# Near supplements and complements in solvable minimax groups

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**Abstract**

Let $\pi$ be a set of primes. Assume that $G$ is a solvable minimax group and $K \lhd G$ such that $G/K$ is $\pi$-minimax. We use cohomology to prove that, if $G/K$ is either finitely generated or virtually torsion-free, then $G$ has a $\pi$-minimax subgroup $X$ such that $[G : KX]$ is finite. In addition, we determine conditions that guarantee that $X$ may be chosen so that $K \cap X = 1$.

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## 1 Introduction

A group is *minimax* if it has a series of finite length in which each factor satisfies either the minimal or maximal condition on subgroups. The most widely studied minimax groups are the solvable ones, which admit a particularly simple characterization. A solvable group $G$ is minimax if and only if it has a series of finite length in which each factor is either cyclic or quasicyclic. The set of primes that correspond to the quasicyclic factors in such a series is an invariant of the group, called the *spectrum* of $G$, and denoted $\mathrm{spec}(G)$. If $\pi$ is a set of primes, then a *$\pi$-minimax group* is a solvable minimax group $G$ such that $\mathrm{spec}(G) \subset \pi$.

Solvable minimax groups occupy a position of central importance within the class of solvable groups. Their special status arises largely from the fact, proved by D. J. S. Robinson [**15**], that every finitely generated solvable group with finite abelian section rank is minimax. Further underscoring the significance of minimax groups, the first author [**8**] generalizes Robinson's theorem by establishing that a finitely generated solvable group without any sections that are wreath products of a finite group with $C_\infty$ must be minimax. Another aspect of solvable minimax groups that has provided an impetus to their study is their relevance to homological algebra. In particular, the class of minimax groups contains the class of constructible solvable groups, whose torsion-free members are precisely those solvable groups that have type FP (see [**9**]).

The present paper attempts to shed light on the structure of solvable minimax groups. Its focus is on detecting the presence of certain types of near supplements and complements. A *near supplement* to a normal subgroup $K$ of a group $G$ is a subgroup $X$ such that $[G : KX]$

is finite. If, in addition, $K \cap X = 1$, then $X$ is referred to as a *near complement* to $K$. We investigate the situation where $G$ is a solvable minimax group and $G/K$ is $\pi$-minimax for a set of primes $\pi$. Our objective is to ascertain conditions that will ensure the existence of a $\pi$-minimax near supplement or complement to $K$. The approach that we adopt employs cohomology and relies on the following elementary observation of Robinson, applied in [**10**, Theorem C] to detect nilpotent near supplements.

**Proposition 1.1.** (Robinson [**11**, Theorem 10.1.15]; [**14**]; [**15**]) *Assume that $G$ is a group and $A$ a $\mathbb{Z}G$-module. Let $\xi \in H^2(G,A)$ and $1 \to A \to E \to G \to 1$ a group extension corresponding to $\xi$. Then $\xi$ has finite order if and only if $E$ contains a subgroup $X$ such that $X \cap A$ is finite and $[E : AX]$ is finite.*

The above property suggests that, in pursuit of our aims, it will be beneficial to identify second cohomology groups that are torsion. For this purpose, we prove the following theorem about the cohomology of $\pi$-minimax groups, which serves as the foundation for our two main results concerning near supplements and complements (Theorems B and D).

**Theorem A.** *Let $\pi$ be a set of primes and $G$ a virtually torsion-free $\pi$-minimax group. Assume that $A$ is a $\mathbb{Z}G$-module whose underlying abelian group is torsion-free and minimax. Suppose further that $A$ does not have any nontrivial $\mathbb{Z}G$-module sections that are torsion-free and $\pi$-minimax as abelian groups. Then $H^n(G,A)$ is torsion for $n \geq 0$.*

Although our group-theoretic results only require the case $n = 2$, we elect to prove Theorem A for every dimension $n$, in the expectation that its general formulation may be of interest independent of the applications described here. Section 3 is devoted to the proof of this theorem and Section 2 to several properties of modules over group rings that are required for the proof. Among these preliminary results on modules, the most important is Proposition A below, which plays a pivotal role in the proof of Theorem A.

**Proposition A.** *Let $G$ be a finitely generated abelian group. Suppose that $A$ and $B$ are $\mathbb{Z}G$-modules whose additive groups are torsion-free minimax abelian groups. Assume further that $A$ and $B$ fail to have a pair of respective rationally irreducible $\mathbb{Z}G$-module sections that are isomorphic. Then, for any $\mathbb{Z}G$-module quotients $\bar{A}$ and $\bar{B}$ of $A$ and $B$, respectively, $\mathrm{Ext}^n_{\mathbb{Z}G}(\bar{A},\bar{B})$ is torsion for $n \geq 0$.*

In § 4 we apply our cohomological findings from § 3 to discern the presence of $\pi$-minimax near supplements. Our principal discovery in this vein is

**Theorem B.** *Let $G$ be a solvable minimax group. Assume that $K$ is normal subgroup of $G$ such that $G/K$ is $\pi$-minimax for some set $\pi$ of primes. If $G/K$ is virtually torsion-free, then there is a $\pi$-minimax subgroup $X$ of $G$ such that $[G : KX]$ is finite.*

The two key ingredients for Theorem B are Theorem A and a result of Robinson [**15**] that describes a situation where the cohomology of a solvable minimax group with torsion coefficients turns out to be torsion. With these results, the proof of Theorem B becomes a rather swift affair, involving the cohomology of $G/K$ with coefficients in the factors of a certain type of series in $K$.

As illustrated in Example 4.2, if $G/K$ is not virtually torsion-free, then $K$ may not have a $\pi$-minimax near supplement. Nevertheless, we prove in Theorem C that it remains possible to find a $\pi$-minimax subgroup $X$ satisfying the weaker condition that $KX$ has the same Hirsch length as $G$. Such a subgroup is referred to in the paper as a *Hirsch-length supplement*. Furthermore, it is an elementary property of finitely generated solvable minimax groups that any subgroup with the same Hirsch length as the group must have finite index (Lemma 4.4). As a result, if $G/K$ is finitely generated, then any Hirsch-length supplement to $K$ must necessarily be a near supplement.

Theorems B and C are both generalizations of [**4**, Proposition I.20], which states that, if $G$ is a solvable minimax group and $K \lhd G$ such that $G/K$ is polycyclic, then $K$ has a polycyclic near supplement. The latter proposition is employed by M. R. Bridson and the first author [**4**] in order to prove a result [**4**, Theorem I.3] on the homological dimension of an abelian-by-polycyclic group. We conclude § 4 by applying Theorem C to generalize Bridson and the first author's theorem to a larger class of groups.

In the final section of the paper, we turn our attention to the detection of near complements to a normal subgroup $K$ of a solvable minimax group $G$ when the quotient $G/K$ is $\pi$-minimax and virtually torsion-free. To ensure the presence of a near complement, we insist on two conditions for $K$. The first is that $K$ should be Noetherian as a $G$-operator group, meaning that $K$ satisfies the maximal condition on $G$-invariant subgroups. For our second hypothesis, we suppose that $K$ has no nontorsion $\pi$-minimax quotients. Denoting the class of groups with the latter property by $\mathfrak{X}_\pi$, we can express the main result of § 5 as follows.

**Theorem D.** *Let $G$ be a solvable minimax group. Assume that $K$ is a normal subgroup of $G$ such that $K$ belongs to the class $\mathfrak{X}_\pi$ and $K$ is Noetherian as a $G$-operator group. Suppose further that $G/K$ is $\pi$-minimax and virtually torsion-free. Then $K$ has a near complement in $G$.*

Before proceeding to the body of the paper, we describe some of the terms and notation that we will be using.

**Notation and terminology.** Throughout the paper, $\pi$ will represent an arbitrary set of primes.

If $p$ is a prime, then $\hat{\mathbb{Z}}_p$ denotes the ring of $p$-adic integers and $\hat{\mathbb{Q}}_p$ the field of $p$-adic rational numbers.

We will observe different conventions for modules and groups with operators, respectively. Unless stated otherwise, all modules will be assumed to be left modules. However, if $G$ is a group, then a $G$-operator group $K$ will always mean a right operator group. In the latter instance, the function $K \times G \to K$ will be denoted by $(x, g) \mapsto x^g$.

Assume that $G$ is a group and $R$ a commutative ring. If $A$ is a $\mathbb{Z}G$-module, then $A^R$ represents the $RG$-module $A \otimes R$.

Let $A$ be a $\mathbb{Z}G$-module. If $A^{\mathbb{Q}}$ is a simple $\mathbb{Q}G$-module, then we will refer to $A$ as *rationally irreducible*. In other words, $A$ is rationally irreducible if and only if the additive group of $A$ is not torsion and, for every submodule $B$ of $A$, either $B$ or $A/B$ is torsion as an abelian group.

Let $G$ be a solvable group. Then $\tau(G)$ represents the torsion radical of $G$, that is, the join of all the torsion normal subgroups of $G$. Furthermore, the join of all the nilpotent normal subgroups of $G$, known as the Fitting subgroup, is denoted by $\mathrm{Fitt}(G)$.

If $G$ is a solvable group of finite torsion-free rank, then $h(G)$ denotes the Hirsch length of $G$. If $G$ is also minimax, then the *minimax rank* of $G$, written $m(G)$, is the number of infinite factors in any series of finite length in which each factor is either cyclic or quasicyclic. Like the Hirsch length, the minimax rank is an invariant of the group; in other words, it doesn't depend on the particular series selected.

## 2 Preliminary results on modules

In this section, we establish an array of results regarding modules, culminating in Proposition A, mentioned in the introduction. Critical to our discussion is the relationship between a module over an integral group ring and its rationalization. It is this connection that is explored in the following two lemmas.

