Supplemental Material

for *Energetic stability of coreless vortices in spin-1 Bose-Einstein condensates with conserved magnetization*

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In this Supplemental Online Material we provide additional information regarding spin-1 mean-field theory, conservation of magnetization, phase imprinting of nonsingular vortices, and construction of analytic spinor wave functions for the coreless textures.

HAMILTONIAN DENSITY, MAGNETIZATION AND LENGTH SCALES

Hamiltonian density: We treat the spin-1 Bose-Einstein condensate (BEC) in the Gross-Pitaevskii mean-field theory. The Hamiltonian density then reads [1, 2]

$$\mathcal{H} = \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\Psi|^2 + \frac{1}{2}m\omega^2 r^2 n + \frac{c_0}{2}n^2 + \frac{c_2}{2}n^2|\langle\hat{\mathbf{F}}\rangle|^2 + g_1 n\langle\mathbf{B}\cdot\hat{\mathbf{F}}\rangle + g_2 n\left\langle(\mathbf{B}\cdot\hat{\mathbf{F}})^2\right\rangle, \tag{S1}$$

where $n = \Psi^\dagger\Psi$ is the atom density and m is the atomic mass. We have assumed an isotropic, harmonic trap of frequency ω. The condensate wave function Ψ is now a three-component spinor,

$$\Psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\zeta(\mathbf{r}) = \sqrt{n(\mathbf{r})}\begin{pmatrix}\zeta_+(\mathbf{r})\\ \zeta_0(\mathbf{r})\\ \zeta_-(\mathbf{r})\end{pmatrix}, \quad \zeta^\dagger\zeta = 1, \tag{S2}$$

in the basis of spin projection onto the z axis. The local condensate spin is the expectation value $\langle\hat{\mathbf{F}}\rangle = \zeta_\alpha^\dagger\hat{\mathbf{F}}_{\alpha\beta}\zeta_\beta$ of the spin operator $\hat{\mathbf{F}}$, defined as a vector of spin-1 Pauli matrices. Linear and quadratic Zeeman shifts, of strength g_1 and g_2 respectively, described by the last two terms of Eq. (S1), may arise from a weak external magnetic field $\mathbf{B}$.

Spin-independent and spin-dependent interaction terms with strengths $c_0 = 4\pi\hbar^2(2a_2 + a_0)/3m$ and $c_2 = 4\pi\hbar^2(a_2 - a_0)/3m$, respectively, arise from the two scattering channels of colliding spin-1 atoms with s-wave scattering lengths a_0 and a_2. Minimization of the interaction energy then leads to the two distinct phases of the spin-1 BEC: $c_2 < 0$ favors the $|\langle\hat{\mathbf{F}}\rangle| = 1$ ferromagnetic (FM) phase (e.g., in ^{87}Rb), while the $|\langle\hat{\mathbf{F}}\rangle| = 0$ polar phase is favored when $c_2 > 0$ (e.g., in ^{23}Na).

Conservation of magnetization: We find stable vortex structures by minimizing the free energy $E = \int d^3r\,\mathcal{H} - \Omega\langle\hat{L}_z\rangle$ in the frame rotating at frequency Ω about the z axis, using imaginary-time propagation of the coupled Gross-Pitaevskii equations. However, the only spin-flip processes possible in s-wave scattering are $2\,|m=0\rangle \rightleftharpoons |m=+1\rangle + |m=-1\rangle$. Therefore s-wave interaction does not change the *longitudinal magnetization*

$$M = f_+ - f_- = \frac{1}{N}\int d^3r\, n(\mathbf{r})F_z(\mathbf{r}), \tag{S3}$$

where $f_\pm = N_\pm/N$. Here the total and the $|m=\pm1\rangle$ level populations are denoted by N and $N_\pm$, respectively. We have also introduced the z-component of the spin $F_z = \hat{\mathbf{z}}\cdot\langle\hat{\mathbf{F}}\rangle$.

Consequently, M is approximately conserved on time scales where s-wave scattering dominates over, e.g., dipolar interactions and collisions with high-temperature atoms. This is the relevant time scale in present experiments with spinor BECs of alkali-metal atoms [3–5]. We take this conservation strictly into account throughout energy relaxation by simultaneously renormalizing N *and* M at each step of imaginary-time evolution.

