}) t_n^* t_n = 1, t_n t_1 = t_{n+1} \rangle = \langle t_1, t_{-1} | t_{-1} t_1 = 1 \rangle.$$

This is a well known example from semigroup theory called the *Bicyclic Monoid.* It is well known also that the translation algebra is the C^*-algebra generated by a unilateral shift; the Toeplitz algebra. Recall that to understand the translation algebra it is enough to study the groupoid $\mathcal{G}_{\widehat{X}}$, where $\widehat{X}$ is a quotient of βX and a subspace of $\widehat{E}$, as outlined in Proposition 102.

So we know that $\widehat{X}$ is the quotient of $\beta\mathbb{N}$ using the family of domains of the maps t_n as n runs though $\mathbb{Z}$, which are in particular cofinite. This leads to the following small but general fact:

CLAIM 218. *If* $E(S) \subseteq$ *Cofin*(X) *then* $\widehat{X} = (H \backslash X)^+$

PROOF. It is enough to remark that in general we have:

$$\widehat{X} = H \backslash X \cup \{\text{Filters arising from nonprincipal ultrafilters in } \beta X\}$$

If we assume $E(S) \subseteq$ Cofin(X) then *all* nonprincipal ultrafilters will agree in the quotient as they only fight over and subsequently differ on infinite subsets with infinite compliments and so are identified in the quotient. Additionally as $E(S)$ has no zero element, the characteristic function of $E(S)$ is a well defined character, which will certainly be maximal. This provides us a complete description of the point at infinity. □

In this example the left stabilizer H is trivial, so $\widehat{X} = \mathbb{N}^+ = \widehat{E}$. So we have that the quotient map $C_r^*(\mathcal{G}_{\widehat{E}}) \to C_r^*(\mathcal{G}_{\widehat{X}})$ is the identity and the semigroup algebra is also the Toeplitz algebra.

We remark that Theorem 216 of Norling now gives a direct computation of the K-groups in this instance, as each idempotent is of the form $t_n t_n^*$ for $n \in \mathbb{N}$ and these are related to 1 via the translation t_{-n}. So it is enough to understand the stabiliser group G_1, which in this instance is trivial. Hence we get that the K-theory groups are those of a point, which is well-known although computed in a different manner [**PV80, BNW09**].

Example 219. (Birget-Rhodes expansion of a group) In this instance there is much more interesting K-theory arising from Theorem 216. The Bridget-Rhodes expansion of a group [**BR84, LMS06**] was first outlined in Section 3 as an example. We recall the construction again for clarity.

In the context of a group G we define a set $S(G)$ with the elements given by pairs: (X, g) for $\{1, g\} \subset X$, where X is a finite subset of G. The set of such (X, g) is then equipped with a product and inverse:

$$(X, g)(Y, h) = (X \cup gY, gh)\ ,\ (X, g)^{-1} = (g^{-1}X, g^{-1})$$

This turns $S(G)$ into a inverse monoid with maximal group homomorphic image G, satisfying a universal covering property for partial G-actions. The partial order on $S(G)$ can be described by reverse inclusion, induced from reverse inclusion on finite subsets of G. It is F-inverse, with maximal elements: $\{(\{1, g\}, g) : g \in G\}$ and idempotents given by $(F, 1)$ where F is a finite subset of G containing 1. We denote the set of finite subsets of G containing 1 by $\mathrm{Fin}_1(G)$.

Using Theorem 216, provided the group has the Haagerup property, we can again compute the K-theory, each finite subset F of G containing 1 admits the partial action by the group elements that arise within the finite subset. This action is given by:

$$g.(F, 1) = (\{1, g^{-1}\}, g^{-1})(F, 1)((\{1, g\}, g)) = (g^{-1}F, 1)$$

defined for every F that contains $\{1, g\}$.

To compute the stabiliser of a given idempotent it is enough to understand the action of a single element of g on $(F, 1)$. In particular $g.(F, 1) = (F, 1)$ if and only if $g^{-1}F = F$ or $F = gF$, from where it follows that g has finite order and the subgroup generated by g also fixes F. Thus, for every F

there exists $g_1, \ldots, g_n \in G$ and a decomposition of F using the left stabiliser H_F of F in G:

$$F = H_F \cup H_F g_1 \cup \ldots \cup H_F g_n$$

Denote the subset of finite subgroups by $FSG(G) \subset Fin(G)_1$.

The main outcome of this is the following calculation using Theorem 216:

$$K_*(C_r^*(S)) \cong \bigoplus_{H \in FSG(G)} \bigoplus_{\substack{[F] \in G \backslash Fin(G)_1 \\ H_F = H}} K_*(C_r^*(G_F))$$

In the light of Theorem 83 this suggests it should be possible, using the long exact sequence, to compute the K-theory of $C_r^*(G)$ from information about its finite subgroups.

If G is finite then it appears as an element in $FSG(G) \subset Fin(G)_1$ and so the sequence provided by Theorem 83 will split at the level of K-theory.

We remark that it is possible to arrive at this inverse monoid in a natural way via a subset of a group known as a universal deep set [**BNW13**]. This subset is universal for partial translation structures, and that it generates this inverse monoid is immediate. However to compute the K-theory of this translation algebra is more complicated than for the reduced C^*-algebra.

We now shift our considerations to the free group on two generators. In this setting, it is possible to get inverse monoids coming from translation structures that are richer in interesting behaviour. We outline some of their natural properties:

CLAIM 220. *Let $X \subset F_2$ be connected and let g and h be words in F_2 such that g does not end in $a_i^{\pm 1}$ and h does not start with $a_i^{\mp 1}$. Then the the translations $\{t_g | g \in F_2\}$ satisfy $t_g t_h = t_{gh}$.*

PROOF. These translations differ only by the fact that the product $t_g t_h$ may contain a relation that is unreduced, whereas t_{gh} acts by the reduced form of gh. As there are no relations other than $a_i a_i^{-1}$ or $a_i^{-1} a_i$ in F_2 whence $t_g t_h$ and t_{gh} agree everywhere they are defined. Lastly because X is connected it is not possible for t_{gh} to be defined unless t_g is defined; hence $t_g t_h = t_{gh}$. □

This property makes working with translation algebras arising from F_2 easier.

