Supplementary Information for: Fu, G., Dawson, R., Khoury, M., & Bullock, S. (2014) Interdependent networks: Vulnerability analysis and strategies to limit cascading failure, *European Physical Journal B.*

1. Performance Comparison of Interdependent and Isolated Systems

We compare the performance of an interdependent system against that of a system which has each of its sub-networks isolated or independent from one another. Fig. S1 plots the relative size, *P*, of the giant component of network *A* and *B*, as function of *q* (the size of attack to network A), for a undirected interdependent system with $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 1.0$ and $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 2$. The results are compared against those of isolated, but otherwise identical, networks, *A'* and *B'*. A failure threshold q_c is observed for the interdependent system. When $q < q_c$, a giant component exists in both networks *A* and *B*; when $q > q_c$, both networks become completely fragmented. The disruption to network *A* results from the initial attack plus the iterative propagation of cascading failure, while the disruption to network *B* is purely caused by cascading failure. Therefore it is not surprising that network *B* performs better than network *A*. While an isolated network *A* undergoes continuous transition at q_c (i.e., P^A is continuous at q_c), an abrupt transition is observed at q_c for an interdependent system. For an interdependent system in which *A* and *B* are fully dependent on each other, P^A and P^B at q_c (denoted as P_c^A and P_c^B) are non-zero, and abruptly drop to zero when $q > q_c$. When either $\mathcal{K}^{A,B} = \mathcal{K}^{B,A}$ is sufficiently large or $\mathcal{F}^{A,B} = \mathcal{F}^{B,A}$ is sufficiently small, the performance of an interdependent system approaches that of a system which has each of its sub-networks isolated or independent from one another.

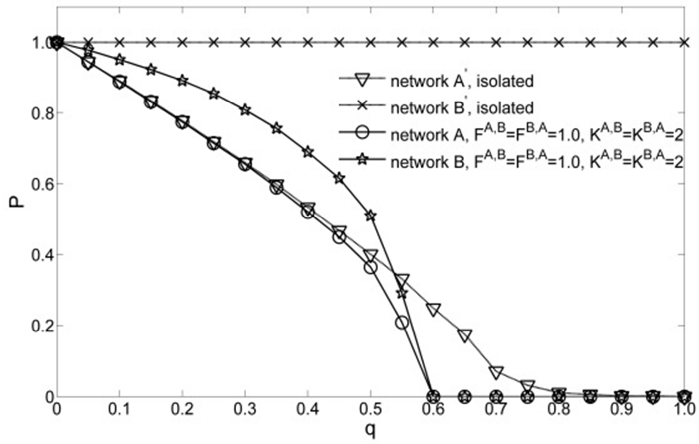

Fig. S1 Performance comparison of interdependent and isolated systems: relative size, *P*, of giant components for networks *A* and *B*, as function of, *q*, the size of initial disruption to network *A*, when $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 1.0$, $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 2$, and all dependencies are undirected. The results are compared against those of isolated, but otherwise identical, networks, *A'* and *B'*.

2. Performance Study of Undirected systems

For an undirected system, as interdependencies are *symmetric*, the following condition must always be satisfied:

$$N^A * \mathcal{F}^{A,B} * \mathcal{K}^{A,B} = N^B * \mathcal{F}^{B,A} * \mathcal{K}^{B,A} \qquad \text{(S1)}$$

That is, the total number of dependences of one network, *A*, on another network, *B*, must always equal the total number of dependences of network *B* on network *A*.

We studied the performance of undirected systems when $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$ take different values, and the results are plotted in Fig. S2(a)[1]. We observed that the ratio $\mathcal{K}^{B,A}:\mathcal{K}^{A,B}$, determined by $\mathcal{F}^{A,B}:\mathcal{F}^{B,A}$, plays an important role in determining system performance. The larger $\mathcal{K}^{B,A}:\mathcal{K}^{A,B}$, the better the system performs when an attack