Characteristic length scales: The interaction terms in Eq. (S1) give rise to the characteristic density and spin healing lengths,

$$\xi_n = l\left(\frac{\hbar\omega}{2c_0 n}\right)^{1/2}, \quad \xi_F = l\left(\frac{\hbar\omega}{2|c_2|n}\right)^{1/2}, \tag{S4}$$

where we have introduced the oscillator length $l = (\hbar/m\omega)^{1/2}$ of the harmonic confinement. The healing lengths determine, respectively, the length scales over which the atomic density $n(\mathbf{r})$ and the spin magnitude $|\langle\hat{\mathbf{F}}\rangle|$ heal around a local perturbation. When magnetization is not conserved, ξ_n and ξ_F determine the core size of singular defects [6, 7]. If ξ_F is sufficiently larger than ξ_n, it becomes energetically favorable to avoid depleting the atomic density, instead accommodating the singularity by exciting the wave function out of its ground-state manifold. The core then expands to the order of ξ_F, instead of the smaller ξ_n that determines the size of a core with vanishing density. The lower gradient energy in the larger core offsets the cost in interaction energy.

Conservation of magnetization introduces a third length scale η_M, which is the size required for a magnetized vortex core in an otherwise unmagnetized condensate to give rise to a given magnetization (S3). Here we specifically study a coreless vortex that is phase-imprinted in the polar BEC. As the energy of the coreless vortex relaxes, the core region remains magnetized, but the spin magnitude sharply decreases outside the core. In order to estimate the magnetization length scale we represent the magnetized core by a cylinder of radius η_M, with $\langle\hat{\mathbf{F}}\rangle = \hat{\mathbf{z}}$ everywhere inside, and $|\langle\hat{\mathbf{F}}\rangle| = 0$ outside. The total magnetization is then

$$M(\eta_M) = \frac{1}{N}\int d^3r\,\Theta(\eta_M - \rho)n_{\mathrm{TF}}(\mathbf{r}), \tag{S5}$$

where $\rho = (x^2 + y^2)^{1/2}$ and Θ is the Heaviside function. We approximate the atomic-density profile by the Thomas-Fermi solution

$$n_{\mathrm{TF}}(r) = \frac{15N}{8\pi R_{\mathrm{TF}}^3}\left(1 - \frac{r^2}{R_{\mathrm{TF}}^2}\right), \quad r \le R_{\mathrm{TF}}, \tag{S6}$$

where $r = (\rho^2 + z^2)^{1/2}$, and

$$R_{\rm TF} = l \left(\frac{15}{4\pi} \frac{N c_{\rm p,f}}{\hbar\omega l^3} \right)^{1/5} \tag{S7}$$

is the Thomas-Fermi radius. Here $c_{\rm p} = c_0$ in a BEC with polar interactions, and $c_{\rm f} = c_0 + c_2$ in the FM regime. Computing the integral in Eq. (S5) and solving for η_M as a function of M, we obtain

$$\eta_M = R_{\rm TF}\sqrt{1-(1-M)^{2/5}}\,. \tag{S8}$$

We also consider a nonsingular coreless vortex formed by a nematic axis. This was experimentally phase imprinted in a polar BEC in Refs. [8, 9]. If the imprinted vortex state carries a finite magnetization (S3), the relaxation of the structure may be described by a spinor wave function that interpolates between the polar and FM phases, so that the core forms a composite topological defect where the interior represents the polar phase. We can then define a length scale $\eta_M^\star$ that describes the size of the polar core at a given magnetization. We estimate $\eta_M^\star$ by taking $\langle\hat{\mathbf{F}}\rangle = \hat{\mathbf{z}}$ everywhere outside a cylindrical core of this radius, and again approximating the density profile by the Thomas-Fermi solution. Then the magnetization is

$$M(\eta_M^\star) = \frac{1}{N}\int d^3r\,\Theta(\rho-\eta_M^\star) n_{\rm TF}(\mathbf{r})\,. \tag{S9}$$

Solving for $\eta_M^\star$ yields

$$\eta_M^\star = R_{\rm TF}\sqrt{1-M^{2/5}}\,. \tag{S10}$$

PHASE IMPRINTING OF NONSINGULAR VORTICES

Two different methods have been demonstrated for controlled preparation of nonsingular vortices. Here we give a brief overview of each.