**Lemma 2.1.** *Assume that $G$ is a group. Let $A$ and $B$ be $\mathbb{Z}G$-modules such that the additive groups of $A$ and $B$ are not torsion and have finite torsion-free rank. If $B^{\mathbb{Q}}$ is a $\mathbb{Q}G$-module homomorphic image of $A^{\mathbb{Q}}$, then $A$ and $B$ must have two respective rationally irreducible $\mathbb{Z}G$-module sections that are isomorphic.*

*Proof.* It suffices to consider the case where the additive groups of both $A$ and $B$ are torsion-free. Let $\phi : A^{\mathbb{Q}} \to B^{\mathbb{Q}}$ be a $\mathbb{Q}G$-module epimorphism. Suppose that $\{x_1, \cdots, x_r\}$ is a basis for $A^{\mathbb{Q}}$ as a vector space over $\mathbb{Q}$, where each $x_i$ is chosen from $A$. Assume that $m$ is a positive integer such that $m\phi(x_i) \in B$ for $1 \leq i \leq r$. Now let $A_0$ be the $\mathbb{Z}G$-submodule of $A$ generated by the $x_i$ and $B_0$ the $\mathbb{Z}G$-submodule of $B$ generated by the $m\phi(x_i)$. Then the restriction of $m\phi$ to $A_0$ defines a $\mathbb{Z}G$-module epimorphism from $A_0$ to $B_0$. Therefore, $A_0/A_0 \cap \mathrm{Ker}\phi$ and $B_0$ are isomorphic nontrivial $\mathbb{Z}G$-module sections of $A$ and $B$, respectively, whose additive groups are torsion-free. It follows, then, that $A$ and $B$ have two rationally irreducible $\mathbb{Z}G$-module sections, respectively, that are isomorphic. □

**Lemma 2.2.** *Let $G$ be an abelian group. Assume that $A$ and $B$ are $\mathbb{Z}G$-modules whose underlying abelian groups have finite torsion-free rank. Suppose further that $A$ and $B$ fail to have a pair of respective rationally irreducible $\mathbb{Z}G$-module sections that are isomorphic. Then $\mathrm{Ext}^n_{\mathbb{Q}G}(A^{\mathbb{Q}}, B^{\mathbb{Q}}) = 0$ for $n \geq 0$.*

*Proof.* The result is trivial if either $A$ or $B$ is torsion *qua* abelian group; hence we assume that both are not. In this case, $A$ and $B$ each have a series of finite length in which all the factors are rationally irreducible. Therefore, it suffices to consider the case where $A$ and $B$ are both rationally irreducible. Let $I$ and $J$ be the annihilator ideals in $\mathbb{Q}G$ of $A^{\mathbb{Q}}$ and $B^{\mathbb{Q}}$, respectively. By the above lemma, the simple $\mathbb{Q}G$-modules $A^{\mathbb{Q}}$ and $B^{\mathbb{Q}}$ are not isomorphic. This means that $I + J = \mathbb{Q}G$. Since $\mathbb{Q}G$ is commutative, $\mathrm{Ext}^n_{\mathbb{Q}G}(A^{\mathbb{Q}}, B^{\mathbb{Q}})$ inherits a $\mathbb{Q}G$-module structure for all $n \geq 0$. With respect to this module structure, both $I$ and $J$ annihilate $\mathrm{Ext}^n_{\mathbb{Q}G}(A^{\mathbb{Q}}, B^{\mathbb{Q}})$, making $\mathrm{Ext}^n_{\mathbb{Q}G}(A^{\mathbb{Q}}, B^{\mathbb{Q}}) = 0$. □

The next three results are of a purely homological nature. The first is well known and entirely elementary; hence we omit its proof.

**Lemma 2.3.** *Let $G$ be a group. Suppose that $R \subset S$ are commutative rings. If $A$ is an $RG$-module and $B$ an $SG$-module, then*

$$\mathrm{Hom}_{RG}(A, B) \cong \mathrm{Hom}_{SG}(A \otimes_R S, B).$$

*Moreover, if $S$ is flat over $R$, then*

$$\mathrm{Ext}^n_{RG}(A, B) \cong \mathrm{Ext}^n_{SG}(A \otimes_R S, B)$$

*for $n \geq 0$.*

In the proposition below, we derive a spectral sequence involving the cohomology of a group $G$ and the functor $\operatorname{Ext}_R^*$, where $R$ is a commutative ring. Although it seems likely that this spectral sequence is already known, we are unaware of any reference to it in the literature and thus include a complete proof. Our sole purpose in introducing the spectral sequence is to extract from it a cohomological isomorphism (Corollary 2.5) that will find frequent application throughout the paper. This isomorphism can also be proved in a direct and elementary fashion; however, we find it illuminating to view it as part of a more general phenomenon.

**Proposition 2.4.** *Let $G$ be a group and $R$ a commutative ring. Suppose that $A$ and $B$ are $RG$-modules and regard $\operatorname{Ext}_R^n(A,B)$ as a $\mathbb{Z}G$-module via the diagonal action for $n \geq 0$. Then there is a first quadrant cohomology spectral sequence whose $E_2$-page is given by*

$$E_2^{pq} = H^p(G, \operatorname{Ext}_R^q(A,B)),$$

*and that converges to $\operatorname{Ext}_{RG}^n(A,B)$.*

*Proof.* This will follow immediately from the Grothendieck spectral sequence in [**7**, Theorem 10.5] provided that we verify that $H^n(G, \operatorname{Hom}_R(A,I)) = 0$ for any injective $RG$-module $I$ and $n \geq 1$. To accomplish this, take a projective $RG$-module resolution $\cdots \to P_1 \to P_0 \to A \to 0$ of $A$. Since $I$ is injective as an $RG$-module, it is also injective as an $R$-module. Thus the sequence

$$0 \to \operatorname{Hom}_R(A,I) \to \operatorname{Hom}_R(P_0,I) \to \operatorname{Hom}_R(P_1,I) \to \cdots \tag{2.1}$$

is exact. We claim that $H^n(G, \operatorname{Hom}_R(P_i,I)) = 0$ for $i \geq 0$ and $n \geq 1$. To show this, it suffices to establish that $H^n(G, \operatorname{Hom}_R(RG,I)) = 0$ for $n \geq 1$. However, this is easily seen to be true because $\operatorname{Hom}_R(RG,I)$ is isomorphic to the coinduced module $\operatorname{Hom}(\mathbb{Z}G,I)$, which has trivial cohomology in every positive dimension. Therefore, (2.1) is an acyclic resolution of $\operatorname{Hom}_R(A,I)$ with respect to the functor $H^0(G, \_\,)$. As a consequence, the groups $H^n(G, \operatorname{Hom}_R(A,I))$ are the cohomology groups of the cochain complex

$$0 \to \operatorname{Hom}_{RG}(P_0,I) \to \operatorname{Hom}_{RG}(P_1,I) \to \cdots .$$

But this complex is acyclic in view of the injectivity of $I$ as an $RG$-module. Hence $H^n(G, \operatorname{Hom}_R(A,I)) = 0$ for $n \geq 1$. ◻

The desired isomorphism yielded by this spectral sequence is described below.

**Corollary 2.5.** *Let $G$ be a group and $R$ a commutative ring. Suppose that $A$ and $B$ are $RG$-modules such that either $A$ is projective or $B$ is injective as an $R$-module. Then, for $n \geq 0$,*

$$\operatorname{Ext}_{RG}^n(A,B) \cong H^n(G, \operatorname{Hom}_R(A,B)),$$

*where $\operatorname{Hom}_R(A,B)$ is viewed as a $\mathbb{Z}G$-module under the diagonal action.*

*Proof.* We apply the spectral sequence above. In view of our assumptions about $A$ or $B$, $E_2^{pq} = 0$ for $q \neq 0$, thus yielding the isomorphism. ◻

**Remark.** The case of Corollary 2.5 for $R = \mathbb{Z}$ and $A$ a free $\mathbb{Z}$-module appears in [**5**, p. 61].

Our first application of Corollary 2.5 is the following lemma, required for the proof of Proposition A.

**Lemma 2.6.** *Let $G$ be a polycyclic group and $A$ and $B$ $\mathbb{Z}G$-modules. Assume that the additive group of $A$ is minimax and that of $B$ torsion and divisible. For each prime $p$, let $B_p$ be the $p$-torsion subgroup of $B$. Then, if $n \geq 0$, $\operatorname{Ext}^n_{\mathbb{Z}G}(A, B)$ is torsion if and only if $\operatorname{Ext}^n_{\mathbb{Z}G}(A, B_p)$ is torsion for each prime $p$.*

*Proof.* The "only if" part is trivial; hence we confine our attention to the "if" direction. Let $\pi$ be the set of all primes outside $\operatorname{spec}(A)$. Since $\operatorname{Ext}^n_{\mathbb{Z}G}(A, \_)$ commutes with finite direct sums, the conclusion will follow if we can establish that $\operatorname{Ext}^n_{\mathbb{Z}G}(A, B_\pi)$ is torsion, where $B_\pi$ is the $\pi$-torsion subgroup of $B$. To accomplish this, we notice that $\operatorname{Ext}^n_{\mathbb{Z}G}(A, B_\pi) \cong H^n(G, \operatorname{Hom}(A, B_\pi))$ by Corollary 2.5. Since $\operatorname{Hom}(A, B_\pi)$ is torsion and $G$ is of type $FP_\infty$, it follows that $\operatorname{Ext}^n_{\mathbb{Z}G}(A, B_\pi)$ is torsion. □

Before establishing Proposition A, we discuss a key tool employed in the proof, namely, the Pontryagin dual of a module, defined below.

**Definition.** Assume that $p$ is a prime. If $A$ is an abelian $p$-group, then the *Pontryagin dual* $A'$ of $A$ is the group $\operatorname{Hom}(A, \mathbb{Z}/p^\infty)$. The dual $A'$ is an abelian pro-$p$ group and, as such, can be regarded as a $\hat{\mathbb{Z}}_p$-module. If $A$ happens to be endowed with a $\mathbb{Z}G$-module structure for a group $G$, then we equip $A'$ with a $G$-action by letting $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for every $g \in G$, $f \in A'$, and $x \in A$. In this way, we may view $A'$ as a $\hat{\mathbb{Z}}_pG$-module.

We are especially interested in the Pontryagin dual of $A$ when $A$ is the direct sum of finitely many copies of $\mathbb{Z}/p^\infty$ for a prime $p$. In this case, $A'$ is the direct sum of the same number of copies of $\hat{\mathbb{Z}}_p$. Another elementary property of Pontryagin duals that will play an important role is described in the following lemma.

**Lemma 2.7.** *Let $p$ be a prime. Suppose that $A$ and $B$ are abelian $p$-groups, and that $B$ has finite rank. Then there is a natural isomorphism*

$$\operatorname{Hom}(A, B) \to \operatorname{Hom}_{\hat{\mathbb{Z}}_p}(B', A').$$

*Proof.* To start, we observe that the finite rank condition implies that $B'$ is topologically finitely generated as a pro-$p$ group. A well-known theorem of J-P. Serre states that, in a topologically finitely generated pro-$p$ group, every subgroup of finite index is open ([**19**, Theorem 4.3.5]). This implies that every homomorphism from a topologically finitely generated pro-$p$ group to a profinite group must be continuous. Hence $\operatorname{Hom}_{\hat{\mathbb{Z}}_p}(B', A')$ coincides with the set of continuous homomorphisms from the pro-$p$ group $B'$ to the pro-$p$ group $A'$. The result follows, then, from [**19**, Theorem 6.4.7].

