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The loop space homotopy type of simply-connected four-manifolds and their generalizations

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A R T I C L E   I N F O

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We determine loop space decompositions of simply-connected four-manifolds, $(n-1)$-connected $2n$-dimensional manifolds provided $n \notin \{4, 8\}$, and connected sums of products of two spheres. These are obtained as special cases of a more general loop space decomposition of certain torsion-free CW-complexes with well-behaved skeleta and some Poincaré duality features.

1. Introduction

The topology of simply-connected four-manifolds is a subject of widespread and enduring interest. They have been classified up to homotopy type by Milnor [19] and up to homeomorphism type by Freedman [13]. Their classification up to diffeomorphism type is one of the great unsolved questions in modern mathematics, with significant advances

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achieved by Donaldson [9] and Seiberg and Witten [28]. They have also been studied in view of their connections to other areas of mathematics, such as knot theory [12] and symplectic geometry [23].

The homotopy theory of simply-connected four-manifolds has continued to attract considerable attention since Milnor's classification. For example, given simply-connected four-manifolds M and N, Cochran and Habegger [6] calculated the group of self-homotopy equivalences of M; Zhao, Gao and Su [29] calculated the homotopy classes of maps $[M, N]$; and Baues [4] has written a monograph entirely devoted to investigating the homotopy theory of M, N and the maps between them.

In another direction, Wall [27] initiated the study of $(n-1)$-connected $2n$-dimensional manifolds as a generalization of simply-connected four-manifolds. Such manifolds have received considerable recent attention as certain families of them arise as intersections of quadrics in geometric topology and moment-angle manifolds in toric topology [5,14]. Another variation is connected sums of products of two spheres, which generalizes the sub-collection of simply-connected four-manifolds that are connected sums of $S^2 \times S^2$. Such connected sums appear in the classification by McGavran [18] of n-torus actions on closed, compact, simply-connected $(n+2)$-manifolds, and they also appear as intersections of quadrics and moment-angle manifolds [5,14].

In this paper we study simply-connected four-manifolds and their generalizations from a new perspective. Let M be a simply-connected manifold. Let ΩM be the space of continuous basepoint preserving maps from the circle to M, called the *(based) loop space* of M. When M is a simply-connected four-manifold, an $(n-1)$-connected $2n$-manifold, or a connected sum of products of two spheres, we aim to give an explicit, integral homotopy decomposition of ΩM as a product of simpler factors.

Decomposing the loops on large classes of manifolds has long been thought to be too hard to do. However, the methods used in the paper are relatively accessible and flexible. Essentially, the starting input is information about the integral cohomology of M derived from Poincaré duality. This is then manipulated by creating appropriate homotopy fibrations involving M which allow one to apply decomposition methods from homotopy theory, in the spirt of [8]. It should be the case that the same methods can be used to investigate the loops on other classes of manifolds.

Such decompositions are useful and to illustrate this, we give three examples. First, in toric topology one associates to a simplicial complex K a space called a moment-angle complex. If K is a simple polytope then this moment-angle complex is a manifold. For example, if K is an n-gon then the moment-angle complex is a connected sum of products of two spheres [5,14]. The combinatorics of the polytope and the geometry of the manifold are deeply connected, but the relationship is not well understood. Decomposing the loops on such connected sums and relating the factors to the combinatorics of the polytope should be insightful. Second, string topology is concerned with properties of the free loop space ΛM of M: the space of continuous unbased maps from the circle to M. There is a fibration $\Omega M \longrightarrow \Lambda M \xrightarrow{e} M$ where e evaluates a map at the basepoint, and e has a section. The section implies that $\pi_m(\Lambda M) \cong \pi_m(M) \oplus \pi_m(\Omega M)$ for $m \geq 2$, so the homotopy groups of ΛM can be determined to the same extent as those of the factors of ΩM. This has implications for counting the geodesics on M (see [26]), and the decomposition of ΩM may help clarify the homology and cohomology of ΛM. The third application is to configuration spaces, which will be discussed in more detail in Section 5. Let $F(M, k)$ be the configuration space of ordered k-tuples of distinct points in the product space M^k. In certain cases, for example if M is a product of two non-trivial manifolds, Cohen and Gitler [7] showed that ΩM is a factor of $\Omega F(M, k)$. A decomposition for ΩM further refines this, and allows for the calculation of a significant subgroup of the homotopy groups of the configuration space.

