=][p

Finiteness conditions in the stable module category

Jonathan Cornick

Department of Mathematics and Computer Science, Queensborough Community College, CUNY 222-05 56th Avenue, Bayside, New York 11364, USA

Ioannis Emmanouil

Department of Mathematics, University of Athens, Athens 15784, Greece

Peter Kropholler

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, UK

Olympia Talelli

Department of Mathematics, University of Athens, Athens 15784, Greece

Abstract

We study groups whose cohomology functors commute with filtered colimits in high dimensions. We relate this condition to projective resolutions which exhibit some finiteness properties in high dimensions, and to the existence of Eilenberg–Mac Lane spaces with finitely many n-cells for all sufficiently large n. To that end we determine the structure of finitary Gorenstein projective modules. The methods are inspired by representation theory and make use of the stable module category in which morphisms are defined through complete cohomology. In order to carry out these methods we need to restrict ourselves to certain classes of hierarchically decomposable groups.

Keywords: cohomology of groups, Gorenstein projective dimension, stable module category

Email addresses: jcornick@qcc.cuny.edu (Jonathan Cornick), emmmanoui@math.uoa.gr (Ioannis Emmanouil), p.h.kropholler@soton.ac.uk (Peter Kropholler), otalelli@math.uoa.gr (Olympia Talelli)

Preprint submitted to Elsevier *March 11, 2014*

Finiteness conditions in the stable module category

Jonathan Cornick

Department of Mathematics and Computer Science, Queensborough Community College, CUNY 222-05 56th Avenue, Bayside, New York 11364, USA

Ioannis Emmanouil

Department of Mathematics, University of Athens, Athens 15784, Greece

Peter Kropholler

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, UK

Olympia Talelli

Department of Mathematics, University of Athens, Athens 15784, Greece

---

Contents

0. Introduction

It follows from work of Bieri and Eckmann that a group G (resp. an R-module M over a ring R) is of type FP_n if and only if the functors $H^i(G, \_)$ (resp. $\mathrm{Ext}^i_R(M, \_)$) commute with filtered colimits (also referred to as inductive limits) of coefficient modules for all i in the range $0 \leq i < n$.

Email addresses: `jcornick@qcc.cuny.edu` (Jonathan Cornick), `emmmanoui@math.uoa.gr` (Ioannis Emmanouil), `p.h.kropholler@soton.ac.uk` (Peter Kropholler), `otalelli@math.uoa.gr` (Olympia Talelli)

Preprint submitted to Elsevier *March 11, 2014*

The FP_n property says that there is a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ (resp. M) in which the 0th through to the nth projective modules are finitely generated. In the group case this property is inherited from the existence of an Eilenberg–Mac Lane space in which there are finitely many cells of dimension $\leq n$. The interconnections between these results are well known and we refer the reader to Bieri's notes [4] for an excellent presentation.

Inspired by work of Rickard [20] and Benson [2] in representation theory of finite groups, a branch of research emerged in which one could study the stable module category for certain classes of infinite groups. The first and third authors of the present work introduced a theory in a series of papers [6, 8, 7, 16] the last one of which builds on the ideas of Benson.

In view of the original Bieri–Eckmann theory, it seems very natural to consider what happens if one requires some or many cohomology functors to commute with filtered colimits. The behaviour of groups for which this kind of continuity of functors begins after some steps was investigated by the third author and then further in a series of papers by Hamilton [17, 11, 13, 12]. The notion of continuity in these works is given the official categorical name of being *finitary* but it is the same Bieri–Eckmann property that is studied.

It is then natural to ask questions of the following kind:

- When does a module possess a projective resolution in which the projective modules are *eventually* finitely generated?

- When does a group possess an Eilenberg–Mac Lane space in which there are only finitely many n-cells for all *sufficiently large* n?

By working *stably*, using methods inspired by Rickard and Benson, we are able to study some of these questions and obtain new results going beyond the work of Hamilton.

Our main conclusion applies to groups in the class $\mathbf{LH}\mathfrak{K}$, where $\mathfrak{K}$ is the class of groups whose group rings are coherent. This is the smallest class of groups containing $\mathfrak{K}$, which is closed under directed unions and such that whenever a group acts on a finite dimensional contractible cell complex (admissibly) with stabilizers already in $\mathbf{LH}\mathfrak{K}$ then G belongs to $\mathbf{LH}\mathfrak{K}$. We need to restrict our attention to groups with finite Gorenstein cohomological dimension $\operatorname{Gcd} G$; this condition allows us to work with complete resolutions of our modules and hence to identify complete cohomology with ordinary cohomology in high dimensions.

Theorem A. *Let G be an $\mathbf{LH}\mathfrak{K}$-group of Gorenstein cohomological dimension $\operatorname{Gcd} G = k < \infty$. Then the following conditions are equivalent:*

(i) *The group G has eventually finitary cohomology (i.e. the cohomology functors $H^n(G, \_)$ commute with filtered colimits of coefficient modules for all sufficiently large n).*
(ii) *The group G has infinitely often finitary cohomology.*
(iii) *In every projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, the kth kernel is a direct summand of a $\mathbb{Z}G$-module which has a projective resolution that is of finite type after one step.*
(iv) *The group G has an Eilenberg–Mac Lane space $K(G, 1)$, which is dominated by a CW-complex with finitely many n-cells for all $n \geq k+2$.*
(v) *The group G has an Eilenberg–Mac Lane space $K(G, 1)$, which is dominated by a CW-complex with finitely many n-cells for all sufficiently large n.*

In proving Theorem A we require a characterization of finitary Gorenstein projective modules. The Gorenstein projective modules are the modules which arise as kernels in complete resolutions. Our characterization may be stated as follows.

Theorem B. *Let G be a $\mathbf{LH}\mathfrak{K}$ of finite Gorenstein cohomological dimension. Then every completely finitary Gorenstein projective $\mathbb{Z}G$-module is a direct summand of a module built up as an iterated sequence of extensions of modules induced from certain special modules induced from completely finitary Gorenstein projective $\mathbb{Z}H$-modules with $H \in \mathfrak{K}$.*

Theorem B generalizes results of Benson [2], and is a special case of Theorem 3.1(ii) below. We do not know a straightforward way to obtain the conclusions Theorem A (iv) and (v) without going via the direct product of the group under consideration with an infinite cyclic group. The trick has a long history: for example a group G is of type FP if and only if the direct product $G \times \mathbb{Z}$ is of type FL. In proving the equivalence of (iv) and (v) with the other conditions in Theorem A we are therefore led to the following additional conclusions.

Theorem C. *Let G be an $\mathbf{LH}\mathfrak{K}$-group of finite Gorenstein cohomological dimension. Then the following conditions are equivalent:*

(i) *The group G has eventually finitary cohomology.*
(ii) *Over the integral group ring of the group $G \times \mathbb{Z}$, the trivial module $\mathbb{Z}$ has a projective resolution $P_* \twoheadrightarrow \mathbb{Z}$ in which P_n is finitely generated for all sufficiently large n.*
(iii) *The group $G \times \mathbb{Z}$ has an Eilenberg–Mac Lane space $K(G \times \mathbb{Z}, 1)$ with finitely many n-cells for all sufficiently large n.*

Note that Theorem A and Theorem B have their parts (i) in common so all the conditions A (i)–(v) and B (i)–(iii) are in fact equivalent. The theorems are brought together and proved as Theorem 3.9 below.

Notations and terminology. If R is a ring and M a left R-module, then we shall denote by ΩM the kernel of a surjective R-linear map $P \longrightarrow M$, where P is a projective module. Even though ΩM is not uniquely determined by M, Schanuel's lemma implies uniqueness up to addition of a projective module. We also consider the iterates $\Omega^n M$, $n \geq 0$, which are defined inductively by letting $\Omega^0 M = M$ and $\Omega^n M = \Omega\Omega^{n-1} M$ for all $n > 0$. If $P_* \longrightarrow M \longrightarrow 0$ is a projective resolution of M, then $\Omega^n M$ is identified with the n-th syzygy module of M, i.e. with the image of the map $P_n \longrightarrow P_{n-1}$ for all $n \geq 1$. If A, B, C are three left R-modules, then we identify the abelian group $\mathrm{Hom}_R(A, B \oplus C)$ with the direct sum $\mathrm{Hom}_R(A, B) \oplus \mathrm{Hom}_R(A, C)$. In this way, an element $(f, g) \in \mathrm{Hom}_R(A, B) \oplus \mathrm{Hom}_R(A, C)$ is identified with the R-linear map $A \longrightarrow B \oplus C$, which is given by $a \mapsto (f(a), g(a))$, $a \in A$. In a similar way, there is an identification of the abelian group $\mathrm{Hom}_R(B \oplus C, A)$ with the direct sum $\mathrm{Hom}_R(B, A) \oplus \mathrm{Hom}_R(C, A)$. For any two elements $f \in \mathrm{Hom}_R(B, A)$ and $g \in \mathrm{Hom}_R(C, A)$ we shall denote by $[f, g] : B \oplus C \longrightarrow A$ the corresponding R-linear map (which is given by $(b, c) \mapsto f(b) + g(c)$, $(b, c) \in B \oplus C$).

1. Preliminary notions

In this section, we collect certain basic notions and preliminary results that will be used in the sequel. These involve:

(i) the relation between complete cohomology and the finiteness of the Gorenstein projective dimension,
(ii) certain basic results concerning finiteness conditions in the stable module category, which are mainly related to the notions of stably compact and completely finitary modules,
(iii) a compactness result concerning poly-$\mathfrak{C}$ and hyper-$\mathfrak{C}$ modules (where $\mathfrak{C}$ is a given class of modules) and
(iv) a concrete description of Rickard's algebraic homotopy colimit associated with a class $\mathfrak{C}$ of modules.

I. Complete cohomology and Gorenstein projective dimension.

The existence of terminal (projective or injective) completions of the ordinary Ext functors in module categories has been studied by Gedrich and Gruenberg in [9]. Using an approach that involves satellites and is heavily influenced by Gedrich and Gruenberg's theory of terminal completions, Mislin defined in [19] for any ring R and any left R-module M complete cohomology functors $\widehat{\mathrm{Ext}}^*_R(M, \_)$ and a natural transformation $\mathrm{Ext}^*_R(M, \_) \longrightarrow \widehat{\mathrm{Ext}}^*_R(M, \_)$, as the projective completion of the ordinary Ext functors $\mathrm{Ext}^*_R(M, \_)$. Equivalent definitions of the complete cohomology functors have been independently formulated by Vogel in [10] (using a hypercohomology approach) and by Benson and Carlson in [3] (using projective resolutions). Using the approach by Benson and Carlson, it follows that the kernel of the canonical map $\mathrm{Hom}_R(M, N) \longrightarrow \widehat{\mathrm{Ext}}^0_R(M, N)$ consists of those R-linear maps $f : M \longrightarrow N$, which are such that the map $\Omega^n f : \Omega^n M \longrightarrow \Omega^n N$ induced by f between the associated n-th syzygy modules factors through a projective left R-module for $n \gg 0$.