□

From Lemma 2.7 we obtain the property below.

**Lemma 2.8.** *Assume that $p$ is a prime and $G$ a group. Let $A$ and $B$ be $\mathbb{Z}G$-modules whose additive groups are $p$-groups. Suppose further that, as an abelian group, $B$ is divisible and has finite rank. Then*

$$\operatorname{Ext}^n_{\mathbb{Z}G}(A, B) \cong \operatorname{Ext}^n_{\hat{\mathbb{Z}}_pG}(B', A')$$

*for $n \geq 0$.*

*Proof.* We have the following chain of isomorphisms.

$$\mathrm{Ext}^n_{\mathbb{Z}G}(A,B) \cong H^n(G, \mathrm{Hom}(A,B)) \cong H^n(G, \mathrm{Hom}_{\hat{\mathbb{Z}}_p}(B',A')) \cong \mathrm{Ext}^n_{\hat{\mathbb{Z}}_pG}(B',A').$$

The first isomorphism in this chain results from the divisibility of $B$ and the last from the fact that $B'$ is free as a $\hat{\mathbb{Z}}_p$-module. □

Armed with the lemmas above, we are finally ready to prove Proposition A.

**Proposition A.** *Let $G$ be a finitely generated abelian group. Suppose that $A$ and $B$ are $\mathbb{Z}G$-modules whose additive groups are both torsion-free and minimax. Assume further that $A$ and $B$ fail to have a pair of respective rationally irreducible $\mathbb{Z}G$-module sections that are isomorphic. Then, for any $\mathbb{Z}G$-module quotients $\bar{A}$ and $\bar{B}$ of $A$ and $B$, respectively, $\mathrm{Ext}^n_{\mathbb{Z}G}(\bar{A},\bar{B})$ is torsion for $n \geq 0$.*

*Proof.* Assume that $A$ and $B$ are both nontrivial. Then $A$ and $B$ each have a series of submodules of finite length in which all the factors are torsion-free and rationally irreducible. This permits a reduction to the case where $A$ and $B$ are rationally irreducible. By Lemmas 2.2 and 2.3, $\mathrm{Ext}^n_{\mathbb{Z}G}(\bar{A},\bar{B}^{\mathbb{Q}}) = 0$ for $n \geq 0$. The conclusion will then follow from the long exact Ext-sequence if we can establish that $\mathrm{Ext}^n_{\mathbb{Z}G}(\bar{A},\bar{B}^{\mathbb{Q}}/\bar{B})$ is torsion for $n \geq 0$. According to Lemma 2.6, this will follow if we manage to show that, for each prime $p$, $\mathrm{Ext}^n_{\mathbb{Z}G}(\bar{A},\bar{B}^{\mathbb{Z}[1/p]}/\bar{B})$ is torsion for $n \geq 0$. With this aim in mind, we take $p$ to be a prime and set $Q = \bar{B}^{\mathbb{Z}[1/p]}/\bar{B}$. The module $\bar{A}$ contains a finitely generated submodule $\bar{A}_0$ such that $\bar{A}/\bar{A}_0$ is torsion. Moreover, since $\mathbb{Z}G$ is Noetherian, $\bar{A}_0$ is of type $FP_\infty$, implying that $\mathrm{Ext}^n_{\mathbb{Z}G}(\bar{A}_0, Q)$ is torsion for $n \geq 0$. Hence the conclusion will follow if we can deduce that, for $n \geq 0$, $\mathrm{Ext}^n_{\mathbb{Z}G}(P,Q)$ is torsion for any $\mathbb{Z}G$-module quotient $P$ of $A$ whose additive group is a divisible $p$-group. Supposing that $P$ is as described, set $P^* = P' \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathbb{Q}}_p$ and $Q^* = Q' \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathbb{Q}}_p$. Let $I^*$ be the annihilator ideal of $P^*$ in the ring $\hat{\mathbb{Q}}_pG$ and $J^*$ that of $Q^*$. Assume, further, that $I$ and $J$ are the respective annihilators of $A^{\mathbb{Q}}$ and $B^{\mathbb{Q}}$ in $\mathbb{Q}G \subset \hat{\mathbb{Q}}_pG$. Also, let $\alpha : \mathbb{Q}G \to \mathbb{Q}G$ be the ring isomorphism such that $\alpha(g) = g^{-1}$ for all $g \in G$. If $x \in I$, then $x = \frac{1}{m}y$, where $m \in \mathbb{Z}$ and $y \in \mathbb{Z}G \cap I$. Because $A$ is torsion-free, $y$ annihilates $P$, and so $\alpha(y)$ must annihilate $P'$. As a result, $\alpha(x) \in I^*$. We have thus shown that $\alpha(I) \subset I^*$. Moreover, by a similar argument, $\alpha(J) \subset J^*$. Since $A^{\mathbb{Q}}$ and $B^{\mathbb{Q}}$ are simple, nonisomorphic $\mathbb{Q}G$-modules, we have $I + J = \mathbb{Q}G$. Hence $I^* + J^* = \hat{\mathbb{Q}}_pG$. As $\mathrm{Ext}^n_{\hat{\mathbb{Q}}_pG}(Q^*,P^*)$ inherits a $\hat{\mathbb{Q}}_pG$-module structure from $Q^*$ and $P^*$, it follows that $\mathrm{Ext}^n_{\hat{\mathbb{Q}}_pG}(Q^*,P^*) = 0$ for $n \geq 0$. Therefore, by Lemma 2.3, $\mathrm{Ext}^n_{\hat{\mathbb{Z}}_pG}(Q',P^*) = 0$ for $n \geq 0$.

In the final step of the proof, we argue that $\mathrm{Ext}^n_{\hat{\mathbb{Z}}_pG}(Q',P')$ is torsion for $n \geq 0$, yielding that $\mathrm{Ext}^n_{\mathbb{Z}G}(P,Q)$ is torsion by Lemma 2.8. The former assertion will follow from the long exact Ext-sequence if we can prove that $\mathrm{Ext}^n_{\hat{\mathbb{Z}}_pG}(Q',P^*/P')$ is torsion for $n \geq 0$. To establish this, we apply Corollary 2.5 to obtain that

$$\mathrm{Ext}^n_{\hat{\mathbb{Z}}_pG}(Q',P^*/P') \cong H^n(G, \mathrm{Hom}_{\hat{\mathbb{Z}}_p}(Q',P^*/P'))$$

for $n \geq 0$. It follows from the classification of finitely generated modules over a principal ideal domain that every torsion $\hat{\mathbb{Z}}_p$-module quotient of $Q'$ is finite. From this observation we infer that $\mathrm{Hom}_{\hat{\mathbb{Z}}_p}(Q',P^*/P')$ must be torsion. Therefore, $\mathrm{Ext}^n_{\hat{\mathbb{Z}}_pG}(Q',P^*/P')$ is torsion for $n \geq 0$, thus completing the argument. □

## 3 Cohomology of $\pi$-minimax groups

The goal of this section is to prove Theorem A. We begin by reminding the reader of three fundamental properties of the Fitting subgroup of a solvable minimax group; proofs of these may be found in [**11**, p. 123].

**Proposition 3.1.** *Let $G$ be a solvable minimax group and $F = \mathrm{Fitt}(G)$. Then the following three statements hold.*

*(i) The subgroup $F$ is nilpotent.*

*(ii) The quotient $G/F$ is virtually abelian.*

*(iii) If $G$ is virtually torsion-free, then $G/F$ is finitely generated.*

Next we examine solvable minimax groups $G$ for which there is a rationally irreducible $\mathbb{Z}G$-module. The properties that we establish for such groups are gleaned from the proofs in [**11**] of parts (ii) and (iii) of the above proposition. For the convenience of the reader, however, we furnish all the details of the arguments.

**Lemma 3.2.** *Let $G$ be a solvable minimax group and $A$ a rationally irreducible $\mathbb{Z}G$-module. Then $G$ has a nilpotent normal subgroup $N$ such that $N \subset C_G(A^{\mathbb{Q}})$ and $G/N$ is a virtually abelian group. Moreover, if $G$ is virtually torsion-free, then $G/N$ is finitely generated.*

*Proof.* Let $F = \mathrm{Fitt}(G)$ and $N = F \cap C_G(A^{\mathbb{Q}})$. Since $G/F$ is virtually abelian, there is a normal subgroup $M$ of $G$ such that $[G : M]$ is finite and $M' \subset F$. There is a well-known theorem of A. Malcev [**11**, Theorem 1.23] that states that every solvable irreducible linear group is virtually abelian. Hence $G/C_G(A^{\mathbb{Q}})$ is virtually abelian. This means that $G$ has a normal subgroup $P$ with finite index such that $P' \subset C_G(A^{\mathbb{Q}})$. Thus $(M \cap P)' \subset N$ and $[G : M \cap P]$ is finite, implying that $G/N$ is virtually abelian.

Finally, we consider the case where $G$ is virtually torsion-free. To see that $G/N$ is finitely generated in this case, notice first that $G/F$ is polycyclic by virtue of Proposition 3.1. Next let $\bar{P}$ be the image of $P$ in $G/C_G(A^{\mathbb{Q}})$. By [**16**, 8.1.3(i)], $A^{\mathbb{Q}}$ is completely reducible as a $\mathbb{Q}\bar{P}$-module. For each simple $\mathbb{Q}\bar{P}$-submodule $S$ of $A^{\mathbb{Q}}$, $S \cong \mathbb{Q}\bar{P}/I_S$, where $I_S$ is a maximal ideal in $\mathbb{Q}\bar{P}$. Further, the ring $\mathbb{Q}\bar{P}/I_S$ is a field whose multiplicative group contains an isomorphic copy of $P/C_G(S)$, namely, the subgroup $\{x + I_S \mid x \in \bar{P}\}$. However, the multiplicative group of a field that is a finite extension of $\mathbb{Q}$ is the direct product of a free abelian group with a finite cyclic group (see [**6**, Theorem 1.27]). Since $P/C_G(S)$ has finite torsion-free rank, this means that $P/C_G(S)$ must be finitely generated and therefore polycyclic. As a result, $\bar{P}$ is polycyclic. Consequently, $G/N$, too, is polycyclic, which completes the proof. □

For our cohomology results, we require the following property of the integral homology of a $\pi$-minimax group. This result is undoubtedly well known and quite straightforward to establish. Nevertheless, for the sake of completeness, we include a proof.