In Refs. [8, 10] a coreless vortex was prepared using a time-dependent magnetic field to induce spin rotations. This technique was first proposed theoretically in Ref. [11] and was also implemented experimentally to prepare singly and doubly quantized vortices in a spin-polarized BEC [12, 13].

The creation of a coreless vortex in the spin-1 BEC begins with a condensate prepared in a fully spin-polarized state, which we take to be $\zeta^1 = (1,0,0)^T$ [14]. The condensate is subject to an external three-dimensional magnetic quadrupole field [8]

$$\mathbf{B} = B'\rho\hat{\boldsymbol{\rho}} + \left[B_z(t) - 2B'z\right]\hat{\mathbf{z}}\,, \tag{S11}$$

where we have introduced cylindrical coordinates (ρ,φ,z). The zero-field point $z = B_z/2B'$ ($\rho = 0$) of the quadrupole field is initially at large z so that $\mathbf{B} \parallel \hat{\mathbf{z}}$ in the condensate.

The coreless-vortex structure is created by linearly sweeping $B_z(t)$ so that the zero-field point passes through the condensate. The changing B_z causes the magnetic field away from the z axis to rotate around $\hat{\boldsymbol{\varphi}}$ from the $\hat{\mathbf{z}}$ to the $-\hat{\mathbf{z}}$ direction. The rate of change of the magnetic field decreases with the distance ρ from the symmetry axis. Where the rate of change is sufficiently slow, the atomic spins adiabatically follow the magnetic field, corresponding to a complete transfer from ζ_+ to ζ_- in the laboratory frame. However, where the rate of change of the magnetic field is rapid, atomic spin rotation is no longer adiabatic. In the laboratory frame, the spins thus rotate through an angle $\beta(\rho)$, given by the local adiabaticity of the magnetic-field sweep, which increases monotonically from zero on the symmetry axis. Linearly ramping $B_z(t)$ thus directly implements the spin rotation

$$\zeta^{\rm i}(\mathbf{r}) = e^{-i\hat{\mathbf{F}}\cdot\beta(\rho)\hat{\boldsymbol{\varphi}}}\zeta^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\cos^2\frac{\beta}{2} \\ e^{i\varphi}\sin\beta \\ \sqrt{2}e^{2i\varphi}\sin^2\frac{\beta}{2} \end{pmatrix}. \tag{S12}$$

The resulting fountainlike spin texture

$$\langle\hat{\mathbf{F}}\rangle = \sin\beta\hat{\boldsymbol{\rho}} + \cos\beta\hat{\mathbf{z}} \tag{S13}$$

that defines the coreless vortex in the spinor BEC is analogous to the Anderson-Toulouse-Chechetkin (ATC) [15, 16] and Mermin-Ho (MH) [17] vortices that arise in the A phase of superfluid liquid ^{3}He. In liquid ^{3}He, a circulation-carrying, nonsingular, fountainlike texture is formed by the local angular-momentum vector $\mathbf{l}$ of the Cooper pairs. In the ATC texture, $\mathbf{l}$ winds by π from the $\hat{\mathbf{z}}$ direction at the center to the $-\hat{\mathbf{z}}$ direction at the edge of the vortex, while the MH texture exhibits a $\pi/2$ rotation.

The first controlled preparation of a nonsingular vortex [10] used a two-dimensional quadrupole field together with an axial bias field. The magnetic field in the trap is then $\mathbf{B}(\rho,\varphi,\theta) = B_z(t)\hat{\mathbf{z}} + B'\rho\left[\cos(2\varphi)\hat{\boldsymbol{\rho}} - \sin(2\varphi)\hat{\boldsymbol{\varphi}}\right]$. By the mechanism described above, ramping of $B_z(t)$ then causes a spin rotation $\zeta(\mathbf{r}) = \exp[-i\hat{\mathbf{F}}\cdot\beta(\rho)\hat{\mathbf{n}}]\zeta^1$ about an axis $\hat{\mathbf{n}}(\varphi) = \sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$. The rotation yields a nonsingular spin texture exhibiting a cross disgyration, instead of the fountainlike structure. The two are topologically equivalent.