There is a special case, where the complete cohomology functors may be computed by means of complexes, as we shall now describe: A left R-module M is said to admit a *complete resolution of coincidence index* n if there exists a doubly infinite acyclic complex of projective left R-modules P_*, which coincides with a projective resolution of M in degrees $\geq n$. If, in addition, P_* remains acyclic after applying the functor $\mathrm{Hom}_R(\_, P)$ for any projective left R-module P, then P_* is called a *complete resolution of* M *in the strong sense*. A left R-module M is called *Gorenstein projective* if it admits a complete resolution in the strong sense of coincidence index 0. The class of Gorenstein projective left R-modules, which will be denoted by $\mathtt{GP}(R)$, contains all projective left R-modules, is closed under arbitrary direct sums and has many interesting properties. In particular, as shown by Holm in [14], $\mathtt{GP}(R)$ is closed under direct summands, extensions and kernels of epimorphisms. As in classical homological algebra, we say that a left R-module M has *finite Gorenstein projective dimension* if there exist a non-negative integer n and an exact sequence of left R-modules

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0,$$

such that M_i is Gorenstein projective for all $i = 0, 1, \ldots, n$. In that case, the *Gorenstein projective dimension* $\mathrm{Gpd}_R M$ of M is defined to be the least such n. It follows from results of Holm [14] that a left R-module M has finite Gorenstein projective dimension if and only if the module M admits a complete resolution in the strong sense. We note that the latter may be chosen to have coincidence index equal to $\mathrm{Gpd}_R M$. Since in the present paper we only consider complete resolutions in the strong sense, for brevity, we shall refer to them as complete resolutions instead. It follows from Mislin's approach to complete cohomology that the functors $\widehat{\mathrm{Ext}}^*_R(M,\_)$ may be computed, in the case where M has a complete resolution P_*, as the cohomology groups of the complex $\mathrm{Hom}_R(P_*,\_)$, (cf. [7], Theorem 1.2). In particular, if M is Gorenstein projective, then the complete cohomology group $\widehat{\mathrm{Ext}}^0_R(M,N)$ may be naturally identified for any left R-module N with the quotient $\underline{\mathrm{Hom}}_R(M,N)$ of the abelian group $\mathrm{Hom}_R(M,N)$ by the subgroup consisting of all R-linear maps $f: M \longrightarrow N$ that factor through a projective left R-module.

In view of the above discussion, it is important to know whether all left R-modules have finite Gorenstein projective dimension, i.e. whether all left R-modules have a complete resolution. The answer to this question involves the cohomological invariants $\mathrm{silp}\, R$ and $\mathrm{spli}\, R$, which were introduced by Gedrich and Gruenberg in [9]. Here, $\mathrm{silp}\, R$ is defined as the supremum of the injective lengths of projective left R-modules, whereas $\mathrm{spli}\, R$ is the supremum of the projective lengths of injective left R-modules. It follows from ([7], Theorem 3.10) that the finiteness of these invariants is equivalent to the assertion that all left R-modules have finite Gorenstein projective dimension.

In the sequel, we shall need the following simple result.

Lemma 1.1. *Let R be a ring and assume that $S \subseteq R$ is a subring, such that R is flat when regarded as a right S-module and projective when regarded as a left S-module. Then, for any Gorenstein projective left S-module M the left R-module $R \otimes_S M$ is Gorenstein projective as well.*

Proof. Let P_* be a complete resolution in the category of left S-modules having M as a syzygy. Since the right S-module R is flat, the induced complex $R \otimes_S P_*$ is acyclic, consists of projective left R-modules and has $R \otimes_S M$ as a syzygy. Since the left S-module R is projective, the restriction of any projective left R-module is a projective left S-module. It follows readily that $R \otimes_S P_*$ is a complete resolution in the category of left R-modules and hence $R \otimes_S M$ is a Gorenstein projective left R-module, as needed. □

In the special case where $R = \mathbb{Z}G$ is the integral group ring of a group G, the finiteness of the Gorenstein projective dimension of all $\mathbb{Z}G$-modules is known to be controlled by the finiteness of the Gorenstein projective dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. In fact, if we define the Gorenstein cohomological dimension $\mathrm{Gcd}\, G$ of G by letting $\mathrm{Gcd}\, G = \mathrm{Gpd}_{\mathbb{Z}G}\mathbb{Z}$, then for any $\mathbb{Z}G$-module M there is an inequality $\mathrm{Gpd}_{\mathbb{Z}G} M \leq \mathrm{Gcd}\, G + 1$ (cf. [1], Proposition 2.4(c)). As shown in [loc.cit., Theorem 2.5], the Gorenstein cohomological dimension $\mathrm{Gcd}\, G$ of G is the supremum of those integers n, for which there exist $\mathbb{Z}G$-modules M and P, with M $\mathbb{Z}$-free and P projective, such that $\mathrm{Ext}^n_{\mathbb{Z}G}(M,P) \neq 0$.

II. Stably compact and completely finitary modules.

The *stable module category* of a ring R is the category whose objects are all left R-modules and whose morphisms $M \longrightarrow N$, where M,N are two left R-modules, are the elements of the complete cohomology group $\widehat{\mathrm{Ext}}^0_R(M,N)$. For later use, we record the following simple lemma.

Lemma 1.2. *Let R be a ring. Then:*

(i) *The composition of morphisms in the stable module category of R is biadditive, i.e. for any three left R-modules A,B,C it is described by means of an additive map*

$$\widehat{Ext}^0_R(B,C) \otimes \widehat{Ext}^0_R(A,B) \longrightarrow \widehat{Ext}^0_R(A,C).$$

(ii) *If $f : B \longrightarrow C$ is an R-linear map and $[f] = [0] \in \widehat{Ext}^0_R(B,C)$, then the additive map*

$$f_* : \widehat{Ext}^0_R(A,B) \longrightarrow \widehat{Ext}^0_R(A,C)$$

is trivial for any left R-module A.

Proof. Assertion (i) is an immediate consequence of the definitions, in view of the biadditivity of the composition of R-linear maps, whereas (ii) follows readily from (i). □

We say that a left R-module M is *stably compact* if the functor $\widehat{\mathrm{Ext}}^0_R(M,\_)$ commutes with filtered colimits. If the functors $\widehat{\mathrm{Ext}}^n_R(M,\_)$ commute with filtered colimits for all $n \in \mathbb{Z}$, then we say that M is *completely finitary.* We denote by $\mathtt{SC}(R)$ (resp. $\mathtt{CF}(R)$) the class of stably compact (resp. completely finitary) left R-modules. We note that all projective left R-modules are stably compact, whereas the class of completely finitary left R-modules is closed under direct summands, extensions, kernels of epimorphisms, and cokernels of monomorphisms. If $\mathfrak{C}$ is any class of left R-modules, then we define the class $\varinjlim \mathfrak{C}$ as the class consisting of those left R-modules, which may be expressed as filtered colimits of modules in $\mathfrak{C}$. We also define the class $\mathfrak{C}^{\perp}$ as the class consisting of those left R-modules N, which are such that

$\widehat{\mathrm{Ext}}^0_R(M,N)=0$ for all $M \in \mathfrak{C}$. It is clear that if $\mathfrak{C},\mathfrak{D}$ are two classes of left R-modules and $\mathfrak{C} \subseteq \mathfrak{D}$, then $\mathfrak{D}^\perp \subseteq \mathfrak{C}^\perp$. In particular, since $\mathtt{CF}(R) \subseteq \mathtt{SC}(R)$, we have an inclusion $\mathtt{SC}(R)^\perp \subseteq \mathtt{CF}(R)^\perp$.

Proposition 1.3. *Let S be a subring of a ring R, such that R is flat when regarded as a right S-module, and consider a class $\mathfrak{C}$ of left S-modules. Then, for any left S-module N contained in $\mathfrak{C}^\perp \cap \varinjlim \mathfrak{C}$ the left R-module $R \otimes_S N$ is contained in $\mathtt{SC}(R)^\perp$.*

Proof. Since $N \in \varinjlim \mathfrak{C}$, we may express N as the colimit of a filtered direct system of left S-modules $(N_i)_i$, where $N_i \in \mathfrak{C}$ for all i; we denote by $f_i : N_i \longrightarrow N$ the canonical maps. Having fixed the index i, we note that our assumption that $N \in \mathfrak{C}^\perp$ implies that the abelian group $\widehat{\mathrm{Ext}}^0_S(N_i,N)$ is trivial; in particular, $[f_i]=[0] \in \widehat{\mathrm{Ext}}^0_S(N_i,N)$. In other words, the S-linear map

$$\Omega^n f_i : \Omega^n N_i \longrightarrow \Omega^n N,$$

which is induced by f_i, factors through a projective left S-module when $n \gg 0$. Since R is flat as a right S-module, the R-linear map

$$\Omega^n(1\otimes f_i) : \Omega^n(R\otimes_S N_i) \longrightarrow \Omega^n(R\otimes_S N),$$

which is induced by $1\otimes f_i : R\otimes_S N_i \longrightarrow R\otimes_S N$, is identified with the R-linear map

$$1\otimes \Omega^n f_i : R\otimes_S \Omega^n N_i \longrightarrow R\otimes_S \Omega^n N.$$

Since the latter map factors through a projective left R-module when $n \gg 0$, it follows that $[1\otimes f_i]=[0] \in \widehat{\mathrm{Ext}}^0_R(R\otimes_R N_i, R\otimes_S N)$. Therefore, Lemma 1.2 implies that for any left R-module M the additive map

$$(1\otimes f_i)_* : \widehat{\mathrm{Ext}}^0_R(M,R\otimes_S N_i) \longrightarrow \widehat{\mathrm{Ext}}^0_R(M,R\otimes_S N) \qquad (1)$$

is trivial.