**Proposition 3.3.** *If $G$ is a nilpotent $\pi$-minimax group, then $H_nG$ is $\pi$-minimax for $n \geq 0$.*

*Proof.* We proceed by induction on the nilpotency class of $G$. Suppose first that $G$ is abelian. Let $T = \tau(G)$ and consider the Lyndon-Hochschild-Serre (LHS) spectral sequence in homology associated to the group extension $1 \to T \to G \to Q \to 1$. Since $Q$ is torsion-free, its integral homology groups are exterior powers of $G$, rendering them $\pi$-minimax. The group $T$ is the direct sum of finitely many quasicyclic groups and finite groups. Furthermore, since integral homology commutes with direct limits, the integral homology groups of a $\pi$-minimax quasicyclic group are $\pi$-minimax. Therefore, $H_qT$ is $\pi$-minimax for $q \geq 0$. Hence $H_p(Q, H_qT) = H_pQ \otimes H_qT$ is $\pi$-minimax for $p,\ q \geq 0$, so that the conclusion follows.

Suppose that $G$ is nonabelian. Let $Z = Z(G)$ and consider the LHS spectral sequence associated to the group extension $1 \to Z \to G \to R \to 1$. By the inductive hypothesis, the integral homology groups of both $R$ and $Z$ are $\pi$-minimax. Invoking the universal coefficient theorem as well as the closure of the class of $\pi$-minimax abelian groups under both tensor and torsion products, we obtain that $H_p(R, H_qZ)$ is $\pi$-minimax for $p,\ q \geq 0$. Hence $H_nG$ is $\pi$-minimax for $n \geq 0$.

□

Next we establish a key proposition *en route* to Theorem A. This proposition will also serve as the basis for Theorem C in § 4, our second near-supplement result.

**Proposition 3.4.** *Let $G$ be a $\pi$-minimax group and $A$ a $\mathbb{Z}G$-module whose underlying abelian group is minimax. Suppose further that $A$ does not have any nontrivial $\mathbb{Z}G$-module sections that are torsion-free and $\pi$-minimax as abelian groups. Then $H^n(G, A^{\mathbb{Q}}) = 0$ for all $n \geq 0$.*

*Proof.* If the underlying abelian group of $A$ is torsion, then the conclusion is trivially true. Hence we assume that $A$ is not torsion. In this case, $A$ has a series of submodules of finite length in which each factor is rationally irreducible. Thus, without any real loss of generality, we can assume that $A$ is rationally irreducible. By virtue of Lemma 3.2, $G$ must then have a nilpotent normal subgroup $N$ such that $N \subset C_G(A^{\mathbb{Q}})$ and $G/N$ is virtually abelian. Let $G_0$ be a normal subgroup of $G$ with finite index such that $N < G_0$ and $Q_0 = G_0/N$ is abelian.

Our plan is to show that $H^n(G_0, A^{\mathbb{Q}}) = 0$ for $n \geq 0$; it will then follow that $H^n(G, A^{\mathbb{Q}}) = 0$ for $n \geq 0$. To investigate the groups $H^n(G_0, A^{\mathbb{Q}})$, we will employ the LHS spectral sequence for the extension $1 \to N \to G_0 \to Q_0 \to 1$. Invoking the universal coefficient theorem, we conclude that $H^q(N, A^{\mathbb{Q}}) \cong \mathrm{Hom}(H_qN, A^{\mathbb{Q}})$ for $q \geq 0$. Thus

$$H^p(Q_0, H^q(N, A^{\mathbb{Q}})) \cong H^p(Q_0, \mathrm{Hom}(H_qN, A^{\mathbb{Q}})) \cong \mathrm{Ext}^p_{\mathbb{Z}Q_0}(H_qN, A^{\mathbb{Q}})$$

for $p,\ q \geq 0$, where the second isomorphism follows from Corollary 2.5. According to Proposition 3.3, $H_qN$ is $\pi$-minimax. Moreover, in view of Lemma 3.5(ii) below, $A$ has no nontrivial torsion-free $\mathbb{Z}G_0$-module sections that are $\pi$-minimax as abelian groups. Therefore, by Lemmas 2.2 and 2.3, $\mathrm{Ext}^p_{\mathbb{Z}Q_0}(H_qN, A^{\mathbb{Q}}) = 0$. Hence $H^n(G_0, A^{\mathbb{Q}}) = 0$ for $n \geq 0$, thus proving the proposition. □

It remains to establish

**Lemma 3.5.** *Let $G$ be a group and $K$ a $G$-operator group.*

*(i) If $M \lhd N < K$ and $N/M$ is torsion-free and $\pi$-minimax, then, for any $g_1, \cdots, g_r \in G$, the group*

$$\bigcap_{i=1}^{r} N^{g_i} \Big/ \bigcap_{i=1}^{r} M^{g_i}$$

*is also torsion-free and $\pi$-minimax.*

*(ii) If $H < G$ with $[G : H] < \infty$ and $K$ has an $H$-invariant section that is torsion-free and $\pi$-minimax, then $K$ also has a $G$-invariant section with the same two properties.*

*Proof.* First we prove statement (i). The torsion-free part is obvious, so we focus only on the $\pi$-minimax conclusion. We proceed by induction on $r$, the case $r = 1$ being immediate. Assume that $r > 1$. Let $P = \bigcap_{i=1}^{r-1} N^{g_i}$ and $Q = \bigcap_{i=1}^{r-1} M^{g_i}$. Since $P \cap N^{g_r}/P \cap M^{g_r}$ embeds in $N^{g_r}/M^{g_r}$, the former quotient is $\pi$-minimax. Similarly, from the fact that $P/Q$ is $\pi$-minimax, we infer that $P \cap M^{g_r}/Q \cap M^{g_r}$ is $\pi$-minimax. Consequently, $P \cap N^{g_r}/Q \cap M^{g_r}$ is $\pi$-minimax. This completes the proof of (i).

To establish (ii), assume that $M \lhd N < K$ such that $M$ and $N$ are $H$-invariant. Suppose further that $N/M$ is torsion-free and $\pi$-minimax. Let $g_1, \cdots, g_r$ be a complete list of right coset representatives of $H$ in $G$. Then $\bigcap_{i=1}^{r} M^{g_i}$ and $\bigcap_{i=1}^{r} N^{g_i}$ are $G$-invariant subgroups of $K$. The conclusion follows, then, from assertion (i). □

In proving Theorem A, we will adduce an argument very similar to the one above for Proposition 3.4. The pivotal step in the proof will be to establish that $\mathrm{Ext}^p_{\mathbb{Z}Q}(H_qN, A)$ is torsion, where $N$ is a torsion-free $\pi$-minimax nilpotent group, $Q$ a finitely generated abelian group, and $A$ a $\mathbb{Z}Q$-module without any nontrivial $\mathbb{Z}Q$-sections that are $\pi$-minimax and torsion-free as abelian groups. As we shall see, this can be accomplished with the aid of Proposition A. First, however, it will be necessary to show that $H_qN$ belongs to a special class of modules, defined below.

**Definition.** Assume that $G$ is a group. Let $\mathfrak{C}(G, \pi)$ be the smallest class of $\mathbb{Z}G$-modules with the following two properties.

(i) The class $\mathfrak{C}(G, \pi)$ contains every $\mathbb{Z}G$-module whose additive group is $\pi$-minimax and torsion-free.

(ii) The class $\mathfrak{C}(G, \pi)$ is closed under forming $\mathbb{Z}G$-module quotients as well as extensions.

As an immediate consequence of the definition, we have that $\mathfrak{C}(G, \pi)$ is also subgroup-closed and therefore section-closed. This can be proved very easily by inducting on the number of closure operations from (ii) required to construct a module in $\mathfrak{C}(G, \pi)$; the details are left to the reader.

**Lemma 3.6.** *For any group $G$, the class $\mathfrak{C}(G, \pi)$ is closed under forming $\mathbb{Z}G$-module sections.*

Below we establish another closure property of $\mathfrak{C}(G, \pi)$.

**Lemma 3.7.** *Assume that $G$ is a group. Suppose that $B$ is a $\mathbb{Z}G$-module whose additive group is torsion-free and $\pi$-minimax. If $A$ is a $\mathbb{Z}G$-module in $\mathfrak{C}(G, \pi)$, then $A \otimes B$ lies in $\mathfrak{C}(G, \pi)$, where $A \otimes B$ is viewed as a $\mathbb{Z}G$-module under the diagonal action.*

*Proof.* First we make the following three observations concerning $B$.
(i) If $M$ is a $\mathbb{Z}G$-module whose additive group is torsion-free and $\pi$-minimax, then $M \otimes B$ is $\pi$-minimax and torsion-free.
(ii) If $M$ is a $\mathbb{Z}G$-module and $M'$ is a $\mathbb{Z}G$-module quotient of $M$, then $M' \otimes B$ is a $\mathbb{Z}G$-module quotient of $M \otimes B$.
(iii) If $M$, $M'$, and $M''$ are $\mathbb{Z}G$-modules such that $M$ is an extension of $M'$ by $M''$, then $M \otimes B$ is a $\mathbb{Z}G$-module extension of $M' \otimes B$ by $M'' \otimes B$.
From these three properties it follows that $A \otimes B$ belongs to $\mathfrak{C}(G, \pi)$ by induction on the number of closure operations required to construct $A$ from $\mathbb{Z}G$-modules that are torsion-free and $\pi$-minimax. Statement (i) establishes the base case, and (ii) and (iii) permit the execution of the inductive step. □

In the following proposition, we furnish an alternative characterization of the modules in $\mathfrak{C}(G,\pi)$.