Another technique for phase imprinting a coreless vortex was recently demonstrated in Ref. [18]. In this experiment, the coreless vortex was created in the $|m=\pm2\rangle$ and $|m=0\rangle$ magnetic sublevels of the spin-2 manifold of ^{87}Rb. The phase imprinting starts with a spin-polarized condensate in the $|m=+2\rangle$ level, with a magnetic field along the z axis. Collinear σ^- and σ^+ polarized laser beams along the symmetry axis then couple $|m=2\rangle$ to the $|m=0\rangle$ and $|m=-2\rangle$ levels. The laser beams have Laguerre-Gaussian (LG) and Gaussian intensity profiles, respectively, so that the population transferred to the $|m=0\rangle$ ($|m=-2\rangle$) level picks up a 2π (4π) phase winding. The intensity minimum of the LG beam leaves a remaining population in $|m=2\rangle$ with no phase winding. The resulting five-component spinor represents a coreless vortex with the spin structure (S13) when the three nonempty levels of the five-component spinor are regarded as a (pseudo)spin-1 system. The bending angle

β is determined by the density profiles of the nonempty spinor components. The laser beams inducing the Raman coupling of the magnetic sublevels can be tailored with a high degree of control, and the vortex structure can therefore be precisely engineered.

By accurately creating specific spin textures, phase imprinting of coreless vortices gives control over the longitudinal magnetization of the cloud, regardless of whether interactions are polar or FM. In the spin-2 coreless-vortex experiment [18], the resulting magnetization in the spin-2 manifold is measured at $M = 0.64$ for an imprinted ATC-like spin texture, and at $M = 0.72$ for a MH-like texture. In the magnetic-field rotation experiment [10] the local magnetization $\mathcal{M}(\mathbf{r}) = [n_+(\mathbf{r}) - n_-(\mathbf{r})]/n(\mathbf{r})$ is reported to be ~ 0.7 at the center of the cloud and ~ -0.5 at the edge. Because of the lower density in the negatively magnetized region, also this vortex can be estimated to carry a positive, nonzero magnetization M.

GENERALIZED VORTEX SOLUTIONS

Coreless vortex

The two phases of the spin-1 BEC have different order-parameter symmetries that support different topological defects. For an overview, see, e.g., Refs. [1, 19]. Here we are interested in nonsingular coreless vortex states that mix the two phases. In order to construct spinor wave functions representing such vortices, we consider first a representative spinor with uniform spin $\langle\hat{\mathbf{F}}\rangle = F\hat{\mathbf{z}}$:

$$\zeta^F = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{1+F} \\ 0 \\ \sqrt{1-F} \end{pmatrix}. \tag{S14}$$

In particular, the limits $F = 0$ and $F = 1$ yield $\zeta^F|_{F=0} = (-1/\sqrt{2}, 0, 1/\sqrt{2})^T$ and $\zeta^F|_{F=1} = (-1, 0, 0)^T$, respectively, corresponding to representative polar and FM spinors.

The spinor wave function (S14) is reminiscent of the stationary mean-field solutions to the spin-1 BEC in the presence of Zeeman energy shifts that were introduced for uniform, vortex-free spinors in Refs. [20–22] for the free-space and optical-lattice systems. The general spinor with $|\langle\hat{\mathbf{F}}\rangle| = F$ can be reached by applying a condensate phase ϕ and a three-dimensional spin rotation $U(\alpha,\beta,\gamma) = \exp(-i\hat{\mathbf{F}}_z\alpha)\exp(-i\hat{\mathbf{F}}_y\beta)\exp(-i\hat{\mathbf{F}}_z\gamma)$, defined by three Euler angles, to Eq. (S14)

$$\zeta = \frac{e^{i\phi}}{2}\begin{pmatrix} \sqrt{2}e^{-i\alpha}\left(e^{i\gamma}D_-\sin^2\frac{\beta}{2} - e^{-i\gamma}D_+\cos^2\frac{\beta}{2}\right) \\ -\left(e^{i\gamma}D_- + e^{-i\gamma}D_+\right)\sin\beta \\ \sqrt{2}e^{i\alpha}\left(e^{i\gamma}D_-\cos^2\frac{\beta}{2} - e^{-i\gamma}D_+\sin^2\frac{\beta}{2}\right) \end{pmatrix}, \tag{S15}$$

where $D_\pm = \sqrt{1 \pm F}$.