In order to show that $R\otimes_S N \in \mathtt{SC}(R)^\perp$, assume that the left R-module M is stably compact. Since the left R-module $R\otimes_S N$ is the colimit of the filtered direct system $(R\otimes_S N_i)_i$, we have

$$\widehat{\mathrm{Ext}}^0_R(M,R\otimes_S N) = \varinjlim_i \widehat{\mathrm{Ext}}^0_R(M,R\otimes_S N_i)$$

and hence the abelian group $\widehat{\mathrm{Ext}}^0_S(M,R\otimes_S N)$ is the union of the images of the maps (1) for all i. The triviality of these maps shows that $\widehat{\mathrm{Ext}}^0_R(M,R\otimes_S N)=0$, as needed. □

In the sequel, we shall use Proposition 1.3, in the form of the following corollary.

Corollary 1.4. *Let S be a subring of a ring R and assume that R is flat as a right S-module. If any left S-module can be expressed as a filtered colimit of completely finitary left S-modules, then for any left S-module $N \in \mathtt{CF}(S)^{\perp}$, the left R-module $R \otimes_S N$ is contained in $\mathtt{CF}(R)^{\perp}$.*

Proof. Since $\mathtt{SC}(R)^{\perp} \subseteq \mathtt{CF}(R)^{\perp}$, this is an immediate consequence of Proposition 1.3, applied in the special case where $\mathfrak{C} = \mathtt{CF}(S)$. □

We shall also need the following property of stably compact modules.

Lemma 1.5. *Let R be a ring and consider a left R-module M, which is stably compact and Gorenstein projective. Then, M is isomorphic to a direct summand of the direct sum $P \oplus N$ of two left R-modules P and N, where P is projective and N is finitely presented.*

Proof. We may express M as the colimit of a filtered direct system $(M_i)_i$ of finitely presented left R-modules and denote by $f_i : M_i \longrightarrow M$ the canonical maps. Since the left R-module M is stably compact and Gorenstein projective, the functor $\underline{\mathrm{Hom}}_R(M, \_)$ commutes with filtered colimits; in particular, there is an isomorphism

$$\underline{\mathrm{Hom}}_R(M, M) \simeq \varinjlim_i \underline{\mathrm{Hom}}_R(M, M_i).$$

Considering the identity map $1_M : M \longrightarrow M$, it follows that there exists an index i and an R-linear map $g_i : M \longrightarrow M_i$, such that $[1_M] = [f_i g_i] \in \underline{\mathrm{Hom}}_R(M, M)$. Then, the endomorphism $1_M - f_i g_i$ of M factors through a projective left R-module P; in other words, there exist R-linear maps $a : M \longrightarrow P$ and $b : P \longrightarrow M$, such that $1_M - f_i g_i = ba$. The composition

$$M \stackrel{(a,g_i)}{\longrightarrow} P \oplus M_i \stackrel{[b,f_i]}{\longrightarrow} M$$

is then equal to the identity map of M and hence M is a direct summand of $P \oplus M_i$. □

III. Poly-$\mathfrak{C}$ and hyper-$\mathfrak{C}$ modules.

Let R be a ring and consider a class $\mathfrak{C}$ of left R-modules. We say that a left R-module E is a poly-$\mathfrak{C}$ module if there exists a non-negative integer n and an ascending filtration $E_0 \subseteq E_1 \subseteq \ldots \subseteq E_n$ of E by submodules, such that $E_0 = 0$, $E_n = E$ and the quotient E_i/E_{i-1} is contained in $\mathfrak{C}$

for all $i = 1, \ldots, n$. In other words, the class of poly-$\mathfrak{C}$ modules consists precisely of the iterated extensions of modules in $\mathfrak{C}$. We say that a left R-module E is a hyper-$\mathfrak{C}$ module if there exists an ordinal α and an ascending filtration of E by submodules E_β, which is indexed by the ordinals $\beta \leq \alpha$, such that $E_0 = 0$, $E_\alpha = E$ and $E_\beta / E_{\beta-1} \in \mathfrak{C}$ (resp. $E_\beta = \bigcup_{\gamma<\beta} E_\gamma$) if $\beta \leq \alpha$ is a successor (resp. a limit) ordinal. Such an ascending chain of submodules $(E_\beta)_{\beta\leq\alpha}$ will be referred to as a continuous ascending chain of submodules with sections in $\mathfrak{C}$. We note that an arbitrary direct sum of modules contained in $\mathfrak{C}$ is a hyper-$\mathfrak{C}$ module, whereas the class of hyper-$\mathfrak{C}$ modules is closed under extensions. In the sequel, we shall use the following compactness result.

Lemma 1.6. *Let R be a ring and consider a class $\mathfrak{C}$ of left R-modules, which is such that:*

(i) *$\mathfrak{C}$ contains all projective left R-modules and*
(ii) *$\mathfrak{C}$ consists of completely finitary Gorenstein projective modules.*

We also consider two left R-modules M and E and assume that:

(iii) *M is completely finitary and Gorenstein projective and*
(iv) *E is a hyper-$\mathfrak{C}$ module.*

Then, any R-linear map $f : M \longrightarrow E$ factors through a poly-$\mathfrak{C}$ module E'.

Proof. In view of (iv), there exists a continuous ascending chain of submodules $(E_\beta)_{\beta\leq\alpha}$ with sections in $\mathfrak{C}$, such that $E = E_\alpha$. We shall prove by transfinite induction on the ordinal $\beta \leq \alpha$ that any R-linear map $f : M \longrightarrow E_\beta$ factors through a poly-$\mathfrak{C}$ module E' for any completely finitary Gorenstein projective left R-module M. Since $E = E_\alpha$, this will complete the proof of the lemma. We note that there is nothing to prove if $\beta = 0$, since $E_0 = 0$. We now proceed with the inductive step:

Case 1: If β is a limit ordinal, then the continuity of the chain implies that $E_\beta = \bigcup_{\gamma<\beta} E_\gamma$. In view of assumption (iii), the functor $\underline{\mathrm{Hom}}_R(M, \_)$ commutes with filtered colimits and hence $\underline{\mathrm{Hom}}_R(M, E_\beta)$ is the colimit of the filtered direct system $(\underline{\mathrm{Hom}}_R(M, E_\gamma))_{\gamma<\beta}$. In particular, there exists an ordinal $\gamma < \beta$ and an R-linear map $g : M \longrightarrow E_\gamma$, such that

$$[f] = [\iota g] \in \underline{\mathrm{Hom}}_R(M, E_\beta),$$

where we denote by ι the inclusion $E_\gamma \hookrightarrow E_\beta$. Then, $[f - \iota g] = [0] \in \underline{\mathrm{Hom}}_R(M, E_\beta)$ and hence there exists a projective left R-module P and R-linear maps $a : M \longrightarrow P$ and $b : P \longrightarrow E_\beta$, such that $f - \iota g = ba$. Therefore, the R-linear map $f = \iota g + ba : M \longrightarrow E_\beta$ factors as the composition

$$M \xrightarrow{(g,a)} E_\gamma \oplus P \xrightarrow{[\iota,b]} E_\beta.$$

Invoking the induction hypothesis, we conclude that $g : M \longrightarrow E_\gamma$ factors through a poly-$\mathfrak{C}$ module E' and hence $f : M \longrightarrow E_\beta$ factors through $E' \oplus P$; since $P \in \mathfrak{C}$ (in view of assumption (i)), the latter module is poly-$\mathfrak{C}$. Hence, we have proved the existence of a factorization of f, as needed.

Case 2a: We now assume that β is a successor ordinal and f is surjective. We consider the pullback N of the diagram

$$\begin{array}{ccc} & & M \\ & & \downarrow f \\ E_{\beta-1} & \xrightarrow{\iota} & E_\beta \end{array}$$

where ι is the inclusion $E_{\beta-1} \hookrightarrow E_\beta$, and the associated commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\jmath} & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & u \downarrow & & f \downarrow & & v \downarrow & & \\ 0 & \longrightarrow & E_{\beta-1} & \xrightarrow{\iota} & E_\beta & \longrightarrow & E_\beta/E_{\beta-1} & \longrightarrow & 0 \end{array} \qquad (2)$$

Since f is surjective, it follows that E_β is the pushout of the diagram

$$\begin{array}{ccc} N & \xrightarrow{\jmath} & M \\ u \downarrow & & \\ E_{\beta-1} & & \end{array}$$

and hence the R-linear map $v : M/N \longrightarrow E_\beta/E_{\beta-1}$ is bijective. Then, assumption (ii) implies that the left R-module M/N is completely finitary and Gorenstein projective; it follows that the left R-module N is completely finitary and Gorenstein projective as well. Hence, invoking the induction hypothesis, we conclude that the R-linear map u factors as the composition

$$N \xrightarrow{u_1} E' \xrightarrow{u_2} E_{\beta-1},$$

where E' is a poly-$\mathfrak{C}$ module. We now consider the pushout E'' of the diagram

$$\begin{array}{ccc} N & \xrightarrow{\jmath} & M \\ u_1 \downarrow & & \\ E' & & \end{array}$$

which fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \stackrel{\jmath}{\longrightarrow} & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & {\scriptstyle u_1} \downarrow & & {\scriptstyle f_1} \downarrow & & {\scriptstyle v_1} \downarrow & & \\ 0 & \longrightarrow & E' & \stackrel{\eta}{\longrightarrow} & E'' & \longrightarrow & E''/E' & \longrightarrow & 0 \end{array}$$

Then, the R-linear map v_1 is bijective and hence the left R-module $E''/E' \simeq M/N \simeq E_\beta/E_{\beta-1}$ is isomorphic with a module in $\mathfrak{C}$. Since E' is a poly-$\mathfrak{C}$ module, it follows that E'' is a poly-$\mathfrak{C}$ module as well. The equalities $f\jmath = \iota u = (\iota u_2)u_1$ and the definition of E'' as a pushout imply that there is a unique R-linear map $f_2 : E'' \longrightarrow E_\beta$, such that $f = f_2 f_1$ and $\iota u_2 = f_2 \eta$. In this way, we obtain a factorization of (2), viewed as a morphism of extensions, as pictured below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \stackrel{\jmath}{\longrightarrow} & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & {\scriptstyle u_1} \downarrow & & {\scriptstyle f_1} \downarrow & & {\scriptstyle v_1} \downarrow & & \\ 0 & \longrightarrow & E' & \stackrel{\eta}{\longrightarrow} & E'' & \longrightarrow & E''/E' & \longrightarrow & 0 \\ & & {\scriptstyle u_2} \downarrow & & {\scriptstyle f_2} \downarrow & & {\scriptstyle v_2} \downarrow & & \\ 0 & \longrightarrow & E_{\beta-1} & \stackrel{\iota}{\longrightarrow} & E_\beta & \longrightarrow & E_\beta/E_{\beta-1} & \longrightarrow & 0 \end{array}$$

In particular, we have shown that the R-linear map f factors as the composition

$$M \stackrel{f_1}{\longrightarrow} E'' \stackrel{f_2}{\longrightarrow} E_\beta,$$

where E'' is a poly-$\mathfrak{C}$ module, as needed.