EXAMPLE 221. (A free group via the Pimsner-Voiculescu method) Let X be the subset of the free group $F_n = \langle a_1, \ldots, a_n \rangle$ consisting of all the words that do not start with an a_1^{-1}. This subset was considered in [**PV82**] and gave rise to a short exact sequence:

$$0 \to \mathcal{K}(\ell^2(X)) \to C^* \mathcal{T}_n \to C_r^* F_n \to 0.$$

This sequence is the translation algebra sequence that arises from Theorem 110. In addition to this sequence the authors of [**PV82**] gave a computation of the K-theory groups associated to $C^* \mathcal{T}_n$

as those for $C^*_r F_{n-1}$, giving an inductive method for computing the K-theory of a free group C^*-algebra. We give a new proof using the generalised short exact sequence from Theorem 83 and Theorem 216.

For this subspace the behaviour is exceptionally like the Toeplitz shift in Example 217, however it is decidedly more complex than in that instance.

Claim 222. *Let $w \in F_n$. Then $t^*_w t_w \approx 1$.*

Proof. We will prove this by induction on the length of w. This clearly holds for the case when the length of w is 1. So now, assume this holds for length equal to n and let w have length $n+1$. Consider $t^*_w t_w$; this will be a word of length $2n+2$ containing a relation of the form in the centre $t^*_{a_i^{\pm 1}} t_{a_i^{\pm 1}}$ for some i. If i is not 1, and this word is not $t_{a_1} t^*_{a_1}$ then we can reduce the idempotent in length to $2n$, represented by the word w_1 with left end removed. So, we can suppose that the centre of $t^*_w t_w$ is $t_{a_1} t^*_{a_1}$. Now, if the outer translation is a $t_{a_i^{\pm 1}}$ for i not 1, then it is a bijection, and we can apply this bijections inverse without disrupting the equivalence relation $\approx$. This again reduces the length of the word by 1. Finally, suppose that w starts and ends with a_1^{-1}. Then $t^*_w t_w$ is certainly less than $t_{a_1} t^*_{a_1}$, at which point we can conjugate by t_a, preserving the relation $\approx$ and reducing our idempotent in length by 2. $\square$

This however does not let us compute for a general e, which will be a product of conjugates of the $t^*_w t_w$.

It is now possible to put together the short exact sequences constructed in Theorem 83 and Theorem 110 into a single diagram:

$$\begin{array}{ccccccccc}
0 & \to & C^*_r(\mathcal{G}_{U\cap\widehat{X}}) & \xrightarrow{\cong} & \ker(q) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^*_r(\mathcal{G}_U) & \longrightarrow & C^*_r(\mathcal{G}_{\widehat{E}}) & \to & C^*_r(F_n) & \to & 0 \\
 & & \downarrow & & \downarrow q & & \downarrow \wr\| & & \\
0 & \to & \mathcal{K}(\ell^2(X)) & \to & C^*_r(\mathcal{G}_{\widehat{X}}) & \to & C^*_r(F_n) & \to & 0
\end{array}$$

Taking K-theory groups and using Theorem 216 we get the following diagram:

$$\begin{array}{ccccccccccc}
\longrightarrow & 0 & \longrightarrow & K_0(C^*_r(\mathcal{G}_{U\cap\widehat{X}^c})) & \xrightarrow{\cong} & K_0(\ker p) & \longrightarrow & 0 & \longrightarrow & K_1(C^*_r(\mathcal{G}_{U\cap\widehat{X}^c})) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \wr\| & \\
\to & K_1(C^*_r(F_n)) & \longrightarrow & K_0(C^*_r(\mathcal{G}_U)) & \longrightarrow & K_0(C^*_r(F_{n-1})) \oplus \bigoplus_{\substack{[e]_\approx \\ e\not\approx 1}} K_*(\mathbb{C}) & \to & K_0(C^*_r(F_n)) & \longrightarrow & K_1(C^*_r(\mathcal{G}_U)) & \longrightarrow \\
 & \downarrow \wr\| & & \downarrow & & \downarrow p & & \downarrow \wr\| & & \downarrow & \\
\to & K_1(C^*_r(F_n)) & \to & K_0(\mathcal{K}(\ell^2(X))) & \longrightarrow & K_0(C^*\mathcal{T}_n) & \longrightarrow & K_0(C^*_r(F_n)) & \longrightarrow & 0 & \longrightarrow
\end{array}$$

This indicates that whilst the translation algebra $C^*\mathcal{T}_n$ does not have the same K-theory as $C^*_r(S)$, the K-theory sequence is split (this relies on the results of Pimsner-Voiculescu [**PV82**]: $K_*(C^*\mathcal{T}_n)) \cong K_*(C^*_r(F_{n-1})))$, so we are picking out the correct K-theory group as well as much more complex information that comes from the product structure on the idempotents. This is connected to regularisation of the generating set for the basis of $\widehat{E}$ [**JC12**], and will be discussed later in the section.

EXAMPLE 223. (Polycyclic monoids, Strong orthogonal completions and the Cuntz extension) Let X be the set of all the positive words within the free group F_n. We consider the inverse monoid that arises from the induced translation structure. In this case the inverse monoid has a zero element and satisfies the relation: $t^*_{a_i}t_{a_j} = \delta_{ij}$. The inverse monoid satisfying this relation is called the *polycyclic monoid* [**Law07**], which is a generalisation of the Bicyclic monoid. We denote this inverse monoid by P_n.

In this case we again have that $E(P_n)$ is in bijection with X, which is a rooted n-ary tree. Hence $\widehat{E}(P_n) \cong \widehat{X}$. However, many of the domains of translation are infinite with infinite compliment and so we have nontrivial ultrafilters to consider. In general we should consider tight filters, as the closure of the ultrafilters, but by work of Lawson [**Law11**] P_n is compactible for each n, that is the ultrafilters are closed in the subspace topology on $\widehat{E}$. As a free group is exact, we can appeal to Theorem 86 to get a short exact sequence:

$$0 \to C^*_r(\mathcal{G}_U) \to C^*_r(\mathcal{G}_{\widehat{E}}) \to C^*_r(\mathcal{G}_{\widehat{E}_\infty}) \to 0.$$

Where $\widehat{E}_\infty$ is set of ultrafilters on E and U is the compliment of $\widehat{E}_\infty$.

