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Models for production and inventory systems for deteriorating items with a supply-chain perspective

by

Yousef Ghiami

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ABSTRACT

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MODELS FOR PRODUCTION AND INVENTORY SYSTEMS FOR DETERIORATING ITEMS WITH A SUPPLY-CHAIN PERSPECTIVE

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This thesis in the field of inventory management for deteriorating items studies the effects of deterioration on optimal policies in inventory and production-inventory models. Four different models are developed and analysed to address some of the gaps identified in this part of the inventory management literature. The first two models have been studied adopting the classic approach towards inventory modelling, where the holding cost is assumed to be proportional to a known exogenous unit holding cost parameter. Taking this approach, first a two-echelon (single-buyer, single-supplier) model is investigated in which the capacity is considered to be limited. In this model also the exact inventory level over time of the supplier is obtained whereas the literature to date has only considered the average inventory level. As the analysis shows, this results in a complex model, and therefore a heuristic is developed. In the second model, a single supplier, multiple buyers system is developed where there is a (in)finite production rate. It has been identified that the literature fails to calculate accurately the average inventory level of the supplier in situations where the production rate is finite. In this model this issue has been addressed, and further analysis reveals the significance of the more accurate modelling approach developed.

The literature evaluates inventory models in different ways in terms of objective function. A body of research is identified in the literature that assumes an equivalence between profit maximisation and cost minimisation, and it often seems logical to discard revenues and minimise the cost function. This equivalence, however, is not always easily established when the objective function is to maximise the Net Present Value (NPV) of the profit function. In the third model of this thesis, this equivalence is analysed in

detail, and it is shown that taking a cost minimisation model instead of a profit maximisation objective in some cases should be done cautiously and in the best case only adopted after establishing the relevant equivalence conditions using NPV Equivalence Analysis (NPVEA). Finally, in a fourth model a two-echelon supply chain with capacity constrains is developed using the NPV criterion. In this model a modification in the inventory level is suggested which makes the model more practical compared to the existing models in cases that the customers are serviced from the own warehouse.

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Declaration of Authorship

I, Yousef Ghiami , declare that the thesis entitled *Models for production and inventory systems for deteriorating items with a supply-chain perspective* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission

Signed: Yousef Ghiami

Date: January 2014

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Symbols/Abbreviation

Chapter 1

EOQ	**E**conomic **O**rder **Q**uantity	
SCM	**S**upply **C**hain **M**anagement	
NPV	**N**et **P**resent **V**alue	
OW	**O**wned **W**arehouse	
RW	**R**ented **W**arehouse	
t	The variable representing time	time unit
T	The inventory period at an inventory system	item
h	The inventory holding cost	£/(item,time unit)
$h(t)$	The inventory holding cost as a function of time	£/(item,time unit)
$E(H)$	The average inventory holding cost of a system	£/time unit
$I(t)$	The inventory level a system	item
$E(I)$	The average inventory level a system	item
D	The constant rate of demand	item/time unit

Chapter 2

GA	**G**enetic **A**lgorithm	
W	The capacity of the OW	unit
t_r	The time at which the inventory level at the RW reaches zero	time unit
t_o	The time at which the inventory	time unit

	level at the OW reaches zero	
t_s	The length of shortage period	time unit
T_R	The inventory period at the retailer	time unit
T_W	The inventory period at the wholesaler	time unit
k	The number of periods (T_R) covered during one T_W	
$I_r(t)$	The inventory level at the RW	item
$I_o(t)$	The inventory level at the OW	item
$I_W(t)$	The generic inventory level at the wholesaler	item
$I_W^i(t)$	The inventory level at the wholesaler during i^{th} interval	item
$D(t)$	The demand rate at the retailer	item/time unit
y	The constant part in the demand function	item/time unit
z	The coefficient of $I_o(t)$ in the demand function	1/time unit
Q_R	The order quantity of the retailer	item
Q_W	The order quantity of the wholesaler	item
θ_r	The deterioration rate at the RW	%
θ_o	The deterioration rate at the OW	%
θ	The deterioration rate at the wholesaler	%
β	The backlogging rate at the retailer	%
Q_D	The total number of deteriorated items at the wholesaler over T_W	item
s_R	The purchasing fixed cost at the retailer	£
s_W	The purchasing fixed cost at the wholesaler	£
p_R	The purchasing price for the retailer	£
p_W	The purchasing price for the wholesaler	£
f_r	The inventory holding cost at the RW	£/(item,time unit)
f_o	The inventory holding cost at the OW	£/(item,time unit)
f	The inventory holding cost at the wholesaler	£/(item,time unit)
b	The backlogging cost	£/(item,time unit)
π	The lost sale cost	£/item
PC_R	The purchasing cost at the retailer during T_R	£
PC_W	The purchasing cost at the wholesaler during T_W	£
HC_{RW}	The holding cost at the RW during T_R	£

HC_{OW}	The holding cost at the OW during T_R	£
HC_R	The holding cost at the retailer during T_R	£
HC_W^i	The holding cost at the wholesaler during i^{th} interval of T_W	£
HC_W	The holding cost at the wholesaler during T_W	£
DC_{RW}	The deterioration cost at the RW during T_R	£
DC_{OW}	The deterioration cost at the OW during T_R	£
DC_R	The deterioration cost at the retailer during T_R	£
DC_W	The deterioration cost at the wholesaler during T_W	£
BC_R	The backlogging cost at the retailer during T_R	£
LC_R	The lost sale cost at the retailer during T_R	£
TC_{SC}	The average cost of the supply chain	£/time unit
$MPOP$	Population size of the GA	
$MGEN$	Maximum number of generations of the GA	
$PREIN$	Probability of reinsertion in the GA	
$PMUT$	Probability of mutation in the GA	
$PCROS$	Probability of cross over in the GA	
X	Vector of decision variables	
CTC	Change in total cost	%

Chapter 3

t_j	The variable representing time, $j = 1, 2, 3$	time unit
T_1	The production cycle at the supplier which appears at the start of T	time unit
T_2	The time interval with no production at the supplier	time unit
T_3	The production cycle at the supplier which appears at the end of T	time unit
T	The inventory period at the supplier, $T = T_1 + T_2 + T_3$	time unit
N	Number of buyers	

n_i	Number of replenishment done by buyer i during T, $i = 1, 2, \dots, N$	
$I_{v1}(t_1)$	The inventory level at the supplier when t_1 is between 0 and T_1	item
$I_{v2}(t_2)$	The inventory level at the supplier when t_2 is between 0 and T_2	item
$I_{v3}(t_3)$	The inventory level at the supplier when t_3 is between 0 and T_3	item
$I_v(t)$	General inventory level function at the supplier	item
I_{mv}	Maximum inventory level at the supplier	item
$I_{bi}(t)$	The inventory level at buyer i when t is between 0 and T/n_i	item
$I_b(t)$	The sum of inventory level of all buyers	item
I_{mi}	Maximum inventory level at buyer i	item
$\widetilde{I}_v(t)$	The echelon stock of the supplier	item
$\widetilde{I}_{vk}(t)$	The echelon stock of the supplier during k^{th} interval $k = 1, 2, 3$	item
$\widetilde{I}_{bi}(t)$	The echelon stock of buyer i	item
$\widetilde{I}_b(t)$	The sum of echelon stock of all buyers	item
θ	The deterioration rate	%
d_i	The demand rate at buyer i, $i = 1, 2, \dots, N$	item/time unit
p	The production rate	item/time unit
$I_{bi}(t)$	The inventory level at buyer i	item
p_v	The unit production cost for the supplier	£/item
p_b	The unit price for the buyers	£/item
C_{sv}	The set-up cost of each production cycle for the supplier	£
C_{sb}	The set-up or ordering cost per order for the buyers	£
F_v	The percentage of holding cost per £ in stock for the supplier	%
F_b	The percentage of holding cost per £ in stock for the buyers	%
TC_v	The total average cost of the supplier	£

TC_b	The sum of total average cost functions of all buyers	£
TC	The total average cost of the supply chain	£

Chapter 4

AP	**A**nchor **P**oint
$NPVEA$	**N**et **P**resent **V**alue **E**quivalence **A**nalysis
CF	**C**ash **F**lows
i.i.d.	**I**ndependent and **I**dentically **D**istributed

p	The sales price per unit of item	£/item
y	The demand rate	item/time unit
β	The backlogging rate in out-of-stock situation	%
R	The production rate	item/time unit
T_1	The production period when stock position is strictly positive	time unit
T_2	The non-production period when stock position is strictly positive	time unit
T_3	The non-production time during out-of-stock period	time unit
T_4	The production time during out-of-stock period	time unit
T	The inventory cycle of the system	time unit
$I(t)$	The inventory level of the system	item
$B(t)$	The shortage level of the system	item
$\theta\gamma t^{\gamma-1}$	Deterioration rate	%
g	Deposit for the backlogged items	£
r	Reduction in price for the backlogged items	£
s	Set-up cost for production	£
c	Production cost per item	£/item
d	Disposal cost per item	£/item
f	Shelf space rent	£/(item,time unit)
π	Lost sale cost	£/item

b	Backlogging cost	£/(item,time unit)
α	Discounting rate	%
R_1	The NPV of revenues earned when the stock position is strictly positive during the first period	£
ASR_1	Annuity stream revenue of all R_1 values	£
ASR_2	Annuity stream revenue earned from g	£
ASR_3	Annuity stream revenue earned from backlogged demand	£
SC	Annuity stream cost of production set-ups	£
PC	Annuity stream cost of production	£
HC_1	The NPV of holding cost during T_1 in the first period	£
HC_2	The NPV of holding cost during T_2 in the first period	£
HC	Annuity stream cost of holding inventory	£
BC_1	The NPV of shortage cost during T_3 in the first period	£
BC_1	The NPV of shortage cost during T_4 in the first period	£
BC	Annuity stream cost of shortage	£
LC	Annuity stream cost of lost sale	£
DC	Annuity stream cost of deterioration	£
ASP	Annuity stream profit function	£
$\mathcal{A}$	Set of feasible scenarios for an activity	
X	A feasible scenario for an activity	
h	Unit holding cost	£/(item,time unit)
h_b	Unit backorder cost	£/(item,time unit)
h_l	Unit lost sale cost	£/item
h_d	Unit deterioration cost	£/item
γ_X	Error between the solutions derived from cost minimisation and profit maximisation	%

Chapter 5

W	The capacity of the OW	unit
$I_o(t)$	The inventory level at the OW	item
$I_r(t)$	The inventory level at the RW	item
y	The constant part in the demand function	item/time unit
z	The coefficient of $I_o(t)$ in the demand function	1/time unit
$D(t)$	The demand rate at the retailer	item/time unit
t_r	The time at which the inventory level at the RW reaches zero	time unit
t_o	The time at which the inventory level at the OW reaches zero	time unit
t_s	The length of shortage period	time unit
T_R	The inventory period at the retailer	time unit
T_W	The inventory period at the wholesaler	time unit
k	The number of periods (T_R) covered during one T_W	
β	The backlogging rate at the retailer	%
Q_R	The order quantity of the retailer	item
θ_r	The deterioration rate at the RW	%
θ_o	The deterioration rate at the OW	%
θ	The deterioration rate at the wholesaler	%
g	Deposit for the backlogged items	£/item
r	Reduction in price for the backlogged items	£/item
p	The sales price per unit of item	£/item
p_R	The purchasing price for the retailer	£/item
s_R	The purchasing fixed cost at the retailer	£
f_r	The inventory holding cost at the RW	£/(item,time unit)
f_o	The inventory holding cost at the OW	£/(item,time unit)
d_R	The deterioration cost at the retailer	£/item
b	The backlogging cost	£/(item,time unit)
π	The lost sale cost	£/item
p_W	The purchasing price for the wholesaler	£
s_W	The purchasing fixed cost at the wholesaler	£

f	The inventory holding cost at the wholesaler	£/(item,time unit)
d_W	The deterioration cost at the wholesaler	£/item
$I_W(t)$	The generic inventory level at the wholesaler	item
$I_W^i(t)$	The inventory level at the wholesaler during i^{th} interval of T_W	item
Q_W	The order quantity of the wholesaler	item
ASR_1	Equivalent annuity stream of revenues earned during $t = 0$ and $t = t_r$ of all inventory cycles over an infinite horizon	£
ASR_2	Equivalent annuity stream of revenues earned during $t = t_r$ and $t = t_o$ of all inventory cycles over an infinite horizon	£
ASR_3	Equivalent annuity stream of revenues earned from deposits during $t = t_o$ and $t = T_R$ of all inventory cycles over an infinite horizon	£
ASR_4	Equivalent annuity stream of revenues earned from backordered demand at $t = T_R$ of all inventory cycles over an infinite horizon	£
ASR_R	Annuity stream of revenues at the retailer	£
SC_R	Annuity stream of all set-up costs at the retailer	£
PC_R	Annuity stream of all purchasing costs at the retailer	£
HC_o	NPV of holding cost of the first inventory period at the OW	£
HC_r	NPV of holding cost of the first inventory period at the RW	£
HC_{OW}	Equivalent annuity stream of HC_o of all inventory cycles	£
HC_{RW}	Equivalent annuity stream of HC_r of all inventory cycles	£
HC_R	Annuity stream of all holding costs at the retailer	£
DC_{OW}	Equivalent annuity stream of all deterioration	£

	costs incurred at the OW	
DC_{RW}	Equivalent annuity stream of all deterioration costs incurred at the RW	£
DC_R	Annuity stream of all deterioration costs at the retailer	£
BC	NPV of shortage cost of the first period	£
BC_R	Annuity stream of all shortage costs at the retailer	£
LC_R	Annuity stream of all lost sale costs at the retailer	£
ASP_R	Annuity stream profit of the retailer	£
ASR_W	Annuity stream of revenues at the wholesaler	£
SC_W	Annuity stream of all set-up costs at the wholesaler	£
PC_W	Annuity stream of all purchasing costs at the wholesaler	£
HC_{W1}	NPV of holding cost of the first inventory period at the wholesaler	£
HC_W	Annuity stream of all holding costs at the wholesaler	£
DC_W	Annuity stream of all deterioration costs at the wholesaler	£
ASP_W	Annuity stream profit of the wholesaler	£
ASP_{SC}	Annuity stream profit of the supply chain	£

Chapter 1

Introduction

1.1 Overview

In the marketplace the goal for many businesses is to have their item/service available with high quality when the demand arises. Failing to do so can result in not only loss in profit that could have been gained by selling the item/service, but may also damage the business's reputation which indeed takes a long time to build up. On the other hand, being able to completely meet the demand for the item/service could impose a considerable level of capital tied up. With this regard, making reasonable trade-offs between service level and tied up capital in businesses have always been a challenge. These concerns and the relevant research works conducted by academics and professionals, have shaped the concepts such as *Inventory Management* and in a more holistic view, *Logistics*.

Increasing competition in the marketplace has made businesses seek any improvement opportunity to secure their market share and even leave the competitors behind. Over decades, competitiveness has resulted in the development of techniques and approaches applied to different processes such as logistics, finance and purchasing (see Lambert et al., 1998) within one organisation or firm. In these *single-echelon* systems, decisions almost at all levels, from strategic to operational, have been made with little concern about how they may affect the suppliers or the downstream business customers.

Later some leading companies took initiatives in areas such as logistics, procurement and information systems that in a way involved other businesses (Ayers, 2006). Successful results of these initiatives conceptualised *Business Partnerships* and *Collaboration* which resulted in the emergence of the *Supply Chain Management* (SCM) concept. In the literature the division between SCM and Logistics has been controversial. Lambert et al. (1998), state the definition for logistics adopted by the Council of Logistics Management as "...that part of the supply chain process that plans, implements, and controls the efficient, effective flow and storage of goods, services, and related information from the point-of-origin to the point-of-consumption in order to meet customers' requirements".

Cooper et al. (1997) and Lambert et al. (1998) introduce a conceptual framework for SCM, and state that the implementation of SCM necessitates in addition to logistics integration, the integration within business units and across their supply chain. This, however, is not an easy task due to differences in priorities and conflicting objectives. Despite the obstacles, some of the efforts towards collaboration and integration have been fruitful and resulted in the establishment of many *multi-echelon* systems in the marketplace.

Ever since Harris (1913) developed the foundation of inventory management by presenting the Economic Order Quantity (EOQ) model, researchers have strived to extend to different inventory systems by fine-tuning certain assumptions in order to get closer to real-world cases. In fact, the long history of research has shaped what inventory management and its many subareas are. To get an overview of key literature, see Clark (1972), Silver (1981), Belobaba (1987), Aksoy and Erenguc (1988), Porteus (1990), Lee and Nahmias (1993), Drexl and Kimms (1997), Chan et al. (2004), and Andersson et al. (2010).

The inventory management literature is largely based on the assumption that an item in stock suffers no loss in quality. This, however, cannot hold in practice for many products. A distinction can be made between an item with a constant quality but limited constant life-time (Nahmias, 1975) and an item for which the quality gradually depletes over time (Mak, 1982). Despite the minor variations, researchers define *deterioration/perishability* in a similar way. Raafat (1991) defines deterioration as a process (of e.g. decay, damage, spoilage, or evaporation) that results in loss of value in an item, hence a decrease in the quality.

In order to address this characteristic, researchers started to incorporate deterioration in modelling to get better interpretation of the relevant systems since the 1950s (Goyal and Giri, 2001). The initial application for such models was to analyse blood-bank systems. Later, other researchers strived to apply these models to different types of deteriorating items. Raafat (1991) argues that over these years this view on inventory theory has grown and found its place within other areas of research.

Goyal and Giri (2001) state that Ghare and Schrader (1963) are the first researchers to model a deteriorating item with *exponentially decaying* pattern. In fact, the item has an exponential life-time which due to the property of being memoryless results in a constant rate of deterioration of on-hand inventory.

Yet, the area of study on deterioration has still a long way to go to reach the sophistication gained by the main stream of the inventory management literature. For instance, just recently a number of researchers started developing models which consider a multi-echelon supply chain for a deteriorating item (see Bakker et al., 2012). This makes the inventory management literature of deteriorating items an interesting area of research.

The literature of inventory management for deteriorating items characterises some factors that researchers make assumptions on while building up models. However as this area is emerging, identifying clusters of studies in this field seems to be difficult. This has resulted in numerous research works with a diverse set of assumptions in the literature.

1.2 Key factors in deteriorating item's modelling

In order to give a clearer overview, here a categorisation of the literature is suggested. The main criteria based on which the literature is categorised are the structure of the supply chain and also the approach taken towards opportunity cost. This categorisation identifies four subsets of the literature and makes a framework for this thesis, see also Figure 1.1.

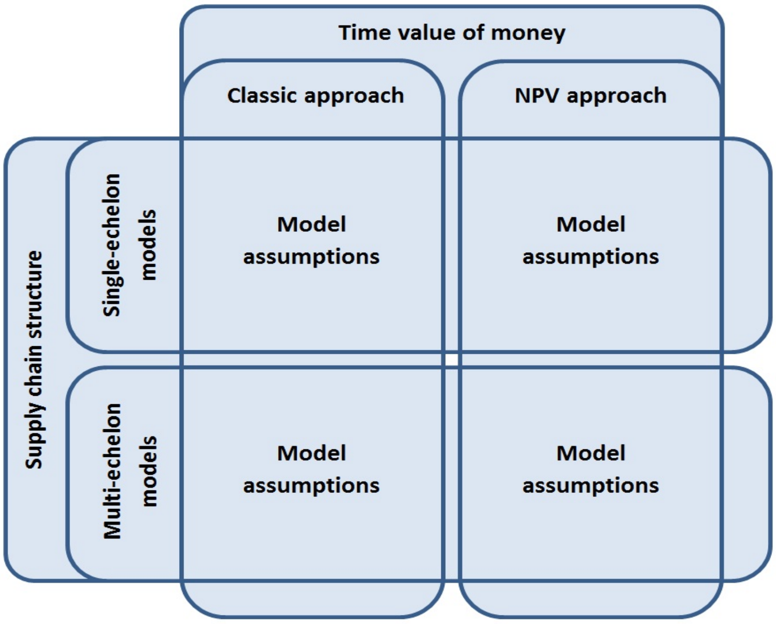

Figure 1.1: A categorisation in the deteriorating item literature

In each of these categories, researchers make assumptions on a group of factors to evaluate and analyse specific models. The key factors considered by the researchers are further described in subsequent sections.

1.2.1 Deterioration rate

The basic assumption in this literature is that the items lose value while kept in stock. The pattern of deterioration is the factor that shows the distribution of the item's life-time.

Deterministic life-time: In some research works it is assumed that the quality of the items stays unchanged for certain units of time (deterministic life-time) after which the item is no longer usable and should be disposed of (see e.g. Bhunia and Maiti, 1998a, Chern et al., 2008).

Stochastic life-time: The life-time for items is often assumed to be negative exponentially distributed, see e.g. Liao et al. (2000), Chung and Huang (2007), and Yang et al. (2010a). Ghare and Schrader (1963) show how this life-time distribution results in decaying of a constant percentage of the on-hand inventory. Another life-time distribution which has been widely used in this literature is Weibull pattern with two parameters. This distribution could be more generic as by assigning different values to the parameters, the rate of deterioration could be decreased or increased over the time, hence it is capable of representing a wide range of items' life-time distribution, see e.g. Wee and Law (2001) and Skouri et al. (2009).

In this thesis, the models developed in Chapters 2, 3, and 5 assume the life-time of the items to be based on a negative exponential distribution. In Chapter 4 a time-dependent deterioration rate is considered of which a special case results in a life-time with Weibull distribution.

1.2.2 Demand

The literature identifies two main streams of research regarding the demand pattern, deterministic or stochastic.

Deterministic demand: The literature of deteriorating items, identifies mainly four types of deterministic demand, namely; constant rate, time-dependent, price-dependent, and stock-dependent.

A large part of the literature considers a *constant rate* for demand, see e.g. Mak (1982), Abad (2000), and Ouyang et al. (2009). In some research works, it is assumed that the demand is *time-dependent*. This could be the case for the grocery retailing industry where the demand for some items varies in different weekdays, see e.g. Chang and Dye (1999) and Lee and Hsu (2009). The model analysed in Chapter 3 of this thesis is assumed to have a constant rate for demand.

In another group of studies, the demand factor is considered to be *ramp type* which is a combination of constant rate and time-dependent demand. This type of function divides the planning horizon into two or more intervals and considers different constant rates of demand in each interval (see e.g. Manna and Chaudhuri, 2006; Panda et al., 2008; Skouri et al., 2009; Agrawal et al., 2013). This demand pattern could be applicable to a new consumer good which is being introduced to the market. The demand for this new product is low at the beginning while it may increase in a stepwise pattern over time.

For the case of price-elastic items, where the demand is influenced by the price, a body of research suggests a *price-dependent* demand function, see e.g. Yang (2004b) and Dye et al. (2007b). This demand function is assumed for the model developed in Chapter 4.

A subset of the literature argues that the demand for some products can be influenced by the amount of the product presented on shelves. To capture this demand pattern, it is suggested to assume a *stock-dependent* demand function, see e.g. Mandal and Phaujdar (1989), Giri et al. (1996), and Hou (2006). This pattern is adopted for the demand in the models developed in Chapters 2 and 5.

Stochastic demand: The literature of deteriorating items with stochastic demand has not been addressed as extensively as the deterministic demand literature. There are a few research works in which the demand is stochastic, however, identifying clusters of research seems to be difficult. A few researchers assume Poisson distribution for the demand function (see e.g. Ravichandran, 1995; Liu and Yang, 1999; Olsson and Tydesjö, 2010). A group of researchers studies models that respect Markov properties, see e.g. Cohen (1976), Manuel et al. (2008), and Lian et al. (2009).

1.2.3 Lead-time

Two main streams of research are identified in the literature regarding lead-times, deterministic or stochastic. It is noted that occasionally a fuzzy lead-time is adopted to model an inventory system, see e.g. Rong et al. (2008).

Deterministic lead-time: In most of models with deterministic lead-time, this factor is assumed to be zero which means that any placed order is replenished instantly (see e.g. Bose et al., 1995; Chen, 1998; Papachristos and Skouri, 2003; Min and Zhou, 2009). A group of research works is conducted assuming a positive lead-time for orders to be replenished (see e.g. Giri et al., 1996; Yang, 2004b; Chung and Huang, 2007; Yang et al., 2010b), however, this assumption may have a limited influence on the results as these studies are conducted with a classic approach. This could no longer be true in an NPV framework, but few studies to date has addressed this case.

Stochastic lead-time: This assumption has drawn little attention in the literature, although in some cases it seems to be more realistic to consider a stochastic time factor between order placement and replenishment as there may be randomness in both production and transportation activities, see e.g. Ravichandran (1995), Liu and Yang (1999), and Manuel et al. (2008).

In all the models developed in this thesis, the lead-time is assumed to be negligible.

1.2.4 Shortages

Depending on the product characteristics, customer loyalty and brand image, businesses adopt different policies regarding shortages.

A body of research has addressed the case that shortages are *not allowed*, see e.g. Wee (1998), Wang and Chen (2001), Teng and Chang (2005), Chung and Huang (2007), Liao (2007), and Yang et al. (2010b).

For the case when shortages are allowed, the unmet demand could be either lost, partially backlogged or completely backlogged. A cluster of research works has been based on the assumption of *lost sale* such as Burnetas and Smith (2000), Chatwin (2000), Lu et al. (2008), and Broekmeulen and Donselaar (2009), while a large number of studies on deteriorating items have considered *complete backlogging* in case of shortages, see e.g. Cohen (1976), Mak (1982), Sarker et al. (1997), Chung and Lin (2001), and Olsson and Tydesjö (2010).

Wee (1993) argues that although the literature has mainly considered either complete backlogging or lost sale when shortages occur, in most cases in real-world problems the demand is *partially backlogged*. Chang and Dye (1999), Moon et al. (2005), Yang (2005), Law and Wee (2006), and Yang and Chang (2013) are of those examples who consider partial backlogging in their models. Apart from Chapter 3 in which shortages are not allowed, in all the other models studied in this thesis in case of shortages it is assumed that the demand is partially backlogged.

1.2.5 Warehouse capacity

Most of the studies in the area of inventory management for deteriorating products assume no limit for warehouse capacity, see e.g. Raafat et al. (1991), Mandal and Maiti (1999), Balkhi and Benkherouf (2004), Dye and Ouyang (2005), and Pal et al. (2006). However, disregarding capacity limits of a warehouse in some real cases may lead to infeasible solutions. Capacity constraint is assumed to model systems analysed in Chapters 3 and 4.

In order to model some practical situations, a group of research works is conducted based on the assumption of capacity constraint for a warehouse. These studies also assume that if the capacity of the *owned warehouse* (OW) is insufficient, the company has the option of using a *rented warehouse* (RW). This assumption has resulted in the emergence of a new category of studies which are known as *two-warehouse models*, see e.g. Yang (2006), Hsieh et al. (2008), Singh et al. (2009), Lee and Hsu (2009), and Agrawal et al. (2013). The inventory systems studied in Chapters 2 and 5 are assumed to have a limited warehouse capacity.

1.2.6 Review policy

Another key factor in modelling an inventory system is how the system is monitored. There are two main policies for reviewing the inventory level, namely continuous and periodic.

Continuous review: This policy is mainly used for high value products that according to their price, keeping them in stock is highly expensive. This policy may also be used for the cases that suppliers are highly flexible with timing and order quantity. There are a number of researchers who adopt the continuous review policy considering the characteristics of the system, see e.g. Mandal and Phaujdar (1989), Ravichandran (1995), Chen (1998), Liu and Yang (1999), Manuel et al. (2007), and Olsson and Tydesjö (2010).

Periodic review: In some real-world problems, companies may prefer to have fixed inventory/production cycles due to high cost of continuous review. The majority of researchers have developed models with a periodic review policy for deteriorating items, see e.g. Mak (1982), Sarker et al. (1997), Yang (2004a), Ouyang et al. (2005), Liao (2007), and Yang and Chang (2013). In all the models analysed in this thesis, it is assumed that the inventory level is reviewed periodically.

1.3 Supply chain structure

A supply chain represents all the stages at which value is added to a (semi-) manufactured product, including the supply of raw materials and intermediate components, finished-goods manufacture, packaging, transportation, warehousing, and logistics (Hall and Potts, 2003). Considering this characteristic of the systems studied, one can divide the literature into two main groups in terms of the structure of the supply chain, namely single-echelon and multi-echelon supply chains.

1.3.1 Single-echelon supply chains

The inventory management theory was founded to address stock-related issues within company boundaries. There are numerous studies to optimise production-inventory processes in a single-echelon model as an independent entity. The absence of a *supply chain concept* was not an issue as businesses could still enjoy high margins by optimising their processes independently (Lummus and Vokurka, 1999). Although taking a single-echelon scope in modelling an inventory system makes it difficult to associate the outcomes to some current real-world situations, it is a base for incrementally building realistic models. This scope of research is still dominating the literature, however, the multi-echelon models are emerging.

In this thesis, this scope of modelling is adopted in Chapter 4 where the aim is to initiate and analyse a basic and fundamental concept which has not been addressed properly in the literature.

1.3.2 Multi-echelon supply chains

As the competition has become more fierce and severe, companies have to think of solutions to sustain competitive advantage in their businesses. Lummus and Vokurka (1999) argue that since the 1980s companies started to incorporate the concept of supply chain management by going beyond their business's boundaries through collaboration with other actors in their supply chain.

Integration in supply chains has been investigated in different ways and levels. Power (2005) indicates that the process of integration in a supply chain is typically a strategic partnership and can only be based on a solid infrastructure of information flows and collaborative arrangements. Only after building up close and long-term partnership, the integration in the supply chain can be implemented at tactical and operational levels.

In this thesis (see Chapters 2, 3, and 5) we look into supply chain integration practices in an operational level where an strategic partnership between partners is well-established.

1.4 Approaches towards opportunity cost

Inventory management is a well-developed area of research. The fundamental property of inventory models is that they seek for the optimal trade-off between inventory holding costs and other costs. The next question would be how to accurately capture, value, and compare these costs.

In order to make an investment, a firm should compare all available alternatives and make sure that it chooses the one which gives the highest return on the investment. Only in this case will the investors be certain that they are going to make the best out of their capital. Brealey and Myers (2003) argue that the next best investment alternative could be a suitable measurement for the *opportunity cost of capital*. With this regard, the capital rate of the next best available option can be used to put a cost on any stock held.

The literature captures the opportunity cost of capital by means of either the Classic or the Net Present Value approach.

1.4.1 Classic models

Harris (1913) introduced the EOQ model mainly on the basis of trade-offs between inventory holding cost and other types of costs in a system. Classic models characterise holding costs as the average cost of stock during an inventory cycle of length T, using a unit holding cost h that represents the cost per stock keeping unit per unit of time, and is often assumed to be a known exogenous and constant parameter. In order to capture the opportunity cost of capital, Silver et al. (1998) define h as follows:

$$h = \alpha v + f, \tag{1.1}$$

where α represents the rate for opportunity cost of capital, v is the money invested per unit of item kept in stock, and f is unit out-of-pocket holding cost. The average holding cost of the system is hence given by:

$$E(H) = \frac{h}{T}\int_0^T I(t)dt = hE(I) \tag{1.2}$$

where $I(t)$ is the inventory level at time t.

Chapters 2 and 3 of this thesis are conducted taking the classic approach towards inventory.

The classic approach is thought to result in fairly accurate solutions in cases that either the opportunity cost of capital is very low or else, that T remains small. If these conditions do not hold then taking a classic approach may cause some classic models not to be an accurate representation of the system. In these cases, one can increase the model's accuracy by accounting for cash-flows and the time they take place.

1.4.2 Net Present Value approach

The objective of inventory (production-inventory) systems in an economic context is arguably to either minimise the total cost or maximise the total profit of future cash-flows. With this regard, the decision makers should be able to evaluate the real value of different costs. In a group of research works it is proven that some classic models are not accurately accounting for the impact of the opportunity cost of capital on the profits of the firm(s), see e.g. Teunter and van der Laan (2002) and Beullens and Janssens (2011, 2013). These researchers assume a discounted cash-flow using a discounting rate which represents the opportunity cost of capital.

In some cases the classic approach falls short to capture the real values of costs and revenues as it is not sensitive to the time that they occur. In order to circumvent the use of (1.2), the net present value (NPV) approach suggests developing profit functions for each party starting from a principle which is valuing cash-flows as they occur.

Considering a capital rate $\alpha \geq 0$, the NPV approach allows for the exact characterisation of cost parameters (unit holding, lost sale, backorder, and deterioration costs) used in the classic approach. As an example, in order to value cash-flow $hI(t)$ (this could represent an out-of-pocket holding cost which is a function of $I(t)$, on-hand inventory), NPV approach discounts the costs as they are incurred using discounting rate α:

$$NPV(H) = \int_0^\infty hI(t)e^{-\alpha t}dt \tag{1.3}$$

The fact that the NPV approach values the costs and the revenues based on the time they take place, enables this approach to precisely measure the values that are put in inventory and also the revenues that are created, hence the capital cost of the system. As the classic approach is not capable of valuing this capital cost, in the literature a value is assigned (see Brealey and Myers, 2003) to capture the capital tied over time. Regarding these shortcomings of the classic approach, Grubbström (1980) shows that in some situations taking the NPV approach to evaluate inventories and work-in-progress results in more accurate measures of their capital cost.