Case 2b: We now assume that β is a successor ordinal and f is any R-linear map. Then, we may choose a projective left R-module L, such that there exists a surjective R-linear map $p : L \longrightarrow E_\beta$, and note that f factors as the composition

$$M \stackrel{(1_M,0)}{\longrightarrow} M \oplus L \stackrel{[f,p]}{\longrightarrow} E_\beta.$$

Since the left R-module $M \oplus L$ is completely finitary and Gorenstein projective, we may apply Case 2a above and conclude that the *surjective* R-linear map $[f,p] : M \oplus L \longrightarrow E_\beta$ factors through a poly-$\mathfrak{C}$ module. It follows that this is also the case for f, as needed. □

IV. Rickard's algebraic homotopy colimit.

We now turn to an important method originally inspired by techniques in algebraic topology and brought into the realm of representation theory by Rickard [20]. Let R be a ring and fix a class $\mathfrak{C}$ of stably compact Gorenstein projective left R-modules. For any left R-module M we shall define below a sequence $(M_n)_n$ of left R-modules and injective R-linear maps

$$M_0 \xrightarrow{\eta_0} M_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{n-1}} M_n \xrightarrow{\eta_n} \cdots ,$$

in such a way that:

(i) $M_0 = M$,

(ii) For any $n \geq 0$ the following condition (which will be referred to below as condition (ii)$_n$) is satisfied: If we denote by $\iota_n : M \longrightarrow M_n$ the composition $M_0 \xrightarrow{\eta_0} M_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{n-1}} M_n$, then there exists a projective left R-module Q_n and an R-linear map $f_n : Q_n \longrightarrow M_n$, such that the map $[\iota_n, f_n] : M \oplus Q_n \longrightarrow M_n$ is surjective and its kernel K_n is a hyper-$\mathfrak{C}$ module.

(iii) If $M_\infty = \varinjlim_n M_n$, then $M_\infty \in \mathfrak{C}^\perp$.

The definition of the above sequence proceeds by induction on n. We begin by letting $M_0 = M$. Having constructed M_k for $k = 0, 1, \ldots, n$ and embeddings $M_0 \xrightarrow{\eta_0} M_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{n-1}} M_n$, in such a way that condition (ii)$_k$ is satisfied for all $k = 0, 1, \ldots, n$, we proceed with the inductive step as follows: We consider the category whose objects are the pairs (C, f), where $C \in \mathfrak{C}$ and $f \in \mathrm{Hom}_R(C, M_n)$, with morphisms the obvious commutative triangles. We choose a set Λ_n of objects of that category, consisting of one object from each isomorphism class therein. We let $C_n = \bigoplus_{(C,f)\in\Lambda_n} C$ and use the maps f associated with each object $(C, f) \in \Lambda_n$, in order to define an R-linear map $f_n : C_n \longrightarrow M_n$. Since $\mathfrak{C}$ consists of Gorenstein projective modules and $\mathtt{GP}(R)$ is closed under arbitrary direct sums, it follows that the left R-module C_n is Gorenstein projective. In particular, there exists a projective left R-module P_n and an embedding $\jmath_n : C_n \longrightarrow P_n$. We now define M_{n+1} as the pushout of the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\jmath_n} & P_n \\ {\scriptstyle f_n}\downarrow & & \\ M_n & & \end{array}$$

In other words, M_{n+1} fits into a commutative diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\jmath_n} & P_n \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \varphi_n} \\ M_n & \xrightarrow{\eta_n} & M_{n+1} \end{array}$$

and there is a short exact sequence of left R-modules

$$0 \longrightarrow C_n \longrightarrow M_n \oplus P_n \stackrel{[\eta_n,\varphi_n]}{\longrightarrow} M_{n+1} \longrightarrow 0.$$

Since $\jmath_n$ is injective, it follows that η_n is injective as well. We now let Q_n be a projective left R-module and $f_n : Q_n \longrightarrow M_n$ an R-linear map, such that $[\iota_n, f_n] : M \oplus Q_n \longrightarrow M_n$ is surjective and its kernel K_n is a hyper-$\mathfrak{C}$ module. Then, the composition

$$M \oplus Q_n \oplus P_n \stackrel{[\iota_n,f_n]\oplus 1_{P_n}}{\longrightarrow} M_n \oplus P_n \stackrel{[\eta_n,\varphi_n]}{\longrightarrow} M_{n+1}$$

is surjective (as both of it factor maps are), coincides with $[\eta_n\iota_n, f_{n+1}] = [\iota_{n+1}, f_{n+1}]$ for a suitable R-linear map $f_{n+1} : Q_n \oplus P_n \longrightarrow M_{n+1}$ (in fact, $f_{n+1} = [\eta_n f_n, \varphi_n]$) and its kernel K_{n+1} is a hyper-$\mathfrak{C}$ module (being an extension of the hyper-$\mathfrak{C}$ module $\ker [\eta_n, \varphi_n] = C_n$ by the hyper-$\mathfrak{C}$ module $\ker ([\iota_n, f_n] \oplus 1_{P_n}) = \ker [\iota_n, f_n] = K_n$[1]). Therefore, condition $(\text{ii})_{n+1}$ is satisfied and this completes the inductive step.

It only remains to show that $M_\infty = \varinjlim_n M_n$ satisfies condition (iii); since $\mathfrak{C} \subseteq \mathtt{GP}(R)$, this amounts to showing that $\underline{\mathrm{Hom}}_R(C, M_\infty) = 0$ for any left R-module $C \in \mathfrak{C}$. To that end, we fix a left R-module $C \in \mathfrak{C}$, a non-negative integer n and consider an R-linear map $f : C \longrightarrow M_n$. Then, by the very definition of the map $f_n : C_n \longrightarrow M_n$, we may write $f = f_n g$ for a suitable R-linear map $g : C \longrightarrow C_n$. In particular, it follows that $\eta_n f = \eta_n f_n g = \varphi_n \jmath_n g$ factors through the projective left R-module P_n and hence $[\eta_n f] = [0] \in \underline{\mathrm{Hom}}_R(C, M_{n+1})$. This being the case for all $f \in \mathrm{Hom}_R(C, M_n)$, we conclude that

$$\eta_{n*} : \underline{\mathrm{Hom}}_R(C, M_n) \longrightarrow \underline{\mathrm{Hom}}_R(C, M_{n+1})$$

is the zero map for all $n \geq 0$. Since $\mathfrak{C}$ consists of stably compact modules and $M_\infty = \varinjlim_n M_n$, the abelian group $\underline{\mathrm{Hom}}_R(C, M_\infty)$ is the direct limit of the system

$$\underline{\mathrm{Hom}}_R(C, M_0) \stackrel{\eta_{0*}}{\longrightarrow} \underline{\mathrm{Hom}}_R(C, M_1) \stackrel{\eta_{1*}}{\longrightarrow} \cdots \stackrel{\eta_{n-1*}}{\longrightarrow} \underline{\mathrm{Hom}}_R(C, M_n) \stackrel{\eta_{n*}}{\longrightarrow} \cdots .$$

Hence, we conclude that $\underline{\mathrm{Hom}}_R(C, M_\infty) = 0$, as needed.

[1] Giver R-linear maps $A \stackrel{a}{\longrightarrow} B \stackrel{b}{\longrightarrow} C$ with a surjective, it is easily seen that there is a short exact sequence of left R-modules $0 \longrightarrow \ker a \longrightarrow \ker ba \stackrel{a|}{\longrightarrow} \ker b \longrightarrow 0$.

2. Orthogonal classes over hierarchically decomposable groups

Let $\mathfrak{C}$ be a class of left R-modules, which contains all projective modules and is closed under direct sums and direct summands. Then, an application of Schanuel's lemma shows that the following two conditions are equivalent for a left R-module M:

(i) There is a projective left R-module P and a surjective R-linear map $f : P \longrightarrow M$, such that $\ker f \in \mathfrak{C}$.

(ii) For any projective left R-module P and any surjective R-linear map $f : P \longrightarrow M$, we have $\ker f \in \mathfrak{C}$.

If these conditions are satisfied, then we say that the first syzygy module of M is contained in $\mathfrak{C}$ and write $\Omega M \in \mathfrak{C}$. We say that $\mathfrak{C}$ is closed under syzygy modules if $\Omega M \in \mathfrak{C}$ for any left R-module $M \in \mathfrak{C}$.

Remark 2.1. Let $\mathfrak{C}$ be a class of left R-modules, which contains all projective modules and is closed under direct sums, direct summands and syzygy modules. We also consider a left R-module N, which fits into an exact sequence

$$0 \longrightarrow N_k \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow N \longrightarrow 0,$$

where $k \geq 0$ and $N_i \in \mathfrak{C}^\perp$ for all $i = 0, 1, \ldots, k$. Then, using induction on k, one can easily show that $N \in \mathfrak{C}^\perp$ as well.