The state (S15) can also be specified by the condensate phase, the spin magnitude F and an orthonormal triad with one vector in the direction of the spin. One of the remaining vectors in the triad forms the nematic axis $\hat{\mathbf{d}}$. (In the polar limit, $\hat{\mathbf{d}}$ fully specifies the state together with the condensate phase [6].) In the representative spinor (S14) we choose the triad such that $\hat{\mathbf{d}} = \hat{\mathbf{x}}$. In the general spinor (S15), $\hat{\mathbf{d}}$ can then be found from the Euler angles.

By allowing $\phi, \alpha, \beta, \gamma$ and the spin magnitude F to vary in space, we can now construct generalized solutions representing the coreless vortex states in a condensate where FM and polar regions coexist and the wave function interpolates smoothly between them. In Eq. (S15) we choose $\phi = \alpha = \varphi$ (where φ is the azimuthal angle) and $\gamma = 0$ to yield [Eq. (3) of the main text]:

$$\zeta^{\mathrm{cl}}(\mathbf{r}) = \frac{1}{2}\begin{pmatrix} \sqrt{2}\left(D_-\sin^2\frac{\beta}{2} - D_+\cos^2\frac{\beta}{2}\right) \\ -e^{i\varphi}(D_- + D_+)\sin\beta \\ \sqrt{2}e^{2i\varphi}\left(D_-\cos^2\frac{\beta}{2} - D_+\sin^2\frac{\beta}{2}\right) \end{pmatrix}. \tag{S16}$$

The spin texture is then given by

$$\langle\hat{\mathbf{F}}\rangle = F(\mathbf{r})[\sin\beta(\mathbf{r})\hat{\boldsymbol{\rho}} + \cos\beta(\mathbf{r})\hat{\mathbf{z}}], \tag{S17}$$

where $\beta(\mathbf{r})$ increases monotonically from zero on the symmetry axis to form the characteristic fountain texture with varying spin magnitude $F(\mathbf{r})$. In the limit $F = 1$, we retrieve the coreless vortex represented by Eqs. (S12) and (S13). In the polar limit $F = 0$, on the other hand, Eq. (S16) represents a singly quantized vortex

$$\zeta^{\mathrm{cl}}|_{F\to 0} = \frac{e^{i\varphi}}{\sqrt{2}}\begin{pmatrix} -e^{-i\varphi}\cos\beta \\ -\sqrt{2}\sin\beta \\ e^{i\varphi}\cos\beta \end{pmatrix}, \tag{S18}$$

where we have explicitly separated out the condensate phase $\phi = \varphi$. The nematic axis forms the texture $\hat{\mathbf{d}} = \cos\beta\hat{\boldsymbol{\rho}} - \sin\beta\hat{\mathbf{z}}$. In general the spin rotation that accompanies the winding of the condensate phase therefore represents a disgyration of the nematic axis.

The vortex (S16) can represent a solution for which F is nonuniform, so that Eqs. (S12) and (S18) are the two limiting solutions. We can form a composite topological defect by setting $F(\rho = 0) = 1$ and $\beta(\rho = 0) = 0$ at the center and letting $F \to 0$ and $\beta \to \pi/2$ as ρ increases. Then the core exhibits a coreless-vortex fountain texture that continuously transforms toward a singular polar vortex as the radius increases.

The mixing of the polar and FM phases in the vortex configuration is also reflected in the superfluid circulation. In the spin-1 BEC, the velocity of superfluid flow is defined in terms of ζ as

$$\mathbf{v} = \frac{\hbar}{2im}\sum_{j=+,0,-}[\zeta_j^*(\boldsymbol{\nabla}\zeta_j) - (\boldsymbol{\nabla}\zeta_j^*)\zeta_j]. \tag{S19}$$

Using the spinor in Eq. (S15), we derive the general expression

$$\mathbf{v} = \frac{\hbar}{m}\boldsymbol{\nabla}\phi - \frac{\hbar F}{m}\left[\boldsymbol{\nabla}\gamma + (\boldsymbol{\nabla}\alpha)\cos\beta\right]. \tag{S20}$$