The models in Chapters 4 and 5 adopt the NPV approach where a discounted cash-flow of future costs and revenues is optimised.

1.5 Structure of the thesis

The main focus of this thesis is the analysis of novel inventory and production-inventory models for deteriorating items. As some of the models developed in the literature of deteriorating items are by far simpler than real-world problems, in this thesis the aim is to relax some of the existing assumptions in order to take a few steps towards real cases.

This thesis follows a paper-based approach. Table 1.1 lists the research papers that have been conducted as part of the thesis.

Table 1.1: List of research papers of this thesis

Chapter of the thesis	Publication
Chapter 2	"A Two-echelon Inventory Model for a Deteriorating Item with Stock-dependent Demand, Partial Backlogging and Capacity Constraints", Ghiami et al. (2013)
Chapter 3	"A Two-echelon Production-Inventory Model for Deteriorating Items with Multiple Buyers", Ghiami and Williams (2013)
Chapter 4	"Net Present Value Equivalence Analysis for an Inventory of a Deteriorating Item with Partial Backlogging and Finite Production Rate", Ghiami and Beullens (2013)
Chapter 5	"A Net Present Value Model for a Two-echelon System of a Deteriorating Item with Capacity Constraints", Beullens and Ghiami (2013)

Chapter 2, analyses a two-echelon inventory model for a deteriorating item with limited capacity taking a classic inventory modelling approach. The supply chain in this model includes one wholesaler and one retailer. The demand rate at the retailer is stock-dependent and shortages are partially backlogged. The supply chain is optimised both with independent and integrated approach and the results are compared. In order to numerically analyse the model a heuristic is developed.

Chapter 3, studies a two-echelon production-inventory model with the classic approach, evaluates the exact inventory level of the supplier, and compares the results with the existing models from the literature. The supply chain in this model consists of single-manufacturer and multiple-buyers. The demand at the retailer is constant and the shortages are not allowed.

Chapter 4, develops a single-echelon production-inventory model with the net present value approach and a profit maximisation objective function. The study compares the model with cost minimisation models in the literature and seeks for the relevant equivalence conditions.

Chapter 5, investigates a similar model as in Chapter 2 with the net present value approach and a modification in the inventory level assumptions with a more practical viewpoint.

Finally, Chapter 6, summarises the contribution of all these research works. It also presents the limitations of this thesis work and suggests directions for future research.

Chapter 2

A Two-echelon Inventory Model for a Deteriorating Item with Stock-dependent Demand, Partial Backlogging and Capacity Constraints

Abstract

This study investigates a two-echelon supply chain model for deteriorating inventory in which the retailer's warehouse has a limited capacity. The system includes one wholesaler and one retailer and aims to minimise the total cost. The demand rate seen by the retailer is stock-dependent and in case of any shortages, the demand is partially backlogged. The warehouse capacity in the retailer (OW) is limited, therefore, the retailer can rent a warehouse (RW) if needed, with a higher cost compared to the OW. The optimisation is done with both independent and integrated approach. In order to solve the problem a genetic algorithm is devised. After developing a heuristic, a numerical example together with sensitivity analysis are presented.

2.1 Introduction and literature review

In the classic inventory model for deteriorating products it is usually assumed that the warehouse has no limits in the capacity. However, in real-life problems the situation could be different. There are a number of factors which influence the optimal solution in different ways. Sometimes these factors may suggest retailers to buy more than their own warehouse (OW) capacity. In these situations, the retailers can benefit from a rented warehouse (RW).

Another assumption that can greatly influence the optimal policies is to take a supply chain perspective when analysing inventory models. In multi-echelon inventory models, actors try to integrate their businesses in order to improve the overall performance of the system (e.g. higher service level, higher profit, or lower cost). Implementing such integrated models, however, remains challenging especially when the actors are independent businesses and should collaborate closely (see Fawcett and Magnan, 2002; Power, 2005). In such cases apart from close collaboration between players in the supply chain, there should be a fair mechanism to distribute the incentives between the actors to encourage the integration. Prajogo and Olhager (2012) argue that establishing any mechanisms for supply chain integration is only possible if there is a long-term relationship between the supply chain partners.

To date, very few studies on deteriorating inventory in two-echelon systems have been carried out (see Nahmias, 1982; Raafat, 1991; Goyal and Giri, 2001; Li et al., 2010 and

Bakker et al., 2012). One can argue that deteriorating product literature is still in its infancy and has a long way to go compared to the maturity acquired by general product literature related to supply chains.

As Cohen (1976) notes, Zyl (1964) is one of the first researchers who addresses deteriorating inventory. Zyl (1964) considers a perishable product with fixed life-time. Cohen (1976) develops a model for a deteriorating item with m-period life-time. In this single echelon model, the demand is stochastic and any shortages are completely backlogged. Mak (1982) considers the same set of assumptions with constant rate of demand. The researcher chooses continuous variables and analyses the system using differential equations, which is the most popular analytical approach in analysing a deteriorating product's inventory level.

When working on models for deteriorating items, researchers consider specific factors based on which they make assumptions. The main factors considered in developing new models include demand pattern, lead-time, deterioration rate, shortages, supply chain structure, reviewing policy, system type (inventory versus production-inventory) and warehouse capacity, see Chapter 1.

The definition of the deteriorating item includes a wide spectrum of products such as food, fruit, blood, flower, medicine and clothes. The demand for these deteriorating items therefore varies depending on the product characteristics and the consumption pattern followed by customers. A large group of researchers considers a constant rate for demand, see Mak (1982), Wee (1993, 1998), Chung et al. (1997), Abad (2000), Yang and Wee (2000), Rau et al. (2003), Dye et al. (2007b), Ouyang et al. (2009), and Yan et al. (2011).

Bhunia and Maiti (1998b), Chung and Tsai (2001), Moon et al. (2005), Yang (2005), and Lee and Hsu (2009) assume the demand to be time-dependent. Examples for application of time-dependent demand can be seen in the grocery retailing industry where the demand for some items varies across weekdays. Another example is clothing industry in which seasonality changes the demand level in different seasons. Wu and Ouyang (2000) and Manna and Chaudhuri (2006) develop models with ramp-type demand which is a combination of constant rate and time-dependent demand. One example for this demand pattern is when a new consumer good is introduced to the market. The demand for this new product (in case of success) increases as the time passes, before it may converge to a specific constant level (Wu and Ouyang, 2000).

Some products are price-elastic, which means that by changing the price of the product, the demand will change, however, the elasticity may vary from one product group to another. Yang (2004a) and Dye (2007) explore systems with price-dependent demand. Changes in demand over different periods mean that retailers that would follow the changes in demand by storing inventory or changing inventory policies frequently may have excessive order costs in their supply chain, in particular if the demand fluctuations are large. Hence, they strive to benefit from the elasticity of demand and influence the demand pattern to minimise their supply chain cost.

Hou (2006) discusses that for some products such as consumer goods the demand may be influenced by the amount of the product displayed on shelves. With this regard a group of researchers have considered stock-dependent demand in their research such as Mandal and Phaujdar (1989), Giri et al. (1996), and Hou (2006). Similar to the case of price-dependent demand, retailers use this characteristic to stimulate the demand rate for a product by displaying a large amount of that item on shelves. Therefore they can increase their sale on the product, which may further result in lower cost.

Researchers make different assumptions regarding policies on shortages when considering the product and market characteristics. Liao (2007) and Yang et al. (2010b) develop models where shortages are not allowed. This is a critical assumption when developing a blood bank model or optimising a distribution system for a group of pharmaceuticals. In this type of models, service level is the objective function rather than cost or benefit as these products are connected to health issues.

For some products when a retailer is out of stock, the demand is lost which means the customer finds the item or a similar one in another store. Hsu (2000) and Lu et al. (2008) study models in which any shortages are lost. This case may happen when there are similar products in the market and differences are negligible such as milk or bread that can be found in every grocery shop.

Sometimes customers tend to wait or have to wait for inventory replenishment in case of shortages. The reason for this can be a specific characteristic, for example the outstanding quality of the product (such as a special type of cheese) or the limited availability of the product elsewhere. Some researchers such as Yang (2004b) and Olsson and Tydesjö (2010) study systems with shortages and consider that any unmet demand is backlogged.

Yang (2005), Law and Wee (2006) and Dye (2007) challenge the last two groups of research and discuss that most of the time the situation lies somewhere in between, where the unsatisfied demand is partially backlogged.

When analysing inventory models for deteriorating items, researchers have mainly considered single-echelon systems and have strived for optimising models from a single business point-of-view. In recent years, however the supply chain perspective for deteriorating items has gained more attention. Even though developing logistics models with a supply chain perspective makes the models perform better, implementing these models is complicated and requires collaboration between the supply chain members. Yang and Wee (2000) and Yang et al. (2010a) develop an inventory model for a two-echelon system with constant rate for demand and deterioration in which shortages are not allowed. Yan et al. (2011) develop a similar model with a difference in input pattern as the model is a production-inventory system. Yang (2004b) studies a two-echelon inventory model, similar to the above mentioned models, in which the demand is a function of price and the lead-time is constant. Another important feature of this study is the influence of the time value of money which is taken into account. Law and Wee (2006) and Lo et al. (2007) investigate a two-echelon system with partial backlogging, two-parameter deterioration rate and constant rate of demand. The former research is conducted allowing for permissible delays in payment while the latter considers inflation rate in its calculations. Zanoni and Zavanella (2007) study a two-echelon system for determining optimal inventory policies with constant rate of demand and no shortages. In this model the perishable product has a constant life-time.

In most of the studies, researchers consider no limit for the capacity of a warehouse. This however, can be an important issue in a real-world situation. Sarma (1987) was first to develop a model while assuming a limited capacity for the owned warehouse (OW). In this model extra capacity can be obtained by renting additional warehouse space (RW). Yang (2004a) studies an inventory system with limited capacity. The demand rate is constant and any shortages are completely backlogged. The item's life-time is based on an exponential function (constant rate of on-hand inventory is deteriorated). In another study, Yang (2006) develops a similar model with partial backlogging. Wee et al. (2005) investigate a model with partial backlogging in which the product life-time is based on a two-parameter Weibull distribution. Pal et al. (2005) explore a deteriorating item with a two-warehouse system in which the demand is time-dependent and shortages are partially backlogged. Lee (2006) develops a production-inventory model in which the inventory level is increased by a finite production rate disregarding the effects of inflation. The other assumptions in this model are similar to the research done by Yang (2004a). Chung and Huang (2007) study an inventory model with no shortages, and permissible delays in payments are assumed. Hsieh et al. (2008) study an inventory system similar to Yang (2004a) and optimise the model taking a net present value approach. Rong et al. (2008) consider an inventory model with fuzzy lead-time and complete backlogging in which the demand is connected to price. Singh et al. (2009) explore a two-warehouse model with complete backlogging and time-dependent demand. This model is analysed and optimised considering permissible delays in payments. Lee and Hsu (2009) develop a production-inventory model with time-dependent demand and no shortages. Gayen and Pal (2009) analyse an inventory model in which shortages are not allowed and the demand is stock-dependent. Liao and Huang (2010) investigate a similar model to Chung and Huang (2007) by adopting a different approach. One can find many examples where a retailer needs to rent a warehouse. A new business on high street where space is very expensive can be one example. In such case the retailer uses the space to display the items and moves the warehousing processes to less costly areas. Only after securing a high demand level, the retailer would think about *owning a warehouse* as a strategic decision if feasible.

Almost in all the deterministic studies on deteriorating items, the objective function is to either maximise the profit or minimise the cost. In stochastic models however there are models with an objective function on service level. Surveys of deteriorating item models are given in Nahmias (1982), Raafat (1991), Goyal and Giri (2001), Li et al. (2010), and Bakker et al. (2012).

All the two-warehouse inventory models are based on a single company's point of view and they fall short of taking an overall supply chain perspective. In this study we consider a two-echelon system consisting of a wholesaler and a retailer in which there is a limit in the retailer's warehouse capacity. The demand is considered to be stock-dependent and the product is deteriorated with a constant rate. An analytical model is developed based on the above mentioned assumptions and the costs incurred by retailer and wholesaler are analysed. Using this optimisation model, the optimal inventory policies for both actors in this two-echelon system are obtained. In order to solve the problem a heuristic method is developed.

2.2 Model description

In this chapter a two-echelon system is considered that delivers a deteriorating item to the end customer. This system consists of one wholesaler and one retailer. The item is supplied to the retailer solely by the mentioned supplier and in the same way the retailer is the only downstream actor to which the deteriorating item is sent. The product has exponential life-time which means there is a constant rate for deterioration. The lead time for both retailer and wholesaler is zero. Shortages are allowed at the retailing level and are partially backlogged while this is not allowed at the wholesaler level.

The retailer has a capacity of W at the OW which is limited; therefore, if the order quantity exceeds this capacity, the retailer rents a temporary warehouse which has higher carrying costs and unlimited capacity. In the case that the retailer uses a RW, the consumption of goods from the OW starts only after the inventory at the RW is depleted at $t = t_r$ (see also Figure 2.1). The inventory level at the OW reaches zero at $t = t_o$, and from this time to the end of the inventory period ($t = T_R$), shortages occur and are partially backlogged until the next replenishment. The wholesaler has unlimited warehouse capacity. During one inventory period at the wholesaler (T_W), k inventory cycles of the retailer are covered ($T_W = kT_R$). The deteriorated items cannot be repaired or replaced. This model is an inventory system with fixed inventory period and order quantity and the optimal solution will specify how much and how often the members of this supply chain should order. The total cost of the system consists of purchasing, holding and deterioration cost for both members and shortage cost for the retailer. Purchasing cost for both members is a linear function of their order quantity including the replenishment cost. The unit deterioration cost and the unit holding cost per unit of time are constant. The unit lost sale cost and the unit shortage cost per unit of time for the backlogged demand are also constant.

The demand arises at the retailer at the rate of $D(t) = y + zI_o(t)$ where y and z are constants and $I_o(t)$ represents the inventory level at the OW. Considering the capacity of the warehouse at the retailer, after receiving an order quantity (Q_R), the first W items are stored at the OW and the rest are put in the RW. In the next section the inventory level and the cost functions at the retailer and the wholesaler are discussed in detail.

2.3 Mathematical model

2.3.1 Inventory level at the retailer (OW and RW)

At the RW the inventory level is depleted due to the deterioration and the demand rate. The following differential equation presents the changes of this level between $t = 0$ and $t = t_r$:

$$\frac{dI_r(t)}{dt} = -zI_o(t) - y - \theta_r I_r(t), \qquad 0 \leq t \leq t_r. \tag{2.1}$$

While the retailer is using the inventory at the RW to meet the demand, the inventory level at the OW goes down with a constant rate (θ_o) due to the deterioration. The following differential equation shows the changes of the inventory level at the OW:

$$\frac{dI_o(t)}{dt} = -\theta_o I_o(t), \qquad 0 \leq t \leq t_r. \tag{2.2}$$

At $t = t_r$ the inventory at the RW reaches zero, therefore the retailer starts using the OW to meet the demand. The inventory level at the OW decreases due to the demand and deterioration until the retailer runs out of stock at $t = t_o$. This change in the inventory level is presented as follows:

$$\frac{dI_o(t)}{dt} = -zI_o(t) - y - \theta_o I_o(t), \qquad t_r \leq t \leq t_o. \tag{2.3}$$

At $t = t_o$ the out-of-stock period starts and lasts until $t = T_R$. During this shortage period the unmet demand is partially backlogged. The following differential equation shows the change in the shortage level:

$$\frac{dB(t)}{dt} = \beta y, \qquad t_o \leq t \leq T_R. \tag{2.4}$$

All the changes of inventory level at the RW and the OW are depicted graphically in Figure 2.1.

In order to solve the presented differential equations, the following boundary conditions should be considered:

$$I_o(0) = W, \tag{2.5}$$

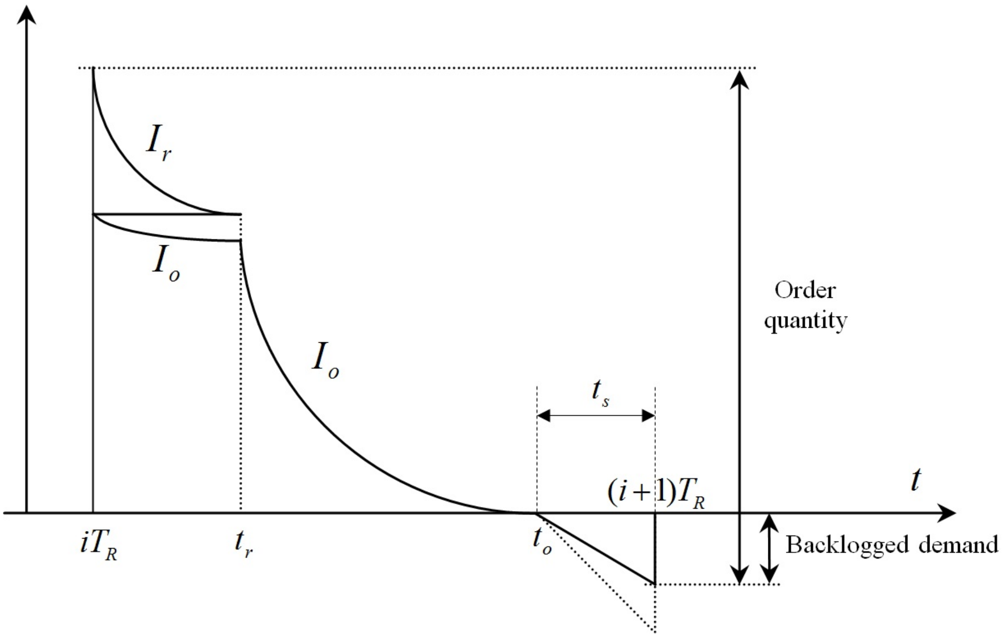

Figure 2.1: The inventory level at the retailer (the RW and the OW)

$$I_o(t_o) = 0, \tag{2.6}$$

and

$$I_r(t_r) = 0. \tag{2.7}$$

By solving the differential equations in (2.1)-(2.4), the inventory and shortage levels at the OW and the RW are obtained:

$$I_r(t) = \frac{y}{\theta_r}(e^{\theta_r(t_r-t)} - 1) + \frac{zWe^{-\theta_o t}}{\theta_r - \theta_o}(e^{(\theta_r-\theta_o)(t_r-t)} - 1), \qquad 0 \le t \le t_r, \tag{2.8}$$

$$I_o(t) = We^{-\theta_o t}, \qquad 0 \le t \le t_r, \tag{2.9}$$

$$I_o(t) = \frac{y}{z+\theta_o}(e^{(z+\theta_o)(t_o-t)} - 1), \qquad t_r \le t \le t_o, \tag{2.10}$$

and

$$B(t) = \beta y(t - t_o), \qquad t_o \le t \le T_R. \tag{2.11}$$

The inventory level at the OW at $t = t_r$ obtained from (2.9) and (2.10) is unique, therefore t_o is a function of t_r:

$$t_o = t_r + \frac{1}{z+\theta_o} ln\left(1 + \frac{z+\theta_o}{y}We^{-\theta_o t_r}\right). \tag{2.12}$$

This means:

$$T_R = t_r + \frac{1}{z+\theta_o} ln \left(1 + \frac{z+\theta_o}{y} W e^{-\theta_o t_r}\right) + t_s. \tag{2.13}$$

The order quantity for the retailer is the sum of the initial inventory level at the RW and the OW and the total backlogged demand during one inventory period:

$$Q_R = \frac{y}{\theta_r}(e^{\theta_r t_r} - 1) + \frac{zW}{\theta_r - \theta_o}(e^{(\theta_r - \theta_o)t_r} - 1) + W + \beta y t_s. \tag{2.14}$$

As shown in (2.13) and (2.14), the inventory policies at the retailer are functions of t_r and t_s.

2.3.2 Inventory level at the wholesaler

The inventory policies for the wholesaler are Q_W and T_W. If it is assumed that T_W is a multiplication of T_R ($T_W = kT_R$), then k should be an integer. If k is not an integer, it means that the wholesaler receives a new batch in their warehouse while there is no demand for new replenishment from the retailer as the retailer still has some items at stock hence not willing to order yet until the end of its inventory period. This means for that fraction of T_R, the wholesaler is carrying inventory and bearing deterioration costs which are not needed until the end of the retailer inventory period. Thus, k should be an integer.

The items held at the wholesaler during one inventory cycle, T_W, is to cover k inventory cycles of the retailer, $T_W = kT_R$. This means after each period of T_R the wholesaler sends a batch of Q_R to the retailer. Therefore, one inventory cycle at the wholesaler consists of k intervals. During each of these intervals the inventory level at the wholesaler is depleted only due to deterioration. The following differential equation shows how the inventory level changes over interval i:

$$\frac{dI_W^i(t)}{dt} = -\theta I_W^i(t), \qquad i = 1, 2, \ldots k-1. \tag{2.15}$$

The order quantity for the wholesaler is equal to the inventory needed for k periods at the retailer, plus the amount of deterioration during the wholesaler inventory cycle. The

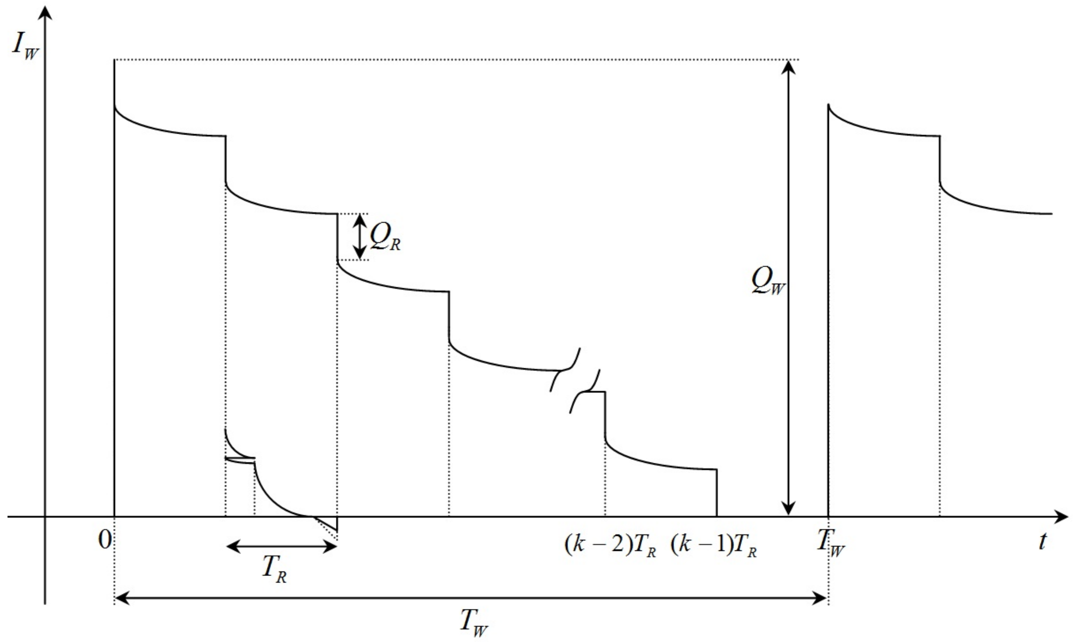

Figure 2.2: The inventory level at the wholesaler

order quantity of the wholesaler is then given by:

$$Q_W = kQ_R + Q_D, \tag{2.16}$$

where Q_D is the number of deteriorated items over one inventory period at the wholesaler.

Figure 2.2 illustrates the inventory level at the wholesaler. It is noted that the inventory level at the wholesaler at $t = (k-1)T_R$ just before sending the last batch to the retailer should be Q_R. This can be used as a boundary condition for the inventory level during the interval between $(k-2)T_R$ and $(k-1)T_R$:

$$I_W^{k-1}(t) = Q_R e^{\theta[(k-1)T_R - t]}, \qquad (k-2)T_R \leq t \leq (k-1)T_R. \tag{2.17}$$

According to (2.17), the inventory level at the wholesaler at $t = (k-2)T_R$, just before sending a batch to the retailer is $Q_R(e^{\theta T_R} + 1)$. Using this inventory level as a boundary condition, the inventory level of the $(k-2)^{th}$ interval is:

$$I_W^{k-2}(t) = Q_R(e^{\theta T_R} + 1)e^{\theta[(k-2)T_R - t]}, \qquad (k-3)T_R \leq t \leq (k-2)T_R. \tag{2.18}$$

The inventory level at the wholesaler during i^{th} interval, hence, is given by:

$$\begin{aligned} I_W^i(t) =& Q_R e^{\theta(iT_R - t)} \sum_{m=0}^{k-i-1} e^{m\theta T_R} \\ =& Q_R e^{\theta(iT_R - t)} \frac{e^{\theta(k-i)T_R} - 1}{e^{\theta T_R} - 1}, \qquad (i-1)T_R \leq t \leq iT_R, \qquad i = 1, 2, ..., k-1. \end{aligned} \tag{2.19}$$

Using (2.19), one can find the inventory level at the wholesaler at $t = 0$ just after sending the first batch to the retailer:

$$I_W^1(0) = Q_R \frac{e^{k\theta T_R} - e^{\theta T_R}}{e^{\theta T_R} - 1}, \tag{2.20}$$

and therefore the wholesaler order quantity is given by:

$$\begin{aligned} Q_W =& I_W^1(0) + Q_R \\ =& Q_R \frac{e^{k\theta T_R} - 1}{e^{\theta T_R} - 1}. \end{aligned} \tag{2.21}$$

Using (2.16) and (2.21), the total number of deteriorated items at the wholesaler over one inventory cycle is obtained:

$$Q_D = Q_R \left(\frac{e^{k\theta T_R} - 1}{e^{\theta T_R} - 1} - k \right). \tag{2.22}$$

As shown in this section, the inventory policies at the wholesaler are functions of the retailer inventory policy hence t_r and t_s. In the next section, the cost functions are analysed.

2.3.3 Cost functions at the retailer

The retailer has five types of cost; purchasing, carrying, deterioration, shortage and lost sale costs. The retailer has to pay s_R as the purchasing fixed cost when placing an order. The item price for the retailer is p_R, therefore the purchasing cost for the retailer in each inventory cycle is:

$$PC_R = s_R + p_R Q_R. \tag{2.23}$$

The retailer incurs an inventory holding cost at the rate (per item per unit of time) of f_r and f_o for the items kept at the RW and the OW respectively:

$$\begin{aligned} HC_{RW} =& f_r \int_0^{t_r} I_r(t)dt \\ =& \frac{f_r z W e^{-\theta_o t_r}}{\theta_r - \theta_o}\left[\frac{1}{\theta_r}(e^{\theta_r t_r}-1) - \frac{1}{\theta_o}(e^{\theta_o t_r}-1)\right] + \frac{f_r y}{\theta_r}\left[\frac{1}{\theta_r}(e^{\theta_r t_r}-1) - t_r\right], \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} HC_{OW} =& f_o \int_0^{t_r} I_o(t)dt + f_o \int_{t_r}^{t_o} I_o(t)dt \\ =& \frac{f_o W}{\theta_o}(1-e^{-\theta_o t_r}) + \frac{f_o y}{z+\theta_o}\left[\frac{1}{z+\theta_o}(e^{(z+\theta_o)(t_o-t_r)}-1) - (t_o - t_r)\right]. \end{aligned} \tag{2.25}$$

Considering (2.24) and (2.25), the total inventory carrying cost at the retailer during one inventory period is as follows:

$$HC_R = HC_{RW} + HC_{OW}. \tag{2.26}$$

It is assumed each item that deteriorates, cost its purchasing price. Therefore, considering inventory levels at the RW and the OW presented in (2.8)-(2.10), the deterioration cost at the RW and the OW are given by:

$$\begin{aligned} DC_{RW} =& p_R \int_0^{t_r} \theta_r I_r(t)dt \\ =& \frac{p_R \theta_r z W e^{-\theta_o t_r}}{\theta_r - \theta_o}\left[\frac{1}{\theta_r}(e^{\theta_r t_r}-1) - \frac{1}{\theta_o}(e^{\theta_o t_r}-1)\right] + p_R y\left[\frac{1}{\theta_r}(e^{\theta_r t_r}-1) - t_r\right], \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} DC_{OW} =& p_R \int_0^{t_r} \theta_o I_o(t)dt + p_R \int_{t_r}^{t_o} \theta_o I_o(t)dt \\ =& p_R W(1-e^{-\theta_o t_r}) + \frac{p_R \theta_o y}{z+\theta_o}\left[\frac{1}{z+\theta_o}(e^{(z+\theta_o)(t_o-t_r)}-1) - (t_o - t_r)\right], \end{aligned} \tag{2.28}$$

hence, the deterioration cost at the retailer in one inventory cycle is:

$$DC_R = DC_{RW} + DC_{OW}. \tag{2.29}$$

During the shortage period, the demand is partially backlogged which results in two types of cost; backlogging cost (b) which is per item that is backlogged per unit of time and lost sale cost (π) which is incurred per unit of demand which is lost. Therefore the shortage cost and lost sale cost at the retailer during one inventory cycle are:

$$\begin{aligned} BC_R =& \int_{t_o}^{T_R} b\beta y(t - t_o)dt \\ =& \frac{1}{2} b\beta y t_s^2, \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} LC_R =& \int_{t_o}^{T_R} \pi(1 - \beta) y dt \\ =& \pi(1 - \beta) y t_s, \end{aligned} \tag{2.31}$$

respectively.

2.3.4 Cost functions at the wholesaler

There are three different types of cost at the wholesaler; purchasing, inventory carrying and deterioration cost.

There is a fixed cost s_W for the wholesaler when placing an order at the beginning of each inventory cycle and the the item price is p_W. With this regard the purchasing cost for each period at the wholesaler is:

$$PC_W = s_W + p_W Q_W. \tag{2.32}$$

The inventory incurs a holding cost at the rate of f at the wholesaler. Using (2.19), the inventory carrying cost for the wholesaler during the i^{th} interval is:

$$\begin{aligned} HC_W^i =& \int_{(i-1)T_R}^{iT_R} f I_W^i(t) dt \\ =& \frac{f Q_R}{\theta} (e^{\theta(k-i)T_R} - 1), \end{aligned} \tag{2.33}$$

therefore the holding cost at the wholesaler over one inventory period is given by:

$$\begin{aligned} HC_W =& \sum_{i=1}^{k-1} HC_W^i \\ =& \frac{f Q_R}{\theta} \left(\frac{e^{\theta k T_R} - 1}{e^{\theta T_R} - 1} - k \right). \end{aligned} \tag{2.34}$$

In a similar way, the deterioration cost over one inventory cycle at the wholesaler can be obtained. It should be noted that each item costs p_W when deteriorates:

$$DC_W = p_W Q_R \left(\frac{e^{\theta k T_R} - 1}{e^{\theta T_R} - 1} - k \right), \tag{2.35}$$

which can also be obtained using (2.22).

Considering the costs incurred at the retailer and the wholesaler, the average total cost of the supply chain is given by:

$$TC_{SC} = \frac{1}{T_R}(PC_R + HC_R + DC_R + BC_R + LC_R) + \frac{1}{T_W}(PC_W + HC_W + DC_W) \tag{2.36}$$

which is a function of t_s, t_r and k. Therefore the cost minimisation problem is:

$$\begin{aligned} &Min && TC_{SC}(t_r, t_s, k) \\ &Subject\ to && t_r, t_s > 0,\ k \in \{1, 2, \dots\}. \end{aligned} \tag{2.37}$$

In the next section a solution method is developed and analysed.

2.4 Optimisation

In order to solve problem (2.37), enumeration can be used as k cannot take a very large value. In this case the problem can be solved for different values of k (e.g. 1, 2, ... 30) and the best optimal solution and the relevant k can be considered as the optimal solution for the problem. Appendix A presents an analytical solution for this problem. This method however can result in an exhaustive search if k tends to get large values. In studies with similar models and objective functions in terms of complexity, researchers develop a heuristic to solve the problem to avoid time consuming solution process (see Yang and Wee, 2002; Pal et al., 2005; Yan et al., 2011). In this research a heuristic is suggested that combines genetic algorithm (GA) and a neighbouring search which searches the feasible area, solves the problem in a short time and gives a near-optimal solution. The steps of the GA are as follow:

Step 1. Deciding about parameters in GA: population, number of generations, the percentage of the next generation which should be generated by mutation, reinsertion and crossover and when to stop the algorithm;

Step 2. Producing the first generation and calculating the fitness function (gen=1);

Step 3. Saving the best solution in the population;

Step 4. If the planned number of generations has been produced go to Step 9;

Step 5. Using the previous generation to produce a new generation by mutation, reinsertion and crossover and calculating the fitness function;

Step 6. Saving the solution in the population;

Step 7. In case of no improvement compare to the previous generation go to Step 9;

Step 8. Go to Step 4;

Step 9. Saving the optimal solution;

Step 10. Stop.

In order to implement this genetic algorithm, similar components as in Maiti et al. (2006) and Gupta et al. (2007) are considered:

- Parameters of genetic algorithm (Population size, maximum number of generations, the probabilities used in the genetic operations);
- Chromosome representation;
- Initial population;
- Fitness function;
- Selection process;
- Genetic operations (crossover, mutation and reinsertion).