Proposition 2.2. *Let G be a group and $\mathfrak{C}$ a class consisting of stably compact $\mathbb{Z}G$-modules, which contains all projective modules and is closed under direct sums, direct summands and syzygy modules. We also consider a class $\mathfrak{X}$ of groups, which is closed under subgroups. Then, the following conditions are equivalent for a $\mathbb{Z}G$-module N:*

(i) *$ind_H^G res_H^G N \in \mathfrak{C}^\perp$ for any subgroup $H \subseteq G$ with $H \in \mathfrak{X}$.*

(ii) *$ind_H^G res_H^G N \in \mathfrak{C}^\perp$ for any subgroup $H \subseteq G$ with $H \in \mathbf{LH}\mathfrak{X}$.*

Proof. We only have to show that (i)→(ii). To that end, we let

$$\mathfrak{H} = \{H : H \text{ is a subgroup of } G \text{ and } \mathrm{ind}_H^G \mathrm{res}_H^G N \in \mathfrak{C}^\perp\}.$$

Our assumption implies that $\mathfrak{H}$ contains all $\mathfrak{X}$-subgroups of G. First of all, we shall prove that $\mathfrak{H}$ contains all $\mathbf{H}\mathfrak{X}$-subgroups of G as well. To that end, it suffices to show that if $H \subseteq G$ is any subgroup for which there exists a finite dimensional contractible H-CW-complex X with cell stabilizers in $\mathfrak{H}$, then we have $H \in \mathfrak{H}$. If $H \subseteq G$ is such a subgroup, then the augmented cellular chain complex of X is an exact sequence of $\mathbb{Z}H$-modules

$$0 \longrightarrow C_m \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Tensoring the exact sequence above with $\mathrm{res}^G_H N$ over $\mathbb{Z}$ from the right and allowing the group G to act diagonally, we obtain an exact sequence of $\mathbb{Z}H$-modules

$$0 \longrightarrow C_m \otimes \mathrm{res}^G_H N \longrightarrow \cdots \longrightarrow C_0 \otimes \mathrm{res}^G_H N \longrightarrow \mathrm{res}^G_H N \longrightarrow 0,$$

which itself yields an exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathrm{ind}^G_H \big[C_m \otimes \mathrm{res}^G_H N \big] \longrightarrow \cdots \longrightarrow \mathrm{ind}^G_H \big[C_0 \otimes \mathrm{res}^G_H N \big] \longrightarrow \mathrm{ind}^G_H \mathrm{res}^G_H N \longrightarrow 0.$$

Since the $\mathbb{Z}H$-module C_i is a permutation module with stabilizers in $\mathfrak{H}$, it follows that the $\mathbb{Z}G$-module $\mathrm{ind}^G_H \big[C_i \otimes \mathrm{res}^G_H N \big]$ is a direct sum of modules of the form $\mathrm{ind}^G_{H'} \mathrm{res}^G_{H'} N$, where H' is an $\mathfrak{H}$-subgroup of G. We note that the latter modules are contained (by the very definition of $\mathfrak{H}$) in $\mathfrak{C}^{\perp}$. Since the class $\mathfrak{C}$ consists of stably compact modules, the orthogonal class $\mathfrak{C}^{\perp}$ is closed under arbitrary direct sums. In particular, we conclude that $\mathrm{ind}^G_H \big[C_i \otimes \mathrm{res}^G_H N \otimes \big] \in \mathfrak{C}^{\perp}$ for all $i = 0, 1, \ldots, m$. Then, Remark 2.1 implies that $\mathrm{ind}^G_H \mathrm{res}^G_H N \in \mathfrak{C}^{\perp}$ and hence $H \in \mathfrak{H}$.

We can now complete the proof and show that all $\mathbf{LH}\mathfrak{X}$-subgroups H of G are contained in $\mathfrak{H}$. Indeed, let $H \subseteq G$ be such a subgroup; then, all finitely generated subgroups H' of H are contained in $\mathbf{H}\mathfrak{X}$. We note that the $\mathbb{Z}G$-module $\mathrm{ind}^G_H \mathrm{res}^G_H N$ is the filtered colimit of the $\mathbb{Z}G$-modules $\mathrm{ind}^G_{H'} \mathrm{res}^G_{H'} N$, where H' runs through the collection of finitely generated subgroups of H. In view of the result that we have just proved for the $\mathbf{H}\mathfrak{X}$-subgroups of G, these latter modules are all contained in $\mathfrak{C}^{\perp}$. Since $\mathfrak{C}$ consists of stably compact modules, the orthogonal class $\mathfrak{C}^{\perp}$ is closed under filtered colimits. It follows that $\mathrm{ind}^G_H \mathrm{res}^G_H N \in \mathfrak{C}^{\perp}$ and hence $H \in \mathfrak{H}$, as needed. □

Definition 2.3. *Let $\mathfrak{Y}$ be the class consisting of those groups G, which are such that any $\mathbb{Z}G$-module may be expressed as a filtered colimit of completely finitary $\mathbb{Z}G$-modules. In other words, if G is a group then $G \in \mathfrak{Y}$ if and only if $\varinjlim \mathtt{CF}(\mathbb{Z}G)$ is the class of all $\mathbb{Z}G$-modules.*

Using this terminology, we may reformulate some of our earlier results. As an example, let G be a group and consider a subgroup $H \subseteq G$ with $H \in \mathfrak{Y}$. Then, Corollary 1.4 implies that for any $\mathbb{Z}H$-module $N \in \mathtt{CF}(\mathbb{Z}H)^{\perp}$, the induced $\mathbb{Z}G$-module $\mathrm{ind}^G_H N$ is contained in $\mathtt{CF}(\mathbb{Z}G)^{\perp}$.

Corollary 2.4. *Let $\mathfrak{X}$ be a subgroup-closed subclass of $\mathfrak{Y}$ and consider an* $\mathbf{LH}\mathfrak{X}$*-group G. Then, the following conditions are equivalent for a $\mathbb{Z}G$-module N:*

(i) $N \in \mathtt{CF}(\mathbb{Z}G)^{\perp}$ *and*
(ii) $res^G_H N \in \mathtt{CF}(\mathbb{Z}H)^{\perp}$ *for any subgroup $H \subseteq G$ with $H \in \mathfrak{X}$.*

Proof. The implication (i)$\rightarrow$(ii) is an immediate consequence of the Eckmann-Shapiro lemma for complete cohomology. We shall now prove that (ii)$\rightarrow$(i). We note that condition (ii) and our assumption that $\mathfrak{X} \subseteq \mathfrak{Y}$ imply (as we noted above) that $\text{ind}^G_H \text{res}^G_H N \in \mathtt{CF}(\mathbb{Z}G)^{\perp}$ for all $\mathfrak{X}$-subgroups $H \subseteq G$. Since G is an $\mathbf{LH}\mathfrak{X}$-group, we may use Proposition 2.2 and conclude that $N = \text{ind}^G_G \text{res}^G_G N \in \mathtt{CF}(\mathbb{Z}G)^{\perp}$, as needed. □

3. Groups of finite Gorenstein cohomological dimension

In this section, we shall obtain certain explicit results concerning completely finitary modules over the integral group ring of certain groups. In particular, these results include:

(i) a description of the structure of completely finitary Gorenstein projective modules over the integral group ring of a hierarchically decomposable group of finite Gorenstein cohomological dimension,
(ii) a study of the extent to which completely finitary modules over the integral group ring of certain hierarchically decomposable groups of finite Gorenstein cohomological dimension have projective resolutions which are eventually of finite type and
(iii) a result relating groups with eventually finitary cohomology to the existence of models for the associated Eilenberg–Mac Lane space with finitely many n-cells for all $n \gg 0$.

I. The structure of completely finitary Gorenstein projective modules.

The Eckmann-Shapiro lemma for complete cohomology implies that induction from a subgroup H of a group G maps completely finitary $\mathbb{Z}H$-modules to completely finitary $\mathbb{Z}G$-modules. Moreover, the class of completely finitary modules is easily seen to be closed under extensions and direct summands. Analogous assertions are valid for the class of Gorenstein projective modules as well. The following result shows that these operations suffice to construct in an economic way all completely finitary Gorenstein projective $\mathbb{Z}G$-modules, in the case where the group G is hierarchically decomposable and has finite Gorenstein cohomological dimension.

Theorem 3.1. *Let $\mathfrak{X}$ be subgroup-closed subclass of the class $\mathfrak{Y}$ introduced in Definition 2.2. We consider an* $\mathbf{LH}\mathfrak{X}$*-group G of finite Gorenstein cohomological dimension and the class*

$$\mathfrak{C} = \{ind_H^G M : H \text{ is an } \mathfrak{X}\text{-subgroup of } G \text{ and } M \in \mathtt{CF}(\mathbb{Z}H) \cap \mathtt{GP}(\mathbb{Z}H)\}.$$

Then:

(i) $\mathtt{CF}(\mathbb{Z}G)^{\perp} = \mathfrak{C}^{\perp}$ *and*

(ii) the class $\mathtt{CF}(\mathbb{Z}G) \cap \mathtt{GP}(\mathbb{Z}G)$ *of completely finitary Gorenstein projective $\mathbb{Z}G$-modules is precisely the class consisting of the direct summands of poly-$\mathfrak{C}$ modules.*

Proof. (i) If H is any subgroup of G and M a completely finitary $\mathbb{Z}H$-module, then the Eckmann-Shapiro lemma for complete cohomology implies that $\mathrm{ind}_H^G M$ is a completely finitary $\mathbb{Z}G$-module. In particular, it follows that $\mathfrak{C} \subseteq \mathtt{CF}(\mathbb{Z}G)$ and hence $\mathtt{CF}(\mathbb{Z}G)^{\perp} \subseteq \mathfrak{C}^{\perp}$. Conversely, let N be a $\mathbb{Z}G$-module contained in $\mathfrak{C}^{\perp}$. In order to show that $N \in \mathtt{CF}(\mathbb{Z}G)^{\perp}$, it suffices (in view of Corollary 2.4) to show that $\mathrm{res}_H^G N \in \mathtt{CF}(\mathbb{Z}H)^{\perp}$ for any $\mathfrak{X}$-subgroup H of G. To that end, let H be an $\mathfrak{X}$-subgroup of G and consider a completely finitary $\mathbb{Z}H$-module M. Since G has finite Gorenstein cohomological dimension, its subgroup H has finite Gorenstein cohomological dimension as well (cf. [9], §5.1(iii)). Then, the $\mathbb{Z}H$-module M has finite Gorenstein projective dimension and hence the functor $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}H}(M, \_)$ is naturally equivalent to the functor $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}H}(M', \_)$, for a suitable completely finitary Gorenstein projective $\mathbb{Z}H$-module M'. (In fact, one may choose M' to be the 0-th syzygy of a complete resolution of M.) Then, we have

$$\widehat{\mathrm{Ext}}^0_{\mathbb{Z}H}\big(M, \mathrm{res}_H^G N\big) = \widehat{\mathrm{Ext}}^0_{\mathbb{Z}H}\big(M', \mathrm{res}_H^G N\big) = \widehat{\mathrm{Ext}}^0_{\mathbb{Z}H}\big(\mathrm{ind}_H^G M', N\big) = 0.$$

In the above chain of equalities, the second one is the Eckmann-Shapiro lemma for complete cohomology, whereas the last one follows since $N \in \mathfrak{C}^{\perp}$. Since the group $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}H}\big(M, \mathrm{res}_H^G N\big)$ is trivial for any $M \in \mathtt{CF}(\mathbb{Z}H)$, it follows that $\mathrm{res}_H^G N \in \mathtt{CF}(\mathbb{Z}H)^{\perp}$, as needed.