[1] Though, in principle, both $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$ may be arbitrary non-negative integers, Eq. (S1) restricts the values that $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$ may take. For example, for a system where $\mathcal{K}^{B,A} = 0.5 * \mathcal{K}^{A,B}$, instead of 1, the minimum value for $\mathcal{K}^{A,B}$ is 2 (so as to ensure an integer value for $\mathcal{K}^{B,A}$). This is reflected in Fig. S2(a) for the system with $\mathcal{F}^{A,B} = 0.5, \mathcal{F}^{B,A} = 1.0$ and $\mathcal{K}^{B,A} = 0.5 * \mathcal{K}^{A,B}$.

is initiated in network *A*. However, systems with the same $\mathcal{K}^{B,A}:\mathcal{K}^{A,B}$ do not deliver the same performance when different values are given to $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$. For systems with the same $\mathcal{K}^{B,A}:\mathcal{K}^{A,B}$, smaller $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$are better for system performance. For example, when $\mathcal{K}^{B,A}:\mathcal{K}^{A,B} = 1.0$, a system with $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 0.5$ performs better than a system with $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 1.0$.

We also studied the performance of undirected systems when $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$ take different values, and results are presented in Fig. S2(b)[2]. A strong negative correlation between *IP* and $\mathcal{F}^{A,B}$ (and $\mathcal{F}^{B,A}$) was identified, i.e., increasing $\mathcal{F}^{A,B}$ (and $\mathcal{F}^{B,A}$) , reduces *IP*. The ratio $\mathcal{F}^{B,A}:\mathcal{F}^{A,B}$, determined by $\mathcal{K}^{A,B}:\mathcal{K}^{B,A}$, plays an important role in deciding system performance. The smaller $\mathcal{F}^{B,A}:\mathcal{F}^{A,B}$, the better a system performs when an attack is initiated in network *A*. However, systems with the same $\mathcal{F}^{B,A}:\mathcal{F}^{A,B}$ ratio do not deliver the same performance when different values are assigned to $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$. For systems with the same $\mathcal{F}^{B,A}:\mathcal{F}^{A,B}$, the larger $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$, the better the system performs. For example, when $\mathcal{F}^{B,A}:\mathcal{F}^{A,B} = 1.0$, a system with $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 2$ performs better than a system with $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 1$.

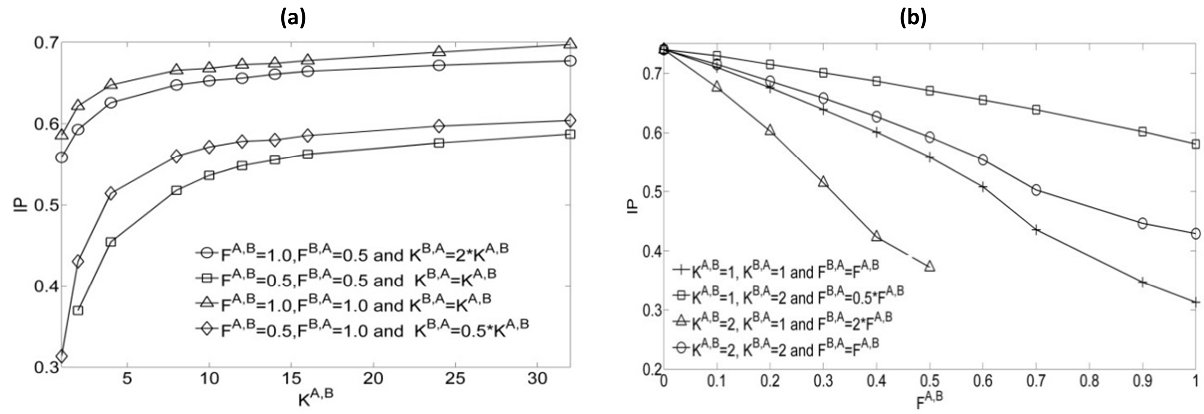

Fig. S2 Undirected systems. (a) aggregate performance *IP* as a function of $\mathcal{K}^{A,B}$ (b) aggregate performance *IP* as a function of $\mathcal{F}^{A,B}$.

The joint effect of the redundancy, $\mathcal{K}$, and extent, $\mathcal{F}$, on the performance of undirected systems are summarised in Fig. S3. By varying $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = K$ and $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = F$ respectively, a wide range of undirected systems were simulated. Results in Fig. S3 show that an undirected system is most vulnerable when $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 1$ and $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 1.0$, and its performance improves when $\mathcal{K}^{A,B} = \mathcal{K}^{B,A}$ increases its value or $\mathcal{F}^{A,B} = \mathcal{F}^{B,A}$ decreased its value.