2.4.1 Parameters of genetic algorithm

Firstly, all the parameters of the genetic algorithm should be defined. These parameters are the population size (MPOP), maximum number of generations (MGEN), probability of reinsertion (PREIN), probability of mutation (PMUT) and probability of cross over (PCROS). In this research the values of the introduced parameters are around the values defined in Maiti et al. (2006): MPOP=200, MGEN=200, PREIN=0.1, PMUT=0.15, PCROS=0.75. Larger population size however is to guarantee high quality in crossovers as in this study a smaller percentage of the population is produced using this operation compared to Maiti et al. (2006). Higher probability for mutation, is to increase the random solutions to give the model the opportunity of searching the feasible area more thoroughly. Setting the parameters as mentioned and running the algorithm for 15 generations, shows that the improvement from generation 14 to 15 is considerable. With this regard, a larger value is assigned as the maximum number of generations in order to give the algorithm a sufficient number of iterations to find a near-optimal solution.

2.4.2 Chromosome representation

A three-dimensional vector, $X = (t_r, t_s, k)$, is used to represent a person in the population (a solution). In this vector t_r and t_s are real numbers and k is an integer.

2.4.3 Initial population

The first generation is generated by assigning random values to the decision variables within the relevant feasible range. This process continues until the desired number of solutions in the population is obtained. In this research for the first two decision variables a uniformly distributed number is used and the value for the third decision variable is chosen randomly from a set of integer values.

2.4.4 Fitness function

After obtaining the initial generation, the quality of each solution, called fitness function, should be evaluated based on the relevant objective function. In this research the objective function is the total cost of the system. In order to use the conventional selection process as in Maiti et al. (2006) and Maiti and Maiti (2007), there is a need for modification as the objective function in their research works is profit maximisation. In this research the fitness function of a solution is considered to be the total cost of the system incurred by that solution to the power of -1 therefore a chromosome with a lower cost function has a higher chance to be selected (higher fitness function).

2.4.5 Selection process

Considering the fitness function, it is guaranteed that the solution with lower cost function has a higher chance to be selected. In this study the selection process used in Maiti et al. (2006) is considered.

2.4.6 Genetic operations

Solutions/persons in the second and any other generation are generated by the ones in the previous generation using three main operations; crossover, mutation and reinsertion. Genetic algorithm continues until one of the termination condition is held; either a specific number of generations are produced or there is no improvement in the optimal solution from one generation to the next.

- *Crossover operation:* The majority of the population in any generation is produced by the crossover operator (predetermined percentage). In this study one point crossover is used. After choosing the parents by selection process, crossover operation generates a random value which shows what part of these two solutions should be exchanged in order to have two new solutions. In the next step, the fitness function for the new solutions is calculated. This process is repeated until reaching the percentage of the population for the new generation.
- *Mutation operation:* A small percentage of the new generation is generated using mutation operation. After choosing a chromosome using the selection process, the mutation operation randomly picks one of the decision variables in the chosen solution and assigns a new value to the decision variable within the bounds. In the next step the corresponding fitness function is found. These steps are repeated until the desired population generated by mutation operation is obtained.
- *Reinsertion operation:* Reinsertion operation creates a very small number of solutions in the new generation. Reinsertion selects one of the solutions randomly using the selection process from the last generation and moves that to the new generation with no change.

Using GA with appropriate generating tools guarantees feasible area coverage but not optimality. Therefore in this study after producing the last generation, each individual in the last generation is studied to see if there is any local optimum in the neighbourhood. To do this, all the neighbouring solutions to each individual are found and evaluated. Each solution has three decision variables and a neighbouring solution is exactly the same as the base solution except for one of the decision variables which has a difference of one unit compared to the base solution. In this study each solution has six neighbouring solutions. After examining all the neighbouring solutions, the best neighbour is considered as the base and in a similar way all the neighbouring solutions to the new base are evaluated. This algorithm is repeated until a local optimum is reached. This local search is conducted for each individual in the last generation. The best local optimum is considered as the solution to problem (2.37).

2.5 Numerical examples

To illustrate the result of the analysis conducted in Section 2.3, the following numerical examples are considered and the heuristic method is applied. In order to see the advantage of the supply chain perspective and the benefits that the retailer and the wholesaler gain through integrated planning, in these examples the inventory system is optimised both with supply chain and single company point of view.

In order to optimise the system of the retailer and the wholesaler independently (non-integrated approach), first the inventory system of the retailer is optimised. Having the inventory policies calculated for the retailer, the inventory system of the wholesaler is optimised using an analytical approach (see Appendix B). The results of this optimisation are also presented in the following numerical examples.

Example 1. Let the time unit to be one day, demand function, $D(t) = 0.2I_o(t) + 200$, warehouse capacity at the OW, 200, the deterioration at the OW, 5%, the deterioration rate at the RW, 8%, the deterioration rate at the wholesaler, 3%, the backlogging rate during shortage period, 50%, fixed ordering cost for the retailer, 1500, fixed ordering cost for the wholesaler, 2500, the retailer purchasing price, 8, the wholesaler purchasing price, 3.5, the unit holding cost at the OW, 0.4, the unit holding cost at the RW, 0.5, the unit holding cost at the wholesaler, 0.3, the unit backorder cost, 4 and the unit lost sale cost, 20.

Table 2.1: Results of the numerical example 1

	k	Q_R	T_R	Q_W	T_W	TC	TC_R	TC_W
Before integration	-	200	0.9	1520	6.3	5021	3522	1499
After integration	3	390	2.8	1275	8.4	4299	3299	1000

As Table 2.1 shows, the total cost of the system is decreased by 14% when optimising the system with a supply chain perspective. As a result of this integration, the total cost of the retailer and the wholesaler drop by 6% and 33% respectively.

The optimal values for t_r and t_s are 0 and 1.9 respectively which means the model does not suggest using the RW. The shortage period is 1.9 days during which 380 units are demanded but only 50% of this demand is backlogged. Therefore the demand for 190 units of the product is backlogged waiting for the new replenishment. As soon as a new batch is received (order quantity= 390), 190 units are used to meet the backlogged demand from previous period and the rest of the order quantity is stored in the warehouse.

Example 2. Let the time unit to be one day, demand function, $D(t) = 0.1I_o(t) + 100$, the OW's warehouse capacity, 50, the OW's deterioration, 5%, the deterioration rate at the RW, 8%, the deterioration rate at the wholesaler, 3%, the backlogging rate during shortage period, 40%, fixed ordering cost for the retailer, 1000, fixed ordering cost for the wholesaler, 2500, the retailer purchasing price, 8, the wholesaler purchasing price, 3.5, the unit holding cost at the OW, 0.4, the unit holding cost at the RW, 0.5, the unit holding cost at the wholesaler, 0.3, the unit backorder cost, 4 and the unit lost sale cost, 30.

The effect of changes in parameters' values on the total cost and decision variables is studied by means of a sensitivity analysis. In order to carry out the analysis, all the parameters in the model are set equal to two different levels apart from the initial value in the example (20% decrease and increase compared to the original level) and the change in total cost (CTC) is calculated as follows:

$$CTC = \frac{TC^{new} - TC^{*}}{TC^{*}} 100\%. \tag{2.38}$$

Table 2.2 presents the results of this numerical example. For the integrated system, it is suggested that the inventory period at the retailer should be 5.0 days and that a quantity of 381 units of the product should be ordered at the beginning of each period. During the first 2.0 days of each cycle, the inventory stored at the RW is used to meet the demand. After this time the inventory at the OW is used as there is no item at the RW. The inventory at the OW is depleted completely at $t = 2.4$ after which there is a shortage period of 2.6 days.

Table 2.2: Results of the numerical example 2

	t_r	t_s	k	t_o	Q_R	T_R	Q_W	T_W	TC	TC_R	TC_W
Before integration	2.0	0	-	2.4	277	2.4	1238	9.6	4572	3677	895
After integration	2.0	2.6	2	2.4	381	5.0	825	10.0	2838	2220	618

The model also suggests that the wholesaler's inventory cycle should be 10.0 days and that at the beginning of each period an order of 825 units should be placed. As can be seen in Table 2.2, the total cost of the system per unit of time is 2838. For further analysis, Table 2.3 presents the changes in different parameters and the effects of these changes on the optimal solution.

Table 2.3: Sensitivity analysis of parameters on integrated total cost

	value	t_r	t_s	k	t_o	Q_R	T_R	Q_W	T_W	TC	$CTC(\%)$
z	0.12	2.0	2.6	2	2.4	383	5.0	829	10.0	2852	0.5
	0.08	2.1	2.5	2	2.5	387	5.0	838	10.0	2824	-0.5
y	120	2.0	2.2	2	2.4	426	4.6	915	9.2	3315	16.8
	80	2.0	3.1	2	2.5	333	5.6	727	11.2	2351	-17.2
W	60	1.2	2.0	3	1.7	273	3.7	921	11.1	2832	-0.2
	40	2.2	2.6	2	2.5	394	5.1	853	10.2	2841	0.1
θ_o	6%	1.9	2.7	2	2.3	373	5.0	804	10.0	2861	0.8
	4%	2.1	2.5	2	2.5	390	5.0	843	10.0	2823	-0.5
θ_r	9.6%	1.3	2.0	3	1.8	275	3.8	928	11.4	2845	0.2
	6.4%	2.1	2.6	2	2.5	389	5.1	844	10.2	2847	0.3
θ	3.6%	2.0	2.5	2	2.4	377	4.9	828	9.8	2848	0.4
	2.4%	1.5	2.0	3	1.9	297	3.9	983	11.7	2825	-0.5
β	48%	1.9	2.6	2	2.3	390	4.9	842	9.8	2788	-1.8
	32%	2.2	2.5	2	2.6	382	5.1	827	10.2	2886	1.7
s_R	1200	2.1	2.7	2	2.5	398	5.2	863	10.4	2877	1.4
	800	1.3	1.8	3	1.8	266	3.6	890	10.8	2788	-1.8
s_W	3000	1.5	2.1	3	1.9	301	4.0	1025	12.0	2884	1.6
	2000	1.9	2.3	2	2.3	357	4.6	767	9.2	2786	-1.8
p_R	9.6	0.6	2.8	3	1.1	226	3.9	767	11.7	3000	5.7
	6.4	2.8	1.4	2	3.2	434	4.6	933	9.2	2602	-8.3
p_W	4.2	1.9	2.7	2	2.3	373	5.0	807	10.0	2899	2.1
	2.8	2.2	2.4	2	2.6	398	5.0	861	10.0	2775	-2.2
f_o	0.48	2.0	2.6	2	2.4	381	5.0	825	10.0	2840	0.1
	0.32	2.0	2.6	2	2.4	381	5.0	825	10.0	2836	-0.1
f_r	0.6	2.0	2.6	2	2.4	381	5.0	825	10.0	2842	0.1
	0.4	2.1	2.5	2	2.5	390	5.0	843	10.0	2833	-0.2
f	0.36	2.0	2.5	2	2.4	377	4.9	815	9.8	2850	0.4
	0.24	1.5	2.0	3	1.9	297	3.9	1008	11.7	2822	-0.6
π	36	2.1	0.6	3	2.5	314	3.1	1036	9.3	2975	4.8
	24	0.6	3.0	3	1.1	234	4.1	799	12.3	2615	-7.9
b	4.8	1.5	1.8	3	1.9	289	3.7	974	11.1	2856	0.6
	3.2	1.9	3.0	2	2.3	385	5.3	837	10.6	2814	-0.8

After examining the effects of the changes on the total cost function, the inventory policies of both the retailer and the wholesaler are considered to investigate how these values react in the case of any changes in parameters (Table 2.4).

Table 2.4: Sensitivity analysis of parameters on inventory policies (%)

	value	Q_R	T_R	Q_W	T_W		value	Q_R	T_R	Q_W	T_W
z	0.12	0.5	0.0	0.5	0.0	s_W	3000	-21.0	-20.0	24.2	20.0
	0.08	1.6	0.0	1.6	0.0		2000	-6.3	-8.0	-7.0	-8.0
y	120	11.8	-8.0	10.9	-8.0	p_R	9.6	-40.7	-22.0	-7.0	17.0
	80	-12.6	12.0	-11.9	12.0		6.4	13.9	-8.0	13.1	-8.0
W	60	-28.3	-26.0	11.6	11.0	p_W	4.2	-2.1	0.0	-2.2	0.0
	40	3.4	2.0	3.4	2.0		2.8	4.5	0.0	4.4	0.0
θ_o	6%	-2.1	0.0	-2.5	0.0	f_o	0.48	0.0	0.0	0.0	0.0
	4%	2.4	0.0	2.2	0.0		0.32	0.0	0.0	0.0	0.0
θ_r	9.6%	-27.8	-24.0	12.5	14.0	f_r	0.6	0.0	0.0	0.0	0.0
	6.4%	2.1	2.0	2.3	2.0		0.4	2.4	0.0	2.2	0.0
θ	3.6%	-1.0	-2.0	0.4	-2.0	f	0.36	-1.0	-2.0	-1.2	-2.0
	2.4%	-22.0	-22.0	19.2	17.0		0.24	-22.0	-22.0	22.2	17.0
β	48%	2.4	-2.0	2.1	-2.0	π	36	-17.6	-38.0	25.6	-7.0
	32%	0.3	2.0	0.2	2.0		24	-38.6	-18.0	-3.2	23.0
s_R	1200	4.5	4.0	4.6	4.0	b	4.8	-24.1	-26.0	18.1	11.0
	800	-30.2	-28.0	7.9	8.0		3.2	1.0	6.0	1.5	6.0

Studying the output of the sensitivity analysis, the following points can be made:

1. As the results in Table 2.3 show, the total cost is most sensitive to y of which a 20% change can make a change of nearly 17% in the same direction. It should be noted that this model is based on the total cost and not profit, therefore this increase in cost cannot be interpreted without considering the total profit of the system.

2. The total cost shows the second highest sensitivity to p_R, π and p_W. According to Table 2.3, all these factors change the total cost in the same direction. As the total cost of the system is sensitive to the purchasing price that both the retailer and the wholesaler pay, applying discounting models in their business is highly recommended. Nevertheless buying larger amount necessitates both retailer and wholesaler to focus more on marketing in order to increase the demand. As mentioned before, the cost function is highly sensitive to demand, which means that the retailer can decrease its margins on the product and influence the demand (in case the product is price elastic) which enables the retailer to purchase in larger quantities and enjoy lower prices.

3. The cost of unmet demand per unit of product which becomes lost sale is very complicated to quantify as it depends on many factors. If there are also competing products on shelves, shortages may cause the product to lose the competition to those

which are available. In some cases when the product obtains especial characteristics, the customer may go to another store in search of the desired item and they may do the whole shopping in the second store and this could be considered as a lost profit for the first store. The shortage of the product may have a negative effect on the image of the product brand or the retailer as well. Usually the cost of lost sale is a managerial evaluation as it needs experience and knowledge about the market and the competition.

4. The total cost is least sensitive to W, f_o and f_r. In the case of a 20% change in these parameters the change in the total cost function is almost zero. This however maybe the case for this numerical example with these specific values assigned to the parameters. As warehouse capacity is a strategic decision which may cause a huge amount of cost or saving for a retailer, studying the influence of warehouse capacity on total cost needs a more detailed sensitivity analysis. Considering the same product with the same market, a change in warehouse capacity from 50 to 150 decreases the total cost by 9%. In case the distribution system is to deliver a product of which the service level is relatively high (from a modelling point of view the cost of shortage per unit of product is considered to be a large number), the total cost of the system would be 3014. In this case (no shortages allowed) the change in warehouse capacity from 50 to 150, decreases the total cost by 14%. However evaluating the trade-offs between the cost of expanding the warehouse capacity and the decrease in total cost in order to make this strategic decision is a managerial task.

5. The effects of changes in the parameters on the inventory policies of the retailer and the wholesaler are presented in Table 2.4. As can be seen a 20% increase in p_R shows 40% and 22% decrease in Q_R and T_R respectively which means that the model suggests the retailer a higher frequency with smaller order quantity.

6. Increase in W, b and θ_r, make a decrease in Q_R and T_R. In case of an increase in W, the model tries to minimise the shortages and the usage of RW by using extra capacity through higher frequency in replenishments. Higher shortage cost (b) has the same influence on the optimal solution as the model tries to avoid shortages. Due to deterioration and the capacity limits the model cannot increase the order quantity, therefore it suggests higher frequency with smaller order quantities. Higher level of deterioration at the RW also motivates the model to adopt lower order quantities and inventory periods for the retailer to minimise the cost of using the RW.

7. A decrease in s_R, θ and f suggests lower order quantity and shorter inventory period for the retailer. For the case of s_R this change is intuitive. In case of a decrease in θ and f, the model tends to keep the inventory at the wholesaler to avoid the higher holding cost and deterioration rate at the retailer level.

8. As presented in Table 2.4, in general the decision variables related to the wholesaler show less sensitivity to the changes in the parameters. In case of a 20% decrease in s_W, f and θ, the model suggests the wholesaler to order larger quantities with less frequency.

9. Some parameters that are related to the retailer may indirectly influence the decision variables of the wholesaler. In case of an increase in π for instance, the model suggests less shortages at the retailer. This decrease in shortage level is obtained through 17% and 38% decrease in the retailer's order quantity and inventory period respectively. In order to meet this demand (which is higher than the initial example) the wholesaler is suggested to have shorter inventory period (a decrease of 7%) but larger order quantity (an increase of 25.6%).

2.6 Conclusions

In this chapter an analytical model for deteriorating inventory is developed considering a limit for warehouse capacity at the retailer. One of the main features which characterises this model is the supply chain perspective where costs of the wholesaler and the retailer are considered and minimised simultaneously. Most of the studies in this area are from a single company point of view. In this model purchasing cost, inventory carrying cost and deterioration cost for both the wholesaler and the retailer and shortage cost at the retailer are taken into account. In the solution part, a heuristic method is developed to find a fairly good solution to avoid time consuming calculations. Another feature of this model is that it is more generic compared to other models; e.g. by setting $z = 0$ the model will be converted to a model with constant demand. In addition, this model can change to a model with complete backlogging or lost sale.

As this model considers a percentage of on-hand inventory getting deteriorated, it cannot be applied to the distribution systems delivering products with fixed life-time. Although this two-echelon model differs greatly from many real-world problems, it requires complex mathematical calculations. This means relaxing more assumptions to get closer to some real cases will result in even more complexity. This shows the importance of using efficient heuristics in solving such models.

In order to make this model more realistic, some extension opportunities are suggested. For further research the inflation rate is suggested to be considered in cost calculation which sometimes is part of the real-life problems. In some cases wholesalers accept delays in payments by retailers in order to motivate them to increase their order quantity.

This model considers and analyses a distribution system which consists of one retailer and one wholesaler. Normally that cannot be the case in reality as distribution systems consist of multiple retailers and multiple wholesalers. Therefore an extension to this model could be considering multi-suppliers and multi-retailers. Few studies have been conducted on deteriorating inventory models for multi-product systems, and as there is competition between products for shelf and warehouse space, this assumption is also suggested for further research. Finally, quantity discount models and transportation models between echelons are recommended for future research.

Chapter 3

A Two-echelon Production-Inventory Model for Deteriorating Items with Multiple Buyers

Abstract

In a production-inventory system, the manufacturer produces the items at a rate, e.g. R, dispatches the order quantities to the customers in specific intervals and stores the excess inventory for subsequent deliveries. Therefore each inventory cycle of the manufacturer can be divided into two phases, first is the period of production, the second is when the manufacturer does not do any production and utilises the inventory that is in stock. One of the challenges in these models is how to obtain the inventory level of the supplier when there is deterioration. Previous literature that considers multi-echelon systems, analyses the deterioration/inventory cost of these echelons in a way that may not be sufficiently accurate for some cases. In this chapter, a production-inventory supply chain including one manufacturer and N retailers is studied. The analysis conducted in this chapter shows that there are situations that the existing literature does not analyse the inventory level at the supplier accurately and hence, the cost of the deterioration and inventory holding.

3.1 Introduction and literature review

The deteriorating item inventory models have increasingly drawn attention in recent years. For an overview of the deteriorating item's literature see Nahmias (1982), Raafat (1991), Goyal and Giri (2001), Li et al. (2010), and Bakker et al. (2012). This literature mainly includes single-echelon inventory models. Very few research works however have addressed a production-inventory multi-echelon supply chain.

One of the challenges in modelling a multi-echelon supply chain of a deteriorating item is how to evaluate the inventory level at the supplier. Ghiami et al. (2013) evaluate the exact inventory level of the supplier of a two-echelon system (a single-buyer, single-supplier model). Assuming finite production rate (production-inventory model) with multiple buyers changes this model in a way that necessitates using another approach to obtain the manufacturer inventory level.

One of the first studies which analyses a multi-echelon supply chain for a deteriorating item with finite production rate is done by Yang and Wee (2000). The authors consider single buyer-single vendor where the vendor produces the items with a finite production rate. The objective is to minimise the total cost function of the supply chain. Later Yang and Wee (2002) extend the work done by Yang and Wee (2000) by considering multi buyers and aim to minimise the total cost of the system. Law and Wee (2006) consider a single-buyer, single-vendor supply chain which produces and delivers a product of which the raw material is livestock. The manufacturer buys young livestock and grows them, the mature livestock is then used to make food. This food is delivered to the buyer in batches. In their investigation, Law and Wee (2006) consider the time value of money by discounting the cost with a specific rate, ultimately minimising the total cost of the system. Taking a discounted cash flow approach, Lo et al. (2007) model a production-inventory system which consists of one buyer and one manufacturer. Similar to the previous research works, Lo et al. (2007) aim to minimise the total cost of the system. The way that the above-mentioned studies calculate the inventory level of the manufacturer is questionable which in some situations it may result in large errors. Yang and Wee (2002) are the first researchers who develop a single-manufacturer, multi-buyer model for a production-inventory system but no further research work is known. With this regard, in this research work, the same model is considered, analysed, and improved upon.

In their research in order to calculate the average inventory cost of the supplier, Yang and Wee (2002) assume that the inventory level of the manufacturer is as shown in Figure 3.1. The inventory cycle at the supplier is $T = T_1 + T_2$, where T_1 is the production period and T_2 is the non-production interval.

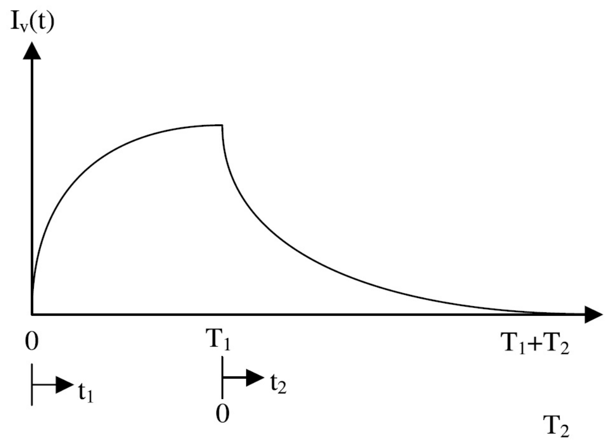

Figure 3.1: Inventory level of the supplier (Yang and Wee, 2002)

This supplier delivers the deteriorating item to N buyers which have a constant demand rate. The inventory level of the buyer i is shown in Figure 3.2.

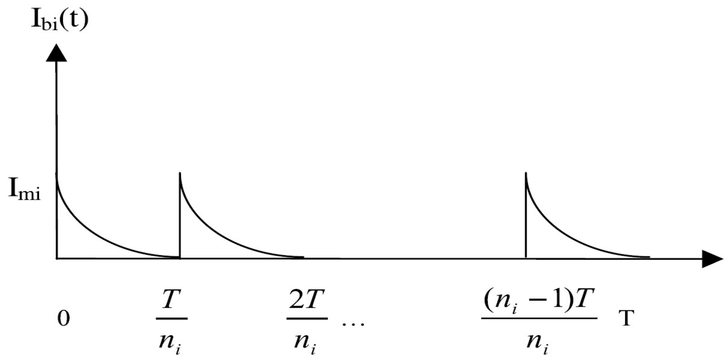

Figure 3.2: Inventory level of buyer i (Yang and Wee, 2002)

Considering the inventory level at the supplier and the buyers, depicted in Figure 3.1 and Figure 3.2, Yang and Wee (2002) calculate the supplier's inventory holding cost as follows:

$$HC_v^{YW} = \frac{p_v F_v}{T}\Big[\int_0^{T_1} I_{v1}(t_1)dt_1 + \int_0^{T_2} I_{v2}(t_2)dt_2 - \sum_{i=1}^{N} n_i \int_0^{\frac{T}{n_i}} I_{bi}(t)dt\Big]. \tag{3.1}$$

In order to calculate the inventory holding cost of the supplier presented in (3.1), the *echelon stock* concept is used as in Joglekar (1988). Echelon stock of an entity in a supply chain refers to the sum of physical inventory in that business unit and all the downstream firms. As buyer i is delivering the product to the end customer, his echelon stock, $\widetilde{I}_{bi}(t)$, is the same as his physical inventory, $I_{bi}(t)$, hence $\widetilde{I}_{bi}(t) = I_{bi}(t)$. The echelon stock of the supplier, $\widetilde{I}_v(t)$, is the sum of the physical inventory of the supplier, $I_v(t)$, and the total echelon stock of the downstream buyers, $\widetilde{I}_b(t)$. Therefore the inventory level of the supplier is:

$$I_v(t) = \widetilde{I}_v(t) - \widetilde{I}_b(t), \qquad 0 \le t \le T, \tag{3.2}$$

where

$$\begin{aligned}\widetilde{I}_b(t) &= \sum_{i=1}^{N} \widetilde{I}_{bi}(t)\\ &= \sum_{i=1}^{N} I_{bi}(t), \qquad 0 \le t \le T.\end{aligned} \tag{3.3}$$

By using (3.1), Yang and Wee (2002) aim to use the echelon stock concept to calculate the manufacturer inventory level and therefore the average holding cost. The researchers however make assumptions on the inventory level of the supplier in a way that in some cases fall short to obtain the average holding cost accurately. This will be discussed in detail in Section 3.2.

Yan et al. (2011) examine an integrated single-buyer, single-supplier model with finite production rate. Although the researchers graphically present the exact inventory level for the supplier, they use an approximation method to calculate the inventory level at both echelons which in some cases fails to capture the real influence of deterioration. Yan et al. (2011) do not use differential equations to find the inventory levels. The researchers assume that a percentage of the order quantity (not the on-hand inventory) deteriorates over the inventory cycle. This assumption overestimates the (average) deterioration at the buyer (considering the number of replenishments fixed). Higher level of deterioration results in larger order quantities which increases the (average) inventory level of the buyer. This approximation however does not bring large error to the buyer's average inventory. On the supplier's side, the method used by Yan et al. (2011) does not present clearly how it captures the effect of the deterioration on the inventory level. The numerical example discussed in Yan et al. (2011) is considered and analysed in Section 3.3 to show the accuracy of the approximation used.

In order to optimise an inventory system when cost and benefit have the first priority, researchers aim to either maximise the total profit or minimise the total cost. However in some cases these two objective functions are not equivalent and result in different optimal solutions. Therefore the decision of which objective function to choose should be made carefully. Only in situations when the revenue is not a function of decision variables can profit maximisation be replaced by cost minimisation. Let TP_v, TP_{bi}, and TP_b to be the total profit function of the supplier, buyer i, and the sum of the profit functions of all buyers respectively. These values can be obtained from relevant revenue and cost functions which are TR_v and TC_v for the supplier, TR_{bi} and TC_{bi} for buyer i and TR_b and TC_b for the group of buyers. The total profit of the supply chain (TP) is the sum of the total profit functions:

$$\begin{aligned} TP =& TP_v + TP_b \\ =& TR_v - TC_v + TR_b - TC_b. \end{aligned} \tag{3.4}$$

In case of constant demand with no shortages (or with complete backlogging), where total sale (in terms of quantity) of the buyer is equal to the total demand and hence, independent of the decision variables (T and n_i) as in this model, minimising the total cost is equivalent to maximising the total profit as the derivatives of TR_b and TR_v with respect to the decision variables are zero (relaxing the integrality assumption on n_i), therefore:

$$\frac{\partial TP}{\partial T} = -\frac{\partial (TC_v + TC_b)}{\partial T}, \tag{3.5}$$

and

$$\frac{\partial TP}{\partial n_i} = -\frac{\partial (TC_v + TC_b)}{\partial n_i}. \tag{3.6}$$

Here also it is implicitly assumed that the opportunity cost of capital is negligible, therefore, taking the classic approach does not result in errors in the optimal solution. This will no longer be true in general when using a Net Present Value (NPV) approach, as it values costs and revenues based on the time they take place, hence captures the opportunity cost of capital accurately.

3.2 Model

In this chapter a supply chain including one manufacturer and N buyers is considered. The assumptions and notations are as in Yang and Wee (2002) (see Appendix C.1). The inventory period of the manufacturer in Yang and Wee (2002) consists of two parts (see Figure 3.1). In the current model, however, the inventory period is assumed to have three parts (see Figure 3.3).

The inventory level at buyer i is as shown in Figure 3.2. The manufacturer's physical inventory dt units of time before dispatching the order quantities to the buyers is equal to the sum of the order quantities ($\sum_{i=1}^{N} I_{mi}$). This level goes to zero after dispatching the order quantities to the buyers. The echelon stock however does not drop to zero (as the batches are now in the buyers' inventory) but gradually decreases due to the demand and the deterioration. In Figure 3.3 the physical inventory (solid line) and the echelon stock (dashed line) of the supplier are shown.

