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FACULTY OF PHYSICAL SCIENCES AND ENGINEERING

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Model Building and Phenomenological Aspects of F-Theory GUTs

by

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ABSTRACT

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In recent years, Grand Unified Theories (GUTs) constructed from F-theory have been extensively studied due to the substantial scope for model building and phenomenology which they provide. This thesis will motivate and introduce the basic tools required for model building in the setting of local F-theory. Starting with GUT groups of E_6, $SO(10)$ and $SU(5)$, a group theoretic dictionary between the three types of theory is formulated, which provides considerable insight into how to build a realistic model. The spectral cover formalism is then applied to each case, enabling the possible low energy spectra after flux breaking of the GUT group to be found. Using these results an E_6 based model is constructed that demonstrates, for the first time, that it is possible to construct a phenomenologically viable model which leads to the MSSM at low energies. In addition to the MSSM model, the E_6 starting point is also used to build F-theory models in which the low energy supersymmetric theory contains the particle content of three 27 dimensional representations of the underlying E_6 gauge group, with the possibility of a gauged U(1) group surviving down to the TeV scale. The models with TeV scale exotics initially appear to be inconsistent due to a splitting of the gauge couplings at the unification scale which is too large, and incompatible with the formalism. However, in E_6 models with flux breaking, there are bulk exotics coming from the 78 dimensional adjoint representation which are always present in the spectrum, and it turns out that a set of these exotics provide a natural way to achieve gauge coupling unification at the one-loop level, even for models with TeV exotics. This motivates a detailed study of bulk exotics, where specific topological formulae determining the multiplicities of bulk states are investigated, and the constraints imposed by these relations applied to the spectra of the models previously studied. In particular, bulk exotics are relevant to the almost miraculous restoration of gauge coupling unification in the case of the models with TeV scale exotics. The consistent local F-theory models with low energy exotics have distinctive characteristics when compared with other, similar models, and so provide potential opportunities to be tested at the LHC.

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Declaration of Authorship

I, James C. E. Callaghan , declare that the thesis entitled *Model Building and Phenomenological Aspects of F-Theory GUTs* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

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Nomenclature

BSM	Beyond the Standard Model
CY	Calabi-Yau
E6SSM	E_6 Minimal Supersymmetric Standard Model
FI	Fayet-Iliopoulos
GWS	Glashow-Weinberg-Salam
GUT	Grand Unified Theory
LH	Left Handed
LSP	Lightest Supersymmetric Particle
MSSM	Minimal Supersymmetric Standard Model
NMSSM	Next to Minimal Supersymmetric Standard Model
NP	Non-perturbative
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
RGE	Renormalisation Group Equation
RH	Right Handed
SM	Standard Model
SSB	Spontaneous Symmetry Breaking
SUSY	Supersymmetry
UV	Ultraviolet
VEV	Vacuum expectation value

"All that is now
All that is gone
All that's to come
And everything under the sun is in tune,
But the sun is eclipsed by the moon."

Pink Floyd

Chapter 1

Introduction

1.1 Motivation and outline

The Standard Model (SM) is one of the great triumphs of modern day physics, successfully explaining many aspects of Electroweak and Strong interactions, confirmed through decades of precise experimental data. Following the announcement in July 2012 of the discovery of a particle whose properties are consistent with those of a Higgs boson, the long awaited missing link of the SM, it would seem that the expected picture of the SM is complete. However, despite the incredible achievements of the theory, there are several theoretical reasons why we expect it not to be the whole story when it comes to a theory describing the physics of our universe. In fact, we expect the SM to be an 'effective theory', valid up to some cutoff scale Λ, where a new 'beyond the Standard Model' (BSM) theory is the correct description of nature.

The first shortcoming of the SM is that gravity is not included, and so it cannot possibly be a Theory of Everything. A second reason for moving beyond the Standard Model is that if the three SM gauge couplings are run up to a high energy, it appears that they may merge at a common scale. This hints at the possibility that the SM gauge group is embedded in a bigger symmetry group, suggesting the existence of a Grand Unified Theory (GUT).

Another issue is the so called 'hierarchy problem' which is concerned with the question of why the weak force is 10^{32} times stronger than the gravitational force. Phrased in another way, this is considered a problem because the Higgs mass squared parameter of the SM is UV sensitive, meaning that the quantum corrections to the Higgs mass squared scale with Λ^2. As such, based on what we know about the Higgs boson, a natural explanation would require $\Lambda \sim 1TeV$, with the SM being replaced by some other physics at the TeV scale. However, this explanation is constrained by Electroweak precision data, as if $\Lambda \sim 1TeV$, we would expect to have already seen evidence of higher dimensional operators constructed from SM fields [4], and as such we are led to consider a non natural explanation. One explanation would be that there is a large fine-tuned cancellation between the radiative corrections and the bare mass of the Higgs, but this 'unnatural' explanation is considered unsatisfactory by physicists. A more satisfactory explanation would be that Λ is higher than a TeV, but there are cancellations due to a symmetry in the UV theory. This is the case if supersymmetry (SUSY) is introduced, where there is a symmetry relating fermions and bosons, with each SM particle having a superpartner whose spin differs by $\frac{1}{2}$. Due to the fact that fermions and bosons with the same gauge quantum numbers give opposite sign contributions to the Higgs mass squared, the quadratic divergences to the Higgs mass exactly cancel.

A BSM framework which incorporates SUSY and GUTs is that of String Theory, where we have ten space-time dimensions where six are compactified and very small, and instead of fundamental point-like particles, we now have one-dimensional strings. A major motivation for string theory is that it provides a consistent formulation of quantum gravity, the effects of which are expected to become important at the Planck scale. With this achievement though, comes the drawback that it is very hard to predict anything about low energy physics, due to the vast numbers of consistent solutions to the string theory equations of motion. If, however, we follow the arguments of [5] and impose the conditions of unification and decoupling on the search for realistic models, the possibilities are severely restricted. Unification refers to the existence of a GUT structure whereby the strong, weak and electromagnetic forces are described by a single gauge group and a single coupling constant at some high energy scale. The fact that gravity is observed to be much weaker than the other forces is linked to the term decoupling, which refers to the existence of a theoretical limit where $\frac{M_{GUT}}{M_{Planck}} \to 0$. A class of models which satisfy both the criteria of unification and decoupling are F-theory GUTs.

Recently there has been considerable activity [6, 7, 8, 9, 10, 11] in the reformulation of GUTs in the context of F-theory (for reviews and related work see e.g. [12, 13, 14, 15, 16]). The reason for the renewed interest is that F-theory provides new opportunities for addressing some of the outstanding issues facing GUTs, such as GUT breaking and Higgs doublet-triplet splitting by flux [9, 8]. In this setting, there has been great progress in both global and local model building in the last few years [17, 18], where global models focus on the construction of elliptically fibered Calabi-Yau four-folds, and local models deal with the effective field theory where the GUT symmetry is realised on a 7-brane wrapping a 4-dimensional surface S. The so called 'semi-local' approach imposes constraints from requiring that S is embedded into a local Calabi-Yau four-fold, which in practice leads to the presence of a local E_8 singularity [19]. All Yukawa couplings originate from this single point of E_8 enhancement, and we can learn about the matter and couplings of the semi-local theory by decomposing the adjoint of E_8 in terms of representations of the GUT group and the perpendicular gauge group. In terms of the local picture, matter is localised on curves where the GUT brane intersects other 7-branes with extra $U(1)$ symmetries associated to them, with this matter transforming in bi-fundamental representations of the GUT group and the $U(1)$. Yukawa couplings are then induced at points where three matter curves intersect, corresponding to a further enhancement of the gauge group.

With this structure in place, there are many possibilities for model building. A considerable amount of this work deals with the reconciliation of F-theory models with the low energy Standard Model and the related phenomenology. These include papers related to fermion mass structure and the computation of Yukawa couplings in the context of F-theory and del Pezzo singularities [17, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. In particular, some interesting mechanisms were suggested to generate Yukawa hierarchy either with the use of fluxes [17, 26] and the notion of T-branes [30] or with the implementation of the Froggatt-Nielsen mechanism [22, 23, 24, 25, 27]. More specifically, in [26] it is argued that when three-form fluxes are turned on in F-theory compactifications, rank-one fermion mass matrices are modified, leading to masses for lighter generations and CKM mixing. Ibanez et al [29] have recently shown that flux and instanton effects can generate a realistic hierarchy of fermion masses. In the F-theory context, such non-perturbative contributions were computed in [31], although the magnitude of such corrections remains somewhat unclear.

Larger GUT groups than $SU(5)$ have also been considered, such as the F-theory E_6 model of ref [32] where non-Abelian fluxes are introduced to break the symmetry. Flipped $SU(5)$ [17, 33, 23, 34, 35] has also been considered, including an attempt using an $SU(4)$ spectral cover [36]. Some examples of $SO(10)$ F-theory models were also considered in [17, 37, 38, 39].

Many (or all) of these models predict exotic states below the unification scale, and the renormalization group (RG) analysis of gauge coupling unification including the effect of such states and flux effects has been discussed in a series of papers [40]-[41]. Other phenomenological issues such as neutrinos from KK-modes[42], proton decay [43] and the origin of CP violation [44] have also been discussed. The possibility of obtaining the Standard Model directly from F-theory [45] has also been considered.

Following this work some generic challenges have been identified that result from the highly constrained nature of the constructions, in particular the constraints related to the compatibility of unification (due to the appearance of exotics), the suppression of proton decay (due to R-parity violating operators and dimension-5 operators), the suppression of the μ term and the generation of realistic Yukawa couplings. These occur when flux is used to break the GUT group and generate doublet-triplet splitting. Prior to the work contained in this thesis, no fully realistic model had been constructed using just the symmetries descending from the underlying unified gauge group [22, 27, 46] and this provides additional motivation for the work presented here.

The layout of the rest of the thesis is as follows. In the remainder of the introduction, the Standard Model is introduced and the theoretical reasons for wanting to move beyond it are explained. Two such extensions are then explored, namely the ideas of Supersymetry and Grand Unification, and issues such as proton decay are discussed in the context of these theories. String theory is then introduced as a framework which combines SUSY and GUTs, and the issues of trying to realise the SM from perturbative D branes are discussed. This motivates the case for moving to F-theory, and the basic tools for model building in the local setting are presented. In Chapter 2, a group theory dictionary between E_6, $SO(10)$ and $SU(5)$ models is established, and the spectral cover formalism is introduced and applied to the case of each GUT group in order to compute the homology classes of matter curves. This information is then utilised in Chapter 3, where an E_6 based model is formulated which demonstrates, for the first time, that it is possible to construct a phenomenologically viable model which leads to the MSSM at low energies. Chapter 4 again deals with models based on E_6, but instead of realising the MSSM, the low energy theories contain the particle content of three 27 dimensional representations of the underlying E_6 gauge group, with the possibility of a gauged $U(1)$ group surviving down to the TeV scale. Chapter 5 is dedicated to the discussion of bulk exotics coming from the adjoint representation of E_6, and these are shown to play an especially crucial role in the context of the models with TeV scale exotics. Finally, Chapter 6 concludes the thesis.

1.2 The Standard Model

The Standard Model is a formulation in terms of gauge theories of three of the four fundamental forces of nature- the strong, weak and electromagnetic interactions. The formalism is based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, where $SU(3)_C$ is the gauge group of Quantum Chromodynamics (QCD) and $SU(2)_L \times U(1)_Y$ is the group of the Electroweak theory. Particles are then classified according to their transformations under these symmetry groups, as well as being grouped into two categories based on their spin- fermions possessing half-odd-integer spins, and bosons possessing integer spins. The fermions of the SM transform in a spin $\frac{1}{2}$ representation of the Lorentz group and interact with each other by exchanging spin 1 vector bosons, while the only Lorentz scalar of the SM is the Higgs boson which is responsible for generating mass for the other particles.

The SM is a chiral theory with left-handed fermions transforming as doublets of $SU(2)_L$ and right-handed fermions transforming as singlets of this group. Associated with $SU(2)_L$ are three gauge bosons- W^+, W^- and Z- which mediate weak interactions and explain the short range of the force due to their large masses relative to those of nucleons. The Electroweak group is broken at low energies to $U(1)_{em}$, the gauge group of Electromagnetism, and this force is mediated by the massless photon. Applying the principles of gauge theory to QCD leads to the notion of colour, where quarks can be 'blue', 'green' or 'red' and gauge transformations are local transformations between quarks of different colours. The gauge bosons of QCD which mediate the Strong interactions are called gluons, and together with the gauge bosons of the Electroweak theory complete the 'force carriers' of the SM. The fermionic matter content of the SM can be divided into three generations, with each member of a generation having greater mass than those of lower generations. Each generation is comprised of 1 left-handed lepton doublet (ν_e, e^-), 1 right-handed lepton e_R^+, 3 left-handed quark doublets $(u,d)_L$, 3 right-handed up type quarks u_R and 3 right-handed down type quarks d_R (the factors of 3 for quarks coming from the existence of 3 colours).

1.2.1 Dirac, Weyl and Majorana spinors

Due to the chiral nature of the SM gauge group, it is convenient to work in a chiral basis of two-component Weyl spinors. A Dirac spinor ψ_D satisfies the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi_D = 0 \tag{1.1}$$

and we can write a Dirac spinor in terms of two Weyl spinors ξ_α and $\chi^{\dagger\dot{\alpha}} \equiv (\chi^\dagger)^{\dot{\alpha}}$

$$\begin{aligned} \psi_D &= \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix} \\ \overline{\psi}_D \equiv \psi_D^\dagger \gamma_0 &= (\chi^\alpha, \xi^\dagger_{\dot{\alpha}}) \end{aligned} \tag{1.2}$$

where $\alpha = 1,2$ and $\dot{\alpha} = 1,2$. Undotted indices denote the first two components of a Dirac spinor and dotted indices denote the second two. This notation is adopted due to the fact that the two types of spinor transform differently under Lorentz transformations, ξ_α being a left-handed Weyl spinor, and $\chi^{\dagger\dot{\alpha}}$ being a right-handed Weyl spinor. This can be seen by introducing the projection operators in the Weyl representation for the gamma matrices:

$$\begin{aligned} P_L &= \frac{1}{2}(1-\gamma_5) \\ P_R &= \frac{1}{2}(1+\gamma_5) \\ \gamma_5 &= \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \end{aligned}$$

Acting with P_L and P_R on a Dirac spinor projects out the left and right-handed parts respectively

$$P_L\psi_D = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \quad P_R\psi_D = \begin{pmatrix} 0 \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}$$

Taking the hermitian conjugate of a left-handed Weyl spinor gives a right-handed Weyl spinor and vice-versa

$$\begin{aligned} (\xi_\alpha)^\dagger &= (\xi^\dagger)_{\dot{\alpha}} \equiv \xi^\dagger_{\dot{\alpha}} \\ (\chi^{\dagger\dot{\alpha}})^\dagger &= \chi^\alpha \end{aligned}$$

Given left handed and right handed Weyl spinors ψ_L and ψ_R, charge conjugation is defined by

$$\psi_L^c = i\sigma^2 \psi_L^*, \qquad \psi_R^c = -i\sigma^2 \psi_R^* \tag{1.3}$$

where σ^2 is the second Pauli matrix, and given $\psi_L = \xi_\alpha$ and $\psi_R = \chi^{\dagger\dot{\alpha}}$, we have $\psi_L^c = \xi^{\dagger\dot{\alpha}}$ and $\psi_R^c = \chi_\alpha$. As such, it can be seen that the charge conjugate of a right-handed field transforms as a left-handed field and vice-versa. This means that we can adopt the notation ψ and ψ^c for right and left-handed fields respectively. Whenever spinor indices are raised or lowered, it is achieved by using the antisymmetric tensor $\varepsilon^{\alpha\beta}$ with $\varepsilon_{12} = 1$. Indices can be omitted provided the contraction of two left-handed spinors is taken to be $\xi\chi = \xi^\alpha \chi_\alpha$, and the contraction of two right-handed spinors to be $\xi^\dagger \chi^\dagger = \xi^\dagger_{\dot{\alpha}} \chi^{\dagger\dot{\alpha}}$.

A four-component Majorana spinor is defined in terms of the Dirac spinor by imposing $\chi = \xi$, leading to

$$\begin{aligned} \psi_M &= \begin{pmatrix} \xi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix} \\ \overline{\psi}_M &= (\xi^\alpha, \xi^\dagger_{\dot{\alpha}}) \end{aligned} \tag{1.4}$$

We can now write the Dirac Lagrangian in terms of two-component Weyl spinors for the cases of the Dirac and Majorana spinors using Eqs. (1.2) and (1.4), giving

$$\begin{aligned} \mathscr{L}_D &= \overline{\psi}_D (i\gamma^\mu \partial_\mu - m_D)\psi_D \\ &= i\xi^\dagger \overline{\sigma}^\mu \partial_\mu \xi + i\chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi - m_D(\xi\chi + \xi^\dagger \chi^\dagger) \end{aligned} \tag{1.5}$$

$$\begin{aligned} \mathscr{L}_M &= \overline{\psi}_M (\frac{i}{2}\gamma^\mu \partial_\mu - \frac{1}{2}m_M)\psi_M \\ &= i\xi^\dagger \overline{\sigma}^\mu \partial_\mu \xi - \frac{1}{2}m_M(\xi\xi + \xi^\dagger \xi^\dagger) \end{aligned} \tag{1.6}$$

where, in the Weyl representation for the gamma matrices

$$\begin{aligned} \gamma^\mu &= \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \\ \sigma^\mu &= (\mathbb{I}_2, \boldsymbol{\sigma}) \\ \overline{\sigma}^\mu &= (\mathbb{I}_2, -\boldsymbol{\sigma}) = \sigma_\mu \end{aligned}$$

with $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, σ^i being the Pauli matrices. It can be seen from Eqs. (1.5) and (1.6) that the Dirac mass term couples fields of different chiralities together, whereas the Majorana mass term couples both left-handed and right-handed fields to themselves.

1.2.2 Abelian gauge transformations and QED

The Dirac Lagrangian density $\mathscr{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ is invariant under complex phase transformations of the fermionic field

$$\psi \to e^{i\omega}\psi, \quad \overline{\psi} \to e^{-i\omega}\overline{\psi}$$

The group of such transformations is $U(1)$, and under infinitesimal transformations of the form $e^{i\omega} = 1 + i\omega + \mathscr{O}(\omega^2)$, the wavefunction transforms as $\psi \to \psi + \delta\psi$ with

$$\delta\psi = i\omega\psi, \quad \delta\overline{\psi} = -i\omega\overline{\psi}$$

The idea behind gauge transformations is to allow an independent symmetry transformation at each point in space-time. As such, the parameter ω will now depend on x^μ and the Lagrangian will no longer be invariant. Under such local transformations, the Lagrangian will become $\mathscr{L} \to \mathscr{L} + \delta\mathscr{L}$ with

$$\delta\mathscr{L} = -\overline{\psi}(x)\gamma_\mu(\partial_\mu \omega(x))\psi(x)$$

However, we can restore the local invariance by introducing the covariant derivative D_μ, and replacing the partial derivative in the Lagrangian by D_μ, where

$$D_\mu = \partial_\mu + ieA_\mu \tag{1.7}$$

The Lagrangian is now invariant under local transformations of the fermion fields, provided that we demand that the vector field A_μ simultaneously transforms as

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \omega(x) \tag{1.8}$$

Interpreting this in the case where the $U(1)$ symmetry is that of Electromagnetism tells us that e corresponds to the electric charge of the fermion field, and the gauge field A_μ corresponds to the photon field. The importance of the gauge field is now clear, as its existence allows us to write down an invariant Lagrangian involving derivatives of ψ. Meanwhile, the presence of the gauge field in the covariant derivative leads to an interaction term in the Lagrangian, and so it can be seen that the invariance only exists if the fermions are not free particles. The construction of a complete locally invariant Lagrangian is not complete though, as we must also write down a kinetic term for A_μ. We can define a gauge invariant field strength by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.9}$$

and add a Lorentz invariant kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian, with the numerical factor ensuring that the equations of motion match up with Maxwell's equations. This is not yet the whole story though as we have to quantise the Electromagnetic field, and in doing so we need to find the Feynman rule for the photon propagator. However, if we go about this in the usual way by looking for the part of the action quadratic in the photon field, we run into trouble due to gauge invariance. In the language of functional integrals of the type $\int \mathscr{D}A e^{iS[A]}$, the integral is badly defined because we are integrating over infinitely many physically equivalent field configurations. In order to solve this problem, Faddeev and Popov invented the trick of taking a function $G(A)$ and setting it to zero as a gauge fixing condition, by means of introducing a delta function $\delta(G(A))$ in the functional integral. In this way, the part of the functional integral is isolated which only counts each physical configuration once. This is equivalent to breaking the gauge symmetry in the Lagrangian by adding a term $-\frac{(\partial^\mu A_\mu)^2}{2\xi}$ in such a way as to preserve the gauge symmetry in observables. In computations a specific value of ξ can be chosen (for example $\xi = 1$ corresponds to Feynman gauge), but the Fadeev-Popov method ensures that the value of a correlation function of gauge invariant operators will be independent of this value [47]. Taking into account the gauge fixing term, the Lagrangian for QED in Feynman gauge is

$$\mathscr{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{1.10}$$

1.2.3 Non-Abelian gauge theories and QCD

The concepts of Abelian gauge symmetries can now be applied to non-Abelian groups. If we consider n free fermionic fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_n \end{pmatrix}$$

the Lagrangian density will be given by

$$\mathscr{L} = \overline{\psi}^i (i\gamma^\mu \partial_\mu - m)\psi_i \tag{1.11}$$

This Lagrangian is invariant under $SU(N)$ transformations in field space acting on ψ, where an arbitrary $SU(N)$ matrix U which mixes up the ψ_i can be written as

$$U = e^{i\omega^a T^a}$$

where T^a are the generators of $SU(N)$ and the ω^a are real parameters. The generators are related to the antisymmetric structure constants f^{abc} by the commutator

$$[T^a, T^b] = i f^{abc} T^c$$

and the normalisation for $SU(N)$ is taken to be

$$Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$$

We now proceed as for the Abelian case by defining the covariant derivative

$$D_\mu = \partial_\mu + igT^a A^a_\mu \tag{1.12}$$

where g is the coupling constant and we now have the same number of gauge fields as generators of $SU(N)$, (N^2-1). Non-Abelian gauge invariance will now exist if we have the following infinitesimal transformation laws for ψ and A^a_μ

$$\begin{aligned}\psi &\to (1 + i\omega^a T^a)\psi \\ A^a_\mu &\to A^a_\mu + \frac{1}{g}\partial_\mu \omega^a + f^{abc} A^b_\mu \omega^c\end{aligned}$$

In order to write down a kinetic term for the gauge bosons, we must again use the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_{\mu\nu} \tag{1.13}$$

meaning that we have the gauge invariant term $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$. Expanding this term out in terms of the gauge fields shows that in non-Abelian gauge theories there are three and four-point interaction terms between them, meaning that the gauge bosons interact with each other, unlike the Abelian case.

1.2.4 Spontaneous symmetry breaking and the GWS theory

Spontaneous symmetry breaking (SSB) is a crucial aspect of the Standard Model, as it is the mechanism responsible for breaking the $SU(2)_L \times U(1)_Y$ Glashow-Weinberg-Salam (GSW) theory of weak interactions to the $U(1)_{em}$ group of Electromagnetism at low energies. The idea is that at some energy scale, a field can take on some non zero global value which can violate a symmetry of the Lagrangian, hence breaking the symmetry of the field theory. As such, even though the Lagrangian may be invariant under a certain set of transformations, the ground state will not be invariant in a spontaneously broken theory. In the context of the SM, the relevant transformations are gauge transformations, and spontaneous breaking will occur if at least one generator of the gauge group does not annihilate the vacuum. Goldstone's theorem states that associated with every broken generator of the group, there is a massless particle called a Goldstone boson, who's quantum numbers mirror those of the corresponding generator.

These concepts can be made more concrete in the context of the $SU(2)_L \times U(1)_Y$ part of the SM gauge group. If we consider a complex scalar Higgs doublet, Φ (with quantum numbers $(2, \frac{1}{2})$ under $SU(2)_L \times U(1)_Y$) and the relevant gauge fields, we can write down the gauge invariant Lagrangian

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \left| D_\mu \Phi \right|^2 - V(\Phi) \tag{1.14}$$

where $F^a_{\mu\nu}$ and $G_{\mu\nu}$ are the field strengths for the $SU(2)_L$ and $U(1)_Y$ gauge bosons, W^a_μ and B_μ respectively, and $D_\mu \Phi$ is the covariant derivative, given by

$$D_\mu \Phi = \partial_\mu + \frac{ig'}{2} B_\mu + \frac{ig}{2} \sigma^a W^a_\mu$$

with σ^a the Pauli matrices, and g, g' the coupling constants of $SU(2)_L$ and $U(1)_Y$. If we take the scalar potential to be

$$V(\Phi) = -\mu^2 \Phi^\dagger_i \Phi^i + \lambda (\Phi^\dagger_i \Phi^i)^2 \tag{1.15}$$

we can see that it has a minimum if $\Phi^\dagger_i \Phi^i = \frac{\mu^2}{2\lambda}$. This equation corresponds to an infinite number of states with the same minimum energy, and the symmetry breaking occurs when a choice is made and one of the minima is picked out to be the true vacuum. It is said that the field Φ acquires a non zero vacuum expectation value (VEV), and in this case we can choose the VEV to be

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v = \frac{\mu}{\sqrt{\lambda}} \tag{1.16}$$

From the four generators of $SU(2)_L \times U(1)_Y$

$$\frac{\sigma_1}{2} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \frac{\sigma_2}{2} = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \quad \frac{\sigma_3}{2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad Y = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{1.17}$$

it can be seen that only one linear combination, $(Y + \frac{\sigma_3}{2})$, annihilates this vacuum, so we have three broken generators and hence three Goldstone bosons. However, as the generators associated with the Goldstone bosons can act on the vacuum to give states inconsistent with the original gauge choice, the Goldstone bosons must not correspond to physical, massless particles in the same way as in the case of a spontaneously broken global symmetry. In order to see what happens to these three degrees of freedom, we can expand Φ^i around the choice of VEV in Eq. (1.16), and write

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 - \phi_2 \\ v + H + i\phi_0 \end{pmatrix}$$

where the ϕ_i and H fields have zero VEV. Putting this into Eq. (1.15) shows that the ϕ_i fields do not acquire mass terms, but the H field does. As such, we identify ϕ_1, ϕ_2 and ϕ_0 with the Goldstone bosons, and H with the Higgs scalar. In fact, we can make a choice of gauge called the 'unitary gauge', where all of the ϕ_i can be set to zero. Writing $\left|D_\mu \Phi\right|^2$ in this gauge shows that we get a mass term for $W_\mu^\pm = \frac{W_\mu^1 \pm i W_\mu^2}{\sqrt{2}}$ and the linear combination $gW_\mu^0 - g' B_\mu$. Introducing the Weinberg angle θ_w, we can diagonalise this system by writing

$$\begin{aligned} B_\mu &= \cos\theta_w A_\mu - \sin\theta_w Z_\mu \\ W_\mu^0 &= \sin\theta_w A_\mu + \cos\theta_w Z_\mu \end{aligned}$$

We now have three massive vector bosons $W_\mu^\pm$ and Z_μ with masses $M_{W^\pm} = \frac{gv}{2} = M_Z \cos\theta_w$, and one massless vector boson, A_μ, which is identified with the photon. The physical interpretation is that whilst a massless vector boson has two degrees of freedom, a massive one has three, and so in this context the three Goldstone bosons provide the three degrees of freedom required to make the W and Z bosons massive. Meanwhile, the charge generator which annihilates the vacuum can be identified with the electric charge, and so with T_3 being the eigenvalue of $\frac{\sigma_3}{2}$ of a state and Y being that of the hypercharge generator, we can write

$$Q = Y + T_3 \tag{1.18}$$

1.2.5 Yukawa couplings and the origins of fermion masses

It can be seen from Eq. (1.5) that we cannot have an explicit mass term for the quarks and charged leptons due to the fact that it would mix right and left handed fermions. However, we can write down gauge invariant couplings involving the Higgs doublet Φ, namely the Yukawa interactions:

$$\mathscr{L}_Y = \lambda_u^{ij}\overline{Q}_{Li}u_{Rj}H_1 + \lambda_d^{ij}\overline{Q}_{Li}d_{Rj}H_2 + \lambda_e^{ij}\overline{L}_{Li}e_{Rj}H_2 + \text{h.c.} \tag{1.19}$$

where $H_2 = i\sigma_2 H_1^\dagger$, i and j are family indices, and λ_u, λ_d and λ_e are the Yukawa couplings. Once the Higgs acquires a VEV of the form in Eq. (1.16), this part of the Lagrangian can be written in terms of mass matrices $m_u = v\lambda_u$, $m_d = v\lambda_d$ and $m_e = v\lambda_e$ as follows

$$\mathscr{L}_M = m_u^{ij}\overline{u}_{Li}u_{Rj} + m_d^{ij}\overline{d}_{Li}d_{Rj} + m_e^{ij}\overline{e}_{Li}e_{Rj} + \text{h.c.} \tag{1.20}$$

It is clear that the Yukawa terms mix quarks and charged leptons of different generations, but the physical particles are those which diagonalise the mass matrices. Focusing on the quarks for now, we can write the three isodoublets of left handed fermions as

$$\begin{pmatrix} u_L \\ \tilde{d}_L \end{pmatrix}, \begin{pmatrix} c_L \\ \tilde{s}_L \end{pmatrix} \text{ and } \begin{pmatrix} t_L \\ \tilde{b}_L \end{pmatrix},$$

where the three u_L^i quarks are linked by the charge-changing weak interactions involving $W^\pm$ bosons to the following unitary rotation of the d_L^i quarks

$$\begin{pmatrix} \tilde{d}_L \\ \tilde{s}_L \\ \tilde{b}_L \end{pmatrix} = V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad V_{CKM} = V_u^L V_d^{L\dagger} \tag{1.21}$$

V_{CKM} is the 'Cabibbo-Kobayashi-Maskawa' (CKM) matrix, and $V_{u,d}^{L,R}$ are the unitary matrices which diagonalise the Yukawa couplings; the basis transformations between weak eigenstates ($u_{L\alpha}$, ...) and mass eigenstates (u_{Li}, ...) being given by

$$u_{L\alpha} = V_{(u)\alpha i}^L u_{Li}, \quad u_{R\alpha} = V_{(u)\alpha i}^R u_{Ri}, \quad d_{L\alpha} = V_{(d)\alpha i}^L d_{Li}, \quad d_{R\alpha} = V_{(d)\alpha i}^R d_{Ri} \tag{1.22}$$

and the diagonal matrices being given by $V_u^L y_u V_u^{R\dagger}$ and $V_d^L y_d V_d^{R\dagger}$. In general, a 3×3 unitary matrix has nine independent parameters, but we can absorb five of these as relative phases between the six quark fields, and so we are left with four parameters of the CKM matrix, which can be interpreted as three mixing angles and a complex phase. In the charged lepton sector, without right handed neutrinos, weak and mass eigenstates can be chosen to coincide meaning that there is no mixing in the leptonic charged current. As such, the nature of neutrinos has to be discussed in order to understand leptonic masses and mixings.

1.2.6 Neutrinos

In the Standard Model, neutrinos are massless, and so unlike the cases of quarks and charged leptons where both right and left handed fields exist, in the neutrino sector there are only left handed neutrinos. However, neutrino oscillations are obsevred in nature whereby a source of neutrinos is produced with a specific flavour, and the probability of finding a neutrino of a different flavour at a suitably large distance from the source is non zero. This implies the existence of neutrino flavour mixing in the leptonic charged current, in an analogous way to the mixing in the quark sector. As such, neutrinos must have masses, with experiments suggesting that these masses are very small- the upper limit being around a factor of 10^6 smaller than the smallest mass in the quark and charged lepton sectors.

The crucial difference between neutrinos and charged fermions when it comes to mass generation is that while we can only write down Dirac mass terms for the charged fermions, we are also able to write down Majorana masses for neutrinos. Nevertheless, the simplest way to introduce neutrino masses is to assume that they are Dirac particles, where the Yukawa interaction is given by

$$\mathscr{L}_Y^{\nu} = \lambda_{\nu}^{ij}\overline{L}_{Li}\nu_{Rj}H_1 + \text{h.c.} \tag{1.23}$$

In order for this term to be gauge invariant, the right handed neutrino must be a singlet under $SU(3)_C$ and $SU(2)_L$ and must also carry zero hypercharge. The mass terms arising from this equation and the equivalent one for charged leptons are

$$\mathscr{L}_m = m_{\nu}^{ij}\overline{\nu}_{Li}\nu_{Rj} + m_e^{ij}\overline{e}_{Li}e_{Rj} + \text{h.c.} \tag{1.24}$$

In a similar way to the quark sector, the mass matrices can now be diagonalised, with the mass eigenstates (with Latin indices) and weak eigenstates (with Greek indices) related by

$$\nu_{L\alpha} = V^L_{(\nu)\alpha i}\nu_{Li}, \quad \nu_{R\alpha} = V^R_{(\nu)\alpha i}\nu_{Ri}, \quad e_{L\alpha} = V^L_{(e)\alpha i}e_{Li}, \quad e_{R\alpha} = V^R_{(e)\alpha i}e_{Ri} \tag{1.25}$$

It then follows that the leptonic charged current written in terms of weak eigenstates $\mathscr{L}_{c.c} \sim \overline{\nu}_{L\alpha}\gamma^{\mu}e_{L\beta}W^+_{\mu}$, can be written in terms of mass eigenstates as

$$\mathscr{L}_{c.c} \sim \overline{\nu}_{Li}\gamma^{\mu}V^{L\dagger}_{(\nu)i\alpha}V^L_{(e)\alpha j}e_{Lj}W^+_{\mu} \tag{1.26}$$

where the matrix describing leptonic mixing, $V_{ij}^{PMNS} = V_{(\nu)i\alpha}^{L\dagger} V_{(e)\alpha j}^{L}$, is the 'Pontecorvo-Maki-Nakagawa-Sakata' (PMNS) matrix.

The problem with Dirac neutrinos is that naturally we would expect the Yukawa coupling to be of the same order as those for the quarks and charged leptons. However, as we know experimentally that neutrino masses are much smaller, a mechanism to explain this fact is desired. One such mechanism is the 'Seesaw mechanism' [48] whereby a Majorana mass term is introduced for the right handed neutrinos, which ends up explaining the light neutrino masses in a natural and beautiful way. The terms of interest in the Lagrangian are now

$$\mathscr{L}_Y^{\nu} = \lambda_{\nu}^{ij} \overline{L}_{Li} \nu_{Rj} H_1 + \frac{1}{2} M^{ij} \overline{\nu^c}_{Ri} \nu_{Rj} + \text{h.c.} \tag{1.27}$$

The key to the Seesaw mechanism is that the Majorana mass term is not generated by the Higgs mechanism, and so can be much larger than the masses for quarks and charged leptons. After spontaneous symmetry breaking, we have both Dirac and Majorana mass terms given by

$$\begin{aligned} \mathscr{L}_M^{\nu} &= m^{ij} \overline{\nu}_{Li} \nu_{Rj} + \frac{1}{2} M^{ij} \overline{\nu^c}_{Ri} \nu_{Rj} + \text{h.c.} \\ &= (\overline{\nu}_L, \overline{\nu^c}_R) \begin{pmatrix} 0 & m \\ m^T & M \end{pmatrix} \begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix} + \text{h.c.} \end{aligned} \tag{1.28}$$

In the approximation that $M >> m$ the matrix in Eq. (1.28) can be diagonalised to give effective Majorana masses for the light neutrinos, once the heavy right handed neutrinos have been integrated out

$$m_\nu = -m\, M^{-1} m^T \tag{1.29}$$

It is clear from this Seesaw mass relation that if $M >> m$, we have $m_\nu << m$, hence giving a natural explanation for the small size of neutrino masses.

1.2.7 Renormalisation and effective field theory

In a quantum field theory (QFT) such as the Standard Model, in addition to tree level Feynman diagrams for a certain process, there will also be loop diagrams of higher order in the coupling constant which can give rise to divergent contributions. The reason for this is that the momenta in a loop are only constrained by momentum conservation at the vertices, and so the computation of the diagram will involve integration over all possible loop momenta. Depending on the form of the integral, this can lead to infinite terms in the calculation of a physical process. Clearly this is unacceptable, so we must find a way of dealing with these infinities in order to 'regularise' the theory.

The essential objects which describe physical observables in QFT are n-point Green functions, and in light of the divergences that can arise, instead of integrating momenta to infinity we could integrate to some cutoff scale, Λ. The Green functions will then depend on Λ, and it follows that the fields, couplings and masses in the Lagrangian will also depend on Λ. As such, we have the notion of a *bare field* $\phi_0(x,\Lambda)$, a *bare coupling* $\lambda_0(\Lambda)$ and a *bare mass* $m_0(\Lambda)$, which all depend on the cutoff scale. As an example, we can consider the Lagrangian for $\lambda\phi^4$ theory

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \tag{1.30}$$

Firstly, we can express the bare field in terms of a renormalised field and a potentially divergent coefficient, Z. The bare field, mass and coupling can then be removed from the Lagrangian by using this redefinition, as well as redefinitions involving the renormalised mass and coupling

$$\phi_0 = \sqrt{Z}\phi_R, \quad \delta_Z = Z-1, \quad \delta_m = Zm_0^2 - m^2, \quad \delta_\lambda = Z^2\lambda_0 - \lambda \tag{1.31}$$

In these definitions, m and λ are the *remormalised* mass and coupling constant, and correspond to the physically measured quantities. In order to make this statement precise, *remormalisation conditions* are chosen, where m^2 is defined to be the location of the pole in the propagator, and λ is defined as the magnitude of the four point scattering amplitude at zero momentum (although we could equally well choose a different momentum scale in this definition). Using these definitions, the Lagrangian can now be written as

$$\begin{aligned}\mathscr{L} &= \left\{\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4\right\} + \left\{\frac{1}{2}\delta_Z\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}\delta_m\phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4\right\} \\ &= \mathscr{L}_R + \mathscr{L}_{CT}\end{aligned} \tag{1.32}$$

where $\mathscr{L}_R$ is the original Lagrangian written in terms of the renormalised field, mass and coupling constant, and $\mathscr{L}_{CT}$ is the *counterterm* Lagrangian, which has absorbed the infinite shifts between the bare parameters and the renormalised ones. In this way, the counterterms introduce a new set of Feynman rules which must be taken into account when computing an amplitude.

Due to loop integrals in diagrams, UV divergences will occur, so we must introduce a *regularisation* procedure in order to deal with the infinities in a consistent way. One gauge invariant method of regularisation is know as 'dimensional regularisation', which involves exploiting the fact that most symmetries do not depend on the number of dimensions. As such, integrals are calculated in d dimensions, where the integral is finite, and an analytic continuation is then made in d by setting $d = 4 - \varepsilon$ and then taking the limit of $\varepsilon \to 0$. The divergences will appear as simple poles at $d = 4$, and so when the integrals are performed, there will be a pole part proportional to $\frac{1}{\varepsilon}$, plus terms which are finite in the limit $\varepsilon \to 0$. Then a scheme such as the 'MS scheme' can be used, where just the pole part is associated with the counterterm. Divergences in a given amplitude can then be cancelled by choosing δ_Z, δ_m and δ_λ in such a way as to maintain the renormalisation conditions, meaning that infinities are cancelled by a lower order diagram with a counterterm insertion.

The key to a renormalisable theory is that we should be able to cancel all the divergences in the theory with a finite number of counterterms. In the $\lambda\phi^4$ example it was outlined how the divergences in the two and four point functions could be cancelled with the introduction of three infinite constants. Now we could go to higher order and examine the six point function and even higher order Green functions. In the Feynman diagrams which contribute to these Green functions, we could potentially have divergent subgraphs whose infinites are already removed by the process of remormalising lower order Green functions. However, if it is not possible to split a diagram into two disconnected pieces by making a cut along a single line, the diagram is called 'one-particle irreducible' (1PI), and can potentially give rise to new divergences. If a new divergence does occur for an n-point function, a new infinite counterterm proportional to ϕ^n must be added to the Lagrangian, and this process is repeated until all the divergences can be cancelled by the counterterms. If this process carries on indefinitely, the theory is called *non-renormalisable*. We should note that the issue with non-renormalisable theories is not that they are mathematically inconsistent, as all the infinities can still be removed. However, in doing so we have to introduce an infinite number of parameters.

In order to determine whether a specific Feynman diagram contains a UV divergence, naively we could simply count powers of momentum. Each loop brings an integral $\int d^4k$, and so could potentially lead to an infinity. However, seeing as each propagator brings either one or two powers of momentum to the denominator, divergences could possibly be avoided. The 'superficial degree of divergence' D, is defined to be the power of momentum in the numerator minus the power in the denominator. In the naive sense, a diagram will diverge if $D \geq 0$, and will be finite if $D < 0$. This statement is not completely true, due to the fact that a diagram with negative D could contain a divergent subgraph, or a superficially divergent diagram could include symmetry cancellations. However, the divergences produced by diagrams with $D < 0$ are cured by the renormalisation of lower order Green functions, and so the criterion for a renormalisable theory is that there should only exist a finite number of Green functions with $D \geq 0$. In the context of $\lambda\phi^n$ theory this criterion translates into the requirement that $n \leq 4$, and due to the fact that the action is dimensionless, this means that terms in the Lagrangian with coefficients (e.g the coupling constant λ) whose mass dimension is greater or equal to zero are renormalisable, whereas terms whose coefficients have negative mass dimension are non-renormalisable. This turns out to be true in general, and can be understood intuitively by considering a coupling constant with dimension $[M]^{-i}$. In this case, each vertex gives a factor of $\frac{1}{M^i}$, which for dimensional reasons means that we expect to find divergences with increasing powers of $\frac{\Lambda^i}{M^i}$. Taking the case where

$i = 2$ and assuming that we just have one typical energy scale, E, we can expand an N-point amplitude up to order λ^{2n} as in [49]

$$A_N(E) = A_N^0(E)\left(1 + c_1\frac{E^2}{M^2} + \cdots + c_n\frac{E^{2n}}{M^{2n}}\right) \tag{1.33}$$

Once all the amplitudes with smaller N have been renormalised, all the c coefficients apart from c_n are finite and calculable. On the surface, it may seem like we have a lack of predictability for the non-renormalisable theory because we have to fix c_n by comparison with experiment. This lack of predictability is only an issue though if E is of the same order of magnitude as M. Otherwise, in the low energy case $E << M$, the c_n term is suppressed by $\frac{E^{2n}}{M^{2n}}$, and the non-renormalisable theory is perfectly acceptable for describing low energy physics. This underlines the fact that non-renormalisable theories have a basic mass scale associated with them, which tells us about their range of validity.

Although the idea of renormalisability is extremely important, the modern perspective on QFT is that if we are only interested in physics at a certain energy scale, it is not important what happens at infinitely high energies, and so the question of whether the theory is renormalisable is not that physically meaninful. For example, QED is a renormalisable theory, but we know that it is not the correct description of nature above the Electroweak scale, and the non-renormalisable Fermi theory, whilst not describing the full Electroweak theory, is still a good approximation when considering low energy processes. As such, modern theoretical physics concerns itself with the idea of *effective field theory*, where we do not necessarily have to have a full, UV complete theory in order to understand physics at lower energies.

An important consequence of the renormalisation procedure is that coupling constants and masses become dependent on the energy scale, and are said to ‘run’ with energy. In the brief discussion of the systematics of renormalisation, we chose a specific renormalisation condition for the scale μ, but noted that we could have equally well chosen another μ at which to renormalise our theory. The renormalised n-point functions Γ_R will depend on this renormalisation scale but have no dependence on the cutoff, whereas conversely the bare n-point functions Γ_0 will depend on the cutoff but not on μ. The relation between Γ_0 and Γ_R is given by (where the coupling in a general theory is now denoted by g)

$$\Gamma_0(p_i, g_0(\Lambda), \Lambda) = Z^{\frac{n}{2}}(g_0(\Lambda), \frac{\Lambda}{\mu})\Gamma_R(p_i, g_R, \mu) \tag{1.34}$$

Assuming that the typical energy scales are much bigger than the masses involved, we can neglect mass terms, and using the fact that Γ_0 is independent of the renormalisation scale, we have

$$0 = \mu\frac{d\Gamma_0}{d\mu} = \left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} + n\gamma(g_R)\right]\Gamma_R(p_i, g_R, \mu) \tag{1.35}$$

where the β and γ functions are defined as

$$\beta(g_R) = \mu \frac{dg_R}{d\mu}, \qquad \gamma(g_R) = \frac{1}{2}\mu \frac{d}{d\mu} \ln Z \tag{1.36}$$

As such, it is the beta functions which encode the information about the running of coupling constants, and one-loop beta functions will be used in the context of real models in later chapters, in order to run the low energy gauge couplings of the SM up to the GUT scale.

1.2.8 Anomalies

In order to prove that a renormalised theory is gauge invariant, it is necessary to use relations between Green functions which follow from gauge invariance (know as 'Ward-Takahashi' identities in QED). When a classical Lagrangian is invariant under a gauge symmetry, a direct consequence is the existence of a conserved current $j^\mu(x)$ which satisfies the conservation equation $\partial_\mu j^\mu(x) = 0$. The Ward identity is then concerned with imposing the symmetry on amplitudes in the quantum theory in such a way that we still have a conserved gauge current at the quantum level. However, there are a set of diagrams which do not satisfy these identities and so would ruin the renormalisability of a gauge theory unless they are eliminated. These diagrams correspond to cases where a symmetry of the classical Lagrangian is not a good symmetry of the quantum theory, and give rise to the phenomenon of *anomalies*.

The simplest example of an anomaly is in the case of chiral transformations of massless Dirac fermions [50]. This anomaly can be understood in terms of the non-invariance of the fermion measure $\int \mathscr{D}\psi\, \mathscr{D}\overline{\psi}$ under these transformations. Under local transformations of the fermion fields $\psi(x) \to U(x)\psi(x)$, where U is a matrix acting on the indices of the gauge group representation, the fermion measure transforms as

$$\mathscr{D}\psi\, \mathscr{D}\overline{\psi} \to (\mathrm{DetU})^{-1}(\mathrm{Det}\overline{\mathrm{U}})^{-1} \mathscr{D}\psi\, \mathscr{D}\overline{\psi} \tag{1.37}$$

If U is of the form $U(x) = e^{i\varepsilon^\alpha(x)t_\alpha}$, where the t_α are generators of the gauge symmetry and a possible global flavour symmetry, it can be seen that $\overline{U} = U^{-1}$ and the fermion measure is invariant. However, if we take the case of a unitary chiral transformation, $U(x) = e^{i\varepsilon^\alpha(x)t_\alpha\gamma_5}$, it can be shown that $\overline{U} = U$, and so the fermion measure transforms with the prefactor $(\mathrm{DetU})^{-2}$. In a non trivial calculation (outlined in [50]), this quantity can be computed to be

$$(\mathrm{DetU})^{-2} = \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4\mathrm{x}\varepsilon^\alpha \mathrm{a}_\alpha(\mathrm{x})}, \qquad \mathrm{a}_\alpha(\mathrm{x}) = -\frac{1}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}_\mathrm{R}\mathrm{t}_\alpha \mathrm{F}_{\mu\nu}(\mathrm{x})\mathrm{F}_{\rho\sigma}(\mathrm{x}) \tag{1.38}$$

where a_α is known as the anomaly function. In this expression, the trace is a matrix trace over the indices of the gamma, gauge and flavour matrices, and the field strengths are given by

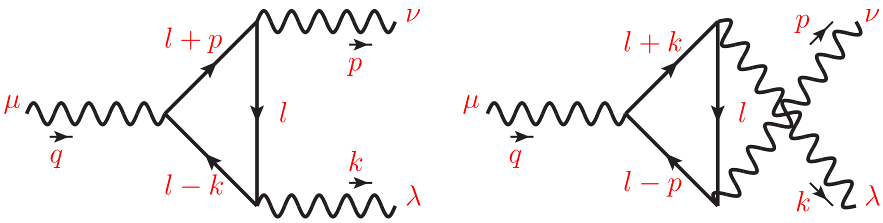

Figure 1.1: Triangle anomaly diagrams

$F_{\mu\nu} = F^{\beta}_{\mu\nu} t_{\beta}$, where the t_α have been defined to include a factor of the gauge coupling constant. As such, it can be seen that for a gauge theory with a single coupling constant g, the anomaly is proportional to g^3. Combined with the Adler-Bardeen theorem which states that the anomaly only occurs at one-loop order, it is clear that the anomaly arises from the triangle diagrams of Figure 1.1, with one of the gauge fields coupling to the axial current, and the other two coupling to the vector current.

As the Standard Model is a chiral theory, we must know about the significance of the axial current in the context of a chiral gauge theory. Considering the case of massless chiral fermions where only the left handed part couples to the gauge field, the matter Lagrangian is given by

$$\mathscr{L} = i\bar{\psi}\partial\!\!\!/ P_R \psi + i\bar{\psi} D\!\!\!\!/ P_L \psi \tag{1.39}$$

This Lagrangian is manifestly not gauge invariant, as the right handed fields do not couple to the gauge field, so to start we can just consider the gauge invariant kinetic term for the left handed fields, $\mathscr{L} = i\bar{\psi} D\!\!\!\!/ P_L \psi$. We can compute the anomaly by considering the triangle diagrams of Figure 1.1, and inserting a projection operator P_L at every vertex. The result of this calculation is that the anomaly is proportional to the coefficient

$$D_R^{abc} = \mathrm{Tr}[\mathrm{T^a}\left\{\mathrm{T^b},\mathrm{T^c}\right\}] \tag{1.40}$$

where the trace is taken over the group generator matrices in representation R. It is useful now to write right handed fields in terms of left handed fields by $\psi'_L = \sigma^2 \psi^*_R$. Rewriting the Lagrangian for the right handed fields and using the fact that generators in the conjugate representation $T^a_{\bar{R}}$, are related to those in R by $T^a_{\bar{R}} = -(T^a_R)^T$, shows that the new fields ψ'_L transform in the conjugate representation to R. This fact means that when R is real (meaning that R is equivalent to $\overline{R}$), the anomaly coefficient is zero and the theory is automatically anomaly free. When a gauge group has complex representations, it must be checked that anomalies are cancelled in order to have a consistent theory, and this can indeed be shown to be true for the Standard Model.

As such, the SM is a renormalisable, anomaly free theory, but for reasons explained in the motivation section for this thesis, there are many questions to which it cannot give satisfactory answers. In accord, we now turn our attention to some BSM topics which will provide the groundwork for building GUT models in an F-theory context.

1.3 Supersymmetry

As hinted at in the motivation for this thesis, the Standard Model with a fundamental Higgs boson suffers from the issue that m_H^2 is quadratically sensitive to heavy mass scales, m_{heavy}, to which the Higgs couples. In fact, even after Λ^2 divergences are subtracted by renormalisation, one still has m_{heavy}^2 terms. This is due to the fact that there is no symmetry in the SM which 'protects' the Higgs mass in the same way, for example, that the photon is ensured to be massless at all orders in perturbation theory by the exact $U(1)$ gauge symmetry of QED.

