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Covers of Acts over Monoids

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ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES
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COVERS OF ACTS OVER MONOIDS

by Alexander Bailey

Since they were first defined in the 1950's, projective covers (the dual of injective envelopes) have proved to be an important tool in module theory, and indeed in many other areas of abstract algebra. An attempt to generalise the concept led to the introduction of covers with respect to other classes of modules, for example, injective covers, torsion-free covers and flat covers. The flat cover conjecture (now a Theorem) is of particular importance, it says that every module over every ring has a flat cover. This has led to surprising results in cohomological studies of certain categories.

Given a general class of objects $\mathcal{X}$, an $\mathcal{X}$-cover of an object A can be thought of a the 'best approximation' of A by an object from $\mathcal{X}$. In a certain sense, it behaves like an adjoint to the inclusion functor.

In this thesis we attempt to initiate the study of different types of covers for the category of acts over a monoid. We give some necessary and sufficient conditions for the existence of $\mathcal{X}$-covers for a general class $\mathcal{X}$ of acts, and apply these results to specific classes. Some results include, every S-act has a strongly flat cover if S satisfies Condition (A), every S-act has a torsion free cover if S is cancellative, and every S-act has a divisible cover if and only if S has a divisible ideal.

We also consider the important concept of purity for the category of acts. Giving some new characterisations and results for pure monomorphisms and pure epimorphisms.

Contents

Author's declaration

I, Alexander Bailey, declare that the thesis entitled *Covers of Acts over Monoids* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: *Covers of acts over monoids and pure epimorphisms*, A. Bailey and J. Renshaw, to appear in Proc. Edinburgh Math. Soc, 2013; *Covers of acts over monoids II*, A. Bailey and J. Renshaw, Semigroup Forum Vol. 87, pp 257–274, 2013.

Signed ....

Date ....

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Introduction

Given a category $\mathcal{C}$, and a subcategory $\mathcal{X} \subseteq \mathcal{C}$, an $\mathcal{X}$-cover can be thought of as the 'best approximation' of an object in $\mathcal{C}$ by an object from $\mathcal{X}$. In particular, covers (and their categorical dual, envelopes), have proved to be an important tool in module theory. This is explained succinctly in the introduction to Göbel and Trlifaj's book *'Approximations and Endomorphism Algebras of Modules'* [30, 31]:

> It is a widely accepted fact that the category of all modules over a general associative ring is too complex to admit classification. Unless the ring is of finite representation type, we must limit attempts at classification to some restricted subcategories of modules. The wild character of the category of all modules, or of one of its subcategories C, is often indicated by the presence of a realisation theorem, that is, by the fact that any reasonable algebra is isomorphic to the endomorphism algebra of a module from C. This results in the existence of pathological direct sum decompositions, and these are generally viewed as obstacles to classification. Realisation theorems have thus turned into important indicators of the "non classification theory" of modules. In order to overcome this problem, the approximation theory of modules has been developed over the past few decades. The idea here is to select suitable subcategories C whose modules can be classified, and then approximate arbitrary modules by those from C. These approximations are neither unique not factorial in general, but there is a rich supply available appropriate to the requirements of various particular applications. Thus approximation theory has developed into an important part of the classification theory of modules.

It was Bass in 1960 who first characterised (right) perfect rings, that is, rings whose (right) modules all have projective covers, as the rings which satisfy the descending chain condition on principal (left) ideals. The classical concepts of projective covers (and dually injective envelopes) of modules then led to the introduction of covers with respect to other classes of modules, for example, injective covers and torsion-free covers. Historically this area has two branches: the covers and envelopes studied by Enochs for arbitrary modules, and the finite dimensional case of Auslanders school under the name of minimal right and left approximations. In this thesis, we primarily imitate Enochs' approach.

In 1963 Enochs showed that over an integral domain every module has a torsion free cover [21]. In 1981 Enochs also showed that every module has an injective cover if and only if the ring is Noetherian [25]. In this same paper he first considered flat covers, showing, for example, that a module has a flat cover if it has a flat precover and conjecturing that every module over every (unital, associative) ring has a flat cover. This came to be known as the flat cover conjecture and much work was done on it over the next two decades. In 1995, J. Xu showed that commutative Noetherian rings with finite Krull dimension satisfied the conjecture and he wrote a book on the problem *'Flat covers of modules'* [57] increasing the conjecture's popularity. In 2001 the conjecture was finally solved independently by Enochs and Bican & El Bashir and published in a joint paper [8]. The two proofs were quite different in their approach, one basically a corollary of a set-theoretic result published by Eklof and Trlifaj and the other a more direct proof with a model-theoretic flavour.

The flat cover conjecture has since been proved in many other categories having surprising applications in (co)homology. To summarise, the existence of flat covers in a category which does not, in general, have enough projectives, allows us to compute homology, i.e. $\mathrm{Tor}_n^{\mathcal{C}}(A,B)$ for right and left $\mathcal{C}$-modules A and B. Using flat resolutions with successive flat precovers means the lifting property from the precovers give well-defined homology groups. This whole area has become known as 'relative homological algebra' and the existence of flat covers is central to the theory (see [26] and [27]). Some categories studied in this area include: modules over a sheaf of rings on a topological space [24], quasi-coherent sheaves over the projective line [23], quasi-coherent sheaves over a scheme [22], arbitrary Grothendieck categories [4] and finitely accessible additive categories [19].

In the same way that rings can be studied by considering their category of modules, monoids can be studied by considering their category of acts. Covers of acts over monoids were first studied by J. Isbell in 1971 [34] and J. Fountain in 1976 [28] who considered projective covers of acts. They gave a complete characterisation of perfect monoids, that is, those monoids where all of their acts have projective covers. A very interesting result showing that like rings, monoids require the descending chain condition on principal left ideals, but unlike rings, an additional ascending chain condition known as Condition (A). Then in 2008, J. Renshaw and M. Mahmoudi extended some of this work to strongly flat and Condition (P) covers of acts. This work was built on in [36] and [6]. The definition they used for covers was not the same as Enochs' definition of flat cover, but was based on the concept of a coessential epimorphism. This definition is equivalent to Enochs definition for the class of projective acts but distinct for flat covers.

It is the purpose of this thesis to initiate the study of Enochs' definition of cover for the category of acts over a monoid with the hope of generalising some of the techniques used in the proof of the flat cover conjecture. Looking at various classes of covers, e.g. free, projective, strongly flat, torsion free, divisible, injective, etc. and asking specifically, for which monoids do all acts have such covers?

In Chapter 1 we cover the preliminary results needed from set theory, category theory and semigroup theory, and in Chapter 2 we give a summary of some known and original results surrounding the category of acts over a monoid. In particular, we give the first proof of the semigroup analogue of the Bass-Papp Theorem, that every directed colimit of injective S-acts is injective when S is Noetherian.

In Chapter 3 we bring to the readers attention another definition of cover, namely a coessential cover of an act. We state some of the results from the literature and how it relates to the definition of cover in this thesis.

Chapter 4 covers the important concept of purity for acts. In particular we give some new necessary and sufficient conditions for pure epimorphisms and pure monomorphisms and discuss how they are connected to the different flatness properties of acts.

Chapter 5 contains the main results on covers. In particular, we show that if a class of S-acts $\mathcal{X}$ is closed under directed colimits, then an S-act A has an $\mathcal{X}$-precover if and only if it has an $\mathcal{X}$-cover. We also give a necessary and sufficient condition for the existence of $\mathcal{X}$-precovers based on the solution set condition, and a sufficient condition based on the 'weakly congruence pure' property. We then completely characterise covers with the unique mapping property.

Chapter 6 contains the application of these results to specific classes of acts. One of the main results is that every S-act has an $\mathcal{SF}$-cover (where $\mathcal{SF}$ is the class of strongly flat acts) if S satisfies Condition (A). We also construct an example of a monoid that has a proper class of indecomposable strongly flat acts. Enochs proved in 1963 that over an integral domain, every module has a torsion free cover, we prove the analogue of this result, that over a (right) cancellative monoid, every act has a $\mathcal{T_F}$-cover (where $\mathcal{T_F}$ is the class of torsion free acts). Enochs also proved in 1981 that every R-module has an injective cover if and only if R is Noetherian. We show that this proof does not carry over in to the category of acts and give a counter example. We finally give a necessary and sufficient condition for the existence of $\mathcal{D}$-covers (where $\mathcal{D}$ is the class of divisible acts), showing in particular that $\mathcal{D}$-covers are monomorphisms rather than epimorphisms.

Chapter 1

Preliminaries

In this Chapter we summarise some of the main definitions and results from set theory, category theory and semigroup theory. The reader familiar with these concepts can feel free to skip to Chapter 2.

1.1 Set theory

Set theory is an area of mathematics that is often avoided by most, leaving the details to the more advanced student, but it plays a prominent role throughout this thesis and so necessitates at least a basic summary of the main ideas. The hope of this section is to give an informal overview of the naive set theory used throughout without getting too bogged down by the rigours of axiomatic set theory. See [35] for more details.

To avoid such contradictions as Russel's paradox, we introduce the term **class** as a collection of **sets**. Every set is a class, and a class which is not a set is called a **proper class**. Informally, a proper class is 'too big' to be a set.

1.1.1 Zorn's Lemma

We say that a binary relation $\leq$ on a set X is a **preorder** if it is

- (reflexive) $x \leq x$ for all $x \in X$, and
- (transitive) $x \leq y$ and $y \leq z$ implies $x \leq z$.

A **partial order** is a preorder that is also

- (antisymmetric) $x \leq y$ and $y \leq x$ implies $x = y$.

A **total order** is a partial order where every pair of elements is comparable, that is, either $x \leq y$ or $y \leq x$ for all $x, y \in X$. We say that $(X, \leq)$ is a **partially ordered set** (resp. **totally ordered set**) if $\leq$ is a partial (resp. total) order on X.

Given a partially ordered set $(X, \leq)$, an element $x \in X$ is called **maximal** (resp. **minimal**) if whenever $x \leq y$ (resp. $x \geq y$) for any $y \in X$, then $x = y$. An element $x \in X$ is called **greatest** (resp. **least**) if $x \geq y$ (resp. $x \leq y$) for all $y \in X$. An element $x \in X$ is called an **upper bound** (resp. **lower bound**) of a subset $S \subseteq X$ if $x \geq y$ (resp. $x \leq y$) for all $y \in S$.

We say that two partially ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ are **order isomorphic** if there exists an order preserving bijection between them, that is, a bijective function $f : X \to Y$ such that $x \leq_X y$ if and only if $f(x) \leq_Y f(y)$.

A **well-ordered** set $(X, \leq)$ is a totally ordered set $(X, \leq)$ such that every non-empty subset of X has a least element.

We assume the truth of the following unprovable statement.

Theorem 1.1 (Zorn's Lemma). *Given a partially ordered set $(X, \leq)$ with the property that every (non-empty) totally ordered subset has an upper bound in X. Then the set X contains a maximal element.*

It is well known that Zorn's Lemma is logically equivalent to the following two statements and we will have occasion to use all three interchangeably.

Theorem 1.2 (Axiom of Choice). *For any indexed family $(X_i)_{i \in I}$ of non-empty sets there exists an indexed family $(x_i)_{i \in I}$ of elements such that $x_i \in X_i$ for all $i \in I$.*

Theorem 1.3 (The Well-Ordering Theorem). *Every set can be well-ordered.*

1.1.2 Ordinal Numbers

A set is **transitive** if every element of S is a subset of S. We define an **ordinal** (or **ordinal number**) to be a transitive set well-ordered by $\in$, that is, we identify an ordinal $\alpha = \{\beta \mid \beta < \alpha\}$ with the set of all ordinals strictly smaller than α. We let **Ord** denote the (proper) class of all ordinals. We (usually) use the symbols α, β, γ to depict arbitrary ordinal numbers.

We define

$$\alpha < \beta \text{ if and only if } \alpha \in \beta.$$

By [35, Fact (1.2.1)], the class of all ordinals **Ord** is well-ordered. Given any ordinal α we define $\alpha + 1 = \alpha \cup \{\alpha\}$ to be the **successor** of α and we say that an ordinal number is a **successor ordinal** if it is the successor of some ordinal. Every (non-zero) finite number is a successor ordinal. An ordinal is called a **limit ordinal** if it is not a successor ordinal. Alternatively, an ordinal α is a limit ordinal if for all ordinals $\beta < \alpha$ there exists an ordinal γ such that $\beta < \gamma < \alpha$. The smallest (non-zero) limit ordinal is $\omega = \mathbb{N}$.

Theorem 1.4 ([35, Theorem 1.2.12])**.** *Every well-ordered set is (order) isomorphic to a unique ordinal.*

The following Theorem is used frequently throughout this thesis:

Theorem 1.5 (Transfinite induction, [35, Theorem 1.2.14])**.** *Given an ordinal γ, and a statement $P(\delta)$ where $\delta \in$ **Ord**, if the following are true*

1. *Base step: $P(0)$;*

2. *Successor step: If $P(\beta)$ is true for $\beta < \gamma$, then $P(\beta + 1)$ is true;*

3. *Limit step: If $0 \neq \beta < \gamma$, β is a limit ordinal and $P(\alpha)$ is true for all $\alpha < \beta$, then $P(\beta)$ is true;*

then $P(\gamma)$ is true.

1.1.3 Cardinal numbers

We first define what it means for two sets to have the same cardinality before we define what a cardinal number is. We say two sets have the same **cardinality** (or **cardinal**) if there exists a bijective function between them. We assume that we can assign to each set X its cardinality, which we denote $|X|$ such that two sets are assigned the same cardinality if and only if there is a bijective function between them. Cardinal numbers can be defined using the Axiom of Choice. We define $|X| \leq |Y|$ if and only if there exists an injective function from X to Y.

Theorem 1.6 ([35, Theorem 1.1.13]). *If X and Y are sets, then either $|X| \leq |Y|$ or $|Y| \leq |X|$.*

Theorem 1.7 (Cantor-Bernstein-Schröeder, [35, Theorem 1.1.14]). *If $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.*

Since every set can be well-ordered by Theorem 1.3 and since every well-ordered set is order isomorphic to a unique ordinal number by Theorem 1.4, to define the cardinality of a set, it is enough to define the cardinality of an ordinal number. Firstly, we say that an ordinal α is a **cardinal** if α is a limit ordinal and for all ordinals β such that $|\beta| = |\alpha|$ then $\alpha \leq \beta$. So given any ordinal number α, we define its cardinality as the least ordinal β such that $|\alpha| = |\beta|$ (this exists since **Ord** is well-ordered). Clearly this is a cardinal number. We (usually) use the symbols κ, λ, μ to denote arbitrary cardinal numbers. The first infinite cardinal $|\omega|$ is denoted $\aleph_0$. Note that if X is a finite set then $|X| = n$ for some $n \in \mathbb{N}$.

We define the successor of a cardinal κ to be the cardinal number λ such that $\lambda > \kappa$ and there does not exist any cardinal μ such that $\kappa > \mu > \lambda$. Note that for infinite cardinals, the cardinal successor differs from the ordinal successor.

We now define cardinal arithmetic which we will use frequently throughout without reference. Given two sets X and Y:

- **Addition** of cardinals $|X| + |Y|$ is defined to be $|X \dot{\cup} Y|$ where $X \dot{\cup} Y$ is the disjoint union.

- **Multiplication** of cardinals $|X| \cdot |Y|$ is defined to be $|X \times Y|$ where $X \times Y$ is the cartesian product.

- **Exponentiation** of cardinals $|X|^{|Y|}$ is defined to be $|X^Y|$ where X^Y is the set of all functions from Y to X.

Theorem 1.8 ([35, Theorem 3.5])**.** *Given two cardinal numbers κ, λ, if either cardinal is infinite (and both are non-zero), then $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

We will have need to make use of the following Lemma in later results.

Lemma 1.9. *Let C be a class of sets and λ a cardinal such that $|X| \leq \lambda$ and $|X| = |Y|$ implies $X = Y$ for all $X, Y \in C$. Then C is a set.*

Proof. Let β be an ordinal such that $|\beta| = \lambda$, then for each cardinal $\mu \leq \lambda$ there exists $\alpha \in \beta+1$ such that $|\alpha| = \mu$. Therefore we can define an injective function $C \to \beta + 1$ and so $|C| \leq |\beta + 1|$. □

1.2 Category theory

Category theoretic methods are used extensively throughout this thesis although they are usually translated explicitly in to the category of acts. In this section we give some categorical motivation as to why covers are important. Namely, we show that covers (and envelopes) are, in a certain sense, 'weak adjoints' of the inclusion functor. The definitions and results in this section can all be found in a standard introduction to category theory, for example [44].

A **category** $\mathcal{C}$ consists of a class of **objects**, denoted $\mathrm{Ob}(\mathcal{C})$, and for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a (possibly empty) set $\mathrm{Hom}(A, B)$ called the **set of morphisms** from A to B such that $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(C, D) = \emptyset$ if $A \neq C$ or $B \neq D$. These are often referred to as the **hom-sets** of $\mathcal{C}$ and the collection of all these sets is denoted $\mathbf{Mor}(\mathcal{C})$. We also require for all objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a composition $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$, $(g, f) \mapsto gf$ satisfying the following properties:

1. for every object $A \in \mathrm{Ob}(\mathcal{C})$, there is an **identity morphism** $id_A \in \mathrm{Hom}(A, A)$ such that $id_B f = f id_A = f$ for all $f \in \mathrm{Hom}(A, B)$.
2. $h(gf) = (hg)f$ for all $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, D)$.

Two examples of categories that will be important later are **Mod-R** the category of (right) R-modules over a ring R with R-homomorphisms, and **Act-S** the category of (right) S-acts over a monoid S with S-maps.

Given a category $\mathcal{C}$, a **subcategory** $\mathcal{D} \subseteq \mathcal{C}$ consists of a subclass of objects $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$, and a subclass of hom-sets $\mathrm{Mor}(\mathcal{D}) \subseteq \mathrm{Mor}(\mathcal{C})$ such that:

1. for all $\mathrm{Hom}(X, Y) \in \mathrm{Mor}(\mathcal{D})$ we have $X, Y \in \mathrm{Ob}(\mathcal{D})$
2. for all $f \in \mathrm{Hom}(Y, Z) \in \mathrm{Mor}(\mathcal{D})$, $g \in \mathrm{Hom}(X, Y) \in \mathrm{Mor}(\mathcal{D})$, we have $fg \in \mathrm{Hom}(X, Z) \in \mathrm{Mor}(\mathcal{D})$
3. for all $X \in \mathrm{Ob}(\mathcal{D})$, we have $1_X \in \mathrm{Hom}(X, X) \in \mathrm{Mor}(\mathcal{D})$.

These conditions ensure that $\mathcal{D}$ is also a category.

A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is **full** if for all $X, Y \in \mathrm{Ob}(\mathcal{D})$, $f \in \mathrm{Hom}(X, Y) \in \mathrm{Mor}(\mathcal{C})$ implies $f \in \mathrm{Hom}(X, Y) \in \mathrm{Mor}(\mathcal{D})$.

1.2.1 Types of morphism

Given a category $\mathcal{C}$ and two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we say that a morphism $f \in \mathrm{Hom}(X, Y)$ is a **monomorphism** if it is left cancellable, that is, for all $V \in \mathrm{Ob}(\mathcal{C})$, $h, k \in \mathrm{Hom}(V, X)$, $fh = fk$ implies $h = k$. We say that $f \in \mathrm{Hom}(X, Y)$ is an **epimorphism** if it is right cancellable, that is, for all $Z \in \mathrm{Ob}(\mathcal{C})$, $h, k \in \mathrm{Hom}(Y, Z)$, $hf = kf$ implies $h = k$. A morphism is a **bimorphism** if it is both a monomorphism and an epimorphism. We say that $f \in \mathrm{Hom}(X, Y)$ is an **isomorphism** if there exists $g \in \mathrm{Hom}(Y, X)$ such that $fg = id_Y$ and $gf = id_X$. We say that $f \in \mathrm{Hom}(X, Y)$ is an **endomorphism** if $X = Y$.

Lemma 1.10. *Let $\mathcal{C}$ be a category, and $X, Y, Z \in Ob(\mathcal{C})$ with morphisms $f \in Hom(X, Y)$, $g \in Hom(Y, Z)$. Then the following are true:*

1. *If gf is a monomorphism then f is a monomorphism.*

2. *If gf is an epimorphism, then g is an epimorphism.*

Proof. 1. If $fh = fk$, for some $V \in \mathrm{Ob}(\mathcal{C})$, $h, k \in \mathrm{Hom}(V, X)$, then $(gf)h = g(fh) = g(fk) = (gf)k$, hence $h = k$ and so f is a monomorphism.

2. The proof is similar. □

Since identity morphisms are clearly bimorphisms we have the following corollary:

Corollary 1.11. *Every isomorphism is a bimorphism.*

Conversely, not every bimorphism is an isomorphism and a category is called **balanced** if all the bimorphisms are isomorphisms.

Lemma 1.12. *Let $\mathcal{C}$ be a category, $X, Y \in Ob(\mathcal{C})$ and $f \in Hom(X, Y)$, $g \in Hom(Y, X)$. If fg and gf are both isomorphisms then f and g are both isomorphisms.*

Proof. By Corollary 1.11, both fg and gf are bimorphisms. Since fg is an isomorphism, there exists $h \in \mathrm{Hom}(Y, Y)$ such that $(fg)h = id_Y = h(fg)$. Therefore $gf = g(fgh)f = gf(ghf)$, and since gf is a monomorphism, we have $id_X = (gh)f$ and since $f(gh) = id_Y$, f is an isomorphism. A similar argument holds for g. □

Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$, if there is a property that X satisfies such that for any other object $Y \in \mathrm{Ob}(\mathcal{C})$ that satisfes the same property there is an isomorphism $f \in \mathrm{Hom}(X, Y)$, then we say that X is **unique up to isomorphism** (with respect to that property) . Similarly, if for any $Y \in \mathrm{Ob}(\mathcal{C})$ that satisfies the property, there is only one isomorphism $f \in \mathrm{Hom}(X, Y)$, then we say that X is **unique up to unique isomorphism**.

An example of a construction unique up to isomorphism but not unique up to unique isomorphism is the algebraic closure of a field, i.e. conjugation of complex numbers.

1.2.2 Terminal and initial objects

Given a category $\mathcal{C}$, we say that an object $X \in \mathrm{Ob}(\mathcal{C})$ is a **terminal object** if for all $A \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $f \in \mathrm{Hom}(A, X)$, that is, $|\mathrm{Hom}(A, X)| = 1$. Similarly, an object $X \in \mathrm{Ob}(\mathcal{C})$ is an **initial object** if for all $A \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $f \in \mathrm{Hom}(X, A)$. An object $X \in \mathrm{Ob}(\mathcal{C})$ is a **zero object** if it is both an initial object and a terminal object.

An example of a zero object is the zero module in the category of modules over a ring.

Lemma 1.13. *Terminal (initial) objects are unique up to unique isomorphism.*

Proof. Given a category $\mathcal{C}$, let X and Y be two terminal (initial) objects in $\mathcal{C}$, then there exist unique morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$, and since both X and Y each have only one endomorphism, $fg = id_Y$ and $gf = id_X$. Therefore both f and g are unique isomorphisms. □

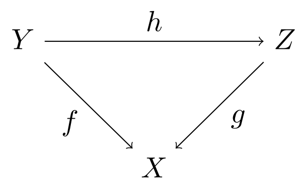
We say that an object $X \in \mathrm{Ob}(\mathcal{C})$ is a **weakly terminal object** if for all $A \in \mathrm{Ob}(\mathcal{C})$, there exists a (not necessarily unique) morphism $f \in \mathrm{Hom}(A, X)$, that is, $\mathrm{Hom}(A, X) \neq \emptyset$. Similarly, an object X is a **weakly initial object** if for all $A \in \mathrm{Ob}(\mathcal{C})$ there exists at least one morphism $f \in \mathrm{Hom}(X, A)$. Slightly adapting terminology from [53], we say that an object is **stable** if all of its endomorphisms are isomorphisms. Clearly terminal and initial objects are both stable, in fact they have only one endomorphism, the identity morphism. We say that an object is **stably weakly terminal** (resp. **stably weakly initial**) if it is stable and weakly terminal (resp. weakly initial). Although weakly terminal (weakly initial) objects need not be unique, we have the following:

Lemma 1.14. *Stably weakly terminal (stably weakly initial) objects are unique up to isomorphism.*

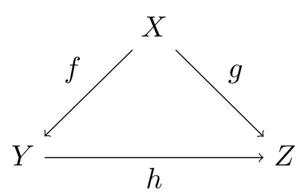
Proof. Let $\mathcal{C}$ be a category, and X, Y be two stably weakly terminal (stably weakly initial) objects in $\mathrm{Ob}(\mathcal{C})$, then there exist morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ such that fg and gf are isomorphisms. Hence by Lemma 1.12, f and g are both isomorphisms. □

1.2.3 Slice and coslice categories

Given a category $\mathcal{C}$, and $X \in \text{Ob}(\mathcal{C})$, we can define a new category $(\mathcal{C} \downarrow X)$ called the **slice category over X** with objects (Y, f) where $Y \in \text{Ob}(\mathcal{C})$, $f \in \text{Hom}(Y, X)$, and homomorphisms $h : (Y, f) \to (Z, g)$, where $h \in \text{Hom}(Y, Z)$ such that the following diagram commutes



Similarly, we can define $(X \downarrow \mathcal{C})$, the **coslice category over X** with objects (Y, f) where $Y \in \text{Ob}(\mathcal{C})$, $f \in \text{Hom}(X, Y)$, and homomorphisms $h : (Y, f) \to (Z, g)$, where $h \in \text{Hom}(Y, Z)$ such that the following diagram commutes



Given a category $\mathcal{C}$, a subcategory $\mathcal{D} \subseteq \mathcal{C}$, and an object $X \in \text{Ob}(\mathcal{C})$, let $(\mathcal{D} \downarrow X)$, the **slice subcategory of $\mathcal{D}$ over X** denote the full subcategory of $(\mathcal{C} \downarrow X)$ consisting of objects (Y, f) where $Y \in \text{Ob}(\mathcal{D})$. Similarly let, $(X \downarrow \mathcal{D})$, the **coslice subcategory of $\mathcal{D}$ over X**, denote the full subcategory of $(X \downarrow \mathcal{C})$ consisting of objects (Y, f) where $Y \in \text{Ob}(\mathcal{D})$.

1.2.4 Functors and adjoints

Given categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** $F : \mathcal{C} \to \mathcal{D}$, assigns each object $X \in \text{Ob}(\mathcal{C})$ to an object $F(X) \in \text{Ob}(\mathcal{D})$, and assigns each morphism $f \in \text{Hom}(X, Y) \in \text{Mor}(\mathcal{C})$, to a morphism $F(f) \in \text{Hom}(F(X), F(Y)) \in \text{Mor}(\mathcal{D})$ such that the following two properties are satisfied:

1. $F(id_X) = id_{F(X)}$ for each $X \in \text{Ob}(\mathcal{C})$,

2. $F(fg) = F(f)F(g)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}(Y, Z)$, $g \in \mathrm{Hom}(X, Y)$.

Given any category $\mathcal{C}$, an important functor $1_{\mathcal{C}}$ is the **identity functor** which sends every object to itself and every morphism to itself. Given a subcategory $\mathcal{D} \subseteq \mathcal{C}$, the **inclusion functor** (or **forgetful functor**) is the functor from $\mathcal{D}$ to $\mathcal{C}$ that sends all objects and morphisms to themselves.

Given two functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$, we can define their composition $GF : \mathcal{A} \to \mathcal{C}$ in the obvious way, each object $X \in \mathrm{Ob}(\mathcal{A})$ is assigned to $G(F(X)) \in \mathrm{Ob}(\mathcal{A})$ and each morphism $f \in \mathrm{Hom}(X, Y) \in \mathrm{Mor}(\mathcal{A})$ is assigned to $G(F(f)) \in \mathrm{Hom}(G(F(X)), G(F(Y))) \in \mathrm{Mor}(\mathcal{A})$.

Given two categories $\mathcal{C}$, $\mathcal{D}$ and two functors $F, G : \mathcal{C} \to \mathcal{D}$, we say that $\sigma : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, $X \mapsto (F(X) \to G(X))$ is a **natural transformation** (from F to G) if for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}(X, Y)$ the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\sigma(X)} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\sigma(Y)]{} & G(Y). \end{array}$$

We write $\sigma : F \to G$ and think of it as a 'morphism of functors'. Furthermore, we say that σ is a **natural equivalence** if $\sigma(X) \in \mathrm{Mor}(\mathcal{D})$ is an isomorphism for all $X \in \mathrm{Ob}(\mathcal{C})$. An important natural equivalence 1_F is the **identity transformation** that sends a functor F to itself, that is, it sends X to $id_{F(X)}$.

Given three functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and two natural transformations $\sigma : F \to G$, $\mu : G \to H$, we define the composition $\mu \circ \sigma$ to be the natural transformation that sends $X \in \mathrm{Ob}(\mathcal{C})$ to $\mu(X) \circ \sigma(X)$.

Given two categories $\mathcal{C}$, $\mathcal{D}$ and two functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, we say that F and G are **adjoint** if there exist two natural transformations $\varepsilon : FG \to 1_{\mathcal{D}}$, $\eta : 1_{\mathcal{C}} \to GF$ such that the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations 1_F and 1_G respectively. If F and G are adjoint then we say that F is **left adjoint** to G and G is **right adjoint** to F. Many important examples of adjoints come from left/right adjoints to the inclusion functor.

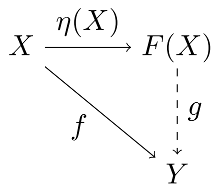
A subcategory is called **reflective** if the inclusion functor has a left adjoint which we call the **reflector map** . We list here a few of the many examples of reflective subcategories:

Reflective subcategory	**Reflector map**
Any category in itself	Identity functor
Unital rings in all rings	Adjoin an identity
Abelian groups in groups	Quotient by the commutator subgroup
Sheaves in presheaves on a topological space	Sheafification
Groups in sets	Free group on set
Fields in integral domains	Field of fractions
Compact spaces in normal Hausdorff topological spaces	Stone-Čech compactification
Groups in inverse semigroups	Quotient by minimum group congruence
Abelian groups in commutative monoids	Grothendieck group construction

It is worth noting that each of these examples gives rise to a universal property. To be precise, we have the following:

Theorem 1.15. *A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is reflective if and only if for all $X \in Ob(\mathcal{C})$, the coslice subcategory $(X \downarrow \mathcal{D})$ has an initial object.*

Proof. Given the inclusion functor $G : \mathcal{D} \to \mathcal{C}$, let $F : \mathcal{C} \to \mathcal{D}$ be a functor left adjoint to G. Then there exist natural transformations $\varepsilon : FG \to 1_{\mathcal{D}}$ and $\eta : 1_{\mathcal{C}} \to GF$ such that $\varepsilon F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta G = 1_G$. Then for any object $X \in \mathrm{Ob}(\mathcal{C})$, it is clear that $(F(X), \eta(X))$ is an initial object in the coslice subcategory $(X \downarrow \mathcal{D})$. That is, for all $Y \in \mathcal{D}$, $f \in \mathrm{Hom}(X, Y)$, there exists a unique $g : F(X) \to Y$ such that the following diagram commutes



Conversely, if every object $X \in \text{Ob}(\mathcal{C})$ has an initial object (Y_X, f_X) in the coslice subcategory $(X \downarrow \mathcal{D})$, then we can define a function $F : \mathcal{C} \to \mathcal{D}$ that sends the object X to Y_X and the morphism $h \in \text{Hom}(X, X') \in \text{Mor}(\mathcal{C})$ to the unique morphism $g_h \in \text{Hom}(Y_X, Y_{X'}) \in \text{Mor}(\mathcal{D})$ such that $f_{X'}h = g_h f_X$. Let η be the function that takes an object $X \in \text{Ob}(\mathcal{C})$ to the morphism $f_X \in \text{Mor}(\mathcal{C})$, and let ε be the function that takes an object $X \in \text{Ob}(\mathcal{D})$ to $id_X \in \text{Mor}(\mathcal{D})$. It is clear that F is in fact a functor, and η and ε are natural transformations from $1_{\mathcal{C}}$ to GF and FG to $1_{\mathcal{D}}$ respectively such that $\varepsilon F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta G = 1_G$, hence F is left adjoint to G. □

A subcategory is called **coreflective** if the inclusion functor has a right adjoint which we call the **coreflector map**, and similarly we have:

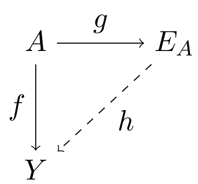
Theorem 1.16. *A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is coreflective if and only if for all $X \in \text{Ob}(\mathcal{C})$, the slice subcategory $(\mathcal{D} \downarrow X)$ has a terminal object.*

Proof. The proof is similar. □

An example of a coreflective subcategory is torsion groups in abelian groups, with the right adjoint being the torsion subgroup.

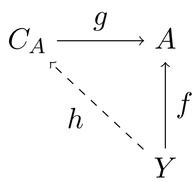
1.2.5 Covers and envelopes

Let $\mathcal{C}$ be a category, $\mathcal{X} \subseteq \mathcal{C}$ a subcategory and $A \in \text{Ob}(\mathcal{C})$. We say that (E_A, g) is an **$\mathcal{X}$-preenvelope of A** if $g \in \text{Hom}(A, E_A)$ such that for all $Y \in \text{Ob}(\mathcal{X})$, $f \in \text{Hom}(A, Y)$ there exists $h \in \text{Hom}(E_A, Y)$ such that the following diagram



commutes. Additionally, if whenever $(Y, f) = (E_A, g)$, h must be an isomorphism, then we say that (E_A, g) is an $\mathcal{X}$**-envelope**.

Similarly, we say that (C_A, g) is an $\mathcal{X}$**-precover of A** if $g \in \mathrm{Hom}(C_A, A)$ such that for all $Y \in \mathrm{Ob}(\mathcal{X})$, $f \in \mathrm{Hom}(Y, A)$ there exists $h \in \mathrm{Hom}(Y, C_A)$ such that the following diagram



commutes. Additionally, if whenever $(Y, f) = (C_A, g)$, h must be an isomorphism, then we say that (C_A, g) is an $\mathcal{X}$**-cover**.

The following Propositions are clear from the definitions.

Proposition 1.17. *Given a category $\mathcal{C}$, an object $A \in Ob(\mathcal{C})$ and a subcategory $\mathcal{X} \subseteq \mathcal{C}$, an $\mathcal{X}$-preenvelope of A is a weakly initial object in the coslice subcategory of $\mathcal{X}$ over A, and $\mathcal{X}$-envelopes are stably weakly initial objects.*

Proposition 1.18. *Given a category $\mathcal{C}$, an object $A \in Ob(\mathcal{C})$ and a subcategory $\mathcal{X} \subseteq \mathcal{C}$, an $\mathcal{X}$-precover of A is a weakly terminal object in the slice subcategory of $\mathcal{X}$ over A, and $\mathcal{X}$-covers are stably weakly terminal objects.*

Hence by Lemma 1.14, envelopes and covers are unique up to isomorphism.

These Propositions therefore give us the following results:

Theorem 1.19. *Given a category $\mathcal{C}$ and a subcategory $\mathcal{X} \subseteq \mathcal{C}$, every $A \in Ob(\mathcal{C})$ has an $\mathcal{X}$-envelope if and only if for all $A \in Ob(\mathcal{C})$, the coslice subcategory $(\mathcal{X} \downarrow A)$ has a stably weakly initial object.*

Theorem 1.20. *Given a category $\mathcal{C}$ and a subcategory $\mathcal{X} \subseteq \mathcal{C}$, every $A \in Ob(\mathcal{C})$ has an $\mathcal{X}$-cover if and only if for all $A \in Ob(\mathcal{C})$, the slice subcategory $(A \downarrow \mathcal{X})$ has a stably weakly terminal object.*

These results indicate why envelopes and covers are important. They say that every object having an $\mathcal{X}$-envelope (resp. $\mathcal{X}$-cover) is a slightly weaker condition than the inclusion functor $\mathcal{X} \subseteq \mathcal{C}$ having a left (resp. right) adjoint. They are unique up to isomorphism, however unlike adjoints, they are not unique up to unique isomorphism.

1.3 Semigroup theory

In this thesis we study the category of acts over a monoid. Therefore some basic semigroup and monoid theory is required, although not much. In particular we give only a few definitions and a simple Lemma. For a more thorough overview, see [33].

A **semigroup** $(S, \cdot)$ is a set S with an associative binary operation $S \times S \to S, (s,t) \mapsto s \cdot t$. A **monoid** $(S, \cdot)$ is a semigroup $(S, \cdot)$ with an identity element $1 \in S$, such that, $1 \cdot s = s \cdot 1 = s$ for all $s \in S$. We usually write $s \cdot t$ as st.

Given any semigroup, we can turn it in to a monoid by adjoining an identity. There are two different ways to do this. Either adjoin an identity precisely when it doesn't already have one, or adjoin an identity even if it does. Clearly these are equivalent for semigroups that are not already monoids.