For the coreless vortex (S16) this reduces to

$$\mathbf{v}^{\rm cl} = \frac{\hbar}{m\rho}\left[1 - F(\rho)\cos\beta(\rho)\right]\hat{\boldsymbol{\varphi}}\,, \tag{S21}$$

when F and β depend only on the radial distance ρ. By considering a circular loop $\mathcal{C}$ at constant ρ enclosing the vortex line, we can then compute the circulation

$$\nu = \int_{\mathcal{C}} d\mathbf{r}\cdot\mathbf{v}^{\rm cl} = \frac{h}{m}\left[1 - F(\rho)\cos\beta(\rho)\right]. \tag{S22}$$

Note that for nonzero F, circulation increases with increasing $\beta(\rho)$, implying that the coreless-vortex texture can adapt to an imposed rotation. This indicates that the spin texture will bend more sharply at faster rotation to provide increased angular momentum. We may regard the integrand of Eq. (S22) as a *circulation density*

$$\mathcal{V}(\mathbf{r}) = \mathbf{v}(\mathbf{r})\cdot\hat{\boldsymbol{\varphi}}\rho \tag{S23}$$

along a cylindrically symmetric path. The circulation of Eq. (S16) continuously interpolates between the polar and FM phases, smoothly connecting the small-distance and large-distance topology of the vortex. Note that it further follows from Eq. (S20) that circulation alone is quantized only in the limit $F \to 0$.

Owing to the effectively two-dimensional structure of the coreless spin texture, it is possible to define a winding number

$$W = \frac{1}{8\pi}\int_{\mathcal{S}} \mathrm{d}\Omega_i \epsilon_{ijk}\hat{\mathbf{n}}_F\cdot\left(\frac{\partial\hat{\mathbf{n}}_F}{\partial x_j}\times\frac{\partial\hat{\mathbf{n}}_F}{\partial x_k}\right). \tag{S24}$$

Here the integral is evaluated over a surface $\mathcal{S}$ crossing the entire vortex core, so that the edge of the surface is outside the core region. We have defined $\hat{\mathbf{n}}_F = \langle\hat{\mathbf{F}}\rangle/|\langle\hat{\mathbf{F}}\rangle|$ as a unit vector in the direction of the local spin vector. The charge W defines a topological invariant if the boundary condition on $\hat{\mathbf{n}}_F$ away from the vortex is fixed. When the asymptotic texture is uniform, W is an integer (representing a mapping of the spin texture on a compactified two-dimensional plane onto the unit sphere).

In the coreless vortex in the spinor BEC, asymptotic behavior of the spin texture is determined by rotation, as the bending of β in Eq. (S16), and therefore the circulation (S22), adapts to minimize the energy. The spin texture away from the vortex line may also be determined by interactions with other vortices, e.g., in the formation of a composite defect. By substituting $\langle\hat{\mathbf{F}}\rangle$ from Eq. (S17) into Eq. (S24), we may evaluate W. Assuming cylindrical symmetry and taking R to be the radial extent of the spin texture, we find

$$W = \frac{1-\cos\beta(R)}{2}\,, \tag{S25}$$

where we have used $\beta = 0$ on the z axis, such that $\hat{\mathbf{n}}_F|_{\rho=0} = \hat{\mathbf{z}}$. The winding number now depends on the asymptotic value of $\beta(\rho)$, such that for $\beta(R) = \pi$ (ATC-like texture) $W = 1$, and for $\beta(R) = \pi/2$ (MH-like texture) $W = 1/2$.

Nematic coreless vortex

From Eq. (S15) we can also construct the spinor for a *nematic coreless vortex* in a magnetized polar BEC. In this case we note that we wish to construct a vortex where

$$\hat{\mathbf{d}} = \sin\beta'\hat{\boldsymbol{\rho}} + \cos\beta'\hat{\mathbf{z}}\,, \tag{S26}$$