This property of the radiative corrections to the Higgs mass is suggestive of a symmetry in the UV theory which manifests itself in the systematic cancellation of quadratic divergences. Indeed, if we have a symmetry relating fermions and bosons, fermions and bosons with the same gauge quantum numbers give opposite sign contributions to the Higgs mass squared, and the quadratic divergences to the Higgs mass exactly cancel (provided that we have equal numbers of bosonic and fermionic degrees of freedom), leaving only logarithmic divergences. The generators of such a 'supersymmetry' (SUSY) will act on a bosonic state to give a fermionic one and vice versa:

$$Q\left|\text{Boson}\right\rangle = \left|\text{Fermion}\right\rangle, \quad Q\left|\text{Fermion}\right\rangle = \left|\text{Boson}\right\rangle \tag{1.41}$$

In 1967 though, the 'no-go' theorem of Coleman and Mandula [51] demonstrated that it is impossible to combine space-time and internal symmetries in any but a trivial way. The implications of this theorem would be that internal symmetries would not be able to relate particles with different masses or spins, clearly seeming to forbid the fundamental starting point of SUSY. However, the Haag-Lopuszański-Sohnius theorem [52] evades this restriction by considering extending the Poincaré group to include symmetries whose generators are fermionic, obeying anticommutation relations. The theorem proved that SUSY is the only possible extension of the Poincaré algebra, with the algebra written in terms of the additional generator Q_α ($\alpha = 1,2$) which can be chosen to transform as a left handed Weyl spinor, and its Hermitian conjugate $\overline{Q}_{\dot{\beta}}$, a right handed Weyl spinor:

$$\left\{Q_\alpha, Q_\beta\right\} = \left\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\right\} = 0 \tag{1.42}$$

$$\left[Q_\alpha, P_\mu\right] = 0 \tag{1.43}$$

$$\left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \tag{1.44}$$

The particles of a supersymmetric theory fall into *supermultiplets* containing both bosonic and fermionic fields, which are known as *superpartners* of each other. As the mass-squared operator $P_\mu P^\mu$ commutes with the operators Q and $\overline{Q}$ which transform different members of a supermultiplet into each other, it follows that the superpartners must have equal masses. Also, as the SUSY generators commute with the generators of gauge transformations, particles in the same supermultiplet transform in the same representation of the SM gauge group.

There are a few important consequences of the anticommutation relation in Eq. (1.44); the first of which can be seen by considering the operator $(-1)^{2s}$ as in [53], where s is the spin quantum number. This operator has an eigenvalue of $+1$ when acting on a bosonic state and -1 when acting on a fermionic state, and due to the fact that the operator Q_α turns bosons into fermions and vice versa, we have

$$(-1)^{2s}Q_\alpha = -Q_\alpha(-1)^{2s}$$

Bearing this anticommutation property in mind, we can consider a finite dimensional representation of the algebra, and take the following trace over the set of states in the supermultiplet

$$\begin{aligned}\mathrm{Tr}\left[(-1)^{2\mathrm{s}}\left\{\mathrm{Q}_\alpha,\overline{\mathrm{Q}}_{\dot\beta}\right\}\right] &= \mathrm{Tr}\left[-\mathrm{Q}_\alpha(-1)^{2\mathrm{s}}\overline{\mathrm{Q}}_{\dot\beta} + (-1)^{2\mathrm{s}}\overline{\mathrm{Q}}_{\dot\beta}\mathrm{Q}_\alpha\right] \\ &= \mathrm{Tr}\left[-\mathrm{Q}_\alpha(-1)^{2\mathrm{s}}\overline{\mathrm{Q}}_{\dot\beta} + \mathrm{Q}_\alpha(-1)^{2\mathrm{s}}\overline{\mathrm{Q}}_{\dot\beta}\right] \\ &= 0\end{aligned}$$

where on the second line the cyclic property of the trace has been used. From Eq. (1.44) this implies that

$$2\sigma^\mu_{\alpha\dot\beta}\mathrm{Tr}\left[(-1)^{2\mathrm{s}}\mathrm{P}_\mu\right] = 0$$

meaning that for a given non-zero P_μ, $\mathrm{Tr}(-1)^{2\mathrm{s}} = 0$. As the operator $(-1)^{2s}$ has equal and opposite value ±1 for bosons and fermions, this implies that in any supermultiplet we must have equal numbers of bosonic and fermionic degrees of freedom

$$n_B = n_F \tag{1.45}$$

Another important consequence of Eq. (1.44) can be seen by multiplying both sides by $(\overline{\sigma}^\nu)^{\dot\beta\alpha}$ and using the relation $\sigma^\mu\overline{\sigma}^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu}$ [54], leading to

$$(\overline{\sigma}^\nu)^{\dot\beta\alpha}\left\{Q_\alpha,\overline{Q}_{\dot\beta}\right\} = 4P^\nu$$

Taking the matrix element of the $\nu = 0$ component gives

$$
\begin{aligned}
\langle\psi| P^0 |\psi\rangle &= \frac{1}{4}\langle\psi| Q_1 Q_{\dot{1}} + Q_{\dot{1}} Q_1 + Q_2 Q_{\dot{2}} + Q_{\dot{2}} Q_2 |\psi\rangle \\
&= \frac{1}{4}\langle\psi| Q_\alpha (Q_\alpha)^* + (Q_\alpha)^* Q_\alpha |\psi\rangle \geq 0
\end{aligned}
\tag{1.46}
$$

As such, the energy of any non-vacuum state is positive definite and the vacuum energy is zero. This has important consequences for the spontaneous breaking of supersymmetry, as for SUSY to be spontaneously broken, the physical vacuum state $|0\rangle$ should not be annihilated by all the SUSY generators. Therefore, whenever SUSY is broken in the vacuum state it has positive energy, and all possible supersymmetric vacuua are degenerate with zero energy. Thus, the effective potential must possess no supersymmetric minimum in order for its global minimum corresponding to the physical vacuum to be non-supersymmetric.

1.3.1 Supermultiplets

From Eq. (1.45), the simplest example of a supermultiplet satisfying this constraint would be the case of a single Weyl fermion and a single complex scalar field. This combination is known as a *chiral* supermultiplet, and all the fermions of the Standard Model fit into such multiplets. The names for the spin-0 partners of the quarks and leptons are ‘squarks’ and ‘sleptons’ respectively, with the right and left handed parts of the fermionic fields having their own complex scalar partners, as shown in Table 1.1. As the Higgs boson is a spin-0 particle, it clearly must reside in a chiral superfield, but as seen from Table 1.1 we actually require two Higgs supermultiplets, H_u and H_d. This is in order to cancel Electroweak gauge anomalies and to give masses to both up and down type quarks as shall be demonstrated later.

The next simplest example of a supermultiplet is the so called *vector* supermultiplet, consisting of a massless spin-1 vector boson and a massless spin-$\frac{1}{2}$ Weyl fermion. The gauge bosons of the Standard Model fit into such multiplets alongside their fermionic partners, the ‘gauginos’, as summarised in Table 1.2. As the gauge bosons transform in the adjoint representation of the associated gauge group, the gauginos must as well, meaning that the left and right handed components of the gaugino fields must have the same transformation properties due to the reality of adjoint representations. It is for this reason that the fermions of the SM all have to be placed in chiral supermultiplets rather than vector ones.

1.3.2 Superspace and superfields

The mechanics of supersymmetry can be conveniently described by introducing the notion of *superspace* (reviewed in, for example, [55, 56]), where instead of fields just being functions of

Chiral supermultiplet	spin 0	spin $\frac{1}{2}$	$SU(3)_C, SU(2)_L, U(1)_Y$
Q	$(\tilde{u}_L, \tilde{d}_L)$	(u_L, d_L)	$(3, 2, \frac{1}{6})$
$\overline{u}$	$\tilde{u}_R^*$	$u_R^\dagger$	$(\overline{3}, 1, -\frac{2}{3})$
$\overline{d}$	$\tilde{d}_R^*$	$d_R^\dagger$	$(\overline{3}, 1, \frac{1}{3})$
L	$(\tilde{\nu}, \tilde{e}_L)$	(ν, e_L)	$(1, 2, -\frac{1}{2})$
$\overline{e}$	$\tilde{e}_R^*$	$e_R^\dagger$	$(1, 1, 1)$
H_u	(H_u^+, H_u^0)	$(\tilde{H}_u^+, \tilde{H}_u^0)$	$(1, 2, \frac{1}{2})$
H_d	(H_d^0, H_d^-)	$(\tilde{H}_d^0, \tilde{H}_d^-)$	$(1, 2, -\frac{1}{2})$

Table 1.1: Chiral supermultiplets in the MSSM

Vector supermultiplet	spin $\frac{1}{2}$	spin 1	$SU(3)_C, SU(2)_L, U(1)_Y$
Gluino, Gluon	$\tilde{g}$	g	$(8, 1, 0)$
Wino, W boson	$\tilde{W}^\pm$, $\tilde{W}^0$	$W^\pm$, W^0	$(1, 3, 0)$
Bino, B boson	$\tilde{B}^0$	B^0	$(1, 1, 0)$

Table 1.2: Vector supermultiplets in the MSSM

the space-time coordinates x^μ, a *superfield* $S(x, \theta, \overline{\theta})$ is also a function of the anticommuting two-component Grassmann variables θ_α and $\overline{\theta}_{\dot{\alpha}}$. Due to the anticommuting nature of the extra coordinates, a general superfield can be expanded as a power series in θ and $\overline{\theta}$, with no more than two powers of θ and $\overline{\theta}$ included in the expansion. In this formalism the fields contained in a particular supermultiplet are united into a single superfield, which is a function of the superspace coordinates. A finite SUSY transformation can now be written as

$$G(x^\mu, \theta, \overline{\theta}) = \exp\left[\mathrm{i}(\theta \mathrm{Q} + \bar{\theta}\bar{\mathrm{Q}} - \mathrm{x}^\mu \mathrm{P}_\mu)\right] \tag{1.47}$$

where the indices have been dropped on the spinors θ, Q, and their conjugates. A superfield $S(x^\mu, \theta, \overline{\theta})$ now transforms under a SUSY transformation as

$$G(a^\mu, \xi, \overline{\xi}) S(x^\mu, \theta, \overline{\theta}) = S(x^\mu + a^\mu - i\xi\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\xi}, \theta + \xi, \overline{\theta} + \overline{\xi}) \tag{1.48}$$

In order to find a representation for the SUSY generators, we can consider infinitesimal transformations of this form

$$\begin{aligned} S(x^\mu + a^\mu - i\xi\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\xi}, \theta + \xi, \overline{\theta} + \overline{\xi}) &= S(x^\mu, \theta, \overline{\theta}) + \delta_S S \\ = S(x^\mu, \theta, \overline{\theta}) + (a^\mu - i\xi\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\xi})\frac{\partial S}{\partial x^\mu} &+ \xi^\alpha \frac{\partial S}{\partial \theta^\alpha} + \overline{\xi}_{\dot{\alpha}} \frac{\partial S}{\partial \overline{\theta}_{\dot{\alpha}}} + \cdots \end{aligned} \tag{1.49}$$

This leads to the following linear representation in terms of differential operators which act on superfields

$$P_\mu = i\partial_\mu \tag{1.50}$$

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu \tag{1.51}$$

$$\overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{1.52}$$

We can now construct covariant derivatives with respect to θ and $\overline{\theta}$:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu \tag{1.53}$$

$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{1.54}$$

These covariant derivatives anticommute with Q and $\overline{Q}$, and also satisfy the algebra

$$\{D_\alpha, D_\beta\} = \left\{\overline{D}_{\dot{\alpha}}, \overline{D}_{\dot{\beta}}\right\} = 0 \tag{1.55}$$

$$\left\{D_\alpha, \overline{D}_{\dot{\beta}}\right\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \tag{1.56}$$

From the anticommutation relations combined with the form of the infinitesimal SUSY transformation in Eq. (1.49), it can be seen that $\delta_S(D_\alpha S) = D_\alpha(\delta_S S)$ and $\delta_S(\overline{D}_{\dot{\alpha}} S) = \overline{D}_{\dot{\alpha}}(\delta_S S)$. As the derivatives commute with SUSY transformations, they are said to be supersymmetric covariant and are useful in defining covariant constraints on superfields.

1.3.3 Chiral superfields

A chiral superfield $\Phi(x, \theta, \overline{\theta})$ is one for which one of the following constraints has been imposed

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \tag{1.57}$$

$$D_\alpha\Phi^\dagger = 0 \tag{1.58}$$

where Eq. (1.57) is said to describe a left chiral superfield, and Eq. (1.58) a right chiral one. If we introduce the coordinates

$$y_l^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta} \tag{1.59}$$

$$y_r^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta} \tag{1.60}$$

it can be seen that Eq. (1.57) is solved by any function of θ and y_l^μ due to the fact that $\overline{D}_{\dot{\alpha}}\theta = \overline{D}_{\dot{\alpha}}y_l^\mu = 0$. As such, the general solution written as an expansion in powers of θ is given by

$$\Phi(y_l^\mu, \theta) = \phi(y_l) + \sqrt{2}\theta\psi(y_l) + \theta\theta F(y_l) \tag{1.61}$$

where the fields ϕ and F are complex scalars, and ψ is a Weyl fermion. Bearing in mind that the coordinate θ has mass dimension of $-\frac{1}{2}$, ϕ and ψ can have the usual dimensions of $+1$ and $+\frac{3}{2}$ respectively, whereas the scalar field F has mass dimension $+2$, and is a non-propagating 'auxiliary' field. The expression for the right chiral superfield $\Phi^\dagger$ is the same, but with θ replaced by $\overline{\theta}$ and y_l^μ replaced by y_r^μ. The SUSY transformations of the component fields in the superfield expansion of Eq. (1.61) can be found by considering the infinitesimal forms

$$\Phi \to \Phi + \delta_S\Phi \tag{1.62}$$

$$\delta_S\Phi = i(\xi Q + \overline{\xi}\overline{Q})\Phi \tag{1.63}$$

$$= \delta_S\phi + \sqrt{2}\theta\delta_S\psi + \theta\theta\delta_S F \tag{1.64}$$

Comparing Eq. (1.63) with Eq. (1.64) leads to the transformations

$$\delta_S\phi = \sqrt{2}\xi\psi \qquad \text{(Boson} \to \text{Fermion)} \tag{1.65}$$

$$\delta_S\psi = \sqrt{2}\xi F - \sqrt{2}\sigma^\mu\overline{\xi}\partial_\mu\phi \qquad \text{(Fermion} \to \text{Boson)} \tag{1.66}$$

$$\delta_S F = -i\sqrt{2}\partial_\mu\psi\sigma^\mu\overline{\xi} \qquad \text{(F} \to \text{Total derivative)} \tag{1.67}$$

1.3.4 Vector superfields

A vector superfield is one which satisfies the constraint

$$V(x,\theta,\overline{\theta}) = V^\dagger(x,\theta,\overline{\theta}) \tag{1.68}$$

Starting with a Lorentz invariant superfield and expanding up to quadratic order in both θ and $\overline{\theta}$, we can use the fact that we can construct the Lorentz scalars $\theta\theta$ and $\overline{\theta\theta}$ and the vector $\theta\sigma^\mu\overline{\theta} = -\overline{\theta}\sigma^\mu\theta$, and impose Eq. (1.68) to get

$$V(x,\theta,\overline{\theta}) = (1+\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu)C(x) + (i\theta+\frac{1}{2}\theta\theta\sigma^\mu\overline{\theta}\partial_\mu)\chi(x) + (-i\overline{\theta}+\frac{1}{2}\bar{\theta}\bar{\theta}\sigma^\mu\theta\partial_\mu)\overline{\chi}(x)$$
$$+\frac{i}{2}\theta\theta\,[M(x)+iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}\,[M(x)-iN(x)] - \theta\sigma^\mu\overline{\theta}A_\mu(x)$$
$$+i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{1.69}$$

where C, M, N and D are real scalar fields, χ and λ are Weyl fermions, and V_μ is a real vector field. We could alternately form a real vector superfield satisfying Eq. (1.68) by looking at the following combination of the left and right chiral superfields

$$i(\Phi-\Phi^\dagger) = i(\phi-\phi^\dagger) + i\sqrt{2}(\theta\psi - \bar{\theta}\bar{\psi}) + i\theta\theta F - i\bar{\theta}\bar{\theta}F^\dagger$$
$$-\theta\sigma^\mu\overline{\theta}\partial_\mu(\phi+\phi^\dagger) - \frac{1}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi + \frac{1}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\overline{\psi}$$
$$-\frac{1}{4}i\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu(\phi-\phi^\dagger) \tag{1.70}$$

By comparing Eqs. (1.69) and (1.70) and noting the similarities in structure, we can consider the transformation of a vector superfield under a $U(1)$ 'supergauge' transformation to be

$$V \to V + i(\Phi - \Phi^\dagger) \tag{1.71}$$

where Φ is a chiral superfield gauge transformation parameter, with the form of Eq. (1.61) in terms of component fields. This transformation ensures the usual gauge transformation for the vector boson field in Eq. (1.69), $A_\mu \to A_\mu + \partial_\mu(\phi+\phi^*)$, but can also be used to eliminate the fields $C(x)$, $\chi(x)$, $M(x)$ and $N(x)$. Since these fields can be supergauged away, they are not physical fields, and in the 'Wess-Zumino gauge' where they are all chosen to be zero, the vector superfield appears in a simpler form

$$V_{WZ}(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{1.72}$$

It can be seen that the field D has mass dimension +2, just like the F field in the chiral superfield of Eq. (1.61). If a SUSY transformation is applied to the case of the vector superfield, it is found that D transforms as a total derivative, also analogous to the F field:

$$\delta_S D = -\xi\sigma^\mu\partial_\mu\overline{\lambda} + \overline{\xi}\sigma^\mu\partial_\mu\lambda \tag{1.73}$$

The fact that the F and D fields transform in this way will help greatly in constructing supersymmetric Lagrangians, as an integral over space-time of a total divergence will vanish provided the fields fall off fast enough at infinity. As such, these terms will be able to provide an action which is invariant under SUSY transformations.

1.3.5 Supersymmetric Lagrangians

The Lagrangian density will be a sum of products of the chiral and vector superfields, and we have seen that in both cases it is the component of the superfield which has the largest number of θ and $\overline{\theta}$ factors which transforms as a total derivative. As such, we can write an invariant action as

$$S = \int d^4x \left(\int d^2\theta \mathscr{L}_F + \int d^2\theta d^2\overline{\theta} \mathscr{L}_D \right) \tag{1.74}$$

Integration over Grassmann variables is defined by

$$\int d\theta_\alpha = 0, \quad \int \theta_\alpha d\theta_\alpha = 1 \tag{1.75}$$

In order to integrate over superspace, we can define

$$d^2\theta = -\frac{1}{4} d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta}, \quad d^2\overline{\theta} = -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \varepsilon^{\dot{\alpha}\dot{\beta}} \tag{1.76}$$

Combining Eqs. (1.75) and (1.76) gives the results

$$\int d^2\theta (\theta\theta) = 1, \quad \int d^2\bar{\theta} (\bar{\theta}\bar{\theta}) \tag{1.77}$$

From this we can see that the result of integrating a superfield over $d^2\theta$ picks out the $\theta\theta$ term; integrating over $d^2\overline{\theta}$ picks out the $\bar{\theta}\bar{\theta}$ term, and integrating over $d^2\theta d^2\overline{\theta}$ picks out the $\theta\theta\bar{\theta}\bar{\theta}$ term. This is just as required in order to select the highest component field of a chiral or vector supermultiplet. Now that the mechanics have been developed for building supersymmetric Lagrangians, possible contributions to the Lagrangian can be analysed by looking at products of superfields.

Firstly, it should be noted that the product of multiple left chiral superfields will always be left chiral (and similarly for right chiral superfields) as there is no $\overline{\theta}$ dependence. Any product of left chiral superfields will always terminate at the $\theta\theta$ term in the expansion due to the fact that $\theta\theta\theta = 0$, and so all these types of terms are of the same type as $\mathscr{L}_F$ in Eq. (1.74). Computing the product of two or three left chiral superfields $\Phi_{i,L} = \phi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i$ and integrating over superspace coordinates gives the following results

$$\int d^2\theta \Phi_{1,L}\Phi_{2,L} = \phi_1 F_2 + \phi_2 F_1 - \psi_1\psi_2 \tag{1.78}$$

$$\int d^2\theta \Phi_{1,L}\Phi_{2,L}\Phi_{3,L} = \phi_1\phi_2 F_3 + \phi_1 F_2\phi_3 + F_1\phi_2\phi_3 - \psi_1\phi_2\psi_3 - \phi_1\psi_2\psi_3 - \psi_1\psi_2\phi_3 \tag{1.79}$$

Note that we cannot have any higher product of superfields of this kind due to the fact that it would give rise to terms of mass dimension greater than 4 in the Lagrangian, leading to non renormalisable interactions. In order to find terms of the same type as $\mathscr{L}_D$ in Eq. (1.74), we can consider the product of a left and a right chiral superfield, $\Phi_L\Phi_L^\dagger$. Due to the fact that the conjugate of a left chiral field is right chiral, this term is clearly a vector superfield, and hence it is of the same type as $\mathscr{L}_D$:

$$\int d^2\theta d^2\overline{\theta}\Phi_L\Phi_L^\dagger = FF^* - \phi\partial_\mu\partial^\mu\phi^* - i\overline{\psi}\sigma_\mu\partial^\mu\psi \tag{1.80}$$

It is useful when considering the F-fields to combine Eqs. (1.78) and (1.79) to form the *superpotential*

$$W(\Phi_i) = \frac{1}{2}M^{ij}\Phi_i\Phi_j + \frac{1}{6}y^{ijk}\Phi_i\Phi_j\Phi_k \tag{1.81}$$

As such, the Lagrangian so far can be written

$$\mathscr{L} = \sum_i (F_iF_i^* + \left|\partial_\mu\phi\right|^2 - i\overline{\psi}_i\sigma_\mu\partial^\mu\psi_i) + \left[\sum_j \frac{\partial W(\phi_i)}{\partial\phi_j}F_j - \frac{1}{2}\sum_{j,k}\frac{\partial^2 W(\phi_i)}{\partial\phi_j\partial\phi_k}\psi_j\psi_k + \text{h.c}\right] \tag{1.82}$$

where the superpotential is now written as a function of the scalar fields ϕ_i rather than the superfields. We can now integrate out the F_i fields using their equations of motion, given by $\frac{\partial\mathscr{L}}{\partial F_i} = 0$, leading to

$$F_i = -\left[\frac{\partial W(\phi_j)}{\partial\phi_i}\right]^* \tag{1.83}$$

Now our Lagrangian consists of kinetic terms for the scalar and fermion components of a chiral supermultiplet but no kinetic term for the auxiliary fields F_i, as well as terms describing both fermion and scalar masses, and Yukawa and scalar interactions. However, the Lagrangian does not yet describe vector (gauge) superfields, and so in order to achieve this, we must consider consider the superspace Lagrangian for a gauge theory. For simplicity, we will take the case of a $U(1)$ theory where the vector superfield transforms as in Eq. (1.71), and we will specialise to the Wess-Zumino gauge where the vector superfield can be written as Eq. (1.72). In order to write down a field strength, the following spinor chiral superfields are defined

$$W_\alpha = -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V, \quad \overline{W}_{\dot{\alpha}} = -\frac{1}{4}(DD)\overline{D}_{\dot{\alpha}}V \tag{1.84}$$

It can be shown that W_α and $\overline{W}^{\dot{\alpha}}$ are supergauge invariant, and so hence is the quantity $W^\alpha W_\alpha$ which is also a left chiral superfield. As such, the F component (i.e. that associated with the $\theta\theta$ factor) of this term may appear in the superspace Lagrangian

$$\frac{1}{32g^2}W_\alpha W^\alpha = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{2}D_a D^a + \left(-\frac{i}{2}\lambda^a\sigma_\mu\partial^\mu\overline{\lambda}_a + \frac{g}{2}\lambda_a\sigma_\mu A^\mu_b\overline{\lambda}_c + \text{h.c.}\right) \tag{1.85}$$

where $F_{\mu\nu}$ is the ordinary field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and λ is the fermionic partner of the gauge boson A_μ, called the *gaugino*. In the $U(1)$ case that has been considered here, due to the fact that the integral over x^μ of the D-term component of the vector superfield is invariant under both SUSY and supergauge transformations, we can add a *Fayet-Iliopoulos* term to the Lagrangian

$$\mathscr{L}_{FI} = -2\kappa[V]_D = -\kappa D \tag{1.86}$$

1.3.6 The Minimal Supersymmetric Standard Model (MSSM)

The matter content in the MSSM is given in Tables 1.1 and 1.2 and the superpotential for the model is given by

$$W_{MSSM} = \overline{u}y_u QH_u - \overline{d}y_d QH_d - \overline{e}y_e LH_d + \mu H_u H_d \tag{1.87}$$

where y_u, y_d and y_e are dimensionless 3×3 Yukawa matrices and the μ term is the supersymmetric generalisation of the SM Higgs mass term. It is the only possible candidate term of this type due to the holomorphicity of the superpotential with the chiral superfields treated as comnplex variables- a fact which also explains the necessity for two Higgs doublets. Another reason for this requirement is that anomalies will not be cancelled with only one Higgs doublet due to gauge anomalies associated with triangle diagrams involving higgsinos. These anomalies are cancelled in the case of two Higgs doublets due to the fact that the two higgsino doublets have opposite hypercharge.

In addition to the terms written in the superpotential of Eq. (1.87) there are additional terms which satisfy all the requirements to be included, but are not contained in the MSSM because they either violate baryon or lepton number conservation. These terms which would be included in the most general superpotential are

$$W_{\Delta L=1} = \frac{1}{2}\lambda LL\bar{e} + \lambda' LQ\bar{d} + \mu' LH_u \tag{1.88}$$

$$W_{\Delta B=1} = \frac{1}{2}\lambda''\bar{u}\bar{d}\bar{d} \tag{1.89}$$

The consequences of the presence of these terms with unsuppressed λ' and λ'' couplings would be that if both $\Delta L = 1$ and $\Delta B = 1$ terms are included, proton decay would be induced at rapid rates, clearly incompatible with current experimental bounds. As such, a new symmetry must be introduced in the MSSM in order to forbid these terms whilst allowing the terms in Eq. (1.87). Such a symmetry is called *R-parity*, which is a discrete symmetry under which all the SM particles and the Higgs boson have $P_R = 1$ and all the SUSY particles have $P_R = -1$

$$P_R = (-1)^{3(B_L)+2s} \tag{1.90}$$

The consequences of R-parity conservation are that sparticles must be produced in even numbers, the lightest supersymmetric particle (LSP) must be stable, and every other SUSY particle must eventually decay into a final state containing an odd number of LSPs. The existence of a stable LSP provides a very attractive candidate for dark matter due to the extremely weak interactions it might have with normal matter.

1.4 Grand Unified Theories (GUTs)

The basic idea behind grand unification is to embed the three gauge groups and associated gauge couplings of the Standard Model into one larger group G, with one coupling constant g. In this way, it is possible to address more of the shortcomings of the SM, in particular by reducing the number of free parameters, putting symmetry restrictions on aspects of SM which are arbitrary, and providing an explanation for the apparent merging of the SM gauge couplings at a high scale. Clearly the first requirement that any potential GUT must satisfy is that the unified group must contain a $SU(3)\times SU(2)\times U(1)$ subgroup and have a rank of at least 4, in order to accommodate the four diagonal generators of the SM. Due to the chiral nature of fermions, it is also clear that G must have complex representations, and because of renormalisability we must have an anomaly free theory. Based on these restrictions, the only rank 4 candidate for a GUT is $SU(5)$, although the rank 5 group $SO(10)$ and the rank 6 group E_6 will also be considered in this thesis. The first proposed GUT was the Georgi-Glashow $SU(5)$ model [57], and it is useful in elucidating some general features of GUTs to look at this model in some detail.

1.4.1 The Georgi-Glashow $SU(5)$ model

$SU(5)$ is the group of 5×5 complex unitary matrices with determinant one. The group elements are given by

$$U = \exp\left(-\mathrm{i}\sum_{\mathrm{j}=1}^{24}\beta^{\mathrm{j}}\mathrm{L}^{\mathrm{j}}\right) \tag{1.91}$$

where the conditions on the group elements of unitarity and unit determinant mean that the 24 generators L^{j} are Hermitian and traceless. As we have 24 generators, we will also have 24 associated gauge bosons, and in order to describe them it is useful to define a basis for the 5×5 matrices where the SU(3) group acts on the first three rows and columns, and the SU(2) group acts on the last two. Normalising the generators such that

$$\mathrm{Tr}(\mathrm{L}^{\mathrm{a}}\mathrm{L}^{\mathrm{b}}) = \frac{1}{2}\delta^{\mathrm{ab}} \tag{1.92}$$

we can embed the SU(3) and SU(2) generators as

$$\mathrm{L}^{\mathrm{a}} = \frac{1}{2}\begin{pmatrix} & & & 0 & 0 \\ & \lambda^a & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{L}^{9,10} = \frac{1}{2}\begin{pmatrix} & & & 0 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\sigma^{1,2}$}} \\ 0 & 0 & 0 & & \end{pmatrix} \tag{1.93}$$

where λ^a with $a = 1,...,8$ are the Gell-Mann matrices, and $\sigma^{1,2}$ are the non diagonal Pauli matrices. Two more diagonal generators (in addition to L^3 and L^8) can be chosen to be proportional to the third component of weak isospin and the hypercharge generator as follows

$$\mathrm{L}^{11} = \mathrm{diag}\frac{1}{2}(0,0,0,1,-1), \quad \mathrm{L}^{12} = \frac{1}{2\sqrt{15}}\mathrm{diag}(-2,-2,-2,3,3) \tag{1.94}$$

Finally we have the non diagonal generators $\mathrm{L}^{13},\mathrm{L}^{14},....,\mathrm{L}^{23},\mathrm{L}^{24}$ which do not correspond to any of the SM gauge group generators. They can be chosen as in the following two matrices, where the other generators are found by following the same pattern.

$$\mathrm{L}^{13} = \frac{1}{2}\begin{pmatrix} & & & 1 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \mathrm{L}^{14} = \frac{1}{2}\begin{pmatrix} & & & i & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ -i & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & 0 & & \end{pmatrix} \tag{1.95}$$

It is useful to describe the $SU(5)$ gauge bosons by a linear combination of these generators

$$\frac{1}{\sqrt{2}}A_\mu = \sum_{a=1}^{24} A_\mu^a L_a \tag{1.96}$$

where it is convenient to introduce the gauge bosons X and Y as

$$\begin{aligned} X_\mu^1 &= \frac{A_\mu^{13}+iA_\mu^{14}}{\sqrt{2}}, \quad X_\mu^2 = \frac{A_\mu^{15}+iA_\mu^{16}}{\sqrt{2}}, \quad X_\mu^3 = \frac{A_\mu^{17}+iA_\mu^{18}}{\sqrt{2}} \\ Y_\mu^1 &= \frac{A_\mu^{19}+iA_\mu^{20}}{\sqrt{2}}, \quad Y_\mu^2 = \frac{A_\mu^{21}+iA_\mu^{22}}{\sqrt{2}}, \quad Y_\mu^3 = \frac{A_\mu^{23}+iA_\mu^{24}}{\sqrt{2}} \end{aligned} \tag{1.97}$$

With these definitions, we have

$$A_\mu = \begin{pmatrix} G_1^1 - \frac{2B}{\sqrt{30}} & G_2^1 & G_3^1 & \overline{X}^1 & \overline{Y}^1 \\ G_1^2 & G_2^2 - \frac{2B}{\sqrt{30}} & G_3^2 & \overline{X}^2 & \overline{Y}^2 \\ G_1^3 & G_2^3 & G_3^3 - \frac{2B}{\sqrt{30}} & \overline{X}^3 & \overline{Y}^3 \\ X_1 & X_2 & X_3 & \frac{W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & \frac{W_1+iW_2}{\sqrt{2}} \\ Y_1 & Y_2 & Y_3 & \frac{W_1-iW_2}{\sqrt{2}} & -\frac{W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{pmatrix} \tag{1.98}$$

In order to place the fermionic content of the Standard Model into $SU(5)$ representations, we must examine some of the lowest dimensional irreducible representations of $SU(5)$. The fundamental 5 dimensional representation is the simplest representation and can be represented by a 5 dimensional column vector. The branching rules for the 5 representation and its conjugate, the $\overline{5}$, when $SU(5)$ is decomposed into $SU(3)\times SU(2)\times U(1)$ are

$$\begin{aligned} 5 &\to (3,1)_{-\frac{1}{3}} + (1,2)_{\frac{1}{2}} \\ \overline{5} &\to (\overline{3},1)_{\frac{1}{3}} + (1,2)_{-\frac{1}{2}} \end{aligned} \tag{1.99}$$

where it has been noted that the $\overline{2}$ of SU(2) is equivalent to the 2, and the $U(1)$ charges have been normalised in such a way to facilitate comparisons with the SM hypercharges, related to the electric charge by Eq. (1.18). Looking at the components of the $\overline{5}$, it can be seen that we can identify the $(\overline{3},1)_{\frac{1}{3}}$ state with the anti-down like quarks and the $(1,2)_{-\frac{1}{2}}$ with the left-handed lepton doublet. Utilising the fact that the charge conjugate of a right-handed field transforms as a left-handed one, we can then express the $\overline{5}$ in terms of left-handed SM states as

$$\overline{5} = \begin{pmatrix} d_b^c \\ d_g^c \\ d_r^c \\ e \\ -\nu_e \end{pmatrix} \tag{1.100}$$

In order to build other representations of $SU(5)$, we can take products of the fundamental representation such as $5 \times 5 = 10 + 15$ [58]. The 10 is the antisymmetric product of two 5s, $x^{ij} = \frac{1}{\sqrt{2}}(a^i a^j - a^j a^i)$, $i, j = 1, ..., 5$, where the a^i are the components of the 5 representations. Under the decomposition to the SM gauge group, we have the following branching rule for the 10

$$10 \to (3,2)_{\frac{1}{6}} + (\overline{3},1)_{-\frac{2}{3}} + (1,1)_1 \tag{1.101}$$

As such, by comparing the components of the decomposition with SM states, we can embed the anti-up like quarks, left handed quark doublets and the positron into the 10 as follows:

$$10 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_r^c & -u_g^c & -u_b & -d_b \\ -u_r^c & 0 & u_b^c & -u_g & -d_g \\ u_g^c & -u_b^c & 0 & -u_r & -d_r \\ u_b & u_g & u_r & 0 & -e^c \\ d_b & d_g & d_r & e^c & 0 \end{pmatrix} \tag{1.102}$$

It can be seen that the hypercharge in these assignments can be written as $Y = CL^{12}$, where C is a normalisation constant. In order to make this normalisation consistent with the SM hypercharges and Eq. (1.18), we can note that the electric charges of the states in the $\overline{5}$ representation are given by $Q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1, 0)$, meaning that

$$\mathrm{Tr}(\mathrm{Q}^2) = \frac{4}{3} = \mathrm{Tr}(\mathrm{L}_{11}^2) + \mathrm{C}^2\mathrm{Tr}(\mathrm{L}_{12}^2) = \frac{1}{2}(1 + \mathrm{C}^2) \tag{1.103}$$

This implies the 'GUT normalised' hypercharge has $C = \sqrt{\frac{5}{3}}$, with the generator given by

$$Y = \frac{1}{2} \begin{pmatrix} -\frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.104}$$

The gauge bosons of Eq. (1.98) can be further understood by decomposing the 24 dimensional adjoint representation of $SU(5)$ under $SU(3) \times SU(2) \times U(1)_Y$

$$24 \to (8,1)_0 + (1,3)_0 + (1,1)_0 + (3,2)_{-\frac{5}{6}} + (\overline{3},2)_{\frac{5}{6}} \tag{1.105}$$

The first three terms simply correspond to the SM gauge bosons, whereas the $(\overline{3},2)_{\frac{5}{6}}$ contains the X_μ and Y_μ bosons (which are the $T_3 = \frac{1}{2}$ and $T_3 = -\frac{1}{2}$ components of the SU(2) doublet respectively), with the $(3,2)_{-\frac{5}{6}}$ containing $\overline{X}_\mu$ and $\overline{Y}_\mu$. Using Eq. (1.18), it can be seen that X_μ has electric charge $Q = \frac{4}{3}$ and Y_μ has $Q = \frac{1}{3}$. As the X and Y bosons possess both colour and charge, they induce interactions between quarks and leptons, and hence mediate baryon and lepton number violating processes. For this reason, they are called *leptoquarks*, and these interactions as well as *diquark* interactions will be important in the later discussion of proton decay.

1.4.2 Anomalies in the $SU(5)$ model

The anomaly in any fermion representation of SU(N) is proportional to

$$D^{abc} = \mathrm{Tr}\left[\left\{\mathrm{T_R^a}, \mathrm{T_R^b}\right\}\mathrm{T_R^c}\right] = \frac{1}{2}\mathrm{A(R)d^{abc}} \tag{1.106}$$

where T_R^a are the generators, A(R) is independent of these generators and d^{abc} is the third order antisymmetric invariant of $SU(N)$. As A(R) is independent of the generator choice, we can choose all the generators in Eq. (1.106) to be the charge generator, leading to

$$\begin{aligned} A(\overline{5}) &= \mathrm{TrQ}_{\overline{5}}^3 = 3(\frac{1}{3})^3 + (-1)^3 = -\frac{8}{9} \\ A(10) &= \mathrm{TrQ}_{10}^3 = 3(-\frac{2}{3})^3 + 3(\frac{2}{3})^3 + 3(-\frac{1}{3})^3 + 1 = \frac{8}{9} \end{aligned} \tag{1.107}$$

As such, $A(\overline{5}) + A(10) = 0$, and as long as we have complete $\overline{5} + 10$ fermion representations we have freedom from anomalies.

1.4.3 Fermion interactions with X and Y gauge bosons

The couplings between fermions and the X and Y bosons can be found by considering the gauge invariant kinetic energy terms for the $\overline{5}$ and 10 representations

$$\mathscr{L}_{X,Y} = i\overline{\psi}^a_{\bar{5}}\gamma^\mu(D_\mu\psi_{\bar{5}})_a + i\overline{\psi}^{ab}_{10}\gamma^\mu(D_\mu\psi_{10})_{ab}$$
$$= \overline{\psi}^a_{\bar{5}}(i\gamma^\mu\partial_\mu\delta^b_a + \frac{g_5}{\sqrt{2}}\gamma^\mu A^b_{\mu a})\psi_{\bar{5}b} + \overline{\psi}^{ab}_{10}(i\gamma^\mu\partial_\mu\delta^c_b + \frac{g_5}{\sqrt{2}}\gamma^\mu A^c_{\mu b})\psi_{10ac} \quad (1.108)$$

The couplings between fermions and the X and Y gauge bosons are given in Figure 1.2.

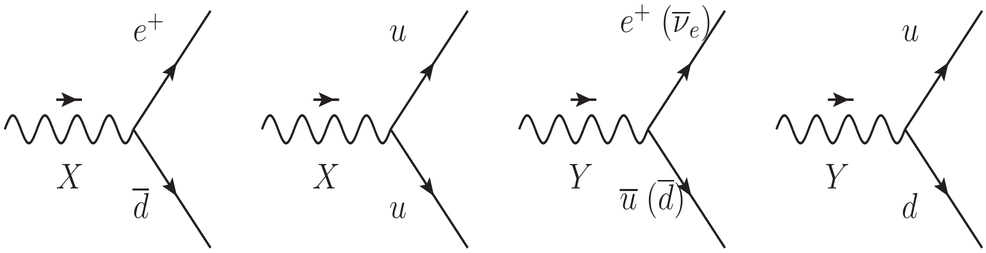

Figure 1.2: X and Y boson couplings in SU(5)

These vertices will lead to Feynman diagrams which violate Baryon and Lepton number. In unbroken $SU(5)$, all fields are massless, and these processes would lead to proton instability which is inconsistent with current experimental data. However, when $SU(5)$ is spontaneously broken, proton decay can be avoided if the breaking occurs at a scale which is high enough above the Electroweak scale. As such, the details of spontaneous symmetry breaking in $SU(5)$ are extremely important in the discussion of a realistic theory.

1.4.4 Spontaneous symmetry breaking

Spontaneous symmetry breaking in $SU(5)$ occurs in two steps- firstly the breaking to the SM gauge group at a scale M_{GUT} where the X and Y bosons become massive, and secondly Electroweak symmetry breaking which gives masses to the $W^\pm$ and Z bosons. The first step is achieved by giving a VEV to an adjoint Higgs multiplet, which we can write as 24 scalar fields, Σ_a, $a = 1,...,24$. The couplings of gauge fields to Σ_a can be found by considering the following kinetic term and covariant derivative

$$\mathscr{L}_k = \frac{1}{2}(D_\mu)^\dagger_a(D^\mu)_a \quad (1.109)$$

$$(D_\mu\Sigma)_a = (\partial_\mu\delta_{ab} - \frac{ig}{2}A^c_\mu(F^c)_{ab})\Sigma_b, \quad a,b,c = 1,...,24 \quad (1.110)$$

where A^c_μ are the 24 gauge fields of $SU(5)$ and $(F^c)_{ab}$ are the 24 generators in the adjoint representation. The form of the adjoint representation can be seen easily by adopting a tensor product notation, where the tensor product of n 5s and m $\overline{5}$s is denoted by $\left|{}^{i_1 \ldots i_m}_{j_1 \ldots j_n}\right\rangle = \left|{}^{i_1}\right\rangle \ldots \left|{}^{i_m}\right\rangle \left|{}_{j_1}\right\rangle \ldots \left|{}_{j_n}\right\rangle$. We can now express an arbitrary state in terms of components in this tensor product space as $|v\rangle = \left|{}^{i_1 \ldots i_m}_{j_1 \ldots j_n}\right\rangle v^{j_1 \ldots j_n}_{i_1 \ldots i_m}$. In this way, it can be seen that we can construct the adjoint representation by taking the product of a 5 (represented by u^i) and a $\overline{5}$ (represented by v_j) as follows:

$$u^i v_j = (u^i v_j - \frac{1}{5}\delta^i_j u^k v_k) + \frac{1}{5}\delta^i_j u^k v_k \tag{1.111}$$

$$5 \times \overline{5} = 24 + 1 \tag{1.112}$$

We can see that the first term in Eq. (1.111) corresponds to a 5×5 matrix which is traceless, hence giving the 24 degrees of freedom of the adjoint, and the second term transforms as a singlet due to the invariant nature of the δ tensor under $SU(5)$ transformations. As the Higgs scalar multiplet Σ_a is adjoint valued, it can be represented by a 5×5 matrix, Σ, which transforms in the same way as the adjoint representation. As such, the covariant derivative is given by

$$D_\mu \Sigma = \partial \Sigma - \frac{ig}{\sqrt{2}} \left[A_\mu, \Sigma\right] \tag{1.113}$$

where A_μ is as given in Eqs. (1.96) and (1.98). When Σ acquires a VEV, the resulting masses for the X and Y bosons can be calculated by inserting this expression into Eq. (1.109), leading to a mass matrix of the following form

$$\frac{1}{4} g^2 \mathrm{Tr}\left[\mathrm{A}_\mu, \langle\Sigma\rangle\right]^2 = \mathrm{m}^2_{\mathrm{ab}} \mathrm{A}^{\mathrm{a}}_\mu \mathrm{A}^{\mu \mathrm{b}} \tag{1.114}$$

In order to understand the possible forms for the VEV in this equation, we must construct a potential for Σ. To this end, we can write down the most general gauge invariant scalar potential with all couplings of dimension 4 or less so as not to spoil renormalisability. The potential can also be simplified by imposing invariance under a discrete Z_2 symmetry, $\Sigma \to -\Sigma$, and with this choice, the most general form is

$$V = -\frac{\mu^2}{2}\mathrm{Tr}\Sigma^2 + \frac{\lambda}{4}\mathrm{Tr}\Sigma^4 + \frac{\lambda'}{4}(\mathrm{Tr}\Sigma^2)^2 \tag{1.115}$$

It follows from Eq. (1.114) that $\langle\Sigma\rangle$ must be diagonal in order to keep the SM gauge bosons massless whilst giving masses to the X and Y bosons. We can always make a gauge transformation which puts Σ in a diagonal form, and after doing this it can be shown that there is a unique minimum for the potential which breaks $SU(5)$ down to the SM group:

$$\langle\Sigma\rangle = \frac{v}{2}\text{diag}(2,2,2,-3,-3) \tag{1.116}$$

It can be seen that this is proportional to the hypercharge generator, and putting this form into Eq. (1.115) and differentiating, shows that there is a non zero solution where we have an extremum of the potential

$$v^2 = \frac{4\mu^2}{7\lambda + 30\lambda'} \tag{1.117}$$

which is a minimum provided that $\mu^2 > 0$ and $7\lambda + 30\lambda' > 0$. Now that a minimum has been found where the VEV for Σ breaks $SU(5)$ to $SU(3)\times SU(2)\times U(1)$, Eq. (1.117) can be inserted into Eq. (1.114) to obtain the masses of the X and Y bosons after this first stage of SSB

$$m_X^2 = m_Y^2 = \frac{25}{8}g^2v^2 \tag{1.118}$$

The second stage of symmetry breaking is now concerned with the Electroweak breaking of the SM by a Higgs field which is a doublet under $SU(2)_L$. Looking at the branching rules for the 5 and $\overline{5}$ representations in Eq. (1.99), it is clear that the simplest possibility is to introduce a Higgs multiplet which is a 5 of $SU(5)$

$$H = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ h^+ \\ -h^0 \end{pmatrix}, \quad \langle H\rangle = v_0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.119}$$

where D is a colour triplet, and the VEV has been chosen to lie in the neutral $SU(2)_L$ direction in order to achieve the required symmetry breaking. In analogy with Eq. (1.15), this VEV could be driven by a potential of the form

$$V(H) = -\frac{1}{2}v^2|H|^2 + \frac{1}{4}\lambda''(|H|^2)^2 \tag{1.120}$$

However, this potential cannot be the whole story as it is not able to give masses to the colour triplet D, and as shall be seen later, this can give rise to dangerously rapid proton decay. Also, we have not included any terms coupling H and Σ to each other, so we can add to Eq. (1.120) the following two gauge invariant terms of mass dimension less than 4

$$V(\Sigma,H) = \alpha|H|^2\,\text{Tr}(\Sigma^2) + \beta \text{H}\Sigma^2\text{H} \tag{1.121}$$

Once Σ has acquired a VEV as in Eq. (1.116), we can write $H = \begin{pmatrix} D \\ h \end{pmatrix}$ and expand Eq. (1.121) to give

$$V(\Sigma,H) = v^2(\frac{15}{2}\alpha+\beta)D^\dagger D + v^2(\frac{15}{2}\alpha+\frac{9}{4}\beta)h^\dagger h \tag{1.122}$$

Since v is of the GUT scale, the first of these terms can give a mass to the colour triplet fields at the GUT scale as required. However, in order to preserve this fact whilst giving the Electroweak Higgs the desired mass, we must adjust the parameters α and β such that the second term is small enough. Making this term equal to zero requires the relation $\alpha = -\frac{9}{30}\beta$, and this precise adjustment is an example of a fine tuning of Lagrangian parameters. This feature is known as *doublet-triplet splitting*, and in minimal $SU(5)$ a fine tuned cancellation is the only way to solve the problem, although we will meet an attractive solution later when discussing breaking $SU(5)$ by hypercharge flux in the context of F-theory.

1.4.5 Yukawa couplings

As in the SM case, masses arise from the gauge invariant couplings of products of fermion fields and Higgs scalars, after spontaneous symmetry breaking. In the minimal case, we can just take a single Higgs in the fundamental representation of $SU(5)$, $H = \begin{pmatrix} D \\ h \end{pmatrix}$, in which case we get the following renormalisable Yukawa couplings

$$\mathscr{L}_Y = (\bar{5}_M)_i Y_5 10_M^{ij} (5_H^*)_j + \frac{1}{4}\varepsilon_{ijklm} 10_M^{ij} Y_{10} 10_M^{kl} 5_H^m \tag{1.123}$$

where Y_5 and Y_{10} are Yukawa matrices in generation space and i,j,k,l,m are $SU(5)$ indices. Picking out the terms involving the light Higgs field h yields

$$\mathscr{L}_Y \to d^c Y_5 Q h^* + L Y_5 e^c h^* + u^c (Y_{10} + Y_{10}^T) Q h \tag{1.124}$$

Due to the structure of $SU(5)$, it can be seen that Y_5 dictates both the down quark and charged lepton masses once h has acquired a VEV. As such, we can diagonalise Y_5 in flavour space and get the following relations between masses at the GUT scale

$$m_d = m_e, \quad m_s = m_\mu, \quad m_b = m_\tau \tag{1.125}$$

As is the case for couplings, radiative corrections cause the masses to run as the scale at which they are measured changes. As such, taking into account this running of the masses, only the prediction $m_b = m_\tau$ is acceptable, and in order to rectify the situation we must go beyond minimal

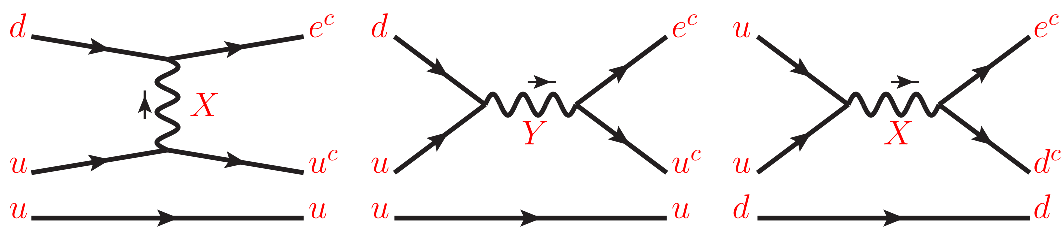

Figure 1.3: Dimension 6 proton decay diagrams

$SU(5)$. One solution is to introduce a Higgs in the 45 representation as well as the fundamental one (as in [59]), which can result in the acceptable GUT mass relations

$$m_e = \frac{1}{3} m_d, \quad m_\mu = 3 m_s, \quad m_\tau = m_b \tag{1.126}$$

1.4.6 SUSY GUTs

Despite the many attractive features of the minimal $SU(5)$ GUT, it has problems which mean that it cannot be a totally realistic model. Firstly, if the SM gauge couplings are run up to the GUT scale it is found that they do not meet, and secondly it predicts massless neutrinos which is in contradiction with experiment. The first of these shortcomings can be addressed if GUTs are combined with SUSY, as if we take the case of the MSSM with all superpartners at the TeV scale, we have the beautiful result that the three couplings unify at a single point around $\mu \approx 10^{16}$GeV [60]. Discussion of possible ways of giving mass to neutrinos in the setting of SUSY GUTs will be postponed until we start looking at semi realistic models resulting from F-theory GUTs. In the meantime, we can note that the Yukawa sector of Eq. (1.123) is modified due to the fact that we now have two Higgs doubles, H_u coming from a 5 and H_d from a $\overline{5}$. As such, we can obtain the Yukawa couplings from the following superpotential

$$W_Y = \overline{5}_M Y_5 10_M \overline{5}_{H_d} + \varepsilon_5 10_M Y_{10} 10_M 5_{H_u} \tag{1.127}$$

1.4.7 Proton decay

As pointed out previously, the couplings of the X and Y bosons to fermions given in Figure 1.2 can lead to proton decay. The dimension 6 operators resulting from the exchange of these bosons with GUT scale masses are shown in Figure 1.3, and illustrate the dominant decay mode by gauge exchange, $p \to e^+ \pi^0$.