We say that a semigroup is **right cancellative** (resp. **left cancellative**) if for every $s, t, c \in S$, $sc = tc$ (resp. $cs = ct$) implies $s = t$. We say that a semigroup is **cancellative** if it is both left cancellative and right cancellative. An example of a cancellative semigroup is the set of natural numbers under addition.

We say that a semigroup is **regular** if for all $s \in S$, there exists $t \in S$ such that $sts = s$ and $tst = t$. We say that a semigroup is **inverse** if for all $s \in S$ there exists a unique $t \in S$ such that $sts = s$ and $tst = t$.

Given any set X, the set of all functions $f : X \to X$ with function composition as a binary operation is called the **full transformation monoid of** X and denoted $\mathcal{T}(X)$. In fact, $\mathcal{T}(X)$ is a regular monoid.

Given any set X the set of all partial bijections on X, that is, bijective functions not everywhere defined, with function composition as a binary operation is called the **symmetric inverse monoid of** X and denoted $\mathcal{I}(X)$. In fact, $\mathcal{I}(X)$ is an inverse monoid.

In fact, every semigroup is isomorphic to a subsemigroup of the full transformation monoid of some set, and every inverse semigroup is isomorphic to a subsemigroup of the symmetric inverse monoid of some set. These are analogous results to Cayley's Theorem for groups.

Given a semigroup S, an **idempotent** is an element $e \in S$ such that $ee = e$, and the set of idempotents is denoted $E(S)$. Every inverse semigroup comes equipped with a natural partial order, $s \leq t$ if and only if there exists $e \in E(S)$ such that $s = et$. For idempotents, this reduces to $e \leq f$ if and only if $e = ef$ as if $e = df$ for some $d \in E(S)$, then $ef = dff = df = e$. On an inverse semigroup S, we can define the **minimum group congruence** to be the relation $\sigma = \{(s,t) \in S \times S \mid es = et \text{ for some } e \in E(S)\}$. It straightforward to check that this is indeed a congruence, and it is the smallest congruence σ such that S/σ is a group, or equivalently S/σ is the maximum group homomorphic image of S. The natural map $\sigma^\natural : S \to S/\sigma$ is a reflector map from the category of inverse semigroups to the full subcategory of groups (see [42, Theorem 2.4.2]).

The following Lemma is straightforward and is used later.

Lemma 1.21. *Given a monoid S, if $xS = S$ for all $x \in S$ then S is a group.*

Proof. We show that every element $x \in S$ has a two-sided inverse. In fact, there exists $t \in S$ such that $xt = 1$ and there exists $u \in S$ such that $tu = 1$. Therefore $u = (xt)u = x(tu) = x$ and t is a two-sided inverse of x. □

Chapter 2

Acts over monoids

In this chapter we give a brief overview of the basic results surrounding S-acts, most of which can be found in [38], although there are some new results as well.

2.1 The category of S-acts

There are many similarities between **Mod-R** the category of (right) modules over a ring and **Act-S** the category of (right) acts over a monoid, but there are also some subtle differences. The first thing to note is that although **Act-S** has a terminal object, it does not have an initial object and hence does not have a zero object and so the category is not additive. This means we need to be more careful when defining homological concepts such as exact sequences or even simple concepts like a kernel.

We start by giving a brief outline of the category **Act-S**.

Let S be a monoid with identity element 1 and let A be a non-empty set. We say that A is a **right S-act** if there is an action

$$A \times S \to A$$
$$(a, s) \mapsto as$$

with the property that for all $a \in A$ and $s, t \in S$

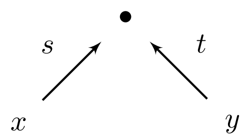
$$a(st) = (as)t \quad \text{and} \quad a1 = a.$$

Given any monoid S, we always have the **one element act** or the **trivial act** denoted Θ_S, that is the one element set $\{a\}$ with the following action $(a, s) \mapsto as = a$ for all $s \in S$. We say that an S-act A has a **fixed point** if it contains the one element S-act as a subact.

Given a right S-act A and $A' \subseteq A$ a non-empty subset of A. We say that A' is a **subact** of A if

$$a's \in A' \quad \text{for all} \quad a' \in A',\, s \in S.$$

Sometimes it is easier to think of acts in a graphical way. Given any S-act X, there is an associated graph (in fact a decorated digraph) where the vertices are the elements of X and the directed edges are labelled $e \in S$ between two vertices $v_1, v_2 \in X$ if $v_1 e = v_2$. For example, if $xs = yt$ then we would have the following graph.



Note that unlike group actions, the edges may not be reversible and the graph may not be connected.

Let A, B be two right S-acts. Then a well-defined function $f : A \to B$ is called a **homomorphism of right S-acts** or just an **S-map** if

$$f(as) = f(a)s \quad \text{for all} \quad a \in A,\, s \in S.$$

The set of all homomorphisms from A to B is denoted by $\mathrm{Hom}(A, B)$. We say that two right S-acts A and B are **isomorphic** and write $A \cong B$ if there is a bijective S-map between them.

Given any monoid S, the category whose objects are right S-acts and whose morphisms are homomorphisms of right S-acts is denoted **Act-S**. It turns out that (unlike the category of semigroups) this category is balanced [38, Proposition 6.15], that is the bimorphisms are isomorphisms. Moreover, the **epimorphisms** and **monomorphisms** are precisely the surjective and injective S-maps respectively.

Similarly we can define the category **S-Act** of left S-acts with homomorphisms between left S-acts in the obvious way. We will be working almost exclusively with the category of right S-acts and so unless otherwise stated an act will always refer to a right S-act.

It is clear that the one element S-act Θ_S is the terminal object in **Act-S** but we do not always have an initial object, hence the category of acts is a non-additive category.

2.2 Congruences

Given an S-act A, a **right S-congruence** on A is an equivalence relation ρ on A (that is, reflexive, symmetric and transitive) such that $x\rho y$ implies $xs\rho ys$ for all $x, y \in A$, $s \in S$. Note, we frequently write $x\rho y$ to mean $(x, y) \in \rho$. Similarly we can define a **left S-congruence** on A. We will be working almost exclusively with right S-congruences and so unless otherwise stated a congruence will refer to a right S-congruence and we use the term **two-sided congruence** to mean an equivalence relation that is both a left S-congruence and a right S-congruence.

Given any S-act, there are always two special congruences on A that we call the **universal relation** defined $A \times A$ and the **diagonal relation** defined $1_A := \{(a, a) : a \in A\}$. These are the greatest and least elements respectively in the partial ordering (by inclusion) of all congruences/equivalence relations on A.

If ρ is a congruence on A then we use the notation A/ρ to denote the set of equivalence classes $\{[a]_\rho : a \in A\}$. It is easy to see that A/ρ is an S-act with the action $[a]_\rho s = [as]_\rho$. We call the canonical surjection

$$\rho^\natural : A \to A/\rho$$
$$a \mapsto [a]_\rho$$

the **natural map** with respect to ρ. We usually write $a\rho$ to mean $[a]_\rho$.

Given any S-act A, it is clear that $A/1_A \cong A$ and $A/(A \times A) \cong \Theta_S$.

Given any S-act A and a set $X \subseteq A \times A$, we write $X^\#$ (read X sharp) to denote the **congruence generated by** X by which we mean the smallest congruence on A that contains X, or equivalently the intersection of all congruences on A containing X. We will frequently use the following Lemma without reference.

Lemma 2.1. *[38, Lemma I.4.37] Let $X \subseteq A \times A$ and $\rho = X^{\#}$. Then for any $a, b \in A$, one has $a \rho b$ if and only if either $a = b$ or there exist $p_1, \ldots, p_n$, $q_1 \ldots, q_n \in A$, $w_1, \ldots, w_n \in S$ where, for $i = 1, \ldots, n$, $(p_i, q_i) \in X \cup X^{op}$, that is, $(p_i, q_i) \in X$ or $(q_i, p_i) \in X$, such that*

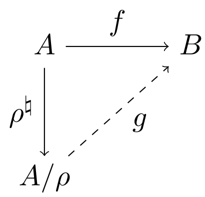
$$a = p_1 w_1, \quad q_1 w_1 = p_2 w_2, \quad q_2 w_2 = p_3 w_3, \quad q_3 w_3 = p_4 w_4, \quad \cdots, \quad q_n w_n = b.$$

Given an S-map $f : A \to B$ between S-acts, we define the **kernel** of f to be

$$\ker(f) := \{(x, y) \in A \times A : f(x) = f(y)\}.$$

It is clear that the kernel of f is a congruence on A. Also note that given any S-act and any congruence ρ on A, $\ker(\rho^\natural) = \rho$.

Theorem 2.2 (Homomorphism Theorem for Acts)**.** *[38, Theorem I.4.21] Let $f : A \to B$ be an S-map and ρ be a congruence on A such that $\rho \subseteq \ker(f)$. Then $g : A/\rho \to B$ with $g(a\rho) := f(a)$, $a \in A$, is the unique S-map such that the following diagram*



commutes. If $\rho = \ker(f)$, then g is injective, and if f is surjective, then g is surjective.

Proof. Suppose $x\rho = y\rho$ for $x, y \in A$, then $(x, y) \in \rho$ and thus $f(x) = f(x')$. Hence g is well-defined. Suppose $g(x\rho) = g(y\rho)$, then $f(x) = f(y)$ and thus $(x, y) \in \ker(f)$. If $\rho = \ker(f)$ then $x\rho = y\rho$ and thus g is injective. □

Corollary 2.3. *If $f : A \to B$ is an S-map, then $\operatorname{im}(f) \cong A/\ker(f)$.*

The following remark will be useful later in the thesis.

Remark 2.4. Let S be a monoid, let A be an S-act and let ρ be a congruence on A. Let σ be a congruence on A/ρ and let $\rho/\sigma = \ker(\sigma^\natural \rho^\natural)$. Then clearly ρ/σ is a congruence on A containing ρ and $A/(\rho/\sigma) = (A/\rho)/\sigma$. Moreover $\rho/\sigma = \rho$ if and only if $\sigma = 1_{F/\rho}$.

2.3 Colimits and limits of acts

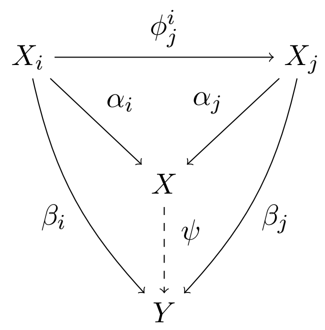
Limits and especially colimits play a prominent role in this thesis and so we here draw particular attention to their definition and some of the more important common constructions for the category of S-acts.

Let I be a (non-empty) set with a preorder (that is, a reflexive and transitive relation). A **direct system** is a collection of S-acts $(X_i)_{i \in I}$ together with S-maps $\phi^i_j : X_i \to X_j$ for all $i \leq j \in I$ such that

1. $\phi^i_i = 1_{X_i}$, for all $i \in I$; and
2. $\phi^j_k \circ \phi^i_j = \phi^i_k$ whenever $i \leq j \leq k$.

The **colimit** of the system (X_i, ϕ^i_j) is an S-act X together with S-maps $\alpha_i : X_i \to X$ such that

1. $\alpha_j \circ \phi^i_j = \alpha_i$, whenever $i \leq j$,
2. If Y is an S-act and $\beta_i : X_i \to Y$ are S-maps such that $\beta_j \circ \phi^i_j = \beta_i$ whenever $i \leq j$, then there exists a unique S-map $\psi : X \to Y$ such that the diagram



commutes for all $i \in I$.

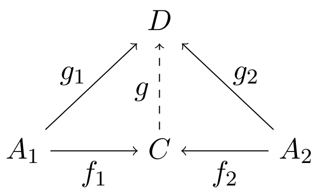
Dually we can also define a **limit** where all the arrows in the previous definitions are reversed, although we do not take the trouble to define them formally as they play a much less prominent role in this thesis than colimits do.

We now describe some of the more important examples of limits and colimits of acts that appear in this thesis.

2.3.1 Coproducts and products

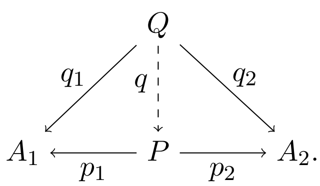
A coproduct (resp. product) is a colimit (resp. limit) where the indexing set is an antichain, that is, no two elements are comparable.

Given a collection of S-acts $(A_i)_{i\in I}$ for some non-empty set I, a pair $(C,(f_i)_{i\in I})$ where C is an S-act and $f_i \in \mathrm{Hom}(A_i, C)$, is called the **coproduct** of $(A_i)_{i\in I}$ if for all S-acts D and all S-maps $g_i \in \mathrm{Hom}(A_i, D)$ there exists a unique S-map $g : C \to D$ such that $gf_i = g_i$ for all $i \in I$. For example, when $|I| = 2$, the following diagram must commute.



We often refer to just C as the coproduct and it is denoted $\coprod_{i\in I} A_i$. It is shown in [38, Proposition II.1.8] that C is in fact just the disjoint union $\dot{\bigcup}_{i\in I} A_i$ with the inherited action and $f_i : A_i \to C$ are the inclusion maps. We will frequently use this fact without reference.

Similarly, given a collection of S-acts $(A_i)_{i\in I}$ for some non-empty set I, a pair $(P,(p_i)_{i\in I})$ where P is an S-act and $p_i \in \mathrm{Hom}(P, A_i)$, is called the **product** of $(A_i)_{i\in I}$ if for all S-acts Q and all S-maps $q_i \in \mathrm{Hom}(Q, A_i)$ there exists a unique S-map $q : Q \to P$ such that $p_i q = q_i$ for all $i \in I$. For example, when $|I| = 2$, the following diagram must commute



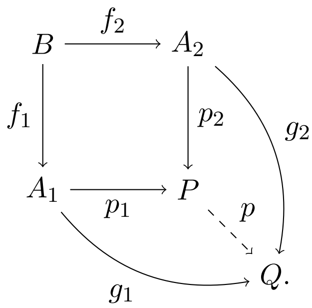
We often refer to just P as the product and it is denoted $\prod_{i\in I} A_i$. It is shown in [38, Proposition II.1.1] that P is in fact just the cartesian product with componentwise action and $p_i : P \to A_i$ are the projection maps $p_i : P \to A_i, (a_j)_{j\in I} \mapsto a_i$.

Note the difference here with **Mod-R** (and indeed any additive category) where finite products and coproducts are the same. Here, disjoint union and cartesian product are always different, even in the finite case.

2.3.2 Pushouts and pullbacks

A pushout (resp. pullback) is a colimit (resp. limit) with a three-element indexing set $i, j, k \in I$ such that $k \leq i, j$ (resp. $i, j \leq k$) and i and j are not comparable.

Given three S-acts A_1, A_2, B and two S-maps $f_i : B \to A_i$, a pair $(P, (p_1, p_2))$ where P is an S-act and $p_i \in \text{Hom}(A_i, P)$ is called the **pushout** of (f_1, f_2) if $p_1 f_1 = p_2 f_2$ and given any S-act Q and any two S-maps $g_i \in \text{Hom}(A_i, Q)$ such that $g_1 f_1 = g_2 f_2$ then there exists a unique S-map $p : P \to Q$ such that $pp_i = g_i$, i.e. the following diagram commutes

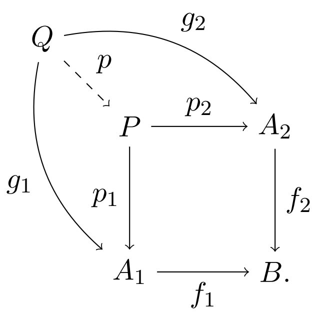


The proof of the following Lemmas are straightforward.

Lemma 2.5. *[38, Proposition II.2.16] Given a pushout as above, $P = (A_1 \amalg A_2)/\rho$ where $\rho = X^\sharp$ is the congruence generated by $X = \{(f_1(b), f_2(b)) \mid b \in B\}$ and $p_i = \rho^\natural u_i$ where $u_i : A_i \to A_1 \amalg A_2$ are the inclusion maps.*

Lemma 2.6. *[49, Lemma I.3.6] Given a pushout as above, if f_1 is surjective (resp. injective) then p_2 is surjective (resp. injective).*

Given three S-acts A_1, A_2, B and two S-maps $f_i : A_i \to B$, a pair $(P, (p_1, p_2))$ where P is an S-act and $p_i \in \text{Hom}(P, A_i)$ is called the **pullback** of (f_1, f_2) if $f_1 p_1 = f_2 p_2$ and given any S-act Q and any two S-maps $g_i \in \text{Hom}(Q, A_i)$ such that $f_1 g_1 = f_2 g_2$ then there exists a unique S-map $p : Q \to P$ such that $p_i p = g_i$, i.e. the following diagram commutes



The proof of the following Lemmas are straightforward.

Lemma 2.7. *[38, Proposition II.2.5] Unlike pushouts, pullbacks do not always exist but they exist if and only if there exists $(a_1, a_2) \in A_1 \times A_2$ such that $f_1(a_1) = f_2(a_2)$. In fact, when they do exist $P = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}$ and $p_i : P \to A_i$, $(a_1, a_2) \mapsto a_i$.*

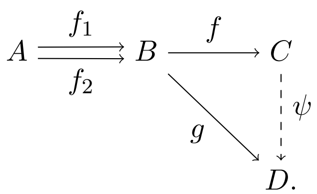
Lemma 2.8. *[49, Lemma I.3.6] Given a pullback as above, if f_1 is surjective (resp. injective) then p_2 is surjective (resp. injective).*

In a similar way, we can define multiple pushouts (resp. pullbacks) over an index set bigger than three, although they are not used in this thesis.

2.3.3 Coequalizers and equalizers

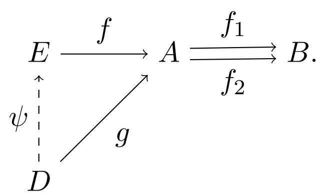
A coequalizer (resp. equalizer) is a pushout (resp. pullback) where $A_1 = A_2$.

Given two S-acts A, B and two S-maps $f_1, f_2 : A \to B$ a pair (C, f) where C is an S-act and $f \in \mathrm{Hom}(B, C)$ is called a **coequalizer** if $ff_1 = ff_2$ and for any S-act D and any S-map $g \in \mathrm{Hom}(B, D)$ such that $gf_1 = gf_2$ there exists a unique S-map $\psi : C \to D$ such that $\psi f = g$, i.e. the following diagram commutes,



Lemma 2.9. *[38, Proposition II.2.21] Given a coequalizer as above, $C = B/\rho$ where $\rho = X^\sharp$ is the congruence generated by $X = \{(f_1(a), f_2(a)) \mid a \in A\}$, and $f = \rho^\natural$.*

Given two S-acts A, B and two S-maps $f_1, f_2 : A \to B$ a pair (E, f) where E is an S-act and $f \in \mathrm{Hom}(E, A)$ is called an **equalizer** if $f_1 f = f_2 f$ and for any S-act D and any S-map $g \in \mathrm{Hom}(D, A)$ such that $f_1 g = f_2 g$ there exists a unique S-map $\psi : D \to E$ such that $f\psi = g$, i.e. the following diagram commutes,



Lemma 2.10. *[38, Proposition II.2.10] Unlike coequalizers, equalizers do not always exist but they exist if and only if there exists $a \in A$ such that $f_1(a) = f_2(a)$. In fact, when they do exist, $E = \{a \in A \mid f_1(a) = f_2(a)\}$ and f is the inclusion map.*

2.3.4 Directed colimits

If the indexing set I satisfies the property that for all $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$ then we say that I is **directed**. In this case we call the colimit a **directed colimit**.

We say that a class $\mathcal{X}$ of S-acts is **closed under (directed) colimits** if every direct system of S-acts in $\mathcal{X}$ has its (directed) colimit in $\mathcal{X}$ as well.

Remark 2.11. A note on terminology: a directed colimit is often referred to as a direct limit in the literature, however some literature (for example [54]) uses the term direct limit to refer to an arbitrary colimit. To avoid ambiguity we will not use the phrase direct limit, but instead directed colimit.

A colimit of S-acts always exists and we can describe it in the following way. Let $\lambda_i : X_i \to \coprod_{i \in I} X_i$ be the natural inclusion and let $\rho = R^\#$ be the right congruence on $\coprod_{i\in I} X_i$ generated by

$$R = \{(\lambda_i(x_i), \lambda_j(\phi^i_j(x_i))) \mid x_i \in X_i, i \leq j \in I\}.$$

Then $X = \left(\coprod_{i\in I} X_i\right)/\rho$ and $\alpha_i : X_i \to X$ given by $\alpha_i(x_i) = \lambda_i(x_i)\rho$ are such that (X, α_i) is the colimit of (X_i, ϕ^i_j). In addition, if the index set I is directed then

$$\rho = \{(\lambda_i(x_i), \lambda_j(x_j)) \mid \exists k \geq i, j \text{ such that } \phi^i_k(x_i) = \phi^j_k(x_j)\}.$$

See ([49, Theorem I.3.1 & Theorem I.3.17]) for more details.

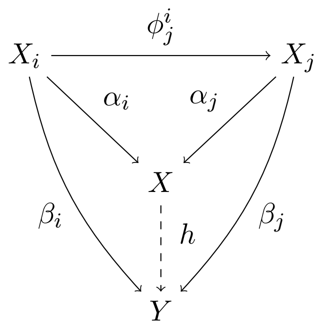
In particular, given a direct system (X_i, ϕ^i_j) with colimit (X, α_i), given any $x \in X$ there exists some $i \in I, x_i \in X_i$ such that $\alpha_i(x_i) = x$.

Lemma 2.12 ([50, Lemma 3.5 & Corollary 3.6])**.** *Let (X_i, ϕ^i_j) be a direct system of S-acts with directed index set and let (X, α_i) be the directed colimit. Then $\alpha_i(x_i) = \alpha_j(x_j)$ if and only if $\phi^i_k(x_i) = \phi^j_k(x_j)$ for some $k \geq i, j$. Consequently α_i is a monomorphism if and only if ϕ^i_k is a monomorphism for all $k \geq i$.*

Directed colimits play a very prominent role in this thesis and there are few references to them in the literature for S-acts, so we here prove some of the more important technical Lemmas which we will use throughout.

Lemma 2.13. *Let S be a monoid, let (X_i, ϕ^i_j) be a direct system of S-acts with directed index set and let (X, α_i) be the directed colimit. Suppose that Y is an S-act and that $\beta_i : X_i \to Y$ are monomorphisms such that $\beta_i = \beta_j\phi^i_j$ for all $i \leq j$. Then there exists a unique monomorphism $h : X \to Y$ such that $h\alpha_i = \beta_i$ for all i.*

Proof. Consider the following commutative diagram



where h is the unique S-map guaranteed by the directed colimit property. Suppose that $h(x) = h(x')$. Then there exists i, j and $x_i \in X_i, x_j \in X_j$ such that $x = \alpha_i(x_i)$ and $x' = \alpha_j(x_j)$. Hence there exists $k \geq i, j$ and so

$$\beta_k \phi_k^i(x_i) = h\alpha_k \phi_k^i(x_i) = h\alpha_i(x_i) = h\alpha_j(x_j) = h\alpha_k \phi_k^j(x_j) = \beta_k \phi_k^j(x_j).$$

Since β_k is a monomorphism then $\phi_k^i(x_i) = \phi_k^j(x_j)$ and so $x = x'$ as required. □

This next construction is often referred to as the **directed union**.

Lemma 2.14. *Let $\{A_i : i \in I\}$ be a set of S-acts partially ordered by inclusion, with the property that for any two acts they are both contained in a larger one, i.e. the index set is directed. Let $\phi_j^i : A_i \hookrightarrow A_j$ be the inclusion map whenever $A_i \subseteq A_j$, so that (A_i, ϕ_j^i) is a direct system over a directed index set. Then $\bigcup_{i \in I} A_i$ is isomorphic to the directed colimit of (A_i, ϕ_j^i).*

Proof. Let (X, α_i) be the directed colimit of (A_i, ϕ_j^i), we intend to show that X is isomorphic to $Y := \bigcup_{i \in I} A_i$. Clearly we can define the inclusion map $\beta_i : A_i \hookrightarrow Y$ so that $\beta_i = \beta_j \phi_j^i$ for all $i \leq j$, hence by Lemma 2.12 there exists a monomorphism $\psi : X \to Y$ such that $\psi\alpha_i = \beta_i$ for all $i \in I$. Now given any $a \in Y$, there must exist some $k \in I$ such that $a \in A_k$. Hence $\psi(\alpha_k(a)) = \beta_k(a) = a$ and ψ is an epimorphism and hence an isomorphism. □

We now prove a similar Lemma for unions of congruences.

Lemma 2.15. *Let S be a monoid, let X be an S-act and let $\{\rho_i : i \in I\}$ be a set of congruences on X, partially ordered by inclusion, with the property that the index set is directed and has a minimum element 0. Let $\phi_j^i : X/\rho_i \to X/\rho_j$ be the S-map defined by $a\rho_i \mapsto a\rho_j$ whenever $\rho_i \subseteq \rho_j$, so that $(X/\rho_i, \phi_j^i)$ is a direct system. Let $\rho = \bigcup_{i \in I} \rho_i$. Then X/ρ is the directed colimit of $(X/\rho_i, \phi_j^i)$.*

Proof. First note that ρ is transitive since I is directed. Clearly we can define S-maps $\alpha_i : X/\rho_i \to X/\rho$, $a\rho_i \mapsto a\rho$ such that $\alpha_i = \alpha_j \phi_j^i$ for all $i \leq j$. Now suppose there exists an S-act Q and S-maps $\beta_i : X/\rho_i \to Q$ such that $\beta_i = \beta_j \phi_j^i$ for all $i \leq j$. Define $\psi : X/\rho \to Q$ by $\psi(a\rho) = \beta_0(a\rho_0)$. To see this is well-defined, let $a\rho = a'\rho$ in X/ρ, that is, $(a, a') \in \rho$ so there must exist some $k \in I$ such that $(a, a') \in \rho_k$ and we get

$$\beta_0(a\rho_0) = \beta_k\phi_k^0(a\rho_0) = \beta_k(a\rho_k) = \beta_k(a'\rho_k) = \beta_k\phi_k^0(a'\rho_0) = \beta_0(a'\rho_0)$$

so $\psi(a\rho) = \psi(a'\rho)$ and ψ is well-defined. It is easy to see that ψ is also an S-map. Because 0 is the minimum element, we have that $\beta_0(a\rho_0) = \beta_i\phi_i^0(a\rho_0) = \beta_i(a\rho_i)$ and so $\psi\alpha_i = \beta_i$ for all $i \in I$. Finally let $\psi' : X/\rho \to Q$ be an S-map such that $\psi'\alpha_i = \beta_i$ for all $i \in I$, then $\psi'(a\rho) = \psi'(\alpha_0(a\rho_0)) = \beta_0(a\rho_0) = \psi(a\rho)$, and we are done. □

Remark 2.16. In particular, this holds when we have a chain of congruences $\rho_1 \subset \rho_2 \subset \dots$ and $\rho = \bigcup_{i\geq 1} \rho_i$.

Example 2.17. If S is an inverse monoid, which we consider as a right S-act, then for any $e \leq f \in E(S)$ it follows that $\ker \lambda_f \subseteq \ker \lambda_e$, where $\lambda_e(s) = es$. Hence there is a set of right congruences on S partially ordered by inclusion, where the identity relation $\ker \lambda_1$ is a least element in the ordering. We can now construct a direct system of S-acts $S/\ker \lambda_f \to S/\ker \lambda_e$, $s \ker \lambda_f \mapsto s \ker \lambda_e$ whose directed colimit, by the previous Lemma, is S/σ where $\sigma = \bigcup_{e\in E(S)} \ker \lambda_e$, which is easily seen to be the minimum group congruence on S.

The following Lemma, which says that a finite family of relations can be lifted from the directed colimit to one of the acts in the direct system, has particular importance for finitely presented acts and pure epimorphisms/monomorphisms, as will be seen later.

Lemma 2.18. *Let S be a monoid, let (X_i, ϕ_j^i) be a direct system of S-acts with directed index set I and directed colimit (X, α_i). For every family $y_1, \dots, y_n \in X$ and relations*

$$y_{j_i} s_i = y_{k_i} t_i \quad 1 \leq i \leq m \quad \textit{and} \quad 1 \leq j_i, k_i \leq n$$

there exists some $l \in I$ and $x_1, \dots, x_n \in X_l$ such that $\alpha_l(x_r) = y_r$ for $1 \leq r \leq n$, and

$$x_{j_i} s_i = x_{k_i} t_i \textit{ for all } 1 \leq i \leq m.$$

Proof. Given $y_1, \dots, y_n \in X$ there exists $p(1), \dots, p(n) \in I$ and $y'_r \in X_{p(r)}$ such that $\alpha_{p(r)}(y'_r) = y_r$ for all $1 \leq r \leq n$. So for all $1 \leq i \leq m$ we have

$$\alpha_{p(j_i)}(y'_{j_i}s_i) = \alpha_{p(j_i)}(y'_{j_i})s_i = \alpha_{p(k_i)}(y'_{k_i})t_i = \alpha_{p(k_i)}(y'_{k_i}t_i)$$

and so there exist $l_i \geq p(j_i), p(k_i)$ such that for all $1 \leq i \leq m$

$$\phi_{l_i}^{p(j_i)}(y'_{j_i})s_i = \phi_{l_i}^{p(j_i)}(y'_{j_i}s_i) = \phi_{l_i}^{p(k_i)}(y'_{k_i}t_i) = \phi_{l_i}^{p(k_i)}(y'_{k_i})t_i.$$

Let $l \geq l_1, \dots, l_m$. Then there exist $\phi_l^{p(1)}(y'_1), \dots, \phi_l^{p(n)}(y'_n) \in X_l$ such that $\alpha_l(\phi_l^{m(r)}(y'_r)) = \alpha_{m(r)}(y'_r) = y_r$ for all $1 \leq r \leq n$ and

$$\phi_l^{p(j_i)}(y'_{j_i})s_i = \phi_l^{l_i}\left(\phi_{l_i}^{p(j_i)}(y'_{j_i})\right)s_i = \phi_l^{l_i}\left(\phi_{l_i}^{p(k_i)}(y'_{k_i})\right)t_i = \phi_l^{p(k_i)}(y'_{k_i})t_i$$

for all $1 \leq i \leq m$ and the result follows. □

2.4 Structure of acts

A (non-empty) subset U of an S-act A is called a **generating set** of A if every element $a \in A$ can be written as $a = us$ for some $u \in U$, $s \in S$ and we write $A = US$ or $A = \langle U \rangle$. We say that A is **finitely generated** if it has a finite generating set. We call A **cyclic** if it is generated by one element and we usually write aS instead of $\{a\}S$.

Proposition 2.19. *[38, Proposition I.5.17] Given a monoid S and a congruence ρ on S, S/ρ is isomorphic to a cyclic S-act, and moreover every cyclic S-act is isomorphic to S/ρ for some congruence ρ on S.*

Proof. Let $A = aS$ be a cyclic S-act, and define an epimorphism $\lambda_a : S \to A$, $s \mapsto as$. By Corollary 2.3, $A \cong S/\ker(\lambda_a)$. Conversely if ρ is any congruence on S then the quotient S/ρ is a cyclic S-act with $[1]_\rho$ the generating element. □

This means we can use congruences as an alternative viewpoint to study cyclic acts.

We say that an S-act A is **decomposable** if there exist two subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \emptyset$. In this case $A = B\dot{\cup}C$ is called a **decomposition** of A. Otherwise A is called **indecomposable**.

Lemma 2.20. *[38, Proposition I.5.8] Every cyclic S-act is indecomposable.*

Proof. If $aS = B \cup C$, where B, C are subacts then $a = a1 \in B$, say, and then $aS \subseteq B$. □

An S-act A is said to be **locally cyclic** if every finitely generated subact is contained within a cyclic subact. This is equivalent to saying that for all $x, y \in A$, there exists $z \in A$ such that $x, y \in zS$.

Lemma 2.21. *[51, Lemma 3.4] Every locally cyclic S-act is indecomposable.*

Proof. Let $A = B \cup C$ be a locally cyclic S-act, the union of two subacts B, C, then given two elements $b \in B$, $c \in C$, without loss of generality there exists $z \in B$, such that $b, c \in zS \subseteq B$ and so $B \cap C \neq \emptyset$. □

Proposition 2.22. *An S-act is locally cyclic if and only if it is the directed colimit of cyclic S-acts.*

Proof. Assume A is a locally cyclic S-act, and take $\{A_i : i \in I\}$ to be the set of cyclic subacts partially ordered by inclusion, since every two cyclic subacts of A both sit inside a third cyclic subact, I is a directed index set and we can apply Lemma 2.14 so that the directed colimit of this direct system is $\bigcup_{i \in I} A_i$ which is clearly equal to A.

Conversely, let (A_i, ϕ^i_j) be any direct system of cyclic S-acts over a directed index set I, and let (A, α_i) be the directed colimit of this system. Given any $x, y \in A$ there exists $a_i \in A_i$, $a_j \in A_j$ such that $\alpha_i(a_i) = x$ and $\alpha_j(a_j) = y$. Since I is directed there exists some $k \in I$ with $i, j \leq k$, and $\phi^i_k(a_i) = a_k s$, $\phi^i_k(a_j) = a_k t$ for some $s, t \in S$, where a_k is the generator for A_k. Then $x, y \in \alpha_k(a_k)S$ and A is locally cyclic. □

Lemma 2.23. *[38, Lemma I.5.9] Let $A_i \subseteq A$, $i \in I$, be indecomposable subacts of an S-act A such that $\bigcap_{i \in I} A_i \neq \emptyset$. Then $\bigcup_{i \in I} A_i$ is an indecomposable subact of A.*

Proof. Clearly $\bigcup_{i \in I} A_i$ is a subact of A. Assume there exists a decomposition $\bigcup_{i \in I} A_i = B \dot{\cup} C$ and take $x \in \bigcap_{i \in I} A_i$ with $x \in B$, say. Then $x \in A_i \cap B$ for all $i \in I$. Since $A_i \cap (B \dot{\cup} C) = (A_i \cap B) \dot{\cup} (A_i \cap C)$ and A_i is indecomposable, it follows that $A_i \cap C = \emptyset$ for all $i \in I$. Thus $\bigcup_{i \in I} A_i = B$, a contradiction. □

Recall from 2.3.1 that disjoint union is in fact the coproduct in the category of S-acts so from here onwards on we use $\amalg$ instead of $\dot\cup$.

We now state one of the most fundamental properties of an S-act.

Theorem 2.24. *[38, Theorem I.5.10] Every S-act A has a unique decomposition $A \cong \coprod_{i\in I} A_i$ into a coproduct of indecomposable subacts A_i.*

Proof. Take $x \in A$. Then xS is indecomposable by Lemma 2.20. Now define

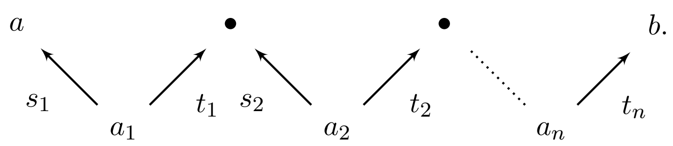
$$U_x := \bigcup\{U \subseteq A : x \in U \text{ and } U \text{ indecomposable}\}$$

and by Lemma 2.23, it is an indecomposable subact of A. For $x, y \in A$ we get that $U_x = U_y$ or $U_x \cap U_y = \emptyset$. Indeed, $z \in U_x \cap U_y$ implies $U_x, U_y \subseteq U_z$. Thus $x \in U_x \subseteq U_z$, $y \in U_y \subseteq U_z$, i.e. $U_z \subseteq U_x \cap U_y$. Therefore $U_x = U_y = U_z$. Denote by A' a representative subset of elements $x \in A$ with respect to the equivalence relation $\sim$ defined by $x \sim y$ if and only if $U_x = U_y$. Then $A = \bigcup_{x\in A'} U_x$ is the unique decomposition of A into indecomposable subacts. □

Alternatively we can think of this in a graphical way. Given an S-act A, define a connectedness relation $\sim$ on A where two elements $a, b \in A$ are connected if there exists a path between a and b in the undirected version of the directed graph associated to the S-act. Equivalently,

$$a \sim b \Leftrightarrow a = a_1 s_1,\ a_1 t_1 = a_2 s_2,\ \ldots\ , a_n t_n = b$$

for some $a_i \in A$, $s_i, t_i \in S$, $i = 1, \ldots, n$, as shown in the following digraph



Then the $\sim$-classes are precisely the connected components of the underlying undirected graph associated to the act. Indecomposable then just means connected (in the underlying undirected graph) and every graph clearly uniquely decomposes in to its connected components.

It is clear that every cyclic act is locally cyclic and every locally cyclic act is indecomposable, but the converses are not true. All indecomposable

S-acts are locally cyclic if and only if all indecomposable S-acts are cyclic if and only if S is a group [51, Lemma 3.2] and all locally cyclic S-acts are cyclic if and only if S satisfies **Condition (A)** (see page 55). By Proposition 2.22 this is also equivalent to the class of cyclic acts being closed under directed colimits. For an overview of results related to Condition (A) see [6].

2.5 Classes of acts

We now attempt to define analogous classes of acts to the well known classes in module theory.