(ii) First of all, we note that $\mathfrak{C}$ is contained in the intersection $\mathtt{CF}(\mathbb{Z}G) \cap \mathtt{GP}(\mathbb{Z}G)$; as we noted above, the inclusion $\mathfrak{C} \subseteq \mathtt{CF}(\mathbb{Z}G)$ follows from the Eckmann-Shapiro lemma for complete cohomology, whereas Lemma 1.1 implies that $\mathfrak{C} \subseteq \mathtt{GP}(\mathbb{Z}G)$. Since both classes $\mathtt{CF}(\mathbb{Z}G)$ and $\mathtt{GP}(\mathbb{Z}G)$ are closed under extensions and direct summands, we conclude that direct summands of poly-$\mathfrak{C}$ modules are completely finitary and Gorenstein projective. Conversely, let M be a completely finitary Gorenstein projective $\mathbb{Z}G$-module.

We consider the $\mathbb{Z}G$-module M_∞ constructed in §1.IV and adopt the notation used therein. Since M is completely finitary and Gorenstein projective, whereas $M_\infty \in \mathfrak{C}^\perp$, it follows from (i) above that $\underline{\mathrm{Hom}}_{\mathbb{Z}G}(M, M_\infty) = 0$. Since M is, of course, stably compact and $M_\infty = \varinjlim_n M_n$, we conclude that

$$\varinjlim_n \underline{\mathrm{Hom}}_{\mathbb{Z}G}(M, M_n) = \underline{\mathrm{Hom}}_{\mathbb{Z}G}(M, M_\infty) = 0.$$

In particular, the class of the identity map of $M = M_0$ maps onto zero under the additive map

$$\iota_{n*} : \underline{\mathrm{Hom}}_{\mathbb{Z}G}(M, M_0) \longrightarrow \underline{\mathrm{Hom}}_{\mathbb{Z}G}(M, M_n)$$

for some $n \gg 0$. It follows that $[\iota_n] = [0] \in \underline{\mathrm{Hom}}_{\mathbb{Z}G}(M, M_n)$ and hence $\iota_n : M \longrightarrow M_n$ factors through a projective $\mathbb{Z}G$-module P for some $n \gg 0$, say as the composition

$$M \stackrel{a}{\longrightarrow} P \stackrel{b}{\longrightarrow} M_n.$$

We now invoke condition $(\mathrm{ii})_n$ in the construction of the M_n's in §1.IV: There is a projective $\mathbb{Z}G$-module Q_n and a $\mathbb{Z}G$-linear map $f_n : Q_n \longrightarrow M_n$, such that $[\iota_n, f_n] : M \oplus Q_n \longrightarrow M_n$ is a surjective $\mathbb{Z}G$-linear map, whose kernel K_n is a hyper-$\mathfrak{C}$ module. Since the latter map factors as the composition

$$M \oplus Q_n \stackrel{a\oplus 1_{Q_n}}{\longrightarrow} P \oplus Q_n \stackrel{[b,f_n]}{\longrightarrow} M_n$$

and $P \oplus Q_n$ is a projective $\mathbb{Z}G$-module, we may invoke ([16], Lemma 3.1) in order to conclude that the $\mathbb{Z}G$-module $M \oplus Q_n$ (and, a fortiori, the $\mathbb{Z}G$-module M) is a direct summand of $K_n \oplus P \oplus Q_n$. We note that the regular module $\mathbb{Z}G$ is obviously contained in $\mathfrak{C}$ and hence all free $\mathbb{Z}G$-modules are hyper-$\mathfrak{C}$. Since K_n is a hyper-$\mathfrak{C}$ module, we conclude that M is a direct summand of a hyper-$\mathfrak{C}$ module E. In other words, there exist $\mathbb{Z}G$-linear maps $f : M \longrightarrow E$ and $g : E \longrightarrow M$, such that $gf = 1_M : M \longrightarrow M$. We now invoke Lemma 1.6 and express f as the composition of two $\mathbb{Z}G$-linear maps $f_1 : M \longrightarrow E'$ and $f_2 : E' \longrightarrow E$, where E' is a poly-$\mathfrak{C}$ module. Since the composition

$$M \stackrel{f_1}{\longrightarrow} E' \stackrel{f_2}{\longrightarrow} E \stackrel{g}{\longrightarrow} M$$

is the identity map of M, it follows that M is a direct summand of the poly-$\mathfrak{C}$ module E'. □

Corollary 3.2. *Let $\mathfrak{X}$ be a subgroup-closed subclass of $\mathfrak{Y}$. We consider an* $\mathbf{LH}\mathfrak{X}$*-group G of finite Gorenstein cohomological dimension and the class*

$$\mathfrak{C} = \{ind_H^G M : H \text{ is an } \mathfrak{X}\text{-subgroup of } G \text{ and } M \in \mathtt{CF}(\mathbb{Z}H) \cap \mathtt{GP}(\mathbb{Z}H)\}.$$

If $P_* \longrightarrow M \longrightarrow 0$ is a projective resolution of a completely finitary $\mathbb{Z}G$-module M, then the image $\Omega^n M$ of the map $P_n \longrightarrow P_{n-1}$ is a direct summand of a poly-$\mathfrak{C}$ module for all $n \geq Gcd\, G+1$.

Proof. The Gorenstein projective dimension of M is $\leq \operatorname{Gcd} G+1$ and hence the image $\Omega^n M$ of the map $P_n \longrightarrow P_{n-1}$ is Gorenstein projective for all $n \geq \operatorname{Gcd} G+1$. Since M is completely finitary, it is clear that $\Omega^n M$ is completely finitary as well. Then, the result follows from Theorem 3.1(ii). □

II. Coherent group rings.

Recall that a ring is *left coherent* if and only if every finitely generated left ideal is finitely presented, or equivalently if every finitely presented module is of type FP_∞. This latter formulation of coherence may be compared with the observation that a ring is Noetherian if and only if every *finitely generated* module is of type FP_∞; therefore this formulation is both relevant to the present article and can help to generalize results and methods from the Noetherian case.

We consider the class $\mathfrak{K}$, consisting of those groups G for which the integral group ring $\mathbb{Z}G$ is (left) coherent. We shall use the following properties of this class of groups below:

(i) $\mathfrak{K}$ is subgroup-closed.

Indeed, let G be a $\mathfrak{K}$-group and consider a subgroup $H \subseteq G$. If I is a finitely generated left ideal of $\mathbb{Z}H$, then $\operatorname{ind}_H^G I$ is a finitely generated left ideal of $\mathbb{Z}G$. Since $G \in \mathfrak{K}$, the $\mathbb{Z}G$-module $\operatorname{ind}_H^G I$ is finitely presented and hence the functor $\operatorname{Hom}_{\mathbb{Z}G}(\operatorname{ind}_H^G I, \_)$ commutes with filtered colimits of $\mathbb{Z}G$-modules. Since any filtered direct system of $\mathbb{Z}H$-modules is a direct summand of the restriction to H of a filtered direct system of $\mathbb{Z}G$-modules, it follows that the functor $\operatorname{Hom}_{\mathbb{Z}H}(I, \_)$ commutes with filtered colimits of $\mathbb{Z}H$-modules. This implies that I is a finitely presented $\mathbb{Z}H$-module. Since this is the case for any finitely generated left ideal $I \subseteq \mathbb{Z}H$, it follows that the group ring $\mathbb{Z}H$ is (left) coherent and hence $H \in \mathfrak{K}$.

(ii) $\mathfrak{K}$ is a subclass of $\mathfrak{Y}$.

Indeed, if $G \in \mathfrak{K}$, then any finitely presented $\mathbb{Z}G$-module is of type FP_∞. Since any module (over any ring) may be expressed as a filtered colimit of finitely presented modules, we conclude that any $\mathbb{Z}G$-module may be expressed (in the case where G is a $\mathfrak{K}$-group) as a filtered colimit of modules of type FP_∞. As modules of type FP_∞ are completely finitary (cf. [15], §4.1(ii)), it follows that $G \in \mathfrak{Y}$.

(iii) If G is a $\mathfrak{K}$-group then any completely finitary Gorenstein projective $\mathbb{Z}G$-module M is a direct summand of a $\mathbb{Z}G$-module M', which possesses a

projective resolution $P'_* \longrightarrow M' \longrightarrow 0$ that is finitely generated after one step, i.e. which is such that the projective $\mathbb{Z}G$-module P'_n is finitely generated for all $n \geq 1$.
Indeed, we know that any completely finitary Gorenstein projective $\mathbb{Z}G$-module M is a direct summand of the direct sum $P \oplus N$ of two $\mathbb{Z}G$-modules P and N, where P is projective and N is finitely presented (cf. Lemma 1.5). As we have already noted above, our assumption that G is a $\mathfrak{K}$-group implies that the $\mathbb{Z}G$-module N is of type FP_∞ and this proves the claim.

At this point it may be worth drawing attention to at least one source of examples of coherent group rings. The following is a natural generalization of ([5], Theorem B (iii) $\Longrightarrow$ (i)).