We would like to understand the algebras that appear within this sequence. Let us begin with the first term; as the left stabiliser of this subspace is trivial we can deduce from Proposition 106 that $\mathcal{G}_U$ is a pair groupoid, hence $C^*_r(\mathcal{G}_U)$ is the compact operators on $\ell^2(X)$. The middle term satisfies the relation $\sum_{i=1}^n t_{a_i}t^*_{a_i} \leq 1$, whence $C^*_r(\mathcal{G}_{\widehat{E}}) \cong C^*_r(S)$ admits a map from E_n, the generalised Cuntz algebra. It is well known that this map is an isomorphism [**Pat99, Ren80**]. It now follows that $C^*_r(\mathcal{G}_{\widehat{E}_\infty})$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$.

We again appeal to Theorem 216 to compute the K-theory. By Proposition 105 we know that the action on $E(P_n)$ translates, via the bijection onto X, to the translation action of F_n on X. It follows that there is only a single orbit under this action as partial translation actions are transitive. The stabiliser is obviously trivial in this case, hence by Theorem 216 we arrive at the computation $K_*(C^*_r(P_n)) \cong K_*(\mathbb{C}) \cong \mathbb{Z}$. All that remains is to compute the maps in the sequence, which are also well known..

We give a direct computation of the K-theory of the final term here, by considering Lawsons *orthogonal completions* of P_n [**Law11**], denoted by $D(P_n)$ and $C(P_n)$ respectively. The main idea of the orthogonal completion $D(P_n$ is to construct a monoid from P_n that has new elements that represent finite sums of orthogonal elements that are compatible with the obvious geometric composition rules. To describe $C(P_n)$ requires a definition:

DEFINITION 224. Let E be a semilattice and let $e, f \in E$. e is *dense* in f if $e \leq f$ and there does not exist $z \in E$ such that $z \leq f$ and $ze = 0$.

The construction of $C(P_n)$ is similar to that of $D(P_n)$, except that certain relations involving dense elements are removed via a congruence [**Law11**].

We remark that any tight representation of a inverse monoid cannot separate dense idempotents [**Exe08**]. This is particularly relevant in this example. It is clear that in P_n the elements $\{t_{a_i}t_{a_i}^* | i = 1, .., n\}$ are pairwise orthogonal, and in the reduced C^*-algebra they have sum that is less than 1. The idea of the orthogonal completion is to capture this C^*-algebraic behaviour in an inverse monoid; in $D(P_n)$ the sum $\bigvee_{i=1}^{n} t_{a_i}t_{a_i}^*$ is defined and is dense in 1, and equal to 1 in $C(P_n)$, which has an underlying tight representation of S.

It follows, from von Neumann equivalence of projections, that each $t_{a_i}t_{a_i}^*$ viewed as an operator in $C_r^*(\mathcal{G}_{\widehat{E}_\infty})$ is equivalent to 1 at the level of K-theory, in particular using Proposition 5.3.1 and Lemma 5.3.2 from [**WO93**] we observe that $C_r^*(\mathcal{G}_{\widehat{E}_\infty})$ is stable and that $[1] = \sum_{i=1}^{n}[1]$. From our calculations above we know that the K-theory group $K_0(C_r^*(P_n))$ is generated by the class [1], hence we know that [1] generates $K_0(C_r^*(\mathcal{G}_{\widehat{E}_\infty})$ also. It follows that $\sum_{i=1}^{n-1}[1] = [0]$, and this gives a homomorphism from C_{n-1}. That this map induces an isomorphism is well known [**BNW09**].

1.2. What happens for the partial translation structure reduction in general? In Example 221 we observed that the inverse monoid generated by the translation structure had K-theory groups that were relatively easy to calculate but much too large. This phenomenon is not uncommon; the same computation using the subspaces present in the work of Lance [**Lan83**] also provide too rich a structure. This additional structure arises as the basis for topology on $\widehat{E}$ that we are using to apply results of Norling and Cuntz-Echerhoff-Li rely on the *regular* basis property. We also observe that the natural elements that contribute to the correct K-theory groups in these instances are precisely the idempotents $t_w^*t_w$ that arise from words $w \in F_n$, as opposed to their products. This essentially says that considering the generating set over the regular basis it generates appears to give the correct answer.

We also remark that the large diagram constructed in Example 221 can be constructed for *any* translation structure. That is we get the following diagram by connecting the short exact sequences from Theorem 83 and Theorem 110 into a single diagram:

$$\begin{array}{ccccccccc}
0 & \to & C_r^*(\mathcal{G}_{U\cap\widehat{X}}) & \xrightarrow{\cong} & \ker(q) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_r^*(\mathcal{G}_U) & \longrightarrow & C_r^*(\mathcal{G}_{\widehat{E}}) & \to & C_r^*(G) & \to & 0 \\
 & & \downarrow & & \downarrow q & & \downarrow \wr\Vert & & \\
0 & \to & \mathcal{K}(\ell^2(X)) & \to & C_r^*(\mathcal{G}_{\widehat{X}}) & \to & C_r^*(G) & \to & 0
\end{array}$$

To apply Theorem 216 however we restrict to discrete groups with the Haagerup property. In that instance, we have the following diagram on K-theory:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(C_r^*(\mathcal{G}_{U\cap\widehat{X}^c})) & \xrightarrow{\cong} & K_0(\ker p) & \longrightarrow & 0 & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K_1(C_r^*(G)) & \longrightarrow & K_0(C_r^*(\mathcal{G}_U)) & \longrightarrow & \bigoplus_{w\in G} K_0(C_r^*(G_{t_w^* t_w})) \oplus \bigoplus_I K_*(C_r^*(G_e)) & \to & K_0(C_r^*(G)) & \to & \\
\downarrow \wr\Vert & & \downarrow & & \downarrow p & & \downarrow \wr\Vert & & \\
K_1(C_r^*(G)) & \to & K_0(\mathcal{K}(\ell^2(X))) & \longrightarrow & K_0(C^*\mathcal{T}) & \longrightarrow & K_0(C_r^*(G)) & \to &
\end{array}$$

Where $I := \{e \in E(S) | e \not\approx t_w^* t_w \forall t_w \in \mathcal{T}\}$.