Based on these diagrams, a crude estimate for the lifetime of the proton can be obtained as $\tau_p \approx \frac{m_X^4}{g^4 m_p^5}$, and as such, the GUT scale can be calculated from the measurement of this lifetime.

In SUSY GUTs the GUT scale is around 3×10^{16}GeV, whereas in the non supersymmetric case it is of order 10^{15}GeV [61, 62, 63]. This leads to a suppression of the dimension 6 operators in SUSY GUTs, and a lifetime of around 10^{34-38} years, which is to be compared with results from the Super-Kamiokande experiment, which give $\tau(p\to e^+\,\pi^0)>5.0\times 10^{33}$yrs. Even though the lifetime in the supersymmetric case is above this bound, this experimental result rules out minimal $SU(5)$ models where the GUT scale is lower.

In addition to these dimension 6 operators, in SUSY GUTs there are dimension 4 and 5 operators which could potentially be dangerous with regards to nucleon decay [61]. The dimension 4 terms can be forbidden as discussed previously by imposing R parity, however there are dimension 5 operators resulting from colour triplet Higgsino exchange which require attention [64]. Starting from the superpotential terms contained in Eq. (1.127), we have the following couplings for the H_u, H_d and associated colour triplet fields D and $\overline{D}$

$$\begin{aligned} W \supset\, & H_d(QY_d d^c + LY_e e^c) + \overline{D}(QY_{ql}L + u^c Y_{ud} d^c) \\ & + H_u Q Y_u u^c + D(QY_{qq}Q + u^c Y_{ue} e^c) \end{aligned} \tag{1.128}$$

where the first line corresponds to the first term in Eq. (1.127), the second line corresponds to the second term, and the Y matrices in flavour space have been left general in order to describe any model. Integrating out the colour triplets leads to the following dimension 5 operators

$$W_5 = \frac{1}{M_D}(Y_{qq}^{ij}Y_{ql}^{mn}Q_iQ_jQ_mL_n + Y_{ue}^{ij}Y_{ud}^{mn}u_i^c e_j^c u_m^c d_n^c) \tag{1.129}$$

These operators lead to proton decay via the diagrams shown in Figures 1.4 and 1.5, where the sparticles are converted to particles by gaugino or Higgsino ‘dressing’, and the dominant decay mode is $p\to K^+\,\overline{\nu}$. The reason for the presence of a kaon in this decay is due to a symmetry argument based on the fact that we are dealing with operators built from four bosonic superfields. We must have all the colour and $SU(2)_L$ indices correctly antisymmetrised for invariance under the SM gauge group, but because of the bosonic nature of the superfields the operators must be totally symmetric under the interchange of all indices. Due to the fact that in both dimension 5 operators we have the appearance of at least two of the same superfields, the family indices cannot be the same on the identical fields, meaning that we must have the presence of a second or third family particle in the final state.

The Super-Kamiokande bounds on the proton lifetime put severe constraints on these dimension 5 operators, and in fact minimal SUSY $SU(5)$ is ruled out [65]. One motivation of moving to F-theory GUTs is that there is the potential for the natural inclusion of additional symmetries which can forbid dimension 5 operators, and also the existence of intricate flux mechanisms which can eliminate Higgs colour triplet states from the particle spectrum of a particular model.

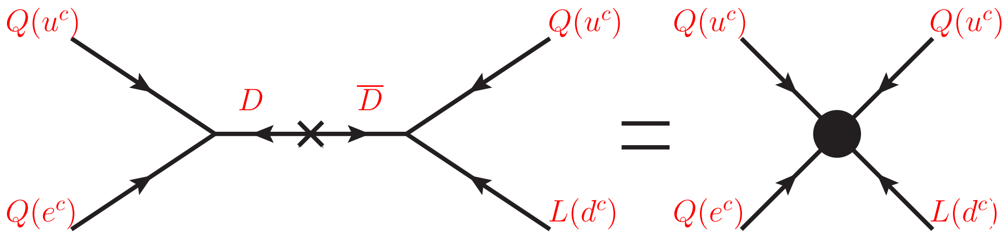

Figure 1.4: Dimension 5 operator from colour triplet Higgsino exchange

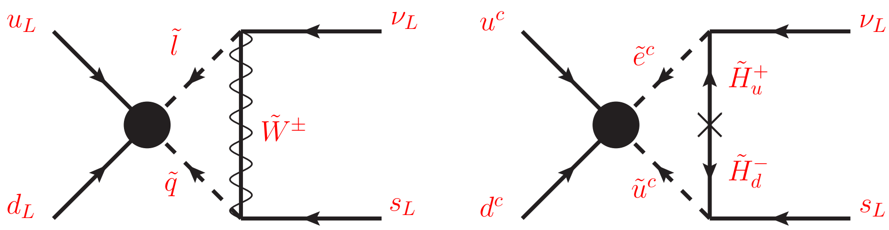

Figure 1.5: Dimension 5 proton decay diagrams

1.4.8 Higher rank GUT groups

The embedding of the Standard Model gauge group into $SU(5)$ is just part of the following larger chain of embeddings of exceptional groups

$$E_3 \times U(1) \supset E_4 \supset E_5 \supset E_6 \supset E_7 \supset E_8 \tag{1.130}$$

This chain is shown nicely in Figure 1.6, where it can be seen that by starting from the exceptional group E_8, the non-abelian part of the SM can be obtained in steps by deleting one node from each diagram. In F-theory GUTs this chain is extremely important, as it is assumed that all the interactions of the theory come from a single point in the internal geometry where the gauge symmetry is enhanced to E_8. In this thesis, in addition to $SU(5)$, GUT groups of $SO(10)$ and E_6 will be studied, where all the SM matter plus the right handed neutrino are embedded into one 16 dimensional representation of $SO(10)$, and this 16 representation is embedded into the fundamental 27 representation of E_6. It turns out that there is a rich phenomenology associated with the embedding of these GUT groups inside the parent E_8 group, and this provides much motivation for model building.

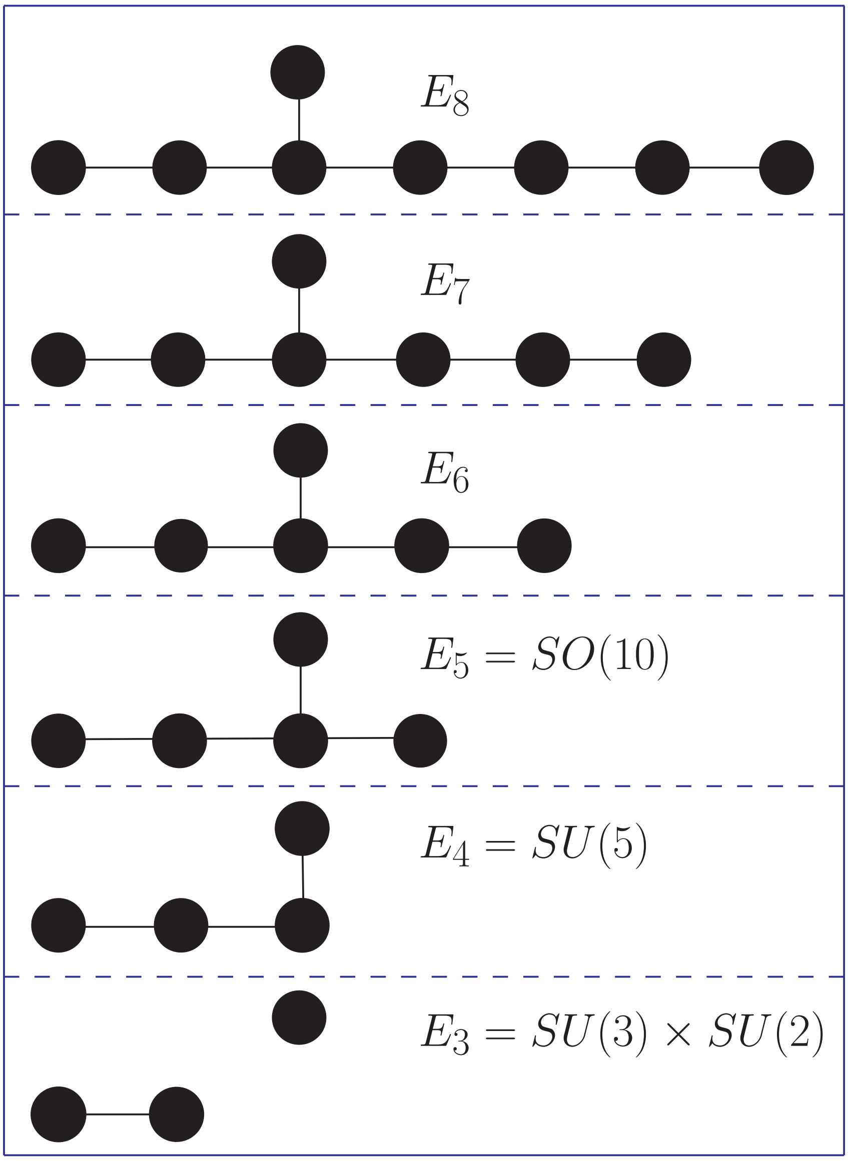

Figure 1.6: Chain of dynkin diagrams from E8 to the Standard Model

1.5 Introducing Strings and Branes

A BSM framework which incorporates SUSY and GUTs is that of String Theory, where we have ten space-time dimensions where six are compactified and very small, and instead of fundamental point-like particles, we now have one-dimensional strings. Just as a point particle sweeps out a worldline in Minkowski space, a string sweeps out a 'worldsheet', which we can parameterise by one timelike coordinate τ, and one spacelike coordinate σ, as depicted in Figure 1.7. The string then sweeps out a surface in spacetime, defining a map from the worldsheet to Minkowski space, $X^{\mu}(\sigma,\tau)$.

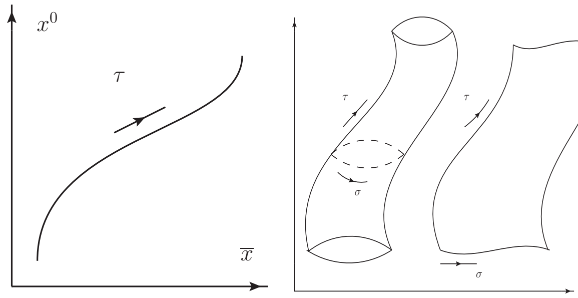

Figure 1.7: The worldline of a particle, and the worldsheets of closed and open strings.

These strings can come in two types- 'open strings' and 'closed strings'- depending on whether we take σ to be periodic, and the strength of interactions between strings is set by the string coupling constant, g_s. Type IIB superstring theory is a variant which includes both open and closed strings, and we will start by considering the case of $g_s << 1$ in this theory. In this perturbative regime, the particles of the Standard Model are described by excitations of open strings, whereas the graviton and gravitino are related to closed strings. Motivated by the weakness of gravity, one could try and formulate the Standard Model by just using open strings.

When dealing with open strings, we must consider the boundary conditions associated with the end points. It turns out that there are two types of boundary conditions we can impose, which are consistent with the string action:

- Neumann boundary conditions where the end points the string are allowed to move freely.
- Dirichlet boundary conditions where the end points are fixed at some position $X^{\mu} = c^{\mu}$.

If we consider some coordinates to have Neumann boundary conditions for $\mu = 0, ..., p$ and Dirichlet boundary consitions for $\mu = p+1, ..., D-1$, the ends of the string are fixed to lie in a (p+1) dimensional hypersurface in spacetime, which is called a Dp brane. The relevant branes for F-Theory are D7 branes. Particles in our four dimensional spacetime now correspond to strings stretched between D branes, and the mass of these particles are related to the tension T, and the distance in the internal dimensions between the branes, D.

$$M = T \times D \tag{1.131}$$

As all the SM particles are effectively massless, this corresponds to $D = 0$, so we are interested in the case where the ends of open strings can coincide in the internal dimensions. If we take N D7 branes filling the same dimensions (a stack of branes), a $U(N)$ gauge theory is realised, with the gauge bosons corresponding to strings which begin and end on any of the D7 branes in the same stack. If, for example, we have a stack of three D7 branes and another stack of two, there can be an intersection of the branes in two of the internal dimensions. In this case, we have a $U(3) \times U(2)$ gauge group, and we can have massless open strings which begin on the $U(3)$ stack and end on the $U(2)$ stack, corresponding to states charged under both gauge groups. These 'bifundamental' states are the matter fields of the theory, and in our simple example can be combined with the $U(3)$ and $U(2)$ gauge bosons into a 5×5 matrix of $U(5)$, with the gauge fields in 3×3 and 2×2 diagonal blocks, and the matter fields in the off diagonal positions. As such, at the intersection of the stack of branes, we can embed all the states into a $U(5)$ group, meaning that we can interpret this set up as a $U(5)$ gauge group at the intersection, broken down to $U(3) \times U(2)$ away from the intersection. If we have a triple intersection of D7 branes, we again have a further enhancement of the gauge group. Using these ideas, we can try and build the SM using perturbative, intersecting D-branes, in an $SU(5)$ GUT setting.

In order to realise the $\overline{5}$ and 10 of $SU(5)$, we can consider the intersection of the $SU(5)$ GUT brane with $U(1)$ branes, so that the symmetry group is enhanced to $SU(6)$ and $SO(10)$ respectively. We can see that these are the enhanced symmetry groups required to realise the $\overline{5}$ and 10 by (purely as a group theory exercise) decomposing the adjoint of the enhanced group into representations of $SU(5) \times U(1)$.

For the $\overline{5}$:

$$SU(6) \to SU(5) \times U(1) \tag{1.132}$$
$$35 \to (24,0) + (1,0) + (\overline{5},6) + (5,-6) \tag{1.133}$$

and for the 10:

$$SO(10) \to SU(5) \times U(1) \tag{1.134}$$
$$45 \to (24,0) + (1,0) + (\overline{10},-4) + (10,4) \tag{1.135}$$

We can now go one step further and try and realise the Yukawa interactions responsible for the masses of the bottom and top quarks. For the bottom quark, the relevant interaction term is

$\overline{5}_H \times \overline{5}_M \times 10_M$, and this corresponds to an enhancement to $SO(12)$ at a point in the geometry. To see this, we look at the decomposition

$$SO(12) \to SU(5) \times U(1)_1 \times U(1)_2 \tag{1.136}$$

$$66 \to (24,0,0) + (1,0,0) + (\overline{10},-4,0) + (10,4,0) + (5,2,2) + (5,2,-2) + (\overline{5},-2,2) + (\overline{5},-2,-2) \tag{1.137}$$

We can now make the identification

$$\overline{5}_H \times \overline{5}_M \times 10_M \sim (\overline{5},-2,2) \times (\overline{5},-2,-2) \times (10,4,0) \tag{1.138}$$

and can see that this operator is invariant under $U(1)_1 \times U(1)_2$. The relevant operator for the top quark is $5_H \times 10_M \times 10_M$, and unfortunately, this term cannot be realised in the same way. This is because we are actually realising a $U(5) = SU(5) \times U(1)_x$ gauge theory, and the $\overline{5}_H \times \overline{5}_M \times 10_M$ term is invariant under this, but the $5_H \times 10_M \times 10_M$ is not. As such, in order to give mass to the top quark, we must go to the case where g_s is of order 1, and F-Theory. In this case, exceptional gauge groups can be realised, which provides us with the correct structure to accomodate the $\overline{5}_H \times \overline{5}_M \times 10_M$ term. The enhancement for this coupling turns out to be E_6, and we can see this in the usual way, from the decomposition

$$E_6 \to SU(5) \times U(1)_1 \times U(1)_2 \tag{1.139}$$

$$78 \to (24,0,0) + (1,0,0) + (1,0,0) + (\overline{10},-4,0) + (10,4,0) + (\overline{10},1,3) + (10,-1,-3) + (5,-3,3) + (\overline{5},3,-3) + (1,-5,-3) + (1,5,3) \tag{1.140}$$

In fact, in F-Theory GUTs, all the interactions come from a point of E_8 enhancement in the geometry.

1.6 Moving to F-Theory

F-theory is a 12 dimensional, strongly coupled formulation of type IIB superstring theory. Formally, F-theory can be defined on a background $R^{3,1} \times X$ where $R^{3,1}$ is 4 dimensional space time, and X is a Calabi-Yau (CY) complex fourfold. It is assumed that X is elliptically fibered with a section over a complex three-fold base, B_3 [7, 9]. The meaning of this can be understood more clearly by considering the example of a fibre bundle, which is a collection of topological spaces E, B, F, and a continuous surjective (onto) map π:

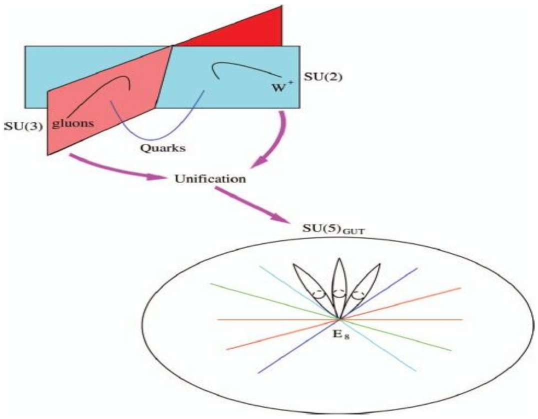

Figure 1.8: The point of E8 enhancement (figure from [5])

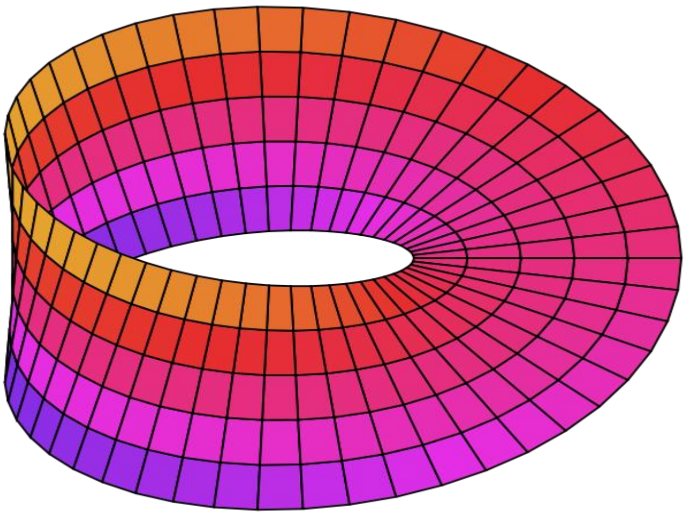

Figure 1.9: The Mobius strip as an example of a fibre bundle

- E is the Total Space
- B is the Base Space
- F is the Fibre
- $\pi : E \to B$ is a map from the total space to the base space

The key is that E must be locally trivial. This means that the total space must locally look like the trivial bundle, $E = B \times F$, but globally, there can be a twist. This can be understood better by looking at the example of the Mobius strip, shown in Figure 1.9. Here the base space is the circle, and the fibres are line segments. Locally, the Mobius strip looks like a cylinder, which is the trivial bundle.

Going back to our case of a CY fourfold, elliptically fibered over a three-fold base, there fibres are no longer line segments, but are two-tori. What this means is that each point of the base

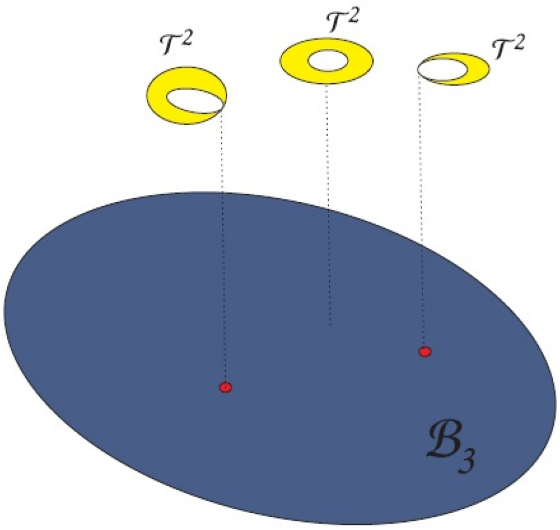

Figure 1.10: CY four-fold, involving an elliptic fibration over a three-fold base, B_3 (figure from [66])

B_3 is represented by a two-torus. The dimensions occupied by the base are the 6 compactified dimensions of type IIB string theory, and the complex modulus of the torus fibre encodes the axion and dilaton (two scalars contained in the bosonic spectrum) at each point on the base:

$$\tau = C_0 + ie^{-\phi} = C_0 + \frac{i}{g_s} \tag{1.141}$$

It is a fact that the presence of D7-branes (filling 7 spatial dimensions and 1 time dimension) affects the profile of the axio-dilaton, τ. As such, the reason that F-theory can be viewed as a 12 dimensional theory is that two dimensions are geometric dimensions which allow us to keep track of the variation of τ over the other ten dimensions. The form of the elliptic fibration can be described in what is known as Weierstrass form as

$$y^2 = x^3 + f(z)x + g(z) \tag{1.142}$$

where x,y,z are complex coordinates on $\mathbb{C}^3$. x and y are coordinates on a two complex dimensional surface S, and z is the coordinate perpendicular to this surface inside the base. The discriminant of the cubic in x is given by

$$\Delta = 4f^3 + 27g^2 \tag{1.143}$$

and when this discriminant vanishes, it indicates the presence of D7-branes, and the elliptic curve becomes singular along a two complex dimensional subspace in B_3. The equation $\Delta = 0$ can factorise into irreducible polynomials in coordinates of the base, $\Delta = \Delta_1 ... \Delta_n = 0$, with each

Group	a_1	a_2	a_3	a_4	a_6	Δ
SU(2n)	0	1	n	n	2n	2n
SU(2n+1)	0	1	n	n+1	2n+1	2n+1
SO(10)	1	1	2	3	5	7
E_6	1	2	2	3	5	8
E_7	1	2	3	3	5	9
E_8	1	2	3	4	5	10

Table 1.3: Classification of Singularities

equation $\Delta_i = 0$ describing the location of a D7-brane. In terms of the torus fibre, the torus degenerates (pinches off) at these points.

1.6.1 D7-branes, Gauge Groups and Singularities

In F-theory, the GUT group is realised on a 7-brane which wraps some 2 complex dimensional surface S. One feature of the analysis of the fibration is that not only does the vanishing discriminant give us the locations of 7-branes, it also gives us information about what gauge group is supported by the 7-brane, depending on the order to which the discriminant vanishes. Much mathematical work has been done on this subject, and a classification of which singularities correspond to which gauge group has been done by Kodaira. In order to describe these singularities, a procedure called Tate's algorithm can be followed [67, 68]. Taking a coordinate on the base, z, such that S is defined by $z = 0$, the coefficients f and g of Eq. (1.143) can be expanded in powers of z.

$$f(z) = \sum_n f_n z^n, \;\; g(z) = \sum_m g_m z^m \tag{1.144}$$

These expansions can then be inserted into Eq. (1.142) and the form of the Weierstrass equation can be studied, depending on to which order in z the discriminant vanishes.

The general Tate form of the Weierstrass equation can be written

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{1.145}$$

Now we can look at a particular singularity by examining the Kodaira classification of singularities, and seeing how the a_i must vary with z in order to produce the required singularity. Some examples of the order to which the coefficients $a_i \sim z^{n_i}$ and the discriminant Δ must vanish in order to give certain gauge groups are shown in Table 1.3.

For example, an $SU(5)$ singularity would be given by the choice

$$a_1 = -b_5, a_2 = b_4 z, a_3 = -b_3 z^2, a_4 = b_2 z^3, a_6 = b_0 z^5 \tag{1.146}$$

leading to the equation

$$y^2 = x^3 + b_0 z^5 + b_2 x z^3 + b_3 y z^2 + b_4 x^2 z + b_5 xy \tag{1.147}$$

where the b_i are fibration coefficients which do not depend on z.

So far, everything that has been stated applies to global F-theory, and in a global model, the structure of the GUT theory is described by the dependence of the b_i on the base coordinates. We can, however, learn a lot by studying so called 'semi-local' models, where the complications of global F-theory are avoided by just looking at regions close to the GUT surface S.

1.6.2 Semi-local F-theory and the Role of E_8

The ideas of local F-theory focus on the submanifold S, where the GUT symmetry is localised. We can consider intersections of the GUT brane (which wraps S) with other 7-branes wrapping surfaces S_i and supporting gauge groups G_i. Along these intersections matter will reside, and so they are known as matter curves, $\Sigma_i = S \cap S_i$. Along the matter curves , the local symmetry group is enhanced to $G_{\Sigma_i} \supset G_S \times G_i$. We can go one step further than this and then study the intersections of matter curves at points in S. When we have an intersection of matter curves, we induce a Yukawa coupling and there is a further enhancement of the local symmetry to $G_Y \supset G_{\Sigma_i} \times G_{\Sigma_j} \times G_{\Sigma_k}$. In order to study Yukawa couplings in the local setup, we can gain information by just considering the local area around the point of intersection on the surface S.

The semi-local approach to F-theory assumes that we have a parent E_8 gauge theory which is broken by a position dependent VEV for an adjoint Higgs field. All the interactions in the theory are assumed to come from a single E_8 point of enhancement. At this point, all the matter curves of the theory meet, and the local symmetry group is enhanced all the way to E_8.

1.6.3 An $SU(5)$ Example and Introducing Monodromy

As an example, we can take the GUT group on S to be $SU(5)$. The breaking of E_8 to the GUT group occurs as

$$E_8 \to SU(5)_{GUT} \times SU(5)_\perp \to SU(5)_{GUT} \times U(1)^4 \tag{1.148}$$

where the commutant of the GUT group inside E_8 is called the perpendicular group, and in this case is $SU(5)_\perp$. The nature of the matter curves of the theory is found by decomposing the adjoint representation of E_8 as follows

$$248 \to (24,1) + (1,24) + (10,5) + (\overline{5},10) + (\overline{5},\overline{10}) + (5,\overline{10}) \tag{1.149}$$

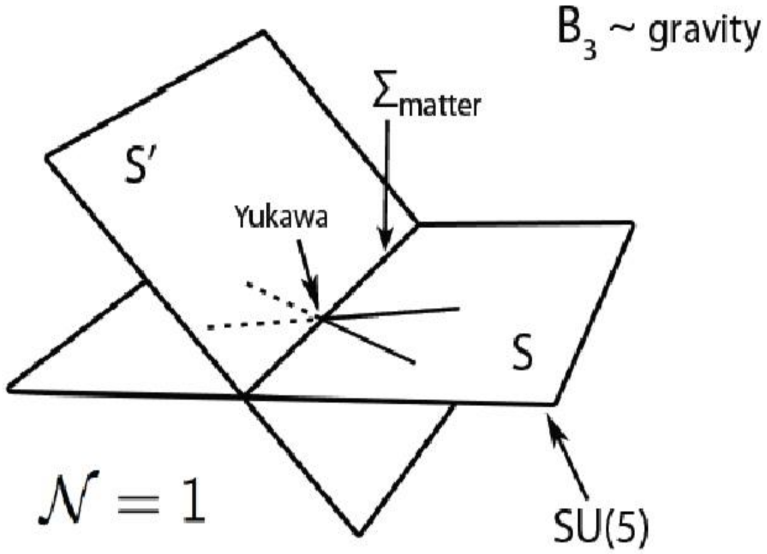

Figure 1.11: Branes intersecting at matter curves, which intersect inducing a yukawa coupling (figure from [69])

This equation shows us that we have twenty four singlet curves (θ_{ij}), five 10 curves, and ten $\overline{5}$ curves. The equations of these curves can be written in terms of the weights t_i ($i = 1, .., 5$, $\sum t_i = 0$), of the 5 representation of $SU(5)_\perp$ as follows

$$\begin{aligned} \Sigma_{10} &: t_i = 0 \\ \Sigma_5 &: -t_i - t_j = 0, i \neq j \\ \Sigma_1 &: \pm(t_i - t_j) = 0, i \neq j \end{aligned} \tag{1.150}$$

The fibration coefficients b_i of Eq. (1.147) are given by the elementary, symmetric polynomials of degree i in the weights. These are non-linear relations, and generally there will be relations identifying some of the t_i. The way in which the t_i can be identified is determined by the 'monodromy group'. As we are working in the semi-local picture, the full Calabi-Yau geometry has been decoupled, and so we must choose the monodromy group by hand. By requiring a tree level top quark Yukawa coupling, we need at least a Z_2 monodromy identifying two of the weights. This is because we need the $5_H \times 10_M \times 10_M$ coupling to be invarient under the perpendicular $U(1)$ symmetries. As the top and anti-top come from the same 10 representation, they both have charge t_i, and the up type Higgs has charge $-t_j - t_k$, meaning that to cancel the charges we must have $2t_i - t_j - t_k = 0$. This can only be the case for $j = k = i$, and so we must have an identification of at least two of the weights. From now on this minimal Z_2 case will be assumed at all times, and we will take $t_1 \leftrightarrow t_2$.

1.6.4 Flux Breaking

So far, we have only seen the case where the gauge symmetry on S is $SU(5)$. However, of course there are other possibilities for the GUT group, and inspired by the Dynkin diagram chain of Figure 1.6, the work in this thesis will focus on the groups E_6 and $SO(10)$ in addition to $SU(5)$. The decomposition of E_8 into the GUT group and perpendicular group in each case is given by

$$\begin{aligned}
E_8 &\supset E_6 \times SU(3)_\perp \\
&\to SO(10) \times U(1)_\psi \times SU(3)_\perp \\
&\to SU(5) \times U(1)_\chi \times U(1)_\psi \times SU(3)_\perp \\
E_8 &\supset SO(10) \times SU(4)_\perp \\
&\to SU(5) \times U(1)_\chi \times SU(4)_\perp \\
E_8 &\supset SU(5) \times SU(5)_\perp
\end{aligned}$$

As we can see from the above breaking chains, even if we start with E_6 or $SO(10)$ as the GUT group, we can always end up with an $SU(5) \times U(1)^4$ structure before we can break down to the Standard Model, by breaking $SU(N)_\perp$ to $U(1)_\perp^{N-1}$. Although we could generally turn on non-Abelian fluxes in the perpendicular groups, in this thesis we will always choose to work with fluxes in $U(1)$s. In this case, the only difference between the three pictures is which $U(1)$s originate from the GUT group and which originate from the perpendicular group, and in the next chapter mappings between the different pictures will be established. Throughout this thesis, we will assume that the GUT group is broken down to $SU(3) \times SU(2) \times U(1)_Y$ via flux breaking.

There are two types of flux that can be turned on: there are fluxes in the $U(1)$s from the perpendicular group which preserve the chirality of complete GUT representations, and there are fluxes that can be turned on in the worldvolume of the 7-brane which break the GUT structure. Whenever we utilise flux breaking we end up with splitting equations which tell us the net number of states in a particular representation, for example, breaking $SU(5)$ down to the Standard Model by turning on a flux in the hypercharge direction gives the following equations for the 10 and 5 representations of $SU(5)$

$$10 = \begin{cases} \text{Rep.} & \# \\ n^1_{3,2} - n^1_{\bar{3},2} & : \; M_{10} \\ n^1_{\bar{3},1} - n^1_{3,1} & : \; M_{10} - N \\ n^1_{\bar{1},1} - n^1_{1,1} & : \; M_{10} + N \end{cases} \qquad 5 = \begin{cases} \text{Rep.} & \# \\ n^1_{3,1} - n^1_{\bar{3},1} & : \; M_5 \\ n^1_{1,2} - n^1_{1,\bar{2}} & : \; M_5 + N \end{cases}$$

We can see from these equations that the flux associated with the integer M respects the GUT structure, and so is a flux in the perpendicular $U(1)$s. The flux associated with the integer N is the hypercharge flux and leads to incomplete $SU(5)$ multiplets. As this breaking is due to the hypercharge flux, the integer N is given by the flux dotted with the homology class of the matter curve in question. As such, we can obtain relations between these N integers (and similar integers for different fluxes) by calculating the homology classes of the matter curves. In order to do this, we will use the spectral cover formalism.

Chapter 2

The Spectral Cover Formalism

2.1 Semi-Local F-theory and Spectral Covers

As described in the Introduction, a global description of an F-theory GUT is given by the geometry of a CY fourfold, elliptically fibred over a threefold base, B_3. A local description, however, focuses on the effective field theory where the GUT symmetry is realised on a 7-brane wrapping a 4-dimensional surface, S. Matter fields are then localised on curves within S, and information about interactions can be found by looking at the points where matter curves intersect.

Another way of looking at F-theory GUTs is the semi-local approach, which involves imposing constraints from the requirement that S is embedded into a local CY fourfold. These constraints mean that the local geometry around 7-branes can be viewed as a deformed E_8 singularity, which is unfolded to a G_{GUT} singularity by the fibration coefficients, b_i, of Eq. (1.147). This unfolding can be viewed as arising from an E_8 gauge theory on $\mathbb{R}^{3,1} \times S$, where the b_is provide information about the adjoint scalar, ϕ, that breaks $E_8 \to G_{GUT}$ (more precisely, the b_is give the eigenvalues of ϕ at different points of S). Generally, the fibration coefficients, b_i, depend on the embedding of S into B_3, but in a semi-local model, this information is not specified.

In order to break the E_8 gauge theory, the Higgs ϕ gets a position dependent VEV in the adjoint of the commutant group of G_{GUT} within E_8. In this thesis, we will study $G_{GUT} = E_6$, $SO(10)$ and $SU(5)$, and look at the breaking

$$E_8 \to G_{GUT} \times SU(N)_\perp \tag{2.1}$$

where $N = 3,4,5$ for E_6, $SO(10)$ and $SU(5)$ respectively. The eigenvalues of ϕ can be diagonalised in a local coordinate patch as

$$\phi \sim \text{diag}(t_1, ..., t_N), \quad \sum_i t_i = 0 \tag{2.2}$$

where the t_i are the weights of the fundamental representation of $SU(N)_\perp$. These weights can be interchanged by the action of monodromies as described in the Introduction.

The *spectral cover* equation for $SU(N)_\perp$ is an N^{th} degree polynomial whose coefficients are the b_i and whose roots are the roots correspond to the local eigenvalues, t_i.

$$\mathscr{C}^N = b_0 s^N + ... + b_k s^{N-k} + ... + b_N = 0 \tag{2.3}$$

As such, whilst the full mathematical language of the E_8 breaking is described by the langauge of Higgs bundles [70], the spectral cover approach simply focuses on the eigenvalues of ϕ at each point on S.

Before a detailed study of the $SU(3)_\perp$, $SU(4)_\perp$ and $SU(5)_\perp$ spectral covers is presented, it is useful to look at some features of spectral cover models by briefly examining the simplest example, $SU(3)_\perp$. In this case, the spectral cover equation is given by

$$\mathscr{C}^3 = b_0 s^3 + b_1 s^2 + b_2 s + b_3 = 0 \tag{2.4}$$

As the roots are given by the $SU(3)_\perp$ weights, t_i, we also have

$$\mathscr{C}^3 = b_0(s+t_1)(s+t_2)(s+t_3) = 0 \tag{2.5}$$

Comparing the equations gives

$$\begin{aligned} b_1 &= b_0(t_1+t_2+t_3) \\ b_2 &= b_0(t_1t_2+t_1t_3+t_2t_3) \\ b_3 &= b_0 t_1 t_2 t_3 \end{aligned} \tag{2.6}$$

As such, we can see that the fibration coefficients, b_i are given by the elementary, symmetric polynomials in the t_i. Also, it should be noted that due to the tracelessness constraint on the weights, b_1 will always be zero in these models.

The reason why the spectral cover approach is so useful is that equations for the matter curves of a given GUT theory can be computed in terms of the b_is. As such, if the homology classes of the fibration coefficients are known, we can work out the homology classes of the matter curves. However, we want to know the equations and homologies of the matter curves in the presence of a monodromy, so the monodromy action must be somehow imposed upon the spectral cover polynomial. In this thesis, the minimal Z_2 monodromy identifying the weights t_1 and t_2 is always assumed, which in the language of an $SU(N)_\perp$ spectral cover corresponds to the polynomial splitting into a quadratic factor and $N-2$ linear factors. As an example, imposing the Z_2 monodromy on the $SU(3)_\perp$ spectral cover equation leads to the following splitting of the polynomial

$$\mathscr{C}^3 = (a_1 + a_2 s + a_3 s^2)(a_4 + a_5 s) \tag{2.7}$$

The b_i coefficients can now be related to the a_is by comparing powers of s, and so the homology classes of the a_i and the equations of matter curves in terms of the a_i can be computed. Putting this information together gives the homology classes of matter curves in the presence of a monodromy, which is the key information required to determine the chiral matter content on a curve after flux breaking, as shall be seen later in this chapter. This is the general spectral cover procedure which will be applied in detail to each of the three GUT groups studied in this thesis, but before this detailed analysis, it will prove incredibly useful for model building to develop a group theory dictionary between the three GUT theories.

2.2 Group Theory Dictionary Between E_6 and $SU(5)$

In this chapter we are concerned with the sequence of rank preserving symmetry breakings induced by flux breaking. Starting from the E_8 point of enhancement in the internal geometry, there exists a variety of breaking patterns to obtain the Standard Model gauge symmetry. A complete classification of these possibilities from the F-theory perspective has been given in the appendix of ref [9]. Here, we shall be interested in the general embeddings discussed in the Introduction, where the adjoint of E_8 decomposes in each case as

$$E_8 \supset E_6 \times SU(3)_\perp \tag{2.8}$$

$$248 \rightarrow (78,1)+(27,3)+(\overline{27},\overline{3})+(1,8) \tag{2.9}$$

$$E_8 \supset SO(10) \times SU(4)_\perp \tag{2.10}$$

$$248 \rightarrow (1,15)+(45,1)+(10,6)+(16,4)+(\overline{16},\overline{4}) \tag{2.11}$$

$$E_8 \supset SU(5) \times SU(5)_\perp \tag{2.12}$$

$$248 \rightarrow (24,1)+(1,24)+(10,5)+(\overline{5},10)+(\overline{5},\overline{10})+(5,\overline{10}). \tag{2.13}$$

In particular, the last case has been extensively studied by many authors including [9, 70, 71, 22, 23]. In this case, as described in the Introduction, the matter content of $SU(5)_{GUT}$ transforms non-trivially under the Cartan subalgebra of $SU(5)_\perp$ with weight vectors $t_{1,\ldots,5}$ satisfying

$$t_1 + t_2 + t_3 + t_4 + t_5 = 0. \tag{2.14}$$

In principle, the superpotential can be maximally constrained by four $U(1)$s according to the breaking pattern

$$E_8 \supset SU(5) \times SU(5)_\perp \rightarrow SU(5) \times U(1)_\perp^4 \tag{2.15}$$

The 5 representation of $SU(5)_\perp$ may be expressed in the conventional basis of the five weight vectors t_i in which the 4 Cartan generators corresponding to $U(1)_\perp^4$ are expressed as:

$$\begin{aligned} H_1 &= \frac{1}{2}\mathrm{diag}(1,-1,0,0,0), \;\; H_2 = \frac{1}{2\sqrt{3}}\mathrm{diag}(1,1,-2,0,0), \\ H_3 &= \frac{1}{2\sqrt{6}}\mathrm{diag}(1,1,1,-3,0), \;\; H_4 = \frac{1}{2\sqrt{10}}\mathrm{diag}(1,1,1,1,-4). \end{aligned} \tag{2.16}$$

In general, however, there is an action on t_is of a non-trivial monodromy group which is a subgroup of the Weyl group $W(SU(5)_\perp) = S_5$. Such subgroups are the alternating groups $\mathscr{A}_n$, the dihedral groups $\mathscr{D}_n$ and cyclic groups $\mathscr{Z}_n$, $n \leq 5$ and the Klein four-group $\mathscr{Z}_2 \times \mathscr{Z}_2$. Throughout this thesis we shall assume the minimal $\mathscr{Z}_2$ monodromy, $t_1 \leftrightarrow t_2$.

It is of interest to consider the possibility of a sequence of flux breaking, which may be associated with different scales. Here we consider the sequence

$$\begin{aligned} E_8 &\to E_6 \times U(1)_\perp^2 && (2.17) \\ &\to SO(10) \times U(1)_\psi \times U(1)_\perp^2 && (2.18) \\ &\to SU(5) \times U(1)_\chi \times U(1)_\psi \times U(1)_\perp^2. && (2.19) \end{aligned}$$

which for the E_6 representations gives

$$\begin{aligned} 78 &\to [24_{(0,0)} + 10_{(4,0)} + \overline{10}_{(-4,0)} + 1_{(0,0)}]_{45} \\ &+ [10_{(-1,-3)} + \overline{5}_{(3,-3)} + 1_{(-5,-3)}]_{16} \\ &+ [\overline{10}_{(1,3)} + 5_{(-3,3)} + 1_{(5,3)}]_{\overline{16}} \\ &+ [1_{(0,0)}]_1 \\ 27 &\to [10_{(-1,1)} + \overline{5}_{(3,1)} + 1_{(-5,1)}]_{16} \\ &+ [5_{(2,-2)} + \overline{5}_{(-2,-2)}]_{\overline{10}} \\ &+ [1_{(0,4)}]_1 \end{aligned} \tag{2.20}$$

where the subscripts refer to the $U(1)_\chi, U(1)_\psi$ charges and $SO(10)$ representation respectively. It is convenient to choose a basis for the weight vectors such that the charge generators have the form

$$\begin{aligned} Q_\chi &\propto \mathrm{diag}[-1,-1,-1,-1,4] \\ Q_\psi &\propto \mathrm{diag}[1,1,1,-3,0] \\ Q_\perp &\propto \mathrm{diag}[1,1,-2,0,0] \end{aligned} \tag{2.21}$$

where $Q_\perp$ is the charge of the $U(1)_\perp$ in the breaking pattern of Eq. (2.17) that remains after imposing the $t_1 \leftrightarrow t_2$ monodromy. This is in fact the same as the conventional basis for the

E_6	$SO(10)$	$SU(5)$	Weight vector
$27_{t_1'}$	16	$\overline{5}_3$	t_1+t_5
$27_{t_1'}$	16	10_M	t_1
$27_{t_1'}$	16	θ_{15}	t_1-t_5
$27_{t_1'}$	10	5_1	$-t_1-t_3$
$27_{t_1'}$	10	$\overline{5}_2$	t_1+t_4
$27_{t_1'}$	1	θ_{14}	t_1-t_4
$27_{t_3'}$	16	$\overline{5}_5$	t_3+t_5
$27_{t_3'}$	16	10_2	t_3
$27_{t_3'}$	16	θ_{35}	t_3-t_5
$27_{t_3'}$	10	5_{H_u}	$-2t_1$
$27_{t_3'}$	10	$\overline{5}_4$	t_3+t_4
$27_{t_3'}$	1	θ_{34}	t_3-t_4

Table 2.1: Complete 27s of E_6 and their $SO(10)$ and $SU(5)$ decompositions. For the $SU(5)$ states we use the notation of ref [22] where indices in $5_i, 10_j$ representations are associated to the corresponding matter curves $\Sigma_{5_i}, \Sigma_{10_j}$.

$SU(5)_\perp$ generators in Eq. (2.16), and the normalisation of the generators is given by identifying,

$$H_1 = H_1', \;\; H_2 = Q_\perp, \;\; H_3 = Q_\psi, \;\; H_4 = -Q_\chi. \tag{2.22}$$

This almost trivial equivalence shows that the $SU(5)_{GUT}$ states in Eq. (2.13) have well defined E_6 charges Q_χ and Q_ψ. For example $SU(5)$ singlets will in general carry Q_χ and Q_ψ charges which originate from E_6 and which may be unbroken. The equivalence will provide insights into both anomaly cancellation and the origin of R-parity for example, in terms of the underlying E_6 structure, in the explicit models discussed later. Throughout this thesis we shall assume the minimal $\mathscr{Z}_2$ monodromy, $t_1 \leftrightarrow t_2$ [24] which trivially corresponds to the minimal $\mathscr{Z}_2$ monodromy, $t_1' \leftrightarrow t_2'$ (where the primes indicate weights of $SU(3)_\perp$). It is clear from Eq. (2.16) that this corresponds to $H_1 = H_1'$ being broken leaving only three independent Cartan symmetries $\{H_2, H_3, H_4\}$ or equivalently $\{Q_\perp, Q_\psi, Q_\chi\}$.

In this basis the weight vectors t_1', t_2', t_3' ($t_1' + t_2' + t_3' = 0$) of $SU(3)_\perp$ are related to the $SU(5)_\perp$ weight vectors by $t_i' = t_i + (t_4 + t_5)/3, \;\; i = 1, 2, 3$. As an example of the use of this dictionary that will play an important role when building a realistic theory we can now connect the two independent representations $27_{t_1'}$ and $27_{t_3'}$ that appear in the E_6 breaking pattern of Eq. (2.17) to the $SU(5)$ representations of Eq. (2.19). These are shown in Table 2.1 with $SU(5)$ states given in the notation of [22].

2.3 Flux breaking and matter content in F-theory GUTs

In this section we determine the light matter content that results if the underlying E_8 is broken to some subgroup by a Higgs bundle on the del-Pezzo surface S [70]. We are interested in the cases that the unbroken gauge group is E_6, $SO(10)$ or $SU(5)$. The reason for studying these models is because they can lead to promising phenomenology based on a high degree of unification, even though they are subsequently further broken by flux to just the Standard Model. A viable model will be presented in the next chapter, which will provide the groundwork for building other related models.

We proceed by studying the spectral cover of the transverse groups for the three cases of interest $E_6 \times SU(3)_\perp$, $SO(10) \times SU(4)_\perp$ and $SU(5) \times SU(5)_\perp$. This will allow us to determine the homology of the matter fields and hence the effect of flux breaking. In dealing with singlets, we note that for a given surface S with associated singularity G_S, there are singlet fields residing on curves that extend away from S and can be affected by $U(1)_\perp$ fluxes not supported by S. There are also singlet fields emerging from the decomposition of GUT representations after the breaking of the covering group G_S by the flux mechanism. The latter singlets localise on curves on the surface S, and as a consequence they are affected by the fluxes breaking G_S. In this case the homologies of the corresponding matter curves can be determined and, as shown in this chapter, certain properties including chirality and multiplicities can be expressed in terms of a few integers parameterising the associated $U(1)$ fluxes.

2.3.1 $SU(3)_\perp$ Spectral Cover

E_6 models are quite attractive and have been extensively studied in compactifications on Calabi-Yau manifolds, in the context of the heterotic superstring with underlying $E_8 \times E_8$ symmetry (see [72, 73, 74] and references therein). Furthermore, recent phenomenological investigations based on string motivated versions with E_6 gauge symmetry have inspired the exceptional supersymmetric standard model [75]. This is distinguished from the minimal one by the appearance of an additional Z' boson and extra matter content at the TeV scale. Interestingly, although these new ingredients are also potentially present in the F-theory E_6-analogue, they are subject to constraints from flux restrictions on matter curves and the topological properties of the compact manifold. We will study two types of model in later chapters, distinguished by whether the Z' boson and extra matter have GUT scale masses or TeV scale masses.

In the context of F-theory in which the GUT group on the brane is E_6, we need to look at the breaking

$$E_8 \to E_6 \times SU(3)_\perp \tag{2.23}$$

We can determine what matter curves arise by decomposing the adjoint of E_8 as follows

$$248 \to (78,1) + (27,3) + (\overline{27},\overline{3}) + (1,8) \tag{2.24}$$

The E_6 content consists of three 27s (and $\overline{27}s$) plus eight singlet matter curves. In terms of the weight vectors t_i, i=1,2,3, of $SU(3)_\perp$ the equations of these curves are

$$\Sigma_{27} : t_i = 0 \tag{2.25}$$

$$\Sigma_1 : \pm(t_i - t_j) = 0 \;\; i \neq j \tag{2.26}$$

The $SU(3)_\perp$ spectral cover polynomial is given by

$$\mathscr{C}^3 = b_0 s^3 + b_2 s + b_3 = 0 \tag{2.27}$$

Letting c_1 be the 1^{st} Chern class of the *tangent* bundle to S_{GUT} and $c_1(NS) = -t$ that of the *normal* bundle, we define for convenience $\eta = 6c_1 - t$ and, we demand that the coefficients b_k are sections of

$$b_k : \qquad [b_k] = \eta - k c_1 \tag{2.28}$$

where k spans the integers $k = 1,2,3,4,5$. The roots of the spectral cover equation

$$0 = b_3 + b_2 s + b_0 s^3 \propto \prod_{i=1}^{3} (s + t_i')$$

are identified as the $SU(3)_\perp$ weight vectors t_i'. In the above the coefficient b_1 is taken to be zero since it corresponds to the sum of the roots which, for $SU(n)$, is always zero, $\sum_i t_i' = 0$.

2.3.1.1 27 and $\overline{27}$ fields

The coefficient b_3 is equal to the product of the roots, i.e. $b_3 = t_1' t_2' t_3'$ and the Σ_{27} curves where the corresponding matter multiplets are localized are determined by its three zeros

$$\Sigma_{27_i}, \quad b_3 = \prod_{i=1}^{3} t_i' = 0 \to t_i' = 0, \quad i = 1,2,3 \tag{2.29}$$

To obtain different curves for 27's we need to split the spectral cover. (If the polynomial is not factorized, there is only one matter curve). There are two possible ways to split a third degree polynomial: either to a binomial-monomial $(2-1)$ or to three monomials $(1-1-1)$. Since we need to impose a monodromy action, we choose this to be $\mathscr{Z}_2$ and therefore we get a $(2-1)$ split. The $\mathscr{Z}_2$ monodromy corresponds to the following split of the spectral cover equation

$$\begin{aligned} 0 = \Pi_3(s) &= (a_1 + a_2 s + a_3 s^2)(a_4 + a_5 s) \\ &= a_1 a_4 + (a_2 a_4 + a_1 a_5)s + (a_2 a_5 + a_3 a_4)s^2 + a_3 a_5 s^3 \end{aligned} \tag{2.30}$$

with $s = U/V$ and a_i coefficients, constituting sections of line bundles each of them being of specific Chern class to be determined.

Matter	Section	Homology
$27_{t_{1,2}}$	a_1	$\eta - 2c_1 - \chi$
27_{t_3}	a_4	$\chi - c_1$

Table 2.2: The three columns show the quantum numbers of matter curves under $E_6 \times U(1)_{t_i}$, the section and the homology class.

The first bracket contains the polynomial factor that corresponds to the $\mathscr{Z}_2$ monodromy $t_1' \leftrightarrow t_2'$, so that the corresponding two Σ_{27} curves lift to a common one in the spectral cover. The Σ_{27} curves are found setting $s = 0$ in the polynomial

$$b_3 \equiv \Pi_3(0) = a_1 a_4 = 0 \quad \rightarrow \quad a_1 = 0, a_4 = 0$$

Thus, after the monodromy action, we obtain two matter curves. When building a realistic theory it is necessary to assign the three families of quarks and leptons and the Higgs to these curves. As there are more than one way to do this, the optimal choice will be dictated by phenomenology.