To present our results, we start with a classification theorem. Assume that homology is taken with integral coefficients and use the symbol "$\simeq$" to denote a homotopy equivalence. By a connected sum of sphere products, we mean a connected sum of products of two spheres.

Theorem 1.1. *The following hold:*

(a) *if M and N are simply-connected four-manifolds, then $\Omega M \simeq \Omega N$ if and only if $H^2(M) \cong H^2(N)$;*
(b) *if M and N are $(n-1)$-connected $2n$-dimensional manifolds and $n \notin \{2, 4, 8\}$, then $\Omega M \simeq \Omega N$ if and only if $H^n(M) \cong H^n(N)$;*
(c) *if M and N are n-dimensional connected sums of sphere products, then $\Omega M \simeq \Omega N$ if and only if $H^m(M) \cong H^m(N)$ for each $m < n$.*

Observe that in each case, the homotopy type of ΩM depends only on the cohomology of M, regarded as a $\mathbb{Z}$-module, in degrees strictly less than the dimension of M. This contrasts with the situation before looping. For example, Milnor [19] proved that two simply-connected four-manifolds M and N are homotopy equivalent if and only if M and N have isomorphic cohomology rings. Theorem 1.1 states that after looping the ring structure in cohomology plays no role, only the rank in degree 2 cohomology does. So looping considerably simplifies the homotopy types. This is interesting because ΩM has the same homotopy groups as M, just shifted down one dimension. We therefore immediately obtain the following corollary.

Corollary 1.2. *The following hold:*

(a) *if M and N are simply-connected four-manifolds, then $\pi_*(M) \cong \pi_*(N)$ if and only if $H^2(M) \cong H^2(N)$;*
(b) *if M and N are $(n-1)$-connected $2n$-dimensional manifolds and $n \notin \{2, 4, 8\}$, then $\pi_*(M) \cong \pi_*(N)$ if and only if $H^n(M) \cong H^n(N)$;*
(c) *if M and N are n-dimensional connected sums of sphere products, then $\pi_*(M) \cong \pi_*(N)$ if and only if $H^m(M) \cong H^m(N)$ for each $m < n$.* □

Part (a) of Corollary 1.2 reproves a theorem of Duan and Liang [10] via more homotopy theoretic methods, while parts (b) and (c) generalize it to wider classes of manifolds. Another proof of part (a), again using more geometric techniques, is given in a recent preprint by Basu and Basu [3]. In fact, they show that the result holds for stable homotopy groups in place of homotopy groups. It would be interesting to see whether this also holds for the generalizations presented here.

Theorem 1.1 is proved by decomposing ΩM into a product of spaces, up to homotopy. Explicitly, we have the following.

Theorem 1.3. *Let M be a simply-connected four-manifold and suppose that $\dim H^2(M) = k$. If $k = 0$ then $M \simeq S^4$, if $k = 1$ then $\Omega M \simeq S^1 \times \Omega S^5$, and if $k \geq 2$ then there is a homotopy equivalence*

$$\Omega M \simeq S^1 \times \Omega\big(S^2 \times S^3\big) \times \Omega\big(J \vee \big(J \wedge \Omega\big(S^2 \times S^3\big)\big)\big)$$

where $J = \bigvee_{i=1}^{k-1}(S^2 \vee S^3)$ if $k > 2$ and $J = *$ if $k = 2$.