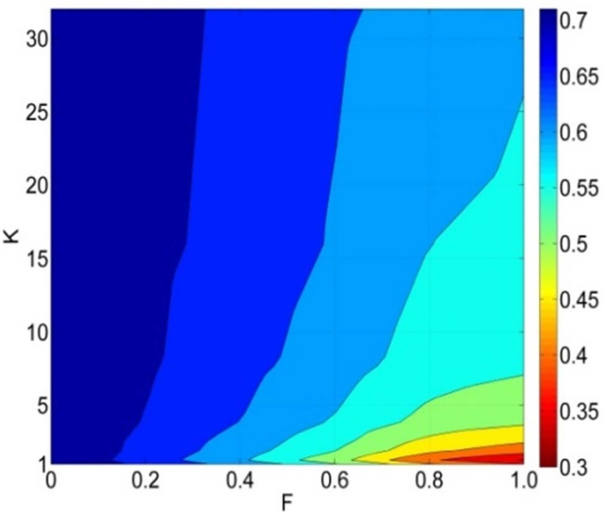

Fig. S3 Aggregated system performance of undirected systems. *IP* is plotted as a function of $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = K$ and $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = F$.

[2] $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$ are in the range of 0.0 to 1.0 . However Eq. (S1) can restrict the values that $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$ may take. For example, for a system with $\mathcal{F}^{B,A} = 2 * \mathcal{F}^{A,B}$, in order for both $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$ to be valid, the range of $\mathcal{F}^{A,B}$ is [0.0, 0.5] and the range of $\mathcal{F}^{B,A}$ is [0.0, 1.0]. This is reflected in Fig. S2(b) for the system with $\mathcal{K}^{A,B} = 2$, $\mathcal{K}^{B,A} = 1$ and $\mathcal{F}^{B,A} = 2 * \mathcal{F}^{A,B}$.

3. Performance Study of Directed systems

By contrast with undirected coupling, dependencies within a directed system are asymmetric. Therefore the number of dependencies of network A on network B does not have to be the same as the number of dependencies of network B on network A. That is, $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$ (or $\mathcal{K}^{B,A}$ and $\mathcal{K}^{A,B}$) do not constrain one another as in an undirected setting. This allows a wide range of network coupling modes to be generated to simulate real world systems. Fig. S4 shows the performance of directed systems where the dependent redundancy and dependent extent from its sub-networks are different from one other. When the initial disruption starts in network A, we observed that:

- $\mathcal{K}^{B,A}$ plays a more dominant role than $\mathcal{K}^{A,B}$. When we fix $\mathcal{K}^{B,A}$ and increase $\mathcal{K}^{A,B}$, *IP* increases, but it does not increase as fast as when we fix $\mathcal{K}^{A,B}$ and increase $\mathcal{K}^{B,A}$.
- $\mathcal{F}^{B,A}$ plays a more important role than $\mathcal{F}^{A,B}$. When we fix $\mathcal{F}^{B,A}$ and increase $\mathcal{F}^{A,B}$, *IP* decreases, but it does not decrease as fast as when we fix $\mathcal{F}^{A,B}$ and increase $\mathcal{F}^{B,A}$.

This is because when the disruption is initiated in network A, $\mathcal{K}^{B,A}$ and $\mathcal{F}^{B,A}$ determines the magnitude of failure propagation from A to B at the first iteration of cascading failure. The failure probability of a network B node is proportional to $\mathcal{F}^{B,A}$ and inversely proportional to $\mathcal{K}^{B,A}$. Therefore, increasing $\mathcal{F}^{B,A}$ directly increases the extent to which failure is cascaded to network B, and consequently increases the chances of failure propagation in the subsequent stages of the cascade. On the other hand, increasing $\mathcal{K}^{B,A}$ directly reduces failure cascaded to network B, which in turn reduces the chances of failure propagation in the further iterations of the cascade. Our experiments reveal that $\mathcal{F}^{A,B}$ impacts on system performance only when $\mathcal{F}^{B,A}$ is sufficient large, and $\mathcal{K}^{A,B}$ impacts on system performance only when $\mathcal{K}^{B,A}$ is sufficient small. When $\mathcal{F}^{B,A}$ is small (or $\mathcal{K}^{B,A}$ is large), varying $\mathcal{F}^{A,B}$(or $\mathcal{K}^{A,B}$) has no significant effect on system performance.