QUESTION 225. *Is the middle column split in both dimensions?*

A positive answer to that question would give us a positive answer to the following:

QUESTION 226. *Is* $K_*(C^*\mathcal{T}) \cong \bigoplus_{w\in G} K_0(C_r^*(G_{t_w^* t_w}))$*; Are the domains and ranges of the* t_w *enough to get a direct computation of the K-theory?*

2. A counterexample to the Boundary Conjecture.

In this section we develop the ideas of Higson, Lafforgue and Skandalis concerning the counterexamples to the coarse Baum-Connes conjecture further, to construct a space of graphs Y that has exceptional properties at infinity. The main idea is to decompose the boundary groupoid further, giving a new short exact sequence at infinity similar to the sequences considered in Chapter 4. From this, we then construct an operator that is not a ghost operator, but is ghostly on certain parts of the boundary. A tracelike argument, similar to those of [**Hig99, WY12a**] then allows us to conclude that the boundary coarse Baum-Connes conjecture fails to be surjective for the space Y.

2.1. The space and its non-ghosts. The space we are going to consider first appeared in [**Wan07**].

Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of finite graphs. Then we construct a space of graphs in the following manner: Let $Y_{i,j} = X_i$ for all $j \in \mathbb{N}$ and consider $Y := \sqcup_{i,j} Y_{i,j}$. We metrize this space using a box metric - that is with the property that $d(Y_{i,j}, Y_{k,l}) \to \infty$ as $i + j + k + l \to \infty$.

Now let $\{X_i\}_i$ be an expander sequence. As discussed in Section 5.2 of Chapter 4, we can construct a ghost operator $p = \prod_i p_i$ on X, the space of graphs of $\{X_i\}_i$. Similarly, we can construct this operator on Y. In this situation we get a projection $q := \prod_{i,j} p_i \in C_u^*Y$, which is a constant operator in the j direction. This was precisely the operator of interest in [**Wan07**], as it can be seen that q is not a ghost operator, as its matrix entries do not vanish in the j direction - a fact proved below in Lemma 227.

Recall that associated to Y we have a short exact sequence of C^*-algebras:

$$0 \longrightarrow \ker(\pi) \longrightarrow C_r^*(G(Y)) \xrightarrow{\pi} C_r^*(G(Y)|_{\partial\beta Y}) \longrightarrow 0.$$

We remark the kernel, $\ker(\pi)$ consists of all the ghost operators in $C_u^*(Y)$, that is those operators with matrix coefficients that tend to 0 in all directions on the boundary.

LEMMA 227. *The projection* $\pi(q) := \pi(\prod_{i,j} p_i) \neq 0 \in C_r^*(G(Y)|_{\partial\beta Y})$. *That is* $q \notin \ker(\pi)$.

PROOF. We first observe that every bounded subset B of Y is contained in some rectangle of the form $R_{i_B,j_B} := \sqcup_{i\leq i_B, j\leq j_B} Y_{i,j}$. So to prove that q is not a ghost operator it suffices to show that there exists an $\epsilon > 0$ such that for all rectangles $R_{i,j}$ there is a pair of points x, y in the compliment of the rectangle such that the norm $\|q_{x,y}\| \geq \epsilon$. To prove this, recall that the projection q is a product of projections p_i on each X_i and fixing j, these projections form a ghost operator.

Fix $\epsilon = \frac{1}{2}$. Then there exists an i_ϵ with the property that $\forall i > i_\epsilon$ and for every $x, y \in \sqcup_i X_i$ we know $\|p_{i,x,y}\| < \epsilon$. We remark that this i_ϵ can be taken to be the smallest such. So for $i \leq i_\epsilon - 1$, we have that $\|p_{i,x,y}\| \geq \frac{1}{2}$. Now let $R_{i_\epsilon-1,\infty}$ be the vertical rectangle $\sqcup_{i\leq i_\epsilon-1,j} Y_{i,j}$.

To finish the proof, consider an arbitrary finite rectangle $R_{i,j}$. This intersects the infinite rectangle $R_{i_\epsilon-1,\infty}$ in a bounded piece. Now pick any pair of points in a fixed box $x, y \in Y_{k,l} \subset R_{i_\epsilon-1,\infty} \setminus R_{i,j}$. Then for those points x, y it is clear that $\|q_{x,y}\| = \|p_{k,x,y}\| \geq \frac{1}{2}$. □

We now describe the boundary $\partial\beta Y$. We are aiming at a decomposition into saturated pieces and with that in mind we construct a map to βX.

Consider the map $\beta Y \twoheadrightarrow \beta X \times \beta\mathbb{N}$ induced by the bijection of Y with $X \times \mathbb{N}$ and the universal property of βY. Now define:

$$f : \beta Y \rightarrow \beta X \times \beta\mathbb{N} \rightarrow \beta X$$

The map f is continuous, hence the preimage of X under projection onto the first factor is an open subset of βY, which intersects the boundary $\partial\beta Y$. In fact, what we can see is that $f^{-1}(X) = \sqcup f^{-1}(X_i)$, where each $f^{-1}(X_i)$ is closed, and therefore homeomorphic to $X_i \times \beta\mathbb{N}$. We can define $U = f^{-1}(X) \cap \partial\beta Y$.

2.2. The boundary groupoid associated to the box space of a discrete group with the Haagerup property. Let Γ be a finitely generated residually finite a-T-menable discrete group, and let $\{N_i\}$ be a family of nested finite index subgroups with trivial intersection. Let $X_i := Cay(\frac{\Gamma}{N_i})$. From now on we will consider the space Y to be defined using these X_i.

LEMMA 228. $U = f^{-1}(X) \cap \partial\beta Y$ *is an open, saturated subset of the boundary* $\partial\beta Y$.