To determine the distribution of families and Higgs on the two matter curves we need to know how the flux restricts on the available curves. To do this we first determine their homology classes $[a_k]$ corresponding to the sections a_k, $k = 1,4$. This can be done comparing the coefficients of Eqs. (2.27,2.30). We get

$$\begin{aligned} b_0 &= a_3 a_5 \\ b_1 &= a_2 a_5 + a_3 a_4 = 0 \\ b_2 &= a_2 a_4 + a_1 a_5 \\ b_3 &= a_1 a_4 \end{aligned}$$

The homology classes $[b_k]$ of the sections b_k are given in Eq. (2.28), while those of a_i can be determined by the system of linear equations in one to one correspondence with the above relations. This linear system consists of four equations with five unknowns $[a_i]$, therefore we can solve the system in terms of one arbitrary parameter. Let a_5 be of some unspecified homology class $[a_5] = \chi$. For the remaining a_i, we find that they are sections of

$$[a_1] = \eta - 2c_1 - \chi, [a_2] = \eta - \chi - c_1,\ [a_3] = \eta - \chi,\ [a_4] = \chi - c_1,\ [a_5] = \chi \tag{2.31}$$

For the two curves we obtain the results of Table 2.2. For the homology classes of the two curves $\mathscr{C}^3 = \mathscr{C}_{t_{1,2}} \mathscr{C}_{t_3}$ from Eq. (2.30) we get

$$\mathscr{C}_{t_{1,2}} = a_1 + a_2 s + a_2 s^2 \tag{2.32}$$

$$\mathscr{C}_{t_3} = a_4 + a_5 s \tag{2.33}$$

so that their homology classes are given by

$$[\mathscr{C}_{t_{1,2}}] = \eta - \chi - 2c_1, \ [\mathscr{C}_{t_3}] = \chi - c_1$$

Using the data of Table 2.2, we can turn on a $\mathscr{F}_{U(1)}$ flux on the external $U(1)$ and find the restriction on the curves of 27's:

$$n_{t_1} = \mathscr{F}_{U(1)} \cdot (\eta - \chi - 2c_1) \quad ; \quad n_{t_3} = \mathscr{F}_{U(1)} \cdot (\chi - c_1) \tag{2.34}$$

These determine the chiral content of states arising from the decomposition of 27's along the matter curves. We have also seen that χ is some unspecified homology class (associated to a_5) and it can be chosen at will. For acceptable choices it can be seen from Table 2.2 that the two curves cannot be of the same homology class. Since the two curves belong to different homology classes, in general flux restricts differently on them. The two conditions can be combined as follows

$$n_{t_3} + n_{t_1} = \mathscr{F}_{U(1)} \cdot (\eta - 3c_1) = \mathscr{F}_{U(1)} \cdot (3c_1 - t) \tag{2.35}$$

From Eq. (2.35) we deduce that if $\mathscr{F}_{U(1)} \cdot (3c_1 - t) = 0$, then $n_{t_3} = -n_{t_1}$ i.e., we get opposite flux restrictions on 27_{t_1} and 27_{t_3}. Notice that the choice $\mathscr{F}_{U(1)} \cdot c_1 \neq 0$ implies that the corresponding gauge boson becomes massive through the Green-Schwarz mechanism. This is not a problem however, for the extra $U(1)$s that do not participate in the hypercharge definition [1].

2.3.1.2 E_6 singlets

Singlet fields are important for the construction of the low energy effective field theory model. Some of them may develop VEVs that can be used to create mass terms for the fermion generations and make massive other potentially dangerous fields mediating proton decay. In certain models, those carrying charges under the weights t_i' undergoing a monodromy action can play the role of the right handed neutrino [42]. The E_6 singlets θ_{ij} lie in the $t_i' - t_j'$ directions of the corresponding Cartan subalgebra, and because of their central role in phenomenology, it would be useful to determine their homology classes. If the worldvolume theory on S has gauge group E_6, these singlets θ_{12}, θ_{13} and θ_{23} are localised on curves which do not lie within the surface S, and as such, spectral cover analysis can no longer sctrictly be used to determine their properties. However, for singlets which carry nontrivial $U(1)$ charges, an index which counts the net number of zero modes can be computed by considering the projection of the singlet curve in B_3 to a curve in S (by setting $z = 0$) [76]. This can be achieved without knowing the precise nature of the zero mode wavefunctions, for which a global description would be necessary. However, in [77], overlap integrals involving singlet fields are computed by using the projections of the curves to the surface S, although it is noted that in order to correctly normalise the singlet wavefunctions,

[1] For a recent work on the $U(1)$ symmetries in F-theory see [18].

a global construction is needed. Nevertheless, there are some calculations in [77] which do not require the correct normalisation of the singlet curves, such as the relation between proton decay and exotic masses.

It should also be noted that the discrete group $\mathscr{Z}_2$ which identifies $t_1 \leftrightarrow t_2$ leads also to the identification of the singlet fields $\theta_{12} \leftrightarrow \theta_{21}$. This will also lead to geometric identification of the corresponding matter curves in the covering theory. Therefore these singlets carry no $U(1)$-charges and are treated as moduli of the spectral cover and differently from the θ_{13} singlet fields, in accordance with previous studies [70, 19].

2.3.2 $SU(4)_\perp$ Spectral Cover

The $SO(10)$ GUT is one of the most promising Unified Theories, and the smallest one incorporating the right-handed neutrino into the same multiplet with the remaining fundamental particles (quarks and leptons). For the case that the GUT group on the brane is $SO(10)$ we need to consider the breaking

$$E_8 \to SO(10) \times SU(4)_\perp \tag{2.36}$$

We can determine which matter curves arise from the decomposition of the adjoint of E_8:

$$248 \to (1,15)+(45,1)+(10,6)+(16,4)+(\overline{16},\overline{4}) \tag{2.37}$$

Thus there are four 16 (and $\overline{16}$) matter curves, six 10 matter curves, and fifteen singlets. The equations for these curves in terms of the weight vectors t_i, i=1,2,3,4, of $SU(4)_\perp$ are

$$\Sigma_{16} : t_i = 0 \tag{2.38}$$
$$\Sigma_{10} : (-t_i - t_j) = 0, \;\; i \neq j \tag{2.39}$$
$$\Sigma_1 : \pm(t_i - t_j) = 0 \;\; i \neq j \tag{2.40}$$

where $\sum_i t_i = 0$. In order to determine how fluxes restrict on these matter curves, taking into account the effects of monodromy, the spectral cover approach is again used. In analogy to the SU(3) spectral cover in the case of an E_6 singularity, the $SU(4)_\perp$ spectral cover polynomial is given by

$$\mathscr{C}^4 = b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4 = 0 \tag{2.41}$$
$$= b_0(s+t_1)(s+t_2)(s+t_3)(s+t_4) = 0 \tag{2.42}$$

where the second line reflects the fact that the t_i are the roots of the polynomial. This polynomial describes the 16 matter curves, which are given by setting s to zero in the above equations, leading to $b_4 = 0$. Equations for the b's in terms of the t's can be found by comparing powers of s in Eqs. (2.41) and (2.42). This leads to the following equations, once t_4 has been eliminated by using the fact that the sum of the t_i is zero:

$$b_1 = -b_0(t_1+t_2+t_3+t_4) = 0 \tag{2.43}$$
$$b_2 = b_0(t_1^2+t_2^2+t_3^2+t_1t_2+t_2t_3+t_1t_3) \tag{2.44}$$
$$b_3 = b_0(t_1+t_2)(t_2+t_3)(t_1+t_3) \tag{2.45}$$
$$b_4 = -b_0t_1t_2t_3(t_1+t_2+t_3) \tag{2.46}$$

It can be seen that the equation $b_4 = 0$ does indeed reproduce Eq. (2.38) for the 16 matter curves in terms of the t_i.

2.3.2.1 $\mathscr{Z}_2$ Monodromy

Imposing a $\mathscr{Z}_2$ monodromy implies the splitting of Eq. (2.41) as follows

$$\mathscr{C}^4 = (a_1+a_2s+a_3s^2)(a_4+a_5s)(a_6+a_7s) \tag{2.47}$$

The first bracket is quadratic in s reflecting the fact that we have chosen a $\mathscr{Z}_2$ monodromy, which in the weight language corresponds to an identification of two weights $t_1 \leftrightarrow t_2$. We can now match powers of s in Eqs. (2.41) and (2.47) to get equations for the b_i in terms of the a_i.

$$b_0 = a_3a_5a_7 \tag{2.48}$$
$$b_1 = a_2a_5a_7+a_3a_5a_6+a_3a_4a_7 \tag{2.49}$$
$$b_2 = a_1a_5a_7+a_2a_4a_7+a_2a_5a_6+a_3a_4a_6 \tag{2.50}$$
$$b_3 = a_1a_4a_7+a_1a_5a_6+a_2a_4a_6 \tag{2.51}$$
$$b_4 = a_1a_4a_6 \tag{2.52}$$

Solving for $b_1 = 0$ gives [2]

$$a_2 = -\gamma(a_5a_6+a_4a_7) \tag{2.53}$$
$$a_3 = \gamma a_5a_7 \tag{2.54}$$

[2] It is understood that some solutions of the $b_1 = 0$ constraint might lead to additional degeneracies. However, for each case in this thesis, we pick up the solution which leads to acceptable factorization, avoiding non-Kodaira singularities. We are also aware that subtleties could in principle appear on split spectral covers. However, we mainly concentrate on general phenomenological issues of F-GUT model building, and it is not our intention to address all these issues in this study.

where γ is unspecified. Now we can demand that the homology classes of the b_n are

$$[b_n] = \eta - nc_1 \tag{2.55}$$

where, as before, $\eta = 6c_1 - t$, c_1 is the first Chern class of the tangent bundle to S_{GUT} and $-t$ is the first Chern class of the normal bundle. We can now determine the homology classes of the a_i coefficients by using Eqs. (2.48)-(2.52), setting the homology class of a given b_n equal to the homology class of each product of a_is on the left hand side of the appropriate equation. This leads to

$$\eta = [a_3] + [a_5] + [a_7] \tag{2.56}$$
$$\eta - c_1 = [a_2] + [a_5] + [a_7] \tag{2.57}$$
$$\eta - 2c_1 = [a_1] + [a_5] + [a_7] \tag{2.58}$$
$$\eta - 3c_1 = [a_1] + [a_4] + [a_7] \tag{2.59}$$
$$\eta - 4c_1 = [a_1] + [a_4] + [a_6] \tag{2.60}$$

As such, we have 5 equations in 7 unknowns, and so we can solve the equations in terms of two free parameters, which we can set as

$$[a_5] = \chi_5 \tag{2.61}$$
$$[a_7] = \chi_7 \tag{2.62}$$
$$\tilde{\chi} = \chi_5 + \chi_7 \tag{2.63}$$

Solving the system of equations gives the homology classes of the remaining a_i

$$[a_1] = \eta - 2c_1 - \tilde{\chi} \tag{2.64}$$
$$[a_2] = \eta - c_1 - \tilde{\chi} \tag{2.65}$$
$$[a_3] = \eta - \tilde{\chi} \tag{2.66}$$
$$[a_4] = -c_1 + \chi_5 \tag{2.67}$$
$$[a_6] = -c_1 + \chi_7 \tag{2.68}$$

We now have determined the homology classes of all the a_i coefficients (which are summarised in Table 2.3), and can use them in order to find the homology classes of the matter curves.

Coefficient	Homology
a_1	$\eta - 2c_1 - \tilde{\chi}$
a_2	$\eta - c_1 - \tilde{\chi}$
a_3	$\eta - \tilde{\chi}$
a_4	$-c_1 + \chi_5$
a_5	χ_5
a_6	$-c_1 + \chi_7$
a_7	χ_7

Table 2.3: Homology classes of the a_i coefficients.

2.3.2.2 Homology of the 16 Matter Curves

As discussed after Eq. (2.42), the 16 matter curves are given by $b_4 = 0$. From Eq. (2.52), this means that the equations of the 16s are

$$a_1 = 0, \;\; a_4 = 0, \;\; a_6 = 0 \tag{2.69}$$

and so the homology classes are

$$[16_1] = \eta - 2c_1 - \tilde{\chi} \tag{2.70}$$

$$[16_2] = -c_1 + \chi_5 \tag{2.71}$$

$$[16_3] = -c_1 + \chi_7 \tag{2.72}$$

2.3.2.3 Homology of the 10 Matter Curves

Just as the correct polynomial to describe the 16 matter curves was the spectral cover polynomial, the polynomial for the 10s is given by

$$\begin{aligned} P_{10} &= b_0^2 \prod_{i<j} (s + t_i + t_j) \\ &= b_0^2 (s - t_1 - t_2)(s + t_1 + t_2)(s - t_1 - t_3)(s + t_1 + t_3)(s - t_2 - t_3)(s + t_2 + t_3) \qquad (2.73) \\ &= s^6 + c_1 s^5 + c_2 s^4 + c_3 s^3 + c_4 s^2 + c_5 s + c_6 \qquad (2.74) \end{aligned}$$

where in Eq. (2.73), t_4 has been eliminated by using $\sum_i t_i = 0$. Comparing coefficients of s between Eqs. (2.73) and (2.74) the following equations for the c_i in terms of the t_i are obtained

$$c_1 = 0 \tag{2.75}$$
$$c_2 = -2(t_1^2 + t_2^2 + t_3^2 + t_1 t_2 + t_1 t_3 + t_2 t_3) b_0^2 \tag{2.76}$$
$$c_3 = 0 \tag{2.77}$$
$$\begin{aligned} c_4 = [&t_1^4 + 2t_1^3(t_2 + t_3) + (t_2^2 + t_2 t_3 + t_3^2)^2 + t_1^2(3t_2^2 + 8t_2 t_3 + 3t_3^2) \\ &+ 2t_1(t_2^3 + 4t_2^2 t_3 + 4t_2 t_3^2 + t_3^3)] b_0^2 \end{aligned} \tag{2.78}$$
$$c_5 = 0 \tag{2.79}$$
$$c_6 = -(t_1 + t_2)^2 (t_1 + t_3)^2 (t_2 + t_3)^2 b_0^2 \tag{2.80}$$

We can now use Eqs. (2.43)-(2.46) to write the c_i coefficients in terms of the b_i. The results are

$$c_2 = -2b_0 b_2 \tag{2.81}$$
$$c_4 = b_2^2 - 4b_4 b_0 \tag{2.82}$$
$$c_6 = -b_3^2 \tag{2.83}$$

Substituting into Eq. (2.74) gives

$$P_{10} = s^6 - 2b_0 b_2 s^4 + (b_2^2 - 4b_4 b_0) s^2 - b_3^2 \tag{2.84}$$

As in the case of the 16 polynomial, the 10 matter curves are found by setting s to zero in this equation, giving $b_3^2 = 0$. In order to know the equations and homology classes for the 10 matter curves when the monodromy is imposed, we must express this equation in terms of the a_i coefficients. From Eq. (2.51), we know b_3 in terms of the a_i. Substituting Eq. (2.53) in for a_2 leads to

$$b_3 = (a_5 a_6 + a_4 a_7)(a_1 - \gamma a_4 a_6) \tag{2.85}$$

As such, the 10 matter curves are defined by the equation

$$(a_5 a_6 + a_4 a_7)(a_1 - \gamma a_4 a_6)(a_5 a_6 + a_4 a_7)(a_1 - \gamma a_4 a_6) = 0 \tag{2.86}$$

We therefore have four 10 matter curves, two of which have homology class $[a_1] = \eta - 2c_1 - \tilde{\chi}$, and two of which have homology class $[a_5 a_6] = [a_5] + [a_6] = -c_1 + \tilde{\chi}$. The information about the homology classes of all the 16 and 10 matter curves is summarised in Table 2.4. For convenience, the following notation is introduced

Matter	Equation	Homology	$U(1)_X$
$16_{t_{1,2}}$	a_1	$\eta - 2c_1 - \tilde{\chi}$	$M - P$
16_{t_3}	a_4	$-c_1 + \chi_5$	P_5
16_{t_4}	a_6	$-c_1 + \chi_7$	P_7
$10_{(t_1+t_3)}$	$a_1 - \gamma a_4 a_6$	$\eta - 2c_1 - \tilde{\chi}$	$M - P$
$10_{(t_1+t_2)}$	$a_5 a_6 + a_4 a_7$	$-c_1 + \tilde{\chi}$	P
$10_{(t_1+t_4)}$	$a_1 - \gamma a_4 a_6$	$\eta - 2c_1 - \tilde{\chi}$	$M - P$
$10_{(t_3+t_4)}$	$a_5 a_6 + a_4 a_7$	$-c_1 + \tilde{\chi}$	P

Table 2.4: 16 and 10 matter curves and their equations and homology classes.

$$M = \mathscr{F}_1 \cdot (\eta - 3c_1) \tag{2.87}$$
$$P = \mathscr{F}_1 \cdot (\chi - c_1) \tag{2.88}$$
$$P_n = \mathscr{F}_1 \cdot (\chi_n - c_1) \tag{2.89}$$
$$C = \mathscr{F}_1 \cdot (-c_1) \tag{2.90}$$

2.3.2.4 Homology of the $SO(10)$ singlets

We have already pointed out that singlet fields can play a decisive role in building the low energy effective model. If the worldvolume theory on S is seen to have gauge group $SO(10)$, then the same argument about singlets applies as before. The $SO(10)$ singlets will reside on curves which extend away from S, forbidding us from computing the homology classes in the local prescription. If we look at a model where the worldvolume group on S is E_6 however, only the singlets θ_{12} and θ_{13} do not live on S. The other $SO(10)$ singlets could then be treated by a polynomial in the usual way, and the resulting homology classes which are computed could be compared with the homologies of the 27 curves which they originate from in the E_6 formalism.

Adopting this technique and using well known theorems, the singlets are given by the equation

$$P_0 = -4b_2^3b_3^2 - 27b_0b_3^4 + 16b_2^4b_4 + 144b_0b_2b_3^2b_4 - 128b_0b_2^2b_4^2 + 256b_0^2b_4^3 = 0 \tag{2.91}$$

When the b_i are expressed in terms of the a_i, the results are

Matter	Charge	Equation	Homology	$U(1)_X$
θ_{14}, θ_{41}	$\pm(t_{1,2}-t_4)$	$a_1a_7-a_6(a_2-\gamma a_5a_6)$	$\eta-2c_1-\chi_5$	$M-P_5$
θ_{34}, θ_{43}	$\pm(t_3-t_4)$	$a_5a_6-a_4a_7$	$-c_1+\chi$	$M+P$

Table 2.5: SU(4) cover singlets and homologies.

$$b_0=\gamma(a_5a_7)^2 \tag{2.92}$$

$$b_1=0 \tag{2.93}$$

$$b_2=a_5a_7(a_1+\gamma a_4a_6)-\gamma(a_5a_6+a_4a_7)^2 \tag{2.94}$$

$$b_3=(a_1-\gamma a_4a_6)(a_5a_6+a_4a_7) \tag{2.95}$$

$$b_4=a_1a_4a_6 \tag{2.96}$$

Factorisation of P_0 leads to

$$\begin{aligned} P_0=&[(a_5a_6+a_4a_7)^2\gamma-4a_1a_5a_7]\times[a_1a_7-a_6(a_2-\gamma a_5a_6)]^2\\ &\times[a_1a_5-a_4(a_2-\gamma a_4a_7)]^2\times[a_5a_6-a_4a_7]^2 \end{aligned} \tag{2.97}$$

As we know the homologies of the a_i, we have the homologies of the singlet curves, which are summarised in Table 2.5.

2.3.3 $SU(5)_\perp$ Spectral Cover

The final investigation contained in this thesis concerns the $SU(5)_{GUT}$. Considering again the maximal symmetry E_8, the spectral cover encoding the relevant information (bundle structure etc) is associated to the commutant of the GUT group, which is $SU(5)_\perp$. Hence, in this case the breaking pattern is

$$E_8\to SU(5)\times SU(5)_\perp \tag{2.98}$$

This case has been extensively studied and the homology of the gauge non-singlets determined. Here we extend the discussion to include the singlets inside an E_6 GUT group. The associated adjoint representation decomposition is

$$248\to(24,1)+(1,24)+(10,5)+(\overline{5},10)+(\overline{5},\overline{10})+(5,\overline{10}) \tag{2.99}$$

Although this case has been analysed by many authors in the recent F-theory model building literature, a detailed examination of the breaking mechanism of the higher intermediate symmetries and possible implications is still lacking. In the following we attempt to implement the constraints obtained from the previous symmetry breaking stages into the $SU(5)_{GUT}$ model.

To start with, we recall that the global model is assumed in the context of elliptically fibered Calabi-Yau compact complex fourfold over a three-fold base. Using Tate's algorithm[67, 78], the $SU(5)$ singularity can be described by the following form of Weierstrass' equation [6]

$$y^2 = x^3 + b_0 z^5 + b_2 x z^3 + b_3 y z^2 + b_4 x^2 z + b_5 x y$$

We determine the corresponding spectral cover by defining homogeneous coordinates

$$z \to U,\ x \to V^2,\ y \to V^3$$

so that the spectral cover equation becomes

$$0 \quad = \quad b_0 U^5 + b_2 V^2 U^3 + b_3 V^3 U^2 + b_4 V^4 U + b_5 V^5$$

We can see this equation as a fifth degree polynomial in terms of the affine parameter $s = U/V$:

$$P_5 = \sum_{k=0}^{5} b_k s^{5-k} \ = \ b_5 + b_4 s + b_3 s^2 + b_2 s^3 + b_1 s^4 + b_0 s^5$$

where we have divided by V^5, so that each term in the last equation becomes a section of $c_1 - t$. The roots of the spectral cover equation.

$$0 = b_5 + b_4 s + b_3 s^2 + b_2 s^3 + b_0 s^5 \propto \prod_{i=1}^{5}(s + t_i)$$

are identified as the $SU(5)$ weights t_i.

In the above, the coefficient b_1 is taken to be zero since it corresponds to the sum of the roots, which for $SU(N)$ is always zero, $\sum t_i = 0$. Also, it can be seen that the coefficient b_5 is equal to the product of the roots, i.e. $b_5 = t_1 t_2 t_3 t_4 t_5$ and the Σ_{10} curves where the corresponding matter multiplets are localized are determined by the five zeros

$$\Sigma_{10_i}, \quad b_5 = \prod_{i=1}^{5} t_i = 0 \to t_i = 0, \quad i = 1,2,3,4,5 \tag{2.100}$$

Following [24], we impose the $\mathscr{Z}_2$ monodromy corresponding to the following splitting of the spectral cover equation

$$0 \quad = \quad (a_1 + a_2 s + a_3 s^2)(a_4 + a_5 s)(a_6 + a_7 s)(a_8 + a_9 s) \tag{2.101}$$

with $s = U/V$ and a_i undetermined coefficients, constituting sections of line bundles each of them being of specific Chern class to be determined. The first bracket contains the polynomial factor which corresponds to the $\mathscr{Z}_2$ monodromy, while the remaining monomials leave three $U(1)$s intact. Expanding, we may determine the homology class for each of the coefficients a_i

Matter	Charge	Equation	Homology	N_Y	$M_{U(1)}$
5_{H_u}	$-2t_1$	$a_8a_5a_7+a_6a_5a_9+a_4a_7a_9$	$-c_1+\tilde{\chi}$	$\tilde{N}$	M_{H_u}
5_1	$-t_1-t_3$	$a_1-ca_4a_8a_7-ca_4a_6a_9$	$\eta-2c_1-\tilde{\chi}$	$-\tilde{N}$	M_{5_1}
5_2	$-t_1-t_4$	$a_1-ca_6a_8a_5-ca_4a_6a_9$	$\eta-2c_1-\tilde{\chi}$	$-\tilde{N}$	M_{5_2}
5_3	$-t_1-t_5$	$a_1-ca_6a_8a_5-ca_4a_8a_7$	$\eta-2c_1-\tilde{\chi}$	$-\tilde{N}$	M_{5_3}
5_4	$-t_3-t_4$	$a_6a_5+a_4a_7$	$-c_1+\chi_5+\chi_7$	N_5+N_7	M_{5_4}
5_5	$-t_3-t_5$	$a_8a_5+a_4a_9$	$-c_1+\chi_5+\chi_9$	N_5+N_9	M_{5_5}
5_6	$-t_4-t_5$	$a_8a_7+a_6a_9$	$-c_1+\chi_7+\chi_9$	N_7+N_9	M_{5_6}
10_M	t_1	a_1	$\eta-2c_1-\tilde{\chi}$	$-\tilde{N}$	$-(M_{5_1}+M_{5_2}$ $+M_{5_3})$
10_2	t_3	a_4	$-c_1+\chi_5$	N_5	M_{10_2}
10_3	t_4	a_6	$-c_1+\chi_7$	N_7	M_{10_3}
10_4	t_5	a_8	$-c_1+\chi_9$	N_9	M_{10_4}
θ_{14}	t_1-t_4	$a_6\left(a_4a_7a_9+a_5\left(a_7a_8+2a_6a_9\right)\right)c$ $+a_1a_7$	$\eta-2c_1-\tilde{\chi}+\chi_7$	0	M_{14}
θ_{15}	t_1-t_5	$2a_5a_7a_8^2c+a_9a_8\left(a_5a_6+a_4a_7\right)c$ $+a_1a_9$	$\eta-2c_1-\tilde{\chi}+\chi_9$	0	M_{15}
θ_{34}	t_3-t_4	$a_5a_6-a_4a_7$	$-c_1+\chi_5+\chi_7$	0	M_{34}
θ_{35}	t_3-t_5	$a_5a_8-a_4a_9$	$-c_1+\chi_5+\chi_9$	0	M_{35}
θ_{45}	t_4-t_5	$a_7a_8-a_6a_9$	$-c_1+\chi_7+\chi_9$	0	M_{45}

Table 2.6: Table showing curves and flux restrictions with $\mathscr{Z}_2$ monodromy $t_1 \leftrightarrow t_2$. $\tilde{N} = N_5+N_7+N_9$. The homologies of the singlet fields θ_{ij} are also shown. Due to monodromy, θ_{12} and θ_{21} do not couple to fluxes so they are not included.

by comparison with the b_k's. Thus,

$$\begin{aligned}
b_0 &= a_3a_5a_7a_9\\
b_1 &= a_3a_5a_7a_8+a_3a_4a_9a_7+a_2a_5a_7a_9+a_3a_5a_6a_9\\
b_2 &= a_3a_5a_6a_8+a_2a_5a_8a_7+a_2a_5a_9a_6+a_1a_5a_9a_7+a_3a_4a_7a_8+a_3a_4a_6a_9+a_2a_4a_7a_9\\
b_3 &= a_3a_4a_8a_6+a_2a_5a_8a_6+a_2a_4a_8a_7+a_1a_7a_8a_5+a_2a_4a_6a_9+a_1a_5a_6a_9+a_1a_4a_7a_9\\
b_4 &= a_2a_4a_8a_6+a_1a_5a_8a_6+a_1a_4a_8a_7+a_1a_4a_6a_9\\
b_5 &= a_1a_4a_6a_8
\end{aligned}$$

We first solve the constraint $b_1 = 0$. We make the Ansatz:

$$a_2 = -c(a_5a_7a_8+a_4a_9a_7+a_5a_6a_9),\ a_3 = ca_5a_7a_9$$

Substituting into b_n's we get

$$\begin{aligned}
b_0 &= c\,a_5^2a_7^2a_9^2\\
b_2 &= a_1a_5a_7a_9-\left(a_5^2a_7^2a_8^2+a_5a_7\left(a_5a_6+a_4a_7\right)a_9a_8+\left(a_5^2a_6^2+a_4a_5a_7a_6+a_4^2a_7^2\right)a_9^2\right)c\\
b_3 &= a_1\left(a_5a_7a_8+a_5a_6a_9+a_4a_7a_9\right)-\left(a_5a_6+a_4a_7\right)\left(a_5a_8+a_4a_9\right)\left(a_7a_8+a_6a_9\right)c\\
b_4 &= a_1\left(a_5a_6a_8+a_4a_7a_8+a_4a_6a_9\right)-a_4a_6a_8\left(a_5a_7a_8+a_5a_6a_9+a_4a_7a_9\right)c\\
b_5 &= a_1a_4a_6a_8
\end{aligned}$$

Singlet	Q_χ	Q_ψ	Representations
θ_{12}	0	0	$SO(10)$ singlet in 78
θ_{13}	0	0	$45 \subset 78$
θ_{14}	0	4	$SO(10)$ singlet in $27_{t_{1,2}}$
θ_{15}	-5	1	$16_{t_{1,2}} \subset 27_{t_{1,2}}$
θ_{34}	0	4	$SO(10)$ singlet in 27_{t_3}
θ_{35}	-5	-1	$16_{t_3} \subset 27_{t_3}$
θ_{45}	-5	-3	$16_{t_4} \subset 78$

Table 2.7: Table showing the E_6 charges and origin of some of the singlets in Table 2.6.

Next, we observe that we have to determine the homology classes of nine unknowns $a_1, \ldots a_9$ in terms of the b_k-classes, which we demand to be $\eta - kc_1$. Three classes are left unspecified which we choose them to be $[a_l] = \chi_l, l = 5,7,9$. The rest are computed easily, and the results are $[a_1] = \eta - 2c_1 - \chi$, $[a_2] = \eta - c_1 - \chi$, $[a_3] = \eta - \chi$, $[a_4] = -c_1 + \chi_5$, $[a_5] = \chi_5$, $[a_6] = -c_1 + \chi_7$, $[a_7] = \chi_7$, $[a_8] = -c_1 + \chi_9$, $[a_9] = \chi_9$.

The Σ_{10} curves are found setting $s = 0$ in the polynomial

$$b_5 \equiv \Pi_5(0) = a_1 a_4 a_5 a_6 = 0 \quad \rightarrow \quad a_1 = 0, a_4 = 0, a_5 = 0, a_6 = 0 \tag{2.102}$$

Thus, after the monodromy action, we obtain four curves (one less) to arrange the appropriate pieces of the three families.

The Σ_5 curves are treated similarly in [24] so we do not present the details here.

2.3.4 Singlets in the $SU(5)_\perp$ Spectral Cover

Akin to the $SO(10)$ case, we will compute the homology classes of the singlets inside E_6 by determining the polynomial $\prod_{i\neq j}(t_i - t_j)$ in terms of b_n's. The results should match the homologies of the corresponding 27s of E_6 which contain the singlets. In analogy with the previous cases, while using the results of Appendix B we find

$$\begin{aligned} P_0 \;=\; & 3125 b_5^4 b_0^5 + 256 b_4^5 b_0^4 - 3750 b_2 b_3 b_5^3 b_0^4 + 2000 b_2 b_4^2 b_5^2 b_0^4 + 2250 b_3^2 b_4 b_5^2 b_0^4 \\ & -1600 b_3 b_4^3 b_5 b_0^4 - 128 b_2^2 b_4^4 b_0^3 + 144 b_2 b_3^2 b_4^3 b_0^3 - 27 b_3^4 b_4^2 b_0^3 + 825 b_2^2 b_3^2 b_5^2 b_0^3 \\ & -900 b_2^3 b_4 b_5^2 b_0^3 + 108 b_3^5 b_5 b_0^3 + 560 b_2^2 b_3 b_4^2 b_5 b_0^3 - 630 b_2 b_3^3 b_4 b_5 b_0^3 \\ & +16 b_2^4 b_4^3 b_0^2 - 4 b_2^3 b_3^2 b_4^2 b_0^2 + 108 b_2^5 b_5^2 b_0^2 + 16 b_2^3 b_3^3 b_5 b_0^2 - 72 b_2^4 b_3 b_4 b_5 b_0^2 \end{aligned}$$

Factorization (via Mathematica) leads to the results which are summarised in the complete $SU(5)$ table (Table 2.6). Since the factorised form is very lengthy we do not exhibit it here. Note that due to the large number of parameters η, c_1, $\tilde{\chi}$, $\tilde{\chi}_{5,7,9}$ there are no constraints between the singlet $M_{U(1)}$s for the case that only the hyperchage is unbroken by flux effects.

The way in which the singlets fit into the E_6 and $SO(10)$ pictures can be found by working out the $U(1)_\chi$ and $U(1)_\psi$ charges using the generators in Eq. (2.21) and matching the charges to the singlets in the decomposition in Eq. (2.20). Putting this information together with the homology classes, leads to the results in Tables 2.6 and 2.7.

In the subsequent model building, if the GUT group on S is taken to be E_6, we cannot know the properties of the singlets θ_{12} and θ_{13} using the spectral cover approach for the reasons discussed previously. If the GUT group is taken to be $SO(10)$ or $SU(5)$, the situation is clearly worse, as then there are more GUT singlets for which we cannot compute homology classes. As such, we can never have a complete knowledge of the singlet properties in a local framework. This means that in model building, we will simply make assumptions about the singlet spectrum (which in turn would ammount to making assumptions about the global completion of the model).

2.4 Singlet VEVs and D- and F-flatness conditions

The homology constraints just discussed can be used to construct models capable of accommodating the Standard Model - an example of this is given in Chapter 3. To obtain a viable model it is usually necessary to remove additional Standard Model 'vectorlike' states by generating mass for them through their coupling to E_6 singlets which acquire VEVs. Any such VEVs should be consistent with F and D flatness conditions and we turn now to a discussion of this. Since, in this chapter, we have assumed all GUT breaking is driven by flux no GUT non-singlet fields acquire VEVs until the electroweak scale and so these VEVs can be ignored when determining high scale VEVs.

In general the superpotential for the massless singlet fields is given by

$$W = \mu_{ijk}\theta_{ij}\theta_{jk}\theta_{ki} \tag{2.103}$$

The F-flatness conditions are given by

$$\frac{\partial W}{\partial \theta_{ij}} = \mu_{ijk}\theta_{jk}\theta_{ki} = 0 \tag{2.104}$$

The D-flatness condition for $U_A(1)$ is

$$\sum_{i,j} Q^A_{ij}(|\langle\theta_{ij}\rangle|^2 - |\langle\theta_{ji}\rangle|^2) = -\frac{TrQ^A}{192\pi^2} g_s^2 M_S^2 \tag{2.105}$$

where the right-hand side (rhs) is the anomalous contribution, Q^A_j are the singlet charges and the trace TrQ^A is over all singlet and non-singlet states. The D-flatness conditions must be checked for each of the $U_A(1)$s.

2.4.1 E_6 case

In this case after the monodromy action there is only a single $U(1)$ and, in the t_i' basis the charge is given by diag$[1,1,-2]$. As both the 27s and the θ_{ij} are charged under the $U(1)$, we must know the number of each after the monodromy action and the flux breaking mechanism in order to compute the trace. The contribution of the 27_{t_i} to TrQ^A is

$$27(q_1 n_1 + q_3 n_3) = 27(n_1 - 2n_3)q_1 \tag{2.106}$$

and the contribution of the θ_{ij} is

$$1 \times [(q_1 - q_2)n_{12} + (q_1 - q_3)n_{13}] = 3n_{13}q_1 \tag{2.107}$$

The multiplicities are given in terms of the flux restrictions as the flux dotted with the homology class, and so we have

$$n_1 + n_3 = \mathscr{F} \cdot (\eta - 3c_1) \tag{2.108}$$

$$n_{12} = n_{13} = \mathscr{F} \cdot (\eta - 2c_1) \tag{2.109}$$

Assuming that only the pair θ_{13}, θ_{31} get VEVs, the flatness condition is

$$q_3(|\langle\theta_{13}\rangle|^2 - |\langle\theta_{31}\rangle|^2) + \frac{9(n_1 - 2n_3) + n_{13}}{64\pi^2} q_1 g_s^2 M_S^2 = 0 \tag{2.110}$$

and as we have $q_3 = -2q_1$

$$|\langle\theta_{13}\rangle|^2 - |\langle\theta_{31}\rangle|^2 = \frac{9(n_1 - 2n_3) + n_{13}}{128\pi^2} g_s^2 M_S^2 \tag{2.111}$$

In order to relate the multiplicities to each other, we define for convenience $\omega = \mathscr{F}_{U(1)} \cdot \eta$, $p = \mathscr{F}_{U(1)} \cdot c_1$ and $x = \mathscr{F}_{U(1)} \cdot \chi$. As such, in this notation, we have

$$n_1 = \omega - 2p - x \tag{2.112}$$

$$= n_{13} - x \tag{2.113}$$

$$n_3 = x - p \tag{2.114}$$

As chirality requires $n_1 > 0$ and $n_3 < 0$, the term $9(n_1 - 2n_3)$ is always positive. If we take the case $n_1 = 4$ and $n_3 = -1$ (i.e. the minimal case of three 27's accommodating the three families and a pair $27_H + \overline{27}_{\bar{H}}$), we have $n_{13} = 3 + p$, and

$$|\langle\theta_{13}\rangle|^2 - |\langle\theta_{31}\rangle|^2 = \frac{54 + n_{13}}{128\pi^2} g_s^2 M_S^2 \tag{2.115}$$

This condition is consistent with $\langle\theta_{13}\rangle \neq 0$ and $\langle\theta_{31}\rangle = 0$ for any $n_{13} > 0$, but not with the case $\langle\theta_{13}\rangle = 0$, $\langle\theta_{31}\rangle \neq 0$ as this would require $n_{13} < -54$.

2.4.2 $SO(10)$ case

Analogous to the E_6 case, the D-flatness condition for the anomalous $U(1)$s is given by Eq. (2.105). In this case there are two $U_A(1)$s with charges that can be taken as $Q^1 = \text{diag}[1,1,1,-3]$ and $Q^2 = \text{diag}[1,1,-2,0]$. For example, for the case of Q_1, using Table 2.4, the trace is given by

$$TrQ_j^1 = 16(n_1^{16} + n_3^{16} - 3n_4^{16}) + 10(2n_{13}^{10} + 2n_{12}^{10} - 2n_{14}^{10} - 2n_{34}^{10}) + 4n_{14}^1 + 4n_{34}^1 \tag{2.116}$$

2.4.3 $SU(5)$ case

In this case there are three $U_A(1)$s with charges given in Eq. (2.21). In the next chapter we discuss F- and D-flatness in detail for a realistic model.

Chapter 3

Model Building: A realistic model based on E_6

3.1 Introduction

In this chapter, a viable low-energy-model is constructed in which the $U(1)$ symmetries and flux effects are utilised to answer all the model building challenges described in the Introduction. Inspired by the elimination of dangerous operators in the MSSM by R-parity, we start with the identification of R-parity in an E_6 GUT. However, after flux breaking the resulting model has some undesirable features, so these must be rectified by relaxing the E_6 constraints on the spectrum. However, even with the slight modifications, the dangerous $R-$parity violating operators are still forbidden. In addition the dimension 5 proton decay operators, allowed by R-parity, are also forbidden due to the $U(1)$ global symmetries of the model.

Due to the flux breaking, the spectrum has additional vector-like states beyond those of the minimal supersymmetric extension of the Standard Model (MSSM). We show that these exotic states get large masses, close to the GUT scale, if certain SM (and $SU(5)$) singlet fields acquire vacuum expectation values (VEVs). We identify the necessary singlet fields and show that these VEVs are needed for F- and D-flatness of the scalar potential, the VEVs being driven close to the GUT scale. Moreover we show that these VEVs do not re-introduce terms that can give rise to rapid proton decay.

Finally we show that the model may have a realistic structure for the quark and charged lepton masses in which the light generation masses and mixings are driven by flux and instanton effects. The neutrinos can get mass from the (type I) 'Seesaw' mechanism through the coupling of the doublet neutrinos to singlet neutrinos that acquire a Majorana mass due to the monodromy.

3.2 Building the model

There are several important ingredients to building a phenomenologically realistic low energy theory. The first is the need to control the baryon and lepton number violating terms in the Lagrangian that generate rapid proton decay. In addition to the dimension 3 and 4 terms (forbidden by R-parity in the MSSM), it is necessary to forbid the dimension 5 nucleon decay terms too. Although the latter are suppressed by an inverse mass factor, this mass must be some 10^7 times the Planck mass, which is unacceptably large in string theory.

A second necessary ingredient is the control of the 'μ term', which is the Higgs supermultiplet mass term in the superpotential, $\mu H_u H_d$. Such a term is allowed by the Standard Model Gauge symmetry but, for a viable theory, its coefficient, μ, must be of order the SUSY breaking scale. In order to explain this fact, the μ term can be effectively generated by a VEV for a singlet field which couples to $H_u H_d$. At the same time the Higgs colour triplets which accompany the Higgs doublets in GUTs must be very heavy - this was described in the Introduction as the 'doublet-triplet splitting' problem.

The final ingredient is that the quark and lepton masses and mixings must be consistent. In particular it is necessary to explain why the quark masses and mixing angles have a hierarchical structure while the leptons must have large mixing angles and a relatively small mass hierarchy to explain the observed neutrino oscillation phenomena.

There has been a significant effort to build F-theory based models that use $U(1)$ symmetries to obtain these ingredients, but prior to the model presented in this chapter (and in [1]), no fully satisfactory model had been obtained, and it was actually speculated that such a model could not exist. Here, using the formalism described in the Introduction and Chapter 2, an explicit example is constructed which demonstrates how the $U(1)$ symmetries alone are sufficient to build a viable theory.

3.2.1 The E_6 inspired model

The first, most important, step in model building is to find a matter and Higgs multiplet assignment that can eliminate rapid nucleon decay. To this end, we find that starting from an underlying unified group is very helpful and we consider the case of E_6. After imposing a $\mathscr{Z}_2$ monodromy there are just two multiplets, $27_{t'_{1,3}}$. The $SU(5)\times SU(5)_\perp$ properties of these multiplets are given in Table 2.1. The only E_6 allowed trilinear term in the superpotential is $27_{t'_1}27_{t'_1}27_{t'_3}$, and as a result, if we assign the quark and lepton supermultiplets to 27_{t_1} and the Higgs supermultiplets to 27_{t_3}, there will be no dimension 3 or dimension 4 baryon- or lepton-number violating terms.

Requiring that anomalies are cancelled in a given model leads to constraints between the number of $SU(5)$ 10 and 5 dimensional representations [24, 79]. These conditions are automatically satisfied for multiplets descending from complete E_6 multiplets. In particular for the E_6 27 dimensional representations we have, in the notation of [24]

$$M_{10_M} = M_{5_1} = -M_{5_2} = -M_{5_3}, \quad (3.1)$$

$$M_{10_2} = -M_{5_4} = -M_{5_5} = M_{5_{H_u}}. \quad (3.2)$$

Furthermore, in the absence of matter in the 78 dimensional representation we have

$$M_{10_3} = M_{10_4} = M_{5_6} = N_8 = N_9 = 0, \quad (3.3)$$

which implies:

$$N_7 = \tilde{N}. \quad (3.4)$$

The resulting states arising from complete 27s are shown in Table 3.1 where we have allowed also for the breaking of $SU(5)$ through hypercharge flux. The SM particle content is also shown in Table 3.1 in the usual notation where a generation of quarks and leptons is Q, u^c, d^c, L, e^c. The Higgs doublets H_u, H_d are accompanied by exotic colour triplets and anti-triplets $D, \overline{D}$. The 27s also contain the CP conjugates of the right-handed neutrinos ν^c and extra singlets S.

E_6	$SO(10)$	$SU(5)$	Weight vector	N_Y	$M_{U(1)}$	SM particle content
$27_{t_1'}$	16	$\overline{5}_3$	$t_1 + t_5$	$\tilde{N}$	$-M_{5_3}$	$-M_{5_3} d^c + (-M_{5_3} + \tilde{N}) L$
$27_{t_1'}$	16	10_M	t_1	$-\tilde{N}$	$-M_{5_3}$	$-M_{5_3} Q + (-M_{5_3} + \tilde{N}) u^c + (-M_{5_3} - \tilde{N}) e^c$
$27_{t_1'}$	16	θ_{15}	$t_1 - t_5$	0	$-M_{5_3}$	$-M_{5_3} \nu^c$
$27_{t_1'}$	10	5_1	$-t_1 - t_3$	$-\tilde{N}$	$-M_{5_3}$	$-M_{5_3} D + (-M_{5_3} - \tilde{N}) H_u$
$27_{t_1'}$	10	$\overline{5}_2$	$t_1 + t_4$	$\tilde{N}$	$-M_{5_3}$	$-M_{5_3} \overline{D} + (-M_{5_3} + \tilde{N}) H_d$
$27_{t_1'}$	1	θ_{14}	$t_1 - t_4$	0	$-M_{5_3}$	$-M_{5_3} S$
$27_{t_3'}$	16	$\overline{5}_5$	$t_3 + t_5$	$-\tilde{N}$	$M_{5_{H_u}}$	$M_{5_{H_u}} d^c + (M_{5_{H_u}} - \tilde{N}) L$
$27_{t_3'}$	16	10_2	t_3	$\tilde{N}$	$M_{5_{H_u}}$	$M_{5_{H_u}} Q + (M_{5_{H_u}} - \tilde{N}) u^c + (M_{5_{H_u}} + \tilde{N}) e^c$
$27_{t_3'}$	16	θ_{35}	$t_3 - t_5$	0	$M_{5_{H_u}}$	$M_{5_{H_u}} \nu^c$
$27_{t_3'}$	10	5_{H_u}	$-2t_1$	$\tilde{N}$	$M_{5_{H_u}}$	$M_{5_{H_u}} D + (M_{5_{H_u}} + \tilde{N}) H_u$
$27_{t_3'}$	10	$\overline{5}_4$	$t_3 + t_4$	$-\tilde{N}$	$M_{5_{H_u}}$	$M_{5_{H_u}} \overline{D} + (M_{5_{H_u}} - \tilde{N}) H_d$
$27_{t_3'}$	1	θ_{34}	$t_3 - t_4$	0	$M_{5_{H_u}}$	$M_{5_{H_u}} S$

Table 3.1: Complete 27s of E_6 and their $SO(10)$ and $SU(5)$ decompositions. The indices of the $SU(5)$ non-trivial states 10, 5 refer to the labeling of the corresponding matter curve (we use the notation of [24]). We impose the extra conditions on the integers N_Y and $M_{U(1)}$ from the requirement of having complete 27s of E_6 and no 78 matter. The $SU(5)$ matter states decompose into SM states as $\overline{5} \to d^c, L$ and $10 \to Q, u^c, e^c$ with right-handed neutrinos $1 \to \nu^c$, while $SU(5)$ Higgs states decompose as $5 \to D, H_u$ and $\overline{5} \to \overline{D}, H_d$, where $D, \overline{D}$ are exotic colour triplets and antitriplets. We identify RH neutrinos as $\nu^c = \theta_{15,35}$ and extra singlets from the 27 as $S = \theta_{14,34}$.

The only undetermined parameters in Table 3.1 are the three integers M_{5_3}, $M_{5_{H_u}}$ and $\tilde{N}$. To maintain the E_6 based suppression of the baryon- and lepton-number violating terms we require that the Higgs should come from $27_{t_3'}$ and the matter from $27_{t_1'}$ and that any states transforming as $H_{u,d}$ in $27_{t_1'}$ be heavy.

We first choose $M_{5_3} = -3$ to get three families of quarks and leptons in $27_{t_1'}$. To get a single pair of Higgs doublets in $27_{t_3'}$ without colour triplet partners we next choose $M_{5_{H_u}} = 0$ and $\tilde{N} = 1$.

According to Table 3.1 this gives the following SM spectrum, grouped according to $SO(10)$ origin:

$$\begin{aligned}
&\left[\overline{5}_3 \to 3d^c + 4L,\ 10_M \to 3Q + 4u^c + 2e^c,\ \theta_{15} \to 3\nu^c\right]_{16},\\
&\left[5_1 \to 3D + 2H_u,\ \overline{5}_2 \to 3\overline{D} + 4H_d\right]_{10},\\
&[\theta_{14} \to 3S]_1,\\
&\left[\overline{5}_5 \to \overline{L},\ 10_2 \to \overline{u^c} + e^c\right]_{16},\\
&\left[5_{H_u} \to H_u,\ \overline{5}_4 \to \overline{H_d}\right]_{10}.
\end{aligned} \qquad (3.5)$$

Note that the matter content is just that contained in 3 complete 27s of E_6: $3[Q,u^c,d^c,L,e^c,\nu^c]_{16}$, $3[H_u,D,H_d,\overline{D}]_{10}$, $3[S]_1$ plus some extra vector pairs $L+\overline{L}, e^c+\overline{e^c}, u^c+\overline{u^c}, H_d+\overline{H_d}$ that may be expected to get a large mass if some of the singlet states acquire large VEVs.

It may be seen that the $U(1)$ flux breaking has resulted in one of the lepton supermultiplets, e^c, being assigned to $27_{t'_3}$ in conflict with our original strategy of assigning all matter states to $27_{t'_1}$. However this does not lead to the dimension 4 R-parity violating superpotential term LLe^c because one of the e^c comes from the 16 of $SO(10)$ and there is no 16^3 coupling allowed by $SO(10)$. In this case it is a combination of the original R-parity and the underlying GUT symmetry that eliminates dangerous baryon and lepton number violating terms. In fact the combination is more effective than R-parity alone for it also forbids the dangerous dimension 5 terms.

More troublesome is the fact that H_d now comes from $27_{t'_1}$ so that down quark masses are forbidden at tree level. However there is an allowed coupling of H_dLe^c for the e^c belonging to $27_{t'_3}$. This discrepancy between down quark and charged lepton masses looks unacceptable even if the remaining masses are generated in higher order through coupling to singlet fields that acquire large VEVs. To avoid this we look at a slightly modified structure choosing

$$\begin{aligned}
M_{10_M} &= -M_{5_3} = 4,\\
M_{5_1} &= -M_{5_2} = 3\\
M_{10_2} &= -M_{5_5} = -1,\\
M_{5_4} &= M_{H_u} = 0,\\
M_{\theta_{15}} &= 2,\\
\tilde{N} &= 1
\end{aligned} \qquad (3.6)$$

This leads to the spectrum given in Table 3.2 where now both the down quarks and leptons originate in $27_{t'_1}$, avoiding the troublesome difference in their mass matrices just discussed.

The difference in the spectrum compared to the previous case is in the vectorlike sector with additional pairs of $L+\overline{L}, Q+\overline{Q}, u^c+\overline{u^c}, d^c+\overline{d^c}$ and $H_d+\overline{H_d}$ and no $e^c+\overline{e^c}$. Provided the vectorlike states are heavy, the absence of the dimension 3 and 4 R-parity violating operators

E_6	$SO(10)$	$SU(5)$	Weight vector	N_Y	$M_{U(1)}$	SM particle content	Low energy spectrum
$27_{t_1'}$	16	$\overline{5}_3$	t_1+t_5	1	4	$4d^c+5L$	$3d^c+3L$
$27_{t_1'}$	16	10_M	t_1	-1	4	$4Q+5u^c+3e^c$	$3Q+3u^c+3e^c$
$27_{t_1'}$	16	θ_{15}	t_1-t_5	0	3	$3\nu^c$	-
$27_{t_1'}$	10	5_1	$-t_1-t_3$	-1	3	$3D+2H_u$	-
$27_{t_1'}$	10	$\overline{5}_2$	t_1+t_4	1	3	$3\overline{D}+4H_d$	H_d
$27_{t_3'}$	16	$\overline{5}_5$	t_3+t_5	-1	-1	$\overline{d^c}+2\overline{L}$	-
$27_{t_3'}$	16	10_2	t_3	1	-1	$\overline{Q}+2\bar{u^c}$	-
$27_{t_3'}$	16	θ_{35}	t_3-t_5	0	0	$-$	-
$27_{t_3'}$	10	5_{H_u}	$-2t_1$	1	0	H_u	H_u
$27_{t_3'}$	10	$\overline{5}_4$	t_3+t_4	-1	0	$\overline{H_d}$	-
$27_{t_3'}$	1	θ_{34}	t_3-t_4	0	1	θ_{34}	-
-	1	θ_{31}	t_3-t_1	0	4	θ_{31}	-
-	1	θ_{53}	t_5-t_3	0	1	θ_{53}	-
-	1	θ_{14}	t_1-t_4	0	3	θ_{14}	-
-	1	θ_{45}	t_4-t_5	0	2	θ_{45}	-

Table 3.2: Complete 27s of E_6 and their $SO(10)$ and $SU(5)$ decompositions. We use the notation of ref [24] for the indices of the $SU(5)$ states and impose the extra conditions on the integers N_Y and $M_{U(1)}$ from the requirement of having complete 27s of E_6 and no 78 matter. The $SU(5)$ matter states decompose into SM states as $\overline{5} \to d^c, L$ and $10 \to Q, u^c, e^c$ with right-handed neutrinos $1 \to \nu^c$, while $SU(5)$ Higgs states decompose as $5 \to D, H_u$ and $\overline{5} \to \overline{D}, H_d$, where $D, \overline{D}$ are exotic colour triplets and antitriplets. We identify RH neutrinos as $\nu^c = \theta_{15}$. The extra singlets are needed for giving mass to neutrinos and exotics and to ensure F and D flatness.

is now guaranteed by the underlying $U(1)$ symmetries[1]. As we shall see, the underlying GUT symmetry still also eliminates the dimension 5 terms that would cause proton decay.