Proposition 3.3. *Let G be the fundamental group of a graph of groups in which the group ring of each edge group is Noetherian and each vertex group is contained in $\mathfrak{K}$. Then G belongs to $\mathfrak{K}$.*

Proof. To prove that $\mathbb{Z}G$ is left coherent it suffices, according to Chase's theorem, to prove that an arbitrary product of flat right $\mathbb{Z}G$-modules is flat (see [18] Theorem 4.47 for example). Let $(F_i)_i$ be a family of flat right $\mathbb{Z}G$-modules and let $F = \prod_i F_i$ be the correspodning direct product. We shall prove that $\mathrm{Tor}_1^{\mathbb{Z}G}(F, M) = 0$ for any left $\mathbb{Z}G$-module M. Let T be a G-tree *with left action of* G that is witness to the hypotheses, with edge set E and vertex set V. Then there is a short exact sequence $0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0$ and this yields a short exact sequence with the diagonal action of G upon tensoring with M, namely $0 \to \mathbb{Z}E \otimes M \to \mathbb{Z}V \otimes M \to M \to 0$. From here we obtain a commutative diagram combining long exact Mayer–Vietoris sequences:

$$\begin{array}{ccccccc}
\mathrm{Tor}_1^{\mathbb{Z}G}(F, \mathbb{Z}V \otimes M) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}G}(F, M) & \longrightarrow & F \otimes_{\mathbb{Z}G} (\mathbb{Z}E \otimes M) & \longrightarrow & F \otimes_{\mathbb{Z}G} (\mathbb{Z}V \otimes M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_i \mathrm{Tor}_1^{\mathbb{Z}G}(F_i, \mathbb{Z}V \otimes M) & \longrightarrow & \prod_i \mathrm{Tor}_1^{\mathbb{Z}G}(F_i, M) & \longrightarrow & \prod_i [F_i \otimes_{\mathbb{Z}G} (\mathbb{Z}E \otimes M)] & \longrightarrow & \prod_i [F_i \otimes_{\mathbb{Z}G} (\mathbb{Z}V \otimes M)] .
\end{array}$$

Let V_0 be a set of orbit representatives for the vertex set V. Then the top left entry of the diagram, $\mathrm{Tor}_1^{\mathbb{Z}G}(F, \mathbb{Z}V \otimes M)$ is isomorphic to $\bigoplus_{v \in V_0} \mathrm{Tor}_1^{\mathbb{Z}G}(F, \mathbb{Z}[G/G_v] \otimes M) \cong \bigoplus_{v \in V_0} \mathrm{Tor}_1^{\mathbb{Z}G_v}(F, M)$ and this is zero because each group ring $\mathbb{Z}G_v$ is coherent. Also the bottom centre-left entries are zero because each F_i is $\mathbb{Z}G_v$-flat. So the diagram simplifies as shown.

$$\begin{array}{ccccccc}
0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}G}(F, M) & \longrightarrow & F \otimes_{\mathbb{Z}G} (\mathbb{Z}E \otimes M) & \longrightarrow & F \otimes_{\mathbb{Z}G} (\mathbb{Z}V \otimes M) \\
 & & \downarrow \alpha & & \downarrow \\
0 & \longrightarrow & \prod_i [F_i \otimes_{\mathbb{Z}G} (\mathbb{Z}E \otimes M)] & \longrightarrow & \prod_i [F_i \otimes_{\mathbb{Z}G} (\mathbb{Z}V \otimes M)] .
\end{array}$$

The desired conclusion $\mathrm{Tor}_1^{\mathbb{Z}G}(F, M) = 0$ follows from the commutativity and exactness properties of this diagram once we show that the map labelled α is injective. The codomain of α may be expressed as $\prod_i [\bigoplus_e (F_i \otimes_{\mathbb{Z}G_e} M)]$ where e runs through a set of orbit representatives of edges of our tree. This in turn sits inside $\prod_i \prod_e (F_i \otimes_{\mathbb{Z}G_e} M) = \prod_e \prod_i (F_i \otimes_{\mathbb{Z}G_e} M)$ and so α can be seen to be the restriction of the product over a set of orbit representatives of edges of the natural maps $F \otimes_{\mathbb{Z}G_e} M \to \prod_i (F_i \otimes_{\mathbb{Z}G_e} M)$. Injectivity follows from the hypothesis that each $\mathbb{Z}G_e$ is Noetherian: it is enough to check injectivity when M is replaced by a finitely generated $\mathbb{Z}G_e$-submodule and therefore finitely presented and in such a case one can appeal to ([18], Proposition 4.44) for example. □

Proposition 3.4. *Let G be an* $\mathbf{LH}\mathfrak{K}$*-group of finite Gorenstein cohomological dimension. Then, any completely finitary Gorenstein projective $\mathbb{Z}G$-module M is a direct summand of a $\mathbb{Z}G$-module M', which itself has a projective resolution that is finitely generated after one step.*

Proof. Let

$$\mathfrak{C} = \{\mathrm{ind}_H^G M : H \text{ is a } \mathfrak{K}\text{-subgroup of } G \text{ and } M \in \mathtt{CF}(\mathbb{Z}H) \cap \mathtt{GP}(\mathbb{Z}H)\}.$$

Since $\mathfrak{K}$ is subgroup-closed (property (i) above) and $\mathfrak{K} \subseteq \mathfrak{Y}$ (property (ii) above), we may invoke Theorem 3.1(ii) and conclude that any completely finitary Gorenstein projective $\mathbb{Z}G$-module is a direct summand of a poly-$\mathfrak{C}$ module. Therefore, it only remains to show that any poly-$\mathfrak{C}$ module M is a direct summand of a $\mathbb{Z}G$-module M', which itself has a projective resolution that is finitely generated after one step. This latter claim in an immediate consequence of the following two assertions:

(a) Any $\mathbb{Z}G$-module contained in $\mathfrak{C}$ is a direct summand of a $\mathbb{Z}G$-module, which itself has a projective resolution that is finitely generated after one step.

(b) The class of those $\mathbb{Z}G$-modules which are direct summands of other $\mathbb{Z}G$-modules, which themselves have a projective resolution that is finitely generated after one step, is extension-closed.

In order to prove assertion (a), we consider a $\mathfrak{K}$-subgroup H of G and a completely finitary Gorenstein projective $\mathbb{Z}H$-module M. Then, M is a direct summand of a $\mathbb{Z}H$-module M', which itself has a projective resolution that is finitely generated after one step (property (iii) above). Hence, the $\mathbb{Z}G$-module $\mathrm{ind}_H^G M$ is a direct summand of the $\mathbb{Z}G$-module $\mathrm{ind}_H^G M'$, which itself has a projective resolution that is finitely generated after one step.

In order to prove assertion (b), we consider a short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow M' \stackrel{\iota}{\longrightarrow} M \stackrel{p}{\longrightarrow} M'' \longrightarrow 0$$

and assume that M' and M'' are direct summands of other $\mathbb{Z}G$-modules, which themselves have projective resolutions that are finitely generated after one step. In other words, we assume that there exist $\mathbb{Z}G$-modules N' and N'', such that the $\mathbb{Z}G$-modules $M' \oplus N'$ and $M'' \oplus N''$ have projective resolutions that are finitely generated after one step. We have to show that M is a direct summand of a $\mathbb{Z}G$-module, which itself has a projective resolution that is finitely generated after one step. To that end, we consider the short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow M' \oplus N' \stackrel{\jmath}{\longrightarrow} M \oplus N' \oplus N'' \stackrel{q}{\longrightarrow} M'' \oplus N'' \longrightarrow 0,$$

where $\jmath(x, y) = (\iota(x), y, 0)$ for all $(x, y) \in M' \oplus N'$ and $q(w, y, z) = (p(w), z)$ for all $(w, y, z) \in M \oplus N' \oplus N''$. Applying the horseshoe lemma to the latter short exact sequence, we conclude that the $\mathbb{Z}G$-module $M \oplus N' \oplus N''$ has a projective resolution that is finitely generated after one step. This proves the claim made for M and finishes the proof of assertion (b). □

Let R be a ring. We say that a projective resolution $P_* \longrightarrow M \longrightarrow 0$ of a left R-module M is finitely generated after k steps if the projective module P_n is finitely generated for all $n \geq k$.

Lemma 3.5. *Let R be a ring and consider a short exact sequence of left R-modules*

$$0 \longrightarrow M' \stackrel{\iota}{\longrightarrow} M \stackrel{p}{\longrightarrow} M'' \longrightarrow 0.$$

If M' (resp. M) has a projective resolution which is finitely generated after k' steps (resp. after k steps), then M'' has a projective resolution which is finitely generated after k'' steps, where $k'' = \max\{k' + 1, k\}$.

Proof. Let $P'_* \longrightarrow M' \longrightarrow 0$ and $P_* \stackrel{\varepsilon}{\longrightarrow} M \longrightarrow 0$ be projective resolutions which are finitely generated after k'- and k steps respectively. We consider a chain map $\iota_* : P'_* \longrightarrow P_*$, which lifts ι, and consider the mapping cone $P''_* = \text{cone}(\iota_*)$. If $\varepsilon'' : P''_0 \longrightarrow M''$ is the composition

$$P''_0 = P_0 \stackrel{\varepsilon}{\longrightarrow} M \stackrel{p}{\longrightarrow} M'',$$

then it is easily seen that $P''_* \stackrel{\varepsilon''}{\longrightarrow} M'' \longrightarrow 0$ is a projective resolution of M''. By its very definition, $P''_n = P_n \oplus P'_{n-1}$ is finitely generated if $n \geq k$ and $n - 1 \geq k'$ and this finishes the proof. □

Corollary 3.6. *Let G be an* $\mathbf{LH}\mathfrak{K}$*-group of finite Gorenstein cohomological dimension. Then, any completely finitary $\mathbb{Z}G$-module M is a direct summand of a $\mathbb{Z}G$-module M', which itself has a projective resolution that is finitely generated after k steps, where $k = Gpd_{\mathbb{Z}G}M + 1$.*

Proof. We proceed by induction on the Gorenstein projective dimension $\text{Gpd}_{\mathbb{Z}G}M$ of M. If M is Gorenstein projective, the result follows from Proposition 3.4. We now assume that $\text{Gpd}_{\mathbb{Z}G}M = m > 0$ and consider a short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow K \stackrel{\iota}{\longrightarrow} P \stackrel{p}{\longrightarrow} M \longrightarrow 0,$$

where P is projective. Then, K is completely finitary and has Gorenstein projective dimension $m-1$. In view of the induction hypothesis, we conclude that there exists a $\mathbb{Z}G$-module L, such that $K \oplus L$ has a projective resolution which is finitely generated after m steps. We now consider the short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow K \oplus L \stackrel{\jmath}{\longrightarrow} P \oplus K \oplus L \stackrel{q}{\longrightarrow} M \oplus K \longrightarrow 0,$$

where $\jmath(x, y) = (\iota(x), 0, y)$ for all $(x, y) \in K \oplus L$ and $q(w, x, y) = (p(w), x)$ for all $(w, x, y) \in P \oplus K \oplus L$. Since $P \oplus K \oplus L$ has clearly a projective resolution which is finitely generated after $\max\{m, 1\} = m$ steps, we may invoke Lemma 3.5 and conclude that $M \oplus K$ has a projective resolution which is finitely generated after $\max\{m+1, m\} = m+1$ steps, as needed. □

Remark 3.7. Let G be an $\mathbf{LH}\mathfrak{K}$-group of finite Gorenstein cohomological dimension. Then, the proof of Corollary 3.6 above shows that for any completely finitary $\mathbb{Z}G$-module M the direct sum $M \oplus \Omega M$ has a projective resolution that is finitely generated after k steps, where $k = \max\{2, \text{Gpd}_{\mathbb{Z}G}M + 1\}$. (Using Schanuel's lemma and Lemma 3.5, it is easily seen that this latter condition does not depend upon the particular choice of the module ΩM.)