**Proposition 3.8.** *Let $G$ be a group. A $\mathbb{Z}G$-module $A$ belongs to the class $\mathfrak{C}(G,\pi)$ if and only if it has a $\mathbb{Z}G$-module series*

$$0 = A_0 \subset A_1 \subset \cdots \subset A_r = A$$

*such that, for each $i = 1, \cdots, r$, $A_i/A_{i-1}$ is a quotient of some $\mathbb{Z}G$-module $M_i$ whose additive group is $\pi$-minimax and torsion-free.*

*Proof.* Let $\mathfrak{B}$ be the class of all $\mathbb{Z}G$-modules with such a series. Clearly, $\mathfrak{B} \subset \mathfrak{C}(G,\pi)$. Moreover, the class $\mathfrak{B}$ is plainly closed under forming extensions and contains every $\mathbb{Z}G$-module whose additive group is $\pi$-minimax and torsion-free. We claim that it is also closed under forming quotients. To show this, let $A$ be a $\mathbb{Z}G$-module with a series as described in the proposition. Take $\bar{A}$ to be a quotient of $A$. Then $\bar{A}$ has a series

$$0 = \bar{A}_0 \subset \bar{A}_1 \subset \cdots \subset \bar{A}_r = \bar{A}$$

of submodules such that $\bar{A}_i/\bar{A}_{i-1}$ is a quotient of $A_i/A_{i-1}$ for $1 \le i \le r$. Thus $\bar{A}$ belongs to $\mathfrak{B}$. Hence $\mathfrak{B}$ is indeed closed under forming quotients. It follows, then, that $\mathfrak{B} = \mathfrak{C}(G,\pi)$. ◻

Combining the above observation with Proposition A gives rise to the following property of the Ext-functor applied to modules in $\mathfrak{C}(G,\pi)$.

**Corollary 3.9.** *Let $G$ be a finitely generated abelian group. Assume that $B$ is a $\mathbb{Z}G$-module whose additive group is torsion-free and minimax. Suppose further that there are no non-trivial $\mathbb{Z}G$-module sections of $B$ that are torsion-free and $\pi$-minimax as abelian groups. If $A$ is a $\mathbb{Z}G$-module in $\mathfrak{C}(G,\pi)$, then $\mathrm{Ext}^n_{\mathbb{Z}G}(A,B)$ is torsion for $n \ge 0$.*

*Proof.* By Proposition 3.8, there is a series of submodules

$$0 = A_0 \subset A_1 \subset \cdots \subset A_r = A$$

such that, for each $i = 1, \cdots, r$, $A_i/A_{i-1}$ is a quotient of some $\mathbb{Z}G$-module $M_i$ whose additive group is $\pi$-minimax and torsion-free. In view of the hypothesis regarding $B$, the modules $B$ and $M_i$ cannot share any rationally irreducible sections. Thus, by Proposition A, $\mathrm{Ext}^n_{\mathbb{Z}G}(A_i/A_{i-1}, B)$ is torsion for $1 \le i \le r$ and $n \ge 0$. As a consequence, $\mathrm{Ext}^n_{\mathbb{Z}G}(A,B)$ is torsion for $n \ge 0$. ◻

Our purpose in defining the class $\mathfrak{C}(G,\pi)$ and enunciating Corollary 3.9 is to apply the corollary to the integral homology of a nilpotent normal subgroup of a torsion-free $\pi$-minimax group in place of the module $A$. To this end, we require the following property.

**Lemma 3.10.** *Assume that $G$ is a group. Let $N$ be a normal subgroup of $G$ that is nilpotent, $\pi$-minimax, and torsion-free. Then, for each $n \ge 0$, the $\mathbb{Z}G$-module $H_nN$ lies in the class $\mathfrak{C}(G,\pi)$.*

*Proof.* We proceed by induction on the nilpotency class of $N$. If $N$ is abelian, then $H_nN$ is torsion-free and $\pi$-minimax for $n \ge 0$, yielding the conclusion immediately. Assume that the nilpotency class of $N$ exceeds one. In this case, set $Z = Z(N)$ and consider the LHS homology spectral sequence associated to the extension $1 \to Z \to N \to N/Z \to 1$. In this spectral sequence, $E^2_{pq} = H_p(N/Z, H_qZ) = H_p(N/Z) \otimes H_qZ$ for $p,\ q \geq 0$. By the inductive hypothesis, $H_p(N/Z)$ lies in $\mathfrak{C}(G,\pi)$. Therefore, by Lemma 3.7, $E^2_{pq}$ belongs to $\mathfrak{C}(G,\pi)$. Since the differentials in the spectral sequence are $\mathbb{Z}G$-module homomorphisms, $E^\infty_{pq}$ can be regarded as a $\mathbb{Z}G$-module section of $E^2_{pq}$. Thus $E^\infty_{pq}$ belongs to $\mathfrak{C}(G,\pi)$. Moreover, $H_nN$ has a series of submodules whose factors are isomorphic to the modules $E^\infty_{pq}$ for $p+q=n$. The conclusion follows, then, from the closure of $\mathfrak{C}(G,\pi)$ with respect to extensions. □

We now finally have everything in place to prove Theorem A.

**Theorem A.** *Let $G$ be a virtually torsion-free $\pi$-minimax group. Assume that $A$ is a $\mathbb{Z}G$-module whose underlying abelian group is torsion-free and minimax. Suppose further that there are no nontrivial $\mathbb{Z}G$-module sections of $A$ that are torsion-free and $\pi$-minimax as abelian groups. Then $H^n(G,A)$ is torsion for $n \geq 0$.*

*Proof.* As in the proof of Proposition A, it suffices to consider the case where $A$ is rationally irreducible. This assumption allows us to apply Lemma 3.2, obtaining a nilpotent normal subgroup $N$ of $G$ such that $N \subset C_G(A)$ and $G/N$ is a finitely generated virtually abelian group. Let $G_0$ be a torsion-free normal subgroup of $G$ such that $G/G_0$ is finite and $G_0/N_0$ is abelian, where $N_0 = N \cap G_0$. The conclusion of the theorem will then follow if we can show that $H^n(G_0,A)$ is torsion for $n \geq 0$. Furthermore, the latter assertion can be deduced from Proposition 3.4 and the long exact cohomology sequence if we manage to prove that $H^n(G_0, A^{\mathbb{Q}}/A)$ is torsion for $n \geq 0$.

Setting $\tilde{A} = A^{\mathbb{Q}}/A$ and $Q_0 = G_0/N_0$, we will employ the LHS spectral sequence for the group extension $1 \to N_0 \to G_0 \to Q_0 \to 1$ to study $H^n(G_0,\tilde{A})$. Invoking the universal coefficient theorem as well as Corollary 2.5, we deduce that

$$H^p(Q_0, H^q(N_0,\tilde{A})) \cong H^p(Q_0, \mathrm{Hom}(H_qN_0,\tilde{A})) \cong \mathrm{Ext}^p_{\mathbb{Z}Q_0}(H_qN_0,\tilde{A})$$

for $p,\ q \geq 0$. In view of Lemma 3.5(ii), $A$ has no nontrivial $\mathbb{Z}G_0$-module sections whose underlying abelian groups are torsion-free and $\pi$-minimax. Thus, by Lemma 3.10 and Corollary 3.9, $\mathrm{Ext}^p_{\mathbb{Z}Q_0}(H_qN_0,A)$ is torsion for $p,\ q \geq 0$. It follows, then, from Lemma 2.2 that $\mathrm{Ext}^p_{\mathbb{Z}Q_0}(H_qN_0,\tilde{A})$ is torsion for $p,\ q \geq 0$. Therefore, $H^n(G_0,\tilde{A})$ is torsion for $n \geq 0$, thus yielding the conclusion. □

# 4 Near and Hirsch-length supplements

In this section, we prove our two theorems regarding the existence of near supplements and Hirsch-length supplements. In addition to Theorem A, we will appeal to the following similar result of Robinson.

**Proposition 4.1.** (Robinson [**15**, Theorem 2.1]) *Let $G$ be a solvable minimax group and $A$ a $\mathbb{Z}G$-module that is both torsion and minimax qua abelian group. If* spec$(A)$ *and* spec$(G)$ *are disjoint, then $H^n(G,A)$ is torsion for all $n \geq 0$.*

Having stated this property, we are ready to proceed with the proof of our main near-supplement theorem.

**Theorem B.** *Let $G$ be a solvable minimax group. Assume that $K$ is a normal subgroup of $G$ such that $G/K$ is $\pi$-minimax and virtually torsion-free. Then there is a $\pi$-minimax subgroup $X$ of $G$ such that $[G:KX]$ is finite.*

*Proof.* Set $Q = G/K$. The result is trivial if $K$ is $\pi$-minimax, so we assume that it is not. It can be deduced by induction on $m(K)$ that $K$ has a $G$-invariant series

$$1 = K_0 < K_1 < \cdots < K_r = K$$

such that, for each $i = 1, \cdots, r$, the following two statements hold:

(i) $K_i/K_{i-1}$ is abelian;
(ii) every $G$-invariant subgroup of $K_i/K_{i-1}$ is either finite or cofinite.

We will prove the theorem by inducting on the length $r$ of such a series. Suppose that $r = 1$; that is, $K$ is abelian, and every $G$-invariant subgroup of $K$ is either finite or cofinite. First we treat the case where $K$ is torsion. Let $K_\pi$ be the $\pi$-torsion part of $K$. Then $K/K_\pi$ is a $\pi'$-group. Therefore, by Proposition 4.1, $H^2(Q, K/K_\pi)$ is torsion. According to Proposition 1.1, this means that there is a subgroup $X$ of $G$ containing $K_\pi$ such that $X \cap K/K_\pi$ is finite and $[G : KX]$ is finite. Then $X$ is $\pi$-minimax, yielding the conclusion. Next assume that $K$ is not torsion. It follows that the torsion subgroup of $K$ must be finite. Hence, without real loss of generality, we can assume that $K$ is torsion-free. Since $K$ is not $\pi$-minimax, none of its infinite $G$-invariant sections are $\pi$-minimax. Therefore, we can apply Theorem A to the $\mathbb{Z}G$-module $K$ and conclude that $H^2(Q, K)$ is torsion. As a result, $G$ must possess a subgroup $X$ such that $X \cap K$ is finite and $[G : KX]$ is finite. Since $X$ is plainly $\pi$-minimax, this completes the argument for the case that $r = 1$.

Suppose now that $r > 1$, and set $L = K_{r-1}$. By the base case of our induction, $G$ has a subgroup $Y$ containing $L$ such that $Y/L$ is $\pi$-minimax and $[G : KY]$ is finite. Applying the inductive hypothesis to $L$ inside $Y$, we obtain a $\pi$-minimax subgroup $X < Y$ such that $[Y : LX]$ is finite. It follows, then, that $[G : KX]$ is finite, thus completing the proof. □

Below we describe an example that demonstrates that the hypothesis that $G/K$ is virtually torsion-free cannot be removed from Theorem B.