3.2.2 Doublet-triplet splitting and vector-like masses.

There remains the doublet-triplet problem of giving large mass to the D and $\overline{D}$ fields and the problem of giving large mass to the vectorlike pairs of fields. Since the D and $\overline{D}$ fields also come in vectorlike pairs these problems are related and are solved by generating mass for vectorlike fields through their coupling to SM singlet fields that acquire large VEVs. For the case in which the vectorlike pairs have components in both the $27_{t_1'}$ and $27_{t_3'}$ multiplets, the extra vector pairs are removed by introducing θ_{31}, an E_6 singlet, with couplings:

$$\theta_{31}27_{t_1'}\overline{27_{t_3'}} = \theta_{31}Q\overline{Q} + \theta_{31}(2u^c)(2\overline{u^c}) + \theta_{31}d^c\overline{d^c} + \theta_{31}(2L)(2\overline{L}) + \theta_{31}H_d\overline{H_d}. \tag{3.7}$$

If θ_{31} gets a large VEV these vector states get large masses as required. We shall discuss how the D-terms associated with the anomalous $U_A(1)$s can require a VEV for this field close to the Planck scale.

[1]Note that these operators do not involve H_d and so the fact that H_d originates in $27_{t_1'}$ does not cause problems.

To remove the remaining exotics we introduce θ_{34} which has the couplings :

$$\theta_{34}5_1\overline{5_2} = \theta_{34}[3D+2H_u][3\overline{D}+3H_d] = \theta_{34}[3(D\overline{D})]+\theta_{34}[2(H_uH_d)]. \tag{3.8}$$

If it too acquires a large VEV it generates large mass to the three copies of $D+\overline{D}$ (solving the doublet-triplet splitting problem) and two families of Higgs H_u, H_d, leaving just the MSSM spectrum as shown in the last column of Table 3.2.

The singlet θ_{14} could also play an important role, as the term $\theta_{14}H_uH_d$ is invariant under the perpendicular $U(1)$ symmetries. As such, the μ term could be effectively generated if θ_{14} acquires a TeV scale VEV. In [1], it is pointed out that the μ term can be generated by non-perturbative effects also. We expect the local $U(1)$ symmetries to be anomalous and the associated gauge bosons to become massive due to the Stueckelberg mechanism, leaving three global $U(1)$ symmetries which act as selection rules in determining the allowed Yukawa couplings [43]. However these global symmetries are only approximate and are explicitly broken by non-perturbative effects [80] with breaking characterised by the Kähler moduli, τ_i, components of the complex fields T_i, whose complex components provide the longitudinal components of the $U(1)$ gauge bosons. These non-perturbative effects will generate an explicit μH_uH_d term with the $\mu = O(M_s e^{-t/M_s})$ where t is the VEV of the appropriate combination of τ_i moduli, and M_s represents the string scale. Due to the exponential dependence on t this term can be of the Electroweak scale as required.

3.2.3 Singlet VEVS

In the model under consideration, in order to determine the large VEVs for the singlets, we consider the F and D flatness conditions. Taking account of the $\mathscr{Z}_2$ monodromy, $t_1 \leftrightarrow t_2$ the D-flatness conditions are of the form given in Eq. (2.105) where there are three $U_A(1)$s with charges given in Eq. (2.21). We wish to show that the D-flatness conditions are satisfied by the massless fields θ_{31}, θ_{34}, θ_{53} needed to give mass to exotics and, as discussed below, to generate viable neutrino masses. Using the spectrum given in Table 3.2 we compute TrQ^A for the three $U_A(1)$s. In a general basis, $Q = \text{diag}[t_1,t_2,t_3,t_4,t_5]$, Eq. (2.105) can be written

$$(t_5-t_3)|\theta_{53}|^2+(t_3-t_4)|\theta_{34}|^2+(t_3-t_1)|\theta_{31}|^2 = -XTrQ^A \tag{3.9}$$

The trace is taken over all states, and is given by

$$TrQ^A = 5\sum n_{ij}(t_i+t_j)+10\sum n_k t_k + \sum m_{ij}(t_i-t_j) \tag{3.10}$$

For our model, this trace is computed to be

$$TrQ^A = 61t_1 - 26t_3 + 14t_4 + 11t_5 \tag{3.11}$$

Applying this to the three $U_A(1)$s using the generators given in Eq. (2.21) leads to

$$
\begin{aligned}
5|\theta_{53}|^2 &= 5X \ (Q_\chi) \\
-|\theta_{53}|^2 + 4|\theta_{34}|^2 &= 7X \ (Q_\psi) \\
2|\theta_{53}|^2 - 2|\theta_{34}|^2 - 3|\theta_{31}|^2 &= -113X \ (Q_\perp)
\end{aligned}
\tag{3.12}
$$

where $X = \frac{g_s^2 M_S^2}{192\pi^2}$. These equations are solved by

$$
\begin{aligned}
|\theta_{53}|^2 &= X \\
|\theta_{34}|^2 &= 2X \\
|\theta_{31}|^2 &= 37X
\end{aligned}
\tag{3.13}
$$

In terms of demonstrating F-flatness, the only allowed superpotential terms that can give a non-zero F-term involves the fields with VEVs plus at most a single additional light field. The only problematic terms have the form $\lambda_{ij}\theta_{53}\theta_{31}^i\theta_{15}^j$ where $i = 1,2,3,4$ and $j = 1,2,3$. The F-terms of θ_{15}^j are potentially non-zero but minimisation of the singlet potential will make $\lambda_{i1}\langle\theta_{31}^i\rangle = 0$ and $\lambda_{i2}\langle\theta_{31}^i\rangle = 0$. This means three independent θ_{31}^i fields have zero VEVs but the fourth one can have a VEV as it decouples from θ_{15}^j. It is this combination that enters in Eqs. (3.12) and (3.13).

3.2.4 Baryon and lepton number violating terms

As discussed above, the R-parity violating superpotential couplings $u^c d^c d^c$, Qd^cL, Le^cL, κLH_u are not allowed because of the underlying $U(1)$ symmetries which play the role of R-parity. Dimension 5 terms in the Lagrangian, corresponding to the superpotential terms $QQQL$ and $u^c u^c d^c e^c$, which would be allowed by usual R-parity, are forbidden by the $U(1)$ symmetries that originate in the underlying E_6.

However, we must clearly be careful that spontaneous symmetry breaking terms coming from SM singlet field VEVs do not allow these dangerous operators to appear. Allowing for arbitrary singlet fields to acquire VEVs the dangerous the baryon and lepton number violating operators arise through the terms $\theta_{15}LH_u$, $(\theta_{31}\theta_{45} + \theta_{41}\theta_{35})10_M\overline{5_3}^2$ and $\theta_{31}\theta_{41}10_M^3\overline{5_3}$. Thus, provided θ_{15}, θ_{41} and θ_{45} do *not* acquire VEVs, these dangerous terms will not arise. One might worry that the non-perturbative effects which can generate an explicit μ term could also generate an explicit κLH_u term due to the similar structure of the two terms. However, this is not a problem for the proton decay operators provided that all of the $\Delta B \neq 0$ are absent.

However this is not sufficient to ensure the absence of baryon and lepton number violating terms because, even in the absence of these VEVs, tree level graphs can generate the dangerous operators at higher order in the singlet fields. The dangerous graph is shown in Fig. 3.1 and is driven by colour triplet exchange coming from the couplings

$$
\begin{aligned}
10_M\,10_M\,5_{H_u} &\rightarrow QQD_h + \ldots \\
5_{H_u}\bar{5}_{\bar{H}_u} &\rightarrow M_D D_h \bar{D}_h + \ldots \\
\theta_{34}5_1\bar{5}_2 &\rightarrow \langle\theta_{34}\rangle\, D_h' \bar{D}_h{}''' + \ldots = \langle\theta_{34}\rangle\, D\bar{D} + \ldots .
\end{aligned}
$$

As may be seen from Table 3.2 only the states D_h' and $\bar{D}_h'''$ appear in the spectrum with mass generated by the singlet VEV $\langle\theta_{34}\rangle$ which from Eq. (3.13) is predicted to be somewhat below the GUT scale. Since the choice of fluxes in Table 3.2 eliminates light colour triplet states D_h arising from 5_{H_u}, and also $D_h{}''$ states arising from 5_4, we assume that if states with the quantum numbers of D_h, D_h'' exist, they will have string scale masses, of $O(M_S)$.

In this case the diagram of Figure 3.1 gives the proton decay operator $QQQL$ with coefficient $1/\Lambda_{eff}$ given by

$$
\frac{1}{\Lambda_{eff}} = \lambda^5 \left(\frac{\langle\theta_{31}\rangle}{M_S}\right)^2 \frac{1}{\langle\theta_{34}\rangle} \tag{3.14}
$$

In (3.14), λ^5 represents the product of the five Yukawa couplings in the relevant diagram and according to ref [25] it is expected to be

$$
\lambda^5 = \lambda_{10\cdot 10\cdot 5}\lambda_{10\cdot\bar{5}\cdot\bar{5}}\lambda^3_{5\cdot\bar{5}\cdot 1} \approx 10^{-3}.
$$

We can further determine the mass ratios by taking into account the solution Eq. (3.13) to the flatness conditions to estimate the effective scale

$$
\Lambda_{eff} \approx 10^3 \left(\frac{M_S}{\langle\theta_{31}\rangle}\right)^2 \frac{\langle\theta_{34}\rangle}{M_S} M_S \approx \frac{8\sqrt{6}\pi}{37 g_s} \times 10^3 M_S \gtrsim 10^3 M_S \tag{3.15}
$$

This, multiplied by the appropriate loop-factor due to higgsino/gaugino dressing and other theoretical factors [81, 82, 83, 84, 65], should be compared to experimental bounds on nucleon

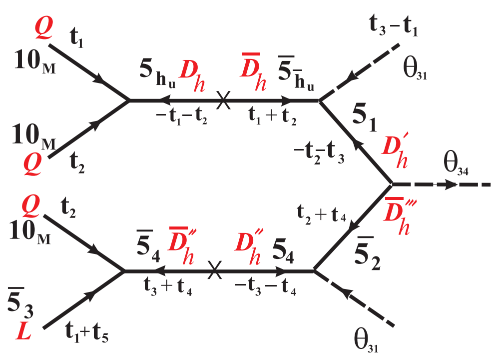

Figure 3.1: The proton decay diagram generating dim. 5 operator $QQQL$.

decay. This bound, relevant to the case that the operator $QQQL$ involves quarks from the two lighter generations only, requires $\Lambda_{\rm eff}^{\rm light} > (10^8 - 10^9)M_S$. Given the large discrepancy between $\Lambda_{\rm eff}^{\rm light}$ and $\Lambda_{\rm eff}$ it is clearly important to determine whether, in the absence of flux, this light quark operator is generated by the diagram of Figure 3.1.

In order to answer this question, the nature of the triple overlap integrals involved in the computation of the diagram must be considered [85]. For the case of trilinear couplings involving light fields only, the calculation involves an integral over the coordinates about the point of intersection, z_i, of the surface on which the matter curves reside. When there are N multiple fields associated with a matter curve the orthogonal wave functions may be chosen proportional to powers of the coordinates, $(z_i)^j$, $j = 1,..,N$. On integration only the coupling involving the fields with $j = 0$ are non-zero, corresponding to a geometric $U(1)_i$ invariance, $z_i \to z_i e^{i\alpha_i}$, as explained in Appendix F. For the case the three families live on the same matter curve this means the mass matrices are rank 1 in the absence of the flux corrections of [29]. Switching on the flux gives a rank 3 mass matrix and generates the mixing between the generations.

In the case where there are vertices involving both light and heavy fields, because the heavy field wave function can involve powers of $\bar{z}_i$ [29], there can be couplings involving light states with $j \neq 0$. As such, as long as the $U(1)$ invariance is intact, higher order operators with only external light fields are generated only if all the external fields have $j = 0$. However, as explained in [85], there are two types of Higgs wavefunction which can be involved in the integral, corresponding to the cases of 'non-vanishing flux density' or 'vanishing flux density'. These cases correspond to the existence of massless colour triplets (or anti-triplets) or vector pairs of triplets and anti-triplets in the spectrum respectively.

The model considered here corresponds to vanishing flux density, and it can be shown that the Higgs wavefunctions in this case have the form of Eq. F.18. Calculating triple overlap integrals involving wavefunctions of this form shows that trilinear couplings in the case of vanishing flux density do not respect the geometric $U(1)$. However, as argued in Appendix F, for the case of a complete diagram the $U(1)$ invariance should be restored due to the presence of the conjugate Higgs wavefunctions. Due to the fact that the $U(1)$ is respected for Figure 3.1, the operator generated by this diagram does not involve the light quarks. As such, its contribution to nucleon decay vanishes in the absence of the non-perturbative flux effects of [29], and hence is significantly suppressed. To estimate this suppression we use the fact that the same flux effects generate the masses and mixings of the light quarks. Using these mixing angles we can convert the heavy quark operator to one involving light quarks. For the least suppressed case involving two down quarks and an up quark, this gives $\Lambda_{\rm eff}^{\rm light} \approx \sqrt{\frac{m_t}{m_u}\frac{m_b}{m_d}}\Lambda_{\rm eff} \approx 10^9 M_S$, consistent with the experimental bound. A similar result applies to the operator involving right handed quarks.

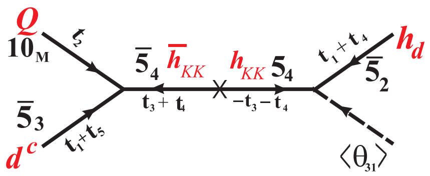

Figure 3.2: Tree-level diagram contributing to the bottom mass.

3.2.5 Quark and charged lepton masses

Up to SM singlets the surviving low energy spectrum is that of the MSSM given by:

$$\begin{aligned}&\left[\overline{5}_3 \to 3d^c + 3L,\ 10_M \to 3Q + 3u^c + 3e^c\right],\\&\left[\overline{5}_2 \to H_d\right]_{10_{t_1}},\\&\left[5_{H_u} \to H_u\right]_{10_{t_3}}.\end{aligned} \tag{3.16}$$

The allowed low energy couplings in the superpotential originate from:

$$\begin{aligned}27_{t_1}27_{t_1}27_{t_3} &\to 16_{t_1}16_{t_1}10_{t_3}\\&\to 10_M 10_M 5_{H_u} + \overline{5}_3\theta_{15}5_{H_u} + \overline{5}_3 10_2 \overline{5}_2\\&\to (3Q)(3u^c)H_u + (3L)(3\nu^c)H_u.\end{aligned} \tag{3.17}$$

A 3×3 up-type and Dirac neutrino mass matrix is allowed at dimension three. In the absence of flux these matrices are rank one. However, as recently shown by Aparicio, Font, Ibanez and Marchesano [29], non-perturbative flux effects can generate an acceptable pattern for the light up quarks.

The down quark and charged leptons acquire mass through the non-renormalisable Yukawa couplings:

$$\begin{aligned}\theta_{31}27_{t_1}27_{t_1}27_{t_1}/M &\to \theta_{31}16_{t_1}16_{t_1}10_{t_1}/M\\&\to \theta_{31}\overline{5}_3 10_M \overline{5}_2/M\\&\to (\theta_{31}(3d^c)(3Q) + \theta_{31}(3L)(3e^c))H_d/M.\end{aligned} \tag{3.18}$$

Note that, from Table 3.2, the relevant graph 3.2 is generated by the exchange of a massive vectorlike pair that is given a mass by $\langle\theta_{31}\rangle$. We have already seen that θ_{31} must have a large VEV to give mass to exotics so this term can lead to down quark and charged lepton Yukawa couplings that are only mildly suppressed relative to the up quark couplings ($\langle\theta_{31}\rangle/M \geq m_b/m_t$). This suppression provides an origin for the relative magnitude of the top quark to the bottom quark. Although the mass matrices for the down quarks and charged leptons coming from Eq. (3.18) are rank one, non-perturbative flux effects will generate the remaining terms and can lead to an acceptable mass structure [29].

3.2.6 Neutrino masses

Due to the $t_1 \leftrightarrow t_2$ monodromy, the conjugate states θ_{12} and θ_{21} are identified, and so we can write down a term $M_M \theta_{12}\theta_{21}$ in the superpotential which corresponds to a Majorana mass for the θ_{12} states [42]. The right handed neutrinos, θ_{15}, couple to the Majorana states through the term $\lambda_{RM}^{ij}\Theta_{51}\theta_{12}^{i}\theta_{15}^{j}$, where $\Theta_{51} = \frac{\theta_{53}\theta_{31}}{M}$. As both θ_{53} and θ_{31} acuire VEVs, Θ_{51} also has a VEV. We allow for an arbitrary number of θ_{12} fields, as the fact that these fields carry no charge under the perpendicular $U(1)$s means that we can have any number of them in the spectrum without affecting flatness conditions etc.

The method of generating masses for the light neutrinos will be a double seesaw mechanism, where the θ_{15} fields will get Majorana masses through their coupling to the Majorana states θ_{12}, and then the light neutrinos will get masses via a seesaw mechanism, made possible by their coupling to the right handed neutrinos θ_{15}. The relevant terms for lepton mass generation are (after the two Higgs doublets have got their VEVs):

$$W_{mass} = \langle H_d \rangle Y_e^{ij} \bar{e}_L^i e_R^j + \langle H_u \rangle \lambda_{LR}^{ia} \bar{\nu}_L^i \theta_{15}^a + \langle \Theta_{51} \rangle \lambda_{RM}^{a\alpha} \theta_{15}^a \theta_{12}^\alpha + M_M^{\alpha\beta} \theta_{12}^\alpha \theta_{21}^\beta \tag{3.19}$$

We can put the notation into a more familiar form by writing

$$M_e^{ij} \equiv \langle H_d \rangle Y_e^{ij},\, m_{LR}^{ia} \equiv \langle H_u \rangle \lambda_{LR}^{ia},\, M_{RM}^{a\alpha} \equiv \langle \Theta_{51} \rangle \lambda_{RM}^{a\alpha}$$

Also, for clarity, we can relabel the fields as $\theta_{15} \equiv \nu_R$, $\theta_{12} \equiv S_R$. Eq. (3.19) can then be written

$$W_{mass} = M_e^{ij} \bar{e}_L^i e_R^j + m_{LR}^{ia} \bar{\nu}_L^i \nu_R^a + M_{RM}^{a\alpha} \nu_R^a S_R^\alpha + M_M^{\alpha\beta} S_R^\alpha S_R^\beta \tag{3.20}$$

In the basis (ν_L, ν_R, S_R), the mass matrix is, in block form

$$M = \begin{pmatrix} 0 & m_{LR} & 0 \\ m_{LR} & 0 & M_{RM} \\ 0 & M_{RM} & M_M \end{pmatrix}$$

Applying the double see-saw mechanism, we have (in matrix notation) for the light left-handed Majorana neutrino masses [86]

$$m_{LL} = m_{LR} M_{RM}^{-1} M_M (M_{RM}^T)^{-1} m_{LR}^T \tag{3.21}$$

We can estimate the magnitude of the resulting neutrino masses by taking the case of just one family of each state. Using Eq. (3.13) and assuming that $M = M_S$ in $\Theta_{51} = \frac{\theta_{53}\theta_{31}}{M}$, we find that $\langle \Theta_{51} \rangle \approx 3 \times 10^{-3} M_S$. Assuming that the Majorana mass M_M is of order the string scale, θ_{15} acquires a Majorana mass M_{RR} though the coupling to θ_{12}, giving

$$M_{RR} \approx -\frac{\langle\Theta_{51}\rangle^2 \lambda_{RM}^2}{M_M} \tag{3.22}$$

In turn, the light neutrinos acquire masses through the seesaw mechanism, due to their couplings to θ_{15}, giving

$$M_\nu \approx \frac{\langle H_u\rangle^2 \lambda_{LR}^2}{\langle\Theta_{51}\rangle^2 \lambda_{RM}^2} M_M \tag{3.23}$$

Assuming that the couplings λ_{LR} and λ_{RM} are $\mathcal{O}(1)$, this gives a neutrino mass of $\mathcal{O}(10^{-1}\text{eV})$ through this very approximate treatment, which is acceptable. However, the size of the λs depends on the proximity of the relevant intersections of the matter curves involved in the couplings, and so a precise calculation would require knowledge of the overlap integrals involved in the coupling computations.

3.2.7 Relation to previous work

In [27] a general analysis was presented of the possible R-symmetries coming from the $U(1)_\perp$ factors in the local analysis of F-theory. Two possibilities were identified but it was shown that it was not possible to realise them in the semi-local picture. The model presented above corresponds to the Matter Parity Case 1 of [27] and we have shown that it is consistent with the semi-local picture. The explanation of the apparent conflict is straightforward. In [27], seeking to generate viable fermion mass matrices without flux effects, the analysis considered only the case that the matter coming from the 10 dimensional representation of $SU(5)$ should come from two matter curves, 10_M and 10_{t_4}. As a result, in order to suppress the dimension 5 nucleon decay operators, a VEV for the field θ_{31} was forbidden and hence, *c.f.* the discussion above, no down-type mass terms could be generated and the Matter Parity Case 1 was ruled out. However in the case of interest here all three generations are assigned to 10_M. As a result a VEV for θ_{31} is allowed without generating dimension 5 nucleon decay operators. Hence a down-type mass matrix proportional to $\langle\theta_{31}\rangle$ is possible and, allowing for flux effects, the resulting mass matrix can be of rank 3.

In [46] a general discussion was presented of the difficulty in obtaining phenomenological viable F-theory models in the semi-local approach. The difficulty of reconciling the exotic spectrum necessitated by flux breaking with the μ-term, the suppression of nucleon decay operators and gauge unification was emphasised and studied in detail for the case of models with one or two $U(1)_\perp$s. The model constructed here has three $U(1)_\perp$s and demonstrates that the problems can be ameliorated but not eliminated. In particular we have shown that the suppression of the dangerous nucleon decay operators is maintained while generating a μ-term. However the constraints following from anomaly cancellation [24, 79] are still severe and lead to an extended exotic spectrum.

3.3 Conclusions

In this chapter, semi-local F-theory GUTs have been considered which arise from a single E_8 point of local enhancement. The study centered around simple GUT gauge groups based on E_6, $SO(10)$ and $SU(5)$ together with $SU(3)$, $SU(4)$ and $SU(5)$ spectral covers, respectively. Assuming the minimal $\mathscr{Z}_2$ monodromy, we determined the homology classes of the spectrum for each case, and the implications for the resultant spectrum after flux breaking.

Using this, and aided by a dictionary relating the E_6, $SO(10)$, $SU(5)$ representations, we constructed a model that leads to the MSSM at low energies. We showed that D-and F-flatness constraints require VEVs for singlet fields, which spontaneously break the $U(1)$ symmetries, and which generate large masses for all the non-MSSM exotic fields. In the absence of flux, the quark and charged lepton mass matrices are of rank one, but when flux and instanton corrections are included, light quark and lepton masses and mixings are generated that can be consistent with their observed values. In the absence of flux, the additional $U(1)$ symmetries descending from E_8 ensure that dangerous baryon and lepton number violating terms are absent up to and including dimension 5, even taking into account the singlet VEVs which break the $U(1)$ symmetries. Including the flux effects, dimension 5 terms involving light quarks are generated but at an acceptable level, and as a result the proton is stable within present limits. The μ term in the theory is also forbidden by the $U(1)$ symmetries but can be generated at the SUSY breaking scale, either effectively as a result of a TeV scale singlet VEV, or again (as is chosen in [1]) through non-perturbative effects which explicitly break the $U(1)$ symmetries. Neutrino masses are generated via the see-saw mechanism, involving singlet neutrinos that acquire large Majorana masses allowed by the monodromy.

In conclusion, we have provided an example of a fully viable F-theory GUT, assuming flux breaking of all symmetries, satisfying the semi-local constraints, and employing only the additional $U(1)$ symmetries descending from the E_8 point of local enhancement.

Chapter 4

Further E_6 Model Building: TeV scale exotics and gauge coupling unification

4.1 Introduction

Although descending from a high energy E_6 group, most of the models studied previously [6, 7, 8, 9, 10, 15, 87, 22, 24, 23, 1, 27] focus on reproducing the minimal supersymmetric standard model (MSSM) at low energies, making it difficult to obtain an experimental link to F-theory. In this chapter we explore F-theory models in which the low energy supersymmetric theory contains the particle content of three 27 dimensional representations of the underlying E_6 gauge group. The resulting low energy models will resemble either the E6SSM [75, 88, 89, 90, 91] or a generalised NMSSM+ [92] depending on whether an additional Abelian gauge group does or does not survive. However there are novel features compared to both these models which, if observed, would provide circumstantial evidence for F-theory.

The F-theory models considered in this thesis all descend from a parent E_8 gauge theory [19]. A crucial question for model construction is whether a gauged $U(1)$ from the E_8 gauge theory can survive down to low energies, where the gauged $U(1)$ may arise from one of the Cartan generators of the non-Abelian gauge group. A clear example of this is the case of hypercharge $U(1)_Y$, arising from $SU(5)$ after flux breaking in many F-theory models [24]. More generally, if we begin with the case of an E_8 gauge theory, we can break E_8 down to an E_6 GUT group with a VEV for an adjoint Higgs, and then break E_6 down to the Standard Model gauge group by turning on flux along three $U(1)$s. In order to label the different $U(1)$s, we can look at the flux breaking from E_6 to the Standard Model as going through the following sequence of rank preserving breakings:

$$E_6 \rightarrow SO(10) \times U(1)_\psi \tag{4.1}$$

$$SO(10) \rightarrow SU(5) \times U(1)_\chi \tag{4.2}$$

$$SU(5) \rightarrow SU(3) \times SU(2) \times U(1)_Y. \tag{4.3}$$

For example, the $U(1)_N$ under which the right handed neutrinos have no charge is given in terms of these $U(1)$s by,

$$U(1)_N = \frac{1}{4}U(1)_\chi + \frac{\sqrt{15}}{4}U(1)_\psi. \tag{4.4}$$

In the F-theory models considered in this chapter, there will either be a surviving gauged $U(1)_N$, or it will be broken at the GUT scale.

The F-theory models with a surviving Abelian gauge group resemble the E6SSM [75, 88, 89] which is a supersymmetric standard model in which precisely such an extra $U(1)_N$ gauge symmetry survives down to the TeV scale. However in the F-theory model, the gauge coupling of the $U(1)_N$ may differ from that in the E6SSM. The matter spectrum is similar to that of the E6SSM, namely three 27s of E_6 which ensures anomaly cancellation. This implies light exotics with the quantum numbers of Higgs doublets and colour triplets of exotic quarks, arising from three $5+\overline{5}$ representations of $SU(5)$, plus three $SU(5)$ singlets which are charged under $U(1)_N$. The coupling one of these singlets to H_uH_d generates an effective μ term after singlet acquires a low scale vacuum expectation value (VEV). Whilst the E6SSM includes an additional pair of doublet states called H' and $\overline{H'}$ in order to achieve gauge coupling unification[89], in the F-theory model the couplings are split at the GUT scale by flux effects. As such, the minimal F-theory version resembles the Minimal E6SSM (ME6SSM) proposed in [90], although as will be shown later, the splitting turns out to be too large to be acceptable. However, instead of having to add in states by hand in order to achieve the desired level of unification, constraints from topology actually lead to the natural emergence of a set of light bulk exotics which have the same effect. These exotics will be the topic of Chapter 5, where the problems of unification will be solved. In this chapter, only the minimal model will be considered.

Proton decay represents another important difference between the E6SSM and the F-theory version. In the F-theory model proton decay is suppressed by the geometric coupling suppression of a singlet state, which effectively suppresses the coupling of the exotic charge $-1/3$ colour triplet state D to quarks and leptons, while in the ME6SSM all proton decay couplings are allowed but with highly suppressed coefficients. This tends to give long lived D decays in the ME6SSM, but prompt D decays in the F-theory model, with large couplings to left-handed quarks and leptons, providing characteristic and striking signatures at the LHC.

If there is no surviving extra Abelian gauge group then the F-theory model resembles the NMSSM+ which also involves three compete 27 dimensional families [92]. However, whereas in the NMSSM+ the $U(1)_N$ is broken by an additional sector close to the GUT scale, in the F-theory model it is simply broken by flux breaking. Another important difference is that the NMSSM+ is a scale invariant theory, involving only trilinear couplings such as the trilinear singlet couplings, while in the case of F-theory there are in addition singlet mass terms arising from non-perturbative effects, giving rise to a generalised version of the NMSSM+.

E_6 based F-theory models have been discussed previously, for example, issues concerning the global resolution of E_6 GUTs in [93, 94], and the models of [32]. It should be noted that here we use Abelian fluxes, whereas [32] uses non-Abelian fluxes.

4.2 E_6 Models from F-theory

We start by looking at the model of Chapter 3 [1] (model 1), which was motivated by the fact that if we build a model based on complete 27s of E_6 with no matter coming from the adjoint (78) representation, we automatically take care of anomaly cancellation. [1] Table 4.1 shows the model building freedom we have in choosing the M and N integers specifying the flux breaking, and how these choices determine the Standard Model particle content of the model. Here we make the same choices for the Ms and Ns as in [1] and these choices are summarised in Table 4.1. In Table 4.1, arbitrary numbers of singlets are allowed in the spectrum for now, so that we can calculate the restrictions on these numbers later on. The final column of Table 4.1 shows the low energy spectrum of the E6SSM that we want to arrive at by eliminating the required exotics from the previous column, which shows the SM particle content after flux breaking. By comparing the final two columns of Table 4.1, we can see that the exotics which we wish to remove are the vector pairs $2(L+\overline{L}), Q+\overline{Q}, 2(u^c+\overline{u^c}), d^c+\overline{d^c}$ and $H_d+\overline{H_d}$. Large masses will be generated for these fields through their coupling to SM singlet fields which acquire large VEVs.

From the E_6 point of view, the only E_6 allowed trilinear term in the superpotential is $27_{t_1}27_{t_1}27_{t_3}$. The vectorlike pairs which we wish to remove from the low energy particle content are those which have components in both the 27_{t_1} and 27_{t_3} multiplets. As such, they are removed by introducing θ_{31}, an E_6 singlet, with couplings:

$$\theta_{31}27_{t_1'}\overline{27_{t_3'}} = \theta_{31}Q\overline{Q} + \theta_{31}(2u^c)(2\overline{u^c}) + \theta_{31}d^c\overline{d^c} + \theta_{31}(2L)(2\overline{L}) + \theta_{31}H_d\overline{H_d}. \tag{4.5}$$

If θ_{31} gets a large VEV, these vector states get large masses as required. The difference between this case and model 1 [1] is that in model 1, θ_{34} also gets a large VEV. This singlet has the following couplings

$$\theta_{34}5_1\overline{5_2} = \theta_{34}[3D+2H_u][3\overline{D}+3H_d] = \theta_{34}[3(D\overline{D})] + \theta_{34}[2(H_uH_d)]. \tag{4.6}$$

In the E6SSM, these exotics are light, and so instead of getting a large VEV, this singlet now must acquire a TeV scale VEV. It needs to be checked that the F and D flatness constraints are satisfied, and that rapid proton decay is forbidden for the realisation of the spectrum.

4.2.1 $U(1)_N$ Charges

The correctly normalised charge generators for $U(1)_\psi$ and $U(1)_\chi$ are

[1] Appendix C shows that there is an anomaly involving two perpendicular $U(1)s$ which is not cancelled by this formalism. However, [95] points out that anomaly cancellation constraints can be relaxed in the case of geometrically massive U(1)s in F-theory, due to GreenSchwarz type cancellation mechanisms.

E_6	$SO(10)$	$SU(5)$	Weight vector	Q_N	N_Y	$M_{U(1)}$	SM particle content	Low energy spectrum
$27_{t'_1}$	16	$\overline{5}_3$	t_1+t_5	$\frac{1}{\sqrt{10}}$	1	4	$4d^c+5L$	$3d^c+3L$
$27_{t'_1}$	16	10_M	t_1	$\frac{1}{2\sqrt{10}}$	-1	4	$4Q+5u^c+3e^c$	$3Q+3u^c+3e^c$
$27_{t'_1}$	16	θ_{15}	t_1-t_5	0	0	n_{15}	$3\nu^c$	-
$27_{t'_1}$	10	5_1	$-t_1-t_3$	$-\frac{1}{\sqrt{10}}$	-1	3	$3D+2H_u$	$3D+2H_u$
$27_{t'_1}$	10	$\overline{5}_2$	t_1+t_4	$-\frac{3}{2\sqrt{10}}$	1	3	$3\overline{D}+4H_d$	$3\overline{D}+3H_d$
$27_{t'_1}$	1	θ_{14}	t_1-t_4	$\frac{5}{2\sqrt{10}}$	0	n_{14}	θ_{14}	θ_{14}
$27_{t'_3}$	16	$\overline{5}_5$	t_3+t_5	$\frac{1}{\sqrt{10}}$	-1	-1	$\overline{d^c}+2\overline{L}$	-
$27_{t'_3}$	16	10_2	t_3	$\frac{1}{2\sqrt{10}}$	1	-1	$\overline{Q}+2\bar{u}^c$	-
$27_{t'_3}$	16	θ_{35}	t_3-t_5	0	0	n_{35}	$-$	-
$27_{t'_3}$	10	5_{H_u}	$-2t_1$	$-\frac{1}{2\sqrt{10}}$	1	0	H_u	H_u
$27_{t'_3}$	10	$\overline{5}_4$	t_3+t_4	$-\frac{3}{2\sqrt{10}}$	-1	0	$\overline{H_d}$	-
$27_{t'_3}$	1	θ_{34}	t_3-t_4	$\frac{5}{2\sqrt{10}}$	0	n_{34}	θ_{34}	θ_{34}
-	1	θ_{31}	t_3-t_1	0	0	n_{31}	θ_{31}	-
-	1	θ_{53}	t_5-t_3	0	0	n_{53}	θ_{53}	-
-	1	θ_{54}	t_5-t_4	$\frac{5}{2\sqrt{10}}$	0	n_{54}	θ_{54}	-
-	1	θ_{45}	t_4-t_5	$-\frac{5}{2\sqrt{10}}$	0	n_{45}	θ_{45}	-

Table 4.1: Complete 27s of E_6 and their $SO(10)$ and $SU(5)$ decompositions. The $SU(5)$ matter states decompose into SM states as $\overline{5} \to d^c, L$ and $10 \to Q, u^c, e^c$ with right-handed neutrinos $1 \to \nu^c$, while $SU(5)$ Higgs states decompose as $5 \to D, H_u$ and $\overline{5} \to \overline{D}, H_d$, where $D, \overline{D}$ are exotic colour triplets and antitriplets. We identify RH neutrinos as $\nu^c = \theta_{15}$. Arbitrary singlets are included for giving mass to neutrinos and exotics and to ensure F and D flatness.

$$Q_\chi = \frac{1}{2\sqrt{10}}\text{diag}[-1,-1,-1,-1,4] \tag{4.7}$$

$$Q_\psi = \frac{1}{2\sqrt{6}}\text{diag}[1,1,1,-3,0] \tag{4.8}$$

As such, from Eq. (5.4), the generator for $U(1)_N$ is given by

$$Q_N = \frac{1}{2\sqrt{10}}\text{diag}[1,1,1,-4,1] \tag{4.9}$$

From this, the $U(1)_N$ charges of all the particles in the spectrum can be computed, and the results are shown in Table 4.1. As required (and described in the introduction), the right handed neutrinos have zero charge under this $U(1)$.

4.2.2 Singlet VEVs and Bad Operators

As in the previous model [1], θ_{31} should get a string scale VEV, and as mentioned earlier θ_{34} now should get a TeV scale VEV to give mass to the exotics. θ_{53} should get a VEV in order to generate neutrino masses (as discussed later), and in order to generate the effective μ term, θ_{14} gets a TeV scale VEV, also discussed later.

The R-parity violating superpotential couplings $u^c d^c d^c$, $Qd^c L$, $Le^c L$, κLH_u as well as the dimension 5 terms in the Lagrangian corresponding to the superpotential terms $QQQL$ and $u^c u^c d^c e^c$, are forbidden by the $U(1)$ symmetries that originate in the underlying E_6. In order to check that spontaneous symmetry breaking terms coming from SM singlet field VEVs do not allow these dangerous operators to appear, we can identify the following terms which could potentially give rise to bad operators if certain singlets acquired VEVs: $\theta_{15}LH_u$, $(\theta_{31}\theta_{45}+\theta_{41}\theta_{35})10_M\overline{5}_3^2$ and $\theta_{31}\theta_{41}10_M^3\overline{5}_3$. As such, taking into account the singlet VEVs that are required, we can see that the dangerous operators do not arise provided θ_{15}, θ_{41} and θ_{45} do *not* acquire VEVs.

However this is not sufficient to ensure the absence of baryon and lepton number violating terms because, even in the absence of these VEVs, tree level graphs can generate the dangerous operators at higher order in the singlet fields. These issues relating to proton decay will be discussed later. Proton decay in the context of F-theory has been previously studied, for example in [43, 85].

4.2.3 The effective μ term

In the E6SSM, the μ term is effectively generated when a singlet which is charged under $U(1)_N$, is coupled to H_uH_d and given a TeV scale VEV. In terms of F-theory model building, the charge of H_uH_d under the perpendicular $U(1)$ symmetries can be seen from Table 4.1 to be $-2t_1+t_1+t_4=-t_1+t_4$. As such, the appropriate singlet which could generate the μ term is θ_{14}. Alternately, we could try and generate the μ term non perturbatively, as in [1], where non perturbative effects which break the perpendicular $U(1)$ symmetries generate an explicit μ term which can naturally be at the electroweak scale. However, as H_uH_d is charged under $U(1)_N$, this method cannot be utilised in the E6SSM, and so we must have a θ_{14} singlet in the spectrum which will acquire an electroweak scale VEV.

4.2.4 D-flatness

In the model under consideration we assume the SUSY breaking soft masses are such that only the SM singlet fields acquire very large VEVs. To determine them we consider the F and D flatness conditions. Taking account of the Z_2 monodromy, $t_1 \leftrightarrow t_2$ the D-flatness conditions are of the form given in Eq. (2.105) where there are three $U_A(1)$s with charges given in Eq. (4.13). We wish to show that the D-flatness conditions are satisfied by the massless fields θ_{31}, θ_{53} needed to give mass to exotics and, as to generate viable neutrino masses. Even though θ_{34} and θ_{14} get VEVs, these VEVs will be at the TeV scale whereas all the other VEVs are at the string scale. As such, the VEV for θ_{34} and θ_{14} will be ignored in the following calculations.

The D-flatness condition for $U_A(1)$ is

$$\sum_j Q^A_{ij}(|\langle\theta_{ij}\rangle|^2 - |\langle\theta_{ji}\rangle|^2) = -\frac{TrQ^A}{192\pi^2} g_s^2 M_S^2$$
$$= -XTrQ^A \quad (4.10)$$

This condition must be checked for all the $U(1)$s, the charge generators of which are given by

$$Q_\chi \propto \text{diag}[-1,-1,-1,-1,4] \quad (4.11)$$
$$Q_\psi \propto \text{diag}[1,1,1,-3,0] \quad (4.12)$$
$$Q_\perp \propto \text{diag}[1,1,-2,0,0] \quad (4.13)$$

In a general basis, $Q = \text{diag}[t_1,t_2,t_3,t_4,t_5]$, and with just θ_{31} and θ_{53} acquiring VEVs, Eq. (4.10) can be written

$$(t_5-t_3)|\theta_{53}|^2 + (t_3-t_1)|\theta_{31}|^2 = -XTrQ^A \quad (4.14)$$

The trace on the right hand side of Eq. 4.10 is taken over all states, and is given by

$$TrQ^A = 5\sum n_{ij}(t_i+t_j) + 10\sum n_k t_k + \sum m_{ij}(t_i-t_j) \quad (4.15)$$

For our model, this trace is computed to be

$$TrQ^A = (60-n_{31}+n_{14}+n_{15})t_1 + (n_{31}+n_{34}-n_{53}-30)t_3 + (15-n_{54}-n_{14}-n_{34})t_4$$
$$+ (15+n_{53}+n_{54}-n_{15})t_5 \quad (4.16)$$

where $n_{ij} \equiv \tilde{n}_{ij} - \tilde{n}_{ji}$ to simplify the notation, with $\tilde{n}_{ij}$ being the absolute number of θ_{ij} singlets. Evaluating the trace for each of the $U(1)$s gives

$$TrQ_\chi = 5(3-n_{15}+n_{53}+n_{54}) \quad (4.17)$$
$$TrQ_\psi = -15+4(n_{14}+n_{34})+n_{15}-n_{53}+3n_{54} \quad (4.18)$$
$$TrQ_\perp = 120+n_{14}+n_{15}-3n_{31}-2n_{34}+2n_{53} \quad (4.19)$$

The flatness conditions with just θ_{31} and θ_{53} getting VEVs then become the three simultaneous equations

$$5|\theta_{53}|^2 = 5(-3+n_{15}-n_{54}-n_{53})X \tag{4.20}$$

$$-|\theta_{53}|^2 = (15-n_{15}-4(n_{14}+n_{34})+n_{53}-3n_{54})X \tag{4.21}$$

$$2|\theta_{53}|^2-3|\theta_{31}|^2 = (-120+3n_{31}-n_{14}-n_{15}+2n_{34}-2n_{53})X \tag{4.22}$$

Putting Eqs. (4.20) and (4.21) together gives the relation

$$n_{14}+n_{34}+n_{54} = 3 \tag{4.23}$$

In order to cancel anomalies, we must have three complete 27s of E_6 and so we must have the following constraint on the absolute number of singlets

$$\tilde{n}_{14}+\tilde{n}_{34} = 3 \tag{4.24}$$

If we have $\tilde{n}_{ij} \neq 0$, in general we will require that $\tilde{n}_{ji} = 0$, as otherwise we would be able to write a mass term $M\theta_{ij}\theta_{ji}$. This is acceptable provided relations, which will be discussed in section 4.2.5, are satisfied. In order to simplify the model, however, we will take the case $\tilde{n}_{ij} \neq 0 \Rightarrow \tilde{n}_{ji} = 0$, and we will take this fact to be implicit from this point onwards. As such, Eqs. (4.23) and (4.24) mean that $n_{54} = 0$. The equation for the θ_{53} VEV then becomes

$$|\theta_{53}|^2 = (n_{15}-n_{53}-3)X \tag{4.25}$$

As θ_{15} corresponds to the right handed neutrino and θ_{53} is required to give neutrino masses, both n_{15} and n_{53} must be positive. Eq. (4.25) then gives a lower limit on the number of right handed neutrinos in the model

$$\tilde{n}_{15} > 3+\tilde{n}_{53} \tag{4.26}$$

Due to the fact that in this model θ_{31} and θ_{53} acquire large VEVs, we require that $\tilde{n}_{31}, \tilde{n}_{53} \geq 1$. Also, we must require $\tilde{n}_{34} > 0$ in order to allow the exotics to get a mass via the term $\theta_{34}D\overline{D}$, and $\tilde{n}_{14} > 0$ in order to generate the μ term. We will take $\tilde{n}_{53} = 1$, meaning that from Eq. (4.26), we must have $\tilde{n}_{15} > 4$. This model will take the minimal case of 5 right handed neutrinos. In order to satisfy Eq. (4.24) we choose $\tilde{n}_{14} = 1$ and $\tilde{n}_{34} = 2$, and we leave $\tilde{n}_{31} > 0$ unspecified for now.

4.2.5 F-flatness

In this model, we have taken the case where no $\theta_{ij}\theta_{ji}$ terms can be written down, so the only terms in the singlet superpotential which could generate a non zero F-term are

$$W_\theta = \lambda_{ij}\theta_{53}\theta_{31}^i\theta_{15}^j \tag{4.27}$$

where j corresponds to the number of right handed neutrinos and runs from 1 to 5, and the range of i represents the number of θ_{31} fields, and is yet unspecified. Minimising the singlet potential leads to

$$\frac{\partial W_\theta}{\partial \theta_{15}^j} = \lambda_{ij}\theta_{53}\theta_{31}^i \Rightarrow \lambda_{ij}\theta_{53}\left\langle \theta_{31}^i \right\rangle = 0 \tag{4.28}$$

As such, seven independent θ_{31} singlets must have zero VEVs. We must have at least one θ_{31} which aquires a non zero VEV in order to satisfy Eq. (4.22), and so we choose $i = \tilde{n}_{31} = 6$. Now we have a full singlet spectrum, consistent with F and D-flatness, where the choices we have made are given by

$$\tilde{n}_{31} = 6, \tilde{n}_{53} = 1, \tilde{n}_{54} = 0, \tilde{n}_{14} = 1, \tilde{n}_{34} = 2, \tilde{n}_{15} = 5$$

4.2.5.1 Singlet mass terms

If we were to drop the requirement that a non zero $\tilde{n}_{ij}$ means having $\tilde{n}_{ji} = 0$, we could have $\theta_{ij}\theta_{ji}$ terms in the superpotential. If, for example, neutrino masses were generated by giving a θ_{51} field a VEV the singlet superpotential would be of the form

$$W_\theta = \lambda_{ijk}\theta_{ij}\theta_{jk}\theta_{ki} + M^{ij}\theta_{15}^i\theta_{51}^j \tag{4.29}$$

Considering the F-term for θ_{15}, the relevant terms in the superpotential are

$$W_\theta = \gamma_{ij}\theta_{15}^i\theta_{53}\theta_{31}^j + M_{ik}\theta_{15}^i\theta_{51}^k \tag{4.30}$$

As such, if a θ_{51} field was to exist in the spectrum and acquire a VEV, the following relation would have to be satisfied

$$\frac{\partial W_\theta}{\partial \theta_{15}^i} = \gamma_{ij}\left\langle \theta_{31}^j \right\rangle \langle \theta_{53} \rangle + M_{ij}\left\langle \theta_{51}^j \right\rangle = 0 \tag{4.31}$$

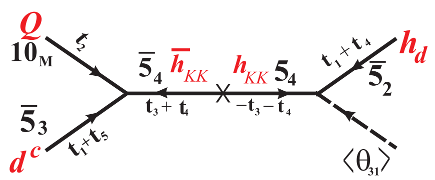

Figure 4.1: Tree-level diagram contributing to the bottom mass.

Similarly, due to the fact that θ_{14} gets a TeV scale VEV to generate the μ term, and θ_{34} acquires a TeV VEV to give masses to the low scale exotics of the E6SSM, the presence of any θ_{43} fields in the spectrum would mean that we would have the analogous relation

$$\frac{\partial W_\theta}{\partial \theta_{43}^i} = \gamma_{ij} \left\langle \theta_{31}^j \right\rangle \langle \theta_{14} \rangle + M_{ij} \left\langle \theta_{34}^j \right\rangle = 0 \tag{4.32}$$

As such, if we were to not impose that $\theta_{ij} \neq 0 \Rightarrow \theta_{ji} = 0$, the model would be consistent with F-flatness provided relations of the type in Eqs. (4.31, 4.32) were satisfied. In our model, we take the simplest case where we do not have equations of this type.

4.2.6 Calculating the singlet VEVs

Now we have a full spectrum for the model, we can calculate the singlet VEVs, giving us information about the scale at which the exotics decouple, neutrino masses etc. From the D-flatness relations, we have

$$|\theta_{53}|^2 = (\tilde{n}_{15} - \tilde{n}_{53} - 3)X \tag{4.33}$$

$$3|\theta_{31}|^2 = 114 + 3(\tilde{n}_{15} - \tilde{n}_{31}) - 2\tilde{n}_{34} + \tilde{n}_{14} \tag{4.34}$$

Putting the number for the singlet spectrum into these equations gives

$$|\theta_{53}|^2 = X \tag{4.35}$$

$$|\theta_{31}|^2 = \frac{118}{3} X \tag{4.36}$$

where $X = \frac{g_s^2 M_S^2}{192\pi^2}$

4.2.7 Quark, charged lepton and exotic masses

From Table 4.1, we can see that the up quark mass matrix (and the Dirac neutrino mass matrix) will originate from the $27_{t_1}27_{t_1}27_{t_3}$ E_6 coupling. These matrices are rank one in the absence of flux, but as demonstrated in [29], the rank can be increased by including non perturbative effects [26]. The down quark and charged lepton mass matrices arise from the non-renormalisable couplings originating at the E_6 level from $\theta_{31}27_{t_1}27_{t_1}27_{t_1}/M$. Figure 4.1 shows the tree-level diagram for the bottom mass, involving the exchange of a massive vectorlike pair. The origin of the difference in magnitude of the top and bottom quark masses can be explained by the fact that the θ_{31} VEV is of the same order as the messenger mass, M, leading to a mild suppression of the down quark Yukawas relative to the up quark couplings.

The terms in the superpotential which are responsible for generating the μ term and the exotic masses are

$$W \sim \lambda_{ij}\theta_{14}H_{di}H_{uj} + \kappa_{ijk}\theta^{i}_{34}\overline{D}_j D_k \tag{4.37}$$

From Table 4.1, it can be seen that both of these couplings originate from the $27_{t_1}27_{t_1}27_{t_3}$ E_6 coupling.

In the standard E6SSM, an approximate Z_2 flavour symmetry is assumed, in order to distinguish the active (third) generations of Higgs doublets from the inert (first and second) generations. However, in this study we do not try and solve problems with flavour, as we can always note that in the absense of flux, matrices are always rank one. As such, we can always pick a basis where the matrix has a one in the position corresponding to the active generation and zeros elsewhere. Also, it should be noted that from Table 4.1, we can see that all three generations of H_d come from the 27_{t_1} curve, whereas the active H_u comes from a different curve (27_{t_3}) than the inert H_us (27_{t_1}). As such, we could generate the up quark masses via the non-renormalisable coupling $\theta_{31}27_{t_1}27_{t_1}27_{t_1}/M$, with H_u coming from the 27_{t_1} matter curve. In this case, the quark masses would arise from diagrams similar to Figure 4.1. H_u will now come from the from the 5_1 curve, and the diagram will involve the coupling $\theta_{31}\overline{5}_{H_u}5_1$. However, this coupling will turn out to be forbidden under a discrete Z_2 symmetry which will be introduced later in order to stabilise the proton, and so quark masses will not be generated in this manner. In any case, it would not pose a problem, due to the fact that the θ_{31} VEV is of the same order as the messenger mass, M.