As can be seen in Figure 3.3, finding the physical inventory level of the supplier is complicated while finding the echelon stock is trivial. The following differential equations represent the change of the echelon stock of the supply chain over T:

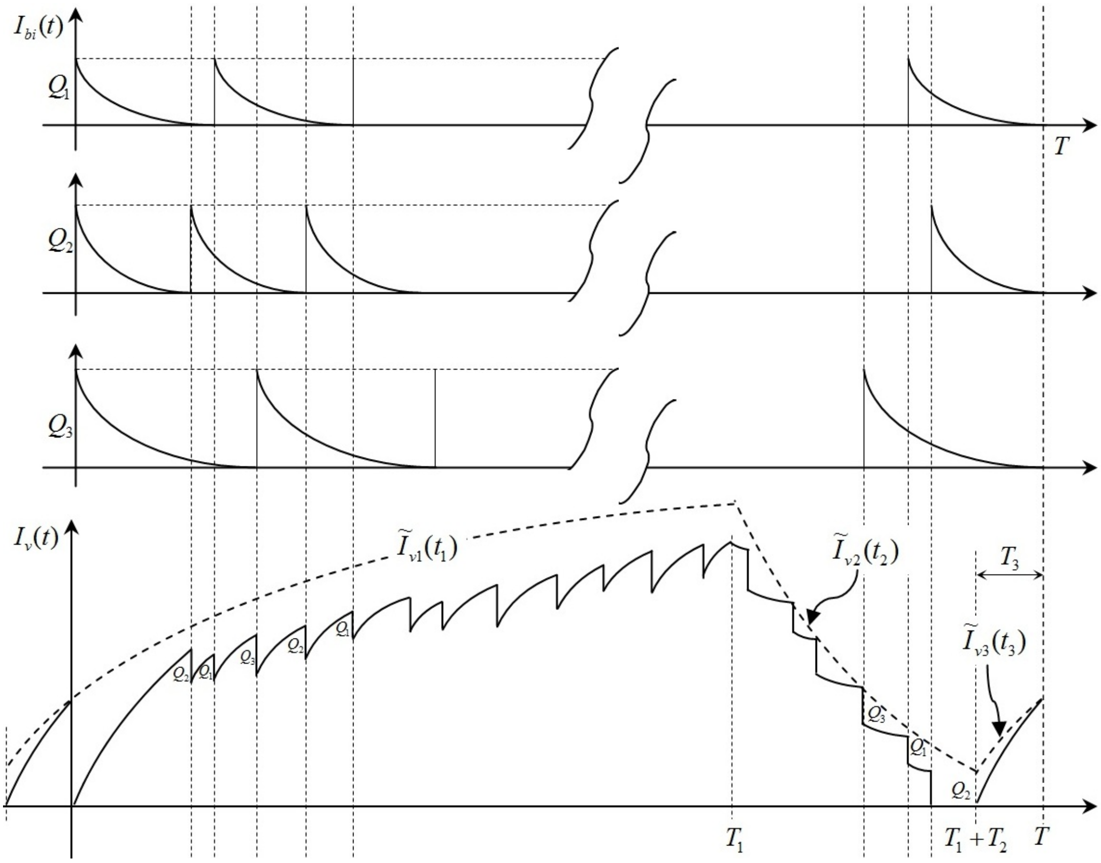

Figure 3.3: Inventory level and echelon stock of the supplier

$$\frac{d\widetilde{I}_{v1}(t_1)}{dt_1} = p - \sum_{i=1}^{N} d_i - \theta \widetilde{I}_{v1}(t_1), \qquad 0 \le t_1 \le T_1, \tag{3.7}$$

$$\frac{d\widetilde{I}_{v2}(t_2)}{dt_2} = - \sum_{i=1}^{N} d_i - \theta \widetilde{I}_{v2}(t_2), \qquad 0 \le t_2 \le T_2, \tag{3.8}$$

and

$$\frac{d\widetilde{I}_{v3}(t_3)}{dt_3} = p - \sum_{i=1}^{N} d_i - \theta \widetilde{I}_{v3}(t_3), \qquad 0 \le t_3 \le T_3. \tag{3.9}$$

In a similar way the following differential equation holds for the inventory level at buyer i:

$$\frac{dI_{bi}(t)}{dt} = -d_i - \theta I_{bi}(t), \qquad 0 \le t \le \frac{T}{n_i}, \qquad i = 1, 2, ...N. \tag{3.10}$$

The inventory level at buyer i reaches zero at T/n_i. Considering this boundary condition together with (3.10), the inventory level of buyer i is as follows:

$$\begin{aligned} I_{bi}(t) =& \frac{d_i}{\theta}\Big[\frac{exp(\frac{\theta T}{n_i}) - exp(\theta t)}{exp(\theta t)}\Big] \\ \approx& d_i\Big(\frac{T}{n_i} - t\Big)\Big[1 + \frac{\theta(\frac{T}{n_i} - t)}{2}\Big], \qquad 0 \le t \le \frac{T}{n_i}, \qquad i = 1, 2, ...N. \end{aligned} \tag{3.11}$$

By setting t equal to zero, the maximum inventory level of buyer i (optimal order quantity) is obtained:

$$I_{mi} = \frac{d_i}{\theta}\Big[exp(\frac{\theta T}{n_i}) - 1\Big], \qquad i = 1, 2, ...N. \tag{3.12}$$

Using Taylor's expansion ($e^x \approx 1 + x + x^2/2$), maximum inventory level of the buyer is:

$$I_{mi} \approx \frac{d_i T}{n_i}\left(1 + \frac{\theta T}{2n_i}\right), \qquad i = 1, 2, ...N. \tag{3.13}$$

In order to find the echelon stock level of the supplier, the relevant boundary conditions should be considered; $\widetilde{I}_{v1}(0) = \sum_{i=1}^{N} I_{mi}$, $\widetilde{I}_{v2}(T - T_1) = 0$ (this is due to the fact that in case production does not start at T_1+T_2, $\widetilde{I}_{v2}$ will reach zero at T) and $\widetilde{I}_{v3}(T-T_1-T_2) = \sum_{i=1}^{N} I_{mi}$. The results drawn from these boundary conditions are:

$$\widetilde{I}_{v1}(t_1) = \frac{p - \sum_{i=1}^{N} d_i}{\theta}(1 - exp(-\theta t_1)) + exp(-\theta t_1)\sum_{i=1}^{N} I_{mi}, \tag{3.14}$$

$$\widetilde{I}_{v2}(t_2) = \frac{\sum_{i=1}^{N} d_i}{\theta}(exp(\theta(T - T_1 - t_2)) - 1), \tag{3.15}$$

and

$$\widetilde{I}_{v3}(t_3) = \frac{p - \sum_{i=1}^{N} d_i}{\theta} + \Big(\sum_{i=1}^{N} I_{mi} - \frac{p - \sum_{i=1}^{N} d_i}{\theta}\Big)exp(\theta(T - T_1 - T_2 - t_3)). \tag{3.16}$$

By using Taylor's expansion and knowing that $\widetilde{I}_{v1}(T_1) = \widetilde{I}_{v2}(0)$, $\widetilde{I}_{v2}(T_2) = \widetilde{I}_{v3}(0)$ and $T = T_1 + T_2 + T_3$ the value of T_1, T_2 and T_3 are obtained as functions of T:

$$T_1 \approx \frac{T}{p}\left(\sum_{i=1}^{N} d_i(1 - \frac{1}{n_i}) + \frac{\theta T}{2}\sum_{i=1}^{N} d_i(1 - \frac{1}{n_i^2})\right), \tag{3.17}$$

$$T_2 \approx T - \frac{T}{p}(1 + \frac{\theta T}{2})\sum_{i=1}^{N} d_i, \tag{3.18}$$

and

$$T_3 \approx \frac{T}{p}\sum_{i=1}^{N} \frac{d_i}{n_i}\left(1 + \frac{\theta T}{2n_i}\right). \tag{3.19}$$

The annual inventory holding cost for all the buyers and the supplier are as shown in (3.20) and (3.21). In order to simplify the calculations, Taylor's expansion is used:

$$HC_b = \frac{p_b F_b}{T}\sum_{i=1}^{N} n_i \int_0^{T/n_i} I_{bi}(t)dt \approx \frac{p_b F_b T}{2}\sum_{i=1}^{N} \frac{d_i}{n_i}(1 + \frac{\theta T}{3n_i}), \tag{3.20}$$

and

$$\begin{aligned} HC_v =& \frac{p_v F_v}{T}\Big[\int_0^{T_1} \widetilde{I}_{v1}(t_1)dt_1 + \int_0^{T_2} \widetilde{I}_{v2}(t_2)dt_2 + \int_0^{T_3} \widetilde{I}_{v3}(t_3)dt_3 - \sum_{i=1}^{N} n_i \int_0^{T/n_i} I_{bi}(t)dt\Big] \\ \approx& \frac{p_v F_v}{T}\Big[\frac{p - \sum_{i=1}^{N} d_i}{2}T_1^2 + (1 - \frac{\theta T_1}{2})T_1 \sum_{i=1}^{N} I_{mi} + T_2(T_3 + \frac{T_2}{2})\sum_{i=1}^{N} d_i \\ &+ \frac{\theta T_2 T_3 (T - T_1)}{2}\sum_{i=1}^{N} d_i + (1 + \frac{\theta T_3}{2})T_3 \sum_{i=1}^{N} I_{mi} - \frac{T_3^2}{2}(1 + \frac{\theta T_3}{3})(p - \sum_{i=1}^{N} d_i) \\ &- \frac{T^2}{2}\sum_{i=1}^{N} \frac{d_i}{n_i}(1 + \frac{\theta T}{3n_i})\Big]. \end{aligned} \tag{3.21}$$

Using (3.13), the deterioration cost for all the buyers and the supplier are as follow:

$$DC_b = \sum_{i=1}^{N} \frac{n_i p_b}{T}\left(I_{mi} - \frac{T d_i}{n_i}\right) \approx \frac{p_b \theta T}{2}\sum_{i=1}^{N} \frac{d_i}{n_i}, \tag{3.22}$$

and

$$DC_v = \frac{p_v}{T}\left(p(T - T_2) - \sum_{i=1}^{N} n_i I_{mi}\right) \approx \frac{p_v \theta T}{2}\sum_{i=1}^{N} d_i \left(1 - \frac{1}{n_i}\right). \tag{3.23}$$

In a cost minimisation model it is necessary to consider these costs, however in a profit maximisation model these costs are captured as the deteriorated items incur purchasing cost while not producing any revenues.

The purchasing cost at the buyers and the supplier are as follow:

$$SC_b = \frac{C_{sb}}{T}\sum_{i=1}^{N} n_i, \tag{3.24}$$

and

$$SC_v = \frac{C_{sv}}{T}. \tag{3.25}$$

The total cost functions of all the buyers, the supplier and the whole supply chain are:

$$TC_b = HC_b + DC_b + SC_b, \tag{3.26}$$

$$TC_v = HC_v + DC_b + SC_v, \tag{3.27}$$

and

$$TC = TC_b + TC_v, \tag{3.28}$$

respectively.

In order to optimise this supply chain the following non-linear program should be solved:

$$\begin{array}{ll} Min & TC(T, n_i) \\ Subject\ \ to & n_i \in \{1, 2, 3, \ldots\}, \ \ for\ \ i \in \{1, 2, \ldots, N\}. \end{array} \tag{3.29}$$

As the number of inventory periods of the buyers within one supplier's inventory period cannot be a large number, enumeration is suggested. In case the number of retailers, N, is large, enumeration may take a long time to find a reasonable solution. With this regard a heuristic similar to Yang and Wee (2002) is developed which can solve the problem when N is relatively large:

Step 1. Assume $n_i = n$ for $i = 1, 2, ..., N$. For a range of values assigned to n (enumeration), find the optimal value for T. As shown in Appendix D, the second derivative of the total cost function with respect to T is positive. Therefore by assuming n to be constant, there is one global optimum for T;

Step 2. In the range of values assigned to n, find the optimal value which results in the lowest cost, and denote this value as n^*;

Step 3. Set all n_i values equal to n^* (n_i values are candidates for the optimal solution);

Step 4. For retailer j, having fixed n_i $(i \neq j)$, find the corrected value of n_j, $j = 1, 2, \dot{;} N$ which satisfies the following inequalities:

$TC(n_j, n_i) \leq TC(n_j - 1, n_i) \qquad \& \qquad TC(nj, n_i) \leq TC(n_1^{\xi} + 1, n_i^{\xi});$

Step 5. If in Steps 4 all n_i values remain unchanged, then $n_i^* = n_i$ and go to Step 6, otherwise repeat Steps 4;

Step 6. Stop.

3.3 Numerical examples and analysis

Example 1. In this part the same example as in Yang and Wee (2002) is considered in which $N = 2$, $p = 20 \times 10^5$, $d_1 = 4 \times 10^4$, $d_2 = 8 \times 10^4$, $F_v = 0.15$, $F_b = 0.17$, $C_{sb} = 200$. $C_{sv} = 5000$, $p_v = 10$, $p_b = 12$, and $\theta = 0.1$. The results of the example are presented in Table 3.1. It should be noted that $T_1 + T_3$ represents the production period in the supplier's inventory cycle.

Table 3.1: Optimal solution

	n_1	n_2	$T_1 + T_3$	T_2	T	TC_b	TC_v	TC
Yang and Wee (2002)	2	2	0.0115	0.1784	0.1899	22,732	38,252	60,983
Current model	2	3	0.0116	0.1784	0.1900	19,650	43,497	63,148

The optimal production-inventory policy for this supply chain is as shown in Table 3.1. As can be seen the results of Yang and Wee (2002) and the ones obtained in this research are different. This difference is due to the error in the supplier's average inventory and also the approximation used in both research works. As mentioned earlier, Yang and Wee (2002) fall short to calculate the supplier's inventory level accurately, hence the optimal solution. For the optimal solution obtained in this research (0.1900,2,3) this miscalculation (the difference between (3.1) and (3.21)) results in an error of -7.03% ($Error = 100 \times (HC_v^{YW} - HC_v)/HC_v$) in the supplier's average holding cost. As expected, ignoring T_3 in Yang and Wee (2002) results in a lower average inventory at the supplier. In this example there is a huge surplus in production capacity at the supplier (due to the deterioration the excess capacity is less than $20 \times 10^5 - (4 + 8) \times 10^4$) which normally does not take place in reality as manufacturers aim to exploit their capacity. By setting the supplier's capacity to 20×10^4 (which leaves a small excess capacity) this error goes to -58.86%. To show the error caused by this miscalculation, the optimal total cost has been calculated for different production rates using the model developed in Section 3.2 and the one in Yang and Wee (2002). The results are presented in Figure 3.4.

Table 3.2 presents the optimal solution using the current model and the model developed by Yang and Wee (2002) for two different cases; when the production rate is 20×10^4 and when the production rate extends infinitely. As can be seen the error of Yang and Wee (2002) is negligible when there is an inventory system (infinite production rate). This shows that for some cases the model of Yang and Wee (2002) results in sufficiently accurate solutions.

Table 3.2: Optimal solution

$p = 20 \times 10^4$	n_1	n_2	T_1	T_2	T_3	T	TC_b	TC_v	TC
Yang and Wee (2002)	3	4	0.2004	0.1327	0	0.3331	22,225	16,244	38,468
Current model	4	5	0.1094	0.0902	0.0299	0.2295	17,520	41,729	59,248
$p \to \infty$									
Yang and Wee (2002)	2	2	ε^*	0.1829	0	0.1829	22,187	41,127	63,314
Current model	2	2	ε^*	0.1835	ε^*	0.1835	22,230	40,985	63,215

*ε is a very small positive value

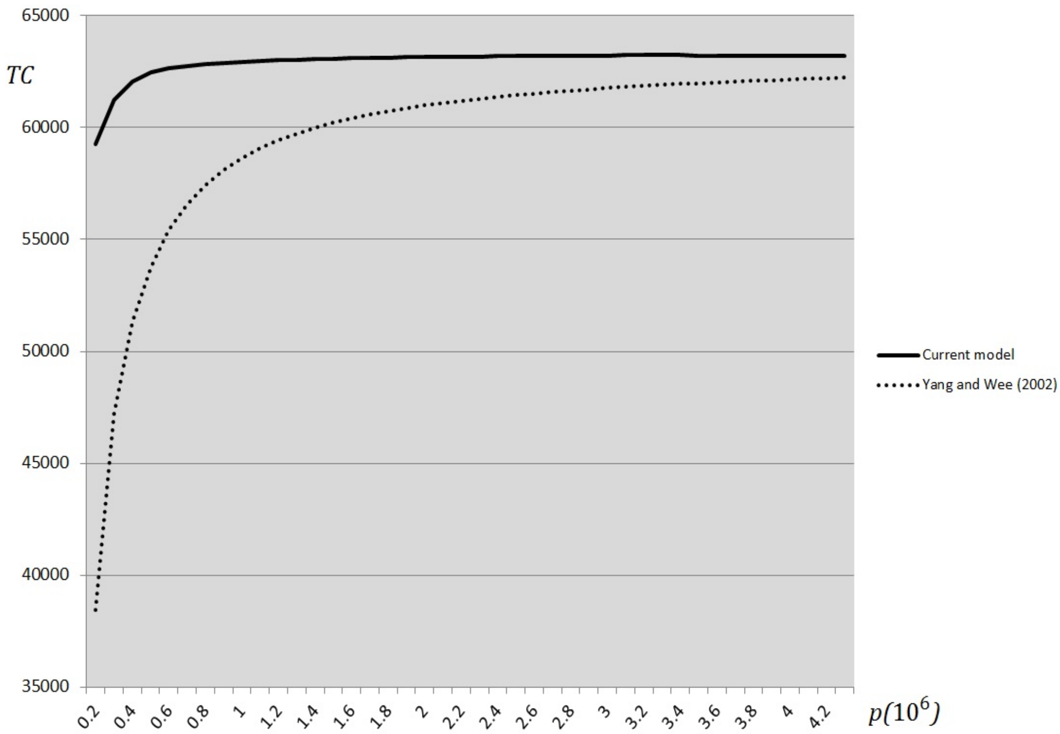

Figure 3.4: Optimal TC when p changes, current model versus Yang and Wee (2002)

Example 2. In this part the example analysed by Yan et al. (2011) is considered (for details of assumptions and notations see Appendix C.2). The data set for this is as follows: $p = 19200$, $C_{sv} = 600$, $p_v = 50$, $F_v = 6/50$, $d = 4800$, $C_{sb} = 25$, $p_b = 50$, $F_b = 7/50$, $F = 50$, $V = 1$, and $\theta = 0$. In order to model the same system as Yan et al. (2011) and include F and V in the calculations, (3.24) should be modified as follows:

$$SC_b = \frac{C_{sb} + F + V\sum_{i=1}^{N} I_{mi}}{T} \sum_{i=1}^{N} n_i. \tag{3.30}$$

Table 3.3 shows the optimal solutions of Yan et al. (2011) and also the results from the current model when the deterioration rate changes. It should be noted that the errors in the analyses done by Yan et al. (2011) take place at the buyer and the supplier in different directions therefore to some extent they cancel out in the integrated model. That is the reason of the small difference between total cost values although the optimal solutions are different.

Table 3.3: A comparison on the optimal solution of current model and Yan et al. (2011)

deterioration rate	0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
Yan et al. (2011)									
Num. of deliveries	3	3	3	3	3	3	3	3	3
Total cycle time	0.2353	0.2148	0.1990	0.1858	0.1751	0.1663	0.1587	0.1518	0.1455
Total cost	11388	12014	12589	13126	13630	14108	14562	14997	15413
Current model									
Num. of deliveries	3	3	3	3	2	2	2	2	2
Total cycle time	0.2353	0.2133	0.1965	0.1832	0.1615	0.1533	0.1463	0.1401	0.1347
Total cost	11388	12065	12684	13258	13772	14250	14705	15140	15557

In order to investigate the accuracy level of the model studied by Yan et al. (2011), first, for each optimal solution presented, the average deterioration (dS_{buy}/T and dS_{sup}/T) and the average inventory (S_{buy}/T and S_{sup}/T) at both the buyer and the supplier for different deterioration rates are calculated. These values are obtained using the model developed in Yan et al. (2011) and presented in Table 3.4. In the next step, for *each optimal solution* obtained by Yan et al. (2011), the average deterioration/inventory costs at the buyer/supplier are calculated more accurately using *current model*, (3.20)-(3.23). It should be noted that (3.20)-(3.23) represent average deterioration/inventory cost and not the level of deterioration/inventory. Table 3.4 shows that as the deterioration rate increases, the model in Yan et al. (2011) shows decrease in average inventory both at the buyer and the supplier. The current model however shows that for those optimal solutions suggested by Yan et al. (2011), the average inventory increases slightly. Similar analysis shows that the method used by Yan et al. (2011) underestimates the increase in average deterioration and hence the relevant cost. The current model however calculates these costs more accurately and suggests shorter inventory period compared to Yan et al. (2011) and also less frequency in replenishment for higher deterioration rates. Figure

3.5 illustrates the level of errors in average deterioration/inventory at the buyer/supplier when using the model developed by Yan et al. (2011).

Table 3.4: The errors in the deterioration/inventory levels at the buyer/supplier

deterioration rate	0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
Yan et al. (2011)									
Buyer:									
Ave. inventory	188.24	172.00	159.50	149.00	140.50	133.50	127.50	122.00	117.00
Ave. deterioration	0	4.30	7.97	11.17	14.05	16.69	19.15	21.35	23.40
Supplier:									
Ave. inventory	329.43	301.00	279.12	260.75	245.87	233.62	223.12	213.50	204.75
Ave. deterioration	0	7.52	13.96	19.56	24.59	29.20	33.47	37.36	40.95
Current model:									
Buyer:									
Ave. inventory	188.24	188.36	188.49	188.61	188.73	188.85	188.98	189.10	189.22
Ave. deterioration	0	4.71	9.41	14.12	18.82	23.53	28.24	32.94	37.65
Supplier:									
Ave. inventory	329.42	329.54	329.66	329.78	329.90	330.03	330.15	330.28	330.41
Ave. deterioration	0	9.41	18.82	28.24	37.65	47.06	56.47	65.88	75.30
Errors (%):									
Buyer:									
Ave. inventory	0	-8.68	-15.38	-21.00	-25.55	-29.31	-32.53	-35.48	-38.17
Ave. deterioration	NA	-8.70	-15.30	-20.89	-25.34	-29.07	-32.19	-35.18	-37.85
Supplier:									
Ave. inventory	0	-8.66	-15.33	-20.93	-25.47	-29.21	-32.42	-35.36	-38.03
Ave. deterioration	NA	-20.08	-25.82	-30.74	-34.69	-37.95	-40.73	-43.29	-45.62

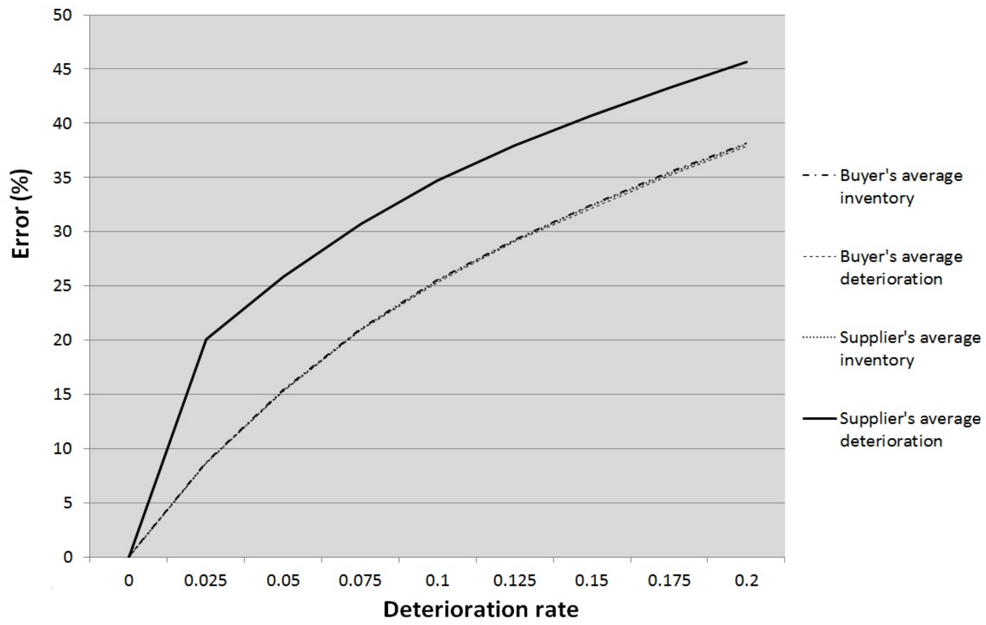

Figure 3.5: Error in the average deterioration/inventory at the buyer/supplier compared to Yan et al. (2011)

3.4 Conclusions

In this chapter a production-inventory model is developed in which a manufacturer (vendor) is delivering a perishable product to N retailers. Similar models have been addressed in the literature however they fall short to accurately calculate the supplier's inventory level for some cases as they use approximation.

In this study the inventory level of the vendor is analysed using the echelon stock concept. Also it is shown how large the errors in the vendor's holding cost can be in some cases when using the Yang and Wee (2000, 2002) method. It is also shown that such models are more accurate when the system is an inventory model or in case of a finite production rate, that the rate is large relative to the total demand rate.

Yan et al. (2011) argue that their model is a simplified version which results in fairly accurate solutions when deterioration rate is low. The model developed in this chapter calculates the exact inventory levels at the echelons while it can be used for the cases with high deterioration rates. The current model shows that the approximation method used by Yan et al. (2011) can result in large errors as it does not take into account the real influence of the deterioration rate on the inventory levels. Using differential equation seems to be the better method when a percentage of the on-hand inventory deteriorates over time.

Implementation of supply chain models remains a huge challenge and highly depends on the relations within the supply chains and the level of collaboration between the partners in the supply chain. This model can be extended in different ways in order to get closer to real-world problems. It is suggested to consider the time value of money in the model as it can have significant impact on optimality, especially when the opportunity cost of capital is high. Also studying the effects of backlogging on the optimal solutions may be interesting in case of high purchasing price which results in a high capital cost.

Chapter 4

Net Present Value Equivalence Analysis for an Inventory of a Deteriorating Item with Partial Backlogging and Finite Production Rate

Abstract

Inventory management finds its roots in the traditional operational research perspective of cost minimisation, and it is thus not unexpected that many inventory models in the NPV literature are cost-minimising models based on the traditional concepts of unit holding costs, unit backorder costs, and unit lost sales costs. The conditions under which we can be assured that these models will maximise the NPV of future profits for the firm need to be studied. Therefore we develop a new profit-maximising NPV model based on cash-flow functions for a situation of a deteriorating item that can be partially backlogged, with a more general payment structure related to backorders. We next apply the technique of NPV Equivalence Analysis (NPVEA), and derive the conditions under which equivalence holds for two cost-minimisation NPV models that characterise two different approaches found in the literature. We find that the cost parameters in these models cannot be arbitrarily chosen but are, in practical applications, interdependent. We discuss the non-triviality of the equivalence results. We present numerical results for further insights.

4.1 Introduction and literature review

The management of an inventory of deteriorating items has been widely addressed in the literature. Surveys of deterministic and stochastic models include Nahmias (1982), Raafat (1991), Goyal and Giri (2001), Li et al. (2010) and Bakker et al. (2012). Most models minimise the average costs. This chapter considers the class of models that use the Net Present Value (NPV) criterion instead. Inventory can be short and demand (partially) backlogged. In comparison to average cost models, NPV offers arguably a more accurate approach to capturing time value of money effects. It should be quite natural, when building NPV models, to maximise profits based on the relevant incoming and outgoing *cash-flows* and their timing. Many NPV models in the current literature, however, are cost-minimising models using the classic inventory concepts of *unit holding costs*, *unit backorder costs*, and *unit lost sales costs*. It is not discussed in the literature what the values of these cost parameters should be, nor if there would be any relationship between them. It is quite generally accepted that quantifying these parameters in practise is difficult (Winston, 1994).

An approach to retrieving insight into unit holding costs is through comparing models using these parameters with NPV models based on cash-flows, see Grubbström (1980), Çorbacioğlu and van der Laan (2007), and Beullens and Janssens (2011), and references therein. In Grubbström (1998), an interpretation of the unit backorder cost is also retrieved. While these results are all valid, one should interpret these findings with some care. Indeed, Beullens and Janssens (2011) illustrate that depending on the boundary conditions imposed on the NPV model, different NPV models and hence different interpretations for the classic model can be arrived at. They formalise this by introducing the Anchor Point (AP) in the supply chain. They prove that in some simple cases a classic inventory model's objective function cannot be made equivalent to optimising the NPV function of the firm, and identify which corrections are needed. Beullens and Janssens (2013) refer to the latter as a case of repairable equivalence. They further argue that since a classic model does not specify the actual payments occurring, nor their actual timing, one should in order to be fair allow its comparison with many potential *reference* NPV models. Those different reference NPV models can be arrived at by making different assumptions, about e.g. the number and types of outside firms with which the firm exchanges cash-flows, about the position of APs, and about payment structures adopted. They formalise the quest for interpretation of (classic) models through investigation of reference NPV models as NPV Equivalence Analysis (NPVEA).

We adopt NPVEA with the aim to test under which conditions the NPV-minimisation of classic cost components in a model with partial backlogging of a deteriorating item is equivalent to the NPV-maximisation of future profits in one fairly general but nevertheless specific reference model. We illustrate how equivalence provides valid interpretations of these cost-minimisation models, and gives an approach as to the calculation of their cost parameters.

Very few papers, studying deterioration with backlogging, model the problem as one of maximising an NPV function whereby revenue streams are explicitly incorporated, and they do so for obvious reasons. Wee and Law (2001) consider price-dependent demand and complete backlogging for an item with a Weibull distributed life-time. Hou (2006) develop a similar model but the demand is now also a function of inventory level; the item's life-time is negative exponentially distributed. Dye et al. (2007b) generalise Wee and Law (2001) in that the deterioration rate is a general function of time. Singh et al. (2009) consider time-dependent demand with backlogging and a capacitated warehouse OW, but any excess inventory can be stocked in another uncapacitated warehouse RW.

Having different deterioration rates in each warehouse, the optimal number of replenishments from RW to OW has to be found as well. Yang et al. (2010b) extend Hou (2006) by assuming partial backlogging, but demand is no longer a function of price. Hsieh and Dye (2010) can be viewed as an extension to Yang et al. (2010b) by considering demand to be price and time dependent. Hou and Lin (2011) develop a model somewhat similar to Hou and Lin (2006) but where demand only depends on price and shortages are not allowed; they consider trade credits and permissible delays in payments. Because of the explicit attention given to the demand, it comes very natural in these models to adopt profit maximisation under NPV.

Many models in the NPV literature are built around the minimisation of discounted classic inventory cost components. Jaggi and Aggarwal (1994) are perhaps first in modelling item deterioration with NPV, and consider three different cases with respect to the use of credit periods, and Aggarwal and Jaggi (1995) the case of permissible delays. A cluster of research grew out of these models, including Liao et al. (2000); Sarker et al. (2000); Chung and Liao (2006); Chang et al. (2010); Liao and Huang (2010); Balkhi (2011). A stream of literature deals with the OW/RW setting, see Yang (2004a, 2006 and 2012), Wee et al. (2005) and Hsieh et al. (2008). A small minority of work concerns inventory management across multiple echelons in the supply chain, see e.g. Law and Wee (2006) and Lo et al. (2007), who are also one of the few that consider item deterioration (amelioration) in a context of a *finite* production rate.

The contributions of this chapter are as follow. First, we model the situation of a *finite* production rate for a deteriorating item with partial backlogging, and compare with the case of batch deliveries (i.e. an infinite rate). The model differs fundamentally from the finite production models of Law and Wee (2006) and Lo et al. (2007) in that we do not minimise a discounted period-average, but minimise the NPV of the cash-flow function as defined in Grubbström (1980). It also models a more general payment structure related to backorders. Second, the model is also unique compared to all existing NPV models on item deterioration with backlogging in the way that we are deliberately avoiding the use of the classic inventory cost parameters of unit holding cost, unit backorder cost, and unit lost sales (or shortage) cost. It illustrates that we do not have to rely on these traditional concepts in the development of an inventory decision model with a clear interpretation. Third, we use this model as a reference to conduct an NPV Equivalence Analysis, and demonstrate how this leads to an interpretation of NPV cost models that use classic inventory cost parameters, with clear specifications on how to set the parameters in those models. As there are several variants of NPV cost minimisation models which all use the same labels for their parameters (as will be further discussed) the analysis illustrates in addition that these parameters may have to be interpreted differently.

Section 4.2 illustrates how a reference NPV model is set up, while Section 4.3 develops the corresponding mathematical equations. In Section 4.4 we investigate under which conditions two different types of cost minimisation modelling approaches from the literature produce the same results as our reference model. Equivalence is found to be subject to a set of conditions from which a useful specification of the classic cost parameters follows. Numerical examples in Section 4.5 highlight the impact of finite production rates and some other novel parameters in the model, and illustrate the gaps between the reference model and the cost minimisation model when classic cost parameters deviate from the identified equivalence conditions.

4.2 The NPV reference model

NPVEA starts from a conjecture about which reference model might produce a result of (repairable) equivalence with an existing model from the literature, and this is then verified through the equivalence analysis (see also Section 4.4). NPVEA differs from traditional (NPV) modelling of systems of inventory in that the reference model is purely based on cash-flow functions: we hence abandon the use of any classic inventory model parameters but instead start from a set of assumptions on how firms involved in the activity reward each other through payment structures superimposed on events that take place in the logistics process. There is no particular reason for preferring one reference model above another one, as long as the reference model leads to (repairable) equivalence, as otherwise there is little insight to be gained from the comparison with the classic model. For a detailed description of the principles of NPVEA, we refer to Beullens and Janssens (2013). We emphasize that our reference model, described below, is only one of many possible useful ones. Therefore, there is no guarantee that equivalence results and interpretations of parameters carry over when comparing with other reference models.

4.2.1 The activity

This is what we assume. Consider a firm involved in the activity of meeting demand for a particular type of item. For the sales price p per (unit of) item, and when the item is in stock, the demand rate y per unit of time is assumed to be a non-increasing function $y(p)$. Whenever the item is not in stock, the demand rate drops to βy, where β $(0 \leq \beta \leq 1)$ is a suitably chosen constant[1]. When the firm places an order for acquiring additional items, a production process is initiated and runs for some time, during which items are generated at a finite production rate R per unit of time. It is assumed that $R > y$; hence the production process is intermittent.

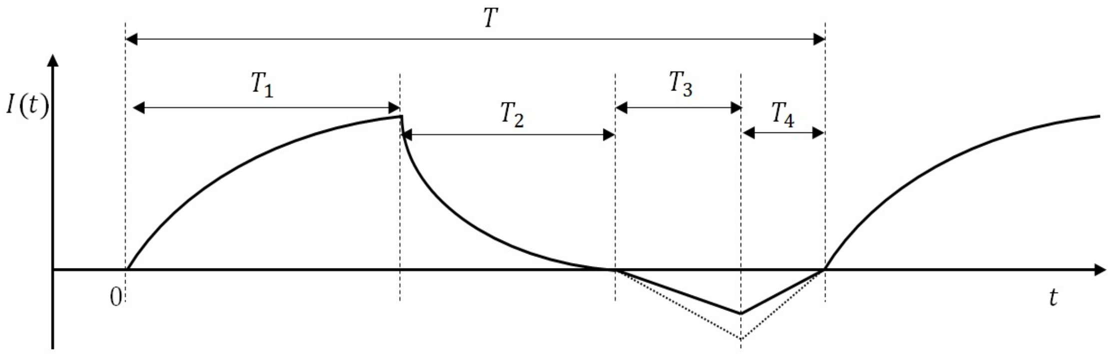

Figure 4.1: Inventory level of the system

The logistics process is assumed to be as follows; see also Figure 4.1, which displays the stock position as a function of time. The system starts at time $t = 0$ with zero inventory and with the initiation of the first production run. From this moment onwards, all future demand has to be met immediately, if possible. This can be achieved during the time of this production run, T_1, and since $R > y$, stock $I(t) > 0$ is built up in addition. At time $t = T_1$ the production stops. For some time period T_2, demand can still be met immediately from stock. At $t = T_1 + T_2$, however, stock has reduced to zero, at which point the demand rate drops to βy, and this demand is backlogged. From this moment, a negative stock position or hence a positive shortage level $B(t) = -I(t) > 0$ is built up. This situation is allowed to proceed for a length of time T_3. At time $t = T_1 + T_2 + T_3$, the production process is initiated a second time. During some length of time T_4, the demand that arisen at rate βy is instantaneously satisfied from this production process, while the excess production capacity $(R - \beta y)$ goes towards satisfying the backlogged demand (in, let's say, a 'first-in-first-satisfied' manner). Hence, while for a length of time T_3 the shortage level $B(t)$ is building up according to rate βy, it is reduced during the

[1]It is conceivable that β can take values larger than one when backorders receive reduced sales prices

length of time T_4 at rate $(R - \beta y)$. At $t = T_1 + T_2 + T_3 + T_4 \equiv T$, the inventory position is back at zero, at which point all demand that occurred in the past has been satisfied. The logistics process is hence back in a state it was in at the beginning $t = 0$ (safe the fact that production has already started), and this process repeats itself at infinitum. We call T the inventory cycle time and say that every additional T time units in the future, a new inventory cycle starts.