**Example 4.2.** Assume that $p$ and $q$ are primes. Define an action of $C_\infty = \langle t \rangle$ on $\mathbb{Z}[1/pq]$ by $t \cdot x = px$ for $x \in \mathbb{Z}[1/pq]$. Let $G = \mathbb{Z}[1/pq] \rtimes \langle t \rangle$ and $K = \mathbb{Z}[1/p]$. Then $K \lhd G$, and $G/K$ is $\{q\}$-minimax but not virtually torsion-free. Moreover, $K$ has no $\{q\}$-minimax near supplement. To see this, suppose that $X$ is a near supplement to $K$. Then $K \cap X \neq 1$; otherwise $X$ would contain a quasicyclic subgroup. Since $K \cap X \lhd KX$, it follows that $K \cap X = K$. Thus $X$ cannot be $\{q\}$-minimax. Notice, however, that $K$ has a $\{q\}$-minimax Hirsch-length supplement, namely, $\langle t \rangle$.

The above example suggests that a normal subgroup of a solvable minimax group whose associated quotient is $\pi$-minimax might at least have a $\pi$-minimax Hirsch-length supplement, if not a near supplement. In Theorem C below, we prove that this is indeed always the case. In addition, we observe that, if the quotient happens to be finitely generated, then any Hirsch-length supplement is necessarily a near supplement. This is due to the fact that, in a finitely generated solvable minimax group, every subgroup with the same Hirsch length as the group must have finite index (Lemma 4.4).

**Theorem C.** *Let $G$ be a solvable minimax group. Assume that $K$ is a normal subgroup of $G$ such that $G/K$ is $\pi$-minimax. Then there is a $\pi$-minimax subgroup $X$ of $G$ such that $h(KX) = h(G)$. Furthermore, if $G/K$ is finitely generated, then $[G : KX]$ is finite.*

Before proceeding with the proof of Theorem C, we need to establish the following property of Hirsch lengths of subgroups.

**Lemma 4.3.** *Let $G$ be a virtually solvable group with finite torsion-free rank and $K$ a normal subgroup of $G$. If $X < Y < G$ and $h(X) = h(Y)$, then $h(KX) = h(KY)$.*

*Proof.* The result follows from the following chain of relations.

$$h(KX) = h(K) + h(X) - h(K \cap X) \geq h(K) + h(Y) - h(K \cap Y) = h(KY).$$

□

**Proof of Theorem C.** Set $Q = G/K$. As in the proof of Theorem B, we suppose that $K$ is not $\pi$-minimax and employ a $G$-invariant series

$$1 = K_0 < K_1 < \cdots < K_r = K$$

with properties (i) and (ii). Again we induct on $r$. Assume that $r = 1$. The case where $K$ is torsion is handled exactly as in the proof of Theorem B. Suppose that $K$ is not torsion. Instead of Theorem A as before, we appeal here to Proposition 3.4, which allows us to conclude that $H^2(Q, K^{\mathbb{Q}}) = 0$. Hence we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & K^{\mathbb{Q}} & \longrightarrow & \bar{G} & \longrightarrow & Q & \longrightarrow & 1
\end{array}$$

with exact rows such that the bottom extension splits. Let $\bar{X}$ be a complement to $K^{\mathbb{Q}}$ in $\bar{G}$, and take $X < G$ to be the inverse image of $\bar{X}$. Then $h(X \cap K) = 0$ and $h(X) = h(Q)$. Thus $h(KX) = h(K) + h(X) = h(G)$. This completes the proof of the base case.

Assume that $r > 1$, and set $L = K_{r-1}$. By the base case of our induction, $G$ has a subgroup $Y$ containing $L$ such that $Y/L$ is $\pi$-minimax and $h(KY) = h(G)$. Applying the inductive hypothesis to $L$ inside $Y$, we obtain a $\pi$-minimax subgroup $X < Y$ such that $h(LX) = h(Y)$. From Lemma 4.3 we infer, then, that $h(KX) = h(G)$. In conclusion, we observe that, if $G/K$ is finitely generated, then $[G : KX]$ is finite. This assertion follows from Lemma 4.4 below. □

The succeeding lemma is surely known to experts on solvable minimax groups. However, since no mention of it appears in the literature, we provide a proof.

**Lemma 4.4.** *Let $G$ be a finitely generated solvable minimax group. If $H < G$ and $h(H) = h(G)$, then $[G : H]$ is finite.*

*Proof.* The proof is by induction on $m(G)$. If $m(G) = 0$, then the conclusion follows immediately. Assume that $m(G) > 0$. Let $A$ be an infinite abelian subgroup with $m(A)$ as small as possible subject to the condition that $[G : N_G(A)]$ is finite. Without any real loss of generality, we may replace $G$ by $N_G(A)$ and $H$ by $H \cap N_G(A)$, rendering $A$ normal in $G$. In this case, we have

$$h(AH/A) = h(H) - h(H \cap A) \geq h(G) - h(A) = h(G/A),$$

implying that $h(AH/A) = h(G/A)$. Hence, by the inductive hypothesis, $AH$ has finite index in $G$.

At this juncture, we distinguish the case of $A$ being torsion from that where $A$ is not torsion. First we suppose that $A$ is torsion. Because $G$ is finitely generated, $AH$ is also finitely generated. Thus there is a finite subset $\mathcal{F}$ of $A$ such that $\langle H \cup \mathcal{F}\rangle = AH$. Since $A$ is a Černikov group, it has a characteristic finite subgroup $F$ containing $\mathcal{F}$. Then $FH = AH$, so that $[AH : H]$ is finite. Hence $[G : H]$ is finite.

Next we treat the case that $A$ is not torsion. Since $H \cap A$ is normal in $AH$, our choice of $A$ ensures that $H \cap A$ is either finite or has finite index in $A$. If $H \cap A$ is finite, then $h(H) = h(G/A) < h(G)$, a contradiction. Therefore, $H \cap A$ has finite index in $A$. It follows, then, that $[AH : H]$ is finite. Thus $[G : H]$ is finite. □

We conclude this section by applying Theorem C to obtain a new result on the homological dimension of solvable groups. It is conjectured in [**4**] that, if $k$ is a field and $G$ a solvable group, then $\mathrm{hd}_k(G)$ is either infinite or equal to the Hirsch length of $G$. This is established by U. Stammbach [**17**] if $k$ has characteristic zero and proved in [**4**] for arbitrary fields in the abelian-by-polycyclic case. We refer the reader to [**4**] for background information on this problem, as well as to [**3**] for the basic facts about homological dimension. Here we employ Theorem C to establish the following generalization of the abelian-by-polycyclic case.

**Proposition 4.5.** *Let $p$ be a prime and $k$ a field of characteristic $p$. Suppose that $G$ is an extension of an abelian group by a $\{p\}'$-minimax group. If $G$ has finite homological dimension over $k$, then* $\mathrm{hd}_k(G) = h(G)$.

*Proof.* That $G$ has finite homological dimension over $k$ yields that $G$ has finite Hirsch length. In addition, it implies that $G$ has no $k$-torsion, meaning that any finite element order in $G$ is invertible in $k$. Let $T = \tau(G)$. Appealing to the LHS spectral sequence, we will argue that $\mathrm{hd}_k(G/T) = \mathrm{hd}_k(G)$. Since $\mathrm{hd}_k(T) = 0$, we deduce right away that $\mathrm{hd}_k(G/T) \geq \mathrm{hd}_k(G)$. To verify the reverse inequality, set $n = \mathrm{hd}_k(G/T)$ and let $M$ be a right $k[G/T]$-module such that $H_n(G/T, M) \neq 0$. Since $H_n(G, M) = H_n(G/T, M)$, it follows that $\mathrm{hd}_k(G) \geq n$. Since $\mathrm{hd}_k(G/T) = \mathrm{hd}_k(G)$, we may, without any real loss of generality, assume that $G$ is torsion-free. As observed in [**3**, Proposition 6.14], there is a finitely generated subgroup of $G$ with the same homological dimension over $k$ as $G$. In addition, there is a finitely generated subgroup with the same Hirsch length as $G$. This means that we can find a finitely generated subgroup enjoying both of these properties. Consequently, it suffices to consider the case where $G$ is finitely generated and therefore minimax.

Invoking Theorem C, and replacing $G$ by a subgroup of finite index if necessary, we may assume that $G$ has a $\{p\}'$-minimax subgroup $X$ and an abelian normal subgroup $A$ such that $AX = G$. The action of $X$ on $A$ by conjugation allows us to construct a semidirect product $A \rtimes X$. Moreover, there is a surjection $\phi : A \rtimes X \to G$, where $B = \mathrm{Ker}\,\phi$ is isomorphic to $A \cap X$. We claim that, if $\mathrm{hd}_k(A \rtimes X) = h(A \rtimes X)$, then $\mathrm{hd}_k(G) = h(G)$. To establish this claim, assume that $\mathrm{hd}_k(A \rtimes X) = h(A \rtimes X)$. From the LHS spectral sequence we know that $\mathrm{hd}_k(A \rtimes X) \leq \mathrm{hd}_k(B) + \mathrm{hd}_k(G)$. Hence $\mathrm{hd}_k(G) \geq h(A \rtimes X) - h(B) = h(G)$. Also, $\mathrm{hd}_k(G) \leq h(G)$ by [**3**, Theorem 7.11]. Therefore, the above claim holds; in other words, we do not really lose any generality in supposing that $G = A \rtimes X$.

The argument advanced by Stammbach [**17**] for fields of characteristic zero also serves to show that $\mathrm{hd}_k(X) = h(X)$ (see [**4**, Lemma I.9]). Let $m$ be the product of all the primes in $\mathrm{spec}(A)$. Then $\tilde{A} = A^{\mathbb{Z}[1/m]}$ is isomorphic to $\mathbb{Z}[1/m]^r$, where $r = h(A)$. Regarding $\tilde{A}$ multiplicatively, let $\theta$ be the automorphism of $\tilde{A}$ defined by $\theta(x) = x^m$. Set $K = \tilde{A} \rtimes \langle\theta\rangle$.