2.5.1 Free acts

A set U of generating elements of an S-act A is said to be a **basis** of A if every element $a \in A$ can be uniquely presented in the form $a = us$, $u \in U$, $s \in S$, i.e. if $a = u_1 s_1 = u_2 s_2$, then $u_1 = u_2$ and $s_1 = s_2$.

If an S-act A has a basis U, then it is called a **free act**. Let $\mathcal{F}r$ denote the class of all free S-acts. Clearly S considered as an S-act over itself is free with basis $\{1\}$. In fact, as the next result shows, all free acts are just coproducts of S.

Theorem 2.25. *[38, Theorem I.5.13] An S-act F is free if and only if $F \cong \coprod_{i \in I} S$ with non-empty set I.*

Corollary 2.26. *An S-act $\coprod_{i \in I} A_i \in \mathcal{F}r$ if and only if $A_i \in \mathcal{F}r$ for each $i \in I$.*

2.5.2 Finitely presented acts

An S-act A is called **finitely presented** if it is the coequalizer $K \rightrightarrows F \to A$, where F is a finitely generated free S-act and K is a finitely generated S-act. We have the following useful characterisation given by Normak. For the sake of completion, we include a slightly more detailed version of this proof in Appendix A.

Proposition 2.27 (Cf. [45, Proposition 4])**.** *An S-act A is finitely presented if and only if there exists a finitely generated free S-act F and a finitely generated congruence ρ on F such that $A \cong F/\rho$.*

One of the most important properties of a finitely presented act A is that $\mathrm{Hom}(A, -)$ commutes with directed colimits, or more precisely,

Proposition 2.28. *[56, Cf. Proposition 4.2] Let S be a monoid, let (X_i, ϕ^i_j) be a direct system of S-acts with directed index set I and directed colimit (X, α_i). Given any finitely presented S-act F and any S-map $h : F \to X$, there exists some $i \in I$ and S-map $g : F \to X_i$ such that $h = \alpha_i g$.*

Proof. Let $F = (A \times S)/\rho$ be a finitely presented S-act, where $A = \{a_1, \ldots, a_n\}$, $\rho = R^{\#}$ and $R = \{((a_{j_i}, s_i), (a_{k_i}, t_i)) \mid 1 \leq i \leq m, 1 \leq j_i, k_i \leq n\}$. For simplicity, we can assume that $R = R^{op}$ by adding in finitely many more relations. Let $y_r = h((a_r, 1)\rho)$ for $1 \leq r \leq n$, so that we have the following family of relations in X for $1 \leq i \leq m$,

$$y_{j_i} s_i = h((a_{j_i}, 1)\rho)s_i = h((a_{j_i}, s_i)\rho) = h((a_{k_i}, t_i)\rho) = h((a_{k_i}, 1)\rho)t_i = y_{k_i} t_i.$$

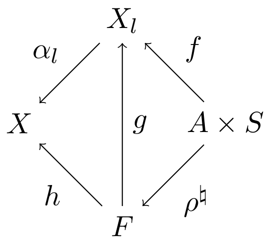
Then by Lemma 2.18 there exists some $l \in I$ and $x_1, \ldots, x_n \in X_l$ such that $\alpha_l(x_r) = y_r$ for $1 \leq r \leq n$ and $x_{j_i} s_i = x_{k_i} t_i$ for $1 \leq i \leq m$. Now define a function $f : A \times S \to X_l, (a_r, s) \to x_r s$, it is clear that this is a well-defined S-map and $\alpha_l f = h\rho^\natural$. Now given any $((a_p, s), (a_q, t)) \in \rho$, either $(a_p, s) = (a_q, t)$ or there exist $(b_1, d_1), \ldots, (b_v, d_v)$ and $w_1, \ldots, w_v \in S$ such that $(b_u, d_u) \in R \cup R^{op} = R$ for $1 \leq u \leq v$ and

$$(a_p, s) = b_1 w_1, \quad d_1 w_1 = b_2 w_2, \quad \ldots \quad d_{v-1} w_{v-1} = b_v w_v, \quad d_v w_v = (a_q, t).$$

Since $(b_u, d_u) \in R$, $(b_u, d_u) = ((a_{j_{c(u)}}, s_{c(u)}), (a_{k_{c(u)}}, t_{c(u)}))$ where $c(u) \in \{1, \ldots, m\}$ for all $1 \leq u \leq v$. Hence we have,

$$\begin{aligned} f((a_p, s)) = f(b_1 w_1) = f(b_1) w_1 &= f((a_{j_{c(1)}}, s_{c(1)})) w_1 = (x_{j_{c(1)}} s_{c(1)}) w_1 \\ &= (x_{k_{c(1)}} t_{c(1)}) w_1 = f((a_{k_{c(1)}}, t_{c(1)})) w_1 = f(d_1) w_1 = f(d_1 w_1) \\ &= f(b_2 w_2) = \cdots = f(d_v w_v) = f((a_q, t)) \end{aligned}$$

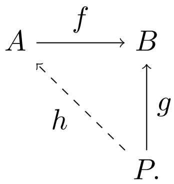
and so $\rho \subseteq \ker(f)$ and by Theorem 2.2 there exists an S-map $g : F \to X_l$ such that $g\rho^\natural = f$. Therefore $(\alpha_l g)\rho^\natural = \alpha_l(g\rho^\natural) = \alpha_l f = h\rho^\natural$ but $\rho^\natural$ is an epimorphism and so $\alpha_l g = h$.



□

2.5.3 Projective acts

An S-act P is called **projective** if given any epimorphism $f : A \to B$, whenever there is an S-map $g : P \to B$ there exists an S-map $h : P \to A$ such that the following diagram commutes



Let $\mathcal{P}$ denote the class of all projective S-acts.

Theorem 2.29. *[3, Theorem 4.1.8] An S-act P is projective if and only if $P \cong \coprod_{i \in I} e_i S$ where for each $i \in I$, $e_i = e_i^2$ is an idempotent.*

Corollary 2.30. *An S-act $\coprod_{i \in I} P_i \in \mathcal{P}$ if and only if $P_i \in \mathcal{P}$ for each $i \in I$.*

2.5.4 Flat acts

In ring theory there are several characterisations of flat modules which are all distinct in **Act-S**. One of the simplest definitions is that a right module M over a ring R is flat if the tensor functor $M \otimes_R -$ preserves short exact sequences, or equivalently preserves monomorphisms. This is the definition we use for a flat S-act.

Let A be a right S-act and B a left S-act. Let $\rho = H^\#$ be the equivalence relation on the set $A \times B$ generated by

$$H = \{((as, b), (a, sb)) \mid a \in A, b \in B, s \in S\}.$$

Then the set $(A \times B)/\rho$ of equivalence classes is called the **tensor product** of A and B, which will be denoted $A \otimes_S B$, or simply $A \otimes B$. For any $a \in A$, $b \in B$, the equivalence class containing (a, b) is denoted $a \otimes b$.

Clearly for any $a \in A$, $b \in B$, $s \in S$, we have $as \otimes b = a \otimes sb$.

A right S-act A is said to be **flat** if given any monomorphism of left S-acts $f : X \to Y$, the induced map $1 \otimes f : A \otimes X \to A \otimes Y$, $a \otimes x \mapsto a \otimes f(x)$, is also a monomorphism. Let $\mathcal{F}$ denote the class of all flat S-acts. Similarly, we define a left S-act B to be flat if $- \otimes B$ preserves monomorphisms.

An important characterisation of flat modules is that one needs only to consider monomorphisms of (finitely generated) left ideals in to the ring. That is, a module M is flat if and only if $M \otimes I \to M \otimes R$ is a monomorphism for all (finitely generated) left deals $I \subseteq R$. However it is not true that we need only consider inclusions of principal ideals. As a counterexample, consider the polynomial ring $R = K[x, y]$ over some field K and let $M = xR + yR$, then $M \otimes Rr \to M \otimes R$ is a monomorphism for all $r \in R$ but M is not a flat R-module [48, Exercise 9.4]. In the category of S-acts these two definitions are both distinct from flat acts and from one another.

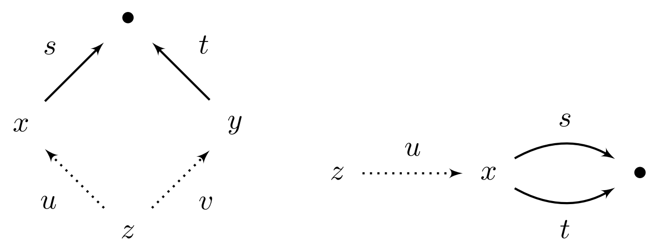
We say that an S-act A is **weakly flat** if $A \otimes I \to A \otimes S$ is a monomorphism for every left ideal $I \subseteq S$, and we say that A is **principally weakly flat** if $A \otimes Ss \to A \otimes S$ is a monomorphism for all $s \in S$. Let $\mathcal{WF}$ and $\mathcal{PWF}$ denote the class of all weakly flat and principally weakly flat S-acts respectively. The following result is obvious as tensor products are preserved under coproducts (see [49, Lemma 4.8]).

Theorem 2.31. *An S-act $F = \coprod_{i \in I} F_i$ is flat (resp. weakly flat, principally weakly flat) if and only if F_i is flat (resp. weakly flat, principally weakly flat).*

In 1969 Lazard gave another characterisation of flat modules being exactly those modules which are directed colimits of finitely generated free modules. In 1971 Stenström showed that the acts which satisfy the same property are again a distinct class of acts. An S-act A is called **strongly flat** if $A \otimes -$ preserves pullbacks and equalizers, rather than all monomorphisms (in fact, it was shown in [13] that equivalently it need only preserve pullbacks). Let $\mathcal{SF}$ denote the class of all strongly flat S-acts. There are several equivalent definitions of strongly flat acts, but one of the most useful is in terms of two 'interpolation' conditions:

An S-act A is said to satisfy **Condition (P)** if whenever $xs = yt$ for some $x, y \in A$, $s, t \in S$ then there exists $z \in A$, $u, v \in S$ such that $x = zu$, $y = zv$ and $us = vt$. Let $\mathcal{CP}$ denote the class of all S-acts satisfying Condition (P).

An S-act A is said to satisfy **Condition (E)** if whenever $xs = xt$ for some $x \in A$, $s, t \in S$ then there exists $z \in A$, $u \in S$ such that $x = zu$, $us = ut$. Let $\mathcal{CE}$ denote the class of all S-acts satisfying Condition (E).



Condition (P) Condition (E)

Then in 1971 Stenström proved the following Theorem,

Theorem 2.32. *[56, Theorem 5.3] Let S be a monoid. Then the following are equivalent for an S-act A:*

1. *A is strongly flat.*
2. *A satisfies Condition (P) and Condition (E).*
3. *A is the directed colimit of finitely generated free S-acts.*

Remark 2.33. We give a proof of part of this Theorem in Appendix B.

We then have the following Theorem, which is also easy to prove:

Theorem 2.34. *[38, Lemma III.9.5] An S-act $F = \coprod_{i \in I} F_i$ satisfies Condition (P) (resp. Condition (E)) if and only if each F_i satisfies Condition (P) (resp. Condition (E)).*

As a Corollary of this, we also have:

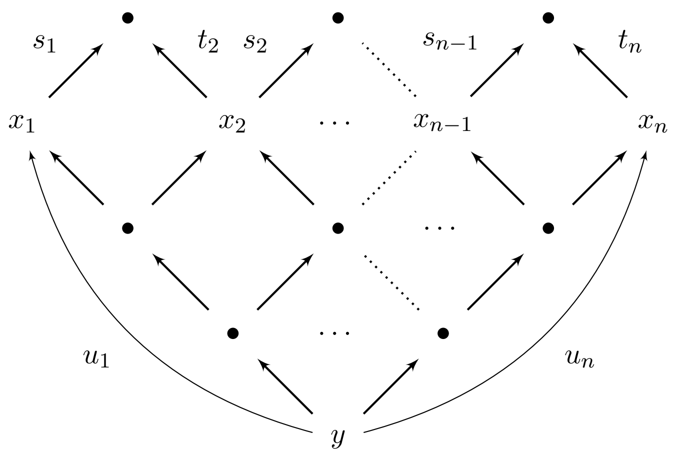
Corollary 2.35. *An S-act $F = \coprod_{i \in I} F_i$ is strongly flat if and only if each F_i is strongly flat.*

The following result we be used later.

Lemma 2.36. *Let S be a monoid and suppose that X satisfies Condition (P) and suppose we have a system of equations*

$$x_1s_1 = x_2t_2,\ x_2s_2 = x_3t_3,\ \ldots,\ x_{n-1}s_{n-1} = x_nt_n$$

where $x_i \in X, s_i, t_i \in S$. Then there exists $y \in X, u_i \in S$ such that $x_i = yu_i$ for $1 \leq i \leq n$ and $u_is_i = u_{i+1}t_{i+1}$ for $1 \leq i \leq n-1$.



Proof. We prove this by induction on n. Firstly, let $n = 2$, then our system is

$$x_1s_1 = x_2t_2$$

and Condition (P) means there exists $y \in X, u_1, u_2 \in S$ with $x_1 = yu_1, x_2 = yu_2$ and $u_1s_1 = u_2t_2$ as required.

Now assume that the result is true for $i \leq n$ and suppose that we have a system of equations

$$x_1s_1 = x_2t_2,\ x_2s_2 = x_3t_3, \ldots,\ x_{n-1}s_{n-1} = x_nt_n,\ x_ns_n = x_{n+1}t_{n+1}.$$

By induction there exists $y \in X, u_i \in S$ such that for $1 \leq i \leq n$ we have $x_i = yu_i$ and for $1 \leq i \leq n-1$, $u_is_i = u_{i+1}t_{i+1}$. In addition, Condition (P) means there exists $y' \in X, u'_n, v'_n \in S$ with $x_n = y'u'_n, x_{n+1} = y'v'_n$ and $u'_ns_n = v'_nt_{n+1}$. But then $x_n = yu_n = y'u'_n$ and so there exists $z \in X, p, q \in S$ with $y = zp, y' = zq$ and $pu_n = qu'_n$. Hence for $1 \leq i \leq n$ it follows that $x_i = z(pu_i)$ and for $1 \leq i \leq n-1$, $(pu_i)s_i = (pu_{i+1})t_{i+1}$. While $x_{n+1} = z(qv'_n)$ and $(pu_n)s_n = qu'_ns_n = (qv'_n)t_{n+1}$ as required. $\square$

Corollary 2.37 (Cf. [51, Theorem 3.7]). *An S-act that satisfies Condition (P) is indecomposable if and only if it is locally cyclic.*

Proof. Let X be an indecomposable S-act satisfying Condition (P). Then for all $x, y \in X$ there exists $x_1, \dots, x_n \in X$, $s_1, \dots, s_n$, $t_1, \dots, t_n \in S$ such that

$$x1 = x_1 s_1, \ x_1 t_1 = x_2 s_2, \ \dots, \ x_n t_n = y1$$

and by Lemma 2.36, there exists $z \in X$, $u, v \in S$ such that $x = zu$, $y = zv$. The converse is obvious as every locally cyclic act is indecomposable. □

2.5.5 Torsion free acts

We say that an S-act A is **torsion free** if for any $x, y \in A$ and any right cancellable element $c \in S$, $xc = yc$ implies $x = y$. Let $\mathcal{T_F}$ denote the class of all torsion free S-acts.

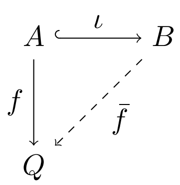
If $A \in \mathcal{T_F}$, then clearly $B \in \mathcal{T_F}$ for every subact $B \subseteq A$.

Lemma 2.38. *$\coprod_{i \in I} A_i \in \mathcal{T_F}$ if and only if $A_i \in \mathcal{T_F}$ for each $i \in I$.*

Proof. Let $A = \coprod_{i \in I} A_i$ and suppose A_i, $i \in I$ are torsion free S-acts. Let $xc = yc$ for some $x, y \in A$, where c is a right cancellative element of S. The equality $xc = yc$ implies x and y are in the same connected component, so there exists some $i \in I$ such that $x, y \in A_i$. Since A_i is torsion free, $x = y$ and A is torsion free. Conversely each A_i is a subact of A and so if A is torsion free, each A_i, $i \in I$ are torsion free. □

2.5.6 Injective acts

An S-act Q is **injective** if for any monomorphism $\iota : A \hookrightarrow B$ and any homomorphism $f : A \to Q$ there exists a homomorphism $\bar{f} : B \to Q$ such that $f = \bar{f}\iota$.



Let $\mathcal{I}$ denote the class of all injective S-acts. Since this definition is unique up to isomorphism, we may assume that ι is an inclusion map and state the definition in the following form.

Lemma 2.39. *[38, Lemma III.1.1] An S-act Q is injective if and only if for any S-act B, for any subact $A \subseteq B$, and for any homomorphism $f : A \to Q$ there exists a homomorphism $\bar{f} : B \to Q$ which extends f, i.e. $\bar{f}|_A = f$.*

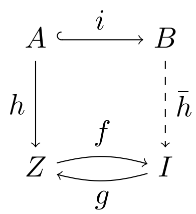
A monoid S is called **left reversible** if for all $s, t \in S$ there exists $p, q \in S$ such that $sp = tq$. Unlike the previous classes, coproducts of injective acts need not always be injective and we have the following:

Proposition 2.40 ([38, Proposition III.1.13])**.** *Let S be a monoid. All coproducts of injective S-acts are injective if and only if S is left reversible.*

Another important result we will require later is that injectivity is closed under the taking of retracts. We include the proof for completeness.

Lemma 2.41. *[38, Proposition I.7.30] Retracts of injective acts are injective.*

Proof. Let I be an injective S-act, and suppose Z is a retract of I, that is, there exist S-maps $g : I \to Z$ and $f : Z \to I$ such that $gf = id_Z$. Given any monomorphism $i : A \hookrightarrow B$ and $h : A \to Z$, using injectivity of I we obtain $\bar{h} : B \to I$ such that $\bar{h}i = fh$, but then $g\bar{h}i = gfh = h$ and Z is injective.



□

Two other important results pertaining to injective acts are:

Lemma 2.42. *[38, Lemma III.1.7] Every injective act contains a fixed point.*

and the Skornjakov-Baer Criterion,

Theorem 2.43. *[38, Theorem III.1.8] Let X be an S-act with a fixed point. Then X is injective if and only if it is injective with respect to all inclusions into cyclic right acts.*

An S-act is called **weakly injective** if it is injective with respect to all inclusions of right ideals in to S. We let $\mathcal{WI}$ denote the class of all weakly injective S-acts.

Similarly an S-act is called **principally weakly injective** if it is injective with respect to all inclusions of principal right ideals in to S. Let $\mathcal{PWI}$ denote the class of all principally weakly injective S-acts.

2.5.7 Divisible acts

A right S-act A is called **divisible** if for all $x \in A$, left cancellable $c \in S$ there exists $y \in A$ such that $x = yc$. Let $\mathcal{D}$ denote the class of divisible S-acts. We now state or prove several basic results about divisible acts which we will require later in the thesis:

Lemma 2.44. *[38, Proposition 2.4]*

1. *$\coprod_{i \in I} A_i \in \mathcal{D}$ if and only if $A_i \in \mathcal{D}$ for each $i \in I$.*
2. *$\mathcal{D}$ is closed under the taking of homomorphic images.*

Proposition 2.45. *[38, Proposition III.2.2] For a monoid S the following statements are equivalent:*

1. *Every S-act is divisible.*
2. *S is divisible.*
3. *All left cancellable elements of S are left invertible.*

Lemma 2.46. *Given an S-act, if it has a divisible subact, then it has a unique maximal divisible subact.*

Proof. Let A be an S-act with a divisible subact. Then consider $D = \bigcup_{i \in I} D_i \subseteq A$, the union of all divisible subacts of A. Clearly D is divisible and contains all divisible subacts. □

2.5.8 Summary

See [38, p217 and p305] for an overview of the following two theorems.

Theorem 2.47. *Given a monoid S, the following inclusions are valid and strict*

$$\mathcal{F}r \subset \mathcal{P} \subset \mathcal{SF} \subset \mathcal{CP} \subset \mathcal{F} \subset \mathcal{WF} \subset \mathcal{PWF} \subset \mathcal{T_F}$$

Theorem 2.48. *Given a monoid S, the following inclusions are valid and strict*

$$\mathcal{I} \subset \mathcal{WI} \subset \mathcal{PWI} \subset \mathcal{D}$$

There is now a very well established branch of semigroup theory that attempts to classify monoids by properties of their acts, in particular it attempts to classify those monoids in which these generally distinct classes of acts actually coincide. This area is often referred to as the homological classification of monoids. See [14] for a good summary of this area, and [38] for a more complete account.

We quote here only a few key results from this area which will be used later.

Theorem 2.49 ([39, Theorem 2.6])**.** *$\mathcal{SF} = \mathcal{F}r$ if and only if S is a group.*

Theorem 2.50 ([16, Corollary 2.2])**.** *If S is a right cancellative monoid, then $\mathcal{T_F} = \mathcal{PWF}$.*

Theorem 2.51 (See Theorem 3.3)**.** *$\mathcal{SF} = \mathcal{P}$ if and only if S is perfect.*

Remark 2.52. Perfect monoids are defined on page 54.

2.5.9 Directed colimits of classes of acts

An important fact in module theory, is that every module is the directed colimit of finitely presented modules (in the language of category theory this says that **Mod-R** is a locally finitely presentable category, see [2]). The following proposition shows us that this is also the case for acts.

Proposition 2.53 ([56, Proposition 4.1])**.** *Every S-act is a directed colimit of finitely presented S-acts.*

We now consider when some of the classes from the previous section are closed under (directed) colimits.

Proposition 2.54 ([56, Proposition 5.2])**.** *$\mathcal{SF}$ is closed under directed colimits.*

Since every strongly flat act is a directed colimit of finitely generated free acts (which are projective) and strongly flat acts are closed under directed colimits. We easily get that the $\mathcal{P}$ is closed under directed colimits if and only if $\mathcal{SF} = \mathcal{P}$, see Theorem 2.51. Therefore,

Proposition 2.55 ([28])**.** *$\mathcal{P}$ is closed under directed colimits if and only if S is perfect.*

The following Proposition is not in the literature, although it is not hard to prove.

Proposition 2.56. *$\mathcal{CP}$ is closed under directed colimits.*

Proof. Let (X_i, ϕ^i_j) be a direct system of S-acts, with directed indexing set and $X_i \in \mathcal{CP}$ for all $i \in I$ and let (X, α_i) be its directed colimit. Suppose that $xs = yt$ in X so that there exists $x_i \in X_i, x_j \in X_j$ with $x = \alpha_i(x_i), y = \alpha_j(x_j)$. Then since I is directed there exists $k \geq i, j$ with $\phi^i_k(x_i)s = \phi^j_k(x_j)t$ in X_k. Consequently there exists $z \in X_k, u, v \in S$ with $\phi^i_k(x_i) = zu, \phi^j_k(x_j) = zv$ and $us = vt$. But then $x = \alpha_i(x_i) = \alpha_k\phi^i_k(x_i) = \alpha_k(z)u$. In a similar way $y = \alpha_k(z)v$ and the result follows. □

Proposition 2.57. *$\mathcal{CE}$ is closed under directed colimits.*

Proof. Similar to previous proof. □

The proof of the following proposition is based on the fact that directed colimits of monomorphisms are monomorphisms (see Lemma 2.12).

Proposition 2.58 ([49, Theorem 5.13])**.** *$\mathcal{F}$ is closed under directed colimits.*

The following Proposition is not in the literature either, but again, it is straightforward.

Proposition 2.59. *$\mathcal{T}_{\mathcal{F}}$ is closed under directed colimits.*

Proof. Let (A_i, ϕ^i_j) be a direct system of torsion free S-acts over a directed index set I with directed colimit (A, α_i). Assume $xc = yc$ where c is a right cancellative element in S and $x, y \in A$. Then there exists $x_i \in A_i$ and $y_j \in A_j$ with $x = \alpha_i(x_i)$, $y = \alpha_j(y_j)$. So $\alpha_i(x_i)c = \alpha_i(x_ic) = \alpha_j(y_jc) = \alpha_j(y_j)c$ and since I is directed, there exists some $k \geq i, j$ such that $\phi^i_k(x_i)c = \phi^i_k(x_ic) = \phi^j_k(y_jc) = \phi^j_k(y_j)c$. Since A_k is torsion free $\phi^i_k(x_i) = \phi^j_k(y_j)$ and $x = \alpha_k\phi^i_k(x_i) = \alpha_k\phi^j_k(y_j) = y$ as required. □

We now consider the question, when is the class of injective acts closed under directed colimits? Before we prove the result, we first recall some basic results about Noetherian monoids.

Let S be a monoid and A an S-act. We say that A is **Noetherian** if every congruence on A is finitely generated, and we say that a monoid S is Noetherian if it is Noetherian as an S-act over itself.

Lemma 2.60 ([45, Proposition 1])**.** *Let S be a monoid and A an S-act. Then A is Noetherian if and only if A satisfies the ascending chain condition on congruences on A.*

Lemma 2.61. *Every Noetherian S-act is finitely generated.*

Proof. Suppose that $x_1, x_2, \ldots$ is an infinite set of generators for X such that for $i \geq 2$, there exists $s_i \in S$ with $x_is_i \notin x_{i-1}S$. Let $X_i = \bigcup_{j\leq i} x_jS$ and define the congruence $\rho_i = (X_i \times X_i) \cup 1_X$ on X and note that $\rho_1 \subsetneq \rho_2 \subsetneq \ldots$ This contradicts the ascending chain condition as required. □

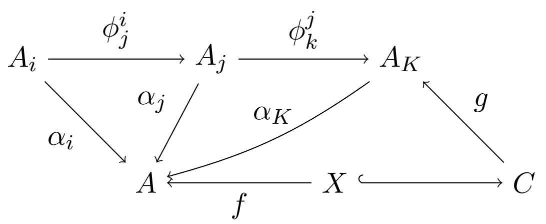
Lemma 2.62 ([45, Proposition 2, Proposition 3, Theorem 3])**.** *Let S be a monoid.*

1. *Every subact and every homomorphic image of a Noetherian S-act is Noetherian.*
2. *All finitely generated S-acts over a Noetherian monoid are Noetherian and finitely presented.*

In the following result we prove the semigroup analogue of what is sometimes called the Bass-Papp Theorem for modules (ca. 1959), although it was known earlier to Cartan and Eilenberg (see [18, p.17 Exercise 8]).

Theorem 2.63. *Let S be a Noetherian monoid, then $\mathcal{I}$ is closed under directed colimits.*

Proof. Let S be a Noetherian monoid, and $(A_i, \phi^i_j)_{i\in I}$ a direct system of injective S-acts with directed index set I and directed colimit (A, α_i). Since A_i is injective it contains a fixed point, by Lemma 2.42, and so A contains a fixed point. Let $X \subseteq C$ be a subact of a cyclic S-act and $f : X \to A$ an S-map. By Theorem 2.43, it is enough to show that f can be extended to C. Since S is Noetherian, by Lemma 2.62, X is Noetherian and hence finitely generated by Lemma 2.61. Therefore $f(X) = \langle a_1, \dots, a_n\rangle$ is a finitely generated subact of A. Since a_i are all elements of the colimit, there exists $m(1), \dots, m(n) \in I$, and $a'_i \in A_{m(i)}$ such that $\alpha_{m(i)}(a'_i) = a_i$ for each $1 \leq i \leq n$. Since I is directed, there exists some $k \in I$ with $k \geq m(1), \dots, m(n)$ and such that $b_i = \phi^{m(i)}_k(a'_i) \in A_k$. Let $B = \langle b_1, \dots, b_n\rangle$ a finitely generated subact of A_k. By Lemma 2.62, B is Noetherian and so every congruence on B is finitely generated. In particular $\ker(\alpha_k|_B) = Z^\#$ is finitely generated, where $Z \subseteq B \times B$ is a finite set. So given any $(x, y) \in \ker(\alpha_k|_B)$, there exists $(p_1, q_1), \dots, (p_m, q_m) \in Z$, $s_1, \dots, s_m \in S$ such that $x = p_1 s_1$, $q_1 s_1 = p_2 s_2$, $\dots$, $q_m s_m = y$. Now, since $\alpha_k(p_j) = \alpha_k(q_j)$, for all $1 \leq j \leq m$, there exists $l(j) \geq k$ such that $\phi^k_{l(j)}(p_j) = \phi^k_{l(j)}(q_j)$. Since I is directed, we can take some $K \in I$ larger than all of the $l(j)$ and we have $\phi^k_K(p_j) = \phi^k_K(q_j)$ for all $1 \leq j \leq m$. Hence $\phi^k_K(x) = \phi^k_K(p_1)s_1 = \phi^k_K(q_1)s_1 = \dots = \phi^k_K(q_n)s_n = \phi^k_K(y)$ and so $\ker(\alpha_k|_B) \subseteq \ker(\phi^k_K)$. Hence $D = \phi^k_K(B)$ is a finitely generated subact of A_K and $\alpha_K|_D$ is a monomorphism. Also, for $1 \leq i \leq n$, $\alpha_K(\phi^k_K(b_i s)) = \alpha_k(b_i s) = \alpha_{m(i)}(a'_i s) = a_i s \in \operatorname{im}(f)$. Conversely given any $a_i s \in \operatorname{im}(f)$, $a_i s = \alpha_{m(i)}(a'_i)s = \alpha_K(\phi^{m(i)}_K(a'_i))s \in \operatorname{im}(\alpha|_D)$ and so $\operatorname{im}(f) = \operatorname{im}(\alpha_K|_D) \cong D$. Since A_K is injective, $\alpha_K^{-1} f$ can be extended to C with some S-map $g : C \to A_K$, and so f can be extended to C with the S-map $\alpha_K g$.



□

Lemma 2.64. *$\mathcal{D}$ is closed under all (not just directed) colimits.*

Proof. Let $(X_i, \phi^i_j)_{i\in I}$ be a direct system of divisible S-acts and let (X, α_i) be the colimit. For each $x \in X$ and left cancellative $c \in S$ there exists $x_i \in X_i$ with $\alpha_i(x_i) = x$ and, since X_i is divisible, there exists $d_i \in X_i$ such that $x_i = d_i c$. So $x = \alpha_i(x_i) = \alpha_i(d_i c) = \alpha_i(d_i)c$ and X is divisible. □

An important categorical idea is when can a class of objects be 'generated by smaller objects'? One such area that makes use of this idea is locally presentable and accessible categories which have had much attention in recent years, see [2]. This idea is especially important with regards to covers.

For example, if we let $\mathcal{F}$ be any class of objects of a Grothendieck category $\mathcal{G}$ closed under coproducts and directed colimits, then it was shown in [20, Theorem 3.2] that every object in $\mathcal{G}$ has an $\mathcal{F}$-cover if there exists a set $S \subseteq \mathcal{F}$ such that every object in $\mathcal{F}$ is a directed colimit of objects from S.

Unfortunately, it is not true that the category of S-acts is a Grothendieck category and the proof does not carry over, but the natural question still arises, which classes of S-acts have this property? We show that $\mathcal{SF}$, $\mathcal{CP}$, $\mathcal{CE}$ and $\mathcal{D}$ all satisfy this property.

Remark 2.65. Note that, given a cardinality κ, there is only a set (i.e. not a proper class) of isomorphic representatives of S-acts A for which $|A| \leq \kappa$. First note that for a fixed cardinality $\lambda \leq \kappa$, let A be a set with $|A| = \lambda$, then any S-act X with $|X| = \lambda$ is uniquely defined up to isomorphism by a function $f : A \times S \to A$ which encodes the action. There are at most $|A^{A\times S}| \leq \kappa^{\kappa|S|}$ such functions for each λ and so the claim follows by Lemma 1.9.

Lemma 2.66. *Given a monoid S, there exists a set $A \subseteq \mathcal{SF}$ such that every strongly flat S-act is a directed colimit of S-acts from A.*

Proof. Let S be a monoid, and let $\alpha := \max\{|S|, \aleph_0\}$, we intend to show that every strongly flat S-act is a directed union of strongly flat subacts of cardinality less than or equal to α and then apply Remark 2.65.. Given any strongly flat S-act X, by Condition (P), whenever $xs = yt$ for $x, y \in X$,

$s, t \in S$, we can find $z \in X$, $u, v \in S$ such that $x = zu$, $y = zv$, and $us = vt$. Also, by Condition (E), whenever $x = y$ we can choose $u = v$. So by the axiom of choice we can define a function,

$$f : X \times X \times S \times S \to X \times S \times S$$
$$(x, y, s, t) \mapsto \begin{cases} (z, u, v) & \text{if } xs = yt \text{ and } x \neq y \\ (z, u, u) & \text{if } xs = yt \text{ and } x = y \\ (x, s, t) & \text{otherwise.} \end{cases}$$

Now given any subset $Y \subseteq X$ with $|Y| \leq \alpha$, define

$$Y_1 := Y \cup \{p_1 f(x, y, s, t) : x, y \in Y, s, t \in S\},$$

where $p_i(a_1, a_2, a_3) := a_i$. Note that Y_1 is a subset of X containing Y also with cardinality at most α as $|Y \cup (Y \times Y \times S \times S)| = \alpha + \alpha^2 \cdot |S|^2 = \alpha$. Similarly we can define

$$Y_{i+1} := Y_i \cup \{p_1 f(x, y, s, t) : x, y \in Y_i, s, t \in S\},$$

for $i \geq 1$ where $Y_i \subseteq Y_{i+1}$ and $|Y_i| \leq \alpha$ for all $i \in \mathbb{N}$. Let $F(Y) := (\bigcup_{i=1}^{\infty} Y_i)S$ be the subact of X generated by the union of all these sets, and note that this has cardinality no greater than $\alpha \cdot \aleph_0 \cdot |S| = \alpha$. We show that $F(Y)$ is a strongly flat subact of X by showing that it satisfies Condition (P) and (E). Let $xs = yt$ for some $x, y \in F(Y)$, $s, t \in S$, then $x \in Y_i$, $y \in Y_j$ for some $i, j \in \mathbb{N}$ and so $x, y \in Y_{\max\{i,j\}}$ and $z := p_1 f(x, y, s, t) \in Y_{\max\{i,j\}+1} \subseteq F(Y)$, $u := p_2 f(x, y, s, t), v := p_3 f(x, y, s, t) \in S$ such that $x = zu$, $y = zv$ and $us = vt$ so that $F(Y)$ satisfies Condition (P). Given $xs = xt$, for some $x \in F(Y)$, $s, t \in S$, then $x \in Y_i$ for some $i \in \mathbb{N}$ and $z := p_1 f(x, x, s, t) \in Y_{i+1} \subseteq F(Y)$, $u := p_2 f(x, x, s, t)$ such that $x = zu$ and $us = ut$ so that $F(Y)$ satisfies Condition (E) and is strongly flat.

Now, given any $x \in X$, it is clearly contained in a subset of X of cardinality less than or equal to α, for example the singleton set $\{x\}$. Hence $X = \bigcup_{i \in I} F_i$ where F_i are all the strongly flat subacts of X of cardinality no greater than α. Moreover, this union is directed in that, given any two strongly flat subacts F_i and F_j of X with cardinality no greater than α, $F_i \cup F_j$ still has cardinality no greater than α and $F(F_i \cup F_j)$ is a strongly flat subact with cardinality no greater than α containing F_i and F_j. $\square$

This result clearly then holds for $\mathcal{CP}$ and $\mathcal{CE}$ as well. A similar construction also holds for divisible S-acts in the obvious way.

Chapter 3

Coessential covers

It is worth noting that there are in fact two different definitions of cover. This arose from the study of projective covers where the two definitions are equivalent (see [57, Theorem 1.2.12] for modules, and Theorem 6.6 for acts). One definition is based on the concept of coessentiality, the other, a categorical definition. But for classes of modules/acts other than projective, these definitions are often distinct. When flat covers of acts were first considered by J. Renshaw and M. Mahmoudi, they studied coessential covers, not the categorical definition. It seems this is not the correct definition for attempting to extend the flat cover conjecture, although it did open up an interesting area of research with several papers expanding on their work. It even led to a new characterisation of Condition (A) based solely on coessential covers (see [6]).

The aim of this thesis is to study the categorical definition with the attempt of extending some of the techniques used by Enochs and others in their work on the flat cover conjecture. But firstly, in this Chapter, we give a brief overview of some of the known results on coessential covers, and how they relate to Enochs' definition of cover, which we will study more thoroughly in Chapter 5.

3.1 Projective coessential covers

Recall that projective coessential covers are equivalent to projective covers. We give a brief overview of the known results for modues and acts.

3.1.1 Projective coessential covers of modules

Let R be a ring, an epimorphism $\phi : P \to M$ of R-modules is called coessential (or superfluous) if $\ker(\phi) + H = P \Rightarrow H = P$ for any submodule $H \subseteq P$. A module P and an epimorphism $\phi : P \to M$ is called a (coessential) projective cover of M if P is projective and ϕ is coessential.

A ring R is called right perfect if all of its right R-modules have projective covers. It was H. Bass who first characterised perfect rings in 1960. He proved the following Theorem:

Theorem 3.1. *[7] For any ring R, the following are equivalent:*

1. *R is (right) perfect.*
2. *R satisfies the descending chain condition on principal (left) ideals.*
3. *Every flat (right) R-module is projective.*

3.1.2 Projective coessential covers of acts

Bass' definition of a coessential epimorphism of modules can be generalised to the act case.