corresponding to the state phase-imprinted by Choi et al. [8, 9]. The angle β' between $\hat{\mathbf{d}}$ and the z axis increases from $\beta' = 0$ at $\rho = 0$ to $\beta' = \pi/2$ ($\beta' = \pi$) at the edge for a MH-like (ATC-like) texture. Note that since the Euler angles in Eq. (S15) represent spin rotations of Eq. (S14), we have $\beta = \beta' + \pi/2$, such that $\beta = \pi/2$ at the center of the vortex. The desired vortex state can then be constructed by additionally choosing $\alpha = \varphi$, $\gamma = \pi$ and $\phi = 0$ to yield [Eq. (5) of the main text]

$$\zeta^{\rm n} = \frac{1}{2}\begin{pmatrix} \sqrt{2}e^{-i\varphi}\left(D_+\cos^2\frac{\beta}{2} - D_-\sin^2\frac{\beta}{2}\right) \\ (D_+ + D_-)\sin\beta \\ \sqrt{2}e^{i\varphi}\left(D_+\sin^2\frac{\beta}{2} - D_-\cos^2\frac{\beta}{2}\right)\end{pmatrix}, \tag{S27}$$

with spin profile $\langle\hat{\mathbf{F}}\rangle = F(\sin\beta\hat{\boldsymbol{\rho}} + \cos\beta\hat{\mathbf{z}})$.

In a magnetized BEC, Eq. (S27) can represent a composite vortex that mixes the FM and polar phases. We consider a solution for which F exhibits a spatial structure interpolating between $F \to 0$ at the center and $F \to 1$ at the edge of the cloud. In the limit $F \to 1$, Eq. (S27) becomes a singular singly quantized FM vortex,

$$\zeta^{\rm n}|_{F\to 1} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2}e^{-i\varphi}\cos^2\frac{\beta}{2} \\ \sin\beta \\ \sqrt{2}e^{i\varphi}\sin^2\frac{\beta}{2}\end{pmatrix}. \tag{S28}$$

We calculate the superfluid circulation by assuming F and β in Eq. (S27) to be functions of the radial distance ρ only:

$$\mathbf{v}^{\rm n} = -\frac{\hbar F}{m\rho}\cos\beta\hat{\boldsymbol{\varphi}}\,. \tag{S29}$$

Similarly, we find the circulation density $\mathcal{V}(\mathbf{r}) = -\hbar F(\rho)\cos\beta(\rho)/m$ and circulation

$$\nu = \int_{\mathcal{C}} d\mathbf{r}\cdot\mathbf{v}^{\rm n} = -\frac{hF}{m}\cos\beta\,, \tag{S30}$$

assuming a circular path $\mathcal{C}$ at constant ρ. It follows that circulation vanishes for the nematic coreless vortex in the pure polar phase $F \to 0$ and becomes nonzero for $F > 0$ ($\cos\beta \neq 0$).

The interpolation between the polar and FM regimes is illustrated in the numerically computed, stable nematic coreless vortex in both spin magnitude and circulation density in Fig. 1 (bottom) of the main text. The vortex is energetically stable only once magnetization is strong enough to deplete ζ_+, enforcing an effective two-component regime. The stable vortex exhibits a MH-like texture in $\hat{\mathbf{d}}$, and a corresponding winding of the spin vector from the $\hat{\boldsymbol{\rho}}$ direction at the center to the $-\hat{\mathbf{z}}$ direction in the FM region away from the vortex line.

Also for the nematic coreless vortex we may define a winding number analogous to Eq. (S24), associated with the fountain texture of the nematic axis $\hat{\mathbf{d}}$, by taking $\hat{\mathbf{n}}_F = \hat{\mathbf{d}}$. Note that due to the equivalence $\hat{\mathbf{d}} \leftrightarrow -\hat{\mathbf{d}}$ the sign of W is no longer well defined. For the cylindrically symmetric fountain texture (S26), the integral in Eq. (S24) can be evaluated to yield

$$W = \frac{1 - \cos\beta'(R)}{2} = \frac{1 - \sin\beta(R)}{2}, \tag{S31}$$

making use of $\beta' = 0$ on the symmetry axis where $\hat{\mathbf{d}} = \hat{\mathbf{z}}$. In the last step we have used the relation $\beta = \beta' + \pi/2$ to rewrite W in terms of the Euler angle β of Eq. (S27). From Eq. (S31), we find $W = 1$ for an ATC-like texture, and $W = 1/2$ for a MH-like texture such as that stabilized in the effective two-component regime (Fig. 1).

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