4.2.8 Neutrino Masses

The discussion of neutrino masses is essentially the same as that of the model of Chapter 3, in Section 3.2.6. The key point is that due to the $t_1 \leftrightarrow t_2$ monodromy, the conjugate states θ_{12} and θ_{21} are identified, and so we can write down a term $M_M\theta_{12}\theta_{21}$ in the superpotential which corresponds to a Majorana mass for the θ_{12} states. As in Chapter 3, we allow for an arbitrary number of θ_{12} fields (the fact that these fields carry no charge under the perpendicular $U(1)$s means that we can have any number of them in the spectrum without affecting flatness conditions etc.), but the difference in this model is that now the number of θ_{15} fields is 5, not 3. (For a reference on models with Z right handed neutrinos, see [96]).

In the same notation as Chapter 3, the relevant terms for lepton mass generation are (after the two Higgs doublets have got their VEVs):

$$W_{mass} = \langle H_d \rangle Y_e^{ij} \bar{e}_L^i e_R^j + \langle H_u \rangle \lambda_{LR}^{ia} \bar{\nu}_L^i \theta_{15}^a + \langle \Theta_{51} \rangle \lambda_{RM}^{a\alpha} \theta_{15}^a \theta_{12}^\alpha + M_M^{\alpha\beta} \theta_{12}^\alpha \theta_{21}^\beta \tag{4.38}$$

where λ_{LR} is a (3×5) matrix of couplings, λ_{RM} is $(5 \times n)$ (where n is the number of θ_{12} states) and M_M is an $(n \times n)$ matrix. The discussion of neutrino masses from a double seesaw mechanism is now entirely analogous to that of Section 3.2.6, with the only difference being the size of the matrices, and the fact that the singlet VEVs are now slightly different, and given by Eqs. (4.35) and (4.36).

4.3 Unification and proton decay

4.3.1 Review of F-theory unification in $SU(5)$

In the case where a $U(1)_Y$ flux mechanism is used to break an $SU(5)$ gauge symmetry down to the Standard Model, there is a splitting of the gauge couplings at the unification scale [40, 97, 98, 99, 41]. The splitting at M_{GUT} is

$$\begin{aligned} \frac{1}{\alpha_3(M_G)} &= \frac{1}{\alpha_G} - y \\ \frac{1}{\alpha_2(M_G)} &= \frac{1}{\alpha_G} - y + x \\ \frac{1}{\alpha_1(M_G)} &= \frac{1}{\alpha_G} - y + \frac{3}{5}x \end{aligned} \tag{4.39}$$

where $x = -\frac{1}{2}\mathrm{Re}S \int c_1^2(\mathcal{L}_Y)$, $y = \frac{1}{2}\mathrm{Re}S \int c_1^2(\mathcal{L}_a)$ $\mathcal{L}_a$ is a non-trivial line bundle and $S = e^{-\phi} + iC_0$ is the axion-dilaton field as discussed in [40]. Combining the above, the gauge couplings at M_{GUT} are found to satisfy the relation

$$\frac{1}{\alpha_Y(M_{GUT})} = \frac{5}{3}\frac{1}{\alpha_1(M_{GUT})} = \frac{1}{\alpha_2(M_{GUT})} + \frac{2}{3}\frac{1}{\alpha_3(M_{GUT})} \tag{4.40}$$

In the E6SSM, however, we have an extra $U(1)_N$ symmetry which survives down to low energies. Accordingly, we must incorporate the $U(1)_N$ gauge coupling into the unification analysis. In order to accomplish this, we can consider how Eq. (4.40) is derived in [100] in terms of

$SU(5)$ group theory, and then generalise the results to E_6 and $SO(10)$, giving us information about $U(1)_\psi$ and $U(1)_\chi$ respectively.

Following [100], we can write the gauge kinetic functions for SU(3), SU(2) and $U(1)_Y$ embedded inside $SU(5)$ in the form

$$f_3 = A + Bc_\alpha,\ \alpha = (1,...,8) \tag{4.41}$$

$$f_2 = A + Bc_\alpha,\ \alpha = (21,22,23) \tag{4.42}$$

$$f_1 = A + Bc_\alpha,\ \alpha = 24 \tag{4.43}$$

where α is an index running from 1 to 24, over all the generators of $SU(5)$, and the missing αs are the generators belonging outside the $SU(3)\times SU(2)\times U(1)$ subgroup of $SU(5)$. A and B are arbitrary gauge invariant functions and the c_α coefficients are given by

$$d_{\alpha\beta 24} = c_\alpha \delta_{\alpha\beta} \tag{4.44}$$

with the index 24 corresponding to the hypercharge generator and the group theory coefficients $d_{\alpha\beta\gamma}$ defined as

$$d_{\alpha\beta\gamma} = 2Tr[\{T_\alpha, T_\beta\}\, T_\gamma] \tag{4.45}$$

As such, in order to calculate the three gauge kinetic functions, we just need $d_{1,1,24}$, $d_{21,21,24}$ and $d_{24,24,24}$, where the generators T_1, T_{21} and T_{24} are given in block matrix notation by

$$T_1 = \begin{pmatrix} \lambda_1/2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1/2 \end{pmatrix}$$

$$T_{24} = \frac{1}{\sqrt{15}} diag(1,1,1,-\frac{3}{2},-\frac{3}{2})$$

where λ_1 refers to the first Gell-Mann matrix, and σ_1 to the first Pauli matrix. These definitions can be used trivially to calculate $c_1 = d_{1,1,24} = \frac{2}{\sqrt{15}}$, $c_{21} = d_{21,21,24} = -\frac{3}{\sqrt{15}}$ and $c_{24} = d_{24,24,24} = -\frac{1}{\sqrt{15}}$, which can be put together with Eqs. (4.41, 4.42, 4.43) giving (after a redefinition of the arbitrary function B)

$$f_3 = A + 2B \tag{4.46}$$

$$f_2 = A - 3B \tag{4.47}$$

$$f_1 = A - B \tag{4.48}$$

The gauge couplings at the unification scale are then related by [100]

$$\alpha_G = \alpha_3(M_G) f_3 = \alpha_2(M_G) f_2 = \alpha_1(M_G) f_1 = \frac{5}{3}\alpha_Y(M_G) f_1 \tag{4.49}$$

Combining this equation with Eqs. (4.46, 4.47, 4.48) gives the following constraint on the gauge kinetic functions

$$f_3 + \frac{3}{2} f_2 = \frac{5}{2} f_1 \tag{4.50}$$

which, when combined with the relations $f_i = \frac{\alpha_G}{\alpha_i(M_G)}$, leads to Eq. (4.40). Comparing this picture with Eq. (4.39), we have the following equations relating x and y to A and B

$$x = -\frac{5B}{\alpha_G},\ y = \frac{1 - A - 2B}{\alpha_G} \tag{4.51}$$

4.3.2 The E_6 and $SO(10)$ cases

We can generalise the $SU(5)$ argument to the breaking patterns

$$\begin{aligned} E_6 &\rightarrow SO(10) \times U(1)_\psi \\ SO(10) &\rightarrow SU(5) \times U(1)_\chi \end{aligned}$$

in order to learn about the $U(1)_N$ gauge coupling $U(1)_N = \frac{1}{4}U(1)_\chi + \frac{\sqrt{15}}{4}U(1)_\psi$. For the E_6 case, the generalisation is the set of equations

$$\alpha_6 = \alpha_{10} f_{10} = \alpha_\psi f_\psi \tag{4.52}$$

$$f_{10} = A' + B' c_\alpha,\ \alpha = (1, ..., 45) \tag{4.53}$$

$$f_\psi = A' + B' c_\alpha,\ \alpha = 78 \tag{4.54}$$

and for the $SO(10)$ breaking, we have

$$\alpha_{10} = \alpha_5 f_5 = \alpha_\chi f_\chi \tag{4.55}$$

$$f_5 = A'' + B'' c_\alpha, \ \alpha = (1, ..., 24) \tag{4.56}$$

$$f_\psi = A'' + B'' c_\alpha, \ \alpha = 45 \tag{4.57}$$

For both E_6 and $SO(10)$ (and indeed for any simple Lie algebra with the exception of SU(N), $N \geq 3$) the $d_{\alpha\beta\gamma}$ and hence the c_α are zero [101] (this can be checked for the E_6 case from Appendix A). Accordingly, we can take the B', B'' in Eqs. (4.53, 4.54, 4.56, 4.57) to be zero. Matching with Eq. (4.51) of the $SU(5)$ case, this clearly leads to $x = 0$, and Eq. (4.39) shows that this corresponds to no relative splitting of the gauge couplings at unification. We can, however, have a shift by the parameter y in all the couplings after each breaking. These parameters will depend on the flux breaking mechanism, and we will leave them as free parameters of the model:

$$\begin{aligned} \frac{1}{\alpha_{10}} &= \frac{1}{\alpha_6} - y' \\ \frac{1}{\alpha_\psi} &= \frac{1}{\alpha_6} - y' \\ \frac{1}{\alpha_5} &= \frac{1}{\alpha_6} - y'' \\ \frac{1}{\alpha_\chi} &= \frac{1}{\alpha_6} - y'' \end{aligned} \tag{4.58}$$

With $\alpha_G = \alpha_5$ in Eq. (4.39), we can then proceed with the analysis as for the $SU(5)$ case. It should be noted that in Eq. (4.58), the signs of y and y' are not known, and so the $U(1)_N$ gauge coupling could be either bigger or smaller than α_5 at unification. This splitting is a free parameter of the model.

4.3.3 The Spectrum, and One Loop Renormalisation Group Analysis

In the considered model we have the following vector pairs of exotics, which get large masses when θ_{31} gets a VEV: $(d + \overline{d}^c)$, $(Q + \overline{Q})$, $(H_d + \overline{H}_d)$, $2(L + \overline{L})$, $2(u^c + \overline{u}^c)$. Below some scale $M_X < M_{GUT}$ these exotics decouple. We then have the extra exotics $3(D + \overline{D})$, $2(H_u, H_d)$ which survive to low energy and decouple at a scale $M_{X'} = 1 TeV$. Below the scale $M_{X'}$, we have the low energy matter content of the MSSM. The low energy values of the gauge couplings are given by the evolution equations

$$\frac{1}{\alpha_a(M_Z)} = \frac{1}{\alpha_a(M_{GUT})} + \frac{b_a^x}{2\pi} \ln \frac{M_{GUT}}{M_X} + \frac{b_a^{x'}}{2\pi} \ln \frac{M_X}{M_{X'}} + \frac{b_a}{2\pi} \ln \frac{M_{X'}}{M_Z} \tag{4.59}$$

where b_a^x is the beta-function above the scale M_X, $b_a^{x'}$ is the beta-function below M_X and b_a is the beta-function below $M_{X'}$. Combining the above equations with Eq. 4.40, we find that the GUT scale is given by

$$M_{GUT} = e^{\frac{2\pi}{\beta \mathscr{A}}\rho} M_Z^{\rho} M_{X'}^{\gamma-\rho} M_X^{1-\gamma} \tag{4.60}$$

where $\mathscr{A}$ is a function of the experimentally known low energy values of the SM gauge coupling constants

$$\begin{aligned} \frac{1}{\mathscr{A}} &= \frac{5}{3}\frac{1}{\alpha_1(M_Z)} - \frac{1}{\alpha_2(M_Z)} - \frac{2}{3}\frac{1}{\alpha_3(M_Z)} \\ &= \frac{\cos(2\theta_W)}{\alpha_{em}} - \frac{2}{3}\frac{1}{\alpha_3(M_Z)} \end{aligned} \tag{4.61}$$

We have also introduced the ratios ρ and γ

$$\rho = \frac{\beta}{\beta_x}\ \gamma = \frac{\beta_{x'}}{\beta_x} \tag{4.62}$$

where $\beta, \beta_{x'}, \beta_x$ are the beta-function combinations in the regions $M_Z < \mu < M_{X'}$, $M_{X'} < \mu < M_X$ and $M_X < \mu < M_{GUT}$ respectively

$$\beta_x = b_Y^x - b_2^x - \frac{2}{3}b_3^x \tag{4.63}$$

$$\beta_{x'} = b_Y^{x'} - b_2^{x'} - \frac{2}{3}b_3^{x'} \tag{4.64}$$

$$\beta = b_Y - b_2 - \frac{2}{3}b_3 \tag{4.65}$$

The beta function coefficients are given by ($b_1 = \frac{3}{5}b_Y$)

$$b_1 = -0 + 2n_f + \frac{3}{10}(n_h + n_L) + \frac{1}{5}n_{d^c} + \frac{1}{10}n_Q + \frac{4}{5}n_{u^c} + \frac{3}{5}n_{e^c} \tag{4.66}$$

$$b_2 = -6 + 2n_f + \frac{1}{2}(n_h + n_L) + 0\,n_{d^c} + \frac{3}{2}n_Q + 0\,n_{u^c} \tag{4.67}$$

$$b_3 = -9 + 2n_f + 0\,(n_h + n_L) + \frac{1}{2}n_{d^c} + n_Q + \frac{1}{2}n_{u^c} \tag{4.68}$$

with $n_f = 3$ the number of families and $n_{h,L,...}$ counting Higgses and exotic matter. For our spectrum, the coefficients are given by

$$b_1 = 6.6,\ b_2 = 1,\ b_3 = -3 \tag{4.69}$$

$$b_1^{x'} = 9,\ b_2^{x'} = 3,\ b_3^{x'} = 0 \tag{4.70}$$

$$b_1^x = 14.6,\ b_2^x = 9,\ b_3^x = 5 \tag{4.71}$$

Plugging these numbers into Eq. (4.60), we see that M_{GUT} becomes independent of the M_X and $M_{X'}$ scales and in fact it is identified with the MSSM unification scale

$$M_U = M_{GUT} \equiv e^{\frac{2\pi}{\beta_{\mathscr{A}}}} M_Z \approx 2\times 10^{16}\text{GeV} \tag{4.72}$$

It should be noted that because of the independence of this relation on the two mass scales, it applies for both the model of this chapter, and that of Chapter 3.

4.3.4 Model Dependence of the Splitting Parameter, x

From Eq. (4.39), the splitting of the standard model gauge couplings is given by

$$x = \frac{1}{\alpha_2(M_G)} - \frac{1}{\alpha_3(M_G)} \tag{4.73}$$

We can now use the evolution equation (4.59) to relate x to the low energy coupling constants α_2 and α_3, giving

$$\left(\frac{1}{\alpha_2} - \frac{1}{\alpha_3}\right)_{M_Z} = x + \frac{b_2^x - b_3^x}{2\pi}\log\left(\frac{M_G}{M_X}\right) + \frac{b_2^{x'} - b_3^{x'}}{2\pi}\log\left(\frac{M_X}{M_{X'}}\right)\frac{b_2 - b_3}{2\pi}\log\left(\frac{M_{X'}}{M_Z}\right) \tag{4.74}$$

Using Eqs. (4.69, 4.70, 4.71, 4.72) and the relations $\alpha_{em} = \alpha_2 sin^2\theta_w$, $\frac{1}{\alpha_Y} = \frac{(1-sin^2\theta_w)}{\alpha_{em}}$ and $\alpha_1 = \frac{5}{3}\alpha_Y$, we arrive at the following expression for x

$$\begin{aligned} x &= \frac{4}{3}\frac{1}{\alpha_2} - \frac{1}{3}\frac{1}{\alpha_Y} - \frac{7}{9}\frac{1}{\alpha_3} - \frac{1}{2\pi}\ln\left(\frac{M_{x'}}{M_x}\right) \\ &= \frac{(5sin^2\theta_w - 1)}{3\alpha_{em}} - \frac{7}{9}\frac{1}{\alpha_3} - \frac{1}{2\pi}\ln\left(\frac{M_{x'}}{M_x}\right) \end{aligned} \tag{4.75}$$

It can be seen that the factors which affect the splitting are the matter content of the spectrum (which manifests itself in the numbers multiplying the Standard Model parameters), and the ratio of the two exotic mass scales. At this point, we can compare the E6SSM model with the E_6 based model of Chapter 3 [1] (model 1), where the E6SSM light exotics are heavy. We can use the above equation for both models as they have the same spectrum, the difference being in the scales at which the exotics decouple. In the E6SSM case we have $M_{X'} = 1TeV$ and from the calculated singlet VEVs, $M_X = 1.44\times 10^{16}GeV$, whilst in model 1, we have $M_{X'} = 0.306\times 10^{16}GeV$ and $M_X = 1.31\times 10^{16}GeV$. Taking the values $\alpha_{em}^{-1}(M_Z) = 127.916$, $sin^2\theta_w(M_Z) = 0.23116$ and $\alpha_3 = 0.1184$, the part of the right hand side of Eq. (4.75) involving these parameters is evaluated as 0.07. Due to the fact that this number is small, in order for x to be close to zero

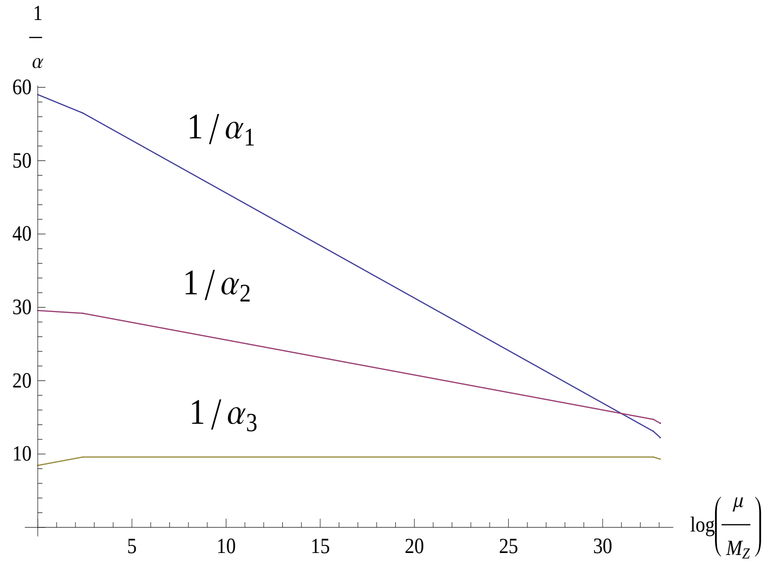

Figure 4.2: The running of α_1, α_2 and α_3 from their SM value at M_Z up to M_{GUT} for the case of the F-theory E6SSM. The large splitting rules this model out in its minimal form, as described in the text.

(corresponding to the usual case of gauge coupling unification) the masses of both sets of exotics need to be close together. This is the case in model 1 where we have x=0.3, but not in the case of the E6SSM model where x=4.9.

The large splitting in this model is actually inconsistent with the formalism, meaning that the model in its present form is not viable. Considering the relations $x = -\frac{1}{2}\text{Re}S \int c_1^2(\mathscr{L}_Y)$, $\mathscr{L}_Y^2 = -2$, and $S = e^{-\phi} + iC_0$, it can be seen that we must have $x < 1$, and so the minimal version of the E6SSM model is ruled out. If we want the couplings to unify such that $x < 1$ in the F-theory E6SSM, we would have to add in extra exotics in such a way to modify the renormalisation group running. However, instead of being arbitrary, there are a particular set of bulk exotics coming from the adjoint of E_6 which are inevitable in models with flux breaking. It turns out that due to topological restrictions on the internal manifold, there is the natural emergence of a low energy set of exotics which when added to the spectrum of this model, forces the gauge couplings to come very close to perfect unification at the one loop level. The issue of these bulk exotics will be discussed in detail in the next chapter.

Taking the low energy values of α_1, α_2 and α_3 and using the one loop remormalisation group equations (RGEs) to run the couplings up to the unification scale (taking into account the presence of the exotic matter) results in Figure 4.2 for the minimal F-theory E6SSM, and Figure 4.3 for model 1. In Figure 4.2, the reciprocals of the gauge couplings are split by approximately 35% (relative to the largest value) at unification, whereas in Figure 4.3 they meet to 1.3% accuracy. The 35% splitting is unacceptably large as discussed above, and the fact that the gauge couplings meet to 1.3% in model 1 means that our spectrum is special for the case of heavy exotics.

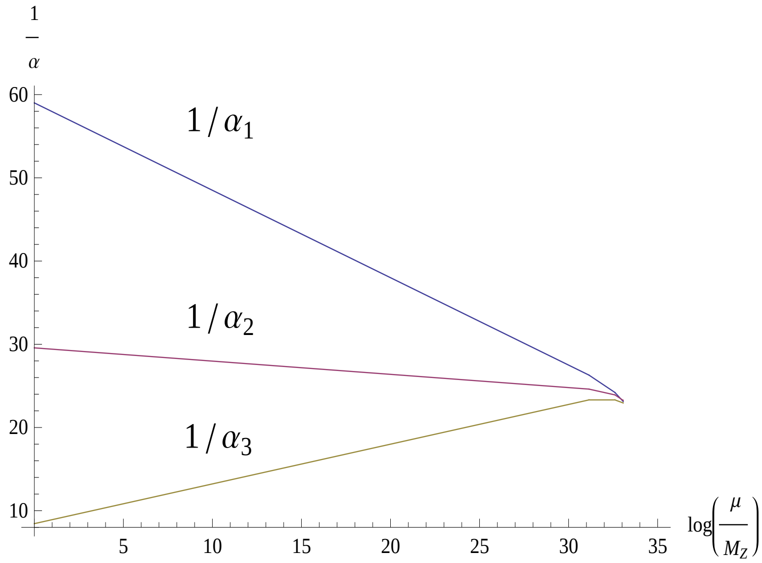

Figure 4.3: The running of α_1, α_2 and α_3 from their SM value at M_Z up to M_{GUT} for the case of model 1, presented in [1].

Another way of looking at Eq. (4.75) is that since x is the dilaton field, $e^{-\phi}$, clearly we must have $x > 0$ which can give a lower bound in α_3 by using α_1 and α_2 as input parameters. In model 1, we have $\langle\theta_{31}\rangle = \sqrt{37X}$ and $\langle\theta_{34}\rangle = \sqrt{2X}$ from Eq. (3.13), meaning that

$$\frac{M_{X'}}{M_X} = \sqrt{\frac{2}{37}} \tag{4.76}$$

As such, we have

$$x = \frac{4}{3}\frac{1}{\alpha_2} - \frac{1}{3}\frac{1}{\alpha_Y} - \frac{7}{9}\frac{1}{\alpha_3} - \frac{1}{4\pi}\ln\left(\frac{2}{37}\right) \tag{4.77}$$

Requiring that $x > 0$ leads to the bound

$$\alpha_3 \geq \frac{7}{9}\frac{1}{\frac{5\sin^2\theta_W - 1}{3\,\alpha_e} - \frac{1}{4\pi}\ln\left(\frac{2}{37}\right)} \approx 0.1130 \tag{4.78}$$

4.3.5 Baryon- and lepton-number violating terms

As discussed in section 4.2.2, it is necessary to forbid VEVs for certain singlet fields in order for the perpendicular $U(1)$ symmetries to prevent dangerous operators. However this is not entirely sufficient to ensure the absence of baryon and lepton number violating terms because, even in the absence of these VEVs, tree level graphs can generate the dangerous operators at higher order in the singlet fields. As such, we must look for graphs of the type shown in Fig. 4.4. In

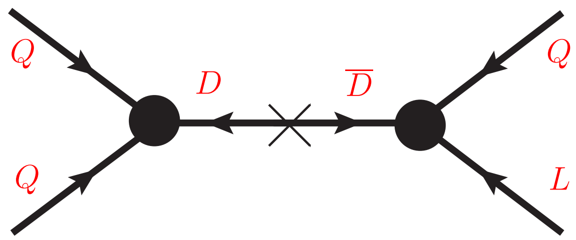

Figure 4.4: The general proton decay diagram generating the dimension 5 operator $QQQL$.

these models, just as in Chapter 3, the dangerous graph is shown in Fig. 4.5 and is driven by colour triplet exchange coming from the couplings

$$\begin{aligned} 10_M\, 10_M\, 5_{H_u} &\rightarrow QQD_h + \ldots \\ 5_{H_u}\bar{5}_{\bar{H}_u} &\rightarrow M_D D_h \bar{D}_h + \ldots \\ \theta_{34} 5_1 \bar{5}_2 &\rightarrow \langle\theta_{34}\rangle\, D'_h \bar{D}_h{}''' + \ldots = \langle\theta_{34}\rangle\, D\bar{D} + \ldots . \end{aligned}$$

The notation has been simplified here by calling the light exotics D'_h and $\bar{D}_h{}'''$ simply D and $\bar{D}$. In Fig. 4.5 the full notation is used, but in Fig. 4.4 and Fig. 4.6 the simplified notation is used, with D representing a light colour triplet.

As may be seen from Table 4.1 only the states D and $\bar{D}$ (i.e. D'_h and $\bar{D}'''_h$ in Fig. 4.5) appear in the spectrum with mass generated by the singlet VEV $\langle\theta_{34}\rangle$ which is at the TeV scale. Since the choice of fluxes in Table 4.1 eliminates light colour triplet states D_h arising from 5_{H_u}, and also $D_h{}''$ states arising from 5_4, we assume that if states with the quantum numbers of D_h, D''_h exist, they will have string scale masses, of $O(M_S)$.

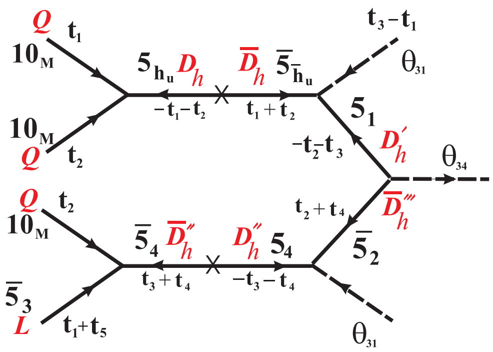

Figure 4.5: The specific proton decay diagram generating the dimension 5 operator $QQQL$ in this model.

With this assumption, the diagram of Fig. 4.5 gives the proton decay operator $QQQL$ with coefficient $1/\Lambda_{eff}$ given by

$$\frac{1}{\Lambda_{eff}} = \lambda^5 \left(\frac{\langle\theta_{31}\rangle}{M_S}\right)^2 \frac{1}{\langle\theta_{34}\rangle} \tag{4.79}$$

In (4.79), λ^5 represents the product of the five Yukawa couplings in the relevant diagram and according to ref [25] it is expected to be

$$\lambda^5 = \lambda_{10\cdot 10\cdot 5}\lambda_{10\cdot\bar{5}\cdot\bar{5}}\lambda^3_{5\cdot\bar{5}\cdot 1} \approx 10^{-3}.$$

This implies

$$\Lambda_{eff} \approx 10^3 \left(\frac{M_S}{\langle\theta_{31}\rangle}\right)^2 \langle\theta_{34}\rangle .$$

In an analogous way to the analysis in Section 3.2.4, this should be compared to experimental bounds on nucleon decay. This bound, relevant to the case that the operator $QQQL$ involves quarks from the two lighter generations only, requires $\Lambda^{\text{light}}_{\text{eff}} > (10^8 - 10^9)M_S$. The difference between this case and the case studied in Chapter 3 is that now $\langle\theta_{34}\rangle \sim TeV \ll M_S$, meaning that there will be a large discrepancy between $\Lambda^{\text{light}}_{\text{eff}}$ and Λ_{eff}, even when the suppression factors for the first and second generations (due to non perturbative flux corrections) are considered (as in section 3.2.4). As such, it is clearly necessary to forbid the light quark operator generated by the diagram of Figure 4.5. One way to do this would be to forbid the coupling $\theta_{31}\overline{5}_{\overline{H}_U}5_1$. Note that all the other vertices in Figure 4.5 are necessary for various phenomenological reasons. For example, the couplings in Figure 4.1 are necessary to generate the bottom quark Yukawa coupling, and so these couplings cannot be set to zero. Similarly the top quark Yukawa coupling originates from the coupling $10_M 10_M 5_{H_U}$. The coupling $\theta_{34}5_1\overline{5}_2$ is necessary to give the exotics a TeV scale mass term $\langle\theta_{34}\rangle D\overline{D}$. In order for the bounds on proton decay to be satisfied, the $\theta_{31}\overline{5}_{\overline{H}_U}5_1$ coupling must be suppressed by a factor of 10^{-12}. This can be seen by looking at Eq. (4.79) and comparing to the bound $\Lambda^{\text{light}}_{\text{eff}} > (10^8 - 10^9)M_S$, whilst taking into account a suppression factor for the light quarks (as in section 3.2.4 [1]).

In fact we only need to forbid the colour triplet components of the $\theta_{31}\overline{5}_{\overline{H}_U}5_1$ coupling. This can be achieved by imposing a discrete Z_2 symmetry with the following set of fields chosen to be odd: $(D'_h, \overline{D}'''_h, D''_h, \overline{D}''_h)$. Either the set (L, e^c) or (Q, d^c, u^c) are also chosen to be odd. All other fields are chosen to be even under Z_2. These assignments forbid the proton decay diagram in Fig. 4.5 but allow the top quark Yukawa coupling.

Note that with these charge assignments the Z_2 symmetry is absolutely conserved. Also Z_2 does not respect $SU(5)$, as for example $\overline{D}''_h(\overline{5}_4)$ must be odd, but the $\overline{H}_d$ state coming from the same curve must be even. This is because it gets a large mass from the coupling $\theta_{31}H_d\overline{H}_d$, and the θ_{31} and H_d fields must be even otherwise Z_2 would be broken leading to cosmological domain walls. The Z_2 symmetry clearly goes beyond the rules of local F-theory, which corresponds to the fact that we are appealing to global F-theory to forbid the colour triplet components of the $\theta_{31}\overline{5}_{\overline{H}_U}5_1$ coupling by a geometric suppression mechanism. However, in the present study this just corresponds to an assumption related to the uncertain nature of singlet fields and their couplings in F-theory. Such assumptions about singlets are always required in any case. In particular, the forbidden coupling involves θ_{31} which does not live in a 27 of E_6, and the Yukawa couplings of such singlets are particularly poorly understood. [2]

4.4 Comparison with known models

4.4.1 E6SSM

The low energy spectrum in Table 4.1 resembles that of the standard E6SSM [75, 88, 89]. The F-theory model with a surviving Abelian gauge group is also a supersymmetric standard model involving the same $U(1)_N$ gauge symmetry surviving down to the TeV scale. However, whereas the E6SSM matter content appears to arise from three 27 representations of E_6, in the F-theory model there is a rather subtle doublet-triplet splitting involved in achieving this spectrum, due to the effects of flux, as indicated in Table 4.1. The light exotics with the quantum numbers of colour triplets D and $\overline{D}$ arise from three 5_1 and three $\overline{5}_2$ representations of $SU(5)$, while the third Higgs doublet H_u arises from a different representation 5_{H_u}.

[2] Note that the θ_{14} and θ_{34} are different types of singlet since they are contained in 27s of E_6.

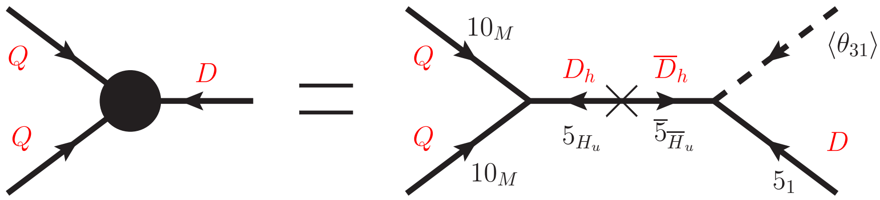

Figure 4.6: Coupling DQQ forbidden by the imposed Z_2 symmetry, where the field D is a TeV scale exotic.

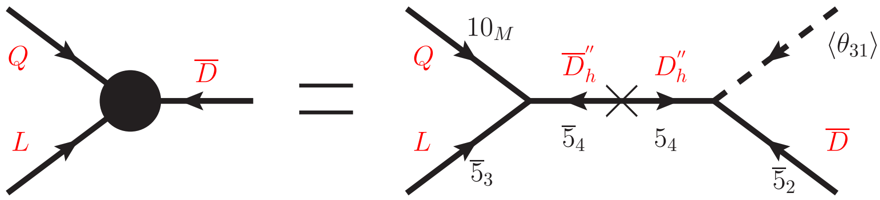

Figure 4.7: Coupling $\overline{D}QL$ allowed by the imposed Z_2 symmetry, where the field $\overline{D}$ is a TeV scale exotic.

The low energy gauge invariant superpotential of the E6SSM can be written

$$W^{\mathrm{E_6SSM}} = W_0 + W_{1,2}, \tag{4.80}$$

where $W_{0,1,2}$ are given by

$$W_0 = W_{\mathrm{Yukawa}} + \lambda_{ijk}\hat{S}_i\hat{H}_{dj}\hat{H}_{uk} + \kappa_{ijk}\hat{S}_i\hat{\bar{D}}_j\hat{D}_k, \tag{4.81}$$

$$W_1 = g^Q_{ijk}\hat{D}_i\hat{Q}_{Lj}\hat{Q}_{Lk} + g^q_{ijk}\hat{\bar{D}}_i\hat{d}^c_{Rj}\hat{u}^c_{Rk}, \tag{4.82}$$

$$W_2 = g^N_{ijk}\hat{N}^c_i\hat{D}_j\hat{d}^c_{Rk} + g^E_{ijk}\hat{D}_i\hat{u}^c_{Rj}\hat{e}^c_{Rk} + g^D_{ijk}\hat{\bar{D}}_i\hat{Q}_{Lj}\hat{L}_{Lk}. \tag{4.83}$$

with $W_{1,2}$ referring to either W_1 or W_2, giving two alternative models in the usual E6SSM. In the E6SSM the three $SU(5)$ singlets S_i which are charged under $U(1)_N$ may be labelled as S_α, $\alpha = 1,2$ and S_3, where the latter couples to exotics, giving them mass and generating the effective μ term after they acquires a non zero VEV. In the F-theory model these are identified as two copies of θ_{34} which give the light exotics mass, and the θ_{14} which generates the μ term in the F-theory model. The other GUT singlets which get VEVs in the F-theory model are θ_{31} (which removes unwanted exotics from the low energy spectrum), and θ_{53} (which helps generate neutrino masses). These singlets acquire string scale VEVs, and are uncharged under the $U(1)_N$ as required. The other important singlet is θ_{12}, as this is the Majorana state which we call S_R. This singlet is uncharged under the perpendicular $U(1)$s and so can get a Majorana mass and play a role in the double see-saw mechanism for generating neutrino masses.

Another difference between the models is that in the E6SSM there are the $H', \overline{H}'$ states coming from incomplete 27 and $\overline{27}$ representations, which are necessary to ensure gauge coupling unification. In F-theory however, we have splitting of the gauge couplings at unification as discussed, although in the minimal case presented in this chapter (resembling the ME6SSM), the splitting is too large. It is only when we consider bulk exotics in the next chapter that we will find a set of exotics which occur naturally due to the flux breaking mechanism, and which ensure the required level of unification. Due to the splitting of the couplings at unification, we cannot know about the size of the $U(1)_N$ gauge coupling. As the normal limits on the Z' come from the assumption of unification, these limits do not apply in the F-theory model.

It should be noted that in the local F-theory version of the E6SSM all the couplings of Eqs.4.82 and 4.83 are forbidden at the level of renormalisable operators due to the perpendicular $U(1)$s. At the level of local F-theory, they are all allowed at the effective level after including one insertion of the θ_{31} field. However at the level of global F-theory we have assumed that not all couplings involving θ_{31} are allowed, and we have described this by imposing a Z_2 symmetry so that certain effective diagrams involving the exchange of heavy colour triplet states are forbidden, in particular those which would lead to proton decay.

The effective DQQ coupling is forbidden by Z_2 since D is odd. In detail, the reason why this operator is forbidden is shown in Fig. 4.6 since D is odd and (D_h, $\overline{D}_h$) are both even. Note that the Z_2 symmetry that we imposed has a global F-theory interpretation as being due to a geometrically suppressed θ_{31} vertex. Similar arguments would forbid the effective $Du^c e^c$ coupling being generated by a diagram analogous to Fig. 4.6. Note that even though a renormalisable $Du^c e^c$ operator would be allowed by Z_2, it is forbidden by the rules of local F-theory.

On the other hand the $\overline{D}QL$ coupling is allowed by Z_2 and can be generated effectively by non-renormalisable operators as shown in Fig. 4.7. All couplings in this diagram are allowed by Z_2 since $\overline{D}$ is odd and in this case also $(D_h'', \overline{D}_h'')$ are odd, as is the combination QL. Thus the effective coupling $\overline{D}QL$ is successfully generated, allowing the $\overline{D}$ to decay as a chiral leptoquark with couplings to left-handed quarks and leptons. Note that the effective $\overline{D}d^c u^c$ coupling is forbidden by Z_2 since $\overline{D}$ is odd while the combination $d^c u^c$ is even. The lifetime of the TeV scale exotic $\overline{D}$ can be estimated by looking at its decay through the diagram of Figure 4.7, and computing

$$\begin{aligned}
\frac{1}{\tau_{\overline{D}}} &= \left|\frac{\lambda^2 \langle\theta_{31}\rangle}{M_S}\right|^2 \times M_D \\
&\sim (\frac{0.08 \times 1.4 \times 10^{16}}{10^{17}})^2 \times 10^3 \mathrm{GeV} \\
&\sim 0.13 \mathrm{GeV} \\
\Rightarrow \tau_{\overline{D}} &\sim 5 \times 10^{-24} \mathrm{s}
\end{aligned} \tag{4.84}$$

where λ^2 represents the product of the two couplings in the diagram, which is estimated to be around 0.08 from [25]. This shows that the decay of the TeV scale exotics in this model is expected to be very rapid.

By contrast, in the ME6SSM all the couplings involving D and $\overline{D}$ are all highly suppressed coefficients. This tends to give long lived D decays in the ME6SSM, but prompt D decays in the F-theory model, with large couplings to left handed quarks and leptons, providing characteristic and striking signatures at the LHC.

In summary, proton decay is suppressed by the geometric coupling suppression of a singlet state θ_{31}, which we interpret in terms of a Z_2 symmetry. This symmetry effectively forbids all the couplings of the exotic charge $-1/3$ colour triplet state D to quarks and leptons, while allowing the coupling involving $\overline{D}QL$. However the coupling $\overline{D}d^c u^c$ is forbidden by Z_2. Thus $\overline{D}$ decays as a chiral leptoquark with couplings to left-handed quarks and leptons, with D coupling to $\overline{D}$ to make a TeV scale Dirac fermion. We emphasise again that the effective coupling $Du^c e^c$ is forbidden, while $\overline{D}QL$ is allowed providing a distinctive signature of chiral leptoquarks.

4.4.2 NMSSM+

The low energy spectrum in Table 4.1 may also apply to a version of the F-theory model in which there is no additional Abelian gauge group present, in other words where the $U(1)_N$ gauge group is broken by flux at the GUT scale. This was the case for the F-theory model in [1].

The difference between the present F-theory model and that in [1] is then mainly in the order of magnitude of the the singlet θ_{34} VEV as determined by the different flatness conditions in the two models. In the previous model the singlet θ_{34} acquired a string scale VEV which gave large masses to the exotic states. In the present model the singlet θ_{34} acquires a TeV scale VEV which remain light in the current model. It was also assumed in [1] that the μ term is generated when the $U(1)$ symmetries are explicitly broken by non-perturbative effects. Here we assume that the singlet θ_{14} acquires an electroweak scale VEV which generates an effective μ term. There will also be non-perturbative corrections which generate trilinear self-couplings and additional electroweak scale masses for θ_{14}, explicitly breaking all global $U(1)$ symmetries.

The resulting F-theory model with the spectrum in Table 4.1 but with no additional Abelian gauge group present, resembles that of the NMSSM+ [92]. However in the F-theory model the $U(1)_N$ is broken by flux at a high scale, whereas in the NMSSM+ it is broken by an explicit sector. Recall that the usual NMSSM is based on the scale invariant superpotential [102],

$$W_{\text{NMSSM}} = W_{\text{Yukawa}} + \lambda S H_u H_d + \frac{1}{3}\kappa S^3, \tag{4.85}$$

where W_{Yukawa} represents the MSSM Yukawa couplings. In the F-theory model we identify the singlet S of the NMSSM with θ_{14}. The trilinear self-coupling and other linear and quadratic terms are generated by non-perturbative corrections, resulting in a generalised NMSSM (GN-MSSM) [103, 104] with superpotential,

$$W_{GNMSSM} = W_{\text{Yukawa}} + (\mu + \lambda S)H_u H_d + \frac{1}{2}\mu_s S^2 + \frac{1}{3}\kappa S^3, \tag{4.86}$$

where the singlet S of the GNMSSM is again identified with θ_{14}. The non-perturbative corrections responsible for these terms are similar to those which were used to generate the μ term in [1].

However the model is more than the usual GNMSSM since it also involves the exotic sector of the NMSSM+, so it more closely resembles a sort of GNMSSM+ with three compete 27 dimensional families [92]. The superpotential terms involving the other exotic states (apart from θ_{14}) are similar to those of the E6SSM in Eq.4.80 and discussed in the preceding subsection. The phenomenological comments also discussed in the preceding subsection concerning unification, proton decay and the $\overline{D}$ couplings at the LHC all apply to this case as well where the $U(1)_N$ is broken.

4.5 Summary and Discussion

In this chapter we have explored F-theory models in which the low energy supersymmetric theory contains the particle content of three 27 dimensional representations of the underlying E_6 gauge group, plus two extra right-handed neutrinos predicted from F and D flatness. Using the techniques of semi-local model building in F-theory, we have shown that it is possible to formulate F-theory models whose TeV scale effective theory resembles either the E6SSM or the NMSSM+, depending on whether an additional Abelian gauge group does or does not survive. However there are novel features compared to both these models as follows:

1. If the additional Abelian gauge group is unbroken then it can have a weaker gauge coupling than in the E6SSM.
2. If the additional Abelian gauge group is broken then non-perturbative effects can violate the scale invariance of the NMSSM+ leading to a generalised model.
3. Unification is achieved in the E6SSM by the addition of doublet states from incomplete 27 representations, whereas in the F-theory case, there is a splitting of the gauge couplings at the unification scale, although bulk exotics from the adjoint representation are always present which can lead to precise unification in a natural way.
4. Proton decay is suppressed by the geometric coupling suppression of a singlet state, which is possible in F-theory, which effectively suppresses the coupling of the exotic charge $-1/3$ colour triplet state D to quarks and leptons.
5. The $\overline{D}$ decays as a chiral leptoquark with couplings to left-handed quarks and leptons, providing characteristic and striking signatures at the LHC.

Model Features	F-MSSM	F-E6SSM	F-NMSSM+
$\langle\theta_{53}\rangle$, $\langle\theta_{31}\rangle$	$\sim M_X$	$\sim M_X$	$\sim M_X$
$\langle\theta_{34}\rangle$	$\sim M_X$	~ 1 TeV	~ 1 TeV
$\langle\theta_{14}\rangle$	0	~ 1 TeV	~ 1 TeV
$U(1)_N$ breaking	Flux $\sim M_X$	$\langle\theta_{34}\rangle \sim 1$TeV	Flux $\sim M_X$
Non perturbative μ term	$\mu^{N.P}H_uH_d$	-	-
Effective μ term	-	$\theta_{14}H_uH_d$	$\theta_{14}H_uH_d$
Non perturbative singlet masses	-	-	$m_s\theta_{14}^2$, $m_s^2\theta_{14}$

Table 4.2: Similarities and differences between different F-theory based models which go beyond the MSSM.

The particle spectrum of the F-theory models is summarized in Table 4.1. The models here may be compared to the F-theory model in [1] in which the singlets θ_{34} acquired a string scale VEV which gave large masses to the exotic states, yielding a low energy theory as in the MSSM, which we can call an F-MSSM. The new models here have a singlet spectrum where the new flatness conditions allow the singlets θ_{34} to have small VEVs resulting in a light exotic mass spectrum. In addition the singlets θ_{14} are used to generate electroweak scale effective μ terms. Five right handed neutrinos, as well as other restrictions on the numbers of certain singlets in the spectrum, are required to make the model consistent with F and D-flatness conditions. If the gauged $U(1)_N$ is broken by flux at the GUT scale then we have either the F-MSSM as discussed previously or the F-NMSSM+ as investigated here, where non-perturbative corrections break all global $U(1)$ symmetries via θ_{14} mass terms. However if the gauged $U(1)_N$ is unbroken then we are led to an F-E6SSM but with the phenomenological differences discussed above. The three different F-theory models are compared in Table 4.2.

In order for proton decay to be controlled, the geometric suppression at the field theory level corresponds to the imposition of a discrete Z_2 symmetry. To understand the origin of this geometric suppression would require knowledge of the GUT singlet matter curves, which in turn requires a knowledge of the global geometry. From our limited understanding of the global aspects of F-theory this just corresponds to an assumption about the global completion of the model.

Chapter 5

Bulk exotics

5.1 Introduction

In Chapter 2, a full classification of how E_6, $SO(10)$ and $SU(5)$ GUT groups arise in the semi-local picture has been presented, where the homology classes of the matter curves were calculated in each case through the spectral cover formalism. However, as well as matter transforming in the fundamental representation of the GUT group localised on curves on S, in all these cases there will also be bulk matter, coming from the adjoint representation of the GUT group. In the case that the GUT group is broken down to the Standard Model (SM) gauge group by flux, there are topological formulae which dictate the multiplicities of these adjoint states [7]. It was demonstrated in [9] that when the GUT group is $SU(5)$, bulk matter with exotic charges under the SM gauge group can be eliminated from the spectrum provided certain topological properties of the manifold are satisfied. However, the same study pointed out that when the GUT group is $SO(10)$ or higher, some bulk exotics must always be present in the low energy spectrum. As such, in order to give these exotics masses, we can look for the topological requirements for them to appear in vector-like pairs, and then turn on VEVs for suitable singlets (such as the 'gluing morphism' of [105]). The presence of these bulk states in the spectrum will clearly affect the running of the gauge couplings and their unification, and in [98] it was shown that states descending from the adjoint of $SU(5)$ with exotic SM charges must be completely removed from the spectrum (in the way of [9]) due to RGE arguments.

In this chapter we will consider models where the GUT group is E_6, and is broken by flux breaking down to the Standard Model gauge group via the sequence of breakings

$$E_6 \rightarrow SO(10) \times U(1)_\psi \tag{5.1}$$

$$SO(10) \rightarrow SU(5) \times U(1)_\chi \tag{5.2}$$

$$SU(5) \rightarrow SU(3) \times SU(2) \times U(1)_Y. \tag{5.3}$$

In addition we shall consider models where $U(1)_\psi$ and $U(1)_\chi$ are both broken near the GUT scale by the vacuum expectation value (VEV) of some scalar field or where a particular linear combination, under which the right handed neutrinos have no charge, survives down to the TeV scale, namely [75, 88],

$$U(1)_N = \frac{1}{4}U(1)_\chi + \frac{\sqrt{15}}{4}U(1)_\psi. \tag{5.4}$$

As the entire breaking of E_6 down to the Standard Model gauge group (perhaps also including a surviving $U(1)_N$) will be achieved by flux breaking, this will necessarily involve bulk exotics appearing below the string scale, which will be the principal concern of this chapter.

We first focus on the bulk exotics coming from the adjoint 78 dimensional representation of E_6, and look at how topological properties of the internal manifold restrict the elimination of these exotics from the spectrum, and dictate the numbers of exotics which cannot be removed. These constraints are then translated into topological restrictions, which then determine the multiplicities of vector-like matter. We impose constraints that exotic matter should appear in vector-like pairs and hence can be eliminated from the low energy spectrum by turning on VEVs for appropriate singlet fields. We show that it is possible that all bulk exotic as well as matter exotics could have masses close to the GUT scale leading to an MSSM type theory somewhat below the GUT scale. However, there is the possibility that some bulk exotics from 5s of $SU(5)$ could get TeV scale masses whereas those from 10s could be near the GUT scale, leading to a characteristic spectrum involving TeV vector-like pairs of d^c-like and H_d-like bulk exotics, with the distinguishing feature that there will always be one more vector pair of H_d-like states than d^c-like states. Although such bulk exotics would by themselves spoil gauge coupling unification, when combined with matter exotics, corresponding to having complete 27 dimensional representations of E_6 at the TeV scale, gauge coupling unification is restored. We emphasise that, without such bulk exotics, the TeV scale matter exotics would lead to an unacceptable splitting of the couplings, and it is only the combination of TeV scale matter exotics from the 27s plus TeV scale bulk exotics from the 78 which (miraculously) restores gauge coupling unification. The resulting TeV scale matter exotics plus bulk exotics is equivalent to four extra $5+\overline{5}$ vector pairs of $SU(5)$, beyond the minimal supersymmetric standard model (MSSM) spectrum. The characteristic prediction of F-theory E_6 GUTs of the matter content of four extra $5+\overline{5}$ vector pairs can be tested at the LHC. This may be compared to the equivalent of three extra $5+\overline{5}$ vector pairs predicted by the E6SSM [75, 88].

The layout of the remainder of the chapter is as follows. In Section 5.2 we review the basic issues related to bulk exotics, including topological formulae from [9], before applying these ideas to the E_6 case, and working out the topological constraints. These constraints are then translated into relations between the multiplicities of bulk exotics which appear in vector-like pairs. Section 5.3 is concerned with gauge coupling unification, including a renormalisation group equation (RGE) analysis taking into account the constraints on exotics, and also the dependencies on the exotic masses of the GUT scale and splitting of the gauge couplings are studied. In Section 5.4 we discuss E_6 models from F-theory, where the bulk exotics are put into the context of two realistic models given in Chapters 3 and 4 ([1] and [2]). In particular we discuss the possibility that some bulk exotics could survive down to the TeV scale, and show how, together with the matter exotics predicted by these models, they restore gauge coupling unification.