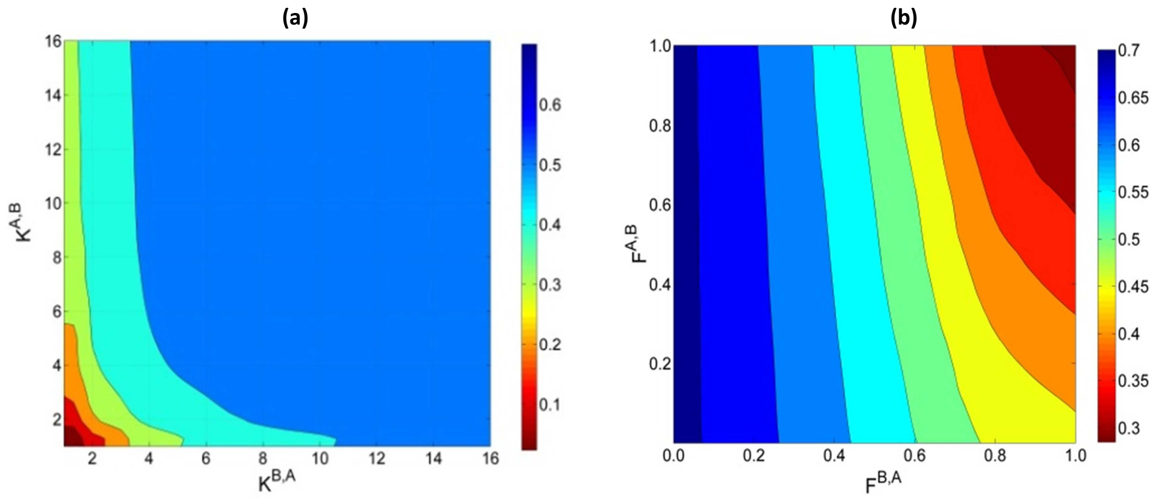

Fig. S4 Performance of directed systems. (a) aggregate performance, *IP*, as a function of $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$, where $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 1.0$ (b) aggregate performance, *IP*, as a function of $\mathcal{F}^{A,B}$ and $\mathcal{F}^{B,A}$, where $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 2$.

A directed system is more vulnerable than an undirected system. Fig. S5 and Fig. 5 (of the main paper) show that a directed system has a smaller failure threshold than an undirected system, and therefore is more vulnerable to cascading failure. The performance of a directed system approaches that of an undirected system when $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = K$ is sufficient large or $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = F$ is sufficient small.

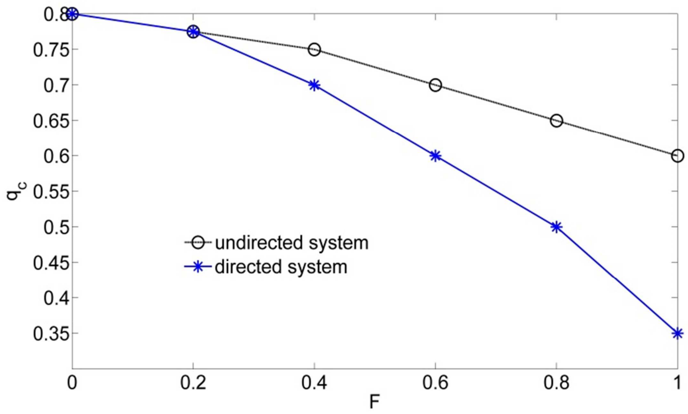

Fig. S5 Performance comparison of undirected and directed systems: failure threshold, q_c, as a function of $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = F$, where $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 2$.