Corollary 3.8. *Let G be an* $\mathbf{LH}\mathfrak{K}$*-group of finite Gorenstein cohomological dimension. Then, any completely finitary $\mathbb{Z}G$-module M is a direct summand of a $\mathbb{Z}G$-module M', which itself has a projective resolution that is finitely generated after k steps, where $k = Gcd\, G + 2$.*

Proof. Since $\text{Gpd}_{\mathbb{Z}G}M \leq \text{Gcd}\, G + 1$, the result follows from Corollary 3.6. □

III. Groups with finitary cohomology.

We say that a group G has infinitely often finitary cohomology if there are infinitely many non-negative integers n, such that the ordinary cohomology functors $H^n(G, \_) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \_)$ commute with filtered colimits. If this latter condition holds for all but finitely many n's, we say that G has eventually finitary cohomology. The following result subsumes both Theorem A and Theorem B as stated in the Introduction.

Theorem 3.9. *Let G be an* $\mathbf{LH}\mathfrak{K}$*-group of Gorenstein cohomological dimension $\mathrm{Gcd}\,G = k < \infty$. Then the following conditions are equivalent:*

- (i) *The cohomology functors $H^n(G, \_)$ commute with direct limits for all $n \geq k + 3$.*
- (ii) *The group G has eventually finitary cohomology.*
- (iii) *The group G has infinitely often finitary cohomology.*
- (iv) *In every projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, the kth kernel is a direct summand of a $\mathbb{Z}G$-module which has a projective resolution that is of finite type after one step.*
- (v) *Over the integral group ring of the group $G \times \mathbb{Z}$, the trivial module $\mathbb{Z}$ has a projective resolution $P_* \twoheadrightarrow \mathbb{Z}$ in which P_n is finitely generated for all sufficiently large n.*
- (vi) *The group $G \times \mathbb{Z}$ has an Eilenberg–Mac Lane space $K(G \times \mathbb{Z}, 1)$ with finitely many n-cells for all sufficiently large $n \geq k + 2$.*
- (vii) *The group G has an Eilenberg–Mac Lane space $K(G, 1)$, which is dominated by a CW-complex with finitely many n-cells for all $n \geq k + 2$.*
- (viii) *The group G has an Eilenberg–Mac Lane space $K(G, 1)$, which is dominated by a CW-complex with finitely many n-cells for all sufficiently large n.*

Proof. The implications (i)→(ii)→(iii), (vi)→(v) and (vii)→(viii) are trivial, whereas the implications (vi)→(vii)→(i) and (vi)→(viii)→(ii) are valid for any group, as shown in ([12], §2.2 and §2.3). In order to show that (iii)→(v)→(vi), we note that condition (iii) implies that the complete cohomology functors $\widehat{H}^n(G, \_) = \widehat{\mathrm{Ext}}^n_{\mathbb{Z}G}(\mathbb{Z}, \_)$ commute with filtered colimits for all $n \in \mathbb{Z}$ (cf. [15], §4.1(ii)), i.e. that the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is completely finitary. If

$$0 \longrightarrow M \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

is any exact sequence of $\mathbb{Z}G$-modules, where F_i is free for all $i = 0, 1, \ldots, k-1$, then M is a completely finitary Gorenstein projective $\mathbb{Z}G$-module. Therefore, M is a direct summand of a $\mathbb{Z}G$-module M', which itself has a projective resolution that is finitely generated after one step (cf. Proposition 3.4) and this shows that (iv) holds. Now suppose that (iv) holds, so that we have a partial resolution as above. Since we wish to think of the additive group $\mathbb{Z}$ as a multiplicative group, we introduce the notation $C = \langle c \rangle$ for it, so that the group algebra $\mathbb{Z}C$ is a Laurent polynomial ring $\mathbb{Z}[c, c^{-1}]$ in one variable. Then we may tensor the exact sequence above with the projective resolution

$$0 \longrightarrow \mathbb{Z}C \xrightarrow{1-c} \mathbb{Z}C \longrightarrow \mathbb{Z} \longrightarrow 0$$

of the trivial $\mathbb{Z}C$-module $\mathbb{Z}$ and obtain an exact sequence of $\mathbb{Z}(G \times C)$-modules

$$0 \longrightarrow M \otimes \mathbb{Z}C \longrightarrow (M \otimes \mathbb{Z}C) \oplus (F_{k-1} \otimes \mathbb{Z}C) \longrightarrow$$
$$(F_{k-1} \otimes \mathbb{Z}C) \oplus (F_{k-2} \otimes \mathbb{Z}C) \longrightarrow \cdots \longrightarrow F_0 \otimes \mathbb{Z}C \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Adding a suitable $\mathbb{Z}(G \times C)$-module in degrees k and $k+1$, we then obtain an exact sequence of $\mathbb{Z}(G \times C)$-modules

$$0 \longrightarrow M' \otimes \mathbb{Z}C \xrightarrow{f} (M' \otimes \mathbb{Z}C) \oplus (F_{k-1} \otimes \mathbb{Z}C) \longrightarrow$$
$$(F_{k-1} \otimes \mathbb{Z}C) \oplus (F_{k-2} \otimes \mathbb{Z}C) \longrightarrow \cdots \longrightarrow F_0 \otimes \mathbb{Z}C \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Since the $\mathbb{Z}G$-module M' admits a projective resolution that is finitely generated after one step, the argument in the proof of Lemma 3.5 shows that there is a $\mathbb{Z}(G \times C)$-projective resolution $Q_* \longrightarrow \operatorname{coker} f \longrightarrow 0$ of $\operatorname{coker} f$, such that Q_0 is a free $\mathbb{Z}(G \times C)$-module and Q_n is finitely generated and free for all $n \geq 2$. By concatenation, we obtain a projective resolution of the trivial $\mathbb{Z}(G \times C)$-module $\mathbb{Z}$

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow (F_{k-1} \otimes \mathbb{Z}C) \oplus (F_{k-2} \otimes \mathbb{Z}C) \longrightarrow \cdots \longrightarrow F_0 \otimes \mathbb{Z}C \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where all modules with the possible exception of Q_1 are free and Q_n is finitely generated (and free) for all $n \geq 2$. Using Eilenberg's trick, we may add a free $\mathbb{Z}(G \times C)$-module to both Q_1 and Q_0 and obtain a resolution of the trivial $\mathbb{Z}(G \times C)$-module $\mathbb{Z}$

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1' \longrightarrow Q_0' \longrightarrow (F_{k-1} \otimes \mathbb{Z}C) \oplus (F_{k-2} \otimes \mathbb{Z}C) \longrightarrow \cdots \longrightarrow F_0 \otimes \mathbb{Z}C \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where all modules are free and Q_n is finitely generated (and free) for all $n \geq 2$. Thus (v) holds. Then, the topological arguments used in the proof of ([12], Theorem 2.20) show that there is a model for the Eilenberg–Mac Lane space $K(G \times C, 1) = K(G \times \mathbb{Z}, 1)$ with finitely many n-cells for all $n \geq k+2$ and we conclude that (vi) holds.

□

References

[1] A. Bahlekeh, F. Dembegioti, and O. Talelli. Gorenstein dimension and proper actions. *Bull. Lond. Math. Soc.*, 41(5):859–871, 2009.

[2] D. J. Benson. Complexity and varieties for infinite groups. I, II. *J. Algebra*, 193(1):260–287, 288–317, 1997.

[3] D. J. Benson and J. F. Carlson. Products in negative cohomology. *J. Pure Appl. Algebra*, 82(2):107–129, 1992.

[4] R. Bieri. *Homological dimension of discrete groups*. Queen Mary College Mathematical Notes. Queen Mary College Department of Pure Mathematics, London, second edition, 1981.

[5] R. Bieri and R. Strebel. Soluble groups with coherent group rings. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 235–240. Cambridge Univ. Press, Cambridge, 1979.

[6] J. Cornick and P. H. Kropholler. Homological finiteness conditions for modules over strongly group-graded rings. *Math. Proc. Cambridge Philos. Soc.*, 120(1):43–54, 1996.

[7] J. Cornick and P. H. Kropholler. On complete resolutions. *Topology Appl.*, 78(3):235–250, 1997.

[8] J. Cornick and P. H. Kropholler. Homological finiteness conditions for modules over group algebras. *J. London Math. Soc. (2)*, 58(1):49–62, 1998.

[9] T. V. Gedrich and K. W. Gruenberg. Complete cohomological functors on groups. *Topology Appl.*, 25(2):203–223, 1987. Singapore topology conference (Singapore, 1985).

[10] F. Goichot. Homologie de Tate-Vogel équivariante. *J. Pure Appl. Algebra*, 82(1):39–64, 1992.

[11] M. Hamilton. Finitary group cohomology and group actions on spheres. *Proc. Edinb. Math. Soc. (2)*, 51(3):651–655, 2008.

[12] M. Hamilton. Finitary group cohomology and Eilenberg-MacLane spaces. *Bull. Lond. Math. Soc.*, 41(5):782–794, 2009.

[13] M. Hamilton. When is group cohomology finitary? *J. Algebra*, 330:1–21, 2011.

[14] H. Holm. Gorenstein homological dimensions. *J. Pure Appl. Algebra*, 189(1-3):167–193, 2004.

[15] P. H. Kropholler. On groups of type $(\mathrm{FP})_\infty$. *J. Pure Appl. Algebra*, 90(1):55–67, 1993.

[16] P. H. Kropholler. Modules possessing projective resolutions of finite type. *J. Algebra*, 216(1):40–55, 1999.

[17] P. H. Kropholler. Groups with many finitary cohomology functors. *to appear in Proceedings of the AMS meeting 4th to 10th December 2010, OSU Columbus*, 2013.

[18] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[19] G. Mislin. Tate cohomology for arbitrary groups via satellites. *Topology Appl.*, 56(3):293–300, 1994.

[20] J. Rickard. Idempotent modules in the stable category. *J. London Math. Soc. (2)*, 56(1):149–170, 1997.