Given a monoid S, an epimorphism $\phi : P \to A$ of S-acts is called **coessential** if there is no proper subact B of P such that $\phi|_B$ is an epimorphism. An S-act P and an epimorphism $\phi : P \to A$ is called a **(coessential) projective cover** of A if P is projective and ϕ is coessential.

Projective covers of acts were first considered by Isbell in his 1971 paper 'Perfect monoids' [34]. **Perfect monoids** are defined analgously as the monoids where all their right acts have projective covers. It was shown that unlike the characterisation for rings you need an extra 'ascending condition' as well.

A submonoid T of a monoid S is called **left unitary** if whenever $ts, t \in T$ then $s \in T$.

A monoid S is said to satisfy **Condition (D)** if every left unitary submonoid of S has a minimal right ideal generated by an idempotent.

A monoid S is said to satisfy **Condition (A)** if every S-act satisfies the ascending chain condition on cyclic subacts, or equivlanetly, if every locally cyclic S-act is cyclic (see [6]).

Theorem 3.2. *[34] For any monoid S, the following are equivalent:*

1. *S is right perfect.*

2. *S satisfies Conditions (D) and (A).*

Fountain then extended this work in his 1976 paper 'Perfect semigroups', by proving Isbell's conjecture that in the presence of Condition (A), a monoid satisfies Condition (D) if and only if it satisfies the descending chain condition on principal left ideals. He also gave an alternative homological characterisation using strongly flat acts.

Theorem 3.3. *[28] For any monoid S, the following are equivalent:*

1. *S is right perfect.*

2. *S satisfies the descending chain condition on principal left ideals and S satisfies Condition (A).*

3. *Every strongly flat S-act is projective.*

In 1996 Kilp gave another characterisation replacing the condition on the ideals with a property based purely on the monoid.

A monoid S is called **left collapsible** if for all $s, t \in S$ there exists $r \in S$ such that $rs = rt$. A monoid S is said to satisfy **Condition (K)** if every left collapsible submonoid of S contains a left zero.

Theorem 3.4. *[37] For any monoid S, the following are equivalent:*

1. *S is right perfect.*

2. *S satisfies Conditions (A) and (K).*

3.2 Flat coessential covers

3.2.1 Flat coessential covers of modules

In 2007, A. Amini et. al. studied flat coessential covers of modules [5]. They called a ring 'generalized perfect' or G-perfect if every module was the coessential epimorphic image of a flat module. Then clearly every G-perfect ring is perfect as every projective cover is a flat coessential cover. However they showed that not every ring is G-perfect. In fact, the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ does not have a (coessential) flat cover.

Therefore the two definitions of cover are clearly distinct for the class of flat modules as it was proved in 2001 that every module has a flat cover (in the Enochs sense).

3.2.2 Flat coessential covers of acts

Renshaw & Mahmoudi first considered flat covers of acts in their 2008 paper 'On covers of cyclic acts over monoids'. In particular they defined strongly flat and Condition (P) covers using the definition of a coessential epimorphism. They gave a characterisation of those monoids whose cyclic acts all have strongly flat and Condition (P) covers.

A monoid S is called **right reversible** if for all $s, t \in S$ there exists $p, q \in S$ such that $ps = qt$.

Theorem 3.5. *[52, Theorem 3.2] Let S be a monoid. Then every cyclic S-act has a strongly flat cover if and only if every left unitary submonoid T of S contains a left collapsible submonoid R such that for all $u \in T$, $uS \cap R \neq \emptyset$.*

Theorem 3.6. *[52, Theorem 4.2] Let S be a monoid. Then every cyclic S-act has a Condition (P) cover if and only if every left unitary submonoid T of S contains a right reversible submonoid R such that for all $u \in T$, $uS \cap R \neq \emptyset$.*

In 2010 Khosravi, Ershad & Sedaghatjoo noticed that by simply adding Condition (A) these results could be extended for acts in general. In fact they proved that given a class of S-acts $\mathcal{X}$ closed under coproducts and decompositions ($\coprod_{i \in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$) if every cyclic S-act has an $\mathcal{X}$ cover and S satsifies Condition (A), then every S-act has an $\mathcal{X}$ cover. They then proved the converse of this result for strongly flat and Condition (P). They thus characterised what they called 'SF-perfect' and '(P)-perfect' monoids.

A monoid S is called right **SF-perfect** (resp. **(P)-perfect**) if every right S-act has a strongly flat (resp. Condition (P)) cover.

Theorem 3.7. *[36, Theorem 2.7] For a monoid S, the following are equivalent:*

1. *S is right SF-perfect.*
2. *S satisfies Condition (A) and every cyclic right S-act has a strongly flat cover.*
3. *S satisfies Condition (A) and every left unitary submonoid T of S contains a left collapsible submonoid R such that for all $u \in T$, $uS \cap R \neq \emptyset$.*

Theorem 3.8. *[36, Theorem 2.8] For a monoid S, the following are equivalent:*

1. *S is right (P)-perfect.*
2. *S satisfies Condition (A) and every cyclic right S-act has a Condition (P) cover.*
3. *S satisfies Condition (A) and every left unitary submonoid T of S contains a right reversible submonoid R such that for all $u \in T$, $uS \cap R \neq \emptyset$.*

All of these results on strongly flat and Condition (P) covers are using the coessential definition of cover. To our knowledge, no one has yet studied Enochs' definition of cover for the category acts. This is the aim of this thesis.

Chapter 4

Purity

Before we study $\mathcal{X}$-covers of acts for different classes of acts $\mathcal{X}$, we first prove some results around the concept of purity. Purity plays an important role in the proof of the flat cover conjecture, because purity is intrinsically connected to flatness. Recall that a short exact sequence of modules is called pure if after tensoring with any module it is still exact (recall that a module is flat if after tensoring with any short exact sequence it is still exact). The relationship between flat modules and pure exact sequences is demonstrated in the following Theorem,

Theorem 4.1. *[41, Theorem 2.4.85] An R module C is flat if and only if any short exact sequence of R-modules*

$$0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$$

is pure.

There are several important characterisations of pure exact sequences of modules, summarised in the following Theorem,

Theorem 4.2. *[41, Theorem 2.4.89] For any short exact sequence of R-modules $\epsilon : 0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$, the following are equivalent:*

1. *ϵ is pure exact.*

2. *If $a_j \in A$ $(1 \leq j \leq n)$, $b_i \in B$ $(1 \leq i \leq m)$ and $s_{ij} \in R$ $(1 \leq i \leq m, 1 \leq j \leq n)$ are given such that $a_j = \sum_i b_i s_{ij}$ for all j, then there exist $a'_i \in A$ $(1 \leq i \leq m)$ such that $a_j = \sum_i a'_i s_{ij}$ for all j.*

3. *Given any commutative diagram of R-modules;*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ R^n & \xrightarrow[\sigma]{} & R^m \end{array}$$

   there exists $\theta \in Hom(R^m, A)$ such that $\theta\sigma = \alpha$.
   (Equivalently, we can replace R^m, R^n with finitely presented modules).

4. *For any finitely presented R-module M, any R-homomorphism $h : M \to C$ can be lifted to an R-homomorphism $f : M \to B$.*

5. *ϵ is the directed colimit of a direct system of split exact sequences*

$$0 \to A \to B_i \to C_i \to 0 \quad (i \in I),$$

   where the C_i's are finitely presented right R-modules.

Purity was first generalised for S-acts in terms of Definition 2 in the previous Theorem, the solvability of equations. In 1971, Stenström introduced the notion of a pure epimorphism $B \twoheadrightarrow C$ of S-acts where every finite system of equations in C, is solvable in B. He then showed that this was equivalent to Definition 4 in the previous Theorem [56]. Then in 1980, Normak introducted the notion of a pure monomorphism. We say that a monomorphism of S-acts $A \hookrightarrow B$ is pure, or $A \subseteq B$ is a pure subact of B, if every finite system of equations with constants from A, which is solvable in B, is solvable in A. He then showed that this is equivalent to a statement similar to Definition 3 in the previous Theorem [46]. Later we give characterisations of pure epimorphisms and pure monomorpihsms in terms of Definition 5 in the previous Theorem.

For the category of modules, every pure monomorphism gives rise to a pure epimorphism (its cokernel) and every pure epimorphism gives rise to a pure monomorphism (its kernel). So we need only talk about pure exact sequences of modules. Unfortunately, this is not the case for the category of S-acts, and we need to consider the two definitions separately. This distinction is made clear in [1] which has been the basis for some of the results in this chapter.

Just to confuse things further, there is another definition of pure monomorphism (called R-pure in [3]), based on tensors, which has been used especially in the area of amalgamation for semigroups (see [50]). For S-acts, this is again distinct from the other definition of pure monomorphism, but Normak proved in [46, Proposition 2], that an R-pure monomorphism is pure. We will not mention this definition again, all of our definitions of purity are based on solvability of equations.

4.1 Pure epimorphisms

Let $\psi : X \to Y$ be an S-map between two S-acts X and Y. We say that ψ is **a pure epimorphism** if for every family $y_1, \ldots, y_n \in Y$ and relations

$$y_{j_i} s_i = y_{k_i} t_i \quad (1 \leq i \leq m)$$

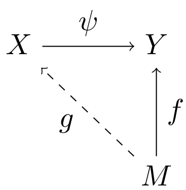
there exists $x_1, \ldots, x_n \in X$ such that $\psi(x_r) = y_r$ for $1 \leq r \leq n$, and

$$x_{j_i} s_i = x_{k_i} t_i \text{ for all } 1 \leq i \leq m.$$

Note that a pure epimorphism is always an epimorphism, as given any $y \in Y$, $y1 = y1$, there exists $x \in X$ such that $\psi(x) = y$.

Stenström showed that this was equivalent to the following:

Theorem 4.3 ([56, Proposition 4.3]). *Let S be a monoid, let X, Y be S-acts and let $\psi : X \to Y$ be an S-map. Then ψ is a pure epimorphism if and only for every finitely presented S-act M and every S-map $f : M \to Y$ there exists $g : M \to X$ such that the following diagram*



commutes.

Example 4.4. Let S be an inverse monoid and σ the minimum group congruence on S as in Example 2.17. Then the right S-map $S \to S/\sigma$ is a pure epimorphism. To see this let $y_1 = x_1\sigma, \dots, y_n = x_n\sigma \in S/\sigma$ and suppose we have relations

$$y_{j_i}s_i = y_{k_i}t_i \quad (1 \leq i \leq m).$$

Then for $1 \leq i \leq m$ we have $(x_{j_i}s_i, x_{k_i}t_i) \in \sigma$ and so there exist $e_i \in E(S), (1 \leq i \leq m)$ such that $e_ix_{j_i}s_i = e_ix_{k_i}t_i$. Now let $e = e_1 \dots e_m$ and note that for $1 \leq i \leq m, ex_{j_i}s_i = ex_{k_i}t_i$ and for $1 \leq l \leq n, \sigma^{\natural}(ex_l) = (ex_l)\sigma = x_l\sigma = y_l$ as required.

It is clear that if the epimorphism ψ splits with splitting monomorphism $\phi : Y \to X$ then $\phi f : M \to X$ is such that $\psi\phi f = f$ and so ψ is pure. The converse is not in general true. For example, let $S = \mathbb{N}$ with multiplication given by

$$n.m = \max\{m, n\} \text{ for all } m, n \in S.$$

Let $\Theta_S = \{a\}$ be the 1-element right S-act and note that $S \to \Theta_S$ is a pure epimorphism by Theorem 4.3. However, as S does not contain a fixed point then it does not split.

Lemma 2.18 gives us,

Corollary 4.5. *Let S be a monoid, let (X_i, ϕ^i_j) be a direct system of S-acts with directed index set I and directed colimit (X, α_i). Then the natural map $\coprod_{i \in I} X_i \to X$ is a pure epimorphism.*

Suppose that (X_i, ϕ^i_j) and (Y_i, θ^i_j) are direct systems of S-acts and S-maps and suppose that for each $i \in I$ there exists an S-map $\psi : X_i \to Y_i$ and suppose (X, β_i) and (Y, α_i), the directed colimits of these systems are such that

$$\begin{array}{ccc} X_i & \xrightarrow{\psi_i} & Y_i \\ {\scriptstyle \beta_i}\downarrow & & \downarrow{\scriptstyle \alpha_i} \\ X & \xrightarrow[\psi]{} & Y \end{array} \qquad \begin{array}{ccc} X_i & \xrightarrow{\psi_i} & Y_i \\ {\scriptstyle \phi^i_j}\downarrow & & \downarrow{\scriptstyle \theta^i_j} \\ X_j & \xrightarrow[\psi_j]{} & Y_j \end{array}$$

commute for all $i \leq j \in I$. Then we shall refer to ψ as the **directed colimit of the** ψ_i (in the language of category theory, this is a directed colimit in the category of arrows). It is shown in [49] that directed colimits of (monomorphisms) epimorphisms are (monomorphisms) epimorphisms.

Proposition 4.6. *Pure epimorphisms are closed under directed colimits.*

Proof. Suppose that (X_i, ϕ_j^i) and (Y_i, θ_j^i) are direct systems and for each $i \in I$ there exists a pure epimorphism $\psi_i : X_i \to Y_i$ and suppose (X, β_i) and (Y, α_i), the directed colimits of these systems are such that

$$\begin{array}{ccc} X_i & \xrightarrow{\psi_i} & Y_i \\ {\scriptstyle \beta_i}\downarrow & & \downarrow{\scriptstyle \alpha_i} \\ X & \xrightarrow[\psi]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X_i & \xrightarrow{\psi_i} & Y_i \\ {\scriptstyle \phi_j^i}\downarrow & & \downarrow{\scriptstyle \theta_j^i} \\ X_j & \xrightarrow[\psi_j]{} & Y_j \end{array}$$

commute for all $i \leq j \in I$.

Given any finitely presented S-act F and any S-map $h : F \to Y$, by Proposition 2.28, there exists some $i \in I$, and S-map $g : F \to Y_i$ such that $h = \alpha_i g$. By the purity of ψ_i there exists $f : F \to X_i$ such that $\psi_i f = g$, therefore $\psi \beta_i f = \alpha_i \psi_i f = \alpha_i g = h$ and ψ is pure.

□

Proposition 4.7. *Pure epimorphisms are closed under pullbacks.*

Proof. Let S be a monoid, let

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow[\psi]{} & D \end{array}$$

be a pullback diagram of S-acts and suppose that ψ is a pure epimorphism. Since ψ is onto, by Lemma 2.8, ϕ is also onto. Suppose that M is finitely presented and that $f : M \to B$ is an S-map. Then there exists an S-map $g : M \to C$ such that $\psi g = \beta f$. Since A is a pullback then there exists a unique $h : M \to A$ such that $\phi h = f$ and $\alpha h = g$. Hence ϕ is also a pure epimorphism. $\square$

Although not every pure epimorphism splits, we can deduce

Theorem 4.8. *Pure epimorphisms are precisely the directed colimits of split epimorphisms.*

Proof. Suppose that $\psi : X \to Y$ is a pure epimorphism. By Proposition 2.53, Y is a directed colimit of finitely presented acts (Y_i, ϕ_j^i) and so let $\alpha_i : Y_i \to Y$ be the canonical maps. For each Y_i let

$$\begin{array}{ccc} X_i & \xrightarrow{\psi_i} & Y_i \\ {\scriptstyle\beta_i}\downarrow & & \downarrow{\scriptstyle\alpha_i} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

be a pullback diagram so that by Proposition 4.7 ψ_i is pure. Hence since Y_i is finitely presented then it easily follows that ψ_i splits. Notice that $X_i = \{(y_i, x) \in Y_i \times X \mid \alpha_i(y_i) = \psi(x)\}, \psi_i(y_i, x) = y_i$ and $\beta_i(y_i, x) = x$ and that since ψ is onto then $X_i \neq \emptyset$.

For $i \leq j$ define $\theta_j^i : X_i \to X_j$ by $\theta_j^i(y_i, x) = (\phi_j^i(y_i), x)$ and notice that $\beta_j \theta_j^i = \beta_i$ and that $\psi_j \theta_j^i = \phi_j^i \psi_i$. Suppose now that there exists Z and $\gamma_i : X_i \to Z$ with $\gamma_j \theta_j^i = \gamma_i$ for all $i \leq j$. Define $\gamma : X \to Z$ by $\gamma(x) = \gamma_i(y_i, x)$ where i and y_i are chosen so that $\alpha_i(y_i) = \psi(x)$. Then γ is well-defined since if $\psi(x) = \alpha_j(y_j)$ then there exists $k \geq i, j$ with $\phi_k^i(y_i) = \phi_k^j(y_j)$ and

$$\gamma_i(y_i, x) = \gamma_k \theta_k^i(y_i, x) = \gamma_k(\phi_k^i(y_i), x) = \gamma_k(\phi_k^j(y_j), x) = \gamma_k \theta_k^j(y_j, x) = \gamma_j(y_j, x).$$

Then γ is an S-map and clearly $\gamma \beta_i = \gamma_i$. Finally, if $\gamma' : X \to Z$ is such that $\gamma' \beta_i = \gamma_i$ for all i, then $\gamma'(x) = \gamma' \beta_i(y_i, x) = \gamma_i(y_i, x) = \gamma(x)$ and so γ is unique. We therefore have that (X, β_i) is the directed colimit of (X_i, θ_j^i) as required.

Conversely, since split epimorphisms are pure then ψ is pure by Proposition 4.6. $\square$

Example 4.9. Let S be as in Example 2.17. Notice that for all $e \in E(S)$, where $\lambda_e : S \to S, s \mapsto es$, the natural map $S \to S/\ker\lambda_e$ splits with splitting map $s\ker\lambda_e \mapsto es$. Moreover

$$\begin{array}{ccc} S & \longrightarrow & S/\ker(\lambda_e) \\ {\scriptstyle id_S}\downarrow & & \downarrow \\ S & \xrightarrow[\sigma^\natural]{} & S/\sigma \end{array}$$

commutes for all $e \in E(S)$ and $\sigma^\natural$ is a directed colimit of split epimorphisms.

4.1.1 n-pure epimorphisms

Recall the following important result,

Theorem 4.10. *[56, Theorem 5.3] Let A be an S-act. The following properties are equivalent:*

1. *A is strongly flat.*
2. *Every epimorphism $B \to A$ is pure.*
3. *There exists a pure epimorphism $F \to A$ where F is free.*
4. *Every morphism $B \to A$, where B is finitely presented, may be factored through a finitely generated free system.*

In [47], Normak defines an S-map $\phi : X \to Y$ to be a **1-pure epimorphism** if for every element $y \in Y$ and relations $ys_i = yt_i, i = 1, \ldots, n$ there exists an element $x \in X$ such that $\phi(x) = y$ and $xs_i = xt_i$ for all i. He proves

Proposition 4.11 ([47, Proposition 1.17])**.** *Let S be a monoid, let X, Y be S-acts, and let $\phi : X \to Y$ be an S-map. Then ψ is 1-pure if and only if for all cyclic finitely presented S-acts C and every morphism $f : C \to Y$ there exits $g : C \to X$ with $f = \phi g$.*

Proposition 4.12 ([47, Proposition 2.2])**.** *Let S be a monoid. Y satisfies condition (E) if and only if every epimorphism $X \to Y$ is 1-pure.*

As a generalisation, we say that an epimorphism $g : B \to A$ of S-acts is n-**pure** if for every family of n elements $a_1, \ldots, a_n \in A$ and every family of m relations $a_{\alpha_i} s_i = a_{\beta_i} t_i$, $\alpha_i, \beta_i \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$ there exist $b_1, \ldots, b_n \in B$ such that $g(b_i) = a_i$ and $b_{\alpha_i} s_i = b_{\beta_i} t_i$ for all i.

We are interested in the cases $n = 1$ and $n = 2$. Clearly pure implies 2-pure implies 1-pure.

Proposition 4.13. *Let S be a monoid and let $\psi : X \to Y$ be an S-epimorphism in which X satisfies condition (E). Then Y satisfies condition (E) if and only if ψ is 1-pure.*

Proof. Suppose that ψ is 1-pure and that $y \in Y, s, t \in S$ are such that $ys = yt$ in Y. Hence there exists $x \in X$ such that $\psi(x) = y$ and $xs = xt$. Since X satisfies condition (E) there exists $x' \in X, u \in S$ such that $x = x'u, us = ut$ and so $y = \psi(x')u, us = ut$ and Y satisfies condition (E).

The converse holds by Proposition 4.12. □

Proposition 4.14. *Let S be a monoid and let $\psi : X \to Y$ be an S-epimorphism in which X satisfies condition (P). If ψ is 2-pure then Y satisfies condition (P).*

Proof. Suppose that ψ is 2-pure and suppose that $y_1, y_2 \in Y, s_1, s_2 \in S$ are such that $y_1 s_1 = y_2 s_2$ in Y. Hence there exists $x_1, x_2 \in X$ with $\psi(x_i) = y_i$ and $x_1 s_1 = x_2 s_2$ in X. Since X satisfies condition (P) then there exists $x_3 \in X, u_1, u_2 \in S$ such that $x_1 = x_3 u_1, x_2 = x_3 u_2$ and $u_1 s_1 = u_2 s_2$. Consequently, $y_1 = \psi(x_3) u_1, y_2 = \psi(x_3) u_2$ and $u_1 s_1 = u_2 s_2$ and so Y satisfies condition (P). □

The converse of this last result is false. For example let $S = (\mathbb{N}, +)$ and let $\Theta_S = \{a\}$ be the 1-element S-act. Let $x = y = a \in \Theta_S$, then $x0 = y0$ and $x0 = y1$ but there cannot exist $x', y' \in S$ such that $x' + 0 = y' + 0$ and $x' + 0 = y' + 1$ and so $S \to \Theta_S$ is not 2-pure, but it is easy to check that Θ_S does satisfy condition (P).

From Theorem 4.10, Proposition 4.13 and Proposition 4.14 we deduce

Corollary 4.15. *Let S be a monoid and let $\psi : X \to Y$ be an S-epimorphism with X strongly flat. The following are equivalent.*

1. *Y is strongly flat;*

2. *ψ is pure;*

3. *ψ is 2-pure.*

Let X be an S-act and ρ a congruence on X. We say that ρ is **pure** (resp. **2-pure**) if $\rho^\natural$ is a pure epimorphism (resp. 2-pure epimorphism). As a corollary to Theorem 4.3 we have

Corollary 4.16. *Let S be a monoid, let X be an S-act and let ρ be a congruence on X. Then ρ is pure if and only if for every family $x_1 \dots, x_n \in X$ and relations*

$$(x_{j_i}s_i, x_{k_i}t_i) \in \rho \quad (1 \leq i \leq m)$$

on X there exists $y_1, \dots, y_n \in X$ such that $(x_i, y_i) \in \rho$ and

$$y_{j_i}s_i = y_{k_i}t_i \text{ for all } 1 \leq i \leq m.$$

Corollary 4.17. *Let ρ be a congruence on a monoid S. Then ρ is pure if and only if S/ρ is strongly flat.*

Example 4.18. It now follows easily from Example 4.4 that if S is an inverse monoid with minimum group congruence σ then S/σ is a strongly flat right S-act.

4.1.2 $\mathcal{X}$-pure congruences

Let A be an S-act and let ρ be a congruence on A. We say that ρ is **$\mathcal{X}$-pure** if $A/\rho \in \mathcal{X}$. So, by Propositions 4.13 and 4.14, Corollary 4.15 and [3, Corollary 4.1.3 and Theorem 4.1.4] we deduce

Corollary 4.19. *Let S be a monoid, let X be an S-act and let ρ be a congruence on X.*

1. *If $X \in \mathcal{CE}$ then ρ is $\mathcal{CE}$-pure if and only if it is 1-pure.*

2. *If $X \in \mathcal{CP}$ then ρ is $\mathcal{CP}$-pure if it is 2-pure.*

3. *If $X \in \mathcal{SF}$ then ρ is $\mathcal{SF}$-pure if and only if it is pure if and only if it is 2-pure.*

4. *If $X \in \mathcal{P}$ then ρ is $\mathcal{P}$-pure if and only if $\rho^\natural$ splits.*

We say that a class of S-acts $\mathcal{X}$ is **closed under chains of $\mathcal{X}$-pure congruences** if given any S-act A, any ordinal β, and any ordinal $\alpha \in \beta$, if ρ_α is an $\mathcal{X}$-pure congruence on A and $\rho_\alpha \subseteq \rho_{\alpha+1}$ then $\bigcup_{\alpha\in\beta} \rho_\alpha$ is also an $\mathcal{X}$-pure congruence on A. Recall from Remark 2.16 that we can immediately deduce the important result,

Proposition 4.20. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts closed under directed colimits. Then $\mathcal{X}$ is closed under chains of $\mathcal{X}$-pure congruences.*

4.2 Pure monomorphisms

Let S be a monoid and A an S-act. We follow the definitions from [32] and [38, Definition III.6.1]. Consider systems Σ consisting of equations of the following three forms

$$xs = xt, \quad xs = yt, \quad xs = a$$

where $s, t \in S$, $b \in A$ and $x, y \in X$ where X is a set. We call x and y variables, s and t coefficients, a a constant and Σ a system of equations with constants from A. Systems of equations will be written as

$$\Sigma = \{xs_i = yt_i : s_i, t_i \in S, 1 \leq i \leq n\}.$$

If we can map the variables of Σ onto a subset of an S-act B such that the equations turn into equalities in B then any such subset of B is called a solution of the system Σ in B. In this case Σ is called solvable in B.

A monomorphism $A \hookrightarrow B$ of S-acts is called a **pure monomorphism**, or $A \subseteq B$ is called a **pure subact** of B if every finite system of equations with constants from A which has a solution in B has a solution in A.

Normak showed this was equivalent to:

Proposition 4.21. *[46, Proposition I] Given a monoid S and a monomorphism $i : A \to B$ of S-acts, then i is pure if and only if for every finitely presented S-act F, for every S-map $g : F \to B$ and for every finite subset*

$T \subseteq F$ such that $g(T) \subseteq \text{im}(i)$, there exists an S-map $h : F \to A$ such that $ih|_T = g|_T$.

We extend this slightly,

Theorem 4.22. *Let S be a monoid, and $i : A \hookrightarrow B$ a monomorphism of S-acts. Then the following are equivalent:*

1. *$i : A \hookrightarrow B$ is a pure monomorphism*

2. *For every finitely presented S-act F, every finitely generated subact $G \subseteq F$, and every commutative diagram*

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ f \uparrow & & \uparrow g \\ G & \hookrightarrow & F \end{array}$$

   there exists an S-map $h : F \to A$ such that $h|_G = f$.

3. *For every finitely presented S-act F, every finitely generated free S-act G, and every commutative diagram*

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ f \uparrow & & \uparrow g \\ G & \xrightarrow[m]{} & F \end{array}$$

   there exists $h : F \to A$ such that $hm = f$.

4. *For any two finitely presented S-acts F and G, and every commutative diagram*

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ f \uparrow & & \uparrow g \\ G & \xrightarrow[m]{} & F \end{array}$$

there exists $h : F \to A$ such that $hm = f$.

Proof. (1) $\Rightarrow$ (2): Let T be some finite generating set of G, then $g(T) = if(T) \subseteq \text{im}(i)$ and so by Proposition 4.21, there exists some $h : F \to A$ such that $ih|_T = g|_T$. Then for every $x = ts \in G$ for some $t \in T$, $s \in S$, $ih(x) = ih(ts) = ih(t)s = g(t)s = g(ts) = g(x)$ and so $ih|_G = g|_G = if$ and since i is a monomorphism $h|_G = f$.
(2) $\Rightarrow$ (4): Note that $m(G)$ is a finitely generated subact of F and so there exists an S-map $h : F \to A$ such that $h|_{m(G)} = f$ and so $hm = f$.
(4) $\Rightarrow$ (3): Every finitely generated free S-act is finitely presented.
(3) $\Rightarrow$ (1): Given any finite subset $T \subseteq F$, let $G = T \times S$ be the free S-act generated by T, and define $m : G \to F$, $(t, s) \mapsto ts$ and $f := gm$. Since $g(T) \subseteq \text{im}(i)$, we have for all $t \in T$, there exists $a_t \in A$ such that $g(t) = i(a_t)$. Now define $f : G \to A$, $(t, s) \mapsto a_t s$, this is well-defined as i is injective. Hence $gm((t, s)) = g(ts) = g(t)s = i(a_t)s = i(a_t s) = if((t, s))$ and so $gm = if$. Therefore there exists an S-map $h : F \to A$ such that $hm = f$ and so $ih(t) = ihm((t, 1)) = if((t, 1)) = i(a_t) = g(t)$ and $ih|_T = g|_T$. □

Remark 4.23. Clearly split monomorphisms are pure monomorphisms.

We now prove some results about pure monomorphisms. But firstly, we need a technical Lemma, which is well known in category theory and says that the arrow category of any locally finitely presentable category is locally finitely presentable, or more specifically,

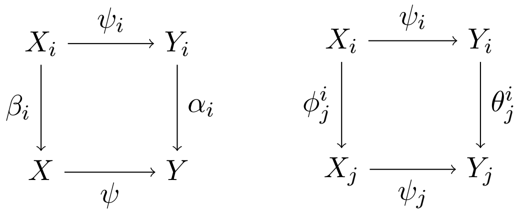
Lemma 4.24. *Every S-map is a directed colimit of S-maps $A_i \to B_i$, where A_i, B_i are finitely presented for all $i \in I$.*

Proof. This follows by Proposition 2.53 and [1, Example 1.55(1)]. □

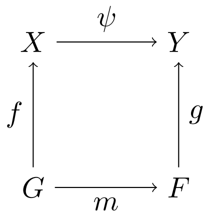
The following three results are adapted from category theoretic results in [1, Proposition 15] and [2, Proposition 2.30].

Proposition 4.25. *Pure monomorphisms are closed under directed colimits.*

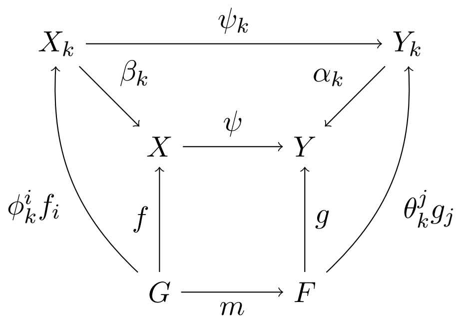
Proof. Suppose that (X_i, ϕ^i_j) and (Y_i, θ^i_j) are direct systems and for each $i \in I$ there exists a pure monomorphism $\psi_i : X_i \to Y_i$ and suppose (X, β_i) and (Y, α_i), the directed colimits of these systems are such that



commute for all $i \leq j \in I$. Now for any two finitely presented S-acts, F and G and any commutative diagram



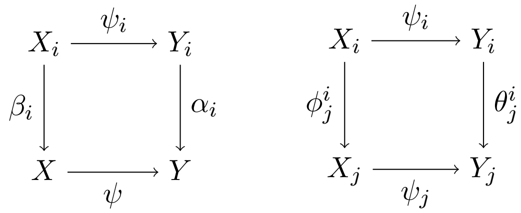
by Proposition 2.28, there exists $i, j \in I$, and $f_i : G \to X_i$, $g_j : F \to Y_j$ such that $\beta_i f_i = f$ and $\alpha_j g_j = g$. Let $k \geq i, j$ so that the following diagram



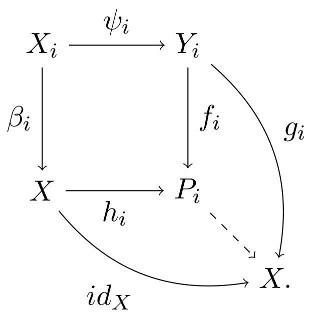
commutes. Since ψ_k is a pure monomorphism, there exists some $h_k : F \to X_k$ such that $h_k m = \phi^i_k f_i$, therefore let $h := \beta_k h_k$ and $hm = \beta_k h_k m = \beta_k \phi^j_k f_i = f$ and ψ is a pure monomorphism. □

Theorem 4.26. *Pure monomorphisms are precisely the directed colimits of split monomorphisms.*

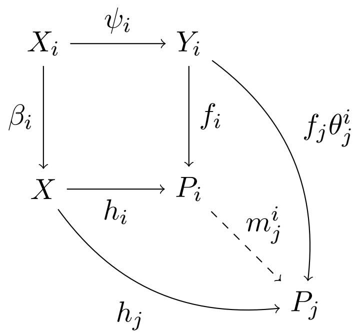
Proof. Suppose that $\psi : X \to Y$ is a pure monomorphism. By Lemma 4.24, ψ is a directed colimit of $\psi_i : X_i \to Y_i$, where X_i, Y_i are finitely presented. That is, (X_i, ϕ^i_j) and (Y_i, θ^i_j) are direct systems and for each $i \in I$ there exists an S-map $\psi_i : X_i \to Y_i$ with X_i, Y_i finitely presented, such that (X, β_i) and (Y, α_i), the directed colimits of these systems are such that



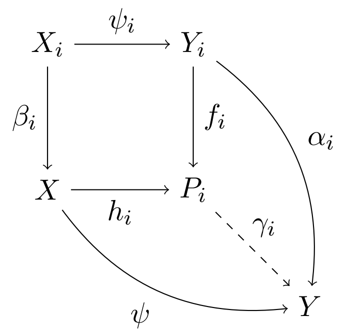
commute for all $i \leq j \in I$. Now since ψ is a pure monomorphism, for each $i \in I$ there exists $g_i : Y_i \to X$ such that $g_i\psi_i = \beta_i$. Now for each $i \in I$, take the pushout $(P_i, (h_i, f_i))$ so that each h_i is a split monomorphism as shown:



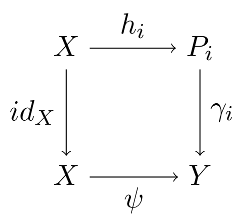
Now $f_j\theta^i_j\psi_i = f_j\psi_j\phi^i_j = h_j\beta_j\phi^i_j = h_j\beta_i$ for all $i \leq j$, so let $m^i_j : P_i \to P_j$ be the unique S-maps that make the following diagram

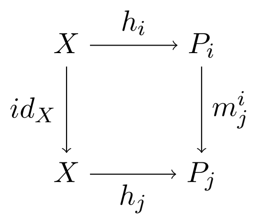


commute. Now let $\gamma_i : P_i \to Y$ be the unique S-maps such that the following diagram



commutes. It is straightforward to check that (Y, γ_i) is the directed colimit of (P_i, m^i_j) and that for $i \leq j$ the following diagrams





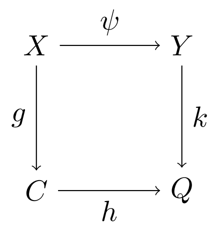
commute and so ψ is the directed colimit of the split monomorphisms h_i.

Conversely, every split monomorphism is a pure monomorphism, and so the result follows by Proposition 4.25. □

Theorem 4.27. *Pure monomorphisms are closed under pushouts.*

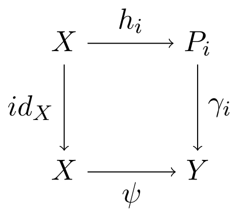
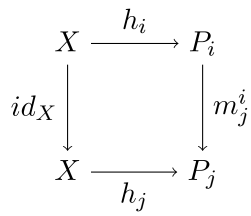
Proof. Firstly, observe that split monomorphisms are closed under pushouts. In fact, let $f : A \to B$ be a split monomorphism with S-map $f' : B \to A$ so that $f'f = id_A$, let $g : A \to C$ be any S-map and let $(P, (p_1, p_2))$ be the pushout of (g, f). By Lemma 2.6, p_1 is a monomorphism. Since $gf'f = g$, there exists some unique S-map $p'_1 : P \to C$ such that $p'_1 p_1 = id_C$ and so p_1 is a split monomorphism.

Now let $\psi : X \to Y$ be a pure monomorphism, $g : X \to C$ any S-map and $(Q, (h, k))$ the pushout of (g, ψ).

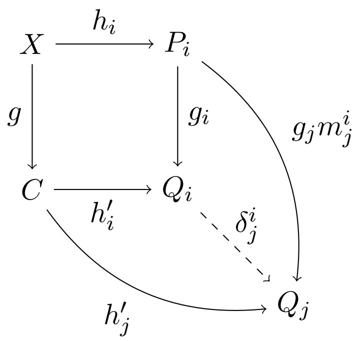


We intend to show that h is a pure monomorphism.