The items deteriorate and this process is modelled, in a deterministic fashion, as follows. Given a non-negative inventory level $I(t)$, where t measures the time from the production run initiation, the rate of deterioration is $\theta\gamma t^{\gamma-1}$, where θ and γ are two suitably chosen constants. Note that although this corresponds to the failure rate of a Weibull distribution $f(t) = \theta\gamma t^{\gamma-1}e^{-\theta t^{\gamma}}$, where θ and γ are the scale and the shape parameter, respectively, *we are not suggesting that we model that the life-times of items on the shelf are i.i.d. random variables.* In case that $\gamma = 1$, we have a constant failure rate which we can view as a characteristic of a negative exponentially distributed deterioration process. Note that as a consequence, the positive inventory level $I(t)$ from $t = 0$ to $t = T_1 + T_2$ increases/decreases in a non-linear fashion due to the item deterioration. In case that $\gamma < 1$, the failure rate becomes a decreasing function over time which may not be relevant in practice.

4.2.2 The cash-flows

It is the firm's objective to manage the inventory process as to maximise the Net Present Value of relevant future profits associated with the activity described in Section 4.2.1. For this purpose, the firm looks at the cash-flows exchanged with relevant others; these are assumed to be customers, a producer, a recycling company, and a parent company, respectively. It is this aspect which is key in NPVEA: payment structures between the firm and its outside world replace in principle all of the classic inventory cost (and revenue) parameters. The payment structure is assumed to be as follows; see also Figure 4.2.

Customers pay whenever they receive an item. For any demand that arises when the stock position is strictly positive, payment of p occurs at the moment that demand arises. Therefore there is an income at the annuity stream level py during the periods associated with T_1 and T_2. Payment is also immediate during the period associated with

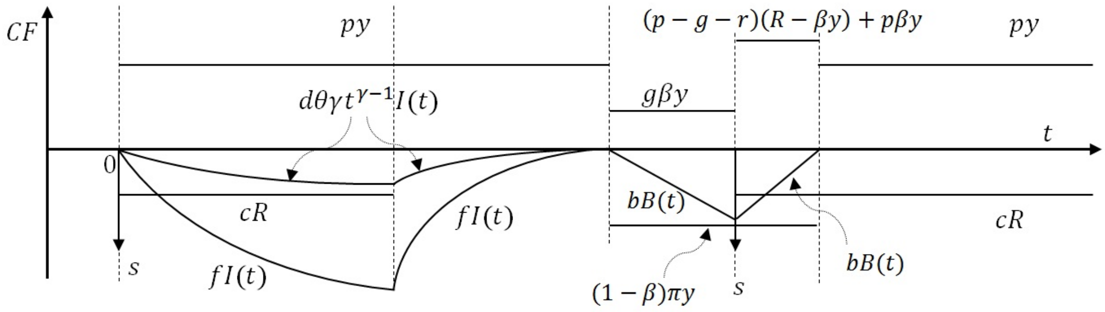

Figure 4.2: Cash-flows of the profit maximisation model

T_4 for the demand generated at rate βy during that period, and there is an income at the annuity stream level $p\beta y$. During the period of stock-out associated with T_3, demand is partially backlogged. Customers pay a deposit g the moment they raise the demand. In addition they are promised a reduction r on the sales price as a compensation for the waiting time. A customer for which a backorder of a product was created will hence pay when receiving the product the amount of $p - g - r$ only. Assume that $0 \leq g + r \leq p$, where g or r can be zero. In T_4, the backlog is reduced at the rate $(R - \beta y)$ and since customers pay $p - g - r$ per item, the annuity stream level is $(p - g - r)(R - \beta y)$.

The producer is paid the set-up cost s at the moment production initiates, and also the unit price c for every item delivered; as this happens at rate R the firm's variable cost is at the annuity stream level cR for a time spanning the period $T_1 + T_2$ in the first production run, and for a time spanning $T_4 + T_1 + T_2$ for any subsequent production run. During the period spanning $T_1 + T_2$, the items that deteriorate at any time t are immediately removed from stock and disposed of for a cost of d per unit of product, immediately paid out to the recycling company.

The firm rents shelf space from the parent company at a price f per unit of item and unit of time, paid out instantaneously. Whenever the inventory position is non-positive, a fine is paid to the parent company of π per unit of demand lost; it is paid as a continuous outflow of cash at the annuity stream level $(1-\beta)\pi y$ during the period spanning T_3+T_4. Also, the parent company will fine the firm for any inventory of backorders by asking the immediate payment of b per unit of outstanding demand and unit of time. The firm's opportunity cost of capital rate, used in calculating the NPV of cash-flows functions, is denoted by α. We consider an infinite horizon.

4.3 Mathematical model

4.3.1 Inventory and shortage levels

At the start of the inventory cycle, production accumulates inventory while deterioration and demand decrease the inventory level. This pattern starts at time zero (for the first period) until the production stops at time T_1. The following differential equation represents the inventory level in this period:

$$\frac{dI(t)}{dt} = R - y - \theta\gamma t^{\gamma-1}I(t), \qquad 0 \le t \le T_1, I(0) = 0. \tag{4.1}$$

By solving the differential equation (4.1), the inventory level in this period is obtained:

$$I(t) = (R-y)e^{-\theta t^\gamma}\int_0^t e^{\theta u^\gamma}du, \qquad 0 \le t \le T_1. \tag{4.2}$$

At time T_1 the production stops and the demand is covered using the produced items; this pattern continues until the inventory level reaches zero at $T_1 + T_2$:

$$\frac{dI(t)}{dt} = -y - \theta\gamma t^{\gamma-1}I(t), \qquad T_1 \le t \le T_1 + T_2, I(T_1+T_2) = 0. \tag{4.3}$$

The inventory level in this interval is hence as follows:

$$I(t) = ye^{-\theta t^\gamma}\left(\int_{T_1}^{T_1+T_2} e^{\theta u^\gamma}du - \int_{T_1}^{t} e^{\theta u^\gamma}du\right), \qquad T_1 \le t \le T_1 + T_2. \tag{4.4}$$

As $I(t)$ takes on a unique value at $t = T_1$, there hence holds a relation between T_1 and T_2 obtained from (4.2) and (4.4); we further consider T_1 as a function of T_2, or $T_1 = g(T_2)$, as implicitly given by this condition (for approximate solutions, see Appendix E).

During the out-of-stock period and before the production starts again, the shortage level increases as a percentage of the demand is backordered:

$$\frac{dB(t)}{dt} = \beta y, \qquad T_1 + T_2 \le t \le T - T_4, B(T_1+T_2) = 0. \tag{4.5}$$

The shortage level within this period is therefore:

$$B(t) = \beta y(t - T_1 - T_2), \qquad T_1 + T_2 \le t \le T - T_4. \tag{4.6}$$

When the production starts at $T - T_4$ as the inventory level is zero a percentage of the demand $(1-\beta)$ is lost and the rest is met with no delay. The excess production capacity is used to satisfy the backorders accumulated between $T_1 + T_2$ and $T - T_4$ and decreases the shortage level:

$$\frac{dB(t)}{dt} = -(R - \beta y), \qquad T - T_4 \le t \le T, B(T) = 0. \tag{4.7}$$

The corresponding shortage level is:

$$B(t) = (R - \beta y)(T - t), \qquad T - T_4 \le t \le T. \tag{4.8}$$

Considering the intersection point of (4.6) and (4.8) at $t = T - T_4$, the following equation is obtained:

$$T_4 = \frac{\beta y}{R - \beta y} T_3, \tag{4.9}$$

therefore

$$T = g(T_2) + T_2 + \frac{R}{R - \beta y} T_3. \tag{4.10}$$

In conclusion, we can take T_2 and T_3 as the independent decision variables for the firm since T_1, T_4, and T then follow from above relationships.

4.3.2 Annuity streams of cash-flows

As shown in Figure 4.2, a revenue at the rate of py is continuously received between 0 and $T_1 + T_2$. As the present value of this revenue for the first inventory cycle only, is given by:

$$\begin{aligned} R_1 =& py \int_0^{T_1+T_2} e^{-\alpha t} dt \\ =& \frac{py}{\alpha}(1 - e^{-\alpha(T_1+T_2)}), \end{aligned} \tag{4.11}$$

The equivalent annuity stream of all such revenues enjoyed over an infinite number of future inventory cycles, is:

$$\begin{aligned} ASR_1 =& R_1 \sum_{i=0}^{\infty} \alpha e^{-i\alpha T} \\ =& py \frac{1 - e^{-\alpha(T_1+T_2)}}{1 - e^{-\alpha T}}. \end{aligned} \tag{4.12}$$

During the interval $T_1+T_2 \leq t \leq T-T_4$, due to the out-of-stock situation, a percentage of the demand is backlogged and the customers pay a deposit g to receive their item later when the production starts again. The corresponding annuity stream of this revenue is hence:

$$\begin{aligned} ASR_2 =& g\beta y \int_{T_1+T_2}^{T-T_4} e^{-\alpha t} dt \sum_{i=0}^{\infty} \alpha e^{-i\alpha T} \\ =& g\beta y \left[\frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}} - \frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}} \right]. \end{aligned} \tag{4.13}$$

Revenues are also received in the first inventory cycle when production restarts at $T-T_4$, and until T. The first part of this revenue is to be associated with the delayed fulfilment of backlogged demand that arose in the interval $T_1+T_2 \leq t \leq T-T_4$, and as this is sold at a discount, it generates revenue at the rate $(p-g-r)(R-\beta y)$; the second part with the immediate fulfilment of demand that arises in $T-T_4 \leq t \leq T$, and thus produces revenue at rate $p\beta y$. The annuity stream of this revenue is given by:

$$\begin{aligned} ASR_3 =& \left((p-g-r)(R-\beta y) + p\beta y\right) e^{-\alpha(T-T_4)} \int_0^{T_4} e^{-\alpha t} dt \sum_{i=0}^{\infty} \alpha e^{-i\alpha T} \\ =& \left((p-g-r)(R-\beta y) + p\beta y\right) \left[1 - \frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}} \right]. \end{aligned} \tag{4.14}$$

The set-up cost of production s is incurred at $t=0$ for the first production run, and then at a time T_4 earlier relative to the start of every subsequent inventory cycle. The relevant annuity stream is as follows:

$$\begin{aligned} SC =& \alpha s (1 + e^{-\alpha(T-T_4)} + e^{-\alpha(T-T_4)} e^{-\alpha T} + e^{-\alpha(T-T_4)} e^{-2\alpha T} + ...) \\ =& \alpha s \left(1 + \frac{e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}} \right). \end{aligned} \tag{4.15}$$

The cost of production at rate cR is incurred between $nT-T_4$ and $nT+T_1$ $(n=1,2,3,...)$ except for the first period in which the production takes place between 0 and T_1. The annuity stream of the production cost is:

$$\begin{aligned} PC =& \alpha c R \int_0^{T_1} e^{-\alpha t} dt + \alpha c R e^{-\alpha(T-T_4)} (1 + e^{-\alpha T} + e^{-2\alpha T} + ...) \int_0^{T_1+T_4} e^{-\alpha t} dt \\ =& cR \left[1 + \frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}} - \frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}} \right]. \end{aligned} \tag{4.16}$$

The cost of renting warehouse space in the first period consists of two parts and considering the inventory level presented in (4.2) and (4.4), the corresponding present value of these costs are (for approximate solutions, see Appendix E):

$$HC_1 = f\int_0^{T_1} I(t)e^{-\alpha t}dt \tag{4.17}$$

and

$$HC_2 = f\int_{T_1}^{T_1+T_2} I(t)e^{-\alpha t}dt. \tag{4.18}$$

The rental cost is incurred at all periods, however, and therefore the equivalent annuity stream of all such future costs is given by:

$$HC = \frac{\alpha}{1-e^{-\alpha T}}(HC_1 + HC_2). \tag{4.19}$$

Outstanding (unsatisfied) demand arises between $T_1 + T_2$ and T, during which a backorder penalty cost b per unit of item and time has to be paid to the parent company. Using the shortage levels presented in (4.6) and (4.8), the present value of this penalty in the first inventory period is:

$$\begin{aligned} BC_1 =& b\beta y\int_{T_1+T_2}^{T-T_4}(t-T_1-T_2)e^{-\alpha t}dt \\ =& \frac{b\beta y}{\alpha}e^{-\alpha(T_1+T_2)}\left[\frac{1}{\alpha}(1-e^{-\alpha T_3}) - T_3e^{-\alpha T_3}\right] \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} BC_2 =& b(R-\beta y)\int_{T-T_4}^{T}(T-t)e^{-\alpha t}dt \\ =& \frac{b(R-\beta y)}{\alpha}e^{-\alpha(T-T_4)}\left[T_4 - \frac{1}{\alpha}(1-e^{-\alpha T_4})\right]. \end{aligned} \tag{4.21}$$

Thus the annuity stream of this cost over an infinite horizon is:

$$\begin{aligned} BC =& \frac{\alpha}{1-e^{-\alpha T}}(BC_1 + BC_2) \\ =& \frac{bR}{\alpha}\left[\frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}}\right] - \frac{b\beta y}{\alpha}\left[\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}}\right] - \frac{b}{\alpha}(R-\beta y). \end{aligned} \tag{4.22}$$

A penalty of π per unit of lost sale is also due to the parent company and with immediate effect. The annuity stream of this lost sale cost is:

$$\begin{aligned} LC =& \frac{\alpha}{1-e^{-\alpha T}} \pi y(1-\beta) \int_{T_1+T_2}^{T} e^{-\alpha t} dt \\ =& \pi y(1-\beta) \left[1 - \frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}} \right]. \end{aligned} \tag{4.23}$$

A deteriorated item incurs a net cost $d \geq 0$ for the firm to cover the disposal or recycling fees, or represents a net revenue $d < 0$ if the deteriorated item retains a salvage value that other industries are willing to pay. The annuity cost for disposal is given by:

$$DC = \frac{\alpha}{1-e^{-\alpha T}} d \int_{0}^{T_1+T_2} \theta \gamma t^{\gamma-1} I(t) e^{-\alpha t} dt. \tag{4.24}$$

Hence, the firm's annuity stream profit function to be maximised is:

$$ASP = ASR_1 + ASR_2 + ASR_3 - (SC + PC + HC + BC + LC + DC), \tag{4.25}$$

where T_2, T_3, and p are the firm's decision variables.

4.3.3 Special case

We present the special case that $\gamma = 1$ (negative exponentially distributed deterioration) for which explicit analytical solutions, to be used in Section 4.4, can be easily obtained. The shortage level equations (4.6) and (4.8) and boundary condition (4.9) remain unaltered. The inventory levels as previously given by (4.2) and (4.4) are now:

$$I(t) = \frac{(R-y)}{\theta}(1-e^{-\theta t}), \qquad 0 \leq t \leq T_1 \tag{4.26}$$

and

$$I(t) = \frac{y}{\theta}(e^{\theta(T_1+T_2-t)} - 1), \qquad T_1 \leq t \leq T_1 + T_2. \tag{4.27}$$

The intersection point of (4.26) and (4.27) at T_1 results in the following boundary condition:

$$e^{\theta T_2} = \frac{R}{y} - (\frac{R}{y} - 1)e^{-\theta T_1}. \tag{4.28}$$

The deterioration cost, previously (4.24), is now:

$$
\begin{aligned}
DC =& \frac{\alpha}{1-e^{-\alpha T}} d\theta \int_0^{T_1+T_2} I(t) e^{-\alpha t} dt \\
=& \frac{d(R-y)}{1-e^{-\alpha T}} \left[1 - e^{-\alpha T_1} - \frac{\alpha}{\alpha+\theta}(1-e^{-(\alpha+\theta)T_1}) \right] \\
&+ \frac{dy}{1-e^{-\alpha T}} \left[\frac{\alpha}{\alpha+\theta} e^{-\alpha T_1} (e^{\theta T_2} - e^{-\alpha T_2}) - e^{-\alpha T_1} + e^{-\alpha(T_1+T_2)} \right] \\
=& \frac{d\theta}{\alpha+\theta} \left[R \frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}} - y \frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}} \right].
\end{aligned}
\tag{4.29}
$$

The annuity stream of the holding cost can be found in a similar way:

$$
HC = \frac{f}{\alpha+\theta} \left[R \frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}} - y \frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}} \right]. \tag{4.30}
$$

The firm's annuity stream profit function is the sum of (4.12), (4.13), (4.14), minus the sum of (4.15), (4.16), (4.22), (4.23), (4.29), and (4.30):

$$
\begin{aligned}
ASP =& (p-c)R - (g+r)(R-\beta y) + \frac{b(R-\beta y)}{\alpha} - \pi(1-\beta)y - \alpha s \left(1 + \frac{e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}} \right) \\
&+ y \left[p - g\beta + \frac{d\theta+f}{\alpha+\theta} + \frac{b\beta}{\alpha} + \pi(1-\beta) \right] \frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}} \\
&- \left[(p-c-g-r)R + r\beta y + \frac{bR}{\alpha} \right] \frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}} - R \left[c + \frac{d\theta+f}{\alpha+\theta} \right] \frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}}.
\end{aligned}
\tag{4.31}
$$

4.4 Equivalence Analysis

Given an activity A and the set $\mathcal{A}$ of feasible scenarios $X \in \mathcal{A}$ of how to undertake it, equivalence analysis seeks to derive if, and if so under which conditions, a reference optimisation model and an existing optimisation model can agree on the optimality of a scenario. In Beullens and Janssens (2013), as the reference model was an NPV model while the existing model was an average cost model with opportunity costs, the conditions included that the linear approximation of the NPV model is sufficiently accurate to represent its optimisation function. As the existing model is now also an NPV model, taking the linear approximation is not needed. Instead, we have to compare a cost minimising NPV model with a profit maximising reference NPV model. Let $ASC(X)$ be the annuity stream function of the former model, and $ASP(X)$ that of the latter. Given the subset $X^* \subset \mathcal{A}$ for which $X^* =\arg\min_{X\in\mathcal{A}} ASC(X)$, it is to be established under which conditions $X^* =\arg\max_{X\in\mathcal{A}} ASP(X)$. If it is assumed that the reference model is a more accurate description of the 'real-world', then we can interpret the conditions derived as providing a valid interpretation of the ASC model. To keep the analysis tractable, we use the special case developed in Section 4.3.3 as the reference model.

4.4.1 Opportunity holding cost models

With the first model we aim to represent models from the literature in which the opportunity cost of investments in stock is incorporated in the unit holding cost h. Next to h, these models introduce a unit backorder cost h_b, a unit lost sales cost h_l, a unit deterioration cost h_d, and a set-up cost s, the latter having a similar meaning as in our reference model. Papers in this stream include Yang (2004a, 2006 and 2012). Note that our model is not capturing all the features of these different models, but merely follows in spirit their common modelling logic.

We construct an NPV cost minimisation model for the activity A that also has been used in the reference model, i.e. as described in Section 4.2.1 for the special case of $\gamma = 1$. Instead of following the logic of Section 4.2.2, however, the function is developed based on the consideration of the annuity stream cost of set-ups as in (4.15), and annuity-stream-like costs of holding stock, backorders, lost sales, and deterioration, using parameters h, h_b, h_l, and h_d, according to:

$$\begin{aligned} HC &= \left[\int_0^{T_1+T_2} hI(t)e^{-\alpha t}dt\right]\sum_{i=0}^{\infty}\alpha e^{-\alpha iT} \\ &= \frac{h}{\alpha+\theta}\left[R\frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}} - y\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}}\right], \end{aligned} \tag{4.32}$$

$$\begin{aligned} BC &= \left[\int_{T_1+T_2}^{T} h_b B(t)e^{-\alpha t}dt\right]\sum_{i=0}^{\infty}\alpha e^{-\alpha iT} \\ &= \frac{h_b R}{\alpha}\left[\frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}}\right] - \frac{h_b\beta y}{\alpha}\left[\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}}\right] - \frac{h_b}{\alpha}(R-\beta y), \end{aligned} \tag{4.33}$$

$$\begin{aligned} LC =& \left[\int_{T_1+T_2}^{T} h_l(1-\beta)ye^{-\alpha t}dt\right]\sum_{i=0}^{\infty}\alpha e^{-\alpha iT} \\ =& h_l y(1-\beta)\left[1-\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}}\right], \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} DC =& \left[\int_{0}^{T_1+T_2} h_d\theta I(t)e^{-\alpha t}dt\right]\sum_{i=0}^{\infty}\alpha e^{-\alpha iT} \\ =& \frac{h_d\theta}{\alpha+\theta}\left[R\frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}}-y\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}}\right]. \end{aligned} \tag{4.35}$$

Hence, the annuity stream cost function that we arrive at is as follows:

$$\begin{aligned} ASC =& h_l(1-\beta)y-\frac{h_b(R-\beta y)}{\alpha}+\alpha s\left(1+\frac{e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}}\right) \\ &-y\left[\frac{h_d\theta+h}{\alpha+\theta}+\frac{h_b\beta}{\alpha}+h_l(1-\beta)\right]\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}} \\ &+\frac{h_bR}{\alpha}\left[\frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}}\right]+\frac{R(h_d\theta+h)}{(\alpha+\theta)}\left[\frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}}\right]. \end{aligned} \tag{4.36}$$

Note that the model (4.36) captures the traditional inventory modelling approach, exemplified in its most simple form by the EOQ model (Harris, 1913), in which neither variable purchasing costs nor revenue streams are explicitly considered. We must hence for obvious reasons assume that final demand y is constant; the decision variables are the cycle times.

Theorem 1. *Sufficient conditions for equivalence are:*

$$h_d\theta+h=(d+c)\theta+\alpha c+f, \tag{4.37}$$

$$h_b=\alpha(p-c)-\alpha g-\alpha r(1-\frac{\beta y}{R})+b, \tag{4.38}$$

$$h_l=(p-c)+r\frac{\beta}{1-\beta}(1-\frac{\beta y}{R})+\pi. \tag{4.39}$$

Proof. The proof is based on a variation to the algebraic derivation method developed in Grubbström (1996), or alternatively, to the difference approach in Grubbström (1998). Let Σ be the sum of (4.31) and (4.36), i.e.:

$$\begin{aligned}\Sigma =&(p-c)R-(g+r)(R-\beta y)-\frac{1}{\alpha}(h_b-b)(R-\beta y)+(h_l-\pi)(1-\beta)y\\ &+y\left[p-g\beta+\frac{(d-h_d)\theta+f-h}{\alpha+\theta}-\frac{(h_b-b)\beta}{\alpha}-(h_l-\pi)(1-\beta)\right]\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}}\\ &-\left[(p-c-g-r)R+r\beta y-\frac{(h_b-b)R}{\alpha}\right]\frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}}\\ &-R\left[c+\frac{(d-h_d)\theta+f-h}{\alpha+\theta}\right]\frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}}.\end{aligned} \tag{4.40}$$

We now seek for the conditions under which Σ is independent of the decision variables. It is observable that this is only true when (4.37), (4.38), and (4.39) hold. Hence, under these conditions it holds for sure that $X^* \equiv \arg\min_{X\in\mathcal{A}} ASC(X) = \arg\max_{X\in\mathcal{A}} ASP(X)$. This ends the proof.

The interpretation of (4.37) to (4.39) corresponds to intuition reasonably well, as explained further below, but may still be very hard to establish if not having been explicitly derived. This hence underlines the value of NPVEA. From (4.37), we see that there is a degree of freedom of how to set h and h_d in relation to each other, but the most obvious solution would be to take $h = \alpha c + f$, as this corresponds to the classic interpretation of this parameter, see Silver et al. (1998), from which it follows that $h_d = d + c$. For every item that deteriorates, the relevant disposal cost should include the initial purchasing cost c. Note that in case that $d < 0$ (a salvage value), its absolute value typically is smaller than the cost price c, and hence $h_d > 0$, as otherwise it would be economical to purchase items for supplying a market of deteriorated items with a net marginal profit. The first component of the unit backorder cost according to (4.38) is $\alpha(p-c)$, i.e. a capital cost of deferred marginal profits. It is re-assuring that this result has previously been suggested to hold in Grubbström (1998) in a stochastic setting, as this conjecture led to the most logical explanation of the different results between a reference NPV model and a classic average cost model. As in our case both models are unapproximated NPV models, we have an exact result in the sense that it is an essential part of a set of sufficient conditions for equivalence. As our reference NPV model is slightly more complex in terms of payment structures, we have in addition to account for the effect of other parameters on h_b: the deposit g, paid in for a backordered item the moment the demand occurs, corrects (i.e. reduces) the capital loss on backordered sales; the loss in sales price by r for backordered items, because it is deferred, also reduces the capital cost of h_b

(since $1-\beta y/R > 0$); and finally, there is a cost b that is similar in interpretation as the f in h: an out-of-pocket backorder cost the moment the backordered demand occurs. Since it must be that $p-g-r-c>0$ for backordering to make economical sense, we find $h_b > 0$. The lost sales cost h_l, as given by (4.39), demonstrates the importance to account for the loss in profits through $p-c$, but shows in addition the somewhat peculiar impact from the cost r that is experienced for items that are sold with backordering. Similar counter-intuitive results also occur in the context of remanufacturing, see e.g. Çorbacioğlu and van der Laan (2007).

Note that, while sufficient, we have not proven the mathematical necessity of the derived conditions (4.37) to (4.39) for equivalence to hold for only optimal solutions X^*. The equivalence conditions derived, although mathematically perhaps too restrictive, do preserve the relative comparison of any solution to an optimum, and are hence arguably of more practical relevance.

4.4.2 Holding cost models excluding opportunity costs

This second model aims to represent the common logic adopted in models in which the timing of the purchasing/production costs are explicitly modelled. Papers in this group include Moon et al. (2005), Jaggi et al. (2006), Law and Wee (2006), Lo et al. (2007); Hsieh et al. (2008), and Chern et al. (2008). Exactly the same calculations (4.32) to (4.35) are used as in Section 4.4.1 for the costs related to parameters h, h_b, h_l, h_d, and (4.15) related to s. In addition, the annuity stream production cost is explicitly modelled as in (4.16). We again assume in our model that y is constant, and that revenue streams are not explicitly modelled.

Theorem 2. *Sufficient conditions for equivalence are:*

$$h_d\theta + h = d\theta + f, \tag{4.41}$$

$$h_b = \alpha p - \alpha g - \alpha r(1 - \frac{\beta y}{R}) + b, \tag{4.42}$$

and

$$h_l = p + r\frac{\beta}{1-\beta}(1 - \frac{\beta y}{R}) + \pi. \tag{4.43}$$

Proof. Details are omitted, as it follows the approach as before based on:

$$\begin{aligned}\Sigma =& pR - (g+r)(R-\beta y) - \frac{1}{\alpha}(h_b - b)(R-\beta y) + (h_l - \pi)(1-\beta)y \\ &+y\left[p - g\beta + \frac{(d-h_d)\theta + f - h}{\alpha+\theta} - \frac{(h_b-b)\beta}{\alpha} - (h_l-\pi)(1-\beta)\right]\frac{1-e^{-\alpha(T_1+T_2)}}{1-e^{-\alpha T}} \\ &-\left[(p-g-r)R + r\beta y - \frac{(h_b-b)R}{\alpha}\right]\frac{1-e^{-\alpha(T-T_4)}}{1-e^{-\alpha T}} \\ &-R\left[\frac{(d-h_d)\theta+f-h}{\alpha+\theta}\right]\frac{1-e^{-\alpha T_1}}{1-e^{-\alpha T}}.\end{aligned} \tag{4.44}$$

End of proof.

Interpretations can be derived as previously in Section 4.4.1. Hence from (4.41) we take $h = f$ and $h_d = d$. The opportunity cost of capital is no longer to be included into the unit holding cost, and the unit deterioration cost has only to capture the disposal cost/salvage value (hence, $h_d < 0$ is now possible). As the impact of purchasing/production is explicitly modelled, its impact should not only be excluded from h, but also from h_b and h_l. Again, it can be argued that the results seem fairly intuitive, but that it is perhaps not so obvious to produce in particular (4.42) and (4.43) purely from intuitive reasoning alone.

4.4.3 The non-triviality of the equivalence results

It should be emphasised that the equivalence results obtained are not at all trivial. Indeed, the general expression for an operation like (4.32) is:

$$\int_0^\infty hI(t)e^{-\alpha t}dt, \tag{4.45}$$

and we have just found that, for equivalence to hold, it must be that $h = \alpha c + f$. This is not conform to the common interpretation of how to calculate the NPV. Besides the intuition that an operation like (4.45) might lead to an effect of double-discounting, there are two oddities in (4.45) when compared to the common definition of NPV, namely (1) that NPV should only be based on real cash-flows, which h by virtue of containing a capital cost αc does not satisfy, and (2) that the NPV should be based on the exact timing of these real cash-flows, which again is not satisfied by (4.45). For $R \to \infty$, for example, the purchasing/production cost occurs at the start of a cycle. It is therefore surprising that these two seemingly wrong assumptions about the real cash-flow structure of the reference model are able to neutralise each other so that the *end result* is still compatible with conventional NPV thinking. We note that the use of (4.45) is not restricted to the literature on deterioration but also occurs in e.g. Moon and Lee (2000) and Yang (2010). Similarly, we should be surprised to have found that a similar operation of the unit backorder cost, as in (4.33), for which it again turns out that $h_b = \alpha(p - c) + ...$, does produce equivalence.

4.4.4 Interdependence of cost parameters

Given the established equivalence conditions for the two representative cost models, we shortly discuss to which degree such models can arbitrarily assume numerical values for their parameters h, h_d, h_b, and h_l. The good news is that, given the conditions as derived, the classic parameters can indeed be set rather independently across a wide valid range. This is particularly so in the presence of the four independent components f, d, b, and π, since by giving these arbitrary values, the interdependence imposed by the other components is weakened. This can offer a justification for the fact that many papers in these two streams of literature have assumed in their numerical evaluations of their models rather arbitrarily specified values. It is hence reassuring that in the *mathematical sense*, the degrees of freedom are not much restricted by the equivalence conditions. There are exceptions. For the model in Section 4.4.1, for example, the special case of $f = \pi = g = r = b = d = 0$, and $R \to \infty$, implies that the range of values becomes restricted by the relationships $h_d = h/\alpha$, and $h_b = \alpha h_l$.

On the other hand, if the model is to be used in a specific context, delivering a specific item to a specific market with a specific production technology, most of the right-hand-side parameters of the equivalence conditions will have to remain rather fixed. For example, varying the values of g and r may influence demand during stock-outs βy, and varying the price p may influence nominal demand y. Experimenting with different values for g and p from some status quo situation, in which their impact on the exogenously assumed β and y is not incorporated, does not lead to any useful practical insights from the cost model. It is in this *practical sense* that the equivalence conditions impose rather strict relationships between the values of the cost parameters.

4.5 Numerical examples

4.5.1 The reference model

The reference model developed in Section 4.3 is to determine the values of cycle time components T_1 to T_4, and sales price $p(y)$, as to maximise the future profits of the firm. We present a limited report on a numerical analysis to provide some basic intuition behind the impact of some parameters.

As the independent parameters are p, T_2, and T_3 only, see Section 4.3.2, we have set up a simple exhaustive search routine in three nested loops, whereby values for these parameters are incremented across a range. While not expecting this to produce the most efficient running times, typical scenarios are solved fairly quick even with small step sizes. We have compared with numerical values reported in Wee and Law (2001) by setting our parameters accordingly, and arrived at a reasonable correspondence.

We report solutions in Table 4.1 for the following data set valid for all examples: $\alpha = 0.08$; $y = 200 - 4p$; $s = 80$; $f = 0.6$; $b = 1.4$; $r = 2$; $c = 5$; $\pi = g = d = 0$; $\theta = 0.05$. The original example takes $\gamma = 1.5$ (an increasing deterioration rate over time), $\beta = 0.5$, and $R = 150$ (having checked that $y^* < 150$ in all cases.) The 7 special cases that follow change one or more values for β, γ, R, and r, as indicated. Special case 6 corresponds most closely to the model and parameter settings of Wee and Law (2001), of which their results are cited in the last row. The small difference in numerical values is thought to be due to the effect of the finite horizon in that model, whereas we have assumed an infinite horizon.