4. Impact of Network Topology and Disruption Strategy on System Performance

Additional to the analysis of coupled ER networks as reported in the main paper, we also consider the performance of interdependent systems that couple two Barabási–Albert scale free (SF) networks, and systems that couple ER and SF networks. The experiments were carried out over networks of the same size and order as coupled ER networks, i.e., each sub-network has 10000 nodes with average node degree of 4. The aggregate performance of these systems under random node attack is presented in Fig. S6. Comparison of these results with Fig. S3 of this document and Fig. 3 of the main paper shows similar patterns in overall performance to those reported for the ER-ER networks.

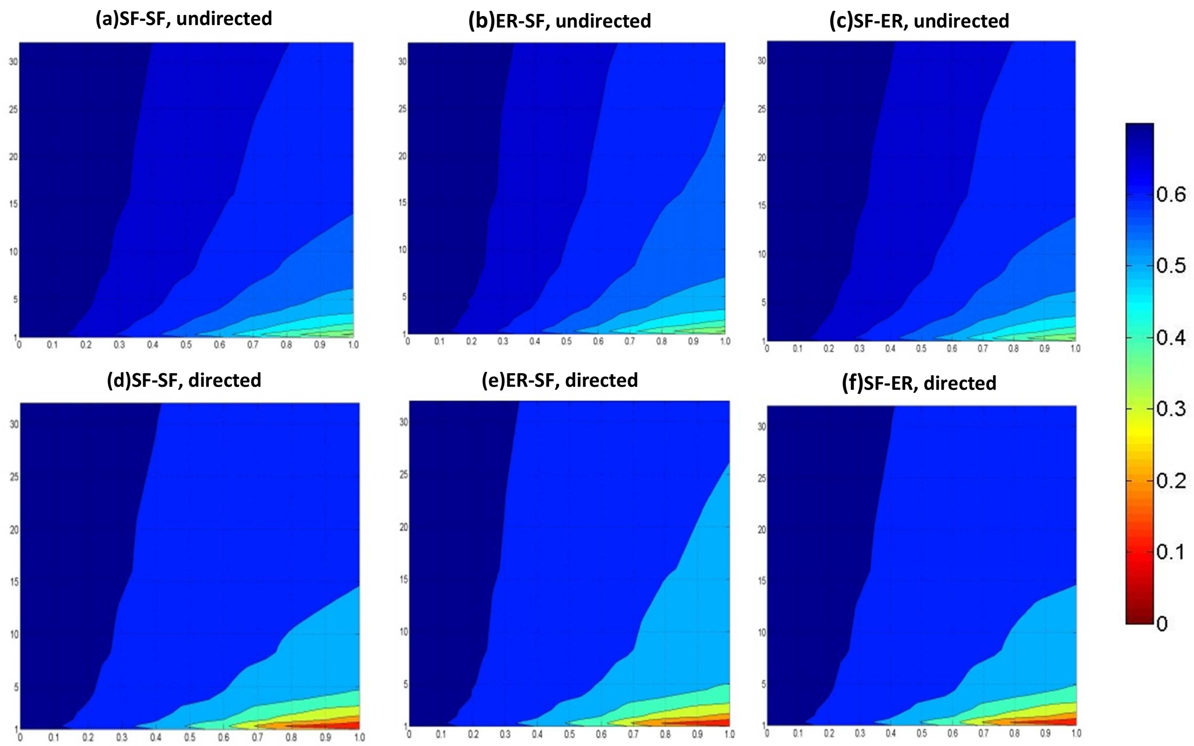

Fig. S6. Simulation results for pairs of coupled networks of varying topology under random attack. Aggregate performance *IP* is plotted as a function of $\mathcal{K}^{\mathrm{A,B}} = \mathcal{K}^{\mathrm{B,A}} = K$ (vertical axis) and $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = F$ (horizontal axis) (a) IP for SF-SF systems with undirected dependencies (b) IP for ER-SF systems with undirected dependencies - where network disruption starts in ER (c) IP for SF-ER systems with undirected dependencies - where network disruption starts in SF (d) IP for SF-SF systems with directed dependencies (e) IP for ER-SF systems with directed dependencies (f) IP for SF-ER systems with directed dependencies.