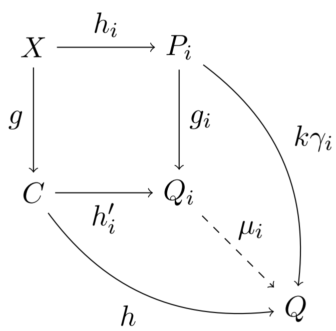
By Theorem 4.26, there exists a direct system (P_i, m_j^i) with directed colimit (Y, γ_i) and splitting monomorphisms $h_i : X \to P_i$ such that for all $i \leq j$, the following diagrams

commute. Now for each $i \in I$, let $(Q_i, (h_i', g_i))$ be the pushout of (g, h_i) so that $g_i h_i = h_i' g$ and note that h_i' are all split monomorphisms. Since $g_j m_j^i h_i = g_j h_j = h_j' g$ for all $i \leq j$, let $\delta_j^i : Q_i \to Q_j$ be the unique S-maps that make the following diagram



commute. Now since $k\gamma_i h_i = k\psi = hg$, let $\mu_i : Q_i \to Q$ be the unique S-maps such that the following diagram



commutes. It is straightforward to check that (Q, μ_i) is the directed colimit of (Q_i, δ_j^i) and that for $i \leq j$ the following diagrams

$$\begin{array}{ccc} C & \xrightarrow{h'_i} & Q_i \\ {\scriptstyle id_C}\downarrow & & \downarrow{\scriptstyle \mu_i} \\ C & \xrightarrow[h]{} & Q \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{h'_i} & Q_i \\ {\scriptstyle id_C}\downarrow & & \downarrow{\scriptstyle \delta^i_j} \\ C & \xrightarrow[h'_j]{} & Q_j \end{array}$$

commute and so h is the directed colimit of the split monomorphisms h'_i and so by Theorem 4.26, is a pure monomorphism. □

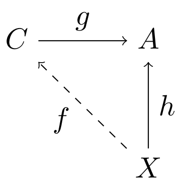
Chapter 5

Covers of acts

Throughout this chapter S will denote a monoid, and $\mathcal{X}$ will refer to a class of S-acts closed under isomorphisms.

We now define an $\mathcal{X}$-cover of an S-act and prove some general results about the existence of such covers.

Let A be an S-act. By an **$\mathcal{X}$-precover** of A we mean an S-map $g : C \to A$ from some $C \in \mathcal{X}$ such that for every S-map $h : X \to A$, for $X \in \mathcal{X}$, there exists an S-map $f : X \to C$ with $h = gf$.



If in addition the $\mathcal{X}$-precover satisfies the condition that each S-map $f : C \to C$ with $gf = g$ is an isomorphism, then we shall call it an **$\mathcal{X}$-cover**. We sometimes refer to just C as the $\mathcal{X}$-precover/cover of A.

The definition of a cover is motivated by attempting to find a weaker version of the right adjoint to the inclusion functor. Recall from Section 1.2.5 that the inclusion functor $\mathcal{X} \subseteq$ **Act-S** has a right adjoint if and only if for all $A \in$ **Act-S**, there exists a terminal object in the slice subcategory $\mathcal{X} \downarrow A$. In this special case we say that every act has an $\mathcal{X}$-cover with the unique mapping property and $\mathcal{X}$ is called a coreflective subcategory of **Act-S** (this is the topic of Section 5.6). However, an $\mathcal{X}$-precover $g : C \to A$ is a weakly terminal object in the slice subcategory $\mathcal{X} \downarrow A$, that is, every object in $\mathcal{X} \downarrow A$ has a map to g which need not be unique. In the case where every act has an $\mathcal{X}$-precover, we say that $\mathcal{X}$ is a weakly coreflective subcategory of **Act-S**. Unlike terminal objects, weakly terminal objects need not be unique and so $\mathcal{X}$-precovers are not necessarily unique. However $\mathcal{X}$-covers are indeed unique up to isomorphism (although not unique up to unique isomorphism). Following the language of Rosický [53], in the case where every act has an $\mathcal{X}$-cover, we say that $\mathcal{X}$ is a stably weakly coreflective subcategory of **Act-S**. Therefore $\mathcal{X}$-covers are very natural objects to study.

For that reason, there is a huge amount of literature on covers, especially for the category of modules over a ring, but also for many other categories. But the most important result is arguably the proof of the flat cover conjecture. This says that every module has a flat cover, which has been generalised to many other categories, with applications in relative homological algebra. But there are also results relating to injective covers, torsion free covers, and various other classes of modules. We intend to imitate some of the proofs in the category of acts. But before we work with any one class, we first proof some general results on $\mathcal{X}$-covers for an arbitrary class of S-acts $\mathcal{X}$. We will then apply these results to specific classes in Chapter 6.

5.1 Preliminary results on $\mathcal{X}$-precovers

Firstly, we show that $\mathcal{X}$-covers are unique up to isomorphism.

Theorem 5.1. *If $g_1 : X_1 \to A$ and $g_2 : X_2 \to A$ are both $\mathcal{X}$-covers of A then there is an isomorphism $h : X_1 \to X_2$ such that $g_2 h = g_1$.*

Proof. By the $\mathcal{X}$-precover property of g_1 there exists $m_1 \in \mathrm{Hom}(X_2, X_1)$ such that $g_1 m_1 = g_2$ and similarly there exists $m_2 \in \mathrm{Hom}(X_1, X_2)$ such that $g_2 m_2 = g_1$, hence $g_1 m_1 m_2 = g_1$ and $g_2 m_2 m_1 = g_2$. Now by the $\mathcal{X}$-cover property of g_2, $m_1 m_2$ must be an isomorphism, and similarly $m_2 m_1$ must be an isomorphism. Hence m_1 and m_2 are both isomorphisms and let $h = m_2$. □

Alternatively, we could have applied Proposition 1.18 and Lemma 1.14.

The following Lemma is obvious.

Lemma 5.2. *An S-act A is an $\mathcal{X}$-cover of itself if and only if $A \in \mathcal{X}$.*

Remark 5.3. So if we have a monoid S where all of the S-acts X satisfy a particular property $X \in \mathcal{X}$, then every S-act has an $\mathcal{X}$-cover. For example every act over an inverse monoid is flat [12] and so every act over an inverse monoid has an $\mathcal{F}$-cover, where $\mathcal{F}$ is the class of flat acts.

Recall from [38, Theorem II.3.16] that an S-act G is called a **generator** if there exists an S-epimorphism $G \to S$.

Proposition 5.4. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts which contains a generator G. If $g : C \to A$ is an $\mathcal{X}$-precover of A then g is an epimorphism.*

Proof. Let $h : G \to S$ be an S-epimorphism. Then there exists an $x \in G$ such that $h(x) = 1$. For all $a \in A$ define the S-map $\lambda_a : S \to A$ by $\lambda_a(s) = as$. By the $\mathcal{X}$-precover property there exists an S-map $f : G \to C$ such that $gf = \lambda_a h$. Hence $g(f(x)) = a$ and so $\mathrm{im}(g) = A$ and g is epimorphic. □

Obviously if every S-act has an epimorphic $\mathcal{X}$-precover, then S has an epimorphic $\mathcal{X}$-precover, which by definition is then a generator in $\mathcal{X}$, so we have the following corollary.

Corollary 5.5. *Let S be a monoid and $\mathcal{X}$ a class of S-acts such that every S-act has an $\mathcal{X}$-precover. Then every S-act has an epimorphic $\mathcal{X}$-precover if and only if $\mathcal{X}$ contains a generator.*

Note that for any class of S-acts containing S then S is a generator in $\mathcal{X}$ and so $\mathcal{X}$-precovers are always epimorphic. In particular this is true for any class containing $\mathcal{F}r$.

The following technical Lemma basically says that the preimage of a decomposable act is decomposable.

Lemma 5.6. *Let $h : X \to A$ be an homomorphism of S-acts where $A = \coprod_{i\in I} A_i$ is a disjoint union of non-empty subacts $A_i \subseteq A$. Then $X =$*

$\coprod_{j\in J} X_j$ where $X_j \subseteq X$ are disjoint non-empty subacts of X and $im(h|_{X_j}) \subseteq A_j$ for each $j \in J \subseteq I$. Moreover, if h is an epimorphism, then $J = I$.

Proof. Let $X_i := \{x \in X \mid h(x) \in A_i\}$ and define $J := \{i \in I \mid X_i \neq \emptyset\}$. For all $x_j \in X_j$, $s \in S$, $h(x_j s) = h(x_j)s \in A_j$ and so $x_j s \in X_j$ and X_j is a subact of X. Since A_j are disjoint and h is a well defined S-map, X_j are disjoint as well and $X = \coprod_{j\in J} X_j$. Clearly $\text{im}(h|_{X_j}) \subseteq A_j$ for each $j \in J$. If h is an epimorphism then none of the X_i are empty and so $J = I$. □

Proposition 5.7. *Let $\mathcal{X}$ be a class of S-acts containing a generator and $g : C \to A$ an $\mathcal{X}$-precover of A, then*

1. *A is cyclic if C is cyclic;*

2. *A is locally cyclic if C is locally cyclic; and*

3. *A is indecomposable if C is indecomposable.*

Proof.

1. Let $g : C \to A$ be an $\mathcal{X}$-precover of A, $C = cS$ a cyclic S-act and let $a = g(c) \in A$. By Proposition 5.4, g is an epimorphism so given any $\alpha \in A$ there exists $\gamma = c\gamma' \in cS = C$ such that $\alpha = g(\gamma) = g(c\gamma') = g(c)\gamma' = a\gamma' \in aS$. So $A = aS$ is cyclic.

2. Let C be locally cyclic, then for all $a, b \in A$, since g is an epimorphism, there exist $x, y \in C$ such that $g(x) = a, g(y) = b$. Now since C is locally cyclic, there exists $z \in C$ such that $x = zs, y = zs'$ for some $s, s' \in S$. So $a = g(zs) = g(z)s, b = g(zs') = g(z)s'$, where $g(z) \in A$ and so A is locally cyclic.

3. Let $C = C_1 \amalg C_2$ be a decomposable S-act, then since g is an epimorphism by Lemma 5.6, $A = A_1 \amalg A_2$ is also decomposable.

□

Conversely, it is not true that a cyclic act must have a cyclic $\mathcal{X}$-cover: for the monoid $S = (\mathbb{N}, +)$ of natural numbers under addition, in 5.2 we show that $\mathbb{Z}$ is a locally cyclic non-cyclic $\mathcal{SF}$-cover of Θ_S.

The following result shows that for well-behaved classes, $\mathcal{X}$-precovers are closed under coproducts and decompositions.

Proposition 5.8. *Let $\mathcal{X}$ satisfy the property that $\coprod_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$. Then each A_i have $\mathcal{X}$-precovers if and only if $\coprod_{i\in I} A_i$ has an $\mathcal{X}$-precover.*

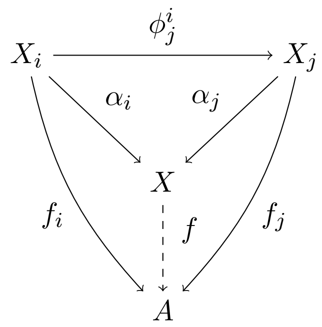
Proof. ($\Rightarrow$) Let $g_i : C_i \to A_i$ be an $\mathcal{X}$-precover of A_i for each $i \in I$. Then define $g : \coprod_{i\in I} C_i \to \coprod_{i\in I} A_i$ by $g|_{C_i} := g_i$ for each $i \in I$. We claim this is an $\mathcal{X}$-precover of $\coprod_{i\in I} A_i$. For all $X \in \mathcal{X}$ with $h : X \to \coprod_{i\in I} A_i$, by Lemma 5.6, there is a subset $J \subseteq I$ such that $X = \coprod_{j\in J} X_j$ and $\mathrm{im}(h|_{X_j}) \subseteq A_j$ for each $j \in J$. Now by the hypothesis $X_j \in \mathcal{X}$ so since C_j is an $\mathcal{X}$-precover of A_j, for each $h|_{X_j} \in \mathrm{Hom}(X_j, A_j)$, there exists $f_j \in \mathrm{Hom}(X_j, C_j)$ such that $h|_{X_j} = g_j f_j$. So define $f : \coprod_{j\in J} X_j \to \coprod_{i\in I} C_i$ by $f|_{X_j} := f_j$ for each $j \in J$ and clearly $gf = h$.
($\Leftarrow$) Let $g : C \to \coprod_{i\in I} A_i = A$ be an $\mathcal{X}$-precover of A. By Lemma 5.6, $C = \coprod_{j\in J} C_j$ for some $J \subseteq I$, and define $C_i := \{c \in C \mid g(c) \in A_i\}$, and $g_i := g|_{C_i}$. For each A_i, given any S-act X with an S-map $h \in \mathrm{Hom}(X, A_i)$, clearly $h \in \mathrm{Hom}(X, A)$ and so by the $\mathcal{X}$-precover property there exists an $f \in \mathrm{Hom}(X, C)$ such that $h = gf$. In fact $g(f(X)) = h(X) \subseteq A_i$ and so $i \in J$ and $f \in \mathrm{Hom}(X, C_i)$ and $h_i = g_i f$. By the hypothesis, $C_i \in \mathcal{X}$, hence $g_i : C_i \to A_i$ is an $\mathcal{X}$-precover of A_i. □

Remark 5.9. Recall, all of the flatness type properties mentioned previously all satisfy $\coprod_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$ (see Corollary 2.26, Corollary 2.30, Corollary 2.35, Theorem 2.34 and Theorem 2.31). So for any of these classes, if we want to show that all S-acts have $\mathcal{X}$-precovers it is enough to show that the indecomposable S-acts have $\mathcal{X}$-precovers.

We now show that colimits of $\mathcal{X}$-precovers are $\mathcal{X}$-precovers. To be more precise

Lemma 5.10. *Let S be a monoid, let $\mathcal{X}$ be a class of S-acts closed under colimits and let A be an S-act. Suppose that $(X_i, \phi_{i,j})$ is a direct system of S-acts with $X_i \in \mathcal{X}$ for each $i \in I$ and with colimit (X, α_i). Suppose also that for each $i \in I$ $f_i : X_i \to A$ is an $\mathcal{X}$-precover of A such that for all $i \leq j$, $f_j \phi_{i,j} = f_i$. Then there exists an $\mathcal{X}$-precover $f : X \to A$ such that $f\alpha_i = f_i$ for all $i \in I$.*

Proof. We have a commutative diagram



and so there exists a unique S-map $f : X \to A$ such that $f\alpha_i = f_i$ for all $i \in I$. If $F \in \mathcal{X}$ and if $g : F \to A$ then for each $i \in I$ there exists $h_i : F \to X_i$ such that $f_i h_i = g$. Choose any $i \in I$ and let $h : F \to X$ be given by $h = \alpha_i h_i$. Then $fh = g$ as required. $\square$

This next result gives us our first necessary condition for the existence of $\mathcal{X}$-(pre)covers.

Lemma 5.11. *An S-act A is an $\mathcal{X}$-precover ($\mathcal{X}$-cover) of the one element S-act Θ_S, if and only if $A \in \mathcal{X}$ and $Hom(X, A) \neq \emptyset$ for all $X \in \mathcal{X}$ (and every endomorphism of A is an isomorphism), that is, A is a weakly terminal object (stably weakly terminal object) in $\mathcal{X}$.*

Proof. Let $A \in \mathcal{X}$, since Θ_S is the terminal object in the category, there exists an S-map $g : A \to \Theta_S$ that sends everything to one element. Given any S-act $X \in \mathcal{X}$ with S-map $h : X \to \Theta_S$, clearly $gf = h$ for every $f \in \text{Hom}(X, A)$ as $g(f(x)) = h(x)$ for all $x \in X$. So A is an $\mathcal{X}$-precover if and only if all the S-acts in $\mathcal{X}$ have an S-map f from X to A, and an $\mathcal{X}$-cover if additionally every endomorphism of A is an isomorphism. $\square$

Corollary 5.12. *If every S-act has an $\mathcal{X}$-cover then there exists a stably weakly terminal object $X \in \mathcal{X}$.*

5.2 Examples of $\mathcal{SF}$-covers

We now give two similar examples of $\mathcal{X}$-covers of the one element S-act Θ_S for the class $\mathcal{X} = \mathcal{SF}$ of strongly flat acts.

5.2.1 The one element act over $(\mathbb{N}, +)$

Let $S = (\mathbb{N}, +)$ be the monoid of natural numbers (with zero) under addition. We will now prove that $\mathbb{Z}$ is a stably weakly terminal object of $\mathcal{SF}$ and hence an $\mathcal{SF}$-cover of Θ_S, the one element S-act.

Lemma 5.13. *$\mathbb{Z}$ is a strongly flat S-act.*

Proof. We show that $\mathbb{Z}$ satisfies Conditions (P) and (E). Let $x, y \in \mathbb{Z}$, $m, n \in S$, and assume $x + m = y + n$. Then without loss of generality we can assume $n \geq m$ and $x - y = n - m = u \in S$. So we have that $x = y + (x - y)$, $y = y + 0$ and $(x - y) + m = (n - m) + m = n = 0 + n$. Hence $\mathbb{Z}$ satisfies Condition (P). Now, if we let $x \in \mathbb{Z}$, $m, n \in S$ and $x + m = x + n$, then $m = n$ and $x = x + 0$ with $0 + m = 0 + n$, so $\mathbb{Z}$ satisfies Condition (E) and is therefore strongly flat. □

Lemma 5.14. *$\mathbb{Z}$ is not cyclic, but is locally cyclic.*

Proof. $\mathbb{Z}$ being cyclic equates to the integers having a least element. It is locally cyclic as given any two integers, they are generated by their minimum. □

Note that $\mathbb{Q}$ is a decomposable S-act, e.g. take the two subacts $A = \mathbb{Z}$ and $B = \mathbb{Q} \setminus \mathbb{Z}$, then $\mathbb{Q} = A \cup B$ and $A \cap B = \emptyset$. Therefore $\mathbb{Q}$ is not locally cyclic.

Lemma 5.15. *$\mathbb{N}$ is not an $\mathcal{SF}$-precover of the one element S-act, Θ_S.*

Proof. Assume there exists a well defined S-map f from $\mathbb{Z}$ to $\mathbb{N}$. So we have $f(x + s) = f(x) + s$ for all $x \in \mathbb{Z}$, $s \in S$. Now by assumption $f(0) = n \in \mathbb{N}$ and so $n = f(0 - n + n)) = f(0 - n) + n$ and $f(0 - n) = 0 \in \mathbb{N}$. But then we have a contradiction $0 = f((0 - n - 1) + 1) = f(0 - n - 1) + 1$ and $f(0 - n - 1) \notin \mathbb{N}$. So by Lemma 5.11 and Lemma 5.13, $\mathbb{N}$ cannot be an $\mathcal{SF}$-precover of Θ_S. □

From now on let X be a strongly flat S-act.

Lemma 5.16. *Define a relation $\leq$ on X by $x \leq y$ if and only if there exists $s \in S$ such that $x + s = y$. Then $(X, \leq)$ is a partial order.*

Proof. 1. For all $x \in X$, $x + 0 = x$, so the relation is reflexive.

2. If $x \leq y$ and $y \leq x$ then there exists $s, t \in S$ such that $x + s = y$ and $y + t = x$, so in particular $x + (s + t) = x$ and Condition (E) tells us there exists $u \in S$ such that $u + (s + t) = u$ so $s + t = 0$ and $s = t = 0$, hence $x = y$ and the relation is antisymmetric.

3. For all, $x, y, z \in X$ with $x \leq y$ and $y \leq z$ we have that there exists $s, t \in S$ such that $x + s = y$ and $y + t = z$, so clearly $x + (s + t) = z$ and so $x \leq z$ and the relation is transitive.

□

Lemma 5.17. *$(X, \leq)$ is a total order if and only if X is indecomposable.*

Proof. $(\Leftarrow)$ Let X be an indecomposable act, since it satisfies Condition (P) it is locally cyclic by Corollary 2.37 and for all $x, y \in A$ there exits $z \in A$, $u, v \in S$ such that $x = z + u$ and $y = z + v$. Now either $u \leq v$ or $v \leq u$, so if we assume $u \leq v$ then $v - u \in S$ and $x + (v - u) = (z + u) + (v - u) = z + (u + v - u) = z + v = y$ and $x \leq y$. Similarly whenever $v \leq u$ we get $y \leq x$. So $(X, \leq)$ is totally ordered.
$(\Rightarrow)$ We take the contrapositive and let X be a decomposable act, then $X = Y \cup Z$ where Y, Z are (non-empty) subacts of X, and $Y \cap Z = \emptyset$. Let $y \in Y$ and $z \in Z$, then $yS \subseteq Y$, $zS \subseteq Z$ are both subacts of X and $yS \cap zS = \emptyset$. Hence neither $y \not\leq z$ nor $z \not\leq y$ and $(X, \leq)$ is not a total order. □

Lemma 5.18. *If X is cyclic then it is isomorphic to $\mathbb{N}$.*

Proof. Let $X = S/\rho$ and since it is strongly flat $s\rho t$ implies there exists $u \in [1]_\rho$ such that $u + s = u + t$ (see [28, Corollary of Result 4]). But this implies $s = t$ so ρ is the identity relation and $X \cong S$. □

Lemma 5.19. *If X is indecomposable but not cyclic then it is isomorphic to $\mathbb{Z}$.*

Proof. We prove this by defining a function from X to $\mathbb{Z}$ and showing it is a well defined bijective S-map.
<u>Function</u>: Let $x \in X$ then for all $y \in X$ by Lemma 5.17 either $y \leq x$ in

which case $x = y + s$ for some $s \in S$ or $x \leq y$ in which case $y = x + t$ for some $t \in S$, with $y \leq x$ and $x \leq y$ only occuring when $y = x$. We now define a function

$$f_x : X \to \mathbb{Z}$$
$$y \mapsto \begin{cases} -s & \text{if } y \leq x \\ t & \text{if } x \leq y. \end{cases}$$

This is well defined when $y = x$ with $f_x(y) = 0$ and for all other $y \in X$, by Condition (E), $x + t_1 = x + t_2$ implies $u + t_1 = u + t_2$ for some $u \in S$ so $t_1 = t_2$, and $y + s_1 = y + s_2$ implies $v + s_1 = v + s_2$ for some $v \in S$ so $-s_1 = -s_2$. Hence for all $y_1, y_2 \in X$, $y_1 = y_2 \Rightarrow f_x(y_1) = f_x(y_2)$ and the function is well defined.

<u>S**-map:**</u> To show f_x is an S-map we consider the two cases. Firstly when $y \leq x$: given any $t \in S$ we have two options, either $s - t \in S$ or $t - s \in S$. When $s-t \in S$, $x = y+s \Rightarrow x = (y+s)+(t-t) = y+(t+s-t) = (y+t)+(s-t)$ and $y + t \leq x$ with $f_x(y + t) = -(s - t) = -s + t = f_x(y) + t$. Otherwise $t - s \in S$, in which case $x + (t - s) = (y + s) + (t - s) = y + (s + t - s) = y + t$ and $x \leq y + t$ with $f_x(y + t) = t - s = -s + t = f_x(y) + t$. Secondly when $x \leq y$, $f_x(y + s) = f_x((x + t) + s) = f_x(x + (t + s)) = t + s = f_x(y) + s$. Hence for all $y \in X, s \in S$, $f_x(y + s) = f_x(y) + s$ and f_x is a well defined S-map.

<u>**Injective:**</u> To show injectivity we first observe that $-s$ only equals t when $s = t = 0$ hence if $y_1 \leq x$ and $f_x(y_1) = f_x(y_2)$ then $y_2 \leq x$ and similarly if $x \leq y_1$ and $f_x(y_1) = f_x(y_2)$ then $x \leq y_2$. Again we consider the two cases: firstly when $y_1 \leq x$, $f_x(y_1) = f_x(y_2) = s$ implies $y_1 + s = y_2 + s$ and by Condition (P) there exists $z \in A$, $u, v \in S$ with $y_1 = z + u$, $y_2 = z + v$ and $u + s = v + s \Rightarrow u = v \Rightarrow y_1 = y_2$. Secondly when $x \leq y_1$, $f_x(y_1) = f_x(y_2)$ clearly implies $y_1 = x + f_x(y_1) = x + f_x(y_2) = y_2$. Hence for all $y_1, y_2 \in X$, $f_x(y_1) = f_x(y_2) \Rightarrow y_1 = y_2$ and f_x is an injective S-map.

<u>**Surjective:**</u> Since f_x is an S-map and $f_x(y + t) = f_x(y) + t$, given the base case $f_x(x) = 0$, by induction $\mathbb{N} \subseteq im(f_x)$. We now need to show that $-\mathbb{N} \subseteq im(f_x)$. Given any $y_i \in X$ we show there exists $y_{i+1} \in X$ with $y_i = y_{i+1} + 1$. Let $y_i \in X$ and since X is not cyclic we can find $z \in X$ with $z \notin y_i S$ which means, by totality of $(X, \leq)$, $y_i = z + t$ for some $t \geq 1$ and

$t-1 \in S$. Now let $y_{i+1} = z+(t-1)$ and $y_i = z+t = (z+(t-1))+1 = y_{i+1}+1$. Hence given any $y_i \in X$ we can find $y_{i+1} \in X$ with $f_x(y_{i+1}) = f_x(y) - 1$. So let $y_0 = x$ and given $f_x(y_0) = 0$, by induction $-\mathbb{N} \subseteq im(f_x)$. Hence $\mathbb{Z} \subseteq im(f_x)$ and f_x is surjective. □

Corollary 5.20. *The only indecomposable strongly flat S-acts are $\mathbb{N}$ and $\mathbb{Z}$.*

Proposition 5.21. *$\mathbb{Z}$ is an $\mathcal{SF}$-precover of Θ_S.*

Proof. Every S-act is a coproduct of indecomposable S-acts by Theorem 2.24, and strongly flat acts decompose into strongly flat acts by Corollary 2.35 which by Corollary 5.20 means every strongly flat S-act is a coproduct of copies of $\mathbb{N}$ and $\mathbb{Z}$, both of which factor through $\mathbb{Z}$ in the obvious way. So send each disjoint copy into $\mathbb{Z}$ and clearly the whole coproduct factors through $\mathbb{Z}$ so by Lemma 5.11 it is an $\mathcal{SF}$-precover of Θ_S. □

Similarly $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ etc are precovers because $\mathbb{Z}$ injects into them. But we now show that $\mathbb{Q}$ is not stable and so cannot be the $\mathcal{SF}$-cover of Θ_S.

Lemma 5.22. *$\mathbb{Q}$ is not an $\mathcal{SF}$-cover of Θ_S.*

Proof. Assume $g : \mathbb{Q} \to \Theta_S$ is an $\mathcal{SF}$-cover of Θ_S. $\mathbb{Z}$ is a proper subact of $\mathbb{Q}$, with inclusion map $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Now let $f : \mathbb{Q} \to \mathbb{Z}$ be the floor function. For all $x \in \mathbb{Q}$, $s \in S$, $f(x+s) = \lfloor x+s \rfloor = \lfloor x \rfloor + s = f(x) + s$, so f is an S-map. Therefore if is an isomorphism which means the inclusion map i is an epirmorphism and so $\mathbb{Z} = \mathbb{Q}$ which is a contradiction.

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{g} & \Theta_S \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle g} \\ \mathbb{Z} & \xleftarrow[f]{} & \mathbb{Q} \end{array}$$

□

We now show that $\mathbb{Z}$ is stable.

Lemma 5.23. *Every S-map from $\mathbb{Z}$ to $\mathbb{Z}$ is an isomorphism.*

Proof. Let $f : \mathbb{Z} \to \mathbb{Z}$ be an S-map, then $f(x+s) = f(x) + s$ for all $x \in \mathbb{Z}$, $s \in S$. Now $f(0) = z$ for some $z \in \mathbb{Z}$, and so for all $x < 0$, $-x \in S$ and $z = f(0) = f(x + (-x)) = f(x) + (-x)$ hence $f(x) = z + x \in \mathbb{Z}$. Similarly when $x \geq 0$, $x \in S$ and so $f(x) = f(0+x) = f(0) + x = z + x$. So whenever $f(x_1) = f(x_2)$ we have $z + x_1 = z + x_2 \Rightarrow x_1 = x_2$ and f is injective. Also, for all $y \in \mathbb{Z}$ we know $f(y - z) = z + (y - z) = y$, hence f is surjective and the map is an isomorphism. □

Theorem 5.24. *$\mathbb{Z}$ is the $\mathcal{SF}$-cover of Θ_S.*

Proof. By Proposition 5.21, $\mathbb{Z}$ is an $\mathcal{SF}$-precover of Θ_S. Now given any S-map $f : \mathbb{Z} \to \mathbb{Z}$, by Lemma 5.23 f is an isomorphism, so it is also an $\mathcal{SF}$-cover. □

5.2.2 The one element act over $(\mathbb{N}, \cdot)$

In the last example we characterised all the strongly flat acts up to isomorphism before we found the $\mathcal{SF}$-cover of Θ_S. Let $S = (\mathbb{N}, \cdot)$ be the monoid of positive integers under multiplication. We now have a very similar set of results, except unlike the previous example, there are infinitely many indecomposable strongly flat acts, but we can show they all inject into $\mathbb{Q}^+$ which is the $\mathcal{SF}$-cover of Θ_S.

Lemma 5.25. *$\mathbb{Q}^+ := \{\frac{a}{b} : a, b \in \mathbb{N}\}$ is a strongly flat S-act.*

Proof. Let $r, s \in \mathbb{Q}^+$, $m, n \in \mathbb{N}$. Whenever $rm = sn$, we have $r = \frac{s}{m}n$ so let $t = \frac{s}{m} \in \mathbb{Q}^+$ and $u = n, v = m \in \mathbb{N}$, then $r = tu, s = tv$ and $um = vn$ so $\mathbb{Q}^+$ satisfies Condition (P). Also $rm = rn \Rightarrow m = n$, so let $t = r, u = m = n$ and Condition (E) is also satisfied. □

Note that $\mathbb{R}^+$ is a decomposable S-act, e.g. take the two subacts $A = \mathbb{Q}^+$ and $B = \mathbb{R}^+ \setminus \mathbb{Q}^+$, then $\mathbb{R}^+ = A \cup B$ and $A \cap B = \emptyset$ so $\mathbb{R}^+$ is not locally cyclic.

Lemma 5.26. *$\mathbb{N}$ is not an $\mathcal{SF}$-precover of Θ_S.*

Proof. Assume there exists a well defined S-map f from $\mathbb{Q}^+$ to $\mathbb{N}$. So we have $f(qm) = f(q)m$ for all $q \in \mathbb{Q}^+$, $m \in S$. Now by assumption $f(1) =$

$n \in \mathbb{N}$ and so $n = f(\frac{1}{n}n) = f(\frac{1}{n})n$ and $f(\frac{1}{n}) = 1 \in \mathbb{N}$. But then we have $1 = f(\frac{1}{2n}2) = f(\frac{1}{2n})2$ and $f(\frac{1}{2n}) \notin \mathbb{N}$ which is a contradiction. So by Lemma 5.11 and Lemma 5.25, $\mathbb{N}$ cannot be an $\mathcal{SF}$-precover of Θ_S. □

From now on let X be a strongly flat S-act.

Lemma 5.27. *Define a relation $\leq$ on X by $x \leq y$ if and only if there exists $t \leq s \in S$ such that $xs = yt$. Then $(X, \leq)$ is a partial order.*

Proof.

1. For all $x \in X$, $x1 = x1$ and $1 \leq 1$, so the relation is reflexive.
2. If $x \leq y$ and $y \leq x$ then there exists $t_1 \leq s_1$ and $s_2 \leq t_2$ in S such that $xs_1 = yt_1$ and $xs_2 = yt_2$. By Condition (P), there exists $z \in X$ and $u, v \in S$ such that $x = zu$, $y = zv$ and $us_1 = vt_1$. Since $t_1 \leq s_1$, this implies $u \leq v$. Now $(zu)s_2 = (zv)t_2$ and so by Condition (E), there exists some $w \in S$ such that $wus_2 = wvt_2$ which implies $us_2 = vt_2$. Again, since $s_2 \leq t_2$, we have $v \leq u$ which implies $u = v$. Therefore $x = zu = zv = y$ and the relation is antisymmetric.
3. For all, $x, y, z \in X$ with $x \leq y$ and $y \leq z$ we have that there exists $t_1 \leq s_1$ and $t_2 \leq s_2$ in S such that $xs_1 = yt_1$ and $ys_2 = zt_2$. Then $xs_1s_2 = yt_1s_2 = ys_2t_1 = zt_2t_1 = zt_1t_2$ and $t_1t_2 \leq s_1s_2$ so the relation is transitive.

□

Lemma 5.28. *$(X, \leq)$ is a total order if and only if X is an indecomposable act.*

Proof. $(\Leftarrow)$ Let X be an indecomposable act, since it satisfies Condition (P) it is locally cyclic by Corollary 2.37 and for all $x, y \in A$ there exits $z \in A$, $u, v \in S$ such that $x = zu$ and $y = zv$. Now $xv = zuv = zvu = yu$ and either $u \leq v$ in which case $x \leq y$ or $v \leq u$ in which case $y \leq x$. So $(X, \leq)$ is totally ordered.
$(\Rightarrow)$ Assume $(X, \leq)$ is a total order. Then given any $x, y \in X$, there exists $s, t \in S$ such that $xs = yt$, and so x is in the same component as y and X is indecomposable. □

Lemma 5.29. *If X is cyclic then it is isomorphic to $\mathbb{N}$.*

Proof. Let $X = S/\rho$ and since it is strongly flat $s\rho t$ implies there exists $u \in [1]_\rho$ such that $us = ut$ (see [28, Corollary of Result 4]). Since u is positive this implies $s = t$ so ρ is the identity relation and $X \cong S$. □

Lemma 5.30. *If X is indecomposable but not cyclic then it injects in to $\mathbb{Q}^+$.*

Proof. We prove this by defining a function from X to $\mathbb{Q}^+$ and showing it is a well defined injective S-map.

Function: Let $x \in X$ then for all $y \in X$ by Lemma 5.28 either $x \leq y$ in which case $xs = yt$ for some $t \leq s \in S$ or $y \leq x$ in which case $xs = yt$ for some $s \leq t \in S$, with $y \leq x$ and $x \leq y$ only occurring when $y = x$. We now define a function

$$\begin{aligned} f_x : X &\to \mathbb{Q}^+ \\ y &\mapsto \frac{s}{t}. \end{aligned}$$

Let $xs = yt$ and $xs' = yt'$, then by Condition (P), there exists some $z \in X$, and $u, v \in S$ such that $x = zu$, $y = zv$ and $us = vt$ so that $\frac{s}{t} = \frac{v}{u}$. Therefore $zus' = zvt'$ and by Condition (E) there exists some $w \in S$ such that $wus' = wvt'$ and $\frac{s'}{t'} = \frac{v}{u} = \frac{s}{t}$ and the function is well defined.

S-map: To show f_x is an S-map let $f_x(y) = \frac{s}{t}$ with $xs = yt$, and consider $f_x(yw) = \frac{s'}{t'}$ with $xs' = (yw)t'$ for some $w \in S$. By Condition (P), there exists some $z \in X$, $u, v \in S$ such that $x = zu$, $y = zv$ and $us = vt$ hence $\frac{s}{t} = \frac{v}{u}$. Now $zus' = zvwt'$ and by Condition (E) there exists some $w' \in S$ such that $w'us' = w'vwt'$ and $\frac{s'}{t'} = \frac{v}{u}w = \frac{s}{t}w$ and f_x is a well defined S-map.

Injective: To show injectivity let $f_x(y) = f_x(y') = \frac{s}{t}$ for some $y, y' \in X$. Then $xs = yt$ and $xs = y't$, so $yt = y't$. By Condition (P), there exists some $z \in X$ and $u, v \in S$ such that $y = zu$, $y' = zv$ and $ut = vt$. Hence $u = v$ so $y = y'$ and f_x is an injective S-map. □

Corollary 5.31. *Every indecomposable strongly flat S-act injects in to $\mathbb{Q}^+$.*

Note that not every non-cyclic indecomposable strongly flat S-act is isomorphic to $\mathbb{Q}^+$. For example the dyadic rationals.

Lemma 5.32. *The dyadic rationals, $\bigcup_{n\geq 0} \frac{\mathbb{N}}{2^n}$, are a strongly flat locally cyclic non-cyclic S-act not isomorphic to $\mathbb{Q}^+$.*

Proof. Let $X = \bigcup_{n\geq 0} \frac{\mathbb{N}}{2^n}$. Firstly, it is clear that multiplication by $\mathbb{N}$ gives rise to a well-defined S-act structure on X. Assume there exists some $x = \frac{a}{2^m} \in X$ such that $X = xS$ then it would not include $\frac{a}{2^{m+1}} \in X$, so X is not cyclic. Given any $x = \frac{a}{2^m}, y = \frac{b}{2^n} \in X$, let $z = \frac{1}{2^{m+n}}$, then $x, y \in zS$, so X is locally cyclic. Given $\frac{a}{2^m}s = \frac{a'}{2^n}t$ take $a'' = \frac{1}{2^{m+n}}$, $u = 2^n a$, $v = 2^m a'$ then $\frac{a}{2^m} = \frac{1}{2^{m+n}}2^n a$, $\frac{a'}{2^n} = \frac{1}{2^{m+n}}2^m a'$ and $2^n as = 2^m a't$ so X satisfies Condition (P). Condition (E) is satisfied since X is left cancellative, so it is strongly flat. Assume there exists an S-map f from $\mathbb{Q}^+$ to X then $f(1) = \frac{a}{2^n}$ for some $a \in \mathbb{N}$, $n \geq 0$. Now since $f(1) = f(\frac{1}{3a}3a) = f(\frac{1}{3a})3a$ we have that $f(\frac{1}{3a}) = \frac{1}{3\cdot 2^n} \notin X$ so $X \ncong \mathbb{Q}^+$. □

It is clear from this last Lemma that there are in fact infinitely many indecomposable strongly flat S-acts, very different from the previous example.