The action of $X$ on $A$ induces an action of $X$ on $\widetilde{A}$. Now define an action of $X$ on $K$ by employing the action of $X$ on $\tilde{A}$ and allowing $X$ to centralize $\theta$. Using this action, form the semidirect product $\tilde{G} = K \rtimes X$. As an ascending HNN extension of $\mathbb{Z}^r$, $K$ is constructible and thus an inverse duality group (see [**2**, Theorem 9]). By [**3**, Theorem 9.4], it follows that $K$ has type FP and $H^n(K, kK)$ is $k$-free. Hence, in view of [**3**, Theorem 5.5], $\mathrm{hd}_k(\widetilde{G}) = \mathrm{hd}_k(K) + \mathrm{hd}_k(X)$. Moreover, according to [**4**, Proposition I.11], $\mathrm{hd}_k(K) = h(K)$. In addition, since $\tilde{G}$ can be viewed as an ascending HNN extension of $G$, the Mayer-Vietoris sequence yields that $\mathrm{hd}_k(\tilde{G}) \le \mathrm{hd}_k(G) + 1$. We may thus argue as follows.

$$\mathrm{hd}_k(G) \ge \mathrm{hd}_k(\tilde{G}) - 1 = h(K) + h(X) - 1 = h(A) + h(X) = h(G).$$

As remarked above, the reverse inequality is already known to hold; hence $\mathrm{hd}_k(G) = h(G)$. □

## 5 Near complements

In this section, we consider the same situation as in Theorem B; that is, we suppose that $G$ is a solvable minimax group and $K \lhd G$ such that $G/K$ is $\pi$-minimax and virtually torsion-free. Here our goal is to determine circumstances under which Theorem B can be strengthened to yield a near complement, rather than just a $\pi$-minimax near supplement. In order to ensure this, it will be necessary to impose two quite stringent conditions on $K$. First, we will assume that $K$ is Noetherian as a $G$-operator group; in other words, $K$ satisfies the maximal condition on $G$-invariant subgroups. Second, we will require that $K$ is, in some sense, the "antithesis" of a torsion-free $\pi$-minimax group.

Before we concern ourselves with the second hypothesis, we point out that the Noetherian property, as well as the condition that the quotient is virtually torsion-free, applies to any normal subgroup in a solvable minimax group that satisfies the maximal condition on normal subgroups. Moreover, solvable minimax groups with the latter attribute are ubiquitous, especially in geometric group theory. In particular, every constructible solvable group enjoys this property. Recall that the class of constructible groups is the smallest class containing the trivial group that is closed under forming finite extensions, generalized free products in which both factors as well as the amalgamated subgroup are constructible, and HNN extensions in which the base group and associated subgroups are constructible. Furthermore, as shown in [**2**], the class of constructible solvable groups is the smallest class of groups that is closed with respect to forming extensions by finite solvable groups and ascending HNN extensions with base group in the class. As well as being of interest to geometers, constructible solvable groups are important in homological algebra since the torsion-free constructible groups are precisely those solvable groups that have type FP (see [**9**]).

Now we turn our attention to the second hypothesis that we will require for our near-complement result, namely, that $K$ fails to be a torsion-free $\pi$-minimax group in some drastic fashion. In order to impart a more precise form to this idea, we introduce the following class.

**Definition.** The class $\mathfrak{X}_\pi$ is the class of all groups $G$ such that every $\pi$-minimax quotient of $G$ is torsion.

To illustrate this definition we consider some examples. Notice that every torsion group belongs to $\mathfrak{X}_\pi$, whereas all $\pi$-minimax groups that are not torsion fall outside $\mathfrak{X}_\pi$. Another elementary observation is that a torsion-free abelian group of rank one is a member of $\mathfrak{X}_\pi$ if and only if it is not $\pi$-minimax. From this example we can see that the class $\mathfrak{X}_\pi$ fails to be closed under taking subgroups.

Although $\mathfrak{X}_\pi$ is not subgroup-closed, the class enjoys the four closure properties described below in Lemma 5.1. Since the proofs of these are straightforward, we leave them to the reader.

**Lemma 5.1.** *(i) If $G$ belongs to $\mathfrak{X}_\pi$, then so does every subgroup of finite index in $G$.*
*(ii) The class $\mathfrak{X}_\pi$ is closed under forming quotients and extensions.*
*(iii) Let $G$ be a group. If $\{N_\alpha : \alpha \in I\}$ is a family of normal subgroups of $G$ such that each $N_\alpha$ lies in $\mathfrak{X}_\pi$, then the join of the $N_\alpha$ also belongs to $\mathfrak{X}_\pi$.*

Statements (ii) and (iii) in Lemma 5.1 make $\mathfrak{X}_\pi$ a radical class in the sense used in [**12**, §1.3]. Employing the terminology from [**12**, §1.3], we define the $\mathfrak{X}_\pi$-radical of a group $G$, denoted $\rho_\pi(G)$, to be the join of all the normal $\mathfrak{X}_\pi$-subgroups of $G$. In studying the properties of $\mathfrak{X}_\pi$ and the $\mathfrak{X}_\pi$-radical, we will draw upon the wealth of information about radical classes contained in [**12**, §1.3].

Our definition of the class $\mathfrak{X}_\pi$ is inspired by the notion of an upper-finite group from [**13**, §10.4]. To shed light on the connection, consider the special case $\pi = \emptyset$. The class $\mathfrak{X}_\emptyset$ is the class of groups all of whose polycyclic quotients are finite. Using the fact that solvable minimax groups are torsion-by-nilpotent-by-polycyclic, it follows that a solvable minimax group is an $\mathfrak{X}_\emptyset$-group if and only if it is upper-finite in the sense employed in [**13**, §10.4]; that is, every finitely generated quotient is finite. Hence the $\mathfrak{X}_\emptyset$-radical of a solvable minimax group coincides with the upper-finite radical from [**13**, §10.4].

The result on near complements that we will prove in this section is stated below.

**Theorem D.** *Let $G$ be a solvable minimax group and $K \lhd G$. Assume that $K$ is a member of $\mathfrak{X}_\pi$ and $K$ is Noetherian as a $G$-operator group. Suppose further that $G/K$ is $\pi$-minimax and virtually torsion-free. Then there exists a near complement to $K$ in $G$.*

Throughout our discussion, we will make use of the following elementary property of Noetherian groups with operators that are solvable and minimax.

**Lemma 5.2.** *Assume that $G$ is a group and $N$ a Noetherian $G$-operator group. If $N$ is solvable and minimax, then $\tau(N)$ is finite.*

The proof of Lemma 5.2 is very easy and therefore left to the reader.

The first step towards proving Theorem D is to establish that the property of belonging to $\mathfrak{X}_\pi$ is inherited by submodules of certain Noetherian modules that lie in $\mathfrak{X}_\pi$.

**Lemma 5.3.** *Let $G$ be an abelian group and $A$ a Noetherian $G$-module that is minimax as an abelian group. If the underlying abelian group of $A$ is in the class $\mathfrak{X}_\pi$, then the same is true for every submodule of $A$.*

*Proof.* We argue by induction on $h(A)$. For $h(A) = 0$, the conclusion is trivially true. Assume that $h(A) \geq 1$. First we show that every rationally irreducible submodule of $A$ lies in $\mathfrak{X}_\pi$. Suppose that there is a rationally irreducible submodule $B$ outside the class $\mathfrak{X}_\pi$. Set $\bar{A} = A/B$. By the inductive hypothesis, every $\mathbb{Z}G$-module section of $\bar{A}$ belongs to $\mathfrak{X}_\pi$. On the other hand, $B$ has no infinite $\mathbb{Z}G$-module sections that lie in $\mathfrak{X}_\pi$. Therefore, by Lemma 2.2, $\mathrm{Ext}^1_{\mathbb{Q}G}(\bar{A}^{\mathbb{Q}}, B^{\mathbb{Q}}) = 0$, implying that $B^{\mathbb{Q}}$ is a direct summand in $A^{\mathbb{Q}}$. According to Lemma 2.3, this means that $B$ has an infinite section in $\mathfrak{X}_\pi$, yielding a contradiction. It follows, then, that every rationally irreducible submodule of $A$ belongs to $\mathfrak{X}_\pi$.

To complete the proof, we let $B$ be an arbitrary $\mathbb{Z}G$-submodule of $A$. If $B$ is finite, then the conclusion follows immediately; hence we assume that $B$ is infinite. Let $C$ be a rationally irreducible submodule of $B$. We have that $C$ belongs to $\mathfrak{X}_\pi$ by what was proved above. Moreover, $B/C$ lies in $\mathfrak{X}_\pi$ by virtue of the inductive hypothesis. Therefore, $B$ is a member of $\mathfrak{X}_\pi$.

□

We wish to discern inheritance properties similar to Lemma 5.3 for normal subgroups of solvable minimax groups. In order to accomplish this, we require the notion of a nilpotent action.

**Definition.** Assume that $G$ is a group and $N$ a $G$-operator group. We define the *lower central $G$-series*

$$\cdots < \gamma_3^G N < \gamma_2^G N < \gamma_1^G N$$

of $N$ as follows: $\gamma_1^G N = N$; $\gamma_i^G N = \langle a(g \cdot b)a^{-1}b^{-1} \mid a \in N, b \in \gamma_{i-1}^G N, g \in G\rangle$ for $i > 1$.

We say that the action of $G$ on $N$ is *nilpotent* if there is a nonnegative integer $c$ such that $\gamma_{c+1}^G N = 1$. The smallest such integer $c$ is called the *nilpotency class* of the action.

In studying nilpotent actions, the following well-known property is exceedingly useful.

**Proposition 5.4.** *Assume that $G$ is a group and $N$ a $G$-operator group. Then, for each $i \geq 1$, there is an epimorphism*

$$\theta_i : \underbrace{G_{\text{ab}} \otimes \cdots \otimes G_{\text{ab}}}_{i-1} \otimes (N/\gamma_2^G N) \to \gamma_i^G N/\gamma_{i+1}^G N.$$

The above epimorphism can be employed to prove the lemma below.