Comparing each time with the original example (unless indicated otherwise), we derive the following insights. Case 1 shows the value of introducing a shortage period when there are no lost sales: the loss of postponed revenues is not as large as the savings made on avoiding deterioration costs. Case 2 and 3 illustrate that when items deteriorate less, there may be value in increasing production cycles and this is rather insensitive to the level of lost sales. Case 4 and 5 illustrate that finite production rates are better that infinite rates (for which obviously $T_1 = T_4 = 0$) in terms of profit. Case 6 and 7 are best compared with cases 4 and 5: they illustrate the large sensitivity to r. It is much more profitable not to provide any discounts for backordered items, and if this can be done, one may have more reasons to introduce a period of backorders. This remains a 'ceteris paribus' exercise, as in practise the value of r might well influence βy (recall the discussion at the end of Section 4.4.4).

Table 4.1: The numerical results of the example presented in Section 4.5.1

	T_1	T_2	T_3	T_4	T	p	Profit (AS)
The original example	1.087	0.674	0	0	1.761	27.393	1934.429
The special case 1 $\beta = 1$	1.087	0.674	0.051	0.077	1.889	27.393	1934.912
The special case 2 $\gamma = 1$	1.200	0.751	0	0	1.951	27.353	1937.969
The special case 3 $\beta = 1$ and $\gamma = 1$	1.200	0.751	0	0	1.951	27.353	1937.969
The special case 4 $R \to \infty$	0	1.163	0	0	1.163	27.838	1883.857
The special case 5 $R \to \infty, \beta = 1$	0	1.166	0	0	1.166	27.830	1883.856
The special case 6 $R \to \infty, \beta = 1, r = 0$	0	0.993	0.416	0	1.409	27.760	1905.546
The special case 7 $R \to \infty, \beta = 1, \gamma = 1, r = 0$	0	0.998	0.422	0	1.420	27.759	1903.900
The original example in Wee and Law (2001)	NA	0.894	0.356	NA	1.250	27.734	1898.765

Table 4.2 illustrates the impact of R in more detail, using instance data almost identical to the original example of Table 4.1 (hence the almost identical results for $R = 150$ and $\beta = 1$). It illustrates our general finding (based on more extensive analysis not reported here) that lowering the production rate increases overall cycle times and boost profits. The waiting time between items produced and sold becomes relatively shorter when R decreases, and hence it is intuitive that the optimal T_1 tends to increase. The optimal value T_2 decreases, but not as drastically. The impact of β on profits remains almost negligible, at least in the range between 0.5 and 1, but planning for backorders becomes part of the optimal policy the higher its value and the lower R. It can be intuitively understood that the lower R the longer it takes to fulfil backorders, and hence that T_4 will increase. The optimal price levels p are only slightly decreasing the lower R and β. (Also, for infinite production rates, see cases 4 and 5 in Table 4.1.)

In another series of experiments (not reported here) we let the value of γ vary in the range between 1.2 and 1.6 for various settings close in range to those of Table 4.1. Items that deteriorate increasingly more over time generate less profits; the optimal cycle times decrease; the introduction of a small period during which there are backorders becomes more profitable, at least for high β values and $r \leq 2$, as the savings in deterioration and holding costs seem to cover for the losses from delayed profits.

Table 4.2: The numerical results when production rate changes

R	β	T_1	T_2	T_3	T_4	T	p	Profit (AS)
130	0.5	1.473	0.582	0	0	2.055	27.212	1945.451
	1	1.445	0.571	0.067	0.158	2.241	27.198	1946.430
140	0.5	1.241	0.630	0	0	1.871	27.300	1939.412
	1	1.241	0.630	0.059	0.109	2.039	27.300	1940.105
150	0.5	1.085	0.670	0	0	1.755	27.346	1934.603
	1	1.085	0.670	0.050	0.076	1.881	27.346	1935.079
160	0.5	0.972	0.710	0	0	1.682	27.418	1930.668
	1	0.972	0.710	0.041	0.053	1.777	27.418	1930.977
170	0.5	0.881	0.742	0	0	1.623	27.479	1927.348
	1	0.881	0.742	0.033	0.037	1.693	27.479	1927.539

4.5.2 Profit maximisation versus cost minimisation

In Section 4.4 we have parametrically obtained the settings for a cost minimisation model for constant demand, so that it will achieve the same optimal results as the reference NPV model, but only for the special case that $\gamma = 1$. We have already made the argument that deriving these relationships purely from intuition seems not trivial, but let us assume that they are known.

We test whether an intuitive setting of the parameters, based on these results, could result in near-optimal solutions for the more general setting of Weibull-based deterioration. We have to assume, however, a constant demand and price. We consider the parameter settings of the original example reported in Section 4.5.1, except that we take $p = 9$, $y = 100$, $f = 1.6$, $r = 0$, $\beta = 0.8$, and $b = 1.72$.

To do the experiment, we first use the algorithm developed for reporting results in Section 4.5.1 directly for maximising the objective function of the reference NPV model, but where we only have to search for the fixed value of p.

Subsequently, we have modified this version of the algorithm so that it works to minimise a cost NPV version of a model similar to the one given by (4.36), but properly adapted for the case of the Weibull-based deterioration model. Based on (4.37) to (4.39), we assume the following relationships $h = \alpha c + f$, $h_d = d + c$, $h_b = \alpha(p - c) + b$, and $h_l = (p - c) + \pi$. We hence guess that $h = 0.08(5) + 1.6 = 2$, $h_d = 0 + 5 = 5$, $h_b = 0.08(9 - 5) + 1.72 = 2.04$, and $h_l = 4 + 0 = 4$. The error terms $\delta_X = 100(X^*_{ASC} - X^*_{ASP})/X^*_{ASP}$ further reported measure the percentage gap of the optimal cycle time value X derived in the cost model to the optimal value of X found with the reference model, where $X \in \{T_2, T_3\}$. We first tested the Weibull algorithm for the exponential case i.e. $\gamma = 1$, ceteris paribus, in order to check that the equivalence relationships derived in Section 4.4.1 can be numerically confirmed. We indeed find a close correspondence as $|\delta_{T_2}| + |\delta_{T_3}| < 0.5\%$.

Table 4.3 and Figure 4.3 illustrate the comparison between the two models when $\gamma = 1.5$. In each scenario, we take $h_b = 2.04$, $h_d = 5$, and $h_l = 4$, but as we cannot be sure of these relationships to hold in case of the Weibull failure rate, we solve for various values of h in the range from 1.6 to 2.8.

Table 4.3: Comparison between profit maximisation and cost minimisation for the case presented in Section 4.5.2

Objective function	h	T_1	T_2	T_3	T_4	T	Profit (AS)	$\delta_{T_2}(\%)$	$\delta_{T_3}(\%)$
Profit max	-	0.924	0.443	0.191	0.218	1.776	290.191	-	-
							Cost (AS)		
Cost min	1.6	1.112	0.526	0.122	0.139	1.899	100.244	19	-36
	1.8	1.041	0.495	0.154	0.176	1.866	104.655	12	-19
	2.0	0.980	0.468	0.182	0.208	1.838	108.641	6	-5
	2.2	0.927	0.444	0.208	0.238	1.817	112.261	0	9
	2.4	0.881	0.423	0.232	0.265	1.801	115.564	-5	21
	2.6	0.837	0.403	0.253	0.289	1.782	118.589	-9	32
	2.8	0.799	0.386	0.273	0.312	1.770	121.370	-13	43

As can be observed, the value for $h = 2$, given the values of the other parameters, is not optimal. Even deviating from this value $h = 2$ does not seem to produce the optimal values for T_2 or T_3 simultaneously for one setting of the parameters. The total costs obviously change, but this gives us no real information. In the absence of a reference model, it is difficult to detect the sub-optimality of the cost model, and more specifically what goes wrong, or how to correct the parameter values. After extensive computational calculations, we have found that the set of parameters that produce the closest results to the profit reference model should be $h = 2.25$, $h_d = 5$, $h_b = 1.95$, and $h_l = 4$. See also Figure 4.4. Hence, there seems to be a more complex relationship between the parameters h and h_b that what is the case for $\gamma = 1$. These numerical errors are not overly great and may in some practical contexts still be acceptable. In the absence of an equivalence result, however, one cannot be sure about how well the cost minimisation model will perform in other settings.

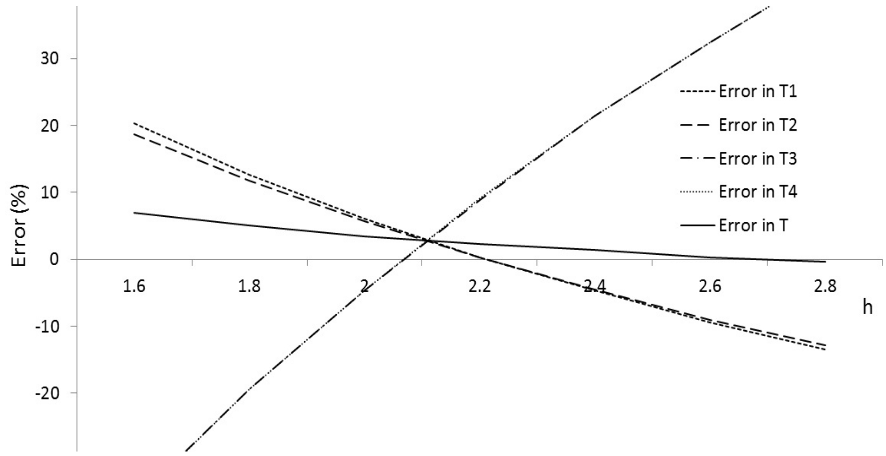

Figure 4.3: Errors in decision variables ($h_l = 4$, $h_d = 5$, $h_b = 2.04$)

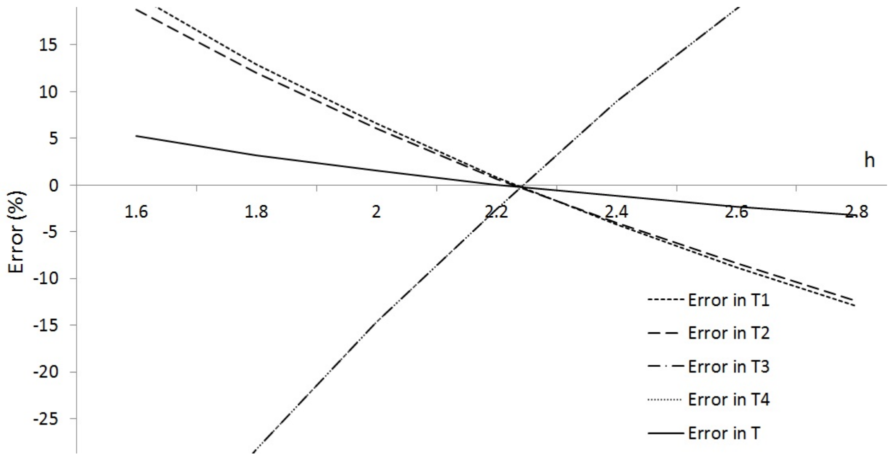

Figure 4.4: Errors in decision variables ($h_l = 4$, $h_d = 5$, $h_b = 1.95$)

4.6 Conclusions

We have presented a profit-maximising NPV model for a Weibull-type deteriorating item and partial backlogging under (in)finite production rates, incorporating various degrees of freedom in terms of payment structures related to backordered items. The model can be used as a starting point for more detailed modelling, for example, in the context of investigating how the payment structure related to backorders may influence the demand during stock-outs, and therefore optimal policies and profits.

Using the model as a reference for the special case of negative exponential deterioration ($\gamma = 1$) allowed us to interpret two strands of cost- minimising NPV models which use, at the surface, the same cost parameters. The results of this analysis are two-sided.

On the one hand, we have proven the equivalence of the basic modelling logic of these two strands of models from the literature in this particular context. It is reassuring that these models can indeed be used to maximise profits in the case of exogenous demand. The equivalence conditions derived illustrate that a practical approach to set their parameters accurately may be identified.

On the other hand, the analysis also reveals potential pitfalls related to cost minimisation under NPV. First, the two strands of models differ, but only slightly, in the settings their parameters should receive. We have argued that this seems in the end reasonably in line with intuition, but nevertheless difficult to arrive at without having conducted the equivalence analysis with a reference profit maximising model. Second, we illustrated numerically that, even when an equivalence result is obtained for a simpler setting, that this may not necessarily carry over when the modelling logic is more complicated. Third, we have pointed out that the straightforward discounting of cost functions from classic inventory theory is not conform to the common understanding that net present values are to be derived from real cash-flows and their timing only. While this procedural difference with conventional NPV modelling did not seem to impact the equivalence results obtained, one cannot be sure that this approach guarantees equivalence to other reference models.

Chapter 5

A Net Present Value Model for a Two-echelon System of a Deteriorating Item with Capacity Constraints

Abstract

Warehouse capacity is one of the main constraints when modelling an inventory system in a real-world problem. Most of the models in the inventory management literature relax this assumption and consider unlimited capacity. Deteriorating item inventory literature as one of the subsets of inventory management literature is not an exception in this regard. A group of researchers modelling inventory systems for deteriorating items, assume a limit for the owned warehouse (OW) capacity and allow the system to be able to rent extra warehouse capacity (RW) if needed (two-warehouse models). Almost all the two-warehouse models in the literature developed for deteriorating items are single-echelon. In this Chapter a single-buyer, single-supplier model is developed where the capacity at the buyer (retailer) is limited. Compared to the literature, in this study a modified version for the inventory level at the OW and the RW is suggested and the effect of this modification is analysed and presented through some numerical examples.

5.1 Introduction and literature review

Deterioration first was incorporated in modelling to address a blood bank system. Blood supply can be kept in stock for certain number of days after which they should be disposed of (Goyal and Giri, 2001). This specific application has caused the emergence of deteriorating item literature as a subset of inventory management area. Ever since a great number of researchers, have considered deterioration in their modelling. In this literature different streams of research can be identified. For an overview of the literature of a deteriorating inventory see Nahmias (1982), Raafat (1991), Goyal and Giri (2001), Li et al. (2010) and Bakker et al. (2012).

In the literature of deteriorating items, mainly models include a single-echelon system of which the total cost is minimised. This study considers the subset of literature where the time value of money is important. Therefore a Net Present Value (NPV) approach is adopted in which all the cash-flows that will take place in future are discounted to the present time.

In many inventory models (for deteriorating products) warehouses are assumed to have unlimited capacity. There are a number of factors which influence the decision model, and hence the optimal inventory policy, when this assumption is relaxed. The literature introduces a new structure named *two-warehouse model* to address this assumption. Researchers argue that it may be beneficial for the retailer to use (rent) an additional warehouse (RW), or warehouse space, as this allows purchasing products in larger batches than the capacity of the retailer's owned warehouse (OW). Therefore, the capacity of the warehouse could play an important role in making trade-offs in these two-warehouse models. Yang (2004b) develops a two-warehouse model for an item with exponential life-time distribution with complete backlogging in the case of shortages. Later, Yang (2006) extends Yang (2004b) by assuming partial backlogging. Wee et al. (2005) study a model with varying rate of deterioration and partial backlogging. Jaggi et al. (2011) investigate a two-warehouse model with partial backlogging and time-dependent demand. Yang (2012) considers a model similar to that in Yang (2006) but where the deterioration rate is based on a three-parameter Weibull distribution. Yang and Chang (2013) extend Yang (2006) by assuming permissible delays in payments.

Law and Wee (2006) analyse a production-inventory model for an item with Weibull distributed life-time in which shortages are partially backlogged. This model is developed considering livestock as the raw material and hence the factor of amelioration is included. Lo et al. (2007) develop a comparable model with time-dependent deterioration rate and no amelioration.

To date, very few studies on deteriorating inventory consider multi-echelon supply chains which indicates that studying such models and the influence of the integration within the supply chains need more attention by researchers.

In all the above-mentioned two-warehouse models the goal is to minimise the average of the discounted cost function of the first inventory cycle. This objective function, however, is hard to interpret as it is not clear what exactly it captures and how well this represents the performance of the system. A group of researchers improved this objective function by minimising (maximising) the present value of total cost (profit) when developing two-warehouse models (see Dey et al., 2008; Hsieh et al., 2008; Singh et al., 2009). These studies are however limited to the models with single-echelon structure. It is noted that the very few research works on two-warehouse multi-echelon models take a cost minimisation approach using the classic inventory modelling parameters of (1) unit holding cost, (2) unit lost sale cost, and (3) unit backorder cost. As Ghiami and Beullens (2013) argue (see also Chapter 4), taking a profit maximisation approach based on the cash-flow structure in a relevant reference NPV model is safer, unless researchers can reasonably justify the equivalence between the cost minimisation model they adopt and the profit maximisation reference model, by appropriately choosing the values of cost parameters (unit holding, lost sale, backorder and deterioration cost). This is while none of the above-mentioned cost minimisation models give a clear explanation on how they assign values to these cost parameters.

In the literature on two-warehouse models, researchers assume that the retailer uses the inventory stored at the RW to meet the demand. This implies that the RW is fairly close to the retailer's market or else, that the cost of delivering from the RW to the OW is negligible. This assumption has been made by Dey et al. (2008) in a single-echelon model, however, the researchers do not bring any discussion on how this assumption could affect the results. Most of researchers assume that both the deterioration rate and the (out-of-pocket) holding cost at the RW are higher than the ones at the OW. Therefore, it seems reasonable to replace what is deteriorated at the OW by fresh items from the RW. This in fact should not be costly due to the implicit assumption in the literature that considers the two warehouses to be relatively close. This replenishment policy keeps the OW inventory level at W that could influence the optimal solution, in particular when the demand is stock dependent.

The contributions of this research work are as follow. Firstly, an integrated inventory model is developed in which due to limits in the warehouse capacity the retailer has the option to use a second warehouse. This is while all the two-warehouse NPV models in the existing literature have a single-echelon structure. The demand is stock-dependent and in case of shortages, unmet demand is partially backlogged. Secondly, the objective is to maximise the annuity stream of the profit function derived from cash-flows. These cash-flows are themselves functions of the payment structures agreed between the firms involved in the logistics process. In the literature in contrast, most researchers minimise cost function using the classic unit holding, lost sale, and backorder cost parameters. This objective function is studied further to illustrate the effects of the integration on this supply chain. Finally, the inventory level at the OW and the RW are modified to a more practical manner and the influence of this change on the optimality is studied.

Section 5.2.2 presents the model assumptions and the cash-flow structure between the retailer and the wholesaler. In Section 5.3 the mathematical model of the integrated system is analysed and the effects of the capacity constraints together with the supply chain property are studied. To investigate how the modified inventory level influences the whole integrated system, in Section 5.4 a similar model is developed in which the inventory level assumptions of the OW and the RW are based on the literature. Some numerical examples are presented in Section 5.5 which are followed with a discussion and conclusion in Section 5.6.

5.2 The NPV reference model

5.2.1 The activity

In this integrated system, a retailer cooperates with its upstream supplier to deliver a deteriorating product to the final customer(s). Figure 5.1 illustrates how the stock position changes at the retailer over time. The demand at the retailer is the function $D(t) = y + zI_o(t)$, where y and z are constants and $I_o(t)$ represents the inventory level at the OW. The retailer places an order to the supplier and receives the batch promptly. The retailer stores W items of the received quantity at time $t = 0$ in the OW and uses the RW for the excess inventory. From $t = 0$ to $t = t_r$ the demand takes place at the rate $D(t) = y + zW$ as the OW is full. At time t_r when the RW inventory level depletes completely, the retailer starts depleting the OW to meet the demand. At $t = t_o$ the OW runs out of inventory and hence the demand rate drops to y and stays at this level until the next replenishment ($t = T_R$). During this shortage period, only a percentage of the demand (β) is backlogged while the rest is lost. On the arrival of the next batch, the backlogged demand is met immediately. The described inventory position between 0 and T_R at the retailer takes place over next intervals of length T_R at infinitum.

Each inventory cycle at the wholesaler is to cover k inventory cycles of the retailer ($T_W = kT_R$). The wholesaler receives the first batch at $t = 0$ and immediately dispatches the first order quantity ($Q_R - \beta y t_s$) to the retailer and stores the rest of the items. As at the end of the first period there will be some backlogged demand at the retailer, the order quantity for the second period and the following periods become Q_R. This results in drops of Q_R at the wholesaler inventory level in T_R intervals. Between each two deliveries to the retailer, the inventory level at the wholesaler decreases due to a constant rate of deterioration (θ). This pattern continues until the inventory level reaches zero at $t = (k-1)T_R$. At time $t = kT_R$ the wholesaler replenishes its inventory and immediately sends the next batch (Q_R) to the retailer. At this point, the inventory

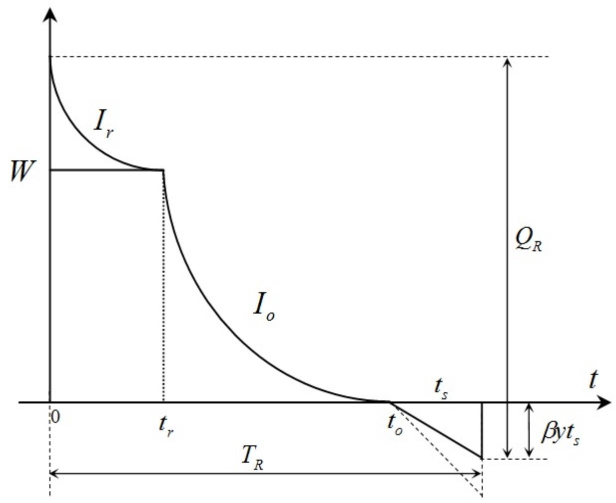

Figure 5.1: Inventory level at the retailer

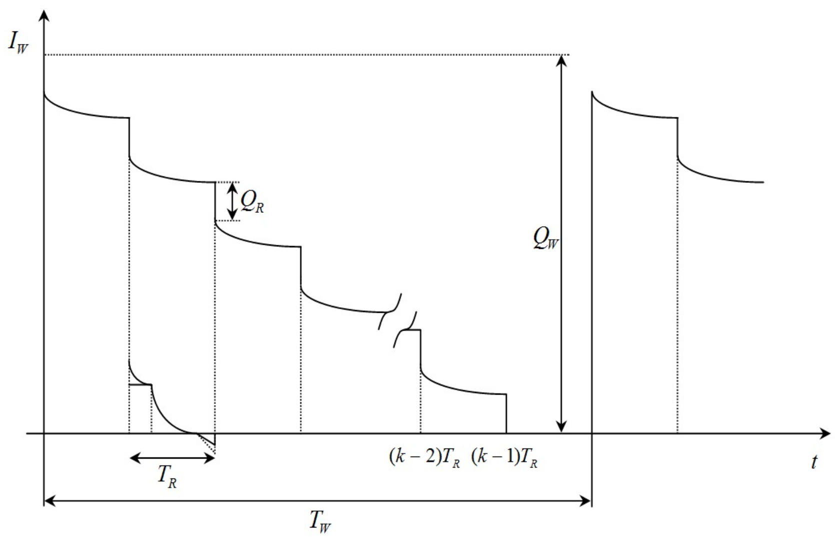

Figure 5.2: Inventory level at the wholesaler

position is exactly as it was at $t = 0$. This logistics pattern repeats itself indefinitely. Figure 5.2 depicts the change of the inventory level at the wholesaler.

5.2.2 The cash-flows

The objective of this integrated supply chain is to maximise the NPV of all future cash-flows. With this regard, the payment structure of the whole supply chain is considered. It is assumed the transfer prices which take place between the two members of this two-echelon model do not affect the optimal solution of the integrated system as the relevant assumptions hold, see Beullens and Janssens (2013). For comparison purposes, however, the transfer prices are considered to evaluate the performance of the both firms independently. The payment structures of the retailer and the wholesaler are illustrated in Figures 5.3 and 5.4. Figure 5.5 shows the payment structure of the whole system.

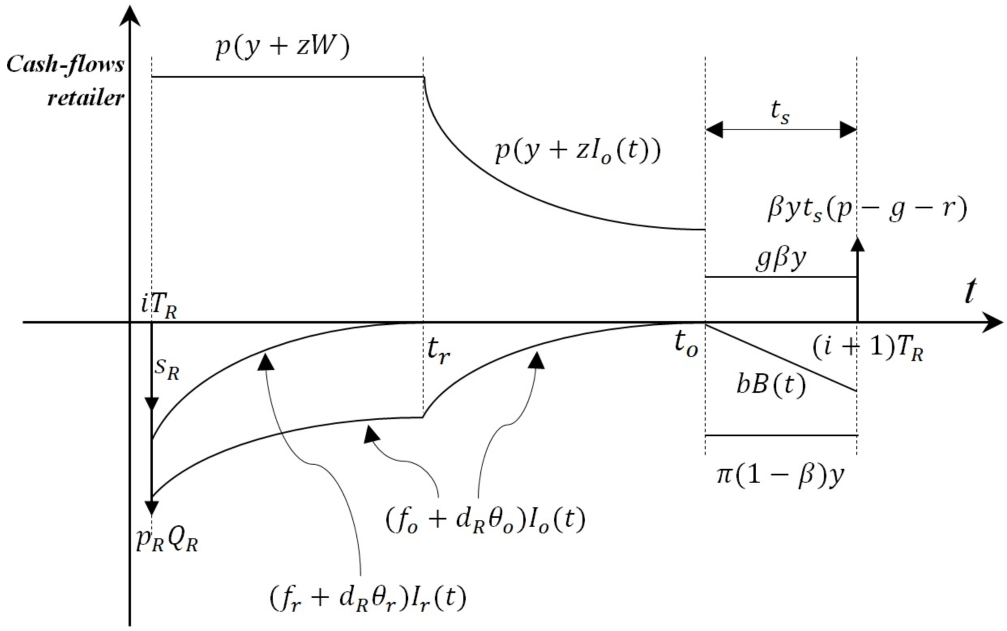

Figure 5.3: Payment structure at the retailer

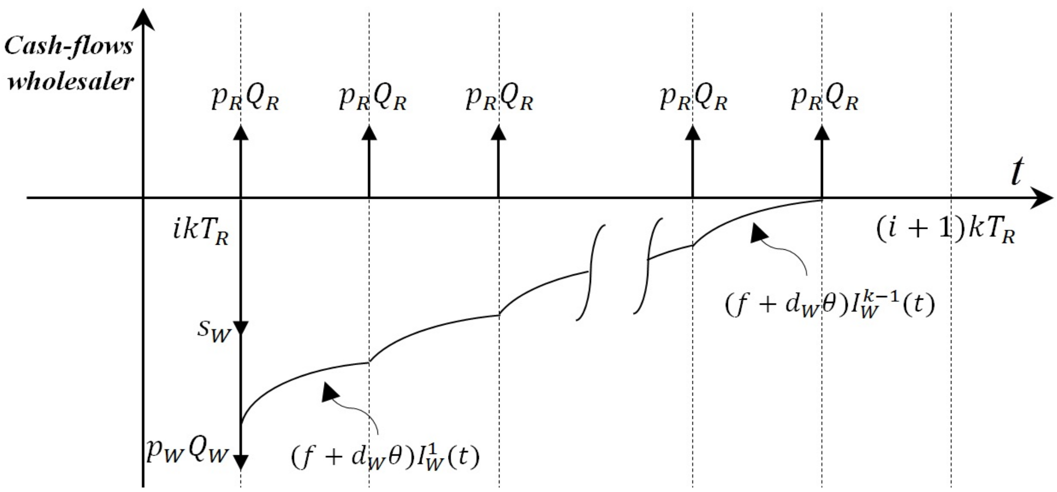

Figure 5.4: Payment structure at the wholesaler

It is assumed that when the demand arises and the retailer's stock position is strictly positive the relevant revenue is immediately enjoyed by the retailer. As the inventory level at the OW is at maximum during the period associated with t_r, the revenue over this period is at the annuity stream level $p(y + zW)$. During the time interval between t_r and t_o the revenue at the retailer is a continuous function of the inventory level, $p(y + zI_o(t))$. Shortage period starts at $t = t_o$ after which the retailer receives a deposit $g \geq 0$ for each unit of the demand which is backlogged. This makes a revenue at the annuity stream level $g\beta y$ between t_o and T_R. At the end of an inventory period when the retailer replenishes its inventory level, the backlogged demand is met immediately and the retailer gives a reduction of $r \geq 0$ per unit of backordered item which creates a revenue of $\beta y t_s(p - g - r)$.

The retailer incurs a set-up cost s_R at the beginning of each inventory period when placing an order to the wholesaler. At the same time the retailer should pay the purchasing price, $p_R Q_R$ (except for the first inventory period where it is $p_R(Q_R - \beta y t_s)$), to the wholesaler. The retailer pays an out-of-pocket holding cost of f_o and f_r per unit of time per unit of item at the OW and the RW respectively. Each deteriorated item at the OW and the RW creates an instantaneous out-of-pocket cost of d_R which is paid to a recycling company to dispose of. During this shortage period the retailer incurs backorder cost b per item per unit of time. The retailer also has to pay a penalty of π for each unit of lost sale which creates an annuity stream cost of $\pi(1 - \beta)y$ during the stock-out period.

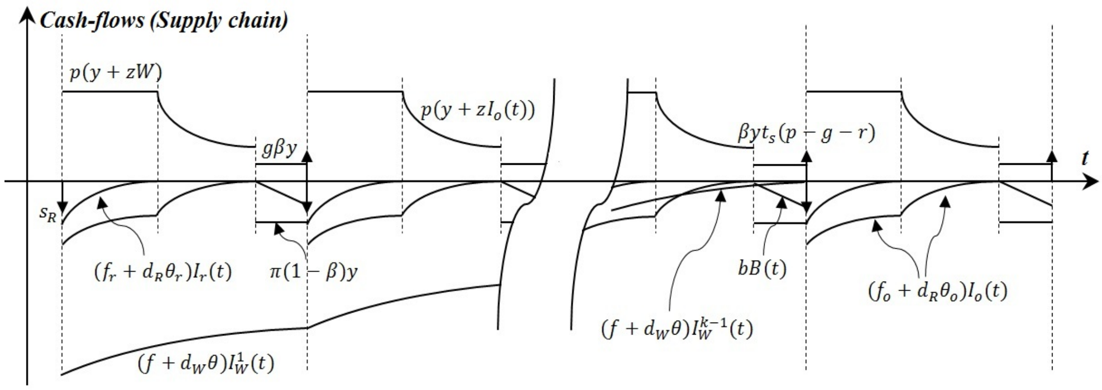

Figure 5.5: Payment structure of the supply chain

The wholesaler receives lump sum revenues of $p_R Q_R$ in T_R intervals, except for the first revenue at $t = 0$ where it is $p_R(Q_R - \beta y t_s)$ only. At the start of each inventory period i.e. at times $t = iT_W$, the wholesaler incurs a fixed set-up cost of s_W to place an order to the upstream supplier. The wholesaler then also pays the purchasing cost proportional to the order quantity Q_W to the external supplier at $t = iT_W$. There is an out-of-pocket holding cost for stock at the wholesaler, f, which should be paid per item per unit of time. During the in-stock period, each deteriorated item creates a disposal cost of d_W for the wholesaler that should be paid to a recycling company.

5.3 Mathematical model

5.3.1 Inventory and shortage levels at the retailer

In this research is is assumed that the demand is a function of on-hand inventory. The following shows the demand function:

$$D(t) = y + zI_o(t). \tag{5.1}$$

As the demand is a function of inventory level at the OW, therefore the retailer should aim to keep the inventory level of the OW at maximum W. To do so, the retailer replaces deteriorated items at the OW with fresh ones from the RW. Other factors which decrease the inventory level at the RW are deterioration at the RW and the demand. This inventory level finally reaches zero at $t = t_r$. This boundary condition, $I_r(t_r) = 0$, is used to find the inventory level. The following differential equation shows the changes of the inventory level at the RW:

$$\frac{dI_r(t)}{dt} = -(zW + y) - \theta_r I_r(t) - \theta_o W, \qquad 0 \le t \le t_r. \tag{5.2}$$

By solving the differential equation presented in (5.2), the inventory level of the RW during this interval is obtained:

$$I_r(t) = \frac{(z + \theta_o)W + y}{\theta_r}(e^{\theta_r(t_r - t)} - 1), \qquad 0 \le t \le t_r. \tag{5.3}$$

There is no change in the level of inventory at the OW between $t = 0$ and $t = t_r$ ($dI_o(t)/dt = 0$) with the initial inventory level of W, hence the inventory level is:

$$I_o(t) = W, \qquad 0 \le t \le t_r. \tag{5.4}$$

At $t = t_r$ the inventory level at the RW is depleted totally and the retailer starts using the items at the OW. The following differential equation shows how this inventory level changes due to the demand and the deterioration until it reaches zero at $t = t_o$:

$$\frac{dI_o(t)}{dt} = -zI_o(t) - y - \theta_o I_o(t), \qquad t_r \le t \le t_o, \tag{5.5}$$

therefore

$$I_o(t) = \frac{y}{z+\theta_o}(e^{(z+\theta_o)(t_o-t)} - 1), \qquad t_r \le t \le t_o. \tag{5.6}$$

The shortage period starts at $t = t_o$ when the retailer runs out of inventory at the OW and continues to the end of the cycle $t = T_R$, during which only a percentage, β, of the demand is backordered. The shortage level is presented by the following differential equation:

$$\frac{dB(t)}{dt} = \beta y, \qquad t_o \le t \le T_R. \tag{5.7}$$

Considering the relevant boundary condition, $B(t_o) = 0$, the shortage level is:

$$B(t) = \beta y(t - t_o), \qquad t_o \le t \le T_R. \tag{5.8}$$

$I_o(t)$ has a unique value at $t = t_r$ and therefore can be obtained using both (5.4) and (5.6):

$$t_o = t_r + \frac{1}{z+\theta_o} ln\left(1 + \frac{z+\theta_o}{y}W\right). \tag{5.9}$$

Having t_o as a function of t_r, the inventory period at the retailer is as follows:

$$T_R = t_r + \frac{1}{z+\theta_o} ln(1 + \frac{z+\theta_o}{y}W) + t_s. \tag{5.10}$$

The retailer's batch size for the second period onward is the sum of the backordered items and the initial inventory level:

$$\begin{aligned} Q_R =& I_r(0) + I_o(0) + B(T_R) \\ =& \frac{(z+\theta_o)W + y}{\theta_r}(e^{\theta_r t_r} - 1) + W + \beta y t_s. \end{aligned} \tag{5.11}$$

5.3.2 Inventory level at the wholesaler

The wholesaler covers k inventory cycles of the retailer during each of its inventory periods T_W. This divides the inventory cycle at the wholesaler into k intervals of length T_R. The stock level at the wholesaler drops by Q_R at the beginning of each of these intervals when a batch is sent to the retailer. During each interval the inventory level at the wholesaler is depleted due to deterioration. It is clear that after sending a batch to the retailer to cover the k^{th} interval at the retailer, it is optimal that the wholesaler does not keep any items in stock until receiving the next order from its upstream supplier.