In the main paper, we have reported on the performance of ER-ER and SF-SF systems under deliberate attacks. We also carried out experiments to investigate the performance of systems that couple ER and SF network under deliberate attacks. Fig. S7 shows the aggregate performance of these systems under two types of deliberate attack: high degree node biased attack (henceforth highBias attack) and low degree node biased attack (henceforth lowBias attack). Results presented in Fig. 12 (of the main paper) and Fig. S7 demonstrate that the SF-SF and SF-ER are most heavily impacted by highBias attacks, but perform better than ER-ER or ER-SF networks subjected to lowBias attacks. The degree distribution of SF networks leads to more highly connected hub nodes compared to ER networks. Consequently highBias attacks lead to more fragmentation in an SF network than in an ER network. On the other hand, due to the existence of a large portion of low degree nodes in SF networks, when lowBias attack is employed, nodes of lower connectivity are preferentially targeted so the SF-SF and SF-ER systems outperform the other two configurations as a result of a lowBias attack.

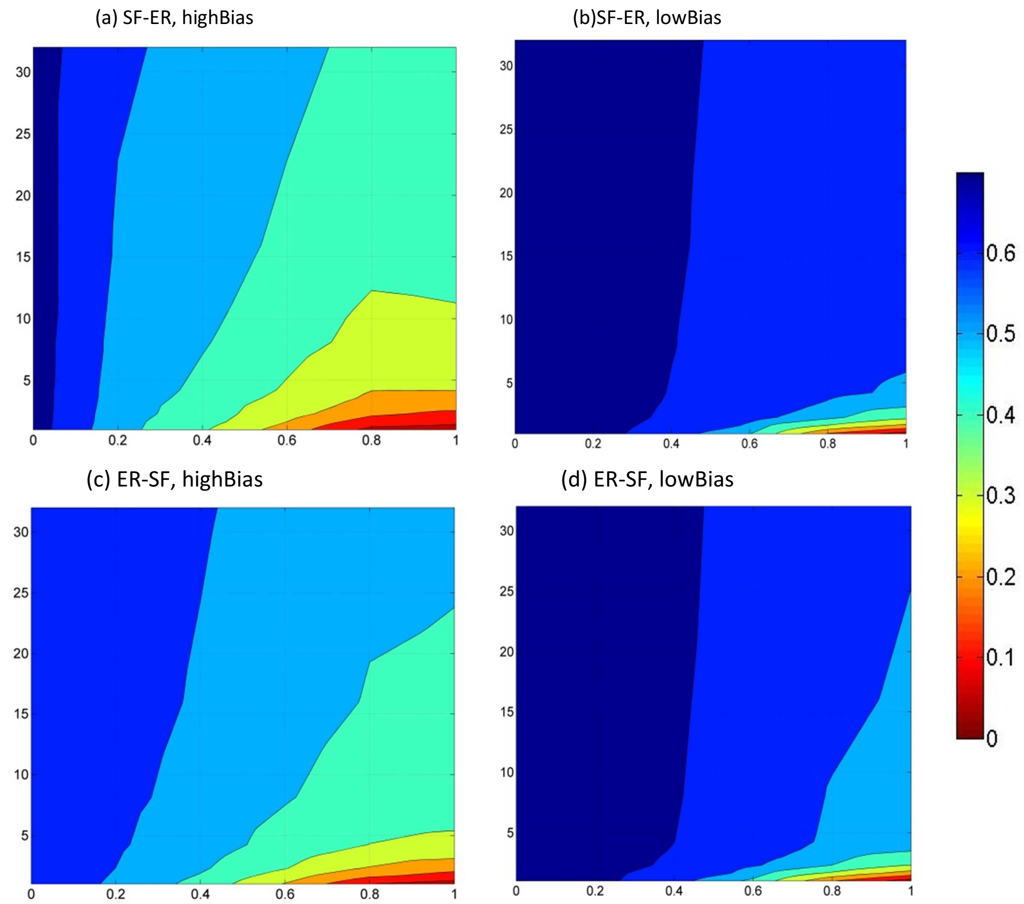

Fig. S7 Results for systems under lowBias and highBias attacks. Aggregate performance *IP* is plotted as a function of $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = K$ (vertical axis) and $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = F$ (horizontal axis), under directed dependency setting (i.e. $\mathcal{D}^{A,B} = \mathcal{D}^{B,A} = 1$).