Theorem 1.4. *Let M be an $(n-1)$-connected $2n$-dimensional manifold and suppose that $\dim H^n(M) = k$. If $k \geq 2$ then there is a homotopy equivalence*

$$\Omega M \simeq \Omega\big(S^n \times S^n\big) \times \Omega\big(J \vee \big(J \wedge \Omega\big(S^n \times S^n\big)\big)\big)$$

where $J = \bigvee_{i=1}^{k-2} S^n$.

Theorem 1.5. *Let M and N be closed oriented $(m-1)$-connected n-dimensional manifolds, with $1 < m \leq n - m$. Suppose that $H_*(M)$ is torsion-free and there is a ring isomorphism $H^*(N) \cong H^*(S^m \times S^{n-m})$. Let $M - *$ and $N - *$ be the punctured manifolds with a single point $*$ removed. Then the following hold:*

(i) *there is a homotopy equivalence*

$$\Omega(M \# N) \simeq \Omega\big(S^m \times S^{n-m}\big) \times \Omega\big((M - *) \vee \big((M - *) \wedge \Omega\big(S^m \times S^{n-m}\big)\big)\big);$$

(ii) *the looped inclusion $\Omega((M - *) \vee \bar{N}) \simeq \Omega((M - *) \vee S^m \vee S^{n-m}) \xrightarrow{\Omega i} \Omega M$ has a right homotopy inverse.*

Consequently, the homotopy type of $\Omega(M \# N)$ is independent of the homotopy type of N, and depends only on the homotopy type of $M - *$.

Theorems 1.4 and 1.5 are consequences of much more general results presented in Theorem 2.6 and Proposition 3.2, both of which are stated in the context of CW-complexes and Poincaré duality spaces.

Observe that in each of these theorems, the decompositions can be further refined. In each case, J is a wedge of simply-connected spheres, so $J \simeq \Sigma J'$ where J' is a wedge of spheres. Therefore, using the facts that $\Omega(X \times Y) \simeq \Omega X \times \Omega Y$, $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y)$ and by [16], $\Sigma\Omega S^s \simeq \bigvee_{i=1}^{\infty} S^{(s-1)i+1}$, we see that $J \wedge \Omega(S^s \times S^t)$ is homotopy equivalent to a wedge of spheres. Thus the factor $\Omega(J \vee (J \wedge \Omega(S^s \times S^t)))$ is homotopy equivalent to the loops on a large wedge of spheres, and the Hilton–Milnor Theorem can be applied to decompose this as a product of loops on spheres of varying dimensions. In particular, in each case, ΩM decomposes as a product of loops on spheres, and so the homotopy groups of M can be determined to the same extent as the homotopy groups of spheres. A similar refinement is possible in Theorem 1.5 when $M - *$ has the homotopy type of a suspension.

From this point of view, Theorems 1.3, 1.4 and 1.5 should be regarded as analogues of the Hilton–Milnor Theorem. As such, these theorems are very practical and should have numerous applications. We have already mentioned how they can be used to determine the homotopy groups of M. As another application described in detail in Section 4, we consider principle G-bundles $P \longrightarrow M$, where M is a simply-connected four-manifold and G is a simply-connected, simple compact Lie group. It is well known that there are $[M, BG] \cong \mathbb{Z}$ distinct equivalence classes of such principle G-bundles. However, after looping the homotopy types of ΩP all coincide as $\Omega M \times \Omega G$.