This means that the inventory level at the wholesaler is zero in the interval between $t = (k-1)T_R$ and $t = kT_R$. The following differential equation shows the change in the stock level over interval i:

$$\frac{dI_W^i(t)}{dt} = -\theta I_W^i(t), \qquad i = 1, 2, \ldots k-1. \tag{5.12}$$

The inventory level at the wholesaler at $t = (k-1)T_R$, just before sending the last batch to the retailer, is Q_R. Considering this boundary condition the inventory level at the wholesaler between $(k-2)T_R$ and $(k-1)T_R$ is given by:

$$I_W^{k-1}(t) = Q_R e^{\theta[(k-1)T_R - t]}, \qquad (k-2)T_R \leq t \leq (k-1)T_R. \tag{5.13}$$

According to (5.13), the inventory level at the wholesaler at $t = (k-2)T_R$ just before sending a batch to the retailer is $Q_R(e^{\theta T_R} + 1)$. Using this inventory level as a boundary condition, the inventory level of $(k-2)^{th}$ interval is:

$$I_W^{k-2}(t) = Q_R(e^{\theta T_R} + 1)e^{\theta[(k-2)T_R - t]}, \qquad (k-3)T_R \leq t \leq (k-2)T_R. \tag{5.14}$$

The inventory level at the wholesaler during i^{th} interval hence is given by:

$$\begin{aligned} I_W^i(t) =& Q_R e^{\theta(iT_R - t)} \sum_{m=0}^{k-i-1} e^{m\theta T_R} \\ =& Q_R e^{\theta(iT_R - t)} \frac{e^{\theta(k-i)T_R} - 1}{e^{\theta T_R} - 1}, \qquad (i-1)T_R \leq t \leq iT_R, \qquad i = 1, 2, ..., k-1. \end{aligned} \tag{5.15}$$

Using (5.15), one can find the inventory level at the wholesaler at $t = 0$ just after sending the first batch to the retailer:

$$I_W^1(0) = Q_R \frac{e^{k\theta T_R} - e^{\theta T_R}}{e^{\theta T_R} - 1}, \tag{5.16}$$

and therefore, the wholesaler order quantity is given by $I_W^1(0) + Q_R$, hence:

$$Q_W = Q_R \frac{e^{k\theta T_R} - 1}{e^{\theta T_R} - 1}. \tag{5.17}$$

In the next two sections, the relevant revenues and costs of the both members are analysed.

5.3.3 Revenues and costs at the retailer

The continuous revenue enjoyed by the retailer during the time interval between $t = 0$ and $t = t_r$ at the beginning of all inventory cycles is $p(y + zW)$. The equivalent annuity stream of all such revenues over an infinite horizon is:

$$ASR_{R1} = p(y + zW)\frac{1 - e^{-\alpha t_r}}{1 - e^{-\alpha T_R}}. \tag{5.18}$$

The revenue function of the retailer changes to $p(y + zI_o(t))$ between t_r and t_o. The equivalent annuity stream of these revenues over an infinite horizon is hence given by:

$$\begin{aligned}ASR_{R2} =& \frac{\alpha}{1 - e^{-\alpha T_R}} \int_{t_r}^{t_o} p(y + zI_o(t))e^{-\alpha t}dt \\ =& pye^{-\alpha t_o}\frac{e^{\alpha(t_o - t_r)} - 1}{1 - e^{-\alpha T_R}} \\ &+ \frac{\alpha pyze^{-\alpha t_o}}{(z + \theta_o)(1 - e^{-\alpha T_R})}\left[\frac{1}{\alpha + z + \theta_o}(e^{(\alpha+z+\theta_o)(t_o-t_r)} - 1) - \frac{1}{\alpha}(e^{\alpha(t_o-t_r)} - 1)\right].\end{aligned} \tag{5.19}$$

In each inventory cycle at the retailer over a shortage period (between $t = t_o$ and $t = T_R$), the retailer receives a deposit of g for each backordered item which creates an annuity revenue of $g\beta y$ in that cycle. The equivalent annuity stream of revenues obtained from deposits in all cycles over an infinite horizon is:

$$ASR_{R3} = g\beta y\frac{e^{-\alpha t_o} - e^{-\alpha T_R}}{1 - e^{-\alpha T_R}}. \tag{5.20}$$

Just after the inventory has been replenished at the retailer, the backordered demand is met immediately, creating a lump sum revenue of $(p - g - r)\beta yt_s$ at the end of each inventory cycle. The equivalent annuity stream of all these lump sum revenues over an infinite horizon is as follows:

$$ASR_{R4} = (p - g - r)\beta yt_s\frac{\alpha e^{-\alpha T_R}}{1 - e^{-\alpha T_R}}. \tag{5.21}$$

The annuity stream of all revenues at the retailer over an infinite horizon is:

$$ASR_R = ASR_{R1} + ASR_{R2} + ASR_{R3} + ASR_{R4}. \tag{5.22}$$

At the start of each inventory cycle, the retailer incurs a set-up cost of s_R. The annuity stream of all set-up costs over an infinite horizon is given by:

$$SC_R = s_R \frac{\alpha}{1 - e^{-\alpha T_R}}. \tag{5.23}$$

The retailer pays the purchasing price when placing an order at the beginning of each inventory cycle. It should be noted that the order quantity of the first inventory cycle is smaller as no backordered demand exists. The annuity stream of all purchasing costs over an infinite horizon is then:

$$PC_R = p_R Q_R \frac{\alpha}{1 - e^{-\alpha T_R}} - \alpha p_R \beta y t_s. \tag{5.24}$$

The present value of out-of-pocket holding cost at the OW and the RW only for the first period are:

$$HC_o = f_o \int_0^{t_o} I_o(t) e^{-\alpha t} dt, \tag{5.25}$$

and

$$HC_r = f_r \int_0^{t_r} I_r(t) e^{-\alpha t} dt, \tag{5.26}$$

respectively.

The equivalent annuity stream of all out-of-pocket holding costs at the retailer over an infinite horizon is then given by:

$$HC_R = HC_{OW} + HC_{RW}, \tag{5.27}$$

where

$$\begin{aligned} HC_{OW} =& \frac{\alpha}{1 - e^{-\alpha T_R}} HC_o \\ =& f_o W \frac{1 - e^{-\alpha t_r}}{1 - e^{-\alpha T_R}} \\ &+ \frac{f_o y}{z + \theta_o} \left[\frac{\alpha}{z + \theta_o + \alpha} \left(\frac{e^{(z+\theta_o)(t_o - t_r) - \alpha t_r} - e^{-\alpha t_o}}{1 - e^{-\alpha T_R}} \right) - \frac{e^{-\alpha t_r} - e^{-\alpha t_o}}{1 - e^{-\alpha T_R}} \right], \end{aligned} \tag{5.28}$$

and

$$\begin{aligned} HC_{RW} =& \frac{\alpha}{1-e^{-\alpha T_R}} HC_r \\ =& \frac{f_r\left((z+\theta_o)W+y\right)}{\theta_r}\left[\frac{\alpha}{\alpha+\theta_r}\left(\frac{e^{\theta_r t_r}-1}{1-e^{-\alpha T_R}}\right)-\frac{\theta_r}{\alpha+\theta_r}\left(\frac{1-e^{-\alpha t_r}}{1-e^{-\alpha T_R}}\right)\right]. \end{aligned} \tag{5.29}$$

The retailer has to pay d_R per unit of deteriorated item to dump/recycle to a recycling company. This cost is incurred immediately after an item deteriorates. The annuity stream of the deterioration cost of all inventory cycles over an infinite horizon is given by:

$$DC_R = DC_{OW} + DC_{RW}, \tag{5.30}$$

where

$$\begin{aligned} DC_{OW} =& d_R\theta_o W \frac{1-e^{-\alpha t_r}}{1-e^{-\alpha T_R}} \\ &+ \frac{d_R\theta_o y}{z+\theta_o}\left[\frac{\alpha}{z+\theta_o+\alpha}\left(\frac{e^{(z+\theta_o)(t_o-t_r)-\alpha t_r}-e^{-\alpha t_o}}{1-e^{-\alpha T_R}}\right)-\frac{e^{-\alpha t_r}-e^{-\alpha t_o}}{1-e^{-\alpha T_R}}\right], \end{aligned} \tag{5.31}$$

and

$$DC_{RW} = d_R\left((z+\theta_o)W+y\right)\left[\frac{\alpha}{\alpha+\theta_r}\left(\frac{e^{\theta_r t_r}-1}{1-e^{-\alpha T_R}}\right)-\frac{\theta_r}{\alpha+\theta_r}\left(\frac{1-e^{-\alpha t_r}}{1-e^{-\alpha T_R}}\right)\right]. \tag{5.32}$$

During the out-of-stock period, each backordered item creates a backorder penalty cost of b per item per unit of time. The present value of this shortage cost for the first period only, is:

$$\begin{aligned} BC =& \int_{t_o}^{T_R} b\beta y(t-t_o)e^{-\alpha t}dt \\ =& \frac{b\beta y}{\alpha}\left(\frac{e^{-\alpha t_o}}{\alpha}(1-e^{-\alpha t_s})-t_s e^{-\alpha T_R}\right), \end{aligned} \tag{5.33}$$

and therefore the equivalent annuity stream of all shortage costs over an infinite horizon is given by:

$$\begin{aligned} BC_R =& \frac{\alpha}{1-e^{-\alpha T_R}} BC \\ =& b\beta y\left(\frac{e^{-\alpha t_o}}{\alpha}\left(\frac{1-e^{-\alpha t_s}}{1-e^{-\alpha T_R}}\right)-t_s\frac{e^{-\alpha T_R}}{1-e^{-\alpha T_R}}\right). \end{aligned} \tag{5.34}$$

A penalty of π per unit of lost sale is paid by the retailer which creates a cost of $\pi y(1-\beta)$ over the time interval between $t = t_o$ and $t = T_R$. The equivalent annuity stream of all lost sale costs over an infinite horizon is:

$$\begin{aligned} LC_R =& \frac{\alpha}{1-e^{-\alpha T_R}} \pi y(1-\beta) \int_{t_o}^{T_R} e^{-\alpha t} dt \\ =& \pi y(1-\beta) \left(\frac{e^{-\alpha t_o} - e^{-\alpha T_R}}{1-e^{-\alpha T_R}} \right). \end{aligned} \tag{5.35}$$

Considering the annuity streams of profit in (5.18) - (5.21) and annuity streams of cost in (5.23), (5.24), (5.27), (5.30), (5.34) and (5.35), the annuity stream profit function at the retailer is given by:

$$ASP_R = ASR_R - (SC_R + PC_R + HC_R + BC_R + LC_R + DC_R). \tag{5.36}$$

5.3.4 Revenues and costs at the wholesaler

The wholesaler receives revenues of $p_R Q_R$ at $t = iT_R$ $(i = 1, 2, 3, ...)$ associated with the batches sent to the retailer. The revenues at $t = 0$, however, is less as there are no backlogged items at the retailer yet. The equivalent annuity stream of all the revenues over an infinite horizon is given by:

$$\begin{aligned} ASR_W =& \alpha p_R (Q_R - \beta y t_s) + \alpha p_R Q_R e^{-\alpha T_R} (1 + e^{-\alpha T_R} + e^{-2\alpha T_R} + ...) \\ =& p_R Q_R \frac{\alpha}{1-e^{-\alpha T_R}} - \alpha p_R \beta y t_s. \end{aligned} \tag{5.37}$$

The set-up cost of purchasing for the wholesaler, s_W, is incurred at the beginning of each of wholesaler's inventory cycle. The equivalent annuity stream of all set-up costs paid over an infinite horizon is given by:

$$SC_W = s_W \frac{\alpha}{1-e^{-\alpha k T_R}}. \tag{5.38}$$

The wholesaler purchases the item in batches of size Q_W at the price of p_W per item. These costs are incurred at the beginning of the wholesaler's inventory cycles. The first wholesaler's batch is $Q_W - \beta y t_s$, due to there being no backorders at the retailer at $t = 0$. The corresponding annuity stream of all purchasing costs at the wholesaler is consequently given by:

$$PC_W = p_W Q_W \frac{\alpha}{1-e^{-\alpha k T_R}} - \alpha p_W \beta y t_s. \tag{5.39}$$

The inventory period at the wholesaler consists of k inventory cycles of the retailer. Considering the inventory level of i^{th} interval presented in (5.15), the present value of holding cost at the wholesaler for the first inventory cycle only, is:

$$\begin{aligned} HC_{W1} =& \sum_{i=1}^{k-1} \int_{(i-1)T_R}^{iT_R} f I_W^i(t) e^{-\alpha t} dt \\ =& \frac{f Q_R (e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)} \left[e^{\theta k T_R} \frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} - \frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{-\alpha T_R}} \right], \end{aligned} \tag{5.40}$$

and therefore the annuity stream of all holding costs at the wholesaler over an infinite horizon is:

$$HC_W = HC_{W1} \frac{\alpha}{1 - e^{-\alpha k T_R}}. \tag{5.41}$$

The deterioration cost at the wholesaler is incurred exactly at the same time as the out-of-pocket holding cost, and occurs at the rate $d_W \theta$. The annuity stream of this cost over an infinite horizon is given by:

$$\begin{aligned} DC_W =& \frac{\alpha d_W \theta Q_R (e^{(\theta+\alpha)T_R} - 1)}{(\theta+\alpha)(e^{\theta T_R} - 1)(1 - e^{-\alpha k T_R})} [e^{\theta k T_R} \frac{e^{-(\theta+\alpha)T_R} - e^{-(\theta+\alpha)kT_R}}{1 - e^{-(\theta+\alpha)T_R}} \\ & - \frac{e^{-\alpha T_R} - e^{-\alpha k T_R}}{1 - e^{-\alpha T_R}}]. \end{aligned} \tag{5.42}$$

Using the revenues and costs presented in (5.37) - (5.39), (5.41) and (5.42) the annuity stream of profit function at the wholesaler is obtained:

$$ASP_W = ASR_W - (SC_W + PC_W + HC_W + DC_W). \tag{5.43}$$

Considering the same capital rate for both the retailer and the wholesaler, the annuity stream profit function of the supply chain is:

$$\begin{aligned} ASP_{SC} =& ASP_R + ASP_W \\ =& ASR_R \\ & - (SC_R + HC_R + BC_R + LC_R + DC_R + SC_W + PC_W + HC_W + DC_W). \end{aligned} \tag{5.44}$$

5.4 A comparison with the existing literature

As discussed in Section 5.1, in the literature, two-warehouse systems are modelled in a different way in terms of replenishments between the OW and the RW. In order to see how this difference can influence the optimality, a model is presented based on the assumptions in the literature. As an example, the model developed by Ghiami et al. (2013) is considered (see Chapter 2). Figure 5.6 graphically illustrates how the inventory level at the OW and the RW change based on the assumptions in the literature.

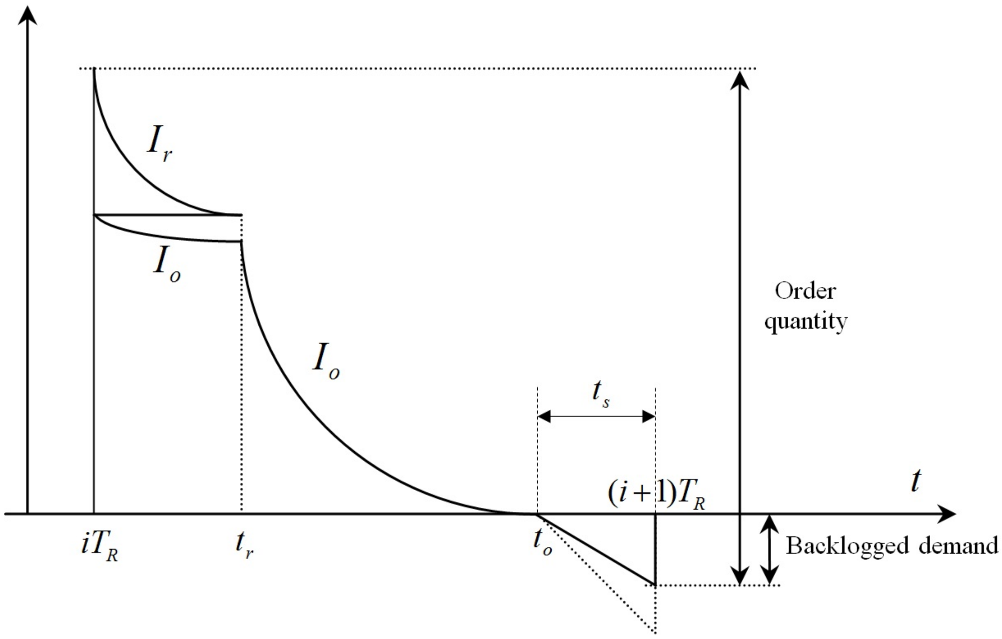

Figure 5.6: The inventory level at the RW and the OW based on the existing literature (Ghiami et al., 2013)

Each inventory period at the retailer, $T_R = t_o + t_s$, is divided into three parts, (1) the start of the inventory when the RW is in use, (2) the interval between t_r and t_o when the stock held at the OW used to meet the demand, and (3) the shortage period.

The following differential equation represent the change in the inventory level at the RW:

$$\frac{dI'_r(t)}{dt} = -\theta_r I'_r(t) - (y + zI'_o(t)), \qquad 0 \le t \le t'_r. \tag{5.45}$$

Considering the boundary condition, $I'_r(t_r) = 0$, for the differential equation presented in (5.45), the inventory level of this time interval is obtained:

$$I'_r(t) = \frac{y}{\theta_r}(e^{\theta_r(t'_r - t)} - 1) + \frac{zWe^{-\theta_o t}}{\theta_r - \theta_o}(e^{(\theta_r - \theta_o)(t'_r - t)} - 1), \qquad 0 \le t \le t'_r. \tag{5.46}$$

In the literature it is assumed that during the time that the RW is in use, the inventory level at the OW goes down due to deterioration:

$$\frac{dI'_o(t)}{dt} = -\theta_o I'_o(t), \qquad 0 \le t \le t'_r. \tag{5.47}$$

Using the boundary condition, $I'_o(0) = W$, the inventory level is obtained:

$$I'_o(t) = We^{-\theta_o t}, \qquad 0 \le t \le t'_r. \tag{5.48}$$

The change in the inventory level at the OW between $t = t'_r$ and $t = t'_o$ is as shown in the following differential equation:

$$\frac{dI'_o(t)}{dt} = -zI'_o(t) - y - \theta_o I'_o(t), \qquad t'_r \le t \le t'_o, \tag{5.49}$$

and therefore:

$$I'_o(t) = \frac{y}{z+\theta_o}(e^{(z+\theta_o)(t'_o-t)} - 1), \qquad t'_r \le t \le t'_o. \tag{5.50}$$

Considering the unique value for $I'_o(t)$ at $t = t'_r$ obtained from (5.48) and (5.50), the value of t'_o is as follows:

$$t'_o = t'_r + \frac{1}{z+\theta_o} ln\left(1 + \frac{z+\theta_o}{y} We^{-\theta_o t'_r}\right), \tag{5.51}$$

and as $T_R = t_o + t_s$, therefore:

$$T'_R = t'_r + \frac{1}{z+\theta_o} ln\left(1 + \frac{z+\theta_o}{y} We^{-\theta_o t'_r}\right) + t'_s. \tag{5.52}$$

The retailer's batch size for the second period onwards is:

$$Q'_R = \frac{y}{\theta_r}(e^{\theta_r t'_r} - 1) + \frac{zW}{\theta_r - \theta_o}(e^{(\theta_r-\theta_o)t'_r} - 1) + W + \beta y t'_s. \tag{5.53}$$

The revenue received by the retailer between $t = 0$ and $t = t'_r$ is $p(y + zI'_o(t))$. The annuity stream of all such revenues over an infinite horizon is hence given by:

$$ASR'_{R1} = py\frac{1 - e^{-\alpha t'_r}}{1 - e^{-\alpha T'_R}} + \frac{\alpha pzW}{\theta_o + \alpha}\left(\frac{1 - e^{-(\theta_o+\alpha)t'_r}}{1 - e^{-\alpha T'_R}}\right). \tag{5.54}$$

The other revenue terms for the retailer are as presented in (5.19) - (5.21):

$$\begin{aligned} ASR'_{R2} =& pye^{-\alpha t'_o}\frac{e^{\alpha(t'_o - t'_r)} - 1}{1 - e^{-\alpha T'_R}} \\ &+ \frac{\alpha pyze^{-\alpha t'_o}}{(z+\theta_o)(1 - e^{-\alpha T'_R})}\left[\frac{1}{\alpha + z + \theta_o}(e^{(\alpha+z+\theta_o)(t'_o - t'_r)} - 1) - \frac{1}{\alpha}(e^{\alpha(t'_o - t'_r)} - 1)\right], \end{aligned} \tag{5.55}$$

$$ASR'_{R3} = g\beta y\frac{e^{-\alpha t'_o} - e^{-\alpha T'_R}}{1 - e^{-\alpha T'_R}}, \tag{5.56}$$

and

$$ASR'_{R4} = (p - g - r)\beta y t'_s\frac{\alpha e^{-\alpha T'_R}}{1 - e^{-\alpha T'_R}}. \tag{5.57}$$

The annuity stream of revenues at the retailer is hence:

$$ASR'_R = ASR'_{R1} + ASR'_{R2} + ASR'_{R3} + ASR'_{R4}. \tag{5.58}$$

Taking the same approach as in Section 5.3.3, the present value of holding cost at the OW and the RW only for the first period are:

$$HC'_o = f_o\int_0^{t'_o} I'_o(t)e^{-\alpha t}dt, \tag{5.59}$$

and

$$HC'_r = f_r\int_0^{t'_r} I'_r(t)e^{-\alpha t}dt. \tag{5.60}$$

The annuity stream of holding cost at the retailer over an infinite horizon is hence given by:

$$HC'_R = HC'_{OW} + HC'_{RW}, \tag{5.61}$$

where

$$\begin{aligned} HC'_{OW} =& \frac{\alpha}{1 - e^{-\alpha T'_R}}HC'_o \\ =& \frac{\alpha f_o W}{\alpha + \theta_o}\left(\frac{1 - e^{-(\alpha+\theta_o)t'_r}}{1 - e^{-\alpha T'_R}}\right) \\ &+ \frac{f_o y}{z + \theta_o}\left[\frac{\alpha}{\alpha + z + \theta_o}\left(\frac{e^{(z+\theta_o)(t'_o - t'_r) - \alpha t'_r} - e^{-\alpha t'_o}}{1 - e^{-\alpha T'_R}}\right) - \frac{e^{-\alpha t'_r} - e^{-\alpha t'_o}}{1 - e^{-\alpha T'_R}}\right], \end{aligned} \tag{5.62}$$

and

$$\begin{aligned} HC'_{RW} =& \frac{\alpha}{1-e^{-\alpha T'_R}} HC'_r \\ =& \frac{\alpha f_r (e^{\theta_r t'_r} - e^{-\alpha t'_r})}{(\alpha+\theta_r)(1-e^{-\alpha T'_R})} \left(\frac{y}{\theta_r} + \frac{zWe^{-\theta_o t'_r}}{\theta_r - \theta_o} \right) \\ & - \frac{f_r y}{\theta_r} \left(\frac{1-e^{-\alpha t'_r}}{1-e^{-\alpha T'_R}} \right) - \frac{\alpha f_r z W}{(\alpha+\theta_o)(\theta_r-\theta_o)} \left(\frac{1-e^{-(\alpha+\theta_o)t'_r}}{1-e^{-\alpha T'_R}} \right). \end{aligned} \tag{5.63}$$

The present value of deterioration cost at the OW and the RW for the first period are:

$$DC'_o = d_R \int_0^{t'_o} \theta_o I'_o(t) e^{-\alpha t} dt, \tag{5.64}$$

and

$$DC'_r = d_R \int_0^{t'_r} \theta_r I'_r(t) e^{-\alpha t} dt. \tag{5.65}$$

Therefore, the annuity stream of deteriorating cost at the retailer over an infinite horizon is:

$$DC'_R = DC'_{OW} + DC'_{RW}, \tag{5.66}$$

where

$$\begin{aligned} DC'_{OW} =& \frac{\alpha}{1-e^{-\alpha T'_R}} DC'_o \\ =& \frac{\alpha d_R \theta_o W}{\alpha+\theta_o} \left(\frac{1-e^{-(\alpha+\theta_o)t'_r}}{1-e^{-\alpha T'_R}} \right) \\ & + \frac{d_R \theta_o y}{z+\theta_o} \left[\frac{\alpha}{\alpha+z+\theta_o} \left(\frac{e^{(z+\theta_o)(t'_o - t'_r) - \alpha t'_r} - e^{-\alpha t'_o}}{1-e^{-\alpha T'_R}} \right) - \frac{e^{-\alpha t'_r} - e^{-\alpha t'_o}}{1-e^{-\alpha T'_R}} \right], \end{aligned} \tag{5.67}$$

and

$$\begin{aligned} DC'_{RW} =& \frac{\alpha}{1-e^{-\alpha T'_R}} DC'_r \\ =& \frac{\alpha d_R \theta_r (e^{\theta_r t'_r} - e^{-\alpha t'_r})}{(\alpha+\theta_r)(1-e^{-\alpha T'_R})} \left(\frac{y}{\theta_r} + \frac{zWe^{-\theta_o t'_r}}{\theta_r - \theta_o} \right) \\ & - \frac{d_R \theta_r y}{\theta_r} \left(\frac{1-e^{-\alpha t'_r}}{1-e^{-\alpha T'_R}} \right) - \frac{\alpha d_R \theta_r z W}{(\alpha+\theta_o)(\theta_r-\theta_o)} \left(\frac{1-e^{-(\alpha+\theta_o)t'_r}}{1-e^{-\alpha T'_R}} \right). \end{aligned} \tag{5.68}$$

The annuity stream of backorder cost and lost sale cost at the retailer are obtained as in (5.34) and (5.35):

$$BC'_R = b\beta y \left(\frac{e^{-\alpha t'_o}}{\alpha} \left(\frac{1 - e^{-\alpha t'_s}}{1 - e^{-\alpha T'_R}} \right) - t'_s \frac{e^{-\alpha T'_R}}{1 - e^{-\alpha T'_R}} \right), \tag{5.69}$$

and

$$LC'_R = \pi y(1 - \beta) \left(\frac{e^{-\alpha t'_o} - e^{-\alpha T'_R}}{1 - e^{-\alpha T'_R}} \right). \tag{5.70}$$

Therefore, the annuity stream of the profit function at the retailer is:

$$ASP'_R = ASR'_R - (SC'_R + PC'_R + HC'_R + DC'_R + BC'_R + LC'_R). \tag{5.71}$$

The annuity stream function of revenue ASR'_W and costs SC'_W, PC'_W, HC'_W, and DC'_W at the wholesaler are the same as (5.37) - (5.39), (5.41) and (5.42). The annuity stream profit function of the supply chain is hence given by:

$$ASP'_{SC} = ASR'_R - (SC'_R + HC'_R + BC'_R + LC'_R + DC'_R + SC'_W + PC'_W + HC'_W + DC'_W). \tag{5.72}$$

In the next section, the difference between the two models presented in Section 5.2 and 5.4 is analysed in more detail.

5.5 Numerical examples

5.5.1 Impact of integration

In order to analyse the influence of integration on this system, first an integrated approach is adopted in analysing the model where the inventory policies of the whole supply chain are optimised. In the next step, the whole system is optimised with an independent sequential approach. In the latter approach first the retailer optimises its inventory policies. Then the wholesaler maximises its profit, using the retailer's optimal policies. The sum of the two profit functions gives the total profit of the supply chain, ASP_{Seq}. The improvement obtained after integration δ_{imp} is then given by:

$$\delta_{imp} = 100 \times \frac{ASP^*_{SC} - ASP^*_{Seq}}{ASP^*_{Seq}}. \tag{5.73}$$

For this purpose, the following data set is assumed in this numerical example: $W = 200$, $y = 200$, $z = 0.2$, $\theta_o = 0.05$, $\theta_r = 0.09$, $\theta = 0.03$, $\beta = 0.7$, $p = 13$, $p_r = 8$, $p_w = 3.5$, $\alpha = 0.05$, $g = 0$, $r = 0$, $s_R = 500$, $s_W = 2000$, $f_o = 0.4$, $f_r = 0.8$, $f = 0.3$, $b = 2$, $\pi = 0$, $d = 0$, $d_W = 0$.

The decision variables are k, t_r and t_s when solving the integrated system and as a maximum can be assumed for these variables, an exhaustive search is conducted. Table 5.1 reports the results of this analysis.

Table 5.1: Numerical results for integrated and sequential approach

	T_R	Q_R	ASP_R	k	T_W	Q_W	ASP_W	ASP_{SC}
Integrated approach	2.49	630	485.613	2	4.98	1309	555.165	1040.778
								ASP_{Seq}
Independent sequential approach	1.99	437	580.667	-	-	-	-	-
	-	-	-	3	5.97	1392	343.126	-
								923.793
δ_{imp} (%)								12.7

As shown in Table 5.1, the integrated approach results in a better performance for the whole supply chain, which is an increase of 12.7% in the total profit. Taking the integrated approach however causes a loss in profit for the retailer. Therefore the success of the integration depends on how the actors in this supply chain distribute the value generated from adopting the integrated policy amongst themselves.

The incentive can be divided between the retailer and the wholesaler by reaching an agreement on the transfer price (p_r). In the current model, whenever the retailer pays for the items (purchasing cost for the retailer), the money is received immediately by the wholesaler (revenue for the wholesaler). This means in the integrated approach for the whole supply chain, these two values cancel out and therefore do not have any influence on the *supply chain* optimal solution. Each of the members however sees their profit function sensitive to p_r. Table 5.2 shows how the profit gained by the retailer and the wholesaler change when different values are assigned to p_r.

Table 5.2: The retailer's and the wholesaler's profit when p_r changes, integrated approach

p_r	T_R	Q_R	ASP_R	k	T_W	Q_W	ASP_W	ASP_{SC}
4	2.49	630	1561.323	2	4.98	1309	-520.545	1040.778
6	2.49	630	1023.468	2	4.98	1309	17.310	1040.778
8	2.49	630	485.613	2	4.98	1309	555.165	1040.778
10	2.49	630	-52.242	2	4.98	1309	1093.020	1040.778

In this numerical example the integration results in 12.7% improvement in the profit function. In order to see the influence of both integration and changes in marginal profit, a range of values is assigned to p, and the values of p_r and p_w are adjusted proportionately. For instance, when $p = 9$, p_r and p_w are set equal to 5.54 and 2.42 respectively, therefore $p - p_w = 6.58$ and for the case $p = 16$, p_r and p_w are assigned values of 9.85 and 4.31 respectively, hence $p - p_w = 11.69$. Figure 5.7 shows that when the margins are tighter and the market is highly competitive, the improvement by integration is greater, while in less competitive market with high margins the increase in the total profit of the supply chain gained after integration tends to be small.