5. Reducing Vulnerability: Outlook and Discussion

5.1 Optimising Interdependency Redundancy

We studied the effect of interdependency redundancy on the performance change of directed systems. Since a directed system with $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 1$ performs worst, we used it as a baseline system to study how the performance of a directed system may be improved. We use R_{IP} to denote the ratio of change from one interdependency setting to another. R_{IP} is calculated as follows, where IP_x and IP_y are aggregate performance measures under interdependency setting *x* and *y*.

$$R_{IP} = (IP_y - IP_x)/IP_x \qquad \text{(S2)}$$

Three strategies were studied and results are presented in Fig. S8. Strategy I fixed $\mathcal{K}^{B,A} = 1$ and increased $\mathcal{K}^{A,B}$. Strategy II fixed $\mathcal{K}^{A,B} = 1$ and increased $\mathcal{K}^{B,A}$. In the third case we increased both $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$ at the same rate. The performance of a directed system improves at the fastest rate in strategy III, but this strategy can be expensive to implement since it requires the creation of additional supporting links from both networks. While strategy II achieves slightly lower performance than strategy III, it is significantly more affordable since it requires only 50% of the additional dependencies. Strategy I is the least desirable in terms of the rate of improvement it achieves. For example, with strategy I, increasing $\mathcal{K}^{A,B}$ from 1 to 4 improves system performance by around 400%. For the same investment (increasing $\mathcal{K}^{B,A}$ from 1 to 4), strategy II improves the system performance by around 600%. Increasing $\mathcal{K}^{A,B}$ and $\mathcal{K}^{B,A}$ simultaneously is the most desirable solution to reduce the vulnerability of directed systems. When the cost of investing in additional redundancy is considered, adding further support from the attacked network to the other network is more effective than the other way around.

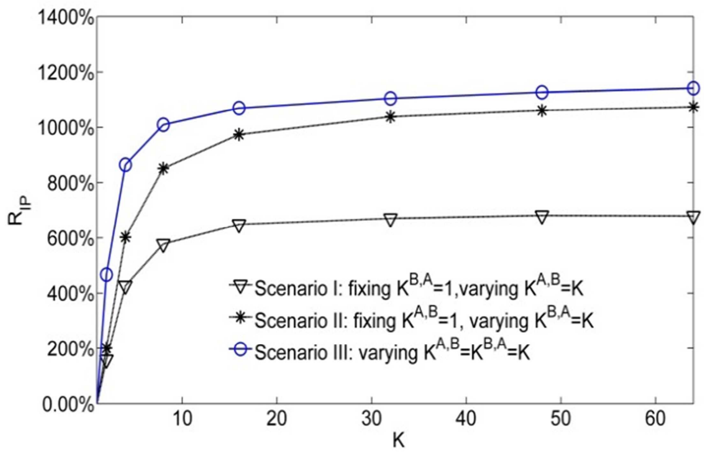

Fig. S8 Reducing vulnerability: ratio of performance change R_{IP} when interdependent redundancy, *K,* was increased under three different strategies.

5.2 Supporting High Degree Nodes

Since the failure of high degree nodes is more likely to lead to large scale network fragmentation, we investigated the effect of *protecting* high degree nodes as a means of improving system performance. One way of doing this is to increase the support of a high degree node and make it less vulnerable to cascading failure. We name the this strategy HDMS (High Degree node getting More Support). Rather than assigning interdependencies randomly across networks, the likelihood that node u in one network (e.g., network A) gains support from a node in another network (e.g., network B) increases exponentially with u's degree, k_u :

$$P(l_u^{A,B}) \propto \alpha_1 * (1 + \alpha_2)^{k_u} \qquad \text{(S3)}$$

Fig. S9 shows how by fixing α_1 and increasing α_2, we can vary the interdependent links for high degree nodes. When α_2 is small, interdependencies are uniformly distributed, i.e., each node has similar support. The support received by high degree nodes increases with increasing α_2.