PROOF. We have already shown above that U is open. To see it is saturated we prove that U^c is saturated. Let g_Y and g_X be the continuous extensions of the map obtained by acting through g on Y and X respectively. We observe that the following diagram commutes:

$$\begin{array}{ccc} \overline{g}_Y : \beta Y & \longrightarrow & \beta Y \\ \downarrow p & & \downarrow p \\ \overline{g_X \times 1} : \beta X \times \beta\mathbb{N} & \longrightarrow & \beta X \times \beta\mathbb{N} \end{array}$$

The projection onto βX is also equivariant under this action. Assume for a contradiction that U^c is not saturated; there exists γ in U^c such that $\overline{g}_Y(\gamma) \in U$. It follows that $\overline{g_X \times 1}(p(\gamma))$ is in $p(U)$, whilst $p(\gamma) \in p(U^c)$, hence $\overline{g_X}(f(\gamma)) \in U$ whilst $f(\gamma) \in U^c$. This is a contradiction as $f(U^c) = \partial\beta X$ is saturated. □

This gives us two natural complimentary restrictions of $G(Y)|_{\partial\beta Y}$ and a short exact sequence of function algebras as in Chapter 4:

$$0 \longrightarrow C_c(G(Y)|_U) \longrightarrow C_c(G(Y)|_{\partial\beta Y}) \longrightarrow C_c(G(Y)|_{U^c}) \longrightarrow 0.$$

We will now show that the corresponding sequence:

$$0 \longrightarrow C_r^*(G(Y)|_U) \longrightarrow C_r^*(G(Y)|_{\partial\beta Y}) \xrightarrow{h} C_r^*(G(Y)|_{U^c}) \longrightarrow 0$$

fails to be exact in the middle. We proceed as in [**WY12a, HLS02**] by using the element $\pi(q)$, which certainly vanishes under the quotient map from $C_r^*(G(Y)|_{\partial\beta Y}) \to C_r^*(G(Y)|_{U^c})$. To show the failure we will show this sequence fails to be exact in the middle at the level of K-theory and for this we will require a firm understanding of the structure of $G(Y)|_U$.

We observe the following facts:

(1) Γ acts on the space $Y := \sqcup_{i,j} Y_{i,j}$ built from $\{X_i\}$.
(2) This action becomes free on piece of the boundary that arises as $i \to \infty$, that is Γ acts freely on U^c.

(3) The group action generates the metric coarse structure on the boundary; the finite sets associated to each $R > 0$ in the decomposition are now finite rectangles. This follows from considerations of the metric on Y.

It follows from the proof of Proposition 167 that the groupoid $G(Y)|_{U^c}$ is isomorphic to $U^c \rtimes \Gamma$ and under the assumption that Γ is a-T-menable we can conclude that the Baum-Connes assembly map for the groupoid $G(Y)|_{U^c}$ is an isomorphism (with any coefficients). We now concern ourselves with $G(Y)|_U$.

LEMMA 229. *The groupoid* $G(Y)|_U$ *is isomorphic to* $\sqcup_i (X_i \times X_i) \times G(\mathbb{N})|_{\partial\beta\mathbb{N}}$, *where* $\mathbb{N}$ *is given the coarse disjoint union metric.*

PROOF. We consider the preimages $f^{-1}(X_i)$. These are clearly invariant subsets of βY that when intersected with the boundary $\partial\beta Y$ are contained within U. The restriction of $(G(Y)|_{\partial\beta Y})|_{f^{-1}(X_i)}$ for each i isomorphic to the closed subgroupoid $G(X_i \times \mathbb{N})|_{\partial\beta(X_i\times\mathbb{N})}$ of $G(Y)|_U$. These groupoids are disjoint by construction and therefore the inclusion $\sqcup_i G(X_i \times \mathbb{N})|_{\partial\beta(X_i\times\mathbb{N})}$ is an open subgroupoid of $G(Y)|_U$. We now prove that:

(1) each $G(X_i \times \mathbb{N})|_{\partial\beta(X_i\times\mathbb{N})}$ is isomorphic to $(X_i \times X_i) \times G(\mathbb{N})|_{\partial\beta\mathbb{N}}$, where $\mathbb{N}$ has the well-spaced metric;
(2) the union $\sqcup_i G(X_i \times \mathbb{N})|_{\partial\beta\mathbb{N}}$ is the entire of $G(Y)|_U$.

To prove (1), observe that the groupoid decomposes as $G(X_i \times \mathbb{N}) = \bigcup_{R>0} \overline{\Delta_R(X_i \times \mathbb{N})}$. For each $R > 0$ we can find a j_R such that $\Delta_R(X_i \times \mathbb{N}) = F_R \cup \bigcup_{j>j_R} \Delta_R(X_i \times \{j\})$, hence for the boundary part of this groupoid it is enough to understand what happens in each piece $Y_{i,j}$, which is constant for each j. Secondly, observe that in the induced metric on a column, the pieces $Y_{i,j}$ separate as $j \to \infty$. This, coupled with the fact that for large enough R, we know that $\Delta_R^j(X_i \times \mathbb{N}) = X_i \times X_i$ allow us to deduce that any behaviour at infinity of this groupoid is a product of $X_i \times X_i$ and the boundary groupoid $G(\mathbb{N})|_{\partial\beta\mathbb{N}}$ where $\mathbb{N}$ has the coarsely disconnected metric. This groupoid is isomorphic to $\partial\beta\mathbb{N}$, from which we can deduce that $G(X_i \times \mathbb{N})|_{\partial\beta(X_i\times\mathbb{N})} = (X_i \times X_i) \times \partial\beta\mathbb{N}$ for each i.

To prove (2) we assume for a contradiction that there is a partial translation t, such that $\bar{t}$ is not an element of the disjoint union. Such an element maps some (x_i, ω) to (x_k, ω), where $i \neq k$. Without loss of generality assume also t has translation length at most R. Then the domain and range of t are both infinite (as the closure is defined in $G(Y)|_U$), and must be contained within a strip of width at most $R > 0$. From the definition of the metric, there are only finitely many $Y_{i,j}$ within such a rectangle, hence $t \in F_R$ and hence $\bar{t}$ is not defined in $G(Y)|_U$, which yields a contradiction. ☐

REMARK 230. Lemma 229 allows us to conclude that $C^*_r(G(Y)|_U) \cong \bigoplus_i M_{|X_i|} \otimes C(\partial\beta\mathbb{N})$

To conclude that $[\pi(q)]$ is not an element of $K_0(C^*_r(G(Y)|_U))$ we construct a trace-like map.