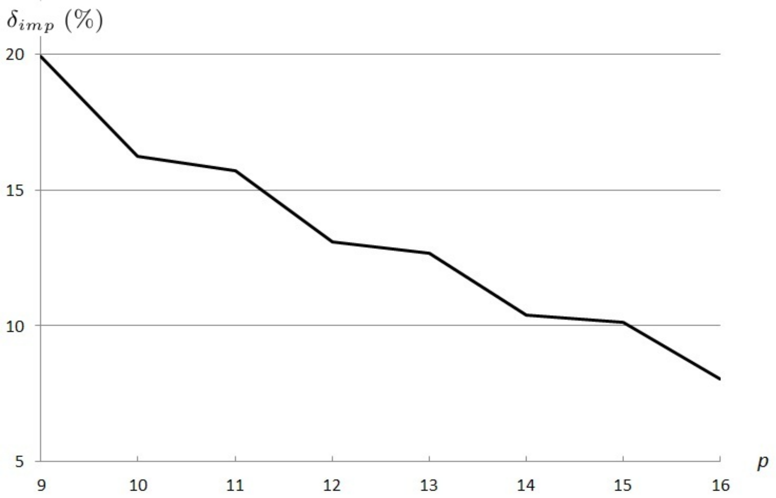

Figure 5.7: Improvement (%) in the ASP after integration

5.5.2 Difference with the model of the literature

This example looks into the difference between the reference model (Model 1) and the literature (Model 2) and examines how this difference influences the optimal solution. For this purpose models developed in Section 5.2 and 5.4 are solved using the following data set: $W = 200$, $y = 50$, $z = 0.7$, $\theta_o = 0.08$, $\theta_r = 0.09$, $\theta = 0.03$, $\beta = 0.7$, $p = 7.5$,

$p_r = 5$, $p_w = 3$, $\alpha = 0.05$, $g = 3$, $r = 1$, $s_R = 300$, $s_W = 1200$, $f_o = 0.4$, $f_r = 0.5$, $f = 0.1$, $b = 2$, $\pi = 0$, $d = 0$ and $d_W = 0$.

Table 5.3 presents the results of this numerical analysis including Δ which captures the difference between the two models:

$$\Delta = 100 \times \frac{ASP^*_{SC}(Model\ 2) - ASP^*_{SC}(Model\ 1)}{ASP^*_{SC}(Model\ 1)}. \tag{5.74}$$

As presented in the table, Model 2 results in an optimal solution with almost 20% lower profit. This is however due to the characteristics of this item and the relevant market. The demand for this item mainly depends on the on-hand inventory (relatively small value for y and large value for z) and the margins are relatively low.

Table 5.3: Numerical results of the comparison between Models 1 and 2

	T_R	Q_R	k	T_W	Q_W	ASP_{SC}
Model 1 (Reference Model)	3.32	531	2	6.63	1117	61.076
Model 2 (Literature)	3.29	512	2	6.59	1077	48.938
Δ (%)	-0.67	-3.58	-	-0.67	-3.61	-19.87

In order to see how the profit margin changes, these two models are solved across a range of values assigned to p. As discussed in Section 5.5.1, the transfer price does not have any influences on the optimal solution, therefore here the profit margin of the supply chain $(p - p_w)$ and the relevant effects on the optimal solution is studied. For this purpose, p_w is first set equal to 3 while p is given a range of values, and the two models are then solved whereby the values of Δ are calculated. The result of this analysis is illustrated in Figure 5.8. As the figure shows, the error can be considerable when the margins are low. This implies that Model 2 underestimates the profits hence is not a very accurate representation for the system studied in this chapter. In the current case, for instance, when $p = 7.1$, Model 1 shows an annuity stream profit of 2.986. This indicates that there is an optimal solution with profit while Model 2 shows no feasible solution for this situation ($ASP_{SC} = -7.402$).

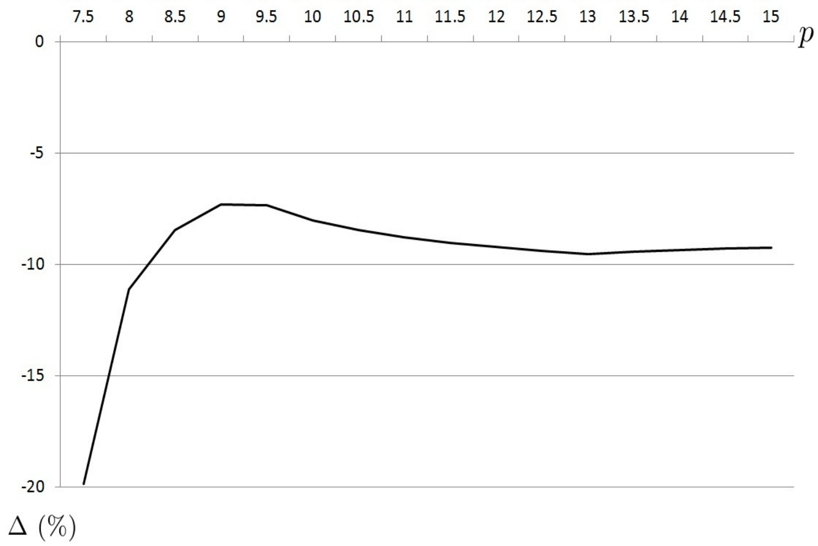

Figure 5.8: The relative difference between both models with respect to p

5.6 Conclusions

In this chapter a distribution system for a perishable item is considered which consists of a retailer and a wholesaler. The retailer has a limited capacity at the owned warehouse (OW) but there is an opportunity of using a second warehouse (RW) with a higher rate of deterioration and of cost. Compared to the two-warehouse models' literature, we have presented a modified version in which the assumptions on the replenishment process between the two parties are revised with a more practical manner.

In this study it is shown numerically that the transfer price does not affect the supply chain optimum. Also it is discussed that although the transfer price is not playing a role in the optimal policies of the supply chain, it can be used as a tool to fairly share the supply chain's gaining between the parties involved.

The analysis conducted in this chapter confirms the benefits of integration discussed in the literature. We show in this study that the significance of these benefits could be small under some circumstances e.g. when the margins are high.

Further in this chapter a comparison is conducted between models developed in the literature and the reference model of this study to see how the model can be modified from a practical viewpoint. The analysis performed in this research work shows that suggested modification in this chapter can result in a better solution. Also numerically it is shown that in some cases in which the models from the literature cannot find any optimum, the modified model presented in this chapter offers an optimum.

Chapter 6

Conclusions

6.1 Overview

The main focus of this thesis work is deteriorating item and both inventory and production-inventory supply chains delivering this type of product to the downstream costumers. First, in Chapter 1 a categorisation of deteriorating item inventory literature is presented which later is used as a framework to identify and address some of the existing gaps. In Chapters 2 - 5 four models are presented and analysed. Chapter 6 briefly outlines the important points of these four models (chapters) and highlights the relevant contributions.

Moreover, this chapter presents the limitations of the models developed in this research work and finally illustrates some directions/topics for future research.

6.2 Chapter 2: A Two-echelon Inventory Model for a Deteriorating Item with Stock-dependent Demand, Partial Backlogging and Capacity Constraints

A two-warehouse supply chain model is considered in which a supplier and a buyer are collaborating. In the literature few models have addressed two-echelon systems (e.g. Yang and Wee, 2002; Rau et al., 2003; Yang, 2004b; Law and Wee, 2006; Lo et al., 2007). These models do not assume any limits for the warehouse capacity. There are, however, some cases that the capacity of the warehouse could be an issue. One of the contributions of this chapter is to model such cases.

Yang (2004b) develops an inventory model (and not a production-inventory model) and appropriately obtains the *average inventory level* (and not the *inventory level*) of the supplier using the concept of echelon stock. In the model developed in Chapter 2, a different method has been used which not only gives the average inventory level but also presents the exact inventory level of the supplier as a function of time. The value of this method is more appreciated when taking the NPV approach as in that case the inventory level is needed. This method later is used to calculate the supplier's (out-of-pocket) holding and deterioration cost with the NPV approach in Chapter 5.

As the model obtained in this chapter is complex and cannot be solved analytically, a heuristic is developed which combines a genetic algorithm and a local search method.

Enumeration can also be used for solving this problem but numerical results are then less accurate. It is shown that for higher accuracy in the decision variable values, the heuristic developed can be much more efficient and faster compared to enumeration.

6.3 Chapter 3: A Two-echelon Production-Inventory Model for Deteriorating Items with Multiple Buyers

In the literature of deteriorating items few researchers have addressed multi-echelon models. Bakker et al. (2012) make a list of research works that model such systems. We argue that some of the papers (e.g. Lee, 2006; Chung and Huang, 2007; Dye et al., 2007a; Hsieh et al., 2008; Rong et al., 2008) categorised in Bakker et al. (2012) as multi-echelon are rather single-echelon systems which analyse two-warehouse models.

The few research works on multi-echelon production-inventory supply chain (e.g. Yang and Wee, 2002; Rau et al., 2003; Lo et al., 2007; Chung and Wee, 2011), define the inventory period at the supplier by dropping some part of the production period. In some of these research works (Yang and Wee, 2002; Law and Wee, 2006; Lo et al., 2007) the production rate is assumed to be considerably greater than the demand rate. This makes the missing part of the production period very small and hence negligible. Having a huge surplus in production capacity, however, is not always feasible nor desirable as it could impose a high tied-up capital. With this regard, in the model developed in Chapter 3 the assumption of having a great surplus in the production capacity is relaxed.

This assumption, however, is not made in the other group of production-inventory multi-echelon models (Yang and Wee, 2003; Rau et al., 2003; Chung and Wee, 2011; Yan et al., 2011). This leaves a difference between the defined production period and the real time of production. In Chapter 3, a single-manufacturer, multi-buyer model is developed in which the production period is properly defined. In the next step the average inventory level of the supplier is obtained using its echelon stock which subsequently is used to calculate the average inventory holding cost. Two numerical examples from the existing literature are analysed to see how the above mentioned method may improve the accuracy of the model. This study shows that in cases with a limited surplus in production capacity, the approach we have developed is considerably more accurate in comparison to the existing models in the literature.

6.4 Chapter 4: Net Present Value Equivalence Analysis for an Inventory of a Deteriorating Item with Partial Backlogging and Finite Production Rate

In Chapters 2 and 3, models are developed and analysed with the classic approach. This approach however may have some shortcomings when for instance the interest rates are high, as in such cases the real value of money (costs and revenues) should be determined based on the time that the cash-flows take place. As a classic approach is unable to do this, the NPV approach has been adopted in Chapter 4.

There are a few research works that model inventory (e.g see Hsieh et al., 2008; Balkhi, 2011) and production-inventory (e.g. see Law and Wee, 2006; Chung and Wee, 2008) systems with the NPV approach. Some of these models minimise the total cost of the system. In a business, in fact, the goal is to maximise the profit rather than to minimise the cost. In basic models of the inventory theory where the demand is constant and shortages are not allowed, the revenues are constant and independent of the decision variables. Therefore, it is mathematically correct to minimise the cost. With this regard, taking a cost minimisation instead of profit maximisation could be done only if this replacement does not affect the optimal solution.

In Chapter 4 an NPV reference model is developed and analysed in which the goal is to maximise the profit. Also a Net Present Value Equivalence Analysis is conducted in order to find conditions under which a cost minimisation model results in the same optimal solution as the reference model. This study shows that there is a clear interpretation for all the cost parameters used in the cost minimisation model in accordance with the reference model. Moreover, it confirms that in some situations deviation from these equivalence conditions could result in differences between the solutions obtained from the two models.

6.5 Chapter 5: A Net Present Value Model for a Two-echelon System of a Deteriorating Item with Capacity Constraints

The NPV models for a deteriorating item inventory with capacity constraints are few (e.g. see Dey et al., 2008; Jaggi et al., 2011; Yang, 2012) and are all single-echelon models. In Chapter 5 a two-echelon model is developed in which the retailer has a limit on the warehouse capacity. The objective function is to maximise the profit to avoid the use of the classic cost parameters (see Chapter 4).

Another contribution of this model is that assumptions regarding the level of the inventory at the OW and the RW are modified with a more practical perspective. The assumption in the literature is that as long as the stock position of the RW is positive (during t_r), the inventory at the OW stays untouched. This means that the level of the inventory at the OW decreases due to the deterioration. The retailer satisfies the demand using the RW and this in fact means that either the RW is at the marketplace (very close to the OW) or the cost of transferring the items (one by one) from the RW to the OW is negligible. Considering the assumption of higher deterioration rate and holding cost at the RW, we suggest to replenish the OW continuously and keep the inventory level at maximum. The results of this study illustrate that for some cases the models in the literature find no feasible solution, while this model shows that an optimal solution can be found.

6.6 Limitations of the thesis and directions for future research

Considering the models developed in this thesis and what is needed in real cases, some points could be highlighted as the limitations of this study and accordingly some areas could be identified for further research.

Firstly, the lead-time has been assumed to be zero in all the models analysed in this thesis. Although that is the assumption of a large part of the literature, some situations would be better modelled with a positive lead-time. This assumption may not affect models developed with the classic approach (Chapters 2 and 3) as this approach is not sensitive to time that cash-flows take place. This, however, is not the case when taking the NPV approach (Chapters 4 and 5) as paying for items and receiving them in a later time changes the NPV of future profits.

Incorporating a positive lead-time either deterministic or stochastic is suggested for models with the NPV approach. Considering real cases, lead-time could be included in different ways when it comes to payment structure. Depending on the agreement between a supplier and a buyer, the price for items could be paid at the time of ordering while the buyer has to wait for some time to receive the batch or the buyer pays for items when the items are delivered. The influence of lead-time on the optimal solution regarding the supply chain structure (single- or multi-echelon) could also be an interesting topic for research. In Chapter 4, it is assumed that transfer prices are received by the supplier immediately when paid by the buyer and in Chapter 5 that the buyer pays for a batch and instantly can sell the items and receive revenues. In both cases assuming a positive lead-time will result in new outcomes which need further research.

Secondly, the demand in the marketplace may be stochastic and could be a function of different factors such as price and time. The demand pattern and the distribution vary from one item to another and can greatly influence the mathematical aspects of these models and hence, the results and the analyses. This assumption, however, has been relaxed in this research. Taking a stochastic demand function would be a better option in modelling some situations with uncertainties that cannot be captured by these deterministic models.

Thirdly, in the retail industry, in particular, inventory systems often distribute a bundle of items rather than a single one. This imposes some constraints to the inventory problem in setting the policies considering the limits on frequency and capacity of vehicles. In this research all the models address systems delivering a single item.

It is suggested to combine transportation and inventory problems. Including transportation into the models developed in this research, results in a system which could give some insights into real distribution models. This transportation system may consist of some types of vehicles with different capacity and cost. From a practical point of view this inventory-transportation problem would be the more interesting in cases of multi-item distribution systems. This added complexity to the model requires more research.

Finally, one of the factors that could directly affect the demand is the service level. Facing out-of-stock situations will result in loss of goodwill besides the loss of revenue. In this thesis all the models allow some levels of shortage and by assigning the relevant cost parameters we have tried to capture the loss of revenue. The loss of goodwill, however, is hard to quantify and should be captured in some other ways. Shortages (low service level) mainly result in a decrease in the future demand function, therefore in order to model the loss of goodwill, the demand functions of these models should be modified in a way that accounts for the service level offered as well. One way towards implementing this in a model would be to link the demand function to service level and also the backlogging rate to waiting time.

A general comment that could be made on the inventory literature of deteriorating items is that it is mainly developed from a theoretical point of view and case studies are scarce in this area. The development and analysis of these models have been done with little contribution from practitioners. For instance, in the case of adopting a deterioration pattern and assigning values to the parameters of Exponential or Weibull functions, it would be more practical to check these values with practitioners to clarify what group of items is exactly being addressed. It may also lead to the identification of special features which are of great relevance to this particular application context and can be incorporated into the models. Therefore, developing some research based on real cases can give a constructive insight into practice in this part of the literature.

Appendix A

Analytical solution for the model presented in Chapter 2

In order to solve the problem (2.37) analytically, first the objective function should be simplified using some approximations. The error of this approximation is acceptable only when T_R and kT_R have relatively small values. (2.21) can be rewritten as follows:

$$Q_W = XQ_R \tag{A.1}$$

where:

$$X = \frac{e^{k\theta T_R} - 1}{e^{\theta T_R} - 1}. \tag{A.2}$$

Using Taylor expansion, the value of X changes to:

$$X \approx (1 + (k-1)\theta T_R)(k + \frac{1}{2}k\theta T_R - \frac{1}{2}k^2\theta T_R). \tag{A.3}$$

The second derivatives of the total cost function with respect to t_s is as follows:

$$\frac{d^2TC}{dt_s^2} \approx \frac{2}{T_R^2}\left(\frac{A}{T_R} - B\right) + \frac{c_{sv}\delta d}{T_R} + \frac{2}{kT_R^2}\left(\frac{C}{T_R} - D\right) + \frac{1}{kT_R}\frac{dD}{dt_s} \tag{A.4}$$

where:

$$A = PC_R + ICC_R + DC_R + SC_R, \tag{A.5}$$

$$B = \frac{dA}{dt_s} = p_R\delta d + c_{sf}(1-\delta)d + c_{sv}\delta dt_s, \tag{A.6}$$

$$C = PC_W + ICC_E W + DC_W, \tag{A.7}$$

$$D = \frac{dC}{dt_S} = \left(\delta dX + Q_R \frac{dX}{dt_s}\right)\left(2p_W + \frac{h_W}{\gamma}\right) - k\delta d\left(p_W + \frac{h_W}{\gamma}\right), \tag{A.8}$$

and

$$\frac{dX}{dt_s} \approx k(k-1)\gamma + \frac{1}{2}k(k+1)\gamma + k(k-1)\gamma^2 T_R. \tag{A.9}$$

By having k and t_r set and changing t_s from zero to a large number that it can possibly take, the second derivative of the total cost function with respect to t_s presented in (A.4), takes a positive value. It however necessitates complex analysis to show the convexity of the total cost function.

An exhaustive search can be conducted to find the global optimum as an upper bound can be assumed for each of the decision variables. There is however a trade-off between the accuracy of the decision variables and the solution time.

In order to conduct an exhaustive search it is assumed that k can get a maximum value of 30. t_r and t_s are assumed to be less than 5 years with the accuracy of $e-01$ (The heuristics is solved with the same accuracy). As shown in Table A.1, the search results in the same optimal solution but the solution time is much longer. Conducting the exhaustive search with a higher accuracy will increase the solution time while this increase in the genetic algorithm time is small (see Table A.1).

Table A.1: Results of the numerical example 2 (Chapter 2) and the run time.

	Accuracy	t_r	t_s	k	t_o	Q_R	T_R	Q_W	T_W	*Run time (seconds)*
Heuristics	$e-01$	2.0	2.6	2	2.4	381	5.0	824	10.1	1.9112
Enumeration	$e-01$	2.0	2.6	2	2.4	381	5.0	824	10.1	148.3035
Heuristics	$e-02$	2.03	2.56	2	2.47	383	5.03	829	10.05	6.0971
Enumeration	$e-02$									M^a

[a] The run time extends infinitely

Appendix B

Independent approach in analysing the system presented in Chapter 2

In this part the inventory system presented in Chapter 2 is optimised with an independent approach. For this purpose, two problems are analysed. Firstly the retailer's inventory system is optimised and the inventory policies are set. In the next step using the retailer's optimal inventory policies, the wholesaler's inventory system is analysed.

Based on analyses done in Section 2.3.3, the inventory cost for retailer per unit of time is as follows:

$$TC_R = \frac{1}{T_R}(PC_R + HC_R + DC_R + BC_R + LC_R) \tag{B.1}$$

The cost function contains T_R and Q_R, therefore partial derivatives of these two terms with respect to t_s and t_r are also needed. Based on (2.13) and (2.14) the derivatives are as follow:

$$\frac{dT_R}{dt_r} = 1 - \frac{W\alpha e^{-\alpha t_r}}{d + (c+\alpha)We^{-\alpha t_r}}, \tag{B.2}$$

$$\frac{dT_R}{dt_s} = 1, \tag{B.3}$$

$$\frac{dQ_R}{dt_r} = cWe^{(\beta-\alpha)t_r} + de^{\beta t_r} \tag{B.4}$$

and

$$\frac{dQ_R}{dt_s} = \delta d. \tag{B.5}$$

The total cost of the retailer is a function of t_s and t_r. To minimise the total cost, the derivatives of the cost function with respect to t_s and t_r are found and set equal to zero,

$$\frac{dTC_R}{dt_s} = 0 \tag{B.6}$$

and

$$\frac{dTC_R}{dt_r} = 0. \tag{B.7}$$

The derivatives of the total cost function with respect to t_s and t_r are as follow:

$$\begin{aligned}\frac{dTC_R}{dt_s} =& \frac{1}{T_R^2}\Big\{p_R\delta dT_R - A_R - p_RQ_R - ICC_R - DC_R + T_R(c_{sf}(1-\delta)d + c_{sv}\delta dt_s)\\ &-c_{sf}(1-\delta)dt_s - \frac{1}{2}c_{sv}\delta dt_s^2\Big\}\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}\frac{dTC_R}{dt_r} =& \frac{1}{T_R^2}\Big\{p_R T_R e^{\beta t_r}(cWe^{-\alpha t_r}+d) - \Big(1-\frac{\alpha We^{-\alpha t_r}}{d+(c+\alpha)We^{-\alpha t_r}}\Big)(A_R+p_R Q_R)\\ &+\frac{h_r cWT_R e^{-\alpha t_r}}{\beta}(e^{\beta t_r}-1)+\frac{h_r dT_R}{\beta}(e^{\beta t_r}-1)+h_o WT_R e^{-\alpha t_r}\\ &-\frac{\alpha h_o dWT_R e^{-\alpha t_r}}{(c+\alpha)\Big(d+(c+\alpha)We^{-\alpha t_r}\Big)}\Big(e^{(c+\alpha)(t_o-t_r)}-1\Big)\\ &-\Big(1-\frac{\alpha We^{-\alpha t_r}}{d+(c+\alpha)We^{-\alpha t_r}}\Big)\Big(\frac{h_r cWe^{-\alpha t_r}}{\beta-\alpha}\Big(\frac{1}{\beta}(e^{\beta t_r}-1)-\frac{1}{\alpha}(e^{\alpha t_r}-1)\Big)\\ &+\frac{h_r d}{\beta}\Big(\frac{1}{\beta}(e^{\beta t_r}-1)-t_r\Big)+\frac{h_o W}{\alpha}(1-e^{-\alpha t_r})\\ &+\frac{h_o d}{c+\alpha}\Big[\frac{1}{d}We^{-\alpha t_r}-\frac{1}{c+\alpha}ln\Big(1+\frac{c+\alpha}{d}We^{-\alpha t_r}\Big)\Big]\Big)\\ &+\alpha p_R T_R We^{-\alpha t_r}\Big(1-\frac{\alpha}{c+\alpha}\Big(1-\frac{d}{d+(c+\alpha)We^{-\alpha t_r}}\Big)\Big)\\ &+\beta p_R T_R e^{\beta t_r}\Big(\frac{d}{\beta}+\frac{cWe^{-\alpha t_r}}{\beta-\alpha}\Big)-\frac{\alpha c p_R T_R We^{-\alpha t_r}(e^{\beta t_r}-1)}{\beta-\alpha}-\frac{\beta cT_R We^{-\alpha t_r}}{\beta-\alpha}-dT_R\\ &-\Big(1-\frac{\alpha We^{-\alpha t_r}}{d+(c+\alpha)We^{-\alpha t_r}}\Big)\Big(p_R W(1-e^{-\alpha t_r})\\ &+\frac{\alpha d p_R}{c+\alpha}\Big(\frac{1}{d}We^{-\alpha t_r}-\frac{1}{c+\alpha}ln\Big(1+\frac{c+\alpha}{d}We^{-\alpha t_r}\Big)\Big)\\ &+p_R(e^{\beta t_r}-1)\Big(\frac{d}{\beta}+\frac{cWe^{-\alpha t_r}}{\beta-\alpha}\Big)+\frac{\beta cW(e^{-\alpha t_r}-1)}{\alpha(\beta-\alpha)}-dt_r\\ &+c_{sf}(1-\delta)dt_s+\frac{1}{2}c_{sv}\delta dt_s^2\Big)\Big\}.\end{aligned} \tag{B.9}$$

As can be seen (B.8) and (B.9) construct a system of non-linear equations which gives the optimal t_s and t_r to minimise the retailer cost. After realisation of t_s and t_r, the optimal inventory period and order quantity for the retailer are calculated. The analysis of the wholesaler inventory cost is exactly the same as what has been done in Section 2.3.4. It is important to note that when the retailer and the wholesaler plan separately, the inventory period and the order quantity of the retailer are not decision variables any more while optimising the wholesaler cost. The cost function of the wholesaler per unit of time is as follows and the only decision variable is k (the number of retailer's inventory period that should be covered with one wholesaler's inventory period):

$$TC_W = \frac{1}{T_W}(PC_W + ICC_W + DC_W). \tag{B.10}$$

The optimal value for k, denoted as k^*, can be derived from the following set of inequalities:

$$TC_W(k^*) \leq TC_W(k^* - 1) \tag{B.11}$$

and

$$TC_W(k^*) \leq TC_W(k^* + 1). \tag{B.12}$$

Appendix C

The assumptions of the models from the literature used in Chapter 3

C.1 Assumptions and notations of the model developed by Yang and Wee (2002)

A single item with constant rate of deterioration in a single-vendor, multi-buyers is assumed.

Shortages are not allowed.

No repair or replacement is done to deteriorated items.

There is a finite production rate which is greater than the sum of the demand at all buyers.

θ	the deterioration rate
N	number of buyers
d_i	the demand rate per year for buyer i, $i = 1, 2, \ldots, N$
p	the production rate per year
T	time length of each cycle, where $T = T_1 + T_2$
T_1	the length of production time in each production cycle T

T_2	the length of non-production time in each production cycle T
$I_{v1}(t_1)$	inventory level for vendor when t_1 is between 0 and T_1
$I_{v2}(t_2)$	inventory level for vendor when t_2 is between 0 and T_2
$I_{bi}(t)$	inventory level for buyer i when t is between 0 and T/n_i
n_i	delivery times per period T for buyer i, where $i = 1, 2, \dots, N$
I_{mv}	the maximum inventory level of vendor
I_{mi}	the maximum inventory level of buyer i
p_v	the unit production cost for vendor
p_b	the unit price for buyer
C_{sv}	the set-up cost of each production cycle for vendor
C_{sb}	the set-up or ordering cost per order for buyer
F_v	the holding cost per dollar per year for vendor
F_b	the holding cost per dollar per year for buyer
VC	the cost of vendor per unit time
BC	the cost of all buyers per unit time
TC	the integrated cost of vendor and all buyer per unit time

C.2 Assumptions and notations of the model developed by Yan et al. (2011)

The operating environment is deterministic.

The suppliers production rate and the demand rate on the buyer are constant.

The inventory item's deterioration is a constant fraction of its on-hand inventory.

The production rate is greater than the demand rate.

The buyer pays transportation and order handling costs.

The cost of the deteriorating item is constant.

Shortages are not allowed.

The deterioration rate is sufficiently small, such that its square or higher powers can be ignored.

N	the number of deliveries per production batch cycle
Q	the production lot size per batch cycle (units)

T	total cycle time of the supplier
T_1	production cycle of the supplier during T
T_2	non-production cycle of the supplier during T
T_3	inventory cycle at the retailer
q	delivery lot size
d	the item's deterioration rate
x	the number of deteriorated units at the retailer during T_3
y	the number of deteriorated units at the supplier during T
C_d	the cost of deterioration per unit($)
p	production rate (units/time unit)
C	set-up cost for a production batch ($/set-up)
H_S	inventory holding cost in $/unit/time unit
S_{sup}	area under the supplier's inventory level curve
D	demand rate in units/time unit
A	ordering cost in $/order
H_B	inventory holding cost in $/unit/time unit
F	fixed transportation cost per delivery ($)
V	unit variable cost for order handling and receiving ($)
S_{buy}	area under the buyer's inventory level curve

The summary of the analysis done by Yan et al. (2011) is as follows:

(a) Set-up cost per unit time for the supplier $= C/T$,

(b) Holding cost per unit time for the supplier $= H_S S_{sup}/T$,

(c) Deterioration cost per unit time for the supplier $= C_d d S_{sup}/T$,

(d) Ordering cost per unit for the buyer $= A/T$,

(e) Holding cost per unit time for the buyer $= H_B S_{buy}/T$,

(f) Transportation and handling cost per unit time for the buyer $= NF + VNq/T$,

(g) Deterioration cost per unit time for the buyer $= C_d d S_{buy}/T$.

The buyer's inventory cost model:

$q = DT_3 + \frac{dqT_3}{2}$

$T = \frac{2Nq}{2D+dq}$

$dS_{buy} = Nq - DT$

$S_{buy} = \frac{Nq-DT}{d}$

therefore:

$TC_{buy}(q,N) = \left(\frac{D}{Nq} + \frac{d}{2N}\right)(A + NF + VNq) + \frac{q}{2}(H_B + C_d d)$

The supplier's inventory cost model:

$y = dS_{sup}$

$y + \frac{dqT}{2} = dT\left(\frac{D}{p}q + \frac{Nq+y}{2}\frac{P-D}{P}\right)$

$S_{sup} = \frac{y}{d} = qT\left(\frac{D}{P} - \frac{1}{2} + \frac{N}{2} - \frac{DN}{2P}\right)$

$TC_{sup}(q,N) = \left(\frac{D}{Nq} + \frac{d}{2N}\right)C + (H_S + C_d d)q\left(\frac{D}{P} - \frac{1}{2} + \frac{N}{2} - \frac{DN}{2P}\right)$

The Integrated inventory cost model:

$TC(q,N) = TC_{buy} + TC_{sup}$

Appendix D

The derivatives of the total cost function of the model presented in Chapter 3

The second derivative of the total cost function presented in (3.28) with respect to T is as follows:

$$\begin{aligned}\frac{\partial^2 TC}{\partial T^2} =& \frac{1}{3} p_b F_b (d-b) + \frac{p_v F_v \theta}{p^2}\left(d^2(p-d) + abp + ad(d-a) + a^3\right) + \frac{2 p_v F_v \theta d^2}{p} \\ &+ \frac{\theta b}{3}\left(p_b F_b - p_v F_v\right) + \frac{2 C_{sb} \sum_{i=1}^{N} n_i}{T^3} + \frac{2 C_{sv}}{T^3}\end{aligned} \tag{D.1}$$

where $a = \sum_{i=1}^{N} \frac{d_i}{n_i}$, $b = \sum_{i=1}^{N} \frac{d_i}{n_i^2}$ and $d = \sum_{i=1}^{N} d_i$.

The second derivative of the total cost function, presented in (D.1), is positive for all the values of T.

Appendix E

An approximated inventory holding cost for the model presented in Chapter 4

One may use the following Maclaurin expansion approximations for (4.2) and (4.4) as in Wee and Law (2001):

$$I(t) = (R-y)e^{-\theta t^{\gamma}}\left(t + \frac{\theta t^{\gamma+1}}{\gamma+1}\right), \qquad 0 \le t \le T_1, \tag{E.1}$$

$$I(t) = ye^{-\theta t^{\gamma}}\left(T_1 + T_2 - t + \frac{\theta}{\gamma+1}\left((T_1+T_2)^{\gamma+1} - t^{\gamma+1}\right)\right), \qquad T_1 \le t \le T_1 + T_2. \tag{E.2}$$

If we use the approximations (E.1) and (E.2), then $T_1 = g(T_2)$ is implicit in:

$$(1-\frac{y}{R})T_1 + \frac{\theta T_1^{\gamma+1}}{\gamma+1} = \frac{y}{R}\left(T_2 + \frac{\theta}{\gamma+1}(T_1+T_2)^{\gamma+1}\right) \tag{E.3}$$

and the values for HC_1 and HC_2 are given by:

$$\begin{aligned} HC_1 =& f\int_0^{T_1} I(t)e^{-\alpha t}dt \\ \approx& f(R-y)\left(\frac{T_1^2}{2T} - \frac{\alpha T_1^3}{3T} - \frac{\theta\gamma T_1^{\gamma+2}}{(\gamma+1)(\gamma+2)T} + \frac{\alpha T_1^2}{4}\right) \end{aligned} \tag{E.4}$$

and

$$
\begin{aligned}
HC_2 =& f\int_{T_1}^{T_1+T_2} I(t)e^{-\alpha t}dt \\
\approx& fy\Big[\frac{T_2(T_1+T_2)}{T} + \frac{\alpha T_2}{2}(T_1+T_2) - \frac{1}{2}\left(\frac{1}{T}+\frac{\alpha}{2}\right)\left((T_1+T_2)^2 - T_1^2\right) \\
&+ \frac{\theta T_2}{T(\gamma+1)}(T_1+T_2)^{\gamma+1} - \frac{\theta}{T(\gamma+1)(\gamma+2)}\left((T_1+T_2)^{\gamma+2} - T_1^{\gamma+2}\right) \\
&- \frac{\alpha(T_1+T_2)}{2T}\left((T_1+T_2)^2 - T_1^2\right) + \frac{\alpha}{3T}\left((T_1+T_2)^3 - T_1^3\right) \\
&- \frac{\theta(T_1+T_2)}{T(\gamma+1)}\left((T_1+T_2)^{\gamma+1} - T_1^{\gamma+1}\right) + \frac{\theta}{T(\gamma+2)}\left((T_1+T_2)^{\gamma+2} - T_1^{\gamma+2}\right)\Big].
\end{aligned}
\tag{E.5}
$$

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