To prove Theorems 1.3, 1.4 and 1.5, we consider a more general class of torsion-free CW-complexes which resemble Poincaré duality spaces. For such a space P of connectivity $m-1$ and dimension n, we assume that the $(n-1)$-skeleton $\bar{P}$ has $S^m \vee S^{n-m}$ as a wedge summand, and that there is a space Q and a map $P \xrightarrow{q} Q$ such that there is a ring isomorphism $H^*(Q) \cong H^*(S^m \times S^{n-m})$ and the composite $S^m \vee S^{n-m} \longrightarrow P \longrightarrow Q$ is onto in cohomology. Taking F to be the homotopy fibre of q, we analyze the homology of F via the Serre spectral sequence, and then use this to determine its homotopy type. This is then fed into a decomposition of ΩP as $\Omega Q \times F$. The decompositions in the three theorems above then follow as special cases of this more general decomposition. All of this goes through provided there are no cup product squares in cohomology, which is the reason for the excluded cases $\{2, 4, 8\}$ in Theorem 1.4. In the case of simply-connected four-manifolds, these difficulties can be overcome through a novel modification. If the simply-connected four-manifold M is mapped into $\mathbb{C}P^\infty$ by representing a cohomology class in degree 2, then the homotopy fibre is a simply-connected 5-dimensional Poincaré duality complex Z which fits into the general class of torsion-free spaces P above. The resulting decomposition of ΩZ is then used to determine the homotopy type of ΩM.

2. A general homotopy decomposition

The loop space functor and localization functors both have effect of simplifying homotopy types while retaining most of the original homotopy theoretic information. At one extreme a conjecture of Anick [2] (for which there is some evidence [22,1,25]) asserts that the loop space of any simply connected finite CW-complex localized away from a predetermined finite set of primes decomposes as a weak product of a certain countable list of indecomposable spaces, while at the other end of the spectrum the loop space homotopy type of a highly connected CW-complex is uniquely determined. This is not difficult to see, for if X and Y and $(2n-2)$-dimensional $(n-1)$-connected, then they are the $2n$-skeletons of $\Sigma\Omega X$ and $\Sigma\Omega Y$ respectively, so a homotopy equivalence $\Omega X \simeq \Omega Y$ would allow one to construct a composite $X \xrightarrow{\text{incl.}} \Sigma\Omega X \xrightarrow{\simeq} \Sigma\Omega Y \xrightarrow{\text{eval.}} Y$ that induces an isomorphism on homology, and is therefore a homotopy equivalence. Recently, a much stronger result of Grbić and Wu [15] shows that if X and Y are simply-connected finite dimensional co-H spaces then $X \simeq Y$ if and only if $\Omega X \simeq \Omega Y$.

This leads to a natural question. Starting with a finite CW-complex $\bar{P}$, and attaching a cell to $\bar{P}$ to form a space P, which homotopy classes of attaching maps yield the same loop space homotopy type for P? By the above remarks, distinct homotopy classes of *co*-H-maps tend to yield distinct loop space homotopy types. Our goal is to provide sufficient cohomological criteria given a few conditions on $\bar{P}$. More precisely, we give a loop space decomposition for a certain class of spaces, which includes certain connected-sums and certain Poincaré duality spaces (both examples to be discussed in more detail in the next section). Looping will have the effect of simplifying homotopy types, and the homotopy types of the loop spaces will be shown to depend only on simple data, often obtained from the homology of the original space in degrees strictly less than the dimension of the space. We begin by defining the class of spaces we have in mind. Throughout, homology is taken with integer coefficients.

Definition 2.1. Let m and n be integers such that $1 < m \leq n - m$. Suppose P is a finite n-dimensional $(m-1)$-connected CW-complex with torsion-free integral homology given by

$$H_*(P) \cong \mathbb{Z}\{a_1, \dots, a_\ell, z\}$$

where

$$1 < m = |a_1| \leq |a_2| \leq \cdots \leq |a_\ell| = n - m < |z| = n.$$

Let $\bar{P}$ be the $(n-1)$-skeleton of P and let $i : \bar{P} \longrightarrow P$ be the skeletal inclusion. Notice that the bottom cell of $\bar{P}$ occurs in dimension m while the top cell occurs in dimension $n-m$.