Theorem 231. *The element* $\pi(q)$ *maps to* 0 *in* $C_r^*(G(Y)|_{U^c})$*, but does not belong to the image of* $K_0(C_r^*(G(Y)|_U))$ *in* $K_0(C_r^*(G(Y)|_{\partial\beta Y})$.

Proof. The first part follows from the definition of $\pi(q)$; the quotient map h kills all operators that are ghostly in the direction $i \to \infty$ and $\pi(q)$ is such an operator.

To prove the second component we remark that each $U_i := f^{-1}(X_i) \cap U$ is a closed saturated subset of $\partial\beta Y$, hence we can consider the reduction to U_i for each i. We consider the product, and the following map:

$$\begin{aligned} \phi : C_r^*(G(Y)|_{\partial\beta Y}) &\to \prod_i C_r^*(G(Y)|_{U_i}) = \prod_i C_r^*(G(X_i \times \mathbb{N})_{\partial\beta(X_i\times\mathbb{N})}) \\ T &\mapsto \prod_i T|_{U_i} \end{aligned}$$

Under the map ϕ, the ideal $C_r^*(G(Y)|_U) = \bigoplus_i M_{|X_i|} \otimes C(\partial\beta\mathbb{N})$ maps to the ideal $\bigoplus_i C_r^*(G(X_i \times \mathbb{N})_{\partial\beta(X_i\times\mathbb{N})})$. So, we can define a tracelike map, in analogy to [**WY12a**, Section 6], by composing with the quotient map τ onto $\frac{\prod_i C_r^*(G(X_i\times\mathbb{N})_{\partial\beta(X_i\times\mathbb{N})})}{\bigoplus_i C_r^*(G(X_i\times\mathbb{N})_{\partial\beta(X_i\times\mathbb{N})})}$. This gives us a map at the level of K-theory:

$$Tr_* = \phi \circ \tau : K_0(C_r^*(G(Y)|_{\partial\beta Y})) \to \frac{\prod_i K_0(C_r^*(G(X_i \times \mathbb{N})_{\partial\beta(X_i\times\mathbb{N})}))}{\bigoplus_i K_0(C_r^*(G(X_i \times \mathbb{N})_{\partial\beta(X_i\times\mathbb{N})}))} = \frac{\prod_i K_0(C(\partial\beta\mathbb{N}))}{\bigoplus_i K_0(C(\partial\beta\mathbb{N}))}$$

By construction, $K_0(C_r^*(G(Y)|_U))$ vanishes under Tr_*. We now consider $[\pi(q)]$ under Tr_*. Recall that $q = \prod_{i,j} p_i$. We define $q_i = \prod_j p_i$ and observe that the operation of reducing to $G(Y)|_{U_i}$ can be performed in two commuting ways: restricting to U then $f^{-1}(X_i)$ or by restricting to $f^{-1}(X_i)$ then U. The second tells us that $q_i = p_i \otimes 1_{\beta\mathbb{N}}$ is constant in the j direction and when restricted to the boundary is $\pi(q_i) = p_i \otimes 1_{\partial\beta\mathbb{N}}$. Hence, $Tr_*([\pi(q)]) = [1_{\partial\beta\mathbb{N}}, 1_{\partial\beta\mathbb{N}}, ...] \neq 0$ and so $[\pi(q)] \notin K_0(C_r^*(G(Y)|_U)$. ∎

So in this case we have the following diagram:

$$\begin{array}{ccccccccc} K_1(C(U^c) \rtimes \Gamma) & \to & K_0(\ker(h)) & \to & K_0(C_r^*(G(Y)|_{\partial\beta Y})) & \to & K_0(C(U^c) \rtimes \Gamma) & \to & K_1(\ker(h)) \\ \wr\| \uparrow & & \uparrow & \nearrow & \mu_{bdry} \uparrow & & \wr\| \uparrow & & \uparrow \\ K_1^{top}(U^c \rtimes \Gamma) & \to & K_0^{top}(G(Y)|_U) & \to & K_0^{top}(G(Y)|_{\partial\beta Y}) & \to & K_0^{top}(U^c \rtimes \Gamma) & \to & K_1^{top}(X \times X) \end{array}$$

Remark 232. We justify the diagonal inclusion of $K_0^{top}(G(Y)|_U)$ into $K_0(C_r^*(G(Y)|_{\partial\beta Y})$. This follows as the groupoid $G(Y)|_U$ is amenable, and hence the assembly map is an isomorphism. The algebra $C_r^*(G(Y)_U) = \bigoplus_i M_{|X_i|} \otimes C(\partial\beta\mathbb{N})$ injects into the product $\prod_i M_{|X_i|} \otimes C(\partial\beta\mathbb{N})$ at the level of K-theory and this inclusion factors through in the inclusion into the kernel of h and into $C_r^*(G(Y)|_{\partial\beta Y})$. These maps provide enough information to conclude injectivity of the assembly map μ_{bdry}.

3. An application to the Exactness of Gromov Monster groups.

It is well known [**HLS02, WY12a**] that any group that contains a coarsely embedded large girth expander does not have Yu's property A and admits coefficients for which the Baum-Connes conjecture fails to be a surjection, but is an injection:

Theorem 233. *Let* G *be a Gromov monster group. Then there exists a (commutative)* G-C^**-algebra* A *such that the Baum-Connes assembly map:*

$$\mu_{r,A,*} : KK_*^G(\underline{E}G, A) \to K_*(A \rtimes_r G)$$

is not surjective, but is injective.

We explore this result from the point of view of the geometry that can be associated to the expander graph that it inherits from the group.