Proposition 5.33. *$\mathbb{Q}^+$ is an $\mathcal{SF}$-precover of Θ_S.*

Proof. Every S-act is the disjoint union of indecomposable S-acts by Theorem 2.24, and strongly flat acts decompose into strongly flat acts by Corollary 2.35 which by Corollary 5.31 means every strongly flat S-act injects in to $\mathbb{Q}^+$ by taking a map for each disjoint S-act into $\mathbb{Q}$. So by Lemma 5.11, $\mathbb{Q}^+$ is an $\mathcal{SF}$-precover of Θ_S. □

Similarly $\mathbb{R}$ and field extensions etc are precovers as $\mathbb{Q}$ injects in to them.

Theorem 5.34. *$\mathbb{Q}^+$ is the $\mathcal{SF}$-cover of Θ_S.*

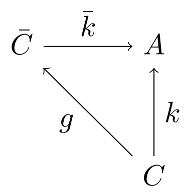
Proof. Since $\mathbb{Q}^+$ is indecomposable by Proposition 5.33 it is enough to show that any S-map $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ is an isomorphism. To see this first note that for all $\frac{a}{b} \in \mathbb{Q}^+$, $f(\frac{a}{b}) = f(1)\frac{a}{b}$, in fact, $f(\frac{a}{b}) = f(\frac{1}{b})a$ and so $\frac{f(\frac{a}{b})}{a} = f(\frac{1}{b}) \Rightarrow f(1) = f(\frac{1}{b})b = f(\frac{a}{b})\frac{b}{a}$ and so $f(\frac{a}{b}) = f(1)\frac{a}{b}$. Therefore whenever $f(\frac{a}{b}) = f(\frac{c}{d}) \Rightarrow f(1)\frac{a}{b} = f(1)\frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d}$ and f is injective. Additionally, for all $\frac{a}{b} \in \mathbb{Q}^+$, $f(\frac{a}{f(1)b}) = f(1)\frac{a}{f(1)b} = \frac{a}{b}$ and f is also surjective. □

In these last two examples, we get an idea of how much harder it is to study $\mathcal{X}$-covers than it is coessential covers. In particular, we need to be able to say something about *all* acts with a certain property rather than just studying one act. The examples considered here are for two very simple monoids and we showed that the one element S-act has an $\mathcal{SF}$-cover by characterising all strongly flat acts. This is not practical for general monoids. In the rest of this Chapter we prove some results for general monoids and classes $\mathcal{X}$ which we can then apply to specific monoids and classes of acts in Chapter 6.

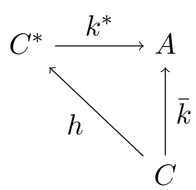
5.3 Precover implies cover

In this section we show that if a class $\mathcal{X}$ is closed under directed colimits, then an S-act A having an $\mathcal{X}$-precover is sufficient for A having an $\mathcal{X}$-cover. The argument used in this proof is similar to the approach first used in Enochs' original paper (see [25, Theorem 3.1] and [57, Theorem 2.2.8]).

Lemma 5.35. *Let S be a monoid, $\mathcal{X}$ a class of S-acts closed under directed colimits and $k : C \to A$ an $\mathcal{X}$-precover of A. Then there exists an $\mathcal{X}$-precover $\bar{k} : \bar{C} \to A$ and a commutative diagram*



such that for any $\mathcal{X}$-precover $k^* : C^* \to A$ and any commutative diagram



$\ker(hg) = \ker(g)$ *(i.e. the kernel of g is in some sense maximal).*

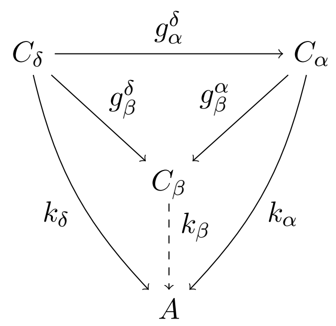
Proof. Let S be a monoid and $k_0 : C_0 \to A$ be an $\mathcal{X}$-precover of A. Assume, by way of contradiction, that for all $\mathcal{X}$-precovers $\bar{k} : \bar{C} \to A$ and S-maps $g : C_0 \to \bar{C}$ with $\bar{k}g = k_0$, there exists an $\mathcal{X}$-precover $k^* : C^* \to A$ and an S-map $h : \bar{C} \to C^*$ with $k^*h = \bar{k}$ such that $\ker(hg) \neq \ker(h)$, that is $\ker(g) \subsetneq \ker(hg)$ as clearly $\ker(hg) \subseteq \ker(g)$.

We intend to show, by transfinite induction, that for each ordinal γ, there is an $\mathcal{X}$-precover (C_γ, k_γ) of A and for all $\beta < \gamma$ there exist S-maps $g_\gamma^\beta : C_\beta \to C_\gamma$ such that $k_\gamma g_\gamma^\beta = k_\beta$ and $g_\gamma^\beta g_\beta^\alpha = g_\gamma^\alpha$ with $\ker(g_\beta^0) \subsetneq \ker(g_\gamma^0)$ for all $\alpha < \beta < \gamma$.

1. Base step: (C_0, k_0) satisfies the statement.

2. Successor step: Assume the statement is true for some $\beta < \gamma$, and let $(\bar{C}, \bar{k}) = (C_\beta, k_\beta)$ and $g = g_\beta^0$ then there exists an $\mathcal{X}$-precover $(C_{\beta+1}, k_{\beta+1}) := (C^*, k^*)$ and an S-map $g_{\beta+1}^\beta := h : C_\beta \to C_{\beta+1}$ with $k_{\beta+1} g_{\beta+1}^\beta = k_\beta$ such that $\ker(g_\beta^0) \subsetneq \ker(g_{\beta+1}^\beta g_\beta^0)$. Now define, $g_{\beta+1}^\alpha := g_{\beta+1}^\beta g_\beta^\alpha$ for all $\alpha < \beta$. Then $k_{\beta+1} g_{\beta+1}^\alpha = k_{\beta+1} g_{\beta+1}^\beta g_\beta^\alpha = k_\beta g_\beta^\alpha = k_\alpha$ and $g_{\beta+1}^\alpha g_\alpha^\delta = g_{\beta+1}^\beta g_\beta^\alpha g_\alpha^\delta = g_{\beta+1}^\beta g_\beta^\delta = g_{\beta+1}^\delta$ with $\ker(g_\alpha^0) \subsetneq \ker(g_\beta^0) \subsetneq \ker(g_{\beta+1}^\beta g_\beta^0) = \ker(g_{\beta+1}^0)$ for all $\delta < \alpha < \beta$. Thus the statement is true for $\beta + 1$.

3. Limit step: If $\beta < \gamma$ is a limit ordinal, assume the statement is true for all $\alpha < \beta$. Let $g_\delta^\delta = id_{C_\delta}$, then $(C_\alpha, g_\alpha^\delta)_{\alpha<\beta}$ is a direct system of S-acts over the directed index set β. Let $(C_\beta, g_\beta^\delta)_{\alpha\in\beta}$ be the directed colimit. Then by the colimit property there exists a unique S-map $k_\beta : C_\beta \to A$ such that the following diagram



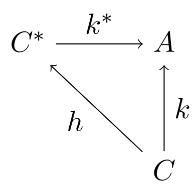
commutes. Since $\mathcal{X}$ is closed under directed colimits, $C_\beta \in \mathcal{X}$, and by Lemma 5.10, (C_β, k_β) is an $\mathcal{X}$-precover of A. Now $g_\beta^0 = g_\beta^\alpha g_\alpha^0$ and so $\ker(g_\alpha^0) \subseteq \ker(g_\beta^0)$ but $\ker(g_\alpha^0) \subsetneq \ker(g_{\alpha+1}^0) \subseteq \ker(g_\beta^0)$. So the statement is true for β.

Therefore, by Theorem 1.5, the statement is true for any ordinal γ and hence

$$\ker(g_1^0) \subsetneq \ker(g_2^0) \subsetneq \cdots \subsetneq \ker(g_\gamma^0) \subsetneq C_0 \times C_0$$

which implies $|C_0 \times C_0| \geq |\gamma|$ which is clearly a contradiction. $\square$

Given an $\mathcal{X}$-precover $k : C \to A$ of A, we say that (C, k) satisfies the **mono-lifting property** if for any $\mathcal{X}$-precover $k^* : C^* \to A$ and any commutative diagram



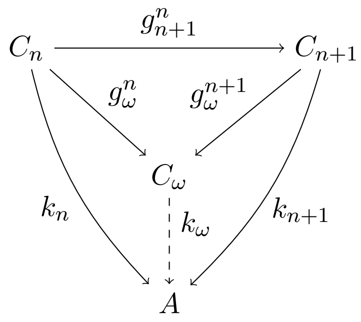
h must be a monomorphism.

Lemma 5.36. *Let S be a monoid, $\mathcal{X}$ a class of S-acts closed under directed colimits. If A has an $\mathcal{X}$-precover, then it has an $\mathcal{X}$-precover with the mono-lifting property.*

Proof. Let (C_0, k_0) be an $\mathcal{X}$-precover of A, then by Lemma 5.35 there exists an $\mathcal{X}$-precover (C_1, k_1) of A and an S-map $g_1^0 : C_0 \to C_1$ with $k_1 g_1^0 = k_0$ such that for any $\mathcal{X}$-precover (C^*, k^*) of A and any S-map $h : C_1 \to C^*$ with $k^* h = k_1$ then $\ker(h g_1^0) = \ker(g_1^0)$, or equivalently, $h|_{\mathrm{im}(g_1^0)}$ is a monomorphism. By way of induction, assume that there is an $\mathcal{X}$-precover $k_n : C_n \to A$ and an S-map $g_n^{n-1} : C_{n-1} \to C_n$ with $k_n g_n^{n-1} = k_{n-1}$ and such that for any $\mathcal{X}$-precover $k^* : C^* \to A$ and any S-map $h : C_n \to C^*$ with $k^* h = k_n$ then $h|_{\mathrm{im}(g_n^{n-1})}$ is a monomorphism.

Then by Lemma 5.35, there exists an $\mathcal{X}$-precover (C_{n+1}, k_{n+1}) of A and an S-map $g_{n+1}^n : C_n \to C_{n+1}$ with $k_{n+1} g_{n+1}^n = k_n$ and such that for any $\mathcal{X}$-precover (C^*, k^*) of A and any S-map $h : C_{n+1} \to C^*$ with $k^* h = k_{n+1}$ then $h|_{\mathrm{im}(g_{n+1}^n)}$ is a monomorphism.

Now let (C_ω, g_ω^n) be the directed colimit of the direct system $(C_n, g_{n+1}^n)_{n \in \mathbb{N}}$ and let $k_\omega : C_\omega \to A$ be the unique S-map that makes the following diagram



commute. Since $\mathcal{X}$ is closed under directed colimits, by Lemma 5.10, (C_ω, k_ω) is an $\mathcal{X}$-precover of A. We claim that this $\mathcal{X}$-precover has the mono-lifting property. So let (C^*, k^*) be any $\mathcal{X}$-precover of A and let $h : C_\omega \to C^*$ be an S-map with $k^*h = k_\omega$. Suppose also that $h(x) = h(y)$ for $x, y \in C_\omega$. Then there exists $m, n \in \mathbb{N}$ and $x_m \in C_m, y_n \in C_n$ such that $g^m_\omega(x_m) = x$ and $g^n_\omega(y_n) = y$. Assume without loss of generality that $m \leq n$ and let $z_n = g^m_n(x_m)$. Then

$$hg^{n+1}_\omega(g^n_{n+1}(z_n)) = hg^n_\omega(z_n) = hg^n_\omega(y_n) = hg^{n+1}_\omega(g^n_{n+1}(y_n)).$$

But $hg^{n+1}_\omega : C_{n+1} \to C^*$ and $hg^{n+1}_\omega|_{\mathrm{im}(g^n_{n+1})}$ is therefore a monomorphism. Hence $g^n_{n+1}(z_n) = g^n_{n+1}(y_n)$ and so

$$x = g^m_\omega(x_m) = g^{n+1}_\omega(g^n_{n+1}(z_n)) = g^{n+1}_\omega(g^n_{n+1}(y_n)) = g^n_\omega(y_n) = y$$

as required. □

Theorem 5.37. *Let S be a monoid, A an S-act and $\mathcal{X}$ a class of S-acts closed under directed colimits. If A has an $\mathcal{X}$-precover then A has an $\mathcal{X}$-cover.*

Proof. By Lemma 5.36, there exists an $\mathcal{X}$-precover (C_0, k_0) of A with the mono-lifting property. We show that A has an $\mathcal{X}$-cover.

Assume, by way of contradiction, that A does not have an $\mathcal{X}$-cover. Then given any $\mathcal{X}$-precover $(\bar{C}, \bar{k})$ of A with the mono-lifting property, there exists $\bar{g} : \bar{C} \to \bar{C}$ with $\bar{k}\bar{g} = \bar{k}$ and such that $\bar{g}$ is a monomorphism but not an epimorphism, and so $\mathrm{im}(\bar{g}) \subsetneq \bar{C}$.

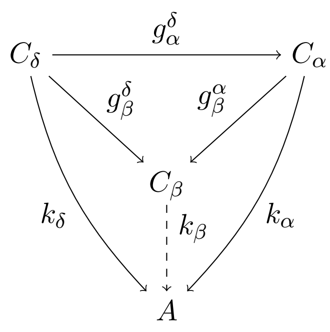
We intend to show, by transfinite induction, that for each ordinal γ there exists an $\mathcal{X}$-precover (C_γ, k_γ) of A with the mono-lifting property with

$C_\gamma \subseteq C_0$, and for all $\beta < \gamma$ there exist S-maps $g_\gamma^\beta : C_\beta \to C_\gamma$ with $k_\gamma g_\gamma^\beta = k_\beta$ which are monomorphisms but not epimorphisms such that $g_\gamma^\beta g_\beta^\alpha = g_\gamma^\alpha$ and so $\mathrm{im}(g_\gamma^\alpha) \subsetneq \mathrm{im}(g_\gamma^\beta)$ for all $\alpha < \beta < \gamma$.

1. Base step: (C_0, k_0) clearly satisfies the statement.

2. Successor step: Assume the statement is true for some $\beta < \gamma$. Now let $(C_{\beta+1}, k_{\beta+1}) := (C_\beta, k_\beta)$, which clearly satisfies the mono-lifiting property and $C_{\beta+1} = C_\beta \subseteq C_0$. By the assumption there exists $g_{\beta+1}^\beta : C_\beta \to C_{\beta+1}$ with $k_{\beta+1} g_{\beta+1}^\beta = k_\beta$ which is a monomorphism but not an epimorphism. For all $\alpha < \beta$, define $g_{\beta+1}^\alpha := g_{\beta+1}^\beta g_\beta^\alpha$, so that $k_{\beta+1} g_{\beta+1}^\alpha = k_{\beta+1} g_{\beta+1}^\beta g_\beta^\alpha = k_\beta g_\beta^\alpha = k_\alpha$ and since both $g_{\beta+1}^\beta$ and all of the g_β^α are monomorphisms but not epimorphisms, then so are all the $g_{\beta+1}^\alpha$. Now $g_{\beta+1}^\alpha g_\alpha^\delta = g_{\beta+1}^\beta g_\beta^\alpha g_\alpha^\delta = g_{\beta+1}^\beta g_\beta^\delta = g_{\beta+1}^\delta$ and so $\mathrm{im}(g_{\beta+1}^\delta) \subsetneq \mathrm{im}(g_{\beta+1}^\alpha)$ for all $\delta < \alpha < \beta$. Thus the statement is true for $\beta + 1$.

3. Limit step: If $\beta < \gamma$ is a limit ordinal, assume the statement is true for all $\alpha < \beta$. Let $g_\delta^\delta = id_{C_\delta}$, then $(C_\delta, g_\alpha^\delta)_{\alpha<\beta}$ is a direct system of S-acts over the directed index set β. Let $(C_\beta, g_\beta^\delta)_{\delta\in\beta}$ be the directed colimit, then by Lemma 2.12, all of the g_β^δ are monomorphisms. By the colimit property there exists a unique S-map $k_\beta : C_\beta \to A$ such that the following diagram



commutes. Since $\mathcal{X}$ is closed under directed colimits, by Lemma 5.10, (C_β, k_β) is an $\mathcal{X}$-precover of A. Given any $\mathcal{X}$-precover (C^*, k^*) of A and any S-map $h : C_\beta \to C^*$ with $k^* h = k_\beta$, then for all $\alpha < \beta$, $h g_\beta^\alpha$ are monomorphisms and so by Lemma 2.13, h must be a monomorphism and (C_β, k_β) satisfies the mono-lifting property. In particular, there is a monomorphism from C_β to C_0. We now show that each g_β^α is not an epimorphism. This is clear, as if there existed an $\alpha < \beta$ with $g_\beta^\alpha = g_\beta^{\alpha+1} g_{\alpha+1}^\alpha$ an epimorphism, then $g_{\alpha+1}^\alpha$ would also be an epimorphism which is a contradiction. So the statement is true for β.

Therefore, by Theorem 1.5, the statement is true for any ordinal γ and hence

$$\mathrm{im}(g_1^0) \subsetneq \mathrm{im}(g_2^0) \subsetneq \cdots \subsetneq \mathrm{im}(g_\gamma^0) \subseteq C_0$$

which implies $|C_0| \geq |\gamma|$ which is a contradiction. □

5.4 Weak solution set condition

We now give a necessary and sufficient condition for existence of $\mathcal{X}$-precovers.

It is clear that a necessary condition for an S-act A to have an $\mathcal{X}$-precover is that there exists $X \in \mathcal{X}$ with $\mathrm{Hom}(X, A) \neq \emptyset$. This condition is always satisfied in the category of modules over a ring (or indeed any category with a zero object), as every Hom-set is always non-empty, but this is not always the case for S-acts.

Let S be a monoid and let $\mathcal{X}$ be a class of S-acts. Borrowing terminology from Freyd's Adjoint Functor Theorem [29], we say that $\mathcal{X}$ satisfies the **(weak) solution set condition** if for all S-acts A there exists a set (rather than a proper class) $S_A \subseteq \mathcal{X}$ such that for all (indecomposable) $X \in \mathcal{X}$ and all S-maps $h : X \to A$ there exists $Y \in S_A$, $f : X \to Y$ and $g : Y \to A$ such that $h = gf$.

Theorem 5.38. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that $\coprod_{i \in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$. Then every S-act has an $\mathcal{X}$-precover if and only if*

1. *for every S-act A there exists an X in $\mathcal{X}$ such that $Hom(X, A) \neq \emptyset$;*

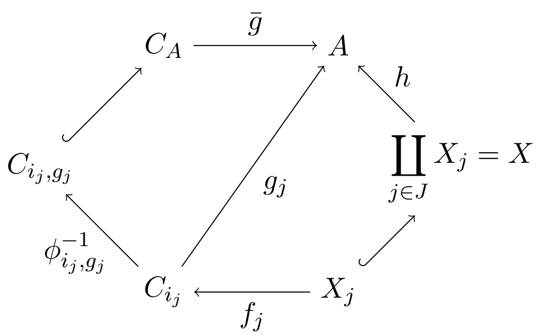
2. *$\mathcal{X}$ satisfies the weak solution set condition;*

Proof. Suppose that $\mathcal{X}$ satisfies the given conditions. Let A be an S-act and let $S_A = \{C_i \mid i \in I\}$ be as given in the weak solution set condition. Notice that by property (1) $S_A \neq \emptyset$. Moreover we can assume that for all $Y \in S_A$, $\text{Hom}(Y, A) \neq \emptyset$ as $S_A \setminus \{Y \in S_A \mid \text{Hom}(Y, A) = \emptyset\}$ will also satisfy the requirements of the solution set condition.

For each $i \in I$ and for each S-map $g : C_i \to A$ let $C_{i,g}$ be an isomorphic copy of C_i with isomorphism $\phi_{i,g} : C_{i,g} \to C_i$ (recall that we are assuming that $\mathcal{X}$ is closed under isomorphisms). Let

$$C_A := \coprod_{\substack{i \in I \\ g \in \text{Hom}(C_i, A)}} C_{i,g}.$$

By hypothesis, $C_A \in \mathcal{X}$ and we can define an S-map $\bar{g} : C_A \to A$ by $\bar{g}|_{C_{i,g}} = g\phi_{i,g}$ for each $i \in I$, $g \in \text{Hom}(C_i, A)$. We claim that $(C_A, \bar{g})$ is an $\mathcal{X}$-precover for A. Let $X \in \mathcal{X}$ and let $h : X \to A$ be an S-map. By the hypothesis $X = \coprod_{j \in J} X_j$ is a coproduct of indecomposable S-acts with $X_j \in \mathcal{X}$ for each $j \in J$. Further, by the hypothesis, there exists $C_{i_j} \in S_A$, $f_j : X_j \to C_{i_j}$ and $g_j : C_{i_j} \to A$ such that $g_j f_j = h|_{X_j}$. Now $\bar{g}|_{C_{i_j,g_j}} \phi^{-1}_{i_j,g_j} = g_j$ and so both squares and the outer hexagon in the following diagram



commutes. So define $f : X \to C_A$ by $f|_{X_j} = \phi^{-1}_{i_j,g_j} f_j$ and note that $\bar{g}f = h$ as required.

Conversely if A is an S-act with an $\mathcal{X}$-precover C_A, then $\text{Hom}(C_A, A) \neq \emptyset$ and on putting $S_A = \{C_A\}$ we see that $\mathcal{X}$ satisfies the (weak) solution set condition. $\square$

Note from the proof of Theorem 5.38 that we can also deduce

Theorem 5.39. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that $X_i \in \mathcal{X}$ for each $i \in I \Rightarrow \coprod_{i\in I} X_i \in \mathcal{X}$. Then every S-act has an $\mathcal{X}$-precover if and only if*

1. *for every S-act A there exists an X in $\mathcal{X}$ such that $Hom(X, A) \neq \emptyset$;*
2. *$\mathcal{X}$ satisfies the solution set condition;*

Corollary 5.40. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that*

1. *$\coprod_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$;*
2. *for every S-act A there exists an X in $\mathcal{X}$ such that $Hom(X, A) \neq \emptyset$;*
3. *there exists a cardinal λ such that for every indecomposable X in $\mathcal{X}$, $|X| < \lambda$.*

Then every S-act has an $\mathcal{X}$-precover.

Proof. By (3) and Remark 2.65 there are only a set $C = \{C_i : i \in I\}$ of isomorphic representatives of indecomposable S-acts in $\mathcal{X}$. Suppose that A is an S-act and let $S_A = C$. If $X \in \mathcal{X}$ is indecomposable and if $h : X \to A$ is an S-map then there exists an isomorphism $\phi : X \to C_i$ for some $C_i \in C$ and we have an S-map $h\phi^{-1} : C_i \to A$ and clearly $h = h\phi^{-1}\phi$ and so $\mathcal{X}$ satisfies the weak solution set condition. □

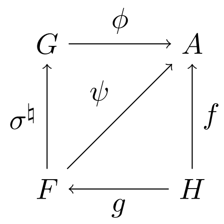
5.5 Weakly congruence pure

The inspiration for some of the following results comes from [57].

Recall the definitions from Section 4.1.2 of $\mathcal{X}$-pure congruences.

Theorem 5.41. *Let S be a monoid, let $\mathcal{X}$ be a class of S-acts closed under chains of $\mathcal{X}$-pure congruences and suppose that A is an S-act such that $\psi : F \to A$ is an $\mathcal{X}$-precover. Then there exists an $\mathcal{X}$-precover $\phi : G \to A$ of A such that there is no non-identity $\mathcal{X}$-pure congruence $\rho \subset \ker(\phi)$ on G.*

Proof. First, if there does not exist a non-identity $\mathcal{X}$-pure congruence $\sigma \subseteq \ker(\psi)$ on F then we let $G = F$ and $\phi = \psi$. Otherwise by assumption any chain of $\mathcal{X}$-pure congruences on F contained in $\ker(\psi)$ has an upper bound and so by Zorn's lemma there is a maximum σ say. Let $G = F/\sigma$ and let $\phi : G \to A$ be the natural map such that $\phi\sigma^\natural = \psi$. Then it is easy to check that $\phi : G \to A$ is an $\mathcal{X}$-precover as if $H \in \mathcal{X}$ and if $f : H \to A$ then there exists $g : H \to F$ such that $\psi g = f$. So $\sigma^\natural g : H \to G$ and $\phi\sigma^\natural g = \psi g = f$ and $\phi : G \to A$ is an $\mathcal{X}$-precover.

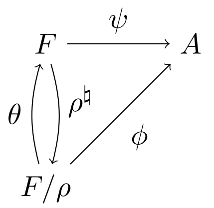


Finally suppose that ρ is an $\mathcal{X}$-pure congruence on G such that $\rho \subset \ker(\phi)$. Then by Remark 2.4, σ/ρ is an $\mathcal{X}$-pure congruence on F containing σ and $\sigma/\rho = \ker(\rho^\natural\sigma^\natural) \subseteq \ker(\psi)$. By the maximality of σ it follows that $\sigma = \sigma/\rho$ and so $\rho = 1_G$, a contradiction as required. □

Following [9, Lemma 1] we can extend this result as follows.

Proposition 5.42. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts. If A is an S-act such that $\psi : F \to A$ is an $\mathcal{X}$-cover then there is no non-identity $\mathcal{X}$-pure congruence $\rho \subset \ker\psi$ on F.*

Proof. Let $\rho \subset \ker\psi$ be an $\mathcal{X}$-pure congruence on F. Then there is an induced S-map $\phi : F/\rho \to A$ such that $\phi\rho^\natural = \psi$. Since (F, ψ) is a precover then there exists an S-map $\theta : F/\rho \to F$ such that $\psi\theta = \phi$. Hence $\psi\theta\rho^\natural = \phi\rho^\natural = \psi$ and so $\theta\rho^\natural$ is an isomorphism of F. Hence $\rho^\natural$ is a monomorphism and so $\rho = 1_A$ as required.



□

Let $\mathcal{X}$ be a class of S-acts. Let us say that $\mathcal{X}$ is **(weakly) congruence pure** if for each cardinal λ there exists a cardinal $\kappa > \lambda$ such that for every (indecomposable) $X \in \mathcal{X}$ with $|X| \geq \kappa$ and every congruence ρ on X with $|X/\rho| \leq \lambda$ there exists a non-identity $\mathcal{X}$-pure congruence $\sigma \subseteq \rho$ of X.

Theorem 5.43. *Let S be a monoid, let $\mathcal{X}$ be a class of S-acts such that*

1. $\coprod_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ *for each* $i \in I$*;*
2. $\mathcal{X}$ *is closed under chains of* $\mathcal{X}$*-pure congruences;*
3. *for every* S*-act* A *there exists an* X *in* $\mathcal{X}$ *such that* $Hom(X, A) \neq \emptyset$*;*
4. $\mathcal{X}$ *is weakly congruence pure.*

Then every S*-act has an* $\mathcal{X}$*-precover.*

Proof. Let A be an S-act, let $\lambda = \max\{|A|, \aleph_0\}$, let κ be as given in the weakly congruence pure condition and let S_A be a set of isomorphic representatives of S-acts of cardinalities less than κ. Suppose that X is an indecomposable S-act and that $h : X \to A$ is an S-map. If $|X| < \kappa$ then let $Y \in S_A$ be an isomorphic copy of X and let $f : X \to Y$ be an isomorphism and define $g : Y \to A$ by $g = hf^{-1}$ so that $h = gf$.

Suppose now that $|X| \geq \kappa$. Then $|X/\ker(h)| = |\mathrm{im}(h)| \leq \lambda$ and so there exists an $\mathcal{X}$-pure congruence $1_X \neq \sigma \subseteq \ker(h)$ on X with $X/\sigma \in \mathcal{X}$. In fact, using a combination of Zorn's lemma and the hypothesis that $\mathcal{X}$ is closed under chains of $\mathcal{X}$-pure congruences, we can assume that σ is maximal with respect to this property. Now let $\bar{h} : X/\sigma \to A$ be the unique map such that $\sigma^\natural h = \bar{h}$. Notice that since $\mathrm{im}(\bar{h}) = \mathrm{im}(h)$ then

$$|(X/\sigma)/\ker(\bar{h})| = |X/\ker(h)| \leq \lambda.$$

Now suppose, by way of contradiction, that $1_{X/\sigma} \neq \rho \subseteq \ker(\bar{h})$ is an $\mathcal{X}$-pure congruence on X/σ so that $(X/\sigma)/\rho \in \mathcal{X}$. Then by Remark 2.4 and since $X \in \mathcal{X}$ it follows that σ/ρ is an $\mathcal{X}$-pure congruence on X containing σ and since $\rho \subseteq \ker(\bar{h})$ it easily follows that $\sigma/\rho \subseteq \ker(h)$. Hence by the maximality of σ we deduce that $\sigma/\rho = \sigma$ and so $\rho = 1_{X/\sigma}$. Therefore it follows that X/σ does not contain a non-identity $\mathcal{X}$-pure congruence contained in $\ker(\bar{h})$

and since by Lemma 5.7, X/σ is indecomposable and since $\mathcal{X}$ is weakly congruence pure we deduce that $|X/\sigma| < \kappa$. Consequently it follows that there exists $Y \in S_A$ and an isomorphism $\bar{f} : X/\sigma \to Y$ and so define $f : X \to Y$ by $f = \bar{f}\sigma^\natural$ and $g : Y \to A$ by $g = \bar{h}\bar{f}^{-1}$ so that $gf = h$.

Hence $\mathcal{X}$ satisfies the weak solution set condition and the result follows from Theorem 5.38. □

A similar condition to this is considered in [8] and forms the basis of one of the proofs of the flat cover conjecture.

5.6 Covers with the unique mapping property

An $\mathcal{X}$-(pre)cover $g : X \to A$ of an S-act A is said to have the **unique mapping property** if whenever there is an S-map $h : X' \to A$ with $X' \in \mathcal{X}$, there is a unique S-map $f : X' \to X$ such that $h = gf$.

Clearly an $\mathcal{X}$-precover with the unique mapping property is an $\mathcal{X}$-cover with the unique mapping property as the unique identity map is an isomorphism.

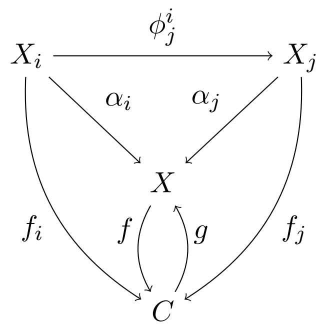
Note that every act having an $\mathcal{X}$-cover with the unique mapping property is equivalent to saying that $\mathcal{X}$ is a coreflective subcategory of the category of all S-acts. That is to say, the inclusion functor has a right adjoint. See Theorem 1.16 or [29, Exercises 3.J and 3.M] for more details and from which some of the next results are based.

Lemma 5.44. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts closed under all (that is, not just directed) colimits. If an S-act has an $\mathcal{X}$-precover then it has an $\mathcal{X}$-cover with the unique mapping property.*

Proof. If an S-act A has an $\mathcal{X}$-precover, then by Theorem 5.37 it has an $\mathcal{X}$-cover, say $g : C \to A$. Let f_1, f_2 be two endomorphisms of C such that $gf_1 = gf_2 = g$, we intend to show that $f_1 = f_2$ and so the unique mapping property holds. Let (h, E) be the coequalizer of f_1 and f_2 in C, so that by Lemma 2.9, $E = C/\rho$ where ρ is the smallest congruence generated by the pairs $\{(f_1(c), f_2(c)) : c \in C\}$. Since $g(f_1(c)) = g(c) = g(f_2(c))$ it is clear that $\rho \subseteq \ker(g)$. Since $\mathcal{X}$ is closed under colimits $E \in \mathcal{X}$ and by Proposition 5.42, $\rho = id_C$ and hence $f_1 = f_2$. □

Lemma 5.45. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts. If every S-act has an $\mathcal{X}$-cover with the unique mapping property then $\mathcal{X}$ is closed under all colimits.*

Proof. Let $(X_i, \phi^i_j)_{i\in I}$ be a direct system of S-acts $X_i \in \mathcal{X}$ with colimit (X, α_i). Let $g : C \to X$ be the $\mathcal{X}$-cover of X so that for each $i \in I$ there exists a unique $f_i : X_i \to C$ with $gf_i = \alpha_i$. Note that if $i \leq j$ then $gf_i = \alpha_i = \alpha_j\phi^i_j = (gf_j)\phi^i_j = g(f_j\phi^i_j)$ and so by the unique mapping property $f_i = f_j\phi^i_j$ for all $i \leq j$. Hence by the colimit property, there exists a unique S-map $f : X \to C$ such that $f\alpha_i = f_i$ for all $i \in I$. Therefore $\alpha_i = gf_i = g(f\alpha_i) = (gf)\alpha_i$ and since, by the colimit property, there exists a unique S-map $h : X \to X$ with $h\alpha_i = \alpha_i$ for all $i \in I$, we clearly have $gf = id_X$. But then $g(fg) = (gf)g = g$ and by the unique mapping property $fg = id_C$ and so X is isomorphic to $C \in \mathcal{X}$.



□

Hence by Theorem 5.38 we have the following

Theorem 5.46. *Let S be a monoid and $\mathcal{X}$ a class of S-acts. Every S-act has an $\mathcal{X}$-cover with the unique mapping property if and only if*

1. *$\mathcal{X}$ is closed under all colimits.*

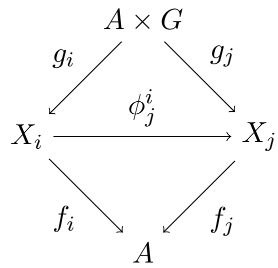
2. *For every S-act A there exists $X \in \mathcal{X}$ such that $Hom(X, A) \neq \emptyset$.*

3. *$\mathcal{X}$ satisfies the solution set condition.*

Recall that an S-act G is called a generator if there exists an epimorphism $G \to S$.

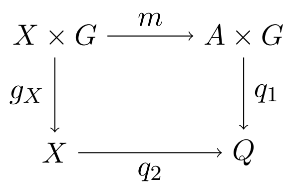
Theorem 5.47. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts containing a generator which is closed under all colimits. Then every S-act has an $\mathcal{X}$-cover with the unique mapping property.*

Proof. Let $G \in \mathcal{X}$ be a generator with S-epimorphism $h : G \to S$. Given any S-act A, let $A \times G$ be the S-act with the action on the right component, so that we have an S-epimorphism $g_A : A \times G \to A$, $(a, y) \mapsto ah(y)$. Notice that $A \times G$ is isomorphic to a coproduct of $|A|$ copies of G and so $A \times G \in \mathcal{X}$. Consider, up to isomorphism, the set $(X_i, g_i, f_i)_{i \in I}$ of all S-acts $X_i \in \mathcal{X}$ and S-epimorphisms $g_i : A \times G \to X_i$ such that there exist $f_i : X_i \to A$ with $f_i g_i = g_A$. Notice that $(A \times G, 1_{A \times G}, g_A)$ is one such triple and so $I \neq \emptyset$, and that this is indeed a set since $|X_i| \leq |A \times G|$. Define an order on this set $(X_i, g_i, f_i) \leq (X_j, g_j, f_j)$ if and only if there exists $\phi^i_j : X_i \to X_j$ with $\phi^i_j g_i = g_j$ and $f_j \phi^i_j = f_i$.

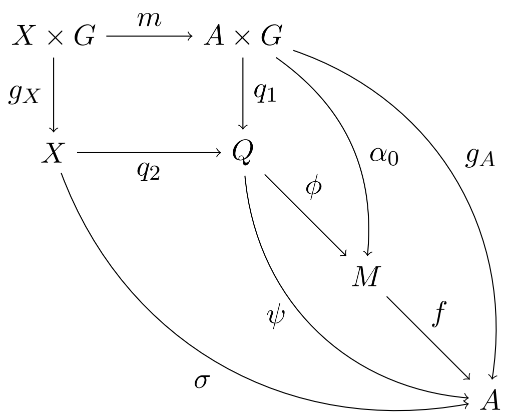


Notice that since g_i is onto then if such a ϕ^i_j exists then it is unique. It is a straightforward matter to check that this is a partial order, and $(X_i, \phi^i_j)_{i \in I}$ is a direct system. In fact, this order has a least element $(X_0, 1_{A \times G}, g_A)$, where $X_0 = A \times G$ and $\phi^0_i = g_i$ for all $i \in I$. Let (M, α_i) be the colimit of this system, since each ϕ^i_j is an epimorphism, so are the α_i and since $\mathcal{X}$ is closed under colimits, $M \in \mathcal{X}$. Since $f_j \phi^i_j = f_i$ for all $i \leq j \in I$ there must exist some $f : M \to A$ such that $f \alpha_i = f_i$ for all $i \in I$. Since $M \in \mathcal{X}$ and α_0 is an epimorphism we see that (M, α_0, f) is in fact a maximal element in the ordering.