**Lemma 5.5.** *Let $G$ be a group such that $G_{\text{ab}}$ has finite torsion-free rank. Assume that $N$ is a $G$-operator group upon which $G$ acts nilpotently. If $N$ belongs to $\mathfrak{X}_\pi$, then $\gamma_i^G(N)$ is in $\mathfrak{X}_\pi$ for all $i \geq 1$.*

*Proof.* The conclusion follows immediately from Proposition 5.4 and Lemma 5.6 below. □

**Lemma 5.6.** *Let $A$ and $B$ be abelian groups. If $A$ belongs to $\mathfrak{X}_\pi$ and $B$ has finite torsion-free rank, then $A \otimes B$ is a member of $\mathfrak{X}_\pi$.*

*Proof.* Choose a free abelian subgroup $C$ of $B$ such that $B/C$ is torsion. Then $C \cong \mathbb{Z}^n$, where $n = h(B)$, and so $A \otimes C \cong \underbrace{A \oplus \cdots \oplus A}_{n}$. The class $\mathfrak{X}_\pi$ is closed under extensions and quotients; hence every quotient of $A \otimes C$ is in $\mathfrak{X}_\pi$ . Also, as a torsion group, $A \otimes B/C$ belongs to $\mathfrak{X}_\pi$ . It follows, then, from the exact sequence $A \otimes C \to A \otimes B \to A \otimes B/C \to 0$ that $A \otimes B$ lies in $\mathfrak{X}_\pi$. □

Lemma 5.5 allows us to establish the following two inheritance properties for solvable minimax $\mathfrak{X}_\pi$-groups. The second of these, Lemma 5.8, will play an essential role in the proof of Theorem D.

**Lemma 5.7.** *Let $M$, $N$, and $P$ be nilpotent normal subgroups of a group $G$ such that $M < N < P$ and the following three conditions hold.*

*(i) The quotient $G/P$ is abelian.*
*(ii) The group $P_{\text{ab}}$ has finite torsion-free rank.*
*(iii) The subgroup $N$ is Noetherian as a $G$-operator group.*

*If $N$ belongs to $\mathfrak{X}_\pi$, then $M$ does too.*

*Proof.* Our proof is by induction on the nilpotency class of the action of $P$ on $N$. If $P$ acts trivially, then we can deduce the conclusion from Lemma 5.3, regarding $N$ and $M$ as $G/P$-modules. Assume that the nilpotency class $c$ of the action exceeds one, and let $A = \gamma_c^P(N)$. Then $A$ is a $\mathfrak{X}_\pi$-group by Lemma 5.5. Consider the chain

$$M/M \cap A < N/A < P/A,$$

the second term of which is in $\mathfrak{X}_\pi$. The action of $P/A$ on $N/A$ is nilpotent of class $c-1$. Consequently, the inductive hypothesis yields that $M/M \cap A$ is in $\mathfrak{X}_\pi$. Next we look at the groups $M \cap A$ and $A$. Treating these groups as $G/P$-modules, it follows from Lemma 5.3 that $M \cap A$ belongs to $\mathfrak{X}_\pi$. Therefore, $M$ is in $\mathfrak{X}_\pi$, as desired.

□

**Lemma 5.8.** *Let $G$ be a solvable minimax group. Suppose that $M$ and $N$ are normal subgroups of $G$ such that $M < N$ and $N$ is Noetherian as a $G$-operator group. If $N$ is in $\mathfrak{X}_\pi$, then so is $M$.*

*Proof.* As a virtually torsion-free solvable minimax group, $N$ can be expressed as an extension of a nilpotent group by a polycyclic one. Since $N$ belongs to the class $\mathfrak{X}_\pi$, this means that $N$ must be virtually nilpotent. By passing to a $G$-invariant subgroup of finite index, we can assume that $N$ is nilpotent. Therefore, the conclusion follows immediately by applying the above lemma to the chain $M < N < \text{Fitt}(G)$.

□

Armed with the above lemma, we can proceed with the proof of Theorem D.

**Proof of Theorem D.** The proof proceeds by induction on the length of the derived series of $K$. First suppose that $K$ is abelian. Invoking Theorem B, we obtain a $\pi$-minimax subgroup $Y$ such that $KY$ has finite index in $G$. In [**18**] it is shown that a Noetherian $G$-operator group is also Noetherian with respect to any subgroup of finite index in $G$. Therefore, no significant loss of generality will result from assuming that $G = KY$. This renders $K \cap Y$ normal in $G$. It follows, then, from Lemma 5.8 that $K \cap Y$ must be in the class $\mathfrak{X}_\pi$. However, $K \cap Y$ is $\pi$-minimax, which means that it must be finite. Since $G/K$ is virtually torsion-free, $Y$ is residually finite. Thus $Y$ has a subgroup $X$ of finite index such that $X \cap K = 1$. The subgroup $X$, then, can serve as the near complement that we seek.

Next assume that the derived length of $K$ is greater than one. By the abelian case, $G$ contains a subgroup $S$ such that $S \cap K = K'$ and $KS$ has finite index in $G$. According to Lemma 5.8, $K'$ belongs to $\mathfrak{X}_\pi$. Hence we can apply the inductive hypothesis to $K'$ inside of $S$ to obtain $X < S$ such that $K' \cap X = 1$ and $K'X$ has finite index in $S$. Then $X$ fulfills our requirements. □

In view of the observations made in this section's second paragraph, Theorem D has the following important special case.

**Corollary 5.9.** *Let $G$ be a solvable minimax group that satisfies the maximal condition on normal subgroups. Assume that $K$ is a normal $\mathfrak{X}_\pi$-subgroup of $G$ such that $G/K$ is $\pi$-minimax. Then $K$ has a near complement in $G$.*

In addition to the assumption that $G/K$ is $\pi$-minimax, Theorem D requires three hypotheses: (1) $G/K$ is virtually torsion-free; (2) $K$ is Noetherian; (3) $K$ belongs to $\mathfrak{X}_\pi$. It is very easy to demonstrate that each of these three conditions is indispensable. To accomplish this, we present three examples in which no near complement is present; in each, one of the three hypotheses is violated, while the other two hold.

**Example 5.10.** Let $p$ be a prime and $\Gamma$ the group consisting of all matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & \dagger & * \\ 0 & 0 & 1 \end{pmatrix},$$

where the entries $*$ above the diagonal are chosen from the ring $\mathbb{Z}[1/p]$ and the diagonal entry $\dagger$ is an integer power of $p$. Let $A$ be the central subgroup generated by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $G = \Gamma/A$ and $K$ be the central subgroup of $G$ generated by the image of

$$\begin{pmatrix} 1 & 0 & p^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $G$ is a finitely generated solvable minimax group and $K \cong C_p$. If $\pi = \{p\}$, then $G/K$ is $\pi$-minimax, yet $K$ lacks a near complement. Notice that $K$ satisfies conditions (2) and (3), but $G/K$ fails to fulfill (1).

**Example 5.11.** Let $p$ be a prime and $G$ the nilpotent minimax group consisting of all matrices of the form

$$\begin{pmatrix} 1 & * & \dagger \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},$$

where the entries $*$ are integers and the entry $\dagger$ is from the ring $\mathbb{Z}[1/p]$. Let $K = Z(G)$; that is, $K$ consists of all the matrices in $G$ whose $*$ entries are 0. For $\pi = \emptyset$, $G/K$ is $\pi$-minimax, and the extension $1 \to K \to G \to G/K \to 1$ satisfies conditions (1) and (3), but not (2). In addition, there is no near complement to $K$ in $G$.

**Remark.** The group $G$ in Example 5.11 is not finitely generated. It would be interesting to discover whether there is a finitely generated example with the same characteristics.

**Example 5.12.** Let $G$ be the group of 3x3 upper unitriangular matrices with integer entries. Set $K = Z(G)$. For $\pi = \emptyset$, $G/K$ is $\pi$-minimax; also, the extension $1 \to K \to G \to G/K \to 1$ satisfies conditions (1) and (2), but not (3). Finally, we observe that $K$ does not have a near complement.

One particular normal subgroup $K$ of a solvable minimax group $G$ that often satisfies the hypotheses of Theorem D is the subgroup $\rho_\pi(G)$. In order to show this, we require the following observation.

**Proposition 5.13.** *Let $G$ be a solvable group such that $\tau(G)$ is finite. If every subnormal abelian subgroup of $G$ is $\pi$-minimax, then $G$ must be $\pi$-minimax.*

In [**11**, §6.1] Robinson provides his own proof of R. Baer's theorem [**1**] that a solvable group is $\pi$-minimax if and only if all its abelian subgroups are $\pi$-minimax. The argument advanced by Robinson for the case where the torsion radical is finite can be invoked to prove Proposition 5.13.

Proposition 5.13 allows us to establish the following lemma and its corollary.

**Lemma 5.14.** *Let $G$ be a solvable minimax group. If $G$ has no nontrivial subnormal $\mathfrak{X}_\pi$-subgroups, then $G$ is $\pi$-minimax.*

*Proof.* Suppose that $G$ is not $\pi$-minimax. We will show that $G$ must contain a nontrivial subnormal $\mathfrak{X}_\pi$-subgroup, proving the lemma. If $G$ is not residually finite, then its torsion radical can serve as the desired subgroup. Assume that $G$ is residually finite. By Proposition 5.13, $G$ must have a subnormal abelian subgroup $A$ that is not $\pi$-minimax. Let $B$ be a subgroup of $A$ with the smallest possible Hirsch length such that $A/B$ is $\pi$-minimax. Then $B$ is a nontrivial group in $\mathfrak{X}_\pi$, and $B$ is subnormal in $G$. □

**Corollary 5.15.** *If $G$ is a solvable minimax group, then $G/\rho_\pi(G)$ is a virtually torsion-free $\pi$-minimax group.*

*Proof.* The $\mathfrak{X}_\pi$-radical of $G/\rho_\pi(G)$ is trivial. Hence $G/\rho_\pi(G)$ has no nontrivial subnormal $\mathfrak{X}_\pi$-subgroups. Therefore, it is $\pi$-minimax by Lemma 5.14. In addition, its torsion radical is trivial, making it virtually torsion-free. □

In light of Corollary 5.15, we can state the following corollary to Theorem D.

**Corollary 5.16.** *Let $G$ be a solvable minimax group such that $\rho_\pi(G)$ is Noetherian as a $G$-operator group. Then $\rho_\pi(G)$ has a near complement in $G$.*

We conclude this section by observing that Corollary 5.16, and *a fortiori* Theorem D, cannot be strengthened to deliver a full complement to $K$.

**Example 5.17.** Assume that $p$ is an odd prime. Let $Q$ be a free abelian group of rank 2. Let $A = \mathbb{Z}[1/p]$ and define an action of $Q$ on $A$ by having both generators act via multiplication by $p$. This makes $A$ into a Noetherian $\mathbb{Z}Q$-module. We have $H^2(Q,A) \cong H_0(Q,A)$ by Poincaré duality. Hence, since $p \neq 2$, $H^2(Q,A) \neq 0$. Therefore, there is at least one nonsplit extension of $A$ by $Q$. In this extension, $A$ is the $\mathfrak{X}_\emptyset$-radical and fails to have a full complement.

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