5.2 Review of issues related to bulk exotics

5.2.1 Formalism and $SU(5)$ example

In F-theory constructions, the appearance of matter is closely related to the topological properties of the internal space. The multiplicities of states are given by specific topological formulae, and therefore are subject to constraints which have to be taken into account. Bulk exotic matter arises from the decomposition of the adjoint of the GUT group G_S. When the gauge group G_S is broken to a group Γ_S by turning on fluxes in a subgroup H_S, with $G_S \supset \Gamma_S \times H_S$, the adjoint of G_S decomposes into representations (τ_j, T_j) of $\Gamma_S \times H_S$,

$$\mathrm{ad(G_S)} \cong \oplus_{\mathrm{j}} \left(\mathrm{\tau_j} \otimes \mathrm{T_j}\right) \tag{5.5}$$

Assuming that S is a del Pezzo surface, the multiplicity of four-dimensional massless fields transforming in a representation τ_j of Γ_S is given in terms of the Euler characteristic by

$$n_j = -\chi(\mathscr{L}_j, S) = -\left(1 + \frac{1}{2} c_1(\mathscr{L}_j) \cdot (c_1(\mathscr{L}_j) + c_1(S))\right) \tag{5.6}$$

where $\mathscr{L}_j$ is a line bundle transforming in the representation T_j of H_S, and the topological quantities $c_1(\mathscr{L}_j)$, $c_1(S)$ are the first Chern classes of $\mathscr{L}_j$ and S. The multiplicity of the conjugate representation can be found by noting that $c_1(\mathscr{L}_j^{-1}) = -c_1(\mathscr{L}_j)$, leading to the equation

$$n_j^* = -\chi(\mathscr{L}_j^{-1}, S) = -\left(1 + \frac{1}{2} c_1(\mathscr{L}_j) \cdot c_1(\mathscr{L}_j) - \frac{1}{2} c_1(\mathscr{L}_j) \cdot c_1(S)\right) \tag{5.7}$$

In the case where we are dealing with states which transform in a representation of H_S corresponding to a direct product of line bundles so that $\mathscr{L}_j = \mathscr{L} \otimes \mathscr{L}'$, we have $n_j = -\chi(\mathscr{L} \otimes \mathscr{L}', S)$ where

$$\begin{aligned} \chi(\mathscr{L} \otimes \mathscr{L}', S) = 1 &+ \frac{1}{2} \left\{c_1(\mathscr{L}) \cdot c_1(S) \oplus c_1(\mathscr{L}') \cdot c_1(S)\right\} \\ &+ \frac{1}{2} \left\{c_1(\mathscr{L}) \cdot c_1(\mathscr{L}) \oplus c_1(\mathscr{L}') \cdot c_1(\mathscr{L}')\right\} \end{aligned} \tag{5.8}$$

Taking for example the exotics coming from the adjoint of $SU(5)$ after hypercharge flux breaking to the Standard Model, we have the decomposition

$$24 \to (8,1)_0 + (1,3)_0 + (3,2)_{-\frac{5}{6}} + (\overline{3},2)_{\frac{5}{6}} \tag{5.9}$$

where the line bundle $\mathscr{L}_Y^{\frac{5}{6}}$ is associated to the hypercharge. This decomposition gives rise to the states $(3,2)_{-\frac{5}{6}}$ and $(\overline{3},2)_{\frac{5}{6}}$ which are in exotic representations of the SM gauge group. It has been shown in [98] that the presence of these exotics lower the unification scale to unacceptable values, so we must require that these states are not present in the spectrum. Using Eqs. (5.6) and (5.7), and labelling the multiplicities of $(3,2)_{-\frac{5}{6}}$ and $(\overline{3},2)_{\frac{5}{6}}$ states by m and m^* respectively, we have

$$m - m^* = -c_1(\mathscr{L}_Y) \cdot c_1(S) \tag{5.10}$$

$$m + m^* = -(2 + c_1(\mathscr{L}_Y) \cdot c_1(\mathscr{L}_Y)) \tag{5.11}$$

If we require there to be only vector-like pairs of bulk exotics in the spectrum, Eq. (5.10) tells us that the following dot product has to be zero

$$c_1(\mathscr{L}_Y) \cdot c_1(S) = 0 \tag{5.12}$$

If we further require the complete elimination of these exotics, then we must demand also that the sum has to be zero, so from Eq. (5.11), we can see that the line bundle has to satisfy

$$c_1(\mathscr{L}_Y) \cdot c_1(\mathscr{L}_Y) = -2 \tag{5.13}$$

which corresponds to the condition for $c_1(\mathscr{L}_Y)$ to correspond to a root of E_N.

5.2.2 E_6 Bulk Exotics and their $SU(5)$ picture

It has been shown in [9] that bulk exotics coming from the adjoint of the GUT group on S cannot be avoided in the case where the gauge group is $SO(10)$ or higher, and the breaking of the GUT group down to the Standard Model is achieved by flux breaking. If we take the GUT group to be E_6, the spectrum can be found by decomposing under the E_8 enhancement

$$\begin{aligned} &E_8 \supset E_6 \times SU(3)_\perp \\ &248 \to (78,1) + (27,3) + (\overline{27},\overline{3}) + (1,8) \end{aligned} \tag{5.14}$$

The SM can be achieved by turning on fluxes in the $U(1)$s contained in the following sequence of rank preserving breakings:

$$\begin{aligned} E_6 &\to SO(10)\times U(1)_\psi \\ &\to SU(5)\times U(1)_\chi \times U(1)_\psi \\ &\to SU(3)\times SU(2)\times U(1)_Y\times U(1)_\chi\times U(1)_\psi \end{aligned} \tag{5.15}$$

In order to discuss the bulk exotics, we must decompose the adjoint of E_6 appearing in Eq. (5.14) under the breaking pattern of Eq. (5.15) as follows

$$\begin{aligned} 78 \to\ & (1,1)_{0,0,0} + \{(1,1)_{0,0,0} + (1,1)_{0,0,0} + (8,1)_{0,0,0} + (1,3)_{0,0,0} + (3,2)_{-5,0,0} + (\overline{3},2)_{5,0,0} \\ & +(3,2)_{1,4,0} + (\overline{3},2)_{-1,-4,0} + (\overline{3},1)_{-4,4,0} + (3,1)_{4,-4,0} + (1,1)_{6,4,0} + (1,1)_{-6,-4,0}\} \\ & + \{(1,1)_{0,-5,-3} + (\overline{3},1)_{2,3,-3} + (1,2)_{-3,3,-3} + (1,1)_{6,-1,-3} + (3,2)_{1,-1,-3} + (\overline{3},1)_{-4,-1,-3}\} \\ & + \{(1,1)_{0,5,3} + (3,1)_{-2,-3,3} + (1,2)_{3,-3,3} + (1,1)_{-6,1,3} + (\overline{3},2)_{-1,1,3} + (3,1)_{4,1,3}\} \end{aligned} \tag{5.16}$$

All representations are charged under three $U(1)$s, and all triplets of $U(1)$ charges can be expressed as a linear combination of the following line bundles

$$\mathscr{L}_1 = (5,0,0),\ \mathscr{L}_2 = (1,4,0),\ \mathscr{L}_3 = (1,-1,-3) \tag{5.17}$$

In Table 5.1 we write down the multiplicities of the exotic states coming from the adjoint of E_6 (where the correct normalisation for the $U(1)_Y$ is given by dividing by 6)

Exotic X_i	Multiplicity n_i	Exotic X_i	Multiplicity n_i
$X_1 = (\overline{3},2)_{\frac{5}{6}}$	$n_1 = -\chi(\mathscr{L}_1, S)$	$X_6 = (\overline{3},1)_{\frac{1}{3}}$	$n_6 = -\chi(\mathscr{L}_2\otimes\mathscr{L}_3, S)$
$X_2 = (3,2)_{\frac{1}{6}}$	$n_2 = -\chi(\mathscr{L}_2, S)$	$X_7 = (1,2)_{-\frac{1}{2}}$	$n_7 = -\chi(\mathscr{L}_1^{-1}\otimes\mathscr{L}_2\otimes\mathscr{L}_3, S)$
$X_3 = (3,1)_{\frac{2}{3}}$	$n_3 = -\chi(\mathscr{L}_1^{-1}\otimes\mathscr{L}_2, S)$	$X_8 = (1,1)_1$	$n_8 = -\chi(\mathscr{L}_1\otimes\mathscr{L}_3, S)$
$X_4 = (1,1)_1$	$n_4 = -\chi(\mathscr{L}_1\otimes\mathscr{L}_2, S)$	$X_9 = (3,2)_{\frac{1}{6}}$	$n_9 = -\chi(\mathscr{L}_3, S)$
$X_5 = (1,1)_0$	$n_5 = -\chi(\mathscr{L}_2^{-1}\otimes\mathscr{L}_3, S)$	$X_{10} = (\overline{3},1)_{-\frac{2}{3}}$	$n_{10} = -\chi(\mathscr{L}_1^{-1}\otimes\mathscr{L}_3, S)$

Table 5.1: E_6 bulk exotics and their multiplicities

We can see where the exotics fit into the $SU(5)$ picture as follows (where the un-normalised $U(1)_Y\times U(1)_\chi\times U(1)_\psi$ charges of the $SU(5)$ states are indicated as subscripts),

$$\begin{aligned}\overline{5}_{3,-3} \rightarrow & (1,2)_{-3,3,-3} + (\overline{3},1)_{2,3,-3} \\ & \quad X_7 \qquad\qquad X_6\end{aligned} \tag{5.18}$$

$$\begin{aligned}10_{4,0} \rightarrow & (1,1)_{6,4,0} + (\overline{3},1)_{-4,4,0} + (3,2)_{1,4,0} \\ & \quad X_4 \qquad\quad \overline{X}_3 \qquad\qquad X_2\end{aligned} \tag{5.19}$$

$$\begin{aligned}10_{-1,-3} \rightarrow & (1,1)_{6,-1,-3} + (\overline{3},1)_{-4,-1,-3} + (3,2)_{1,-1,-3} \\ & \quad X_8 \qquad\qquad X_{10} \qquad\qquad\qquad X_9\end{aligned} \tag{5.20}$$

$$\begin{aligned}24_{0,0} \rightarrow & (1,1)_{0,0,0} + (8,1)_{0,0,0} + (1,3)_{0,0,0} + (3,2)_{-5,0,0} + (\overline{3},2)_{5,0,0} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{X}_1 \qquad\quad X_1\end{aligned} \tag{5.21}$$

5.2.3 Removing bulk exotics

When breaking the adjoint of a high gauge group there are always representations beyond those of the SM spectrum. These extraneous matter fields may be classified according to their charges in two categories: the ones that carry charges like the SM fields and those which have fractional charges other than those of the SM quarks. It can be seen that the exotics $\overline{X}_3$ and X_{10} have the same SM quantum numbers as u^c, X_2 and X_9 have the same as Q, and X_4 and X_8 the same as e^c, with one set of states coming from Eq. (5.19) and the other coming from Eq. (5.20). X_1 has exotic charges under the SM gauge group, and so we wish to remove these states from the spectrum. X_6 and X_7 have the same SM quantum numbers as d^c and H_d respectively, and if present in the spectrum, we must require that they appear in vector pairs, and get mass via the couplings

$$\begin{aligned}1_{0,0} \cdot \overline{5}_{3,-3} \cdot 5_{-3,3} &\rightarrow SX_6\overline{X}_6 + SX_7\overline{X}_7 \\ 24_{0,0} \cdot \overline{5}_{3,-3} \cdot 5_{-3,3} &\rightarrow S'X_6\overline{X}_6 + S'X_7\overline{X}_7\end{aligned} \tag{5.22}$$

Requiring that X_6 and X_7 occur in vector pairs corresponds to imposing the conditions $n_6 - n_6^* = n_7 - n_7^* = 0$. Using Table 5.1, this leads to the following topological constraints

$$c_1(S) \cdot c_1(\mathscr{L}_2) = -c_1(S) \cdot c_1(\mathscr{L}_3) \tag{5.23}$$

$$c_1(S) \cdot c_1(\mathscr{L}_1) = 0 \tag{5.24}$$

Presence of the X_1 states with exotic SM charges in the spectrum has been shown to lower the unification scale to unacceptable values, so requiring that these states are completely removed imposes the constraints (from Appendix D)

$$\begin{aligned} c_1(S)\cdot c_1(\mathscr{L}_1) &= 0 \\ c_1(\mathscr{L}_1)^2 &= -2 \end{aligned} \tag{5.25}$$

From Eq. (5.17) and Table 5.1 along with the decompositions in the $SU(5)$ picture, it can be seen that $\mathscr{L}_1$ corresponds to the hypercharge bundle. As such, Eq. (5.25) simply corresponds to the normal $SU(5)$ condition $c_1(\mathscr{L}_Y)^2 = -2$.

If we were to impose that each type of exotic came in vector pairs individually (i.e $n_i = n_i^*$ for i=1,...,10), from Appendix D we would be lead to the case of

$$c_1(S)\cdot c_1(\mathscr{L}_1) = c_1(S)\cdot c_1(\mathscr{L}_2) = c_1(S)\cdot c_1(\mathscr{L}_3) = 0 \tag{5.26}$$

After imposing Eq. (5.25), we can see that the only further choices we can make to eliminate some exotics (without getting negative numbers for any multiplicities) is

$$c_1(\mathscr{L}_2)^2 = -2 \tag{5.27}$$

$$c_1(\mathscr{L}_3)^2 = -2 \tag{5.28}$$

This ensures that the exotics X_2 and X_9 are completely removed, in addition to X_1. All other exotics are present in vector pairs in this case, with their multiplicities given by

$$n_7 = n_7^* = 2, \quad n_i = n_i^* = 1 \quad (i = 3,4,5,6,8,10)$$

5.2.4 A more general case

As we have seen, we have two different 10 representations of $SU(5)$, with different charges under $U(1)_\psi$ and $U(1)_\chi$, and so we can either give masses to the exotics contained in these 10s by couplings of the type

$$\begin{aligned} 1_{0,0}\cdot 10_{4,0}\cdot \overline{10}_{-4,0} &\to SX_2\overline{X}_2 + SX_3\overline{X}_3 + SX_4\overline{X}_4 \\ 24_{0,0}\cdot 10_{4,0}\cdot \overline{10}_{-4,0} &\to S'X_2\overline{X}_2 + S'X_3\overline{X}_3 + S'X_4\overline{X}_4 \end{aligned} \tag{5.29}$$

or

$$1_{-5,-3} \cdot 10_{4,0} \cdot \overline{10}_{1,3} \rightarrow X_5(X_2\overline{X}_9) + X_5(\overline{X}_3 X_{10}) + X_5(X_4\overline{X}_8) \quad (5.30)$$

where X_5 is a singlet exotic (corresponding to the 'gluing morphism' of [105]) coming from the 16 of $SO(10)$, inside the 78 of E_6. As such, we can more generally impose

$$\begin{aligned} n_2 + n_9 &= n_2^* + n_9^* \\ n_4 + n_8 &= n_4^* + n_8^* \\ n_3^* + n_{10} &= n_3 + n_{10}^* \end{aligned} \quad (5.31)$$

It can be seen that all three of these constraints are satisfied by imposing Eq. (5.23). As such, with Eqs. (5.24, 5.25) also imposed, the multiplicities can be written in terms of the dot products

$$\begin{aligned} A &= c_1(S) \cdot c_1(\mathscr{L}_2) = -c_1(S) \cdot c_1(\mathscr{L}_3) \\ B &= c_1(\mathscr{L}_2)^2 \\ C &= c_1(\mathscr{L}_3)^2 \end{aligned} \quad (5.32)$$

$n_1 = n_1^* = 0$	$n_6 = -1 - \frac{B}{2} - \frac{C}{2}$
$n_2 = -1 - \frac{A}{2} - \frac{B}{2}$	$n_7 = -\frac{B}{2} - \frac{C}{2}$
$n_3 = \frac{A}{2} - \frac{B}{2}$	$n_8 = \frac{A}{2} - \frac{C}{2}$
$n_4 = -\frac{A}{2} - \frac{B}{2}$	$n_9 = -1 + \frac{A}{2} - \frac{C}{2}$
$n_5 = -1 + A - \frac{B}{2} - \frac{C}{2}$	$n_{10} = \frac{A}{2} - \frac{C}{2}$

Table 5.2: Multiplicities of the E_6 exotics in terms of the topological numbers A, B, C (see text).

The multiplicities are then given in Table 5.2 where when dealing with the conjugate representations, A changes sign, but B and C keep the same sign. We can now think about different combinations of exotic matter which satisfy these constraints, and consider the effect on gauge coupling unification. The multiplicities of exotic matter are as follows

$$\begin{aligned} n_Q &= n_2 + n_9 + n_2^* + n_9^* = -(B+C) - 4 = \gamma - 4 \\ n_{u^c} &= n_3 + n_{10} + n_3^* + n_{10}^* = -(B+C) \quad = \gamma \\ n_{e^c} &= n_4 + n_8 + n_4^* + n_8^* = -(B+C) \quad = \gamma \\ n_{d^c} &= n_6 + n_6^* = -(B+C) - 2 \quad = \gamma - 2 \\ n_{H_d} &= n_7 + n_7^* = -(B+C) \quad = \gamma \end{aligned} \quad (5.33)$$

where we see that everything can be expressed in terms of the parameter γ, given in terms of Chern classes by

$$\gamma = -c_1(\mathscr{L}_2)^2 - c_1(\mathscr{L}_3)^2 \tag{5.34}$$

It can be seen from Table 5.2 that requiring $n_5 = n_5^*$ for the singlet X_5 leads us to the case $A = 0$. As such, all the exotic matter will satisfy $n_i = n_i^*$, although we will still be able to get masses from both Eqs. (5.29) and (5.30). It is important to note that as everything comes in conjugate pairs, anomalies are always cancelled. We can now work out the contributions to the beta functions due to the exotic matter, and discuss gauge coupling unification. Note that in order to satisfy the requirement that all multiplicities are positive, we must have $\gamma \geq 4$, with the minimal value being taken in the case where the line bundles satisfy the condition $c_1(\mathscr{L}_2)^2 = c_1(\mathscr{L}_3)^2 = -2$, meaning that $c_1(\mathscr{L}_2)$ and $c_1(\mathscr{L}_3)$ correspond to roots of E_8.

5.3 Gauge Coupling Unification

5.3.1 The effect of bulk exotics at a single mass scale M_X

It has been shown in [40] that in the context of an $SU(5)$ GUT, the splitting at M_{GUT} due to hypercharge flux is

$$\begin{aligned}
\frac{1}{\alpha_3(M_{GUT})} &= \frac{1}{\alpha_G} - y \\
\frac{1}{\alpha_2(M_{GUT})} &= \frac{1}{\alpha_G} - y + x \\
\frac{1}{\alpha_1(M_{GUT})} &= \frac{1}{\alpha_G} - y + \frac{3}{5}x
\end{aligned} \tag{5.35}$$

where $x = -\frac{1}{2}\mathrm{Re}S \int c_1^2(\mathscr{L}_Y)$ and $y = \frac{1}{2}\mathrm{Re}S \int c_1^2(\mathscr{L}_a)$ associated with a non-trivial line bundle $\mathscr{L}_a$ and $S = e^{-\phi} + iC_0$ the axion-dilaton field. It is argued in Section 4.3.2 that the $U(1)_\psi$ and $U(1)_\chi$ fluxes do not lead to any relative splittings of the gauge couplings at unification, although there could be a constant shift in all the couplings at each breaking. As such, Eq. (5.35) can be used in the case of interest here, and combining the three equations shows that the gauge couplings at M_{GUT} are found to satisfy the relation

$$\frac{1}{\alpha_Y(M_{GUT})} = \frac{5}{3}\frac{1}{\alpha_1(M_{GUT})} = \frac{1}{\alpha_2(M_{GUT})} + \frac{2}{3}\frac{1}{\alpha_3(M_{GUT})} \tag{5.36}$$

If we assume that the bulk exotics all decouple at a single mass scale M_X, the low energy values of the gauge couplings are given by the evolution equations

$$\frac{1}{\alpha_a(M_Z)} = \frac{1}{\alpha_a(M_{GUT})} + \frac{b_a^x}{2\pi}\ln\frac{M_{GUT}}{M_X} + \frac{b_a}{2\pi}\ln\frac{M_X}{M_Z} \tag{5.37}$$

where b_a^x are the beta functions above the scale M_X, and b_a are the beta functions below this scale, i.e. those of the MSSM. Combining Eqs. (5.36) and (5.37) leads to the relation for the GUT scale

$$M_{GUT} = e^{\frac{2\pi}{\beta\mathscr{A}}\rho} \left(\frac{M_X}{M_Z}\right)^{1-\rho} M_Z \tag{5.38}$$

where $\mathscr{A}$ is a function of the experimentally known low energy values of the SM gauge coupling constants

$$\begin{aligned} \frac{1}{\mathscr{A}} &= \frac{5}{3}\frac{1}{\alpha_1(M_Z)} - \frac{1}{\alpha_2(M_Z)} - \frac{2}{3}\frac{1}{\alpha_3(M_Z)} \\ &= \frac{\cos(2\theta_W)}{\alpha_{em}} - \frac{2}{3}\frac{1}{\alpha_3(M_Z)} \end{aligned} \tag{5.39}$$

Here use has been made of the relations $\alpha_Y = \alpha_e/(1-\sin^2\theta_W)$ and $\alpha_2 = \alpha_e/\sin^2\theta_W$. We have also introduced the ratio ρ

$$\rho = \frac{\beta}{\beta_x} \tag{5.40}$$

where β, β_x are the beta-function combinations in the regions $M_Z < \mu < M_X$ and $M_X < \mu < M_{GUT}$ respectively

$$\beta_x = b_Y^x - b_2^x - \frac{2}{3}b_3^x \tag{5.41}$$

$$\beta = b_Y - b_2 - \frac{2}{3}b_3 \tag{5.42}$$

Recall now the beta-function coefficients ($b_1 = \frac{3}{5}b_Y$)

$$b_1 = 6 + \frac{3}{10}(n_h + n_L) + \frac{1}{5}n_{d^c} + \frac{1}{10}n_Q + \frac{4}{5}n_{u^c} + \frac{3}{5}n_{e^c} \tag{5.43}$$

$$b_2 = \frac{1}{2}(n_h + n_L) + \frac{3}{2}n_Q \tag{5.44}$$

$$b_3 = -3 + \frac{1}{2}n_{d^c} + n_Q + \frac{1}{2}n_{u^c} \tag{5.45}$$

where $n_{h,L,...}$ counts the number of Higgses and exotic matter.

Below M_X we have only the MSSM spectrum, thus $n_G = 3, n_h = 2$ and all extra matter contributions are zero, $n_i = 0$, thus

$$\{b_Y, b_2, b_3\} = \{11, 1, -3\} \quad \to \quad \beta = b_Y - b_2 - \frac{2}{3}b_3 = 12$$

Above M_X we have the extra matter given in Eq. (5.33) in addition to the two Higgses of the MSSM, giving for the beta functions

$$\begin{aligned} b_Y^x &= \frac{1}{3}(29+10\gamma) \\ b_2^x &= 2\gamma-5 \\ b_3^x &= 2(\gamma-4) \\ \beta_x &= 20 \end{aligned} \tag{5.46}$$

As such, we can see that the beta function combination β_x does not depend on the parameter γ and so the choice of this parameter will not affect the unification scale. Putting the numbers into Eq. (5.38) gives

$$M_{GUT} = \left(\frac{M_X}{2.09\times10^{16}\text{ GeV}}\right)^{\frac{2}{5}} 2.09\times10^{16}\text{ GeV} \tag{5.47}$$

Clearly, if we take $M_X = 2.09\times10^{16}$GeV, we also get $M_{GUT} = 2.09\times10^{16}$GeV. We can see how different values of M_X change the GUT scale in the graph of Figure 5.1.

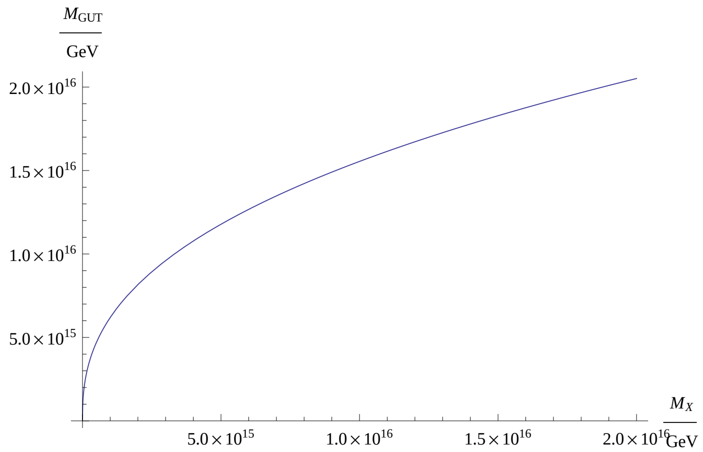

Figure 5.1: Graph of how the bulk exotic mass scale M_X impacts on the GUT scale M_{GUT}.

5.3.2 The splitting parameter, x

Combining Eqs. (5.35) and (5.37) leads to the following expression for the parameter x

$$\begin{aligned} x &= \left(\frac{1}{\alpha_2}-\frac{1}{\alpha_3}\right)_{M_{GUT}} \\ &= \left(\frac{1}{\alpha_2}-\frac{1}{\alpha_3}\right)_{M_Z} + \frac{b_3^x-b_2^x}{2\pi}\log\left(\frac{M_{GUT}}{M_X}\right) + \frac{b_3-b_2}{2\pi}\log\left(\frac{M_X}{M_Z}\right) \\ &= \frac{26\sin^2\theta_W-3}{20\alpha_{em}} - \frac{9}{10\alpha_3} - \frac{11}{10\pi}\log\left(\frac{M_X}{M_Z}\right) \end{aligned} \tag{5.48}$$

It can be seen that the splitting of the gauge couplings at the unification scale does not depend on the parameter γ. It should also be noted that as x is given by $x = -\frac{1}{2}\mathrm{Re}S \int c_1^2(\mathscr{L}_Y)$ with $S = e^{-\phi} + iC_0$, it must take a value between 0 and 1.

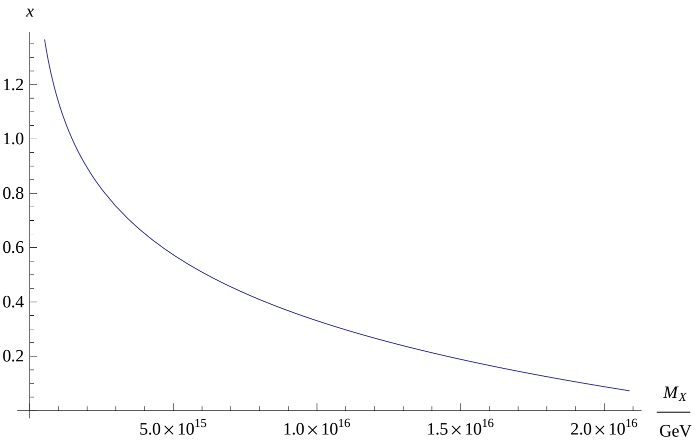

Figure 5.2: The dependence of the splitting parameter x on the bulk exotic mass scale M_X. Only values of $x \leq 1$ are acceptable, leading to the approximate lower bound on the bulk exotic mass scale $M_X \geq 2 \times 10^{15}$ GeV. Note that this bound assumes that no matter exotics are present.

5.4 E_6 Models from F-theory

5.4.1 Matter exotics only

We start by looking at the class of models proposed in Chapters 3 and 4, which were motivated by the fact that any model involving complete 27s of E_6, with no matter coming from the adjoint 78 representation, automatically satisfies anomaly cancellation involving most of the extra $U(1)$s. Here we make the same choices for the M and N integers specifying the flux breaking as in Chapters 3 and 4, and these choices are summarised in Table 5.3. Although the SM particle content is equivalent to having three complete 27s, it is clear that the particles are originating from incomplete multiplets of several different 27s. The $U(1)_N$ charges of all the particles in the spectrum can be computed, and the results are shown in Table 5.3. As required, the right handed neutrinos have zero charge under $U(1)_N$. The final column of Table 5.3 shows the low energy spectrum of the models of Chapter 4 (i.e the E6SSM) obtained by eliminating the required exotics from the previous column, which shows the SM particle content after flux breaking. By comparing the final two columns of Table 5.3, we can see that the matter exotics which we wish to remove are the vector pairs $2(L+\overline{L}), Q+\overline{Q}, 2(u^c+\overline{u^c}), d^c+\overline{d^c}$ and $H_d+\overline{H_d}$. Large masses will be generated for these fields through their coupling to SM singlet fields which acquire large VEVs.

From the E_6 point of view, the only E_6 allowed trilinear term in the superpotential is $27_{t_1}27_{t_1}27_{t_3}$. The vectorlike pairs which we wish to remove from the low energy particle content are those which have components in both the 27_{t_1} and 27_{t_3} multiplets. As such, they are removed by introducing θ_{31}, an E_6 singlet, with couplings:

$$\theta_{31}27_{t_1'}\overline{27_{t_3'}} = \theta_{31}Q\overline{Q} + \theta_{31}(2u^c)(2\overline{u^c}) + \theta_{31}d^c\overline{d^c} + \theta_{31}(2L)(2\overline{L}) + \theta_{31}H_d\overline{H_d}. \quad (5.49)$$

If θ_{31} gets a large VEV these vector states get large masses as required. The difference between this case and model 1 (in Chapter 3 is that in model 1, θ_{34} also gets a large VEV. This singlet has the following couplings

$$\theta_{34}5_1\overline{5_2} = \theta_{34}[3D+2H_u][3\overline{D}+3H_d] = \theta_{34}[3(D\overline{D})] + \theta_{34}[2(H_uH_d)]. \quad (5.50)$$

In the E6SSM, these matter exotics are light, and so instead of getting a large VEV, this singlet now must acquire a TeV scale VEV. It was checked that the F and D-flatness constraints are satisfied, and that rapid proton decay is forbidden for the realisation of the spectrum [1].

E_6	$SO(10)$	$SU(5)$	Weight vector	Q_N	N_Y	$M_{U(1)}$	SM particle content	Low energy spectrum
$27_{t_1'}$	16	$\overline{5}_3$	t_1+t_5	$\frac{1}{\sqrt{10}}$	1	4	$4d^c+5L$	$3d^c+3L$
$27_{t_1'}$	16	10_M	t_1	$\frac{1}{2\sqrt{10}}$	-1	4	$4Q+5u^c+3e^c$	$3Q+3u^c+3e^c$
$27_{t_1'}$	16	θ_{15}	t_1-t_5	0	0	n_{15}	$3\nu^c$	-
$27_{t_1'}$	10	5_1	$-t_1-t_3$	$-\frac{1}{\sqrt{10}}$	-1	3	$3D+2H_u$	$3D+2H_u$
$27_{t_1'}$	10	$\overline{5}_2$	t_1+t_4	$-\frac{3}{2\sqrt{10}}$	1	3	$3\overline{D}+4H_d$	$3\overline{D}+3H_d$
$27_{t_1'}$	1	θ_{14}	t_1-t_4	$\frac{5}{2\sqrt{10}}$	0	n_{14}	θ_{14}	θ_{14}
$27_{t_3'}$	16	$\overline{5}_5$	t_3+t_5	$\frac{1}{\sqrt{10}}$	-1	-1	$\overline{d^c}+2\overline{L}$	-
$27_{t_3'}$	16	10_2	t_3	$\frac{1}{2\sqrt{10}}$	1	-1	$\overline{Q}+2\overline{u^c}$	-
$27_{t_3'}$	16	θ_{35}	t_3-t_5	0	0	n_{35}	$-$	-
$27_{t_3'}$	10	5_{H_u}	$-2t_1$	$-\frac{1}{2\sqrt{10}}$	1	0	H_u	H_u
$27_{t_3'}$	10	$\overline{5}_4$	t_3+t_4	$-\frac{3}{2\sqrt{10}}$	-1	0	$\overline{H_d}$	-
$27_{t_3'}$	1	θ_{34}	t_3-t_4	$\frac{5}{2\sqrt{10}}$	0	n_{34}	θ_{34}	θ_{34}
-	1	θ_{31}	t_3-t_1	0	0	n_{31}	θ_{31}	-
-	1	θ_{53}	t_5-t_3	0	0	n_{53}	θ_{53}	-
-	1	θ_{54}	t_5-t_4	$\frac{5}{2\sqrt{10}}$	0	n_{54}	θ_{54}	-
-	1	θ_{45}	t_4-t_5	$-\frac{5}{2\sqrt{10}}$	0	n_{45}	θ_{45}	-

Table 5.3: Complete 27s of E_6 and their $SO(10)$ and $SU(5)$ decompositions. The $SU(5)$ matter states decompose into SM states as $\overline{5} \to d^c, L$ and $10 \to Q, u^c, e^c$ with right-handed neutrinos $1 \to \nu^c$, while $SU(5)$ Higgs states decompose as $5 \to D, H_u$ and $\overline{5} \to \overline{D}, H_d$, where $D, \overline{D}$ are exotic colour triplets and antitriplets. We identify RH neutrinos as $\nu^c = \theta_{15}$.

Clearly the matter exotics $(d+\overline{d}^c)$, $(Q+\overline{Q})$, $(H_d+\overline{H}_d)$, $2(L+\overline{L})$, $2(u^c+\overline{u}^c)$ get masses and decouple at some scale $M_{\theta_{31}} < M_{GUT}$ due to the couplings in Eq. (5.49). The matter exotics $3(D+\overline{D})$, $2(H_u,H_d)$ get masses and decouple at a scale $M_{\theta_{34}} < M_{\theta_{31}}$ due to the couplings in Eq. (5.50). In [1, 2] (which we will call models 1 and 2 respectively from now on) two different classes of model were proposed only distinguished by the mass scales of the matter exotics. The scales of the two models are summarised below.

In model 1 ("MSSM"):

$$M_{\theta_{31}}^{(1)} = 1.31 \times 10^{16}\text{GeV}$$
$$M_{\theta_{34}}^{(1)} = 0.306 \times 10^{16}\text{GeV}$$

In model 2 ("E6SSM"):

$$M_{\theta_{31}}^{(2)} = 1.44 \times 10^{16}\text{GeV}$$
$$M_{\theta_{34}}^{(2)} = 1 \times 10^{3}\text{GeV}$$

The main difference between the two models is clearly that in model 1 the θ_{34} matter exotics are computed to be almost as heavy as the θ_{31} exotics, whereas in model 2 the θ_{34} matter exotics are kept light, getting TeV scale masses. We see that model 1 reproduces the MSSM somewhat below the GUT scale since only the MSSM spectrum survives below $M_{\theta_{34}}$, whereas model 2 corresponds to the so called E6SSM above the TeV scale (or NMSSM+ if the $U(1)_N$ gauge group is broken at high energy). However strictly speaking the spectrum of model is not quite that of the E6SSM since it only contains the matter content of three 27 dimensional representations of E_6 and does not contain the extra vector-like matter usually denoted as H' and $\overline{H'}$ which is required for gauge coupling unification. As we shall see shortly, the role of the extra H' and $\overline{H'}$ will be played by bulk exotics.

5.4.2 High scale bulk exotics

The above analysis does not so far include the effect of bulk exotics. However, as we have seen earlier in this chapter, such bulk exotics are an inevitable consequence of the flux breaking of E_6. As remarked above, such additional bulk exotics at the TeV scale, not included in the spectrum so far, are able to provide the extra vector-like matter to enable gauge unification to be achieved for the E6SSM. However the resulting spectrum will differ somewhat from that of the E6SSM, providing a distinctive experimental signature of the F-theory model at the LHC.

In both the above models, the beta function combination given in Eq. (5.42) is given by $\beta = 12$ (the MSSM value) in all of the regions $M_Z < \mu < M_{\theta_{34}}$ and $M_{\theta_{34}} < \mu < M_{\theta_{31}}$ and $M_{\theta_{31}} < \mu < M_{GUT}$. As such, assuming that the bulk exotics get masses M_X, somewhere between $M_{\theta_{31}}$ and M_{GUT}, we will have an equation analogous to Eq. (5.38)

$$M_{GUT} = e^{\frac{2\pi}{\beta\mathscr{A}}\rho} M_Z^{\rho} M_{\theta_{34}}^{\eta-\rho} M_{\theta_{31}}^{\lambda-\eta} M_X^{1-\lambda} \tag{5.51}$$

where in the same way as for Eq. (5.38), $\rho = \eta = \lambda = \frac{3}{5}$. As such, the GUT scale only depends on the mass of the bulk exotics, and is still given by Eq. (5.47). If we take $M_X = M_{GUT}$, the RGE analysis is obviously unchanged from that of [1, 2], however if we take $M_X = M_{\theta_{31}}$, the GUT scale is lowered slightly by Eq. (5.47)

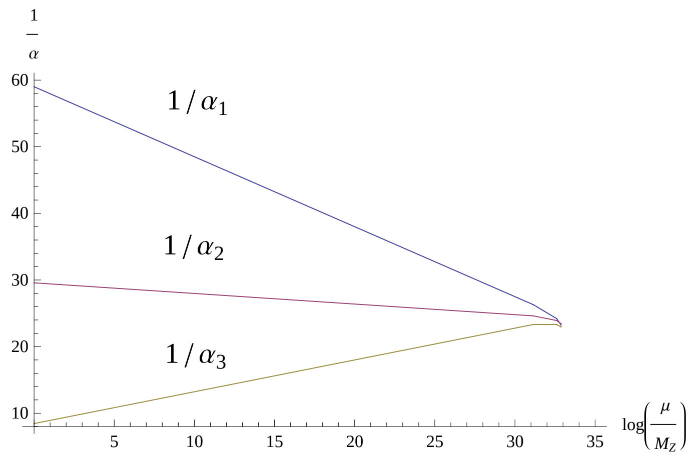

Figure 5.3: Gauge coupling unification in model 1 (MSSM) with high scale bulk exotics.

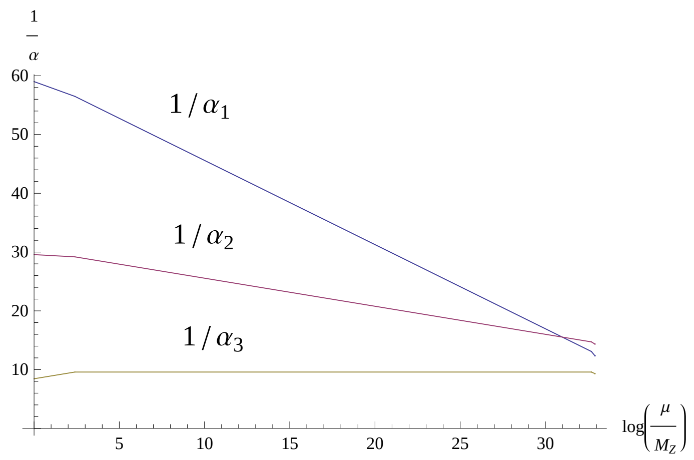

Figure 5.4: Gauge couplings fail to unify in model 2 (E6SSM) with high scale bulk exotics.

$$M_{GUT}^{(1)} = 1.73 \times 10^{16}\text{GeV}$$
$$M_{GUT}^{(2)} = 1.80 \times 10^{16}\text{GeV}$$

For model 1 (MSSM) the one loop running of the couplings is shown in Figure 5.3. This takes into account the modification of the beta functions due to the bulk exotics above the scale $M_X = M_{\theta_{31}}$. In this case the couplings are split by 2 percent (compared to 1.3 percent when the bulk exotics are not taken into account), and it can be seen that the effect of bulk exotics near the GUT scale on the splitting of the gauge couplings is small (0.5-1 percent depending on the model).

For model 2 (E6SSM) the splitting is 35 percent (compared to 34.5 percent in the case with no bulk exotics), which would correspond to $x \sim 5$. This is shown in Figure 5.4. As pointed out before, x must take a value between 0 and 1 and so model 2 must be ruled out in the case where the bulk exotics get masses near the GUT scale.

5.4.3 Low scale bulk exotics

We have seen that as long as the bulk exotics get masses close to the GUT scale, the GUT scale is not lowered drastically. However, due to the fact that the bulk exotic spectrum ensures anomaly cancellation, the gauge groups $U(1)_\chi$ and $U(1)_\psi$ and the bulk exotics could in principle survive to the TeV scale. We will now look at this possibility that at least some of the bulk exotics are light. From Eqs. (5.43, 5.44, 5.45) we have

$$\begin{aligned} \beta &= 12 + n_{u^c} + n_{e^c} - 2n_Q \\ \delta\beta &= \delta n_{u^c} + \delta n_{e^c} - 2\delta n_Q \end{aligned} \tag{5.52}$$

where $\delta\beta = \beta_x - \beta$ is the difference in β as we move a higher energy scale where a number of exotics (δn_{u^c}, δn_{e^c} and δn_Q) join with the massless spectrum. In models 1 and 2 there is no exotic e^c type matter and the only Q and u^c exotics get the same mass, near the GUT scale. In both models, there is twice as much u^c-like exotic matter as there is Q-like, and so $\delta\beta = 0$ when we do not take into account contributions from the bulk exotics. For the bulk exotics, Eq. (5.33) gives

$$\delta\beta = n_{u^c} + n_{e^c} - 2n_Q = 8 \tag{5.53}$$

Previously, we looked at the case where $M_X \geq M'$ and we found that the GUT scale is slightly lowered. If we now consider the case where $M_X \leq M_{\theta_{34}}$, Eq. (5.51) gets modified to

$$M_{GUT} = e^{\frac{2\pi}{\beta \mathscr{A}}\rho} M_Z^{\rho} M_X^{\eta-\rho} M_{\theta_{34}}^{\lambda-\eta} M_{\theta_{31}}^{1-\lambda} \tag{5.54}$$

with

$$\begin{aligned} \rho &= \frac{\beta}{\beta_{\theta_{31}}} = \frac{3}{5} \\ \eta &= \frac{\beta_x}{\beta_{\theta_{31}}} = 1 \\ \lambda &= \frac{\beta_{\theta_{34}}}{\beta_{\theta_{31}}} = 1 \end{aligned} \tag{5.55}$$

Again, we end up with Eq. (5.47) for the GUT scale, with the bulk exotic mass (the mass of those coming from a 10 of $SU(5)$ if we allow the 5s and 10s to get different masses) being the only exotic mass entering the equation. As such, apart from the possibility that all bulk exotics get masses near the GUT scale (as previously discussed), we have two other possibilities:

- *All bulk exotics at the TeV scale:* In this case Eq. (5.54) tells us that $M_{GUT} \sim 1 \times 10^{11}$GeV. It may seem at first sight that such a low unification scale would lead to dangerous dimension 6 operators giving proton decay rates which are much faster than experimentally observed. However, in [106] a method has been pointed out for suppressing proton-decay in F-theory $SU(5)$ with hypercharge flux breaking. The idea is that since the dangerous operators involve the $SU(5)$ gauge bosons X,Y in trilinear couplings such as XQu^c, a computation would consist of firstly computing the trilinear coupling by using the wave-function overlap techniques of eg. [85], and then integrating out X,Y. The key is that the $SU(5)$ gauge bosons need not be localised on a matter curve, but can be spread out over S. As such, these fields feel the effect of hypercharge flux in a different way to those on matter curves, and this gives rise to a suppression of the integral. This way, we can in principle avoid rapid proton decay, even with a seemingly low unification scale. Even though this is the case, when all the bulk exotics are at the TeV scale the splitting of the gauge couplings is large, and $x > 1$. As such, this possibility must be ruled out and we must look at the next case.

- *Bulk exotics from 10s heavy, but those from 5s light:* As the singlets S and S' which give the 5 state mass through Eq. (5.22) can also give the 10s mass through Eq. (5.29), we reject the possibility of heavy 5s and light 10s. However, since the 10s can get mass from a different singlet in Eq. (5.30), it would seem that there is a possibility of giving this singlet a much bigger VEV, and keeping the 10s heavy whilst the 5s could be TeV exotics. If this was the case, we would once again have $M_{GUT} \sim 2 \times 10^{16}$GeV due to the fact that the 5s do not contribute to $\delta\beta$ of Eq. (5.52). If the splitting parameter x is calculated for this case with the spectrum of model 1, it turns out to be negative so again we must rule this case out. This means that for model 1, high energy bulk exotics are the only possibility, but on the contrary we will see that for model 2 these low energy bulk exotics are the only possibility. As pointed out previously, model 2 which has TeV scale exotics in its spectrum cannot be compatible with bulk exotics with masses close to the GUT scale, as $x > 1$ which is forbidden. However, if we have the bulk exotics which belong to 5s of $SU(5)$ at the TeV scale as described above, it turns out that the multiplicities of exotic states forced upon us by topological constraints make the couplings unify. If we take the mass of the exotics from 10s to be M_{GUT}, we find $x \sim 0.01$, corresponding to a splitting of approximately 0.2 percent. This effect is illustrated in Figure 5.5, which shows how the low energy bulk exotics are precisely what is needed to make the couplings unify. In addition to the $3(D+\overline{D})$, $2(H_u, H_d)$ exotics which are also at the TeV scale, this leads to a characteristic spectrum involving TeV vector-like pairs of d^c and H_d exotics, with the distinguishing feature that there will always be one more vector pair of H_d states than d^cs. (In the $\gamma = 4$ case, we have one pair of d^c states and two pairs of H_d states).

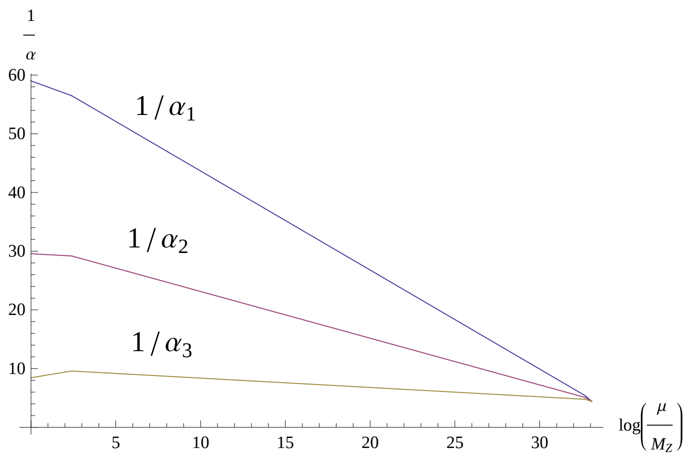

Figure 5.5: Gauge coupling unification in model 2 (E6SSM) with TeV scale bulk exotics.

In the presence of the large VEV for X_5, the F and D flatness equations of [2] must be modified accordingly. It can be shown (see Appendix E that there is a solution to the flatness relations for this model where X_5 gets a large VEV without giving rise to dangerous operators. In this section we have taken $\langle X \rangle = M_{GUT}$ for simplicity and to illustrate its effects, although in a full study it will be slightly lower, depending on the model building choices.

5.5 Conclusions

We have considered gauge coupling unification in E_6 F theory Grand Unified Theories (GUTs) where E_6 is broken to the Standard Model (SM) gauge group using fluxes. In such modes there are two types of exotics that can affect gauge coupling unification, namely matter exotics from the matter curves in the 27 dimensional representation of E_6 and the bulk exotics from the adjoint 78 dimensional representation of E_6. We then explored the conditions required for either the complete or partial removal of bulk exotics from the low energy spectrum.

We have examined the conditions for the removal from the low energy spectrum of bulk exotic matter from the adjoint of E_6 in terms of topological properties of the manifold. These conditions led to the fact that all vector-like pairs come in multiplicities which depend on one topological parameter, γ. We studied how the bulk exotics affect the one loop RGE anaylsis, and it was shown that both the GUT scale and the splitting of the gauge couplings depend on the mass of the exotics, but not on γ, meaning that the results are general for any E_6 F-theory model using fluxes to break the GUT group.

We then considered two realistic models E_6 proposed in [1, 2], which we called model 1 (MSSM) and model 2 (E6SSM). Both these model involve matter exotics and only differ by the mass scale of the matter exotics. These models were then supplemented by either high or low scale bulk exotics. It was demonstrated that if the bulk exotics acquire GUT scale masses, only model 1 (of [1]) is viable, with M_{GUT} being lowered slightly (<15 percent), and the splitting of the gauge couplings being increased by less than 1 percent.

It was then proposed that the bulk exotics from 5s of $SU(5)$ could get TeV scale masses whereas those from 10s could be near the GUT scale due to a large VEV for a singlet charged under $U(1)_\psi$ and $U(1)_\chi$. In this case model 1 cannot be realised, but it turns out to be the perfect solution to unify the gauge couplings of model 2 [2], which without the bulk exotics would be ruled out due to an unacceptable splitting of the couplings. Here we have a characteristic spectrum involving TeV vector-like pairs of d^c and H_d exotics, with the distinguishing feature that there will always be one more vector pair of H_d states than d^cs.

In summary, it is possible that all bulk exotic as well as matter exotics could have masses close to the GUT scale leading to an MSSM type theory somewhat below the GUT scale. However, there is the possibility that the bulk exotics from 5s of $SU(5)$ could get TeV scale masses whereas those from 10s could be near the GUT scale. Although such bulk exotics would by themselves spoil gauge coupling unification, when combined with matter exotics, corresponding to model 2 with complete 27 dimensional representations of E_6 at the TeV scale, gauge coupling unification is restored. We emphasise that, without such bulk exotics, the TeV scale matter exotics of the E6SSM would lead to an unacceptable splitting of the couplings, and it is only the combination of TeV scale matter exotics from the 27s plus TeV scale bulk exotics from the 78 which, almost miraculously, restores gauge coupling unification.

In the case of the E6SSM with TeV scale bulk exotics the resulting TeV scale matter exotics plus bulk exotics is equivalent to four extra $5+\overline{5}$ vector pairs of $SU(5)$, beyond the minimal supersymmetric standard model (MSSM) spectrum. The characteristic prediction of F-theory E_6 GUTs of the matter content of four extra $5+\overline{5}$ vector pairs can be tested at the LHC. This may be compared to the equivalent of three extra $5+\overline{5}$ vector pairs predicted by the E6SSM [75, 88].

Chapter 6

Conclusions

This thesis has been concerned with the study of semi-local F-theory GUTs arising from a single E_8 point of local enhancement, leading to simple GUT groups based on E_6, $SO(10)$ and $SU(5)$. In Chapter 2 the $SU(3)$, $SU(4)$ and $SU(5)$ spectral covers associated with these GUT groups were analysed. Assuming the minimal $\mathscr{Z}_2$ monodromy, the homology classes and associated spectra after flux breaking were determined for each case. Furthermore, a group theory dictionary was established between the different GUTs, providing a very useful tool for model building.

Using these results, in Chapter 3 an E_6 based model was constructed that demonstrates, for the first time, that it is possible to construct a phenomenologically viable model which leads to the MSSM at low energies. In this model, the exotics that result from flux breaking all get a large mass when singlet fields acquire vacuum expectation values driven by F and D flatness. Due to the underlying GUT symmetry and the $U(1)$s descending from E_8, baryon- and lepton-number violating terms are forbidden up to and including dimension 5, and as a result, nucleon decay is naturally suppressed below present bounds. The μ-term can be generated by non-perturbative $U(1)$ breaking effects. After including the effect of flux and instanton corrections, acceptable quark and charged lepton masses and mixing angles can be obtained. Neutrinos get a mass from the see-saw mechanism through their coupling to singlet neutrinos that acquire large Majorana mass as a result of the monodromy.

In Chapter 4, F-theory models were explored in which the low energy supersymmetric theory contains the particle content of three 27 dimensional representations of the underlying E_6 gauge group, plus two extra right-handed neutrinos predicted from F and D flatness. The resulting TeV scale effective theory resembles either the E6SSM or the NMSSM+, depending on whether an additional Abelian gauge group does or does not survive. However there are novel features compared to both these models as follows:
(i) If the additional Abelian gauge group is unbroken then it can have a weaker gauge coupling than in the E6SSM;
(ii) If the additional Abelian gauge group is broken then non-perturbative effects can violate the

scale invariance of the NMSSM+ leading to a generalised model;
(iii) Unification is achieved in the E6SSM by the addition of doublet states from incomplete 27 representations, whereas in the F-theory case, there is a splitting of the gauge couplings at the unification scale, although bulk exotics from the adjoint representation are always present which can lead to precise unification in a natural way;
(iv) Proton decay is suppressed by the geometric coupling suppression of a singlet state, which effectively suppresses the coupling of the exotic charge $-1/3$ colour triplet state D to quarks and leptons;
(v) The $\overline{D}$ decays as a chiral leptoquark with couplings to left-handed quarks and leptons, providing characteristic and striking signatures at the LHC.

In Chapter 5 the issues of bulk exotics were studied, which Chapter 4 hinted at as being essential for acceptable gauge coupling unification in models with light exotics. The chapter considered the general setting of gauge coupling unification in E_6 F-Theory GUTs where E_6 is broken to the Standard Model gauge group using fluxes. In such models there are two types of exotics that can affect gauge coupling unification, namely matter exotics from the matter curves in the 27 dimensional representation of E_6, and the bulk exotics from the adjoint 78 dimensional representation of E_6. Exploring the conditions required for either the complete or partial removal of bulk exotics from the low energy spectrum leads to the conclusion that there are always bulk exotics present. The multiplicities of these exotics are constrained by topological properties of the internal manifold, and exotic spectra were presented which are compatible with the models of Chapters 3 and 4. It was shown that (almost miraculously) gauge coupling unification may be possible even if there are bulk exotics at the TeV scale. Indeed in some cases it is necessary for bulk exotics to survive to the TeV scale in order to cancel the effects coming from other TeV scale matter exotics which would by themselves spoil gauge coupling unification. The combination of matter and bulk exotics in these cases can lead to precise gauge coupling unification which would not be possible with either type of exotics considered by themselves. The combination of matter and bulk exotics at the TeV scale represents a unique and striking signature of E_6 F-theory GUTs that can be tested at the LHC.