We claim that $f : M \to A$ is an $\mathcal{X}$-precover of A. Given any $X \in \mathcal{X}$ with S-map $\sigma : X \to A$, let $g_X : X \times G \to X$, $(x, y) \mapsto xh(y)$ be an S-epimorphism. As before, observe that $A \times G, X \times G \in \mathcal{X}$. Define $m : X \times G \to A \times G$ by $m(x, y) = (\sigma(x), y)$ and consider the pushout diagram



Since g_X is an epimorphism then, by Lemma 2.6, so is q_1 and since $\mathcal{X}$ is closed under colimits then $Q \in \mathcal{X}$. By Lemma 2.5, $Q = (X \amalg (A \times G))/\rho$ where $\rho = \{(m(z), g_X(z)) : z \in X \times G\}^{\#}$. Since $g_A m = \sigma g_X$ then there exists a unique $\psi : Q \to A$ such that $\psi q_1 = g_A, \psi q_2 = \sigma$ and so by the maximality of (M, α_0, f) there exists an S-map $\phi : Q \to M$ such that $\phi q_1 = \alpha_0$.



It is straightforward to check that $f\phi q_2 = \sigma$, and so $f : M \to A$ is an $\mathcal{X}$-precover of A. Since $\mathcal{X}$ is closed under colimits, we can apply Lemma 5.44. □

So by Corollary 5.5 we get the following result

Corollary 5.48. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts. Every S-act has an epimorphic $\mathcal{X}$-cover with the unique mapping property if and only if $\mathcal{X}$ contains a generator and is closed under colimits.*

Chapter 6

Applications to specific classes

We now apply our results concerning existence of $\mathcal{X}$-covers to specific classes of S-acts.

6.1 Free covers

Let S be a monoid and $f : C \to A$ be an S-epimorphism. Recall from Chapter 3, we call f coessential if there is no proper subact B of C such that $f|_B$ is onto.

Lemma 6.1. *[52, Cf. Theorem 5.7] Let S be a monoid and let A an S-act. Then $g : C \to A$ is a $\mathcal{F}r$-cover of A if and only if f is a coessential epimorphism with $C \in \mathcal{F}r$.*

Proof. Suppose that g is a $\mathcal{F}r$-cover of A. Then by Proposition 5.4, g is an epimorphism. Let B be a subact of C such that $g|_B$ is an epimorphism. Then since C is projective, there exists an S-map $h : C \to B$ with $(g|_B)h = g$. Then we get easily that $g = g\iota h$, where $\iota : B \to C$ is the inclusion map. Now, by hypothesis, ιh must be an isomorphism which gives $B = C$.

Conversely let $g : C \to A$ be a coessential epimorphism and suppose that $C \in \mathcal{F}r$. Then g is a $\mathcal{F}r$-precover since every free S-act is projective. To prove that it is a $\mathcal{F}r$-cover, let $f : C \to C$ be an S-map with $g = gf$. Then, $g|_{\mathrm{im}(f)}$ is onto, and so $\mathrm{im}(f) = C$. Thus f is an epimorphism, and since C

is projective, there exists an S-map $h : C \to C$ such that $fh = 1_C$. So h is a monomorphism and $gh = (gf)h = g(fh) = g$. Thus, $g|_{\mathrm{im}(h)}$ is onto, and hence $\mathrm{im}(h) = C$. Therefore, h is an epimorphism and so an isomorphism. Hence f is an isomorphism. □

Lemma 6.2. *Let S be a monoid. Then every S-act has a $\mathcal{F}r$-precover.*

Proof. Let A be an S-act. Take $A \times S$ the free S-act generated by A with the S-map $g : A \times S \to A$, $(a, s) \mapsto as$. Then g is an S-epimorphism and so every free S-act (which is also projective) factors through it. □

Theorem 6.3. *Given any monoid S, the following are equivalent:*

1. *Every S-act has an $\mathcal{F}r$-cover.*
2. *Every S-act has a (coessential) free cover.*
3. *The one element S-act Θ_S has an $\mathcal{F}r$-cover.*
4. *S is a group.*

Proof. (1) ⇔ (2) by Lemma 6.1.
(1) ⇒ (3) is a tautology.
(3) ⇒ (4) If $g : C \to \Theta_S$ is a $\mathcal{F}r$-cover of Θ_S then $C = A \times S$ for some set A. Let $a \in A$ and define $f : C \to C$ by $f(x, s) = (a, s)$ for $x \in A$. Then $gf = g$ and so f is an isomorphism. Hence $|A| = 1$ and so $C \cong S$. Now let $x \in S$ and consider $h : S \to S$ given by $h(s) = xs$. Then $gh = g$ and so h is an isomorphism and hence $S = xS$ for all $x \in S$. Hence S is a group by Lemma 1.21.
(4) ⇒ (1) By Theorem 2.49, S is a group if and only if every strongly flat S-act is free. In particular, since by Proposition 2.54, the strongly flat S-acts are closed under directed colimits, the free S-acts are also closed under directed colimits and the result follows from Lemma 6.2 and Theorem 5.37. □

6.2 Projective covers

Theorem 6.4. *Let S be a monoid, then every S-act has a $\mathcal{P}$-precover.*

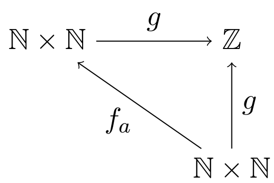
Proof. The same proof as Lemma 6.2 □

We now give an example of a $\mathcal{P}$-precover that is not a $\mathcal{P}$-cover.

Example 6.5. Let $S = (\mathbb{N}, +)$ be the monoid of natural numbers (with zero) under addition, and consider the countably generated free S-act $\mathbb{N} \times \mathbb{N}$ with the following action:

$$(\mathbb{N} \times \mathbb{N}) \times S \to \mathbb{N} \times \mathbb{N}$$
$$(m, n) + s \mapsto (m + s, n).$$

We now define a function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, $(m, n) \mapsto m - n$ and we can see this is an S-map, as $g((m, n) + s) = g((m + s, n)) = (m + s) - n = (m - n) + s = g((m, n)) + s$. Also given any $z \geq 0$, $g((z, 0)) = z$ and for any $z < 0$, $g((0, -z)) = z$ so g is an epimorphism. Since $\mathbb{N} \times \mathbb{N}$ is free it is also projective. Now given any other projective S-act with an S-map to $\mathbb{Z}$, by definition of projectivity this factors through $\mathbb{N} \times \mathbb{N}$ since g is epimorphic. Hence $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ is a $\mathcal{P}$-precover of $\mathbb{Z}$, but it is not a $\mathcal{P}$-cover as there are an infinite number of S-maps $f_a : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $(m, n) \mapsto (m + a, n + a)$ for each $a > 0$ that complete the following commutative diagram



but $\{(0, 0), \ldots, (a - 1, a - 1)\} \notin \text{im}(f_a)$ so f_a is not even an epimorphism.

Recall from Chapter 3, that an S-act A has a **coessential projective cover** if there exists an S-act $P \in \mathcal{P}$ and an S-epimorphism $g : P \to A$ such that for any subact $P' \subseteq P$, $g|_{P'}$ is not an epimorphism. Coessential projective covers of acts were studied in [34] and [28] and those monoids where every S-act has a coessential projective cover were characterised. Analogously to rings, these have been named **perfect** monoids.

Lemma 6.6. *([52, Theorem 5.7]) An S-map $g : P \to A$, with $P \in \mathcal{P}$, is a coessential projective cover of A if and only if it is a $\mathcal{P}$-cover.*

Proof. The same proof as Lemma 6.1. □

Corollary 6.7. *Every S-act has a $\mathcal{P}$-cover if and only if S is perfect.*

It is worth mentioning that one characterisation of perfect monoids is those monoids where the projective acts are closed under directed colimits, or equivalently, when all the strongly flat acts are projective (see [28]). This would mean we could apply Theorems 5.37 and 6.4 to get the same result.

Proposition 6.8. *([38, Proposition 17.24]) If an S-act A is the union of an infinite, strictly ascending chain of cyclic subacts then A does not have a projective cover.*

Proof. Suppose $A = \bigcup_{n\in\mathbb{N}} a_nS$ and

$$a_1S \subset a_2S \subset \cdots \subset a_nS \subset \cdots ,$$

where all inclusions are strict, is an ascending chain of cyclic subacts of A and assume A has a projective cover P with coessential epimorphism $f : P \to A$. Now $P = \coprod_{i\in I} e_iS$ for some idempotents $e_i \in S$, $i \in I$ by Theorem 2.29. But if $|I| > 1$ and $f(e_iS) \subseteq a_nS$ for some $n \in \mathbb{N}$ then $a_{n+1} \in \operatorname{im}(f|_{P\setminus e_iS})$ and so $f|_{P\setminus e_iS}$ is still an epimorphism and thus P cannot be a cover of A. Finally, if $|I| = 1$ then the image of f lies in one of the subacts a_nS and thus f cannot be an epimorphism. □

Recall from Example 6.5, that $\mathbb{Z} = \bigcup_{i\in\mathbb{N}}(-i+\mathbb{N})$ is a union of an infinite strictly ascending chain of cyclic subacts, so by Proposition 6.8, $\mathbb{Z}$ doesn't have a projective cover, and so by Lemma 6.6 doesn't have a $\mathcal{P}$-cover.

6.3 Strongly flat covers

Recall from Theorem 2.54 that $\mathcal{SF}$ is closed under directed colimits and from Corollary 2.35 that $\coprod_{i\in I} X_i \in \mathcal{SF} \Leftrightarrow X_i \in \mathcal{SF}$ for each $i \in I$. Also note that $S \in \mathcal{SF}$ and so for any S-act A, $\operatorname{Hom}(S, A) \neq \emptyset$. Therefore, by Proposition 4.20, Theorem 5.43 and Corollaries 4.19 and 5.40 we have the following results:

Theorem 6.9. *If for each cardinal λ there exists a cardinal $\kappa > \lambda$ such that for every indecomposable $X \in \mathcal{SF}$ with $|X| \geq \kappa$ and every congruence ρ on X with $|X/\rho| \leq \lambda$ there exists a non-identity pure (or 2-pure) congruence $\sigma \subseteq \rho$ on X, then every S-act has an $\mathcal{SF}$-cover.*

Theorem 6.10. *Given a monoid S, if there exists a cardinal λ such that every indecomposable S-act $A \in \mathcal{SF}$ satisfies $|A| \leq \lambda$, then every S-act has an $\mathcal{SF}$-cover.*

Monoids embeddable in groups

Lemma 6.11. *Let S be a monoid that embeds in a group G. Then every S-act has an $\mathcal{SF}$-cover.*

Proof. We show that every indecomposable strongly flat S-act embeds in G and so can apply Theorem 6.10. Let X be an indecomposable strongly flat S-act, then it is locally cyclic by Corollary 2.37. Pick some some $x \in X$, then for all $y \in X$, there exists $z \in X$, $s, t \in S$ such that $x = zs$ and $y = zt$, and we can define a function

$$\begin{aligned} f_x : X &\to G \\ y &\mapsto s^{-1}t. \end{aligned}$$

We first check that this is well-defined. Let $x = z's'$ and $y = z't'$, then $zs = z's'$ and by Condition (P) there exists $z'' \in X$, $u, v \in S$ such that $z = z''u$, $z' = z''v$ and $us = vs'$. Then $z''ut = z''vt'$ and by Condition (E) there exists $z''' \in X$, $w \in S$ such that $z'' = z'''w$ and $wut = wvt'$, so we have

$$s^{-1}t = s^{-1}(wu)^{-1}(wu)t = (s^{-1}u^{-1})w^{-1}(wvt') = (us)^{-1}vt' = (vs')^{-1}vt' = s'^{-1}t.'$$

This is clearly an S-map as $f(yr) = s^{-1}(tr) = (s^{-1}t)r = f(y)r$, and if we let $s^{-1}t = s'^{-1}t'$, then

$$y = z''ut = z''u(ss^{-1})t = z''(us)(s^{-1}t) = z''(vs')(s'^{-1}t') = z''vt' = y'$$

so f is also injective. $\square$

Condition (A)

Recall that a monoid S is said to satisfy condition (A) if every locally cyclic right S-act is cyclic.

Proposition 6.12. *Let S be a monoid that satisfies condition (A). Then every S-act has an $\mathcal{SF}$-cover.*

Proof. By Corollary 2.37, the indecomposable acts in $\mathcal{SF}$ are the locally cyclic acts but since S satisfies Condition (A) all the locally cyclic acts are cyclic. If S/ρ is cyclic then clearly $|S/\rho| \leq |S|$ and the result follows from Theorem 6.10. □

It is well known that not every monoid that satisfies condition (A) is perfect and so we can then deduce that $\mathcal{P}$-covers are in general different from $\mathcal{SF}$-covers, and by Theorem 3.7 coessential strongly flat covers are different from $\mathcal{SF}$-covers.

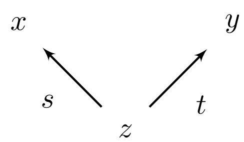
Weak finite geometric type

We say that a monoid S has **weak finite geometric type** if for all $s \in S$ there exists $k \in \mathbb{N}$ such that for all $m \in S$, $|\{p \in S \mid ps = m\}| \leq k$.

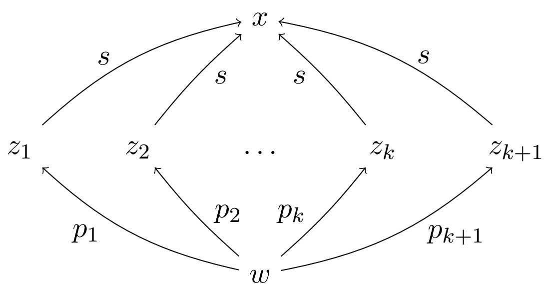
The following was suggested to us by Philip Bridge [10]. For a version involving more general categories see [11].

Proposition 6.13 (Cf. [11, Theorem 5.21])**.** *Let S be a monoid having weak finite geometric type. Then every S-act has an $\mathcal{SF}$-cover.*

Proof. Let X be an indecomposable strongly flat S-act, then by Corollary 2.37, it is locally cyclic and so for all $x, y \in X$ there exists $z \in X, s, t \in S$ such that $x = zs, y = zt$.



We now fix $x \in X$ and consider how many possible $y \in X$ could satisfy these equations. Firstly we take a fixed $s \in S$ and consider how many possible $z \in X$ could satisfy $x = zs$. By the hypothesis, there exists $k \in \mathbb{N}$ such that for any $m \in S$, $|\{p \in S \mid ps = m\}| \leq k$. Let us suppose that there are at least $k+1$ distinct z such that $x = zs$. That is, $x = z_1 s = z_2 s = \ldots = z_{k+1} s$. Then by Lemma 2.36 there exists $w \in X, p_1, \ldots, p_{k+1} \in S$ such that $p_1 s = \ldots = p_{k+1} s$ and $z_i = wp_i$ for each $i \in \{1, \ldots, k+1\}$.



However, by the hypothesis this means at least two p_i are equal and hence at least two z_i are equal which is a contradiction. So given some fixed $s \in S$ there are at most k possible z such that $x = zs$. Hence, there are no more than $\aleph_0|S|$ possible $z \in X, s \in S$ such that $x = zs$. Similarly, given a fixed $z \in X$, there are at most $|S|$ possible $t \in S$ such that $zt = y$ and hence there are no more than $\aleph_0|S|^2$ possible elements in X and we apply Theorem 6.10 □

A finitely generated monoid that satisfies this property is said to have **finite geometric type** (see [55]). They are precisely the semigroups with locally finite Cayley graphs.

Note that setting $k = 1$ in the weak finite geometric type property is the definition of a right cancellative monoid. But it is a much larger class of monoids, for example the bicyclic monoid has finite geometric type. In fact, let B be the bicyclic monoid and let $(s,t) \in B$. Suppose that $(m,n) \in B$ is fixed and suppose that $(p,q) \in B$ is such that $(p,q)(s,t) = (m,n)$. We count the number of solutions to this equation. Recall that

$$(p,q)(s,t) = (p - q + \max(q,s), t - s + \max(q,s)) = (m,n).$$

If $q \geq s$ then $(p,q) = (m, n - (t - s))$ and there is at most one solution to the equation. Otherwise $(p,q) = (m - s + q, q)$ where q ranges between 0 and $s-1$. There are therefore at most $s+1$ possible values of (p,q) that satisfy the equation and so B has finite geometric type.

So from [36, Example 2.9, Example 2.10] and [6, Corollary 3.13] and the previous remarks we can deduce,

Theorem 6.14. *For the following classes of monoid every act has an $\mathcal{SF}$-cover.*

1. *Monoids having weak finite geometric type;*

   right cancellative monoids,

   the Bicylic monoid,

2. *Monoids satisfying Condition* (A)*;*

   finite monoids,

   rectangular bands with a 1 adjoined,

   right groups with a 1 adjoined,

   right simple semigroups with a 1 adjoined,

   the semilattice $(\mathbb{N}, \max)$*,*

   completely simple and completely 0-simple semigroups with a 1 adjoined.

The previous results rely on us showing that the indecomposable strongly flat S-acts are bounded in size and hence the class of (isomorphic representatives of) indecomposable strongly flat S-acts forms a set. We show there exists a monoid S with a proper class of indecomposable strongly flat acts by constructing an indecomposable strongly flat act of arbitrarily large cardinality.

Counterexample of set of indecomposable $\mathcal{SF}$-acts

We now show that the full transformation monoid of an infinite set does not have a set of indecomposable strongly flat acts.

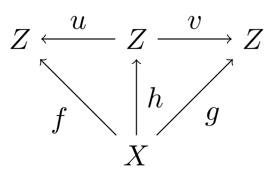
Lemma 6.15. *Given an infinite set Z, there is a well-defined bijective function $\phi : Z \times Z \to Z$.*

Proof. For infinite sets, $|Z \times Z| = |Z| \cdot |Z| = \max\{|Z|, |Z|\} = |Z|$. □

An example of such a pairing function for the natural numbers is $\phi((x, y)) := \frac{1}{2}(x+y)(x+y+1)+x$. This function, which is due to Cantor, maps a diagonal path across the $\mathbb{N} \times \mathbb{N}$ lattice and is well known to be bijective [17, see p.494]

Example 6.16. We show there exists a monoid with a proper class of (isomorphic representatives of) indecomposable strongly flat acts by constructing an indecomposable strongly flat act of arbitrarily large cardinality.

Let Z be an infinite set, let $S = \mathcal{T}(Z)$ be the full transformation monoid of Z and by Lemma 6.15, let $\phi : Z \times Z \to Z$ be a bijective function.. Given any cardinal $\lambda > 0$, let X be a set with $|X| = \lambda$ and let $Z^X = \{f : X \to Z\}$ be the set of all functions from X to Z. We can make Z^X into an S-act with the action $S \times Z^X \to Z^X$, $(f, g) \mapsto fg$ (note, it is much more convenient to consider Z^X as a left S-act since the action is composition of maps). Given any $f, g \in Z^X$, let $h \in Z^X$ be defined as $h(x) = \phi((f(x), g(x))$. Then define $u, v \in S$ to be $u = p_1\phi^{-1}$ and $v = p_2\phi^{-1}$, where $p_i((a_1, a_2)) = a_i$. Therefore $f = uh$, $g = vh$ and Z^X is locally cyclic (hence indecomposable) and has cardinality $|Z|^{|X|} > |X| = \lambda$.

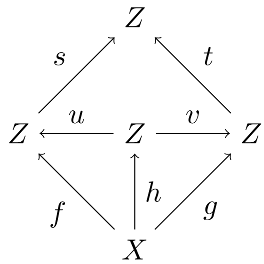


We now show Z^X is strongly flat. Let $f, g \in Z^X$, $s, t \in S$ such that $sf = tg$. Define $h \in Z^X$ as before, pick some $x \in X$ and define $u_x, v_x \in S$ by

$$u_x(n) := \begin{cases} u(n) & \text{if } n \in \text{im}(h) \\ f(x) & \text{otherwise} \end{cases}$$

$$v_x(n) := \begin{cases} v(n) & \text{if } n \in \text{im}(h) \\ g(x) & \text{otherwise.} \end{cases}$$

Then $f = u_x h$, $g = v_x h$. Therefore $su_x h = sf = tg = tv_x h$. To see that $su_x(n) = tv_x(n)$ for all $n \in Z$, consider the two cases: it is obvious when $n \in \text{im}(h)$, otherwise $su_x(n) = sf(x) = tg(x) = tv_x(n)$ and so Z^X satisfies Condition (P).



Let $f \in Z^X$, $s, t \in S$ such that $sf = tf$. Pick some $x \in X$ and define $w \in S$,

$$w(n) := \begin{cases} n & \text{if } n \in \text{im}(f) \\ f(x) & \text{otherwise.} \end{cases}$$

Then $f = wf$ and $sw = tw$, so Z^X satisfies Condition (E) and is strongly flat.

The following example which proves that not every S-act has an $\mathcal{SF}$-cover is essentially due to Kruml [40], my contribution being to translate the example from the language of varieties to the language of S-acts.

Example 6.17. Let $T = \langle a_0, a_1, a_2 \cdots \mid a_i a_j = a_{j+1} a_i \text{ for all } i \leq j \rangle$ and $S = T^1$, then the one element S-act Θ_S does not have an $\mathcal{SF}$-precover.

Proof. We first note that S is left cancellative. In fact, every word $w \in T$ has a unique normal form $w = a_{\alpha(1)} \cdots a_{\alpha(n)}$ where $\alpha(i) \leq \alpha(i+1)$ for all $1 \leq i \leq n-1$, and given any $a_{\alpha(n+1)}, a_{\beta(n+1)}$, it is easy to see that $wa_{\alpha(n+1)} = wa_{\beta(n+1)}$ implies $\alpha(n+1) = \beta(n+1)$. Hence every S-endomorphism $h : S \to S$ is injective, as $h(s) = h(t)$ implies $h(1)s = h(1)t$.

Assume Θ_S does have an $\mathcal{SF}$-precover, then by Lemma 5.11, $\mathcal{SF}$ contains a weakly terminal object, say T. By Theorem 2.32, let $(T, \alpha_i)_{i \in I}$ be the directed colimit of finitely generated free S-acts $(T_i, \phi^i_j)_{i \in I}$. Let X be any set with $|X| > \max\{|I|, \aleph_0, |S|\}$, by Theorem 1.3, put a total order on X and let $\mathrm{Fin}(X)$ denote the set of all finite subsets of X. We now define a direct system indexed over $\mathrm{Fin}(X)$ partially ordered by inclusion, where every object S_Y is isomorphic to S and a map from an $n-1$ element subset Y into an n element subset $Y \cup \{z\}$ is defined to be the endomorphism $\lambda_{a_i} : S \to S$, $s \mapsto a_i s$, where $i = |\{y \in Y \mid y < z\}|$. It follows from the presentation of S that this is indeed a direct system, that is, adding in i then adding in j is the same as adding in j then adding in i. Let $(F, \beta_Y)_{Y \in \mathrm{Fin}(X)}$ be the directed colimit of this direct system, which by Proposition 2.54, is a strongly flat act. Therefore, there exists an S-map $t : F \to T$. Now for each singleton $\{x\} \in \mathrm{Fin}(X)$, by Proposition 2.28, there exists some $i \in I$ and $\theta_i \in \mathrm{Hom}(S_{\{x\}}, T_i)$ such that $t\beta_{\{x\}} = \alpha_i\theta_i$. So by the axiom of choice we can define a function $h : X \to Z, x \mapsto (i, \theta_i(1))$ where $Z := \{(i, x) \in \{i\} \times T_i \mid i \in I\}$ and $|Z| \leq \max\{|I|, \aleph_0, |S|\}$. Since $|X| > |Z|$, h cannot be an injective function and so there exist $x \neq y \in X$ with $h(x) = h(y)$. Since θ_i is determined entirely by the image of 1, we have that $t\beta_{\{x\}} = \alpha_i\theta_i = t\beta_{\{y\}}$. Without loss of generality, assume $x < y$ in X, then $\beta_{\{x,y\}}\lambda_{a_1} = \beta_{\{x\}}$ and $\beta_{\{x,y\}}\lambda_{a_0} = \beta_{\{y\}}$. Again, by Proposition 2.28, there also exists $j \in I$, $\theta_j \in \mathrm{Hom}(S_{\{x,y\}}, T_j)$ such that $t\beta_{\{x,y\}} = \alpha_j\theta_j$. Therefore we have

$$\begin{aligned} & \alpha_i\theta_i = t\beta_{\{x\}} = t\beta_{\{x,y\}}\lambda_{a_1} = \alpha_j\theta_j\lambda_{a_1} \\ \Rightarrow \quad & \alpha_i\left(\theta_i(1)\right) = \alpha_j\left(\theta_j\lambda_{a_1}(1)\right) \end{aligned}$$

and so by Lemma 2.12 there exists some $k \geq i, j$ such that $\phi_k^i\left(\theta_i(1)\right) = \phi_k^j\left(\theta_j\lambda_{a_1}(1)\right)$ which implies $\phi_k^i\theta_i = \phi_k^j\theta_j\lambda_{a_1}$. Similarly

$$\begin{aligned} & \alpha_i\theta_i = t\beta_{\{y\}} = t\beta_{\{x,y\}}\lambda_{a_0} = \alpha_j\theta_j\lambda_{a_0} = \alpha_k\phi_k^j\theta_j\lambda_{a_0} \\ \Rightarrow \quad & \alpha_i\left(\theta_i(1)\right) = \alpha_k\left(\phi_k^j\theta_j\lambda_{a_0}(1)\right) \end{aligned}$$

which again, implies there exists some $m \geq i, k$ such that $\phi_m^i\theta_i = \phi_m^k\phi_k^j\theta_j\lambda_{a_0} = \phi_m^j\theta_j\lambda_{a_0}$. Therefore

$$\phi_m^j\theta_j\lambda_{a_1} = \phi_m^k\phi_k^j\theta_j\lambda_{a_1} = \phi_m^k\phi_k^i\theta_i = \phi_m^i\theta_i = \phi_m^j\theta_j\lambda_{a_0}.$$

Since both T_j and T_m are finitely generated free S-acts, and $S_{\{x,y\}}$ is a cyclic S-act, it is clear that $\phi_m^j\theta_j$ is an endomorphism of S and so a monomorphism. Therefore $\lambda_{a_0} = \lambda_{a_1}$ which implies $a_0 = a_1$ which is a contradiction. $\square$

6.4 Condition (P) covers

From Theorem 2.56 we have that $\mathcal{CP}$ is closed under directed colimits and from Corollary 2.34 that $\coprod_{i\in I} X_i \in \mathcal{CP} \Leftrightarrow X_i \in \mathcal{CP}$ for each $i \in I$. Also note that $S \in \mathcal{CP}$ and so for any S-act A, $\mathrm{Hom}(S, A) \neq \emptyset$. Therefore, by Proposition 4.20, Theorem 5.43 and Corollaries 4.19 and 5.40 we have the following results:

Theorem 6.18. *If for each cardinal λ there exists a cardinal $\kappa > \lambda$ such that for every indecomposable $X \in \mathcal{CP}$ with $|X| \geq \kappa$ and every congruence ρ on X with $|X/\rho| \leq \lambda$ there exists a non-identity 2-pure congruence $\sigma \subseteq \rho$ on X, then every S-act has a $\mathcal{CP}$-cover.*

Theorem 6.19. *Given a monoid S, if there exists a cardinal λ such that every indecomposable S-act $A \in \mathcal{CP}$ satisfies $|A| \leq \lambda$, then every S-act has a $\mathcal{CP}$-cover.*

By observing the proofs, it is clear that both Propositions 6.13 and 6.12 in the previous section clearly also hold for S-acts satisfying Condition (P) and so we also have

Theorem 6.20. *For the following classes of monoid every act has a $\mathcal{CP}$-cover.*

1. *Monoids having weak finite geometric type;*

   right cancellative monoids,

   the Bicylic monoid,

2. *Monoids satisfying Condition (A);*

   finite monoids,

   rectangular bands with a 1 adjoined,

   right groups with a 1 adjoined,

   right simple semigroups with a 1 adjoined,

   the semilattice $(\mathbb{N}, \max)$,

   completely simple and completely 0-simple semigroups with a 1 adjoined.

Example 6.16 is also an example of a monoid that does not have a set of indecomposable acts satisfying Condition (P).

6.5 Condition (E) covers

From Theorem 2.57 we have that $\mathcal{CE}$ is closed under directed colimits and from Corollary 2.34 that $\coprod_{i\in I} X_i \in \mathcal{CE} \Leftrightarrow X_i \in \mathcal{CE}$ for each $i \in I$. Also note that $S \in \mathcal{CE}$ and so for any S-act A, $\mathrm{Hom}(S, A) \neq \emptyset$. Therefore, by Proposition 4.20, Theorem 5.43 and Corollaries 4.19 and 5.40 we have the following results:

Theorem 6.21. *If for each cardinal λ there exists a cardinal $\kappa > \lambda$ such that for every indecomposable $X \in \mathcal{CE}$ with $|X| \geq \kappa$ and every congruence ρ on X with $|X/\rho| \leq \lambda$ there exists a non-identity 1-pure congruence $\sigma \subseteq \rho$ on X, then every S-act has a $\mathcal{CE}$-cover.*

Theorem 6.22. *Given a monoid S, if there exists a cardinal λ such that every indecomposable S-act $A \in \mathcal{CE}$ satisfies $|A| \leq \lambda$, then every S-act has a $\mathcal{CE}$-cover.*

Example 6.16 is also an example of a monoid that does not have a set of indecomposable acts satisfying Condition (E).

6.6 Flat covers

From Theorem 2.58 we have that $\mathcal{F}$ is closed under directed colimits and from Corollary 2.31 that $\coprod_{i\in I} X_i \in \mathcal{F} \Leftrightarrow X_i \in \mathcal{F}$ for each $i \in I$. Also note that $S \in \mathcal{F}$ and so for any S-act A, $\mathrm{Hom}(S, A) \neq \emptyset$. Therefore, by Proposition 4.20, Theorem 5.43 and Corollaries 4.19 and 5.40 we have the following results:

Theorem 6.23. *Given a monoid S, if $\mathcal{F}$ is weakly congruence pure, then every S-act has a $\mathcal{F}$-cover.*

Theorem 6.24. *Given a monoid S, if there exists a cardinal λ such that every indecomposable S-act $A \in \mathcal{F}$ satisfies $|A| \leq \lambda$, then every S-act has a $\mathcal{F}$-cover.*

6.7 Torsion free covers

From Theorem 2.59 we have that $\mathcal{T_F}$ is closed under directed colimits and from Lemma 2.38 that $\coprod_{i\in I} X_i \in \mathcal{T_F} \Leftrightarrow X_i \in \mathcal{T_F}$ for each $i \in I$. Also note that $S \in \mathcal{T_F}$ and so for any S-act A, $\text{Hom}(S, A) \neq \emptyset$. Therefore, by Proposition 4.20, Theorem 5.43 and Corollaries 4.19 and 5.40 we have the following results:

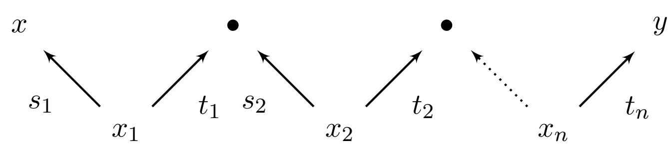
Theorem 6.25. *Given a monoid S, if $\mathcal{T_F}$ is weakly congruence pure, then every S-act has a $\mathcal{T_F}$-cover.*

Theorem 6.26. *Given a monoid S, if there exists a cardinal λ such that every indecomposable S-act $A \in \mathcal{T_F}$ satisfies $|A| \leq \lambda$, then every S-act has a $\mathcal{T_F}$-cover.*

In 1963 Enochs proved that over an integral domain, every module has a torsion free cover [21]. We give a proof of the semigroup analogue of Enochs' result that over a right cancellative monoid, every right act has a torsion free cover.

Theorem 6.27. *Let S be a right cancellative monoid, then every S-act has a $\mathcal{T_F}$-cover.*

Proof. Let A be an indecomposable torsion free S-act. For each $xs = x's \in A$, $s \in S$, since s is right cancellative, $x = x'$. Hence for each $x \in A$, $s \in S$ there is no more than one solution to $x = ys$. Now let $x, y \in A$ be any two elements. Since A is indecomposable there exist $x_1, \ldots, x_n \in A$, $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ such that $x = x_1s_1$, $x_1t_1 = x_2s_2$, $\ldots$, $x_nt_n = y$, as shown below.



If we can show there is a bound on the number of such paths, then there is a bound on the number of elements in A. Now, by the previous argument, there are only $|S|$ possible $x_1 \in A$ such that $x = x_1s_1$ for some $s_1 \in S$. In a similar manner, given x_1 there are only $|S|$ possible x_1t_1 for some $t_1 \in S$.

Continuing in this fashion we see that the number of such paths of length $n \in \mathbb{N}$ is bounded by $|S|^{2n}$, and so $|A| \leq |S|^{\aleph_0}$. So by Theorem 6.26 every S-act has a $\mathcal{T_F}$-cover. □

Example 6.16 is also an example of a monoid that does not have a set of indecomposable torsion free acts.

6.8 Principally weakly flat covers

By Theorem 2.50 over a right cancellative monoid, an act is torsion free if and only if it is principally weakly flat, so we get the following corollary from the last result.

Corollary 6.28. *Every act over a right cancellative monoid has a $\mathcal{PWF}$-cover.*

6.9 Injective covers

In 1981 Enochs proved that every module over a ring has an injective cover if and only if the ring is Noetherian [25, Theorem 2.1]. The situation for acts is not so straightforward. In particular if R is a Noetherian ring then there exists a cardinal λ such that every injective module is the direct sum of indecomposable injective modules of cardinality less than λ. We give an example later to show that this is not so for monoids.

It is worth noting that by Lemma 2.42, every injective S-act has a fixed point and that if an S-act A has an $\mathcal{I}$-precover then there exists $C \in \mathcal{I}$ such that $\mathrm{Hom}(C, A) \neq \emptyset$.

We have the following necessary conditions on S for all S-acts to have an $\mathcal{I}$-precover.

Lemma 6.29. *Let S be a monoid. If every S-act has an $\mathcal{I}$-precover then*

1. *S is a left reversible monoid.*

2. *S has a left zero.*

Proof.

1. Let A_i, $i \in I$ be any collection of injective S-acts, $B = \coprod_{i\in I} A_i$ their coproduct, and $g : C \to B$ the $\mathcal{I}$-precover of B. For each $j \in I$ and inclusion $h_j : A_j \to B$ there exists an S-map $f_j : A_j \to C$ such that $gf_j = h_j$. Hence we can define an S-map $f : B \to A$ by $f|_{A_j} = f_j$ so that $gf = id_B$ and B is a retract of C. Therefore by Proposition 2.41, B is an injective S-act and by Proposition 2.40, S is left reversible.

2. Let $g : I \to S$ be an $\mathcal{I}$-precover of S. Since I is injective it has a fixed point z and so $g(z)$ is a left zero in S. □

Remark 6.30. In particular if every S-act has an $\mathcal{I}$-precover then there is a left zero $z \in S$ such that for all $s \in S$ there exists $t \in S$ with $st = z$. Obviously both conditions above are satisfied if S contains a (two-sided) zero.

Notice also that if S contains a left zero z then every S-act contains a fixed point since if A is a right S-act and $a \in A$ then $(az)s = az$ for all $s \in S$. Consequently all Hom-sets are non-empty.

Lemma 6.31. *Let S be a left reversible monoid with a left zero. Then $\coprod_{i\in I} A_i \in \mathcal{I}$ if and only if $A_i \in \mathcal{I}$ for each $i \in I$.*

Proof. Since S is left reversible if each A_i are injective then $\coprod_{i\in I} A_i$ is injective by Proposition 2.40. Conversely, assume $A = \coprod_{i\in I} A_i$ is injective, and first notice that since S has a left zero each A_i has a fixed point say $z_i \in A_i$. Given any $j \in I$ and monomorphism $\iota : X \to Y$ and any homomorphism $f : X \to A_j$, clearly $f \in \text{Hom}(X, A)$ and so there exists $\bar{f} : Y \to A$ such that $\bar{f}|_X = f$. Now let $K_j = \{y \in Y \mid \bar{f}(y) \in A_j\}$ and notice that $X \subseteq K_j$ and that $y \in K_j$ if and only if $ys \in K_j$ for all $s \in S$. Now define a new function $h : Y \to A_j$ by

$$h(y) = \begin{cases} \bar{f}(y) & y \in K_j \\ z_j & \text{otherwise} \end{cases}$$

Since z_j is a fixed point, h is a well-defined S-map with $h|_X = f$ and so A_j is injective. □

Therefore, by Lemma 6.31, and the fact that when S contains a left zero, $\text{Hom}(\Theta_S, A) \neq \emptyset$ for all S-acts A and Θ_S is injective, we can apply

Corollary 5.40, Proposition 4.20, and Theorems 2.63, 5.37 and 5.43 to have the following results:

Theorem 6.32. *Let S be a monoid, then every S-act has an $\mathcal{I}$-precover if and only if S is left reversible, has a left zero and $\mathcal{I}$ satisfies the weak solution set condition.*

Theorem 6.33. *Let S be a left reversible Noetherian monoid with a left zero. If $\mathcal{I}$ is weakly congruence pure then every S-act has an $\mathcal{I}$-cover.*

Theorem 6.34. *Let S be a left reversible Noetherian monoid with a left zero. If there is a cardinal λ such that every indecomposable injective S-act X is such that $|X| \leq \lambda$ then every S-act has an $\mathcal{I}$-cover.*

We now give a counterexample to the conditions of the previous Theorem.