Define $\mathcal{P}$ as the collection of all such spaces P which also satisfy the following two properties:

(1) there is a homotopy equivalence $\bar{P} \simeq J \vee (S^m \vee S^{n-m})$ for some space J;
(2) if Q is the homotopy cofibre of the composite $J \hookrightarrow \bar{P} \xrightarrow{i} P$, then there is a ring isomorphism $H^*(Q) \cong H^*(S^m \times S^{n-m})$.

To analyze ΩP for $P \in \mathcal{P}$, some observations and notation are required.

Observations:

(1) If X is a space and $H_*(X)$ is torsion-free, an element $x \in H_*(X)$ has a dual class in $H^*(X)$ which we label as x^*. In our case, since $H_*(P)$ is torsion-free, whenever $|a_i| + |a_j| = n$, the cup product $a_i^* a_j^*$ is some multiple of z^*; define the integer c_{ij} by $a_i^* a_j^* = c_{ij} z^*$.

(2) Observe that the homological description of P implies that there is a homotopy cofibration

$$S^{n-1} \xrightarrow{\alpha} \bar{P} \xrightarrow{i} P$$

where α attaches the top cell to P. A basis for $H_*(\bar{P})$ is given by the elements $\{a_1, \ldots, a_\ell\}$.

(3) The homotopy decomposition of $\bar{P}$ lets us define composites

$$s : J \hookrightarrow \bar{P} \xrightarrow{i} P$$
$$s' : S^m \vee S^{n-m} \hookrightarrow \bar{P} \xrightarrow{i} P.$$

Let $\iota_t \in H_t(S^t)$ represent a generator. Without loss of generality we may assume that the basis for $H_*(P)$ has been chosen so that $(s')_*(\iota_m) = a_1$ and $(s')_*(\iota_{n-m}) = a_\ell$. Then the decomposition $\bar{P} \simeq J \vee (S^m \vee S^{n-m})$ implies that s_* induces an injection onto $\{a_2, \ldots, a_{\ell-1}\}$.

(4) The definition of Q also lets us define a map q by the homotopy cofibration

$$J \xrightarrow{s} P \xrightarrow{q} Q.$$

As this cofibration induces a long exact sequence in homology, the fact that $\bar{P} \simeq J \vee (S^m \vee S^{n-m})$ is the $(n-1)$-skeleton of P implies that the composite $S^m \vee S^{n-m} \xrightarrow{s'} P \xrightarrow{q} Q$ induces an injection in homology.

(5) The ring isomorphism $H^*(Q) \cong H^*(S^m \times S^{n-m})$ implies that

$$H_*(Q) \cong \mathbb{Z}\{x, y, e\},$$

where $|x| = m$, $|y| = n - m$, $|e| = n$ and the generators can be chosen so that $(x^*)^2 = (y^*)^2 = 0$ and $y^* x^* = e^*$. Further, since $(q \circ s')_*$ is an injection, we have $q_*(a_1) = x$, $q_*(a_2) = y$ and $q_*(z) = e$; and as $q \circ s$ is null homotopic we have $q_*(a_i) = 0$ for $2 \leq i \leq \ell - 1$.

(6) The description of q_* on the generators of $H_*(P)$ implies that $c_{\ell 1} = 1$, $c_{1\ell} = (-1)^{m(n-m)}$, and $c_{11} = c_{\ell\ell} = 0$.

We are aiming for the homotopy decomposition of ΩP stated in Theorem 2.6. To get started, we begin with an initial decomposition. Define the space F and the maps f and δ by the homotopy fibration sequence

$$\Omega Q \xrightarrow{\delta} F \xrightarrow{f} P \xrightarrow{q} Q.$$

We first calculate the homology of ΩQ and relate it to the homology of $\Omega(S^m \vee S^{n-m})$. By the Bott–Samelson theorem, there is an algebra isomorphism

$$H_*\big(\Omega\big(S^m \vee S^{n-m}\big)\big) \cong T(u,v)$$

where $T(u,v)$ is the free tensor algebra on generators u and v of degrees $m-1$ and $n-m-1$ respectively. Let $\mathbb{Z}[u,v]$ be the polynomial algebra generated by u and v.