Recall the coarse Baum-Connes conjecture for any uniformly discrete bounded geometry space X can be phrased as a conjecture with coefficients in a certain groupoid $G(X)$ associated to X [**STY02**]. This groupoid admits a transformation groupoid decomposition [**STY02**, Lemma 3.3b)], giving an easy description of $G(X)$ when it is possible to get a handle on the generators of the metric coarse structure on X. When X is coarsely embedded into a group, this is certainly the case; the concept of a partial translation structure [**BNW07**] gives any space coarsely embedded into a group a nice collection of generators, as well as a locally compact, Hausdorff, second countable groupoid that implements the transformation decomposition.

On the other hand, the question of when a groupoid admits a transformation groupoid decomposition, up to Morita equivalence, has been well studied for the class of groupoids that are constructed from suitable inverse semigroups [**KS02, MS11**]. This is related to the problem of globalisation of a partial action of Γ on a space X. The result would be a space Y, with a true action of Γ such that $X \hookrightarrow Y$ is a topological embedding and the restriction of the Γ action to X induces the original partial action.

The problem of globalisation of partial actions of groups was discussed in Section 3 of Chapter 2. We recall following definition from Chapter 2:

Definition 234. Let X be a topological space and let G be a group acting partially on X. Then we denote by Ω the *Morita evelope* of the action of G on X, which is constructed as follows:

Consider the space $X \times G$, equipped with the product topology. Then define $\sim$ on $X \times G$ by $(x, g) \sim (y, h)$ if and only if $x(h^{-1}g) = y$. We define Ω as the quotient of $X \times G$ by $\sim$ with the quotient topology.

G acts on Ω using right multiplication by inverses on the group factor of the equivalence classes. Clearly the map that sends $x \in X$ to $[1, x] \in \Omega$ is a topological injection. The main result of [**KL04**] is that this new topological space is minimal amongst globalisations of X.

3.1. Some remarks about the coarse groupoid. From earlier work in Chapter 3 any space that coarsely embeds into a group admits a group-like partial translation structure. We equip the expander sequence coarsely embedded in our Gromov monster group with this translation structure. The results of Chapter 3 then tell us that there is a groupoid $\mathcal{G}(\mathcal{T})$, such that the translation algebra is isomorphic to the reduced groupoid C^*-algebra $C^*_r(\mathcal{G}(\mathcal{T}))$.

Using Claim 28 the translations of $\mathcal{T}$ are the only elements we need to be understand when working with $\mathcal{G}(\mathcal{T})$. From the definition of $\mathcal{T}$ we can think of a translation structure as providing us an excellent generating set for the metric coarse structure on the space X; the groupoid $\mathcal{G}(\mathcal{T})$ acts freely on βX, and we can generate now the coarse groupoid using this data:

LEMMA 235. *The coarse groupoid* $G(X) \cong \beta X \rtimes \mathcal{G}(\mathcal{T})$.

PROOF. We observe that the set of $[t_g, \widehat{D}_{t_g^* t_g}]$ covers $G(X)$; hence the collection $\mathcal{T}$ forms an admissible pseudogroup in the terminology of [**STY02**]. The groupoid it generates is $\mathcal{G}(\mathcal{T})$. Then the result follows from Lemma 3.3b) [**STY02**]. ☐

Now, we prove the following interesting Lemma:

LEMMA 236. *Let* $X \subset \Gamma$ *be a metric space with a group-like partial translation structure* $\mathcal{T}$ *induced from* Γ. *Then the inverse submonoid* $S = \langle \mathcal{T} \rangle \subset I_b(X)$ *is strongly* 0*-F-inverse.*

PROOF. It was proved that the monoid was 0-F-inverse earlier in Lemma 97. It follows that it is strongly 0-F-inverse as the conditions of Proposition 196 are satisfied for S. ☐

COROLLARY 237. *Let* X *be a metric space and* G *be a group such that* X *is coarsely embedded into* G. *Then the translation groupoid* $\mathcal{G}(\mathcal{T})$ *admits a (T,C,F)-cocycle onto* G.

PROOF. This follows immediately from Lemma 236 and Corollary 57. ☐

3.2. Non-exactness of a Gromov Monster. We begin with a definition:

DEFINITION 238. A finitely generated discrete group Γ is a *Gromov monster group* if there exists a large girth expander X with vertex degree uniformly bounded above and a coarse embedding $f : X \hookrightarrow \Gamma$.

These groups were shown to exist by Gromov [**Gro03**], with a detailed proof by Arzhantseva, Delzant [**AD08**]. The construction is technical and we require no details beyond those presented in the definition.

The rest of this chapter is dedicated to proving the following theorem:

THEOREM. *Let Γ be a Gromov monster group. Then there are locally compact Hausdorff topological Γ-spaces Y_i, $i \in \{1,2,3\}$ and a short exact sequence:*

$$0 \to C_0(Y_1) \to C_0(Y_2) \to C_0(Y_3) \to 0$$

such that

$$0 \to C_0(Y_1) \rtimes_r G \to C_0(Y_2) \rtimes_r G \to C_0(Y_3) \rtimes_r G \to 0$$

fails to be exact in the middle.

We proceed first by analysing the situation from [**WY12a**, Section 8]. The primary idea is to globalise C^*X in C^*G. Fix a left invariant proper metric on G.

Let X_n be the n-neighbourhood of X in G. Then we can form the C^*-algebras $\ell^\infty(X_n) \subseteq \ell^\infty(G)$. Being commutative algebras in this case, we could consider the dual picture by taking spectra, getting $C(\beta X_n) \subset C(\beta G)$. It is clear that $X_n \subset X_{n+1}$, so the algebras $\ell^\infty(X_n) \subset \ell^\infty(X_{n+1})$. The remark here is that the inclusion of $X_n \subset G$ is not G-equivariant, but the system is G-equivariant; the action of G on X_n on the right by translations will send points in X_n into a most $X_{n+l(g)}$ for each $g \in G$. Hence, the limit of the $\ell^\infty(X_n)$ over n is a G-algebra, and so we can form the crossed product algebra $(\varinjlim \ell^\infty(X_n)) \rtimes_r G$. Lemma 8.4 from [**WY12a**] provides us the following isomorphisms:

LEMMA 239. *Let X_n as above. Then $(\varinjlim \ell^\infty(X_n)) \rtimes G \cong \varinjlim C^*_u(X_n)$ and $(\varinjlim \ell^\infty(X_n, \mathcal{K})) \rtimes G \cong \varinjlim C^*(X_n)$.* □

Let the coefficients $\varinjlim \ell^\infty(X_n, \mathcal{K})$ be denoted by A. We appeal to the fact that each X_n is coarsely equivalent to X. As these limits are functorial in coarse maps, we conclude:

PROPOSITION 240. *Let G be a Gromov monster group and X the coarsely embedded large girth expander. Then we have $K_*(A \rtimes_r G) \cong K_*(C^*X)$.* □

We develop a semigroup theoretic approach; the procedure we will follow will be a geometric analogue of this argument using translation structures and Theorem 53 of Milan and Steinberg, which relies on the information about the coarse groupoid given above as well as the fact the the inverse semigroups associated to the coarse groupoid are strongly 0-F-inverse.