Example 6.35. Let $S = \{1, 0\}$ be the trivial group with a zero adjoined. Given any set X, choose and fix $y \in X$ and define an S-action on X by $x \cdot 1 = x$ and $x \cdot 0 = y$. Given any $x, x' \in X$, $x \cdot 0 = x' \cdot 0$ and so it is easy to see that X is an indecomposable S-act. It is clear that the only cyclic S-acts are the one element S-act Θ_S and S itself. Therefore since y is a fixed point in X, by Theorem 2.43, to show X is an injective S-act it suffices to show that any S-map $f : \Theta_S \to X$ extends to S. This is straightforward as the image of f is a fixed point. We can therefore construct arbitrarily large indecomposable injective S-acts.

Since the monoid given in the previous Example is finite then it is clearly Noetherian. Hence it is an example of a Noetherian left reversible monoid with a left zero with arbitrarily large indecomposable injective acts. Consequently, unlike in the ring case, not every monoid satisfies the conditions given in Theorem 6.34.

6.10 Divisible covers

As mentioned previously, an obvious necessary condition for an S-act A to have an $\mathcal{X}$-cover is the existence of an S-act $C \in X$ such that $\mathrm{Hom}(C, A) \neq$

$\emptyset$. It is fairly obvious that if $\mathcal{X}$ includes all the free acts then this condition is always satisfied. We consider here the class of divisible acts where this condition is not always satisfied and where the covers, when they exist, are monomorphism rather than epimorphisms.

Recall from Theorem 5.47, that if $\mathcal{X}$ is a class of S-acts containing a generator and closed under colimits, then every S-act has an $\mathcal{X}$-cover. Although by Lemma 2.64, $\mathcal{D}$ is closed under colimits, it does not always contain a generator. In fact we have the following

Lemma 6.36. *Let S be a monoid, then the following are equivalent*

1. *$\mathcal{D}$ has a generator.*
2. *S is divisible.*
3. *All left cancellative elements of S are left invertible.*
4. *Every S-act is divisible.*
5. *Every S-act has an epimorphic $\mathcal{D}$-cover.*

Proof. The equivalence of $(2), (3)$ and (4) follows by Proposition 2.45.

$(1) \Rightarrow (2)$ If $G \in \mathcal{X}$ is a generator, then there exists an epimorphism $g : G \to S$. Hence S is the homomorphic image of a divisible S-act and so is divisible by Lemma 2.44.

$(4) \Rightarrow (5)$ Every S-act is its own epimorphic $\mathcal{D}$-cover.

$(5) \Rightarrow (1)$ The epimorphic $\mathcal{D}$-cover of S is a generator in $\mathcal{D}$. □

Recall from Lemma 2.46 that if an S-act A contains a divisible subact, then it has a unique largest divisible subact $D_A = \bigcup_{i \in I} D_i$ where $\{D_i \mid i \in I\}$ is the set of all divisible subacts of A.

Theorem 6.37. *Let S be a monoid and A an S-act. Then the following are equivalent:*

1. *$g : D \to A$ is a $\mathcal{D}$-cover of A.*
2. *$g : D \to A$ is a $\mathcal{D}$-cover of A with the unique mapping property.*
3. *$D = D_A$ is the largest divisible subact of A and g is the inclusion map.*

Proof. Clearly (2) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2) By Lemmas 5.44 and 2.64, A has a $\mathcal{D}$-cover $g' : D' \to A$ with the unique mapping property. By Theorem 5.1, there exists an isomorphism $\psi : D' \to D$ such that $g' = g\psi$. Given any $X \in \mathcal{D}$ with $h : X \to A$, since D is a $\mathcal{D}$-cover of A, there exists some $f : X \to D$ such that $gf = h$. Assume there exists another S-map $f' : X \to D$ such that $gf' = h$. Since $g'(\psi^{-1}f) = g\psi(\psi^{-1}f) = gf = h = gf' = g\psi(\psi^{-1}f') = g'(\psi^{-1}f')$, and $g' : D' \to A$ has the unique mapping property, then $\psi^{-1}f = \psi^{-1}f'$ which implies $f = f'$ and $g : D \to A$ also has the unique mapping property.
(3) $\Rightarrow$ (1) Let X be a divisible S-act, and let $h : X \to A$ be an S-map. By Lemma 2.44, $\mathrm{im}(h)$ is a divisible subact of A and so $\mathrm{im}(h) \subseteq D$. Therefore $h : X \to D$ is a well-defined S-map obviously commuting with the inclusion map. Hence D is a $\mathcal{D}$-precover of A. It is clear that this is also a $\mathcal{D}$-cover as any map $f : D \to D$ commuting with the inclusion map is an automorphism.
(2) $\Rightarrow$ (3) Let $g : D \to A$ be a $\mathcal{D}$-cover of A. The image of g is a divisible subact of A, and so A has a largest divisible subact D_A. Let $i : D_A \to A$ be the inclusion map, then by the $\mathcal{D}$-cover property there exists some $h : D_A \to D$ such that $hg = i$, hence $\mathrm{im}(g) = D_A$. Since $g(hg) = g$, by the unique mapping property, $hg = id_D$ and g is a monomorphism. □

We therefore have the following result

Theorem 6.38. *Let S be a monoid. Then the following are equivalent*

1. *Every S-act has a $\mathcal{D}$-precover.*

2. *Every S-act has a $\mathcal{D}$-cover.*

3. *Every S-act has a divisible subact.*

4. *S contains a divisible right ideal K.*

Proof. (1) and (2) are equivalent by Lemma 5.44.
The equivalence of (2) and (3) is obvious by the last theorem.
If every S-act has a divisible subact then clearly S has a divisible subact, which is a right ideal. Conversely if K is a divisible subact of S, then given any S-act X, it has a divisible subact XK. Hence (3) and (4) are equivalent. □

For example, if S is any monoid with a left zero z, then $K = \{z\}$ is a divisible right ideal of S and so every S-act has a $\mathcal{D}$-cover.

Notice that not every S-act has a $\mathcal{D}$-cover. For example, let $S = (\mathbb{N}, +)$ and consider S as an S-act over itself. For every $n \in S$, $n + 1$ is a left cancellable element in S, but there does not exist $m \in S$ such that $n = m + (n + 1)$. Therefore S does not have have any divisible right ideals.

Chapter 7

Open Problems and Further Work

We list here a few open problems and suggestions for further work surrounding this area.

1. What are the necessary and sufficient conditions on a monoid S for every S-act to have an $\mathcal{SF}$-cover? (and similarly for other classes of acts).

2. If $\mathcal{Y} \subseteq \mathcal{X}$ is a subclass of a class of S-acts, both closed under isomorphisms, how does an S-act having a $\mathcal{Y}$-cover relate to an S-act having an $\mathcal{X}$-cover?

3. What can be said about $\mathcal{X}$-envelopes, the categorical dual notion? How do these relate to divisible extensions, principally weakly injective extensions and other known constructions?

Appendix A

Normak's Theorem

The following Theorem was first proved by P. Normak, although his original paper is in Russian and quite difficult to get hold of. I thank Christopher Hollings for providing a translation of this paper. For completeness sake I include the proof here, although written in my own style.

Proposition A.1 (Cf. [45, Proposition 4])**.** *An S-act A is finitely presented if and only if there exists a finitely generated free S-act F and a finitely generated congruence ρ on F such that $A \cong F/\rho$.*

Proof. Let A be a finitely presented S-act. Then there exists an exact sequence

$$K \underset{\beta}{\overset{\alpha}{\rightrightarrows}} F \xrightarrow{\gamma} A$$

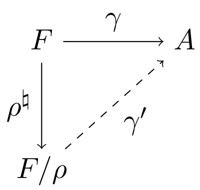
where K is finitely generated and F is finitely generated free. Let $k_1, \ldots, k_r$ be a set of generators for K and let ρ be the congruence on F generated by the pairs $(\alpha(k_i), \beta(k_i))$ for $i = 1, \ldots, r$. Then $(x, y) \in \rho$ if and only if $x = y$ or there exists $\alpha(k_{i_1}), \ldots, \alpha(k_{i_n}), \beta(k_{i_1}), \ldots, \beta(k_{i_n}) \in F$, $s_1, \ldots, s_n \in S$ with

$$\begin{array}{ccccccc} x & = & \alpha(k_{i_1})s_1 & \beta(k_{i_2})s_2 & = & \alpha(k_{i_3})s_3 & \cdots \beta(k_{i_n})s_n = y \\ & & \beta(k_{i_1})s_1 & = & \alpha(k_{i_2})s_2 & \beta(k_{i_3})s_3 & = \alpha(k_{i_4})s_4 \cdots \end{array}$$

where $i_1, \ldots, i_n \in \{1, \ldots, r\}$. Since $\gamma\alpha = \gamma\beta$,

$$\gamma(x) = (\gamma\alpha)(k_{i_1}s_1) = (\gamma\beta)(k_{i_1}s_1) = (\gamma\alpha)(k_{i_2}s_2) = \cdots = (\gamma\beta)(k_{i_n}s_n) = \gamma(y)$$

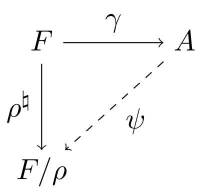
and so $\rho \subseteq \ker(\gamma)$. Hence we can apply Theorem 2.2 to find an S-map $\gamma' : F/\rho \to A$ such that the following diagram



commutes. Now given any $k \in K$, $k = k_i s$ for some $k_i \in \{k_1, \ldots, k_r\}$, $s \in S$. So we have

$$\rho^\natural \alpha(k) = \rho^\natural \alpha(k_i s) = \rho^\natural \alpha(k_i)s = \rho^\natural \beta(k_i)s = \rho^\natural \beta(k_i s) = \rho^\natural \beta(k)$$

and so $\rho^\natural \alpha = \rho^\natural \beta$ and by exactness there also exists an S-map $\psi : A \to F/\rho$ such that the following diagram



commutes. Therefore $\gamma = \gamma' \rho^\natural$ and $\rho^\natural = \psi\gamma$ so that $\rho^\natural = \psi\gamma'\rho^\natural$ and $\gamma = \gamma'\psi\gamma$. Now γ is an epimorphism (by the uniqueness requirement in the definition of coequalizers), and clearly $\rho^\natural$ is an epimorphism so we get $1_{F/\rho} = \psi\gamma' = \gamma'\psi$ and so γ' and ψ are mutually inverse and $A \cong F/\rho$.

Conversely, assume $A \cong F/\rho$, where F is finitely generated free and ρ is generated by pairs (a_i, b_i), $i \in R = \{1, \ldots, r\}$. Now let $K := R \times S$ be the finitely generated free S-act with r generators and define $\alpha : K \to F$, $(i, s) \mapsto a_i s$ and $\beta : K \to F$, $(i, s) \mapsto b_i s$. Clearly these are well defined S-maps. Now for any $(i, s) \in K$ we have

$$\left(\rho^\natural \alpha\right)(i,s) = \rho^\natural(\alpha(i,s)) = \rho^\natural(a_i s) = \rho^\natural(b_i s) = \rho^\natural(\beta(i,s)) = \left(\rho^\natural \beta\right)(i,s)$$

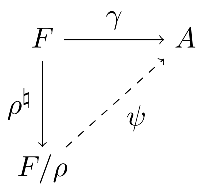
so $\rho^\natural \alpha = \rho^\natural \beta$. Now let $\gamma\alpha = \gamma\beta$ for some $\gamma : F \to A$. Now $(x, y) \in \rho$ if and only if $x = y$ or there exists $a_{i_1}, \ldots, a_{i_n}, b_{i_1}, \ldots, b_{i_n} \in F$, $s_1, \ldots, s_n \in S$ with

$$\begin{array}{ccccccccc} x & = & a_{i_1}s_1 & & b_{i_2}s_2 & = & a_{i_3}s_3 & & \cdots b_{i_n}s_n = y \\ & & b_{i_1}s_1 & = & a_{i_2}s_2 & & b_{i_3}s_3 & = & a_{i_4}s_4 \cdots \end{array}$$

where $i_1, \ldots, i_n \in R$. Therefore

$$\begin{aligned}\gamma(x) &= \gamma(a_{i_1}s_1) = (\gamma\alpha)\,(i_1, s_1) = (\gamma\beta)\,(i_i, s_1) = \gamma(b_{i_1}s_1)\\ &= \gamma(a_{i_2}s_2) = (\gamma\alpha)\,(i_2, s_2) = \qquad \cdots \qquad = \gamma(b_{i_n}s_n) = \gamma(y)\end{aligned}$$

and so $\rho \subseteq \ker(\gamma)$ and we can apply Theorem 2.2 to find a homomorphism $\psi : F/\rho \to A$ such that the following diagram



commutes. Hence $K \rightrightarrows F \to F/\rho$ is exact and A is finitely presented. $\square$

Appendix B

Govorov-Lazard Theorem

D. Lazard proved that every flat module is a directed colimit of finitely generated free modules in his Thesis [43]. Govorov also independently proved the result. B. Stenström proved the semigroup analogue of this result, that every strongly flat act is a directed colimit of finitely generated free acts in his 1971 paper [56]. Towards the end of his proof he claims the rest "is done exactly as in the additive case", although I struggled to replicate the method. For completeness sake, I include my version of the proof, based somewhat on Stenström's proof and also on Bulman-Fleming's proof for the category of S-posets [15].

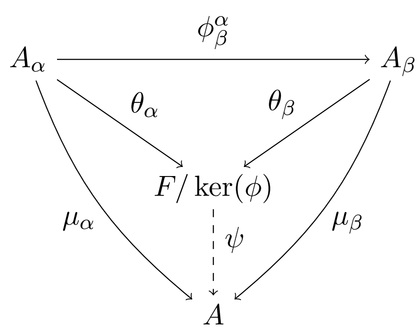
Theorem B.1. *Every strongly flat act is a directed colimit of finitely generated free acts.*

Proof. For any S-act A, let $F := A \times \mathbb{N} \times S$ be the free S-act generated by $A \times \mathbb{N}$, and let $\phi : F \to A$ be the epimorphism that sends (a, n, s) to as, so by Theorem 2.2, there is an isomorphism $\rho : F/\ker(\phi) \to A$ that sends $(a, n, s)\ker(\phi)$ to as. We shall define a set I as follows. An element $\alpha \in I$ is a pair $\alpha = (L_\alpha, K_\alpha)$, where L_α is a finite subset of $A \times \mathbb{N}$, and K_α is a finitely generated congruence on F_α contained in $\ker(\phi)$, where $F_\alpha := L_\alpha \times S$ is the free subact of F generated by L_α. For $\alpha, \beta \in I$, we define $\alpha \leq \beta$ if $L_\alpha \subseteq L_\beta$ and $K_\alpha \subseteq K_\beta$. Let A_α be the finitely presented S-act F_α/K_α. For $\alpha \leq \beta$, we have a natural S-map $\phi^\alpha_\beta : A_\alpha \to A_\beta$, $(a, n, s)K_\alpha \mapsto (a, n, s)K_\beta$, so we get a direct system $(A_\alpha, \phi^\alpha_\beta)$.

We now intend to show that I is a directed index set. Given any α,

$\beta \in I$, let $L_\gamma := L_\alpha \cup L_\beta$ and this is a finite subset of $A \times \mathbb{N}$ containing L_α and L_β. Let Z_α, Z_β be finite sets of generators for K_α, K_β respectively, and define $Z_\gamma := Z_\alpha \cup Z_\beta$. Take the congruence generated by Z_γ on F_γ, the free subact of F generated by L_γ, and call it K_γ. Now the congruence generated by Z_γ on F_γ is clearly contained in the congruence generated by Z_γ on F, which in turn must be contained in $\ker(\phi)$ as it is, by definition, the smallest congruence on F containing Z_γ. Hence K_γ is a finitely generated congruence on F_γ contained in $\ker(\phi)$. So there must exist some $\gamma \in I$ such that $\gamma = (L_\gamma, K_\gamma)$, giving $\alpha, \beta \leq \gamma$ and hence I is directed.

Let (X, θ_α) be the directed colimit of $(A_\alpha, \phi^\alpha_\beta)$. There are natural S-maps $\mu_\alpha : A_\alpha \to F/\ker(\phi)$, $(a, n, s)K_\alpha \mapsto (a, n, s)\ker(\phi)$ which commute with ϕ^α_β for all $\alpha \leq \beta$, so by the property of colimits, there exists an S-map $\psi : X \to F/\ker(\phi)$ such that $\psi\theta_\alpha = \mu_\alpha$ for all $\alpha \in I$. We now intend to show that ψ is an isomorphism and hence $F/\ker(\phi)$ is a directed colimit of finitely presented S-acts.



For all $(a, n, s)\ker(\phi) \in F/\ker(\phi)$, $L_\delta := \{(a, n)\}$ is a finite (singleton) subset of $A \times \mathbb{N}$ and $F_\delta := L_\delta \times S$ is a free subact of F. Then $K_\delta := 1_{F_\delta}$, the identity relation on F_δ, and K_δ is a finitely generated congruence on F_δ contained in $\ker(\phi)$. Hence, there must exist some $\delta \in I$ such that $\delta = (L_\delta, K_\delta)$. Therefore $\psi\left(\theta_\delta\left((a, n, s)1_{F_\delta}\right)\right) = \mu_\delta\left((a, n, s)1_{F_\delta}\right) = (a, n, s)\ker(\phi)$ and ψ is an epimorphism. Now given any $x, x' \in X$ such that $\psi(x) = \psi(x')$, there exist α, $\beta \in I$, $(a, n, s)K_\alpha \in A_\alpha$, $(a', n', s')K_\beta \in A_\beta$ such that $\theta_\alpha\left((a, n, s)K_\alpha\right) = x$ and $\theta_\beta\left((a', n', s')K_\beta\right) = x'$. Now since $\psi(x) = \psi(x')$ we get that

$$\begin{aligned}(a,n,s)\ker(\phi) &= \theta_\alpha\left((a,n,s)K_\alpha\right) = \psi\mu_\alpha\left((a,n,s)K_\alpha\right) = \psi(x) = \psi(x')\\ &= \psi\mu_\beta\left((a',n',s')K_\beta\right) = \theta_\beta\left((a',n',s')K_\beta\right) = (a',n',s')\ker(\phi)\end{aligned}$$

so $((a,n,s),(a',n',s')) \in \ker(\phi)$. Now let Z_α, Z_β be finite generating sets for K_α, K_β respectively, and define $Z_\gamma := Z_\alpha \cup Z_\beta \cup \{((a,n,s),(a',n',s'))\}$ which is contained within $\ker(\phi)$. Let $L_\gamma := L_\alpha \cup L_\beta$ and K_γ be the congruence generated by Z_γ on F_γ, the free subact of F generated by L_γ. Clearly K_γ is a finitely generated congruence on F_γ and since $Z_\gamma \in \ker(\phi)$, by the same argument as before, the congruence generated by Z_γ on F_γ is contained within the congruence generated by Z_γ on F which is contained within $\ker(\phi)$. So there must exist some $\gamma \in I$ such that $\gamma = (L_\gamma, K_\gamma)$, giving α, $\beta \leq \gamma$ and $(a,n,s)K_\gamma = (a',n',s')K_\gamma$. Finally, we get that

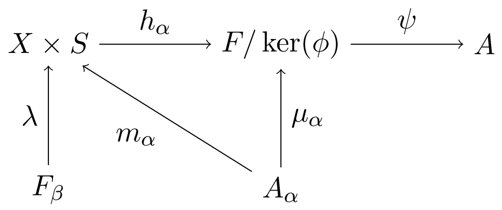
$$\begin{aligned}x &= \theta_\alpha((a,n,s)K_\alpha) = \theta_\gamma\phi_\gamma^\alpha((a,n,s)K_\alpha) = \theta_\gamma((a,n,s)K_\gamma)\\ &= \theta_\gamma((a',n',s')K_\gamma) = \theta_\gamma\phi_\gamma^\beta((a',n',s')K_\beta) = \theta_\beta((a',n',s')K_\beta) = x'\end{aligned}$$

and ψ is a monomorphism and hence an isomorphism. So A is a directed colimit of finitely presented S-acts as $A \cong F/\ker(\phi)$.

For the next part of the proof we show that when A is strongly flat, the set $I_0 := \{\beta \in I : A_\beta \text{ is (finitely generated) free}\}$ is cofinal in I (see [54, Exercise 2.43]), for then A is the directed colimit of the finitely generated free S-acts $\{A_\beta : \beta \in I_0\}$.

Let $\alpha \in I$, and given the S-map $\mu_\alpha : A_\alpha \to F/\ker(\phi)$, since $F/\ker(\phi) \cong A$ is strongly flat and A_α is finitely presented, by Theorem 4.10, there exists a finitely generated free S-act, which we represent as $X \times S$ where $X := \{x_1, \ldots, x_k\}$, and S-maps $m_\alpha : A_\alpha \to X \times S$, $h_\alpha : X \times S \to F/\ker(\phi)$ such that $h_\alpha m_\alpha = \mu_\alpha$. Let $a_i = \rho h_\alpha((x_i, 1))$ for each $x_i \in X$ where $\rho : F/\ker(\phi) \to A, (a,n,s)\ker(\phi) \mapsto as$ is an isomorphism, and define $L := \{(a_1,n_1), \ldots, (a_k,n_k)\}$, where $n_1, \ldots, n_m$ are chosen to be distinct and such that $(a_i, n_i) \notin L_\alpha$. Note that $L_\beta := L_\alpha \cup L$ is a finite subset of $A \times \mathbb{N}$.

Let $F_\beta := L_\beta \times S$ be the free subact of F generated by L_β and define $\lambda : F_\beta \to X \times S$ by $\lambda|_{F_\alpha} := m_\alpha K_\alpha^\natural$ and $\lambda((a_i,n_i,s)) := (x_i, s)$ for all $(a_i, n_i) \in L$, $s \in S$.



Now for each $(a_i, n_i, s) \in L \times S$ we have

$$(h_\alpha \lambda)((a_i, n_i, s)) = h_\alpha((x_i, s)) = h_\alpha((x_i, 1))s = \rho^{-1} a_i s = (a_i, n_i, s) \ker(\phi)$$

and for each $(a, n, s) \in L_\alpha \times S$ we have

$$(h_\alpha \lambda)((a, n, s)) = h_\alpha(m_\alpha((a, n, s)K_\alpha)) = \mu_\alpha((a, n, s)K_\alpha) = (a, n, s) \ker(\phi)$$

so that $h_\alpha \lambda = \ker(\phi)^\natural|_{F_\beta}$. So if we let $x, y \in F_\beta$, then it is clear that

$$\lambda(x) = \lambda(y) \Rightarrow h_\alpha \lambda(x) = h_\alpha \lambda(y) \Rightarrow \ker(\phi)^\natural(x) = \ker(\phi)^\natural(y) \Rightarrow (x, y) \in \ker(\phi)$$

and so $\ker(\lambda) \subseteq \ker(\phi)$. We now wish to show that $\ker(\lambda)$ is finitely generated.

Given any $(a, n) \in L_\alpha$, let $(x_{(a,n)}, s_{(a,n)}) := m_\alpha((a, n, 1)K_\alpha)$, and then take $(a_{(a,n)}, n_{(a,n)})$ to be the unique pair in L such that $a_{(a,n)} = \rho h_\alpha((x_{(a,n)}, 1))$. We can then define Z to be all the pairs $((a, n, 1), (a_{(a,n)}, n_{(a,n)}, s_{(a,n)}))$ where $(a, n) \in L_\alpha$, and note that Z is finite. Now let Z_α be a finite generating set for K_α and define $Z_\beta := Z_\alpha \cup Z$. We claim that $\ker(\lambda)$ is equivalent to K_β, the congruence generated by Z_β, and hence is finitely generated.

Given any pair in Z_β, it is either in Z_α or it is in Z, so we consider two cases. Firstly, let $((a, n, s), (a', n', s')) \in Z_\alpha$, then $(a, n, s)K_\alpha = (a', n', s')K_\alpha$ and

$$\begin{aligned} & K_\alpha^\natural((a, n, s)) = K_\alpha^\natural((a', n', s')) \\ \Rightarrow\ & \lambda((a, n, s)) = m_\alpha K_\alpha^\natural((a, n, s)) = m_\alpha K_\alpha^\natural((a', n', s')) = \lambda((a', n', s')). \end{aligned}$$

Secondly, let $((a, n, 1), (a_{(a,n)}, n_{(a,n)}, s_{(a,n)}) \in Z$, then

$$\lambda((a, n, 1)) = m_\alpha K_\alpha^\natural((a, n, 1)) = (x_{(a,n)}, s_{(a,n)}) = \lambda((a_{(a,n)}, n_{(a,n)}, s_{(a,n)})).$$

Therefore, given any pair $(p_i, q_i) \in Z_\beta \cup Z_\beta^{op}$, $\lambda(p_i) = \lambda(q_i)$. So given any pair $((a, n, s), (a', n', s'))$ in K_β, either $(a, n, s) = (a', n', s')$ in which case

$\lambda((a,n,s)) = \lambda((a',n',s'))$ or

$$(a,n,s) = p_1w_1, \quad q_1w_1 = p_2w_2, \quad q_2w_2 = p_3w_3, \quad \cdots \quad q_nw_n = (a',n',s')$$

where $w_1,\ldots,w_n \in S$ and $(p_i,q_i) \in Z_\beta \cup Z_\beta^{op}$, so that

$$\begin{aligned}\lambda((a,n,s)) &= \lambda(p_1w_1) = \lambda(p_1)w_1 \\ &= \lambda(q_1)w_1 = \lambda(q_1w_1) = \cdots = \lambda(q_nw_n) = \lambda((a',n',s')).\end{aligned}$$

Hence $K_\beta \subseteq \ker(\lambda)$. Now we intend to show that $\ker(\lambda) \subseteq K_\beta$.

Let $((a,n,s),(a',n',s')) \in \ker(\lambda)$, since $F_\beta := F_\alpha \amalg (L \times S)$, without loss of generality we can consider three cases;

(i) $(a,n,s),(a',n',s') \in F_\alpha$; or

(ii) $(a,n,s),(a',n',s') \in L \times S$; or

(iii) $(a,n,s) \in F_\alpha$, $(a',n',s') \in L \times S$.

For each case we show that $((a,n,s),(a',n',s')) \in K_\beta$.

(i) Let $(a,n,s),(a',n',s') \in F_\alpha$ and $\lambda((a,n,s)) = \lambda((a',n',s'))$. Then $\lambda((a,n,1))s = \lambda((a',n',1))s$ and $(x_{(a,n)}, s_{(a,n)})s = (x_{(a',n')}, s_{(a',n')})s'$, hence $x_{(a,n)} = x_{(a',n')}$ and $s_{(a,n)}s = s_{(a',n')}s'$, therefore we also have $a_{(a,n)} = a_{(a',n')}$ and $n_{(a,n)} = n_{(a',n')}$. Hence

$$\begin{aligned}(a,n,s) =& (a,n,1)s, \\ & (a_{(a,n)}, n_{(a,n)}, s_{(a,n)})s \quad = \quad (a_{(a',n')}, n_{(a',n')}, s_{(a',n')})s', \\ & (a',n',1)s' = (a',n',s')\end{aligned}$$

and $((a,n,s),(a',n',s')) \in K_\beta$.

(ii) Let $(a_i,n_i,s),(a_j,n_j,s') \in L \times S$ and $\lambda((a_i,n_i,s)) = \lambda((a_j,n_j,s'))$. Then $(x_i,s) = (x_j,s')$ and $a_i = \rho h_\alpha(x_i,1) = \rho h_\alpha(x_j,1) = a_j$ so $n_i = n_j$ as well. Therefore $((a_i,n_i,s),(a_j,n_j,s')) \in K_\beta$ since it is reflexive.

(iii) Let $(a,n,s) \in F_\alpha$, $(a_i,n_i,s') \in L \times S$, and $\lambda((a,n,s)) = \lambda((a_i,n_i,s'))$. Then $\lambda((a,n,1))s = \lambda((a_i,n_i,s')$ and $(x_{(a,n)}, s_{(a,n)})s = (x_i,s')$. Therefore $a_{(a,n)} = \rho h_\alpha(x_{(a,n)},1) = \rho h_\alpha(x_i,1) = a_i$ and so $n_{(a,n)} = n_i$. Now,

$$\begin{aligned}(a,n,s) =& (a,n,1)s \\ & (a_{(a,n)}, n_{(a,n)}, s_{(a,n)})s = (a_i, n_i, s_{(a,n)}s) = (a_i,n_i,s')\end{aligned}$$

and $((a,n,s),(a_i,n_i,s')) \in K_\beta$. Therefore $\ker(\lambda) \subseteq K_\beta$ and $\ker(\lambda) = K_\beta$.

Since K_β is generated by $Z \cup Z_\alpha$ it clearly contains K_α which is generated by Z_α, so there must exist some $\beta = (L_\beta, K_\beta) \in I$ such that $\alpha \leq \beta$. Finally $\beta \in I_0$, since $A_\beta := F_\beta / K_\beta = F_\beta / \ker(\lambda) \cong X \times S$, which is free. $\square$

Bibliography

[1] J. Adámek and J. Rosický. On pure quotients and pure subobjects. *Czechoslovak Math. J.*, 54(129)(3):623–636, 2004.

[2] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[3] J. Ahsan and Z. Liu. *A homological approach to the theory of monoids*. Science Press, 2008.

[4] S. Tempest Aldrich, Edgar E. Enochs, J. R. García Rozas, and Luis Oyonarte. Covers and envelopes in Grothendieck categories: flat covers of complexes with applications. *J. Algebra*, 243(2):615–630, 2001.

[5] A. Amini, B. Amini, M. Ershad, and H. Sharif. On generalized perfect rings. *Comm. Algebra*, 35(3):953–963, 2007.

[6] Alex Bailey, Victoria Gould, Miklós Hartmann, James Renshaw, and Lubna Shaheen. Covers for S-acts and Condition (A) for a monoid S. To appear.

[7] Hyman Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.*, 95:466–488, 1960.

[8] L. Bican, R. El Bashir, and E. Enochs. All modules have flat covers. *Bull. London Math. Soc.*, 33(4):385–390, 2001.

[9] Ladislav Bican. Some properties of precovers and covers. *Algebr. Represent. Theory*, 12(2-5):311–318, 2009.

[10] Philip Bridge. Private communication. PhD Student in Manchester, November 2010.

[11] Philip Bridge. *Essentially algebraic theories and localizations in toposes and abelian categories*. PhD thesis, School of Mathematics, University of Manchester, January 2012.

[12] S. Bulman-Fleming and K. McDowell. Absolutely flat semigroups. *Pacific J. Math.*, 107(2):319–333, 1983.

[13] Sydney Bulman-Fleming. Pullback-flat acts are strongly flat. *Canad. Math. Bull.*, 34(4):456–461, 1991.

[14] Sydney Bulman-Fleming. The classification of monoids by flatness properties of acts. In *Semigroups and applications (St. Andrews, 1997)*, pages 18–38. World Sci. Publ., River Edge, NJ, 1998.

[15] Sydney Bulman-Fleming and Valdis Laan. Lazard's theorem for S-posets. *Math. Nachr.*, 278(15):1743–1755, 2005.

[16] Sydney Bulman-Fleming and Kenneth McDowell. Monoids over which all weakly flat acts are flat. *Proc. Edinburgh Math. Soc. (2)*, 33(2):287–298, 1990.

[17] Georg Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Math. Ann.*, 46(4):481–512, 1895.

[18] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[19] Septimiu Crivei, Mike Prest, and Blas Torrecillas. Covers in finitely accessible categories. *Proc. Amer. Math. Soc.*, 138(4):1213–1221, 2010.

[20] Robert El Bashir. Covers and directed colimits. *Algebr. Represent. Theory*, 9(5):423–430, 2006.

[21] Edgar Enochs. Torsion free covering modules. *Proc. Amer. Math. Soc.*, 14:884–889, 1963.

[22] Edgar Enochs, S. Estrada, J. R. García-Rozas, and L. Oyonarte. Flat cotorsion quasi-coherent sheaves. Applications. *Algebr. Represent. Theory*, 7(4):441–456, 2004.

[23] Edgar Enochs, S. Estrada, J. R. García Rozas, and L. Oyonarte. Flat covers in the category of quasi-coherent sheaves over the projective line. *Comm. Algebra*, 32(4):1497–1508, 2004.

[24] Edgar Enochs and Luis Oyonarte. Flat covers and cotorsion envelopes of sheaves. *Proc. Amer. Math. Soc.*, 130(5):1285–1292, 2002.

[25] Edgar E. Enochs. Injective and flat covers, envelopes and resolvents. *Israel J. Math.*, 39(3):189–209, 1981.

[26] Edgar E. Enochs and Overtoun M. G. Jenda. *Relative homological algebra. Volume 1*, volume 30 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, extended edition, 2011.

[27] Edgar E. Enochs and Overtoun M. G. Jenda. *Relative homological algebra. Volume 2*, volume 54 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2011.

[28] John Fountain. Perfect semigroups. *Proc. Edinburgh Math. Soc. (2)*, 20(2):87–93, 1976.

[29] Peter Freyd. *Abelian categories. An introduction to the theory of functors*. Harper's Series in Modern Mathematics. Harper & Row Publishers, New York, 1964.

[30] Rüdiger Göbel and Jan Trlifaj. *Approximations and endomorphism algebras of modules. Volume 1*, volume 41 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, extended edition, 2012. Approximations.

[31] Rüdiger Göbel and Jan Trlifaj. *Approximations and endomorphism algebras of modules. Volume 2*, volume 41 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, extended edition, 2012. Predictions.

[32] Victoria Gould. The characterisation of monoids by properties of their S-systems. *Semigroup Forum*, 32(3):251–265, 1985.

[33] John M. Howie. *Fundamentals of semigroup theory*, volume 12 of *London Mathematical Society Monographs. New Series.* The Clarendon Press Oxford University Press, New York, 1995. Oxford Science Publications.

[34] John R. Isbell. Perfect monoids. *Semigroup Forum*, 2(2):95–118, 1971.

[35] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[36] R. Khosravi, M. Ershad, and M. Sedaghatjoo. Strongly flat and Condition (P) covers of acts over monoids. *Comm. Algebra*, 38(12):4520–4530, 2010.

[37] Mati Kilp. Perfect monoids revisited. *Semigroup Forum*, 53(2):225–229, 1996.

[38] Mati Kilp, Ulrich Knauer, and Alexander V. Mikhalev. *Monoids, acts and categories*, volume 29 of *de Gruyter Expositions in Mathematics.* Walter de Gruyter & Co., Berlin, 2000. With applications to wreath products and graphs, A handbook for students and researchers.

[39] Ulrich Knauer and Mario Petrich. Characterization of monoids by torsion-free, flat, projective and free acts. *Arch. Math. (Basel)*, 36(4):289–294, 1981.

[40] David Kruml. On flat covers in varieties. *Comment. Math. Univ. Carolin.*, 49(1):19–24, 2008.

[41] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1999.

[42] Mark V. Lawson. *Inverse semigroups.* World Scientific Publishing Co. Inc., River Edge, NJ, 1998. The theory of partial symmetries.

[43] Daniel Lazard. Autour de la platitude. *Bull. Soc. Math. France*, 97:81–128, 1969.

[44] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[45] P. Normak. Noetherian and finitely presented polygons. *Tartu Riikl. Ül. Toimetised*, (Vih. 431):37–46, 1977.

[46] Peeter Normak. Purity in the category of M-sets. *Semigroup Forum*, 20(2):157–170, 1980.

[47] Peeter Normak. On equalizer-flat and pullback-flat acts. *Semigroup Forum*, 36(3):293–313, 1987.

[48] Donald S. Passman. *A course in ring theory*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991.

[49] James Renshaw. *Flatness, extension and amalgamation in monoids, semigroups and rings*. PhD thesis, Department of Pure Mathematics, St. Andrews, May 1985.

[50] James Renshaw. Extension and amalgamation in monoids and semigroups. *Proc. London Math. Soc. (3)*, 52(1):119–141, 1986.

[51] James Renshaw. Monoids for which Condition (P) acts are projective. *Semigroup Forum*, 61(1):45–56, 2000.

[52] James Renshaw and Mojgan Mahmoudi. On covers of cyclic acts over monoids. *Semigroup Forum*, 77(2):325–338, 2008.

[53] J. Rosický. On projectivity in locally presentable categories. *J. Algebra*, 272(2):701–710, 2004.

[54] Joseph J. Rotman. *An introduction to homological algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[55] Pedro V. Silva and Benjamin Steinberg. A geometric characterization of automatic monoids. *Q. J. Math.*, 55(3):333–356, 2004.

[56] Bo Stenström. Flatness and localization over monoids. *Math. Nachr.*, 48:315–334, 1971.

[57] Jinzhong Xu. *Flat covers of modules*, volume 1634 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

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