In conclusion, the local F-theory models studied here provide a good framework for phenomenology, and the existence of consistent models with low energy exotics can even give interesting glimpses into possible LHC signatures.

Appendix A

Group theory of E_6

With the nodes of the dynkin diagram of E_6 labelled as in Figure A.1, the simple roots of E_6 are [107]

$$
\begin{aligned}
\alpha_1 &= (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}) \\
\alpha_2 &= (-1, 1, 0, 0, 0, 0) \\
\alpha_3 &= (0, -1, 1, 0, 0, 0) \\
\alpha_4 &= (0, 0, -1, 1, 0, 0) \\
\alpha_5 &= (0, 0, 0, -1, 1, 0) \\
\alpha_6 &= (1, 1, 0, 0, 0, 0)
\end{aligned}
\tag{A.1}
$$

The Cartan matrix is given by $A_{ij} = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j}$ where α_i and α_j are simple roots. (Positive roots are defined by the first non zero entry from the right being positive).

$$
A = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
$$

The fundamental weights μ_i are a dual basis to the simple roots, defined by $\frac{\mu_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} = \frac{1}{2}\delta^{ij}$. They are given by

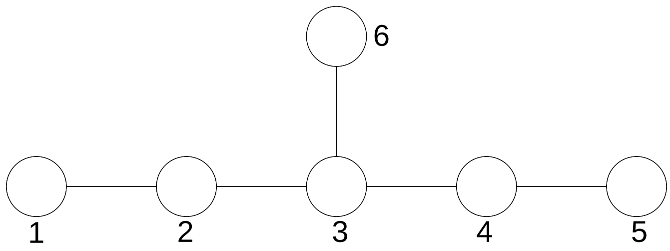

Figure A.1: Dynkin diagram for E_6 with labelled simple roots

$$\begin{aligned}
\mu_1 &= (0,0,0,0,0,\frac{2}{\sqrt{3}}) \\
\mu_2 &= (-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{2\sqrt{3}}) \\
\mu_3 &= (0,0,1,1,1,\sqrt{3}) \\
\mu_4 &= (0,0,0,1,1,\frac{2}{\sqrt{3}}) \\
\mu_5 &= (0,0,0,0,1,\frac{1}{\sqrt{3}}) \\
\mu_6 &= (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{\sqrt{3}}{2})
\end{aligned} \tag{A.2}$$

There is an $SU(2)$ algebra associated with each root, and this is encoded in the master formula

$$\frac{\mu\cdot\alpha}{\alpha\cdot\alpha} = -\frac{1}{2}(p-q) \tag{A.3}$$

where p represents the maximum number of times the root α can be added to the weight μ to get another weight, and q represents how many times it can be subtracted. The highest weight of the 27 representation of E_6 is such that $\mu+\alpha_i$ is not a root for any simple root, and as such the highest weight is $w_1 = \mu_1$. Now the master formula can be used to build up a picture of the weight space of the 27 representation

		Fundamental weight basis	Normal basis
w_1	μ_1	$(1,0,0,0,0,0)$	$(0,0,0,0,0,\frac{2}{\sqrt{3}})$
$w_2 = w_1 - \alpha_1$	$\mu_1 - \alpha_1$	$(-1,1,0,0,0,0)$	$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_3 = w_2 - \alpha_2$	$\mu_1 - \alpha_1 - \alpha_2$	$(0,-1,1,0,0,0)$	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_4 = w_3 - \alpha_3$	$\mu_1 - \alpha_1 - \alpha_2 - \alpha_3$	$(0,0,-1,1,0,1)$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_5 = w_4 - \alpha_6$	$\mu_1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_6$	$(0,0,0,1,0,-1)$	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_6 = w_4 - \alpha_4$	$\mu_1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	$(0,0,0,-1,1,1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_7 = w_6 - \alpha_6$ $= w_5 - \alpha_4$	$\mu_1 - \alpha_1 - \alpha_2$ $-\alpha_3 - \alpha_4 - \alpha_6$	$(0,0,1,-1,1,-1)$	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_8 = w_6 - \alpha_5$	$\mu_1 - \alpha_1 - \alpha_2$ $-\alpha_3 - \alpha_4 - \alpha_5$	$(0,0,0,0,-1,1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_9 = w_7 - \alpha_3$	$\mu_1 - \alpha_1 - \alpha_2$ $-2\alpha_3 - \alpha_4 - \alpha_6$	$(0,1,-1,0,1,0)$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{10} = w_8 - \alpha_6$ $= w_7 - \alpha_5$	$\mu_1 - \alpha_1 - \alpha_2 - \alpha_3$ $-\alpha_4 - \alpha_5 - \alpha_6$	$(0,0,1,0,-1,-1)$	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{11} = w_9 - \alpha_2$	$\mu_1 - \alpha_1 - 2\alpha_2$ $-2\alpha_3 - \alpha_4 - \alpha_6$	$(1,-1,0,0,1,0)$	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{12} = w_{10} - \alpha_3$ $= w_9 - \alpha_5$	$\mu_1 - \alpha_1 - \alpha_2 - 2\alpha_3$ $-\alpha_4 - \alpha_5 - \alpha_6$	$(0,1,-1,1,-1,0)$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{13} = w_{11} - \alpha_1$	$\mu_1 - 2\alpha_1 - 2\alpha_2$ $-2\alpha_3 - \alpha_4 - \alpha_6$	$(-1,0,0,0,1,0)$	$(0,0,0,0,1,-\frac{1}{\sqrt{3}})$
$w_{14} = w_{11} - \alpha_5$ $= w_{12} - \alpha_2$	$\mu_1 - \alpha_1 - 2\alpha_2 - 2\alpha_3$ $-\alpha_4 - \alpha_5 - \alpha_6$	$(1,-1,0,1,-1,0)$	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{15} = w_{12} - \alpha_4$	$\mu_1 - \alpha_1 - \alpha_2 - 2\alpha_3$ $-2\alpha_4 - \alpha_5 - \alpha_6$	$(0,1,0,-1,0,0)$	$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{16} = w_{13} - \alpha_5$ $= w_{14} - \alpha_1$	$\mu_1 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$ $-\alpha_4 - \alpha_5 - \alpha_6$	$(-1,0,0,1,-1,0)$	$(0,0,0,1,0,-\frac{1}{\sqrt{3}})$
$w_{17} = w_{14} - \alpha_4$ $= w_{15} - \alpha_2$	$\mu_1 - \alpha_1 - 2\alpha_2 - 2\alpha_3$ $-2\alpha_4 - \alpha_5 - \alpha_6$	$(1,-1,1,-1,0,0)$	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{18} = w_{16} - \alpha_4$ $= w_{17} - \alpha_1$	$\mu_1 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$ $-2\alpha_4 - \alpha_5 - \alpha_6$	$(-1,0,1,-1,0,0)$	$(0,0,1,0,0,-\frac{1}{\sqrt{3}})$
$w_{19} = w_{17} - \alpha_3$	$\mu_1 - \alpha_1 - 2\alpha_2 - 3\alpha_3$ $-2\alpha_4 - \alpha_5 - \alpha_6$	$(1,0,-1,0,0,1)$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{20} = w_{18} - \alpha_3$ $= w_{19} - \alpha_1$	$\mu_1 - 2\alpha_1 - 2\alpha_2 - 3\alpha_3$ $-2\alpha_4 - \alpha_5 - \alpha_6$	$(-1,1,-1,0,0,1)$	$(0,1,0,0,0,-\frac{1}{\sqrt{3}})$
$w_{21} = w_{19} - \alpha_6$	$\mu_1 - \alpha_1 - 2\alpha_2 - 3\alpha_3$ $-2\alpha_4 - \alpha_5 - 2\alpha_6$	$(1,0,0,0,0,-1)$	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2\sqrt{3}})$
$w_{22} = w_{21} - \alpha_1$ $= w_{20} - \alpha_6$	$\mu_1 - 2\alpha_1 - 2\alpha_2 - 3\alpha_3$ $-2\alpha_4 - \alpha_5 - 2\alpha_6$	$(-1,1,0,0,0,-1)$	$(-1,0,0,0,0,-\frac{1}{\sqrt{3}})$
$w_{23} = w_{20} - \alpha_2$	$\mu_1 - 2\alpha_1 - 3\alpha_2 - 3\alpha_3$ $-2\alpha_4 - \alpha_5 - \alpha_6$	$(0,-1,0,0,0,1)$	$(1,0,0,0,0,-\frac{1}{\sqrt{3}})$
$w_{24} = w_{23} - \alpha_6$ $= w_{22} - \alpha_2$	$\mu_1 - 2\alpha_1 - 3\alpha_2 - 3\alpha_3$ $-2\alpha_4 - \alpha_5 - 2\alpha_6$	$(0,-1,1,0,0,-1)$	$(0,-1,0,0,0,-\frac{1}{\sqrt{3}})$
$w_{25} = w_{24} - \alpha_3$	$\mu_1 - 2\alpha_1 - 3\alpha_2 - 4\alpha_3$ $-2\alpha_4 - \alpha_5 - 2\alpha_6$	$(0,0,-1,1,0,0)$	$(0,0,-1,0,0,-\frac{1}{\sqrt{3}})$
$w_{26} = w_{25} - \alpha_4$	$\mu_1 - 2\alpha_1 - 3\alpha_2 - 4\alpha_3$ $-3\alpha_4 - \alpha_5 - 2\alpha_6$	$(0,0,0,-1,1,0)$	$(0,0,0,-1,0,-\frac{1}{\sqrt{3}})$
$w_{27} = w_{26} - \alpha 5$	$\mu_1 - 2\alpha_1 - 3\alpha_2 - 4\alpha_3$ $-3\alpha_4 - 2\alpha_5 - 2\alpha_6$	$(0,0,0,0,-1,0)$	$(0,0,0,0,-1,-\frac{1}{\sqrt{3}})$

Table A.1: The weights of the 27 representation of E_6

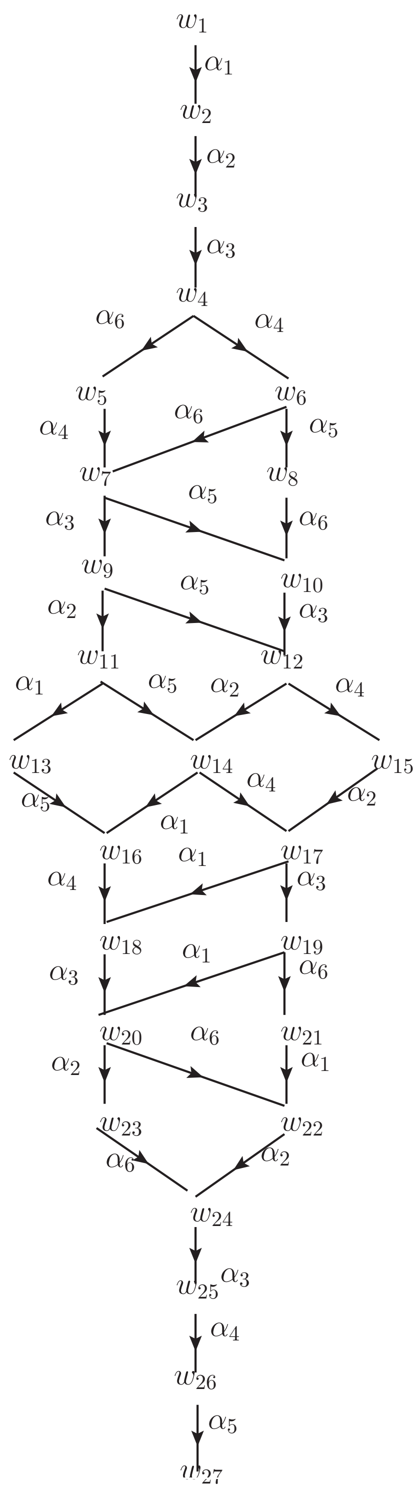

Figure A.2: The weights of the 27 representation of E_6

Appendix B

The homology classes of the Singlets

This Appendix is taken directly from [1]. In order to determine the homology classes for the singlets of a particular $SU(n)$, we first need to express the product of the the differences of the roots $t_i - t_j$ of the n^{th} degree polynomial $P_n = b_k s^{n-k}$ in terms of its coefficients b_k.

Consider first the simplest case $b_0 s^2 + b_1 s + b_2 = 0$. If t_1, t_2 are the roots, we know

$$(t_1 - t_2)(t_2 - t_1) \propto -\Delta = -b_1^2 + 4b_0 b_2$$

Note that the same result is obtained from the determinant

$$\frac{1}{b_0} \begin{vmatrix} b_0 & b_1 & b_2 \\ 2b_0 & 1b_1 & 0b_2 \\ 0 & 2b_0 & 1b_1 \end{vmatrix} = -b_1^2 + 4b_0 b_2 \tag{B.1}$$

We can easily repeat this procedure for the cubic roots.

Consider now the generalization for the $SU(4)$ case: According to known theorems (see theorem 2.5 of [108]) the required quantity is given by the Sylvester formula

$$\begin{vmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 \\ 4b_0 & 3b_1 & 2b_2 & b_3 & 0 & 0 & 0 \\ 0 & 4b_0 & 3b_1 & 2b_2 & b_3 & 0 & 0 \\ 0 & 0 & 4b_0 & 3b_1 & 2b_2 & b_3 & 0 \\ 0 & 0 & 0 & 4b_0 & 3b_1 & 2b_2 & b_3 \end{vmatrix} \tag{B.2}$$

If these are the roots of $SU(4)$, we have $b_1 = \sum_i t_i = 0$ and we get the result (2.91).

The extension to $SU(5)$ is straightforward. It can be computed form the determinant

$$\begin{vmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ 5b_0 & 4b_1 & 3b_2 & 2b_3 & b_4 & 0 & 0 & 0 & 0 \\ 0 & 5b_0 & 4b_1 & 3b_2 & 2b_3 & b_4 & 0 & 0 & 0 \\ 0 & 0 & 5b_0 & 4b_1 & 3b_2 & 2b_3 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 5b_0 & 4b_1 & 3b_2 & 2b_3 & b_4 & 0 \\ 0 & 0 & 0 & 0 & 5b_0 & 4b_1 & 3b_2 & 2b_3 & b_4 \end{vmatrix} \tag{B.3}$$

Setting $b_1 = 0$ we obtain the result quoted in the text.

Appendix C

Anomaly cancellation in E6 models

It has been noted in [109] that in models with multiple perpendicular U(1) symmetries, there is a $U(1)_Y - U(1) - U(1)$ anomaly which is not automatically cancelled through the spectral cover approach. In order for this anomaly to be cancelled, the following condition is required:

$$3\sum_{C_{10}^i}(Q_{10}^i)^A(Q_{10}^i)^B N_{10}^i + \sum_{C_5^j}(Q_5^j)^A(Q_5^j)^B N_5^j = 0 \tag{C.1}$$

where the sums are over all the 10 and 5 matter curves, Q denotes the charge under either the U(1) labelled A or the one labelled B (allowing for mixed anomalies in the case of multiple U(1)s), and the Ns refer to the chirality induced by hypercharge flux. In the models considered in this thesis, we have 3 U(1)s, with generators

$$Q_\chi = \frac{1}{2\sqrt{10}}\text{diag}(-1,-1,-1,-1,4) \tag{C.2}$$

$$Q_\psi = \frac{1}{2\sqrt{6}}\text{diag}(1,1,1,-3,0) \tag{C.3}$$

$$Q_\perp = \frac{1}{2\sqrt{3}}\text{diag}(1,1,-2,0,0) \tag{C.4}$$

As such, we can tabulate the U(1) charges of all the 5 and 10 curves in the model

We can now check if Eq. (C.1) holds for all the combinations of $A,B = \chi,\psi,\perp$ in $U(1)_Y - U(1)^A - U(1)^B$. Plugging in the charges and the N_Y values from Table C.1 into the left hand side of Eq. (C.1) gives

Curve	Q_χ	Q_ψ	$Q_\perp$	N_Y
10_M	$-\frac{1}{2\sqrt{10}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2\sqrt{3}}$	-1
10_2	$-\frac{1}{2\sqrt{10}}$	$\frac{1}{2\sqrt{6}}$	$-\frac{1}{\sqrt{3}}$	1
5_{H_u}	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{3}}$	1
5_1	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{6}}$	$\frac{1}{2\sqrt{3}}$	-1
5_2	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{2\sqrt{3}}$	-1
5_3	$-\frac{3}{2\sqrt{10}}$	$-\frac{1}{2\sqrt{6}}$	$-\frac{1}{2\sqrt{3}}$	-1
5_4	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	1
5_5	$-\frac{3}{2\sqrt{10}}$	$-\frac{1}{2\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	1

Table C.1: U(1) charges of the 10 and 5 matter curves

$$A=\chi, B=\chi \to 3[-\frac{1}{40}+\frac{1}{40}]+[\frac{1}{10}-\frac{1}{10}-\frac{1}{10}-\frac{9}{40}+\frac{1}{10}+\frac{9}{40}]=0$$
$$A=\psi, B=\psi \to 3[-\frac{1}{24}+\frac{1}{24}]+[\frac{1}{6}-\frac{1}{6}-\frac{1}{6}-\frac{1}{24}+\frac{1}{6}+\frac{1}{24}]=0$$
$$A=\chi, B=\psi \to 3[\frac{1}{4\sqrt{60}}-\frac{1}{4\sqrt{60}}]+[-\frac{1}{\sqrt{60}}+\frac{1}{\sqrt{60}}-\frac{1}{\sqrt{60}}-\frac{3}{4\sqrt{60}}+\frac{1}{\sqrt{60}}+\frac{3}{4\sqrt{60}}]=0$$

This shows that the relation is indeed obeyed for the cases $U(1)_Y - U(1)^\chi - U(1)^\chi$, $U(1)_Y - U(1)^\psi - U(1)^\psi$ and $U(1)_Y - U(1)^\chi - U(1)^\psi$. (This was to be expected, as $U(1)_\chi$ and $U(1)_\psi$ are both embedded in E6). However, for the 3 anomalies involving $U(1)_\perp$, Eq. (C.1) is not satisfied, meaning that the anomalies involving $U(1)_\perp$ are not cancelled. However, it should be noted that [95] points out that anomaly cancellation constraints can be relaxed in the case of geometrically massive U(1)s in F-theory.

Appendix D

Topological relations arising from the elimination of bulk exotics

The requirement that each type of exotic matter occurs in vector pairs is given by $n_j - n_j^* = 0$. The extra requirement which would mean that this type of exotic is completely eliminated from the spectrum is $n_j + n_j^* = 0$. These requirements are given here for each type of exotic. Note that not all of these relations can be satisfied at once, and are written here on the assumption that a subset of them will be satisfied.

$$n_1 - n_1^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_1) = 0 \tag{D.1}$$

$$n_1 + n_1^* = 0 \Rightarrow c_1(\mathscr{L}_1)^2 = -2 \tag{D.2}$$

$$n_2 - n_2^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_2) = 0 \tag{D.3}$$

$$n_2 + n_2^* = 0 \Rightarrow c_1(\mathscr{L}_2)^2 = -2 \tag{D.4}$$

$$n_3 - n_3^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_1) = c_1(S) \cdot c_1(\mathscr{L}_2) \tag{D.5}$$

$$n_3 + n_3^* = 0 \Rightarrow c_1(\mathscr{L}_1)^2 + c_1(\mathscr{L}_2)^2 = -2 \tag{D.6}$$

$$n_4 - n_4^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_1) = -c_1(S) \cdot c_1(\mathscr{L}_2) \tag{D.7}$$

$$n_4 + n_4^* = 0 \Rightarrow c_1(\mathscr{L}_1)^2 + c_1(\mathscr{L}_2)^2 = -2 \tag{D.8}$$

$$n_5 - n_5^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_3) = c_1(S) \cdot c_1(\mathscr{L}_2) \tag{D.9}$$
$$n_5 + n_5^* = 0 \Rightarrow c_1(\mathscr{L}_2)^2 + c_1(\mathscr{L}_3)^2 = -2 \tag{D.10}$$

$$n_6 - n_6^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_2) = -c_1(S) \cdot c_1(\mathscr{L}_3) \tag{D.11}$$
$$n_6 + n_6^* = 0 \Rightarrow c_1(\mathscr{L}_2)^2 + c_1(\mathscr{L}_3)^2 = -2 \tag{D.12}$$

$$n_7 - n_7^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_1) = c_1(S) \cdot c_1(\mathscr{L}_2) + c_1(S) \cdot c_1(\mathscr{L}_3) \tag{D.13}$$
$$n_7 + n_7^* = 0 \Rightarrow c_1(\mathscr{L}_1)^2 + c_1(\mathscr{L}_2)^2 + c_1(\mathscr{L}_3)^2 = -2 \tag{D.14}$$

$$n_8 - n_8^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_1) = -c_1(S) \cdot c_1(\mathscr{L}_3) \tag{D.15}$$
$$n_8 + n_8^* = 0 \Rightarrow c_1(\mathscr{L}_1)^2 + c_1(\mathscr{L}_3)^2 = -2 \tag{D.16}$$

$$n_9 - n_9^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_3) = 0 \tag{D.17}$$
$$n_9 + n_9^* = 0 \Rightarrow c_1(\mathscr{L}_3)^2 = -2 \tag{D.18}$$

$$n_{10} - n_{10}^* = 0 \Rightarrow c_1(S) \cdot c_1(\mathscr{L}_1) = c_1(S) \cdot c_1(\mathscr{L}_3) \tag{D.19}$$
$$n_{10} + n_{10}^* = 0 \Rightarrow c_1(\mathscr{L}_1)^2 + c_1(\mathscr{L}_3)^2 = -2 \tag{D.20}$$

Appendix E

F and D flatness conditions with bulk exotics

In the language of Table 4.1, the singlets X_5 and $\overline{X}_5$ correspond to θ_{45} and θ_{54} respectively. As these singlets get GUT scale VEVs in the E6SSM model, we must check that this is compatible with the F- and D-flatness conditions. The D-flatness condition for $U_A(1)$ is

$$\begin{aligned} \sum Q^A_{ij}(\left|\langle\theta_{ij}\rangle\right|^2 - \left|\langle\theta_{ji}\rangle\right|^2) &= -\frac{TrQ^A}{192\pi^2}g_s^2 M_S^2 \\ &= -XTrQ^A \end{aligned} \tag{E.1}$$

This condition must be checked for all the U(1)s, the charge generators of which are given in the form $Q = \text{diag}[t_1, t_2, t_3, t_4, t_5]$ by

$$Q_\chi \propto \text{diag}[-1,-1,-1,-1,4] \tag{E.2}$$

$$Q_\psi \propto \text{diag}[1,1,1,-3,0] \tag{E.3}$$

$$Q_\perp \propto \text{diag}[1,1,-2,0,0] \tag{E.4}$$

We can see immediately that if $\langle\theta_{45}\rangle = \langle\theta_{54}\rangle = M_{GUT}$, the presence of these VEVs will not affect the D-flatness relations due to the relative minus sign in Eq. (E.1). As such, it is only necessary to check the conditions for F-flatness. As in the E6SSM model θ_{31} and θ_{53} get large VEVs while θ_{34} gets a TeV scale VEV, the only new problematic terms in the superpotential are

$$W_\theta = \lambda_{ijk}\theta^i_{45}\theta^j_{53}\theta^k_{34} + M_{ab}\theta^a_{45}\theta^b_{54} \tag{E.5}$$

As such, the F-flatness equations will be satisfied provided the following conditions are satisfied

$$\frac{\partial W_\theta}{\partial \theta_{34}^k} = \lambda_{ijk} \left\langle \theta_{45}^i \right\rangle \left\langle \theta_{53}^j \right\rangle = 0$$
$$\frac{\partial W_\theta}{\partial \theta_{53}^j} = \lambda_{ijk} \left\langle \theta_{45}^i \right\rangle \left\langle \theta_{34}^k \right\rangle = 0$$
$$\frac{\partial W_\theta}{\partial \theta_{45}^i} = \lambda_{ijk} \left\langle \theta_{53}^j \right\rangle \left\langle \theta_{34}^k \right\rangle + M_{ib} \left\langle \theta_{54}^b \right\rangle = 0$$

Due to the model building freedom we have in the number of singlet fields and the fact that the number of θ_{45} and θ_{54} fields in the spectrum can be changed by looking at topological relations where $\gamma > 4$ in Eq. (5.34), these F-flatness relations can always be satisfied in realisations of the E6SSM-like model.

Appendix F

Overlap integrals

As in [85], the equations for a massless 4-dimensional fermionic field are

$$\mathscr{D}\psi = 0 \tag{F.1}$$

with

$$\mathscr{D}\psi = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & D_3^\dagger & -D_2^\dagger \\ -D_2 & -D_3^\dagger & 0 & D_1^\dagger \\ -D_3 & D_2^\dagger & -D_1^\dagger & 0 \end{pmatrix} \psi = 0 \ , \psi = \begin{pmatrix} \sqrt{2}\eta \\ \psi_{\bar{1}} \\ \psi_{\bar{1}} \\ \chi \end{pmatrix}$$

The covariant derivatives are given by

$$D_i = \frac{M}{R_\parallel}(\partial_i - \frac{1}{2}q_a M^a_{ij}\bar{z}_j) \tag{F.2}$$

$$D_i^\dagger = \frac{M}{R_\parallel}(\partial_i + \frac{1}{2}q_a (M^a_{ij})^* z_j) \tag{F.3}$$

$$D_3 = -MR_\perp m^a_i \bar{z}_i \tag{F.4}$$

$$D_3^\dagger = mR_\perp (m^a_i)^* z_i \tag{F.5}$$

where i=1,2

The equations for massive modes are given by

$$\mathscr{D}^\dagger \mathscr{D} = |m_\lambda|^2 \psi \tag{F.6}$$

Working out $\mathscr{D}^\dagger\mathscr{D}$ gives

$$\mathscr{D}^\dagger\mathscr{D} = -\sum_i D_i^\dagger D_i \mathbb{I} + \mathbb{B} \tag{F.7}$$

where $\mathbb{B}$ is given by

$$\mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & [D_2^\dagger D_2] + [D_3^\dagger D_3] & [D_2, D_1^\dagger] & [D_3, D_1^\dagger] \\ 0 & [D_1, D_2^\dagger] & [D_1^\dagger D_1] + [D_3^\dagger D_3] & [D_3, D_2^\dagger] \\ 0 & [D_1, D_3^\dagger] & [D_2, D_3^\dagger] & [D_1^\dagger D_1] + [D_2^\dagger D_2] \end{pmatrix} \tag{F.8}$$

Given that $[D_3^\dagger D_3] = 0$, this reduces to Eq.(2.28) of [85]. A basis change can now be made as in Eq.(2.29), and so the rotated $\mathbb{D}$ is given by Eq.(2.30)

$$\tilde{\mathbb{D}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_{1,1} & \varepsilon_{1,2} & \varepsilon_{1,3} \\ 0 & \varepsilon_{2,1} & \varepsilon_{2,2} & \varepsilon_{2,3} \\ 0 & \varepsilon_{3,1} & \varepsilon_{3,2} & \varepsilon_{3,3} \end{pmatrix} \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & D_3^\dagger & -D_2^\dagger \\ -D_2 & -D_3^\dagger & 0 & D_1^\dagger \\ -D_3 & D_2^\dagger & -D_1^\dagger & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} \\ 0 & \varepsilon_{1,2} & \varepsilon_{2,2} & \varepsilon_{3,2} \\ 0 & \varepsilon_{1,3} & \varepsilon_{2,3} & \varepsilon_{3,3} \end{pmatrix}$$

As an example, the (2,3) element of $\tilde{\mathbb{D}}$ can be calculated as follows

$$\begin{aligned} \tilde{\mathbb{D}}_{2,3} &= \varepsilon 1,1(\varepsilon_{2,2} D_3^\dagger - \varepsilon_{2,3} D_2^\dagger) + \varepsilon 1,2(-\varepsilon_{2,1} D_3^\dagger + \varepsilon_{2,3} D_1^\dagger) + \varepsilon 1,3(\varepsilon_{2,1} D_2^\dagger - \varepsilon_{2,2} D_1^\dagger) \\ &= D_1^\dagger(\varepsilon_{1,2}\varepsilon_{2,3} - \varepsilon_{1,3}\varepsilon_{2,3}) + D_2^\dagger(\varepsilon_{1,3}\varepsilon_{2,1} - \varepsilon_{1,1}\varepsilon_{2,3}) + D_3^\dagger(\varepsilon_{1,1}\varepsilon_{2,2} - \varepsilon_{1,2}\varepsilon_{2,1}) \\ &= \sum_k \varepsilon_{3,k} D_k^\dagger \end{aligned} \tag{F.9}$$

where $\varepsilon_{1,2,3}$ are the normalised eigenvectors of $\mathbb{B}$. Performing similar analysis for the other elements gives the matrix $\tilde{\mathbb{D}}$ as

$$\tilde{\mathbb{D}} = \begin{pmatrix} 0 & \tilde{D}_1 & \tilde{D}_2 & \tilde{D}_3 \\ -\tilde{D}_1 & 0 & \tilde{D}_3^\dagger & -\tilde{D}_2^\dagger \\ -\tilde{D}_2 & -\tilde{D}_3^\dagger & 0 & \tilde{D}_1^\dagger \\ -\tilde{D}_3 & \tilde{D}_2^\dagger & -\tilde{D}_1^\dagger & 0 \end{pmatrix}$$

where $\tilde{D_p}$ are given as in Eq.(2.31) of [85], by

$$\tilde{D}_p = \sum_k \varepsilon_{p,k} D_k \tag{F.10}$$

Putting this result for $\tilde{\mathbb{D}}$ into Eq.(2.33) gives the result

$$\begin{aligned} \tilde{D}_1 \varphi &= 0 \\ \tilde{D}_2^\dagger \varphi &= 0 \\ \tilde{D}_3^\dagger \varphi &= 0 \end{aligned} \tag{F.11}$$

This defines how we should define the raising and lowering operators. As φ is annihilated by $\tilde{D}_1$, but by the daggered operators $\tilde{D}_2^\dagger$ and $\tilde{D}_3^\dagger$, the lowering operators $a_{1,2,3}$ are identified in Eq.(2.34) and the raising operators by the conjugates. This is the reason why in Eq.(2.37) the action of three raising operators on the ground state involves one dagger, and two non-daggers.

Using Eq.(2.32) (and bearing in mind that $\lambda_1 + \lambda_2 + \lambda_3 = 0$), for fields transforming in a particular representation of the gauge group, the operator $\tilde{\mathbb{D}}^\dagger \tilde{\mathbb{D}}$ is given by

$$\tilde{\mathbb{D}}^\dagger \tilde{\mathbb{D}} = -(\tilde{D}_1^\dagger \tilde{D}_1 + \tilde{D}_2^\dagger \tilde{D}_2 + \tilde{D}_3^\dagger \tilde{D}_3)\mathbb{I} + \tilde{\mathbb{B}} \tag{F.12}$$

$$= -(\tilde{D}_1^\dagger \tilde{D}_1 + \tilde{D}_2 \tilde{D}_2^\dagger + \tilde{D}_3 \tilde{D}_3^\dagger)\mathbb{I} + ([\tilde{D}_2, \tilde{D}_2^\dagger] + [\tilde{D}_3, \tilde{D}_3^\dagger])\mathbb{I} + \tilde{\mathbb{B}} \tag{F.13}$$

$$= (a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3)\mathbb{I} + (\frac{M}{R_\parallel})^2(\lambda_2 + \lambda_3)\mathbb{I} - (\frac{M}{R_\parallel})^2 diag(0, \lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2) \tag{F.14}$$

$$= (a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3)\mathbb{I} + (\frac{M}{R_\parallel})^2 diag(-\lambda_1, 0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1) \tag{F.15}$$

As described in [85], massive wavefunctions involve both chiralities, and so muct have components which transform in the conjugate representation. This can be taken care of by changing the sign of the charges q_a in the definitions of the covariant derivatives.

F.1 The Set Up

In order to compute this diagram, we need to rotate the wavefunctions for the $\overline{5}_M$ and the 10_M curve from the z_1, z_2 basis, into the u,w basis used in the computation of the $\overline{5}_H$ wavefunction. Our model corresponds to the case of vanishing flux density, and so the wavefunctions are (leaving the normalisation factors out):

$$\psi^l_{\bar{5}_M} \sim (w-u)^l exp[-\frac{1}{2}((\rho_1+\tilde{M}_1)\,|w|^2+(\rho_1+\tilde{M}_1)\,|u|^2+(\rho_1-\tilde{M}_1)w\overline{u}+(\rho_1-\tilde{M}_1)u\overline{w})] \tag{F.16}$$

$$\psi^m_{10_M} \sim (-w-u)^m exp[-\frac{1}{2}((\rho_2+\tilde{M}_2)\,|w|^2+(\rho_2+\tilde{M}_2)\,|u|^2-(\rho_2-\tilde{M}_2)w\overline{u}-(\rho_2-\tilde{M}_2)u\overline{w})] \tag{F.17}$$

$$\psi^k_{\bar{5}_H} \sim exp[\frac{1}{\sqrt{v_+^2+v_-^2}}(-\frac{R}{v_+v_-}\,|v_+u+v_-w|^2+2i\mathrm{Im}(\mathrm{k}(-\mathrm{v}_-\mathrm{u}+\mathrm{v}_+\mathrm{w})))] \tag{F.18}$$

As the 5_H is on the same curve as the $\overline{5}_H$, its wavefunction is given by the complex conjugate:

$$\psi^k_{5_H} \sim exp[\frac{1}{\sqrt{v_+^2+v_-^2}}(-\frac{R}{v_+v_-}\,|v_+u+v_-w|^2-2i\mathrm{Im}(\mathrm{k}(-\mathrm{v}_-\mathrm{u}+\mathrm{v}_+\mathrm{w})))] \tag{F.19}$$

For the case of non-vanishing flux density, the 5_H wavefunction does not include the $2i\mathrm{Im}(\mathrm{k}(-\mathrm{v}_-\mathrm{u}+\mathrm{v}_+\mathrm{w}))$ part inside the exponential. In this case, there is no conserved KK momentum, and the exponential factor of the wavefunction is invariant under the global U(1):

$$u \to e^{i\theta}u,\ w \to e^{i\theta}w \tag{F.20}$$

Clearly the exponentials in the wavefunctions of our case of vanishing flux density are not invariant under this geometric U(1). However, we aim to show that by taking the example of Figure 3.2, for a complete diagram the U(1) is restored (at least approximately). In order to understand this in more detail, it is instructive to derive the 5_H wavefunction.

F.2 Derivation of the Vanishing Flux Density 5_H Wavefunction

In the notation of CDP, the covariant derivatives are

$$D_u = \frac{M_\star}{R_\parallel}\partial_u,\ D_w = \frac{M_\star}{R_\parallel}\partial_w,\ D_3 = -M_\star R_\perp(\frac{\overline{w}}{v_+}+\frac{\overline{u}}{v_-})$$
$$D_u^\dagger = \frac{M_\star}{R_\parallel}\overline{\partial_u},\ D_w^\dagger = \frac{M_\star}{R_\parallel}\overline{\partial_w},\ D_3^\dagger = M_\star R_\perp(\frac{w}{v_+}+\frac{u}{v_-})$$

The matrix $\mathbb{B}$ is given by (with $R = R_\parallel R_\perp$)

$$\mathbb{B} = \frac{M_\star^2}{R_\parallel^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{R}{v_-} \\ 0 & 0 & 0 & \frac{R}{v_+} \\ 0 & \frac{R}{v_-} & \frac{R}{v_+} & 0 \end{pmatrix} \tag{F.21}$$

with eigenvalues

$$\lambda_0 = \lambda_3 = 0,\ \lambda_1 = -\frac{M_\star^2}{R_\parallel^2} R\sqrt{\frac{1}{v_+^2} + \frac{1}{v_-^2}},\ \lambda_2 = \frac{M_\star^2}{R_\parallel^2} R\sqrt{\frac{1}{v_+^2} + \frac{1}{v_-^2}} \tag{F.22}$$

and corresponding eigenvectors

$$\xi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \xi_1 = \begin{pmatrix} 0 \\ v_+ \\ v_- \\ -\sqrt{v_+^2 + v_-^2} \end{pmatrix},\ \xi_2 = \begin{pmatrix} 0 \\ v_+ \\ v_- \\ \sqrt{v_+^2 + v_-^2} \end{pmatrix},\ \xi_3 = \begin{pmatrix} 0 \\ -v_- \\ v_+ \\ 0 \end{pmatrix} \tag{F.23}$$

In the basis where the matrix $\mathbb{B}$ is diagonal, the covariant derivatives are

$$\tilde{D}_1 = \frac{1}{|\xi_1|}\frac{M_\star}{R_\parallel}[v_+\partial_u + v_-\partial_w + R\sqrt{v_+^2 + v_-^2}(\frac{\overline{w}}{v_+} + \frac{\overline{u}}{v_-})] \tag{F.24}$$

$$\tilde{D}_2^\dagger = \frac{1}{|\xi_2|}\frac{M_\star}{R_\parallel}[v_+\overline{\partial}_u + v_-\overline{\partial}_w + R\sqrt{v_+^2 + v_-^2}(\frac{w}{v_+} + \frac{u}{v_-})] \tag{F.25}$$

$$\tilde{D}_3^\dagger = \frac{1}{|\xi_3|}\frac{M_\star}{R_\parallel}[-v_-\overline{\partial}_u + v_+\overline{\partial}_w] \tag{F.26}$$

with the following equations for the ground state wavefunction

$$\tilde{D}_1\varphi = 0, \quad \tilde{D}_2^\dagger\varphi = 0, \quad \tilde{D}_3^\dagger\varphi = \overline{k_{kk}}\varphi \tag{F.27}$$

where the appearance of k_{kk} in the third equation corresponds to the fact that $[\tilde{D}_3^\dagger, \tilde{D}_3] = 0$, and so there is a conserved quantum number associated with this commutator instead of a quantum harmonic oscillator. We can solve these equations by taking a general form for φ

$$\varphi = exp[-p_1^H|w|^2 - p_2^H|u|^2 + p_3^H\overline{w}u + p_4^H\overline{u}w + au + bw + c\overline{u} + d\overline{w}] \tag{F.28}$$

Plugging this form into the three above equations gives the result we used in the previous section, but with k_{kk} scaled by

$$k_{kk} \to \frac{R_{\parallel}}{M_{\star}} k_{kk} \tag{F.29}$$

F.3 Triple Wavefunction Overlaps

As the right hand vertex of the digram coorresponds to an E8 point of enhancement, the wavefunctions for the $\overline{5}_2$ and θ_{31} curves will have the same form (with some suitable approximation for the singlet wavefunction as a projection on the del Pezzo surface), but the cooeficients inside the exponentials will be different. Taking into account the fact that there are multiple states on the same curves, the $\overline{5}_2$ and θ_{31} wavefunctions will have some 'family' dependence. As such, we can write these wavefunctions, with general coefficients, as

$$\psi^{p}_{\overline{5}_2} \sim (a_w w + a_u u)^p exp[-A|w|^2 - B|u|^2 + Cw\overline{u} + Du\overline{w})] \tag{F.30}$$

$$\psi^{q}_{\theta_{31}} \sim (b_w w + b_u u)^q exp[-E|w|^2 - F|u|^2 + Gw\overline{u} + Hu\overline{w})] \tag{F.31}$$

The first thing to note is that we can calculate the triple wavefunction overlaps at each point

$$\int_S \psi^{l}_{\overline{5}_M} \psi^{m}_{10_M} \psi^{k}_{\overline{5}_H} \sim \int_S d^2u \wedge d^2w (w-u)^l (-w-u)^m exp[\alpha|w|^2 + \beta|u|^2 + \gamma w\overline{u} + \delta u\overline{w} + 2i\mathrm{Im}(\mathrm{k}(-\mathrm{v}_{-}\mathrm{u} + \mathrm{v}_{+}\mathrm{w}))] \tag{F.32}$$

$$\int_S \psi^{p}_{\overline{5}_2} \psi^{q}_{\theta_{31}} \psi^{k}_{\overline{5}_H} \sim \int_S d^2u \wedge d^2w (a_w w + a_u u)^p (b_w w + b_u u)^q [\alpha'|w|^2 + \beta'|u|^2 + \gamma' w\overline{u} + \delta' u\overline{w} - 2i\mathrm{Im}(\mathrm{k}(-\mathrm{v}_{-}\mathrm{u} + \mathrm{v}_{+}\mathrm{w}))] \tag{F.33}$$

The problem with the k dependent part of each expression is that it contains factors of u and w, which carry non zero $U(1)$ charge. If we were to expand each exponential, we would see that both even and odd terms involving the KK momentum k would cancel provided that $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$ and $\delta = \delta'$. However, this will generally not be true, due to the fact that the enhancement groups at each vertex are different, and so the wavefunction coefficients will also be different. The dangerous terms will always vanish however to first order.

F.4 Calculating the Diagram as a Six Wavefunction Overlap

If the coordinates of the SO(12) point are centered on the origin of S, the coordinates of the other vertex are centered on the point $r = (a,b)$ as in Figure F.1.

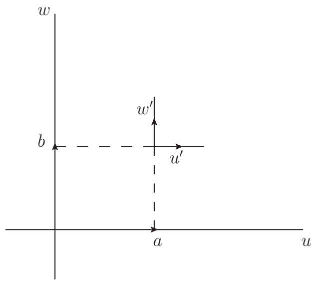

Figure F.1: Coordinates of the two vertices

As such, we can calculate the effective diagram for the operator $10_M\overline{5}_M\overline{5}_H\theta_{31}$ by writing down the position space Feynman rules, where the Feynman rule for each vertx is the triple overlap integral, and the propagator in position space is $\frac{e^{-M|r|}}{|r|}$. As such, we have integrals over S (u and w) and S' ($u' = u+a$ and $w' = w+b$).

$$
\begin{aligned}
G^{l,m,p,q} \sim & \int_S d^2u \wedge d^2w \int_{S'} d^2u' \wedge d^2w' (w-u)^l(-w-u)^m(a_w w' + b_u u')^p(b_w w' + b_u u')^q \\
& \times exp[\alpha |w|^2 + \beta |u|^2 + \gamma w\overline{u} + \delta u\overline{w} + 2i\mathrm{Im}(\mathrm{k}(-\mathrm{v_- u + v_+ w}))] \times \frac{\mathrm{e}^{-\mathrm{M}|(\mathrm{u'-u,w'-w})|}}{|(\mathrm{u'-u,w'-w})|} \\
& \times [\alpha' |w'|^2 + \beta' |u'|^2 + \gamma' w'\overline{u'} + \delta' u'\overline{w'} - 2i\mathrm{Im}(\mathrm{k}(-\mathrm{v_- u' + v_+ w'}))] \qquad \text{(F.34)}
\end{aligned}
$$

The Yukawa potential appears as it is the Fourier transform of the propagator

$$
\frac{e^{-M|r|}}{|r|} \sim \int d^4k \frac{1}{k^2+M^2} e^{ikr} \qquad \text{(F.35)}
$$

$$
\sim \int d^4k \frac{e^{ikr}}{M^2}\left(1 - \frac{k^2}{M^2} + ...\right) \qquad \text{(F.36)}
$$

where on the second line we have expanded for large M. We can now insert this expression into Eq. (F.34), and as such, we can perform the integrals over the two sets of coordinates as well as k. As an example, we will take the case where the wavefunction coefficients are $\alpha = \alpha' = -2$, $\beta = \beta' = -2$, $\gamma = \gamma' = 1$ and $\delta = \delta' = 1$ (which corresponds to both vertices looking like the usual picture of one curve along the u and w axes, and one along the line $u = w$). We can also take the case of the third family, so we do not have to worry about powers of u and w premultiplying the exponential, and matching these by introducing covariant derivatives. Looking at the term in Eq. (F.34) which corresponds to the first term in the expansion of Eq. (F.36), the integral over k will give a delta function, leading to

$$
\begin{aligned}
G^{(0)} &= \frac{1}{M^2}\int_S d^2u\wedge d^2w \int_{S'} d^2u'\wedge d^2w' exp[-2\,|w|^2-2\,|u|^2+w\bar{u}+u\overline{w} \\
&-2\left|w'\right|^2-2\left|u'\right|^2+w'\overline{u'}+u'\overline{w'}+2i\mathrm{Im}(\mathrm{k_{kk}}(-\mathrm{v}_-(\mathrm{u}-\mathrm{u}')+\mathrm{v}_+(\mathrm{w}-\mathrm{w}')))] \\
&\times(2\pi)^4\delta(u-u',w-w') \\
&= \frac{\pi^2}{3M^2}\times(2\pi)^4 \qquad \text{(F.37)}
\end{aligned}
$$

where the notation k_{kk} has now been introduced for the KK momentum, to distinguish it from the k in the Fourier transform of the propagator. Since the parts of the wavefunctions which violate the geometric $U(1)$ cancel, the $U(1)$ will be respected by these first order terms, when we consider families other than the third. We can now get the first order correction to this result by looking at the expression associated with the second term in the expansion of the propagator in Eq. (F.36). Now we consider a general family dependence:

$$
\begin{aligned}
G^{(1)} &= -\frac{1}{M^4}\int d^4k\int_S d^2u\wedge d^2w\int_{S'} d^2u'\wedge d^2w'\; k^2e^{ikr}(w-u)^l(-w-u)^m exp[-2\,|w|^2-2\,|u|^2+w\bar{u}+u\overline{w} \\
&-2\left|w'\right|^2-2\left|u'\right|^2+w'\overline{u'}+u'\overline{w'}+2i\mathrm{Im}(\mathrm{k_{kk}}(-\mathrm{v}_-(\mathrm{u}-\mathrm{u}')+\mathrm{v}_+(\mathrm{w}-\mathrm{w}')))] \qquad \text{(F.38)}
\end{aligned}
$$

In this expression, we need to look at the e^{ikr} factor more closely. Here, r is given by

$$
r=(a,b)=(u'-u,w'-w) \qquad \text{(F.39)}
$$

where u,w,... are complex coordinates. As such, we could write r as a four component vector with real components

$$
r=(\mathrm{Re}(\mathrm{u}'-\mathrm{u}),\mathrm{Im}(\mathrm{u}'-\mathrm{u}),\mathrm{Re}(\mathrm{w}'-\mathrm{w}),\mathrm{Im}(\mathrm{w}'-\mathrm{w})) \qquad \text{(F.40)}
$$

Of course we can also write k as a real four component vector, $k=(k_1,k_2,k_3,k_4)$. As such, we have

$$
\begin{aligned}
e^{ikr} &\to exp[i(k_1\mathrm{Re}(\mathrm{u}'-\mathrm{u})+\mathrm{k}_2\mathrm{Im}(\mathrm{u}'-\mathrm{u})+\mathrm{k}_3\mathrm{Re}(\mathrm{w}'-\mathrm{w})+\mathrm{k}_4\mathrm{Im}(\mathrm{w}'-\mathrm{w}))] \qquad \text{(F.41)} \\
&= exp[\frac{1}{2}(u(ik_2-k_1)+u'(k_1-ik_2)+w(ik_4-k_3)+w'(k_3-ik_4) \\
&-\overline{u}(k_1+ik_2)+\overline{u}'(k_1+ik_2)-\overline{w}(k_3+ik_4)+\overline{w}'(k_3+ik_4))] \qquad \text{(F.42)}
\end{aligned}
$$

Putting this into Eq. (F.38) gives

$$G^{(1)l,m} = -\frac{1}{M^4}\int dk_1 dk_2 dk_3 dk_4 \int_S d^2u \wedge d^2w \int_{S'} d^2u' \wedge d^2w' \; (k_1^2+k_2^2+k_3^2+k_4^2)(w-u)^l(-w-u)^m$$
$$exp[-2|w|^2 - 2|u|^2 + w\overline{u} + u\overline{w} + \frac{1}{2}[(ik_2 - k_1 - 2v_- k_{kk})u$$
$$+ (ik_4 - k_3 + 2v_+ k_{kk})w + (-ik_2 - k_1 + 2v_-\overline{k_{kk}})\overline{u} - (ik_4 + k_3 + 2v_+\overline{k_{kk}})\overline{w}]$$
$$exp[-2|w'|^2 - 2|u'|^2 + w'\overline{u'} + u'\overline{w'} + \frac{1}{2}[(k_1 - ik_2 + 2v_- k_{kk})u'$$
$$+ (k_3 - ik_4 - 2v_+ k_{kk})w' + (ik_2 + k_1 - 2v_-\overline{k_{kk}})\overline{u}' + (ik_4 + k_3 + 2v_+\overline{k_{kk}})\overline{w}'] \qquad \text{(F.43)}$$

Leaving the integrals over k for now, we can see that the result is the product of two integrals of the form

$$\int_S d^2u \wedge d^2w \; u^{n_1}\overline{u}^{n_2} w^{n_3}\overline{w}^{n_4} e^{-p_1|u|^2 - p_2|w|^2 + p_3 w\overline{u} + p_4 u\overline{w} + a_1 u + a_2 w + b_1\overline{u} + b_2\overline{w}}$$
$$= R_\parallel^4 \partial_{a_1}^{n_1}\partial_{b_1}^{n_2}\partial_{a_2}^{n_3}\partial_{b_2}^{n_4}[\frac{\pi^2}{p_1p_2 - p_3p_4} e^{\frac{a_2b_2p_1 + a_1b_1p_2 + a_1b_2p_3 + a_2b_1p_4}{p_1p_2 - p_3p_4}}] \qquad \text{(F.44)}$$

In order to see if this first order correction is invariant under the geometric $U(1)$, we can evaluate it for l and m zero, and then non zero. If the $U(1)$ is respected, the only non zero case should be $l = 0$, $m = 0$, as we are not acting on the Higgs wavefunction with any covariant derivatives. Using Mathematica, we get

$$G^{(1)0,0} = -\frac{1}{M^4}\frac{\pi^4 R_\parallel^8}{9}\int dk_1 dk_2 dk_3 dk_4 (k_1^2+k_2^2+k_3^2+k_4^2)$$
$$\times exp[\frac{1}{3}(k_1^2+k_2^2+k_3^2+k_4^2 + k_1k_3 + k_2k_4 + (k_1 + ik_2 - k_3 - ik_4)k_{kk}$$
$$+ (k_3 + ik_2 - k_1 - ik_4 - 4k_{kk})\overline{k_{kk}})] \qquad \text{(F.45)}$$

$$G^{(1)1,0} = -\frac{1}{M^4}\frac{\pi^4 R_\parallel^8}{54}\int dk_1 dk_2 dk_3 dk_4 (k_1^2+k_2^2+k_3^2+k_4^2)(k_1 + ik_2 - k_3 - ik_4 - 4\overline{k_{kk}})$$
$$\times exp[\frac{1}{3}(k_1^2+k_2^2+k_3^2+k_4^2 + k_1k_3 + k_2k_4 + (k_1 + ik_2 - k_3 - ik_4)k_{kk}$$
$$+ (k_3 + ik_2 - k_1 - ik_4 - 4k_{kk})\overline{k_{kk}})] \qquad \text{(F.46)}$$

Provided that the terms with l and m non-zero are small compared to Eq. (F.37), the geometric $U(1)$ will be preserved for a complete diagram. In this thesis we will assume that this is true, although to prove this would require further work.

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