THEOREM 241. *Let* G *be a Gromov monster group. Then there are locally compact Hausdorff topological* G*-spaces* Y_i, $i \in \{1, 2, 3\}$ *and a short exact sequence:*

$$0 \to C_0(Y_1) \to C_0(Y_2) \to C_0(Y_3) \to 0$$

such that

$$0 \to C_0(Y_1) \rtimes_r G \to C_0(Y_2) \rtimes_r G \to C_0(Y_3) \rtimes_r G \to 0$$

fails to be exact in the middle.

PROOF. To construct the complete sequence we use Lemma 84 to get $Y_1 := (X \times G)/\sim$, $Y_2 := (\beta X \times G)/\sim$ and $Y_3 := (\partial\beta X \times G)/\sim$. We then get the short exact sequence of G-algebras:

$$0 \to C_0(Y_1) \to C_0(Y_2) \to C_0(Y_3) \to 0.$$

Now we consider the crossed product algebras $C_0(Y_i) \rtimes G$. Then the sequence above gives us six terms at the level of K-theory:

$$\begin{array}{ccccccc} \dots \to & K_0(C_0(Y_1) \rtimes G) & \to & K_0(C_0(Y_2) \rtimes G) & \to & K_0(C_0(Y_3) \rtimes G) & \to \dots \\ & \wr\| \uparrow & & \wr\| \uparrow & & \wr\| \uparrow & \\ \dots \to & K_0(\mathcal{K}) & \to & K_0(C_r^*(G(X))) & \to & K_0(C_r^*(G(X)|_{\partial\beta X})) & \to \dots \end{array}$$

And the bottom line is not exact on K-theory by the work of Chapter 4. It follows therefore that the sequence:

$$0 \to C_0(Y_1) \rtimes_r G \to C_0(Y_2) \rtimes_r G \to C_0(Y_3) \rtimes_r G \to 0$$

is not exact in the middle term. □

This idea can be extended to connect this proof of failure to be exact to the geometric one that is outlined above from [**Hig99, WY12a**].

We connect this geometric approach using groupoids to the analytic approach outlined in the previous section. Let B_n be the Morita enveloping action associated to the monster group G partially acting on X_n for each n.

PROPOSITION 242. *Let* $X = X_0$ *and* X_n *as above. Then the globalisations of* $G(X_n)$ *given by* $B_n \rtimes G$ *that come from the translation groupoid action of Lemma 235 are all Morita equivalent.*

PROOF. As X_0 is coarsely equivalent to X_n for all n, it follows that $G(X)$ is Morita equivalent to $G(X_n)$ for all n. Using Lemma 235 we can see that each of the groupoids $G(X_n)$ is isomorphic to a transformation groupoid $\beta(X_n) \rtimes \mathcal{G}_{\widehat{X}_n}$ and therefore admits a (T,C,F)-cocycle onto the monster group G. Using Theorem 55 (or Theorem 53) we can conclude that each $G(X_n)$ is also Morita equivalent to $B_n \rtimes G$. Subsequently $B_n \rtimes G$ are Morita equivalent for each n, induced by the natural inclusions that extend $B_n \to B_{n+1}$. □

Lemma 239 is naturally a corollary to Proposition 242.

3.3. Boundary Coefficients for a Gromov Monster. We extend the ideas in the previous section using the results from Chapter 4. In that chapter we proved that the boundary groupoid $G(X)|_{\partial\beta X}$ of a large girth sequence with uniformly bounded vertex degree decomposes as $\partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$, where $\mathcal{G}_{\widehat{X}}$ has the Haagerup property. We extend these ideas by considering the impact this has on a Gromov monster group that contains such a large girth expander. To this end we prove:

THEOREM 243. *There exists a locally compact Hausdorff space* Z *such that the groupoid* $Y_3 \rtimes G$ *is Morita equivalent to* $Z \rtimes F_k$.

This result relies on many aspects of Chapter 4.

LEMMA 244. *The boundary groupoid* $G(X)|_{\partial\beta X}$ *admits a (T,C,F)-cocycle onto* F_k.

PROOF. We remark that this follows directly from the fact that the coarse boundary groupoid $G(X)|_{\partial\beta X}$ has a decomposition as $\partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$, and that S_{inf} is strongly 0-F-inverse. □

PROOF. (Of Theorem 243). Recall from the proof of Theorem 241 that the groupoid $G(X)|_{\partial\beta X}$ is Morita equivalent to $Y_3 \rtimes G$. Using Lemma 244 we know also that $Z := (\partial\beta X \times F_k)/\sim$ is a locally compact Hausdorff space, arising from a (T,C,F)-cocycle onto F_k. This enables us to again appeal to either [**MS11**, Theorem 6.14] (Theorem 53) or [**KS02**, Theorem 1.8] (Theorem 57) to conclude that $G(X)|_{\partial\beta X}$ is Morita equivalent to $Z \rtimes F_k$. The Theorem then follows from transitivity of Morita equivalence. □

Theorem 243 has an important Corollary, as the Baum-Connes conjecture with coefficients is a Morita invariant:

COROLLARY 245. *Let* G *be a finitely generated group that coarsely contains a large girth expander. Then the Baum-Connes conjecture for* G *with coefficients in any* $(Y_3 \rtimes G)$*-*C^**-algebra is an isomorphism.* □

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