Lemma 2.2. *There is a coalgebra isomorphism $H_*(\Omega Q) \cong \mathbb{Z}[u,v]$ which can be chosen so that the map $\Omega(S^m \vee S^{n-m}) \xrightarrow{\Omega(q\circ s')} \Omega Q$ induces in homology the abelianization $T(u,v) \longrightarrow \mathbb{Z}[u,v]$.*

Proof. First, consider the homology Serre spectral sequence for the path-loop homotopy fibration $\Omega(S^m \vee S^{n-m}) \longrightarrow * \longrightarrow S^m \vee S^{n-m}$. Let $\iota_k \in H_k(S^k)$ represent a generator. Then the elements $\iota_m, \iota_{n-m} \in H_*(S^m \vee S^{n-m})$ transgress to the elements $u, v \in T(u,v)$, and the element $[u,v] \in T(u,v)$ arises in the spectral sequence as the element $u \otimes \iota_{n-m} + (-1)^{|u||\iota_{n-m}|}\iota_m \otimes v$.

Next, consider the homology Serre spectral sequence for the path-loop homotopy fibration $\Omega Q \longrightarrow * \longrightarrow Q$. By Observation (5), $H_*(Q) \cong \mathbb{Z}\{x,y,e\}$ where $|x| = m$, $|y| = n-m$, $|e| = n$ and the cohomology duals satisfy $x^*y^* = e^*$. Thus in homology, the reduced diagonal $\overline{\Delta}(e)$ equals $x \otimes y + y \otimes x$. Thus in the Serre spectral sequence for the path-loop homotopy fibration, we have x and y transgressing to elements a and b respectively, and $d^n(e) = a \otimes y + (-1)^{|a||y|}x \otimes b$. It is now a straightforward calculation to show that there is an isomorphism of vector spaces $H_*(\Omega Q) \cong \mathbb{Z}[a,b]$ where $|a| = m-1$ and $|b| = n-m-1$.

Now consider the homotopy commutative diagram of path-loop homotopy fibrations

$$\begin{array}{ccccc} \Omega(S^m \vee S^{n-m}) & \longrightarrow & * & \longrightarrow & S^m \vee S^{n-m} \\ \downarrow{\scriptstyle \Omega(q\circ s')} & & \downarrow & & \downarrow{\scriptstyle q\circ s'} \\ \Omega Q & \longrightarrow & * & \longrightarrow & Q. \end{array}$$

This induces a morphism of Serre spectral sequences between the two path-loop homotopy fibrations. By Observation (5), the map $(q \circ s')_*$ is an isomorphism in degrees $< n$. Therefore, comparing Serre spectral sequences, $(\Omega(q \circ s'))_*$ is an isomorphism in degrees $< n-1$. In particular, $(\Omega(q \circ s'))_*$ is an isomorphism in degrees $m-1$ and $n-m-1$. Thus, up to sign, $(\Omega(q \circ s'))_*$ sends $u, v \in T(u,v)$ to $a, b \in \mathbb{Z}[u,v]$. Comparing spectral sequences, we also have the element $u \otimes \iota_{n-m} + (-1)^{|u||\iota_{n-m}|}\iota_m \otimes v$ sent to $a \otimes y + (-1)^{|a||y|}x \otimes b$, which is the image of the differential $d^n(e)$. That is, $[u,v] \in T(u,v)$ is sent to $0 \in \mathbb{Z}[a,b]$. Further, it is straightforward to see that once the

d^n differential is taken into account and we move to E^{n+1}, that the E^{n+1} page for the fibration $\Omega(S^m \vee S^{n-m}) \longrightarrow * \longrightarrow S^m \vee S^{n-m}$ maps onto the E^{n+1} page for the fibration $\Omega Q \longrightarrow