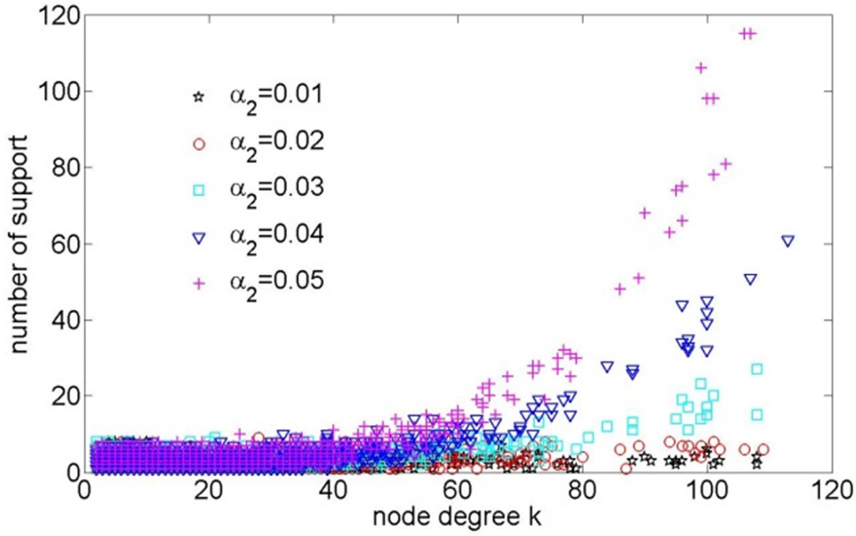

Fig. S9 The relation between node degree and the support that a node receives in a SF-SF system with HDMS protection strategy. We fix $\alpha_1 = 0.02$ and vary α_2 in the range 0.01 to 0.05. The plot is generated for a directed system with 2000 nodes and with average intra-network degree of 4, where $\mathcal{K}^{A,B} = \mathcal{K}^{B,A} = 2$ and $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 1.0$.

Our experimental results show that for a range of HDMS strategies of increasing α_2 (and consequently the number of supporting interdependent links) there is little gain in aggregate system performance. However Fig. S10 shows that increasing α_2 does reduce the variability of P and leads to a more consistent failure response. This can help network operators better plan for a respond to failure.

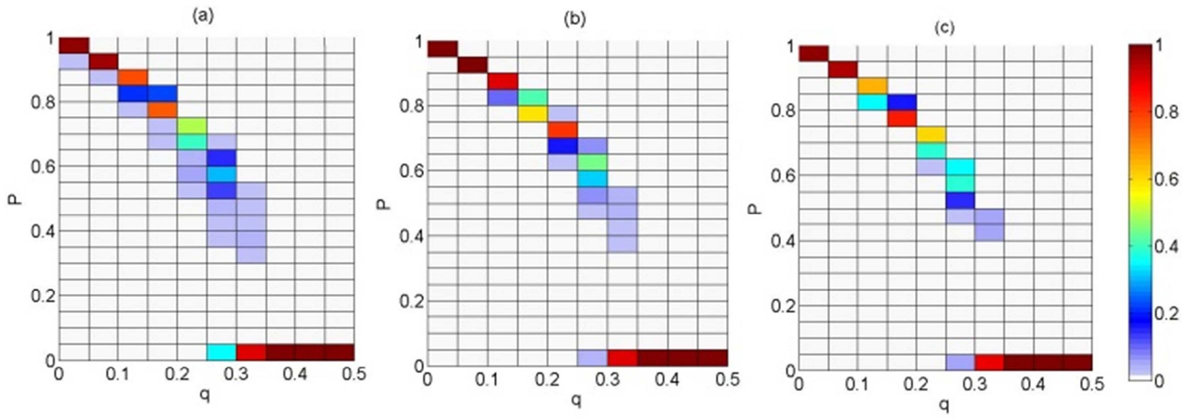

Fig. S10 Simulation results for SF-SF HDMS systems where inter-network dependencies are directed and $\mathcal{F}^{A,B} = \mathcal{F}^{B,A} = 1.0$ and $\mathcal{K}^{\mathrm{A,B}} = \mathcal{K}^{\mathrm{B,A}} = 2$. Frequency of relative size of giant component P is plotted as a function of q and P. (a) $\alpha_2 = 0.01$. (b) $\alpha_2 = 0.02$. (c) $\alpha_2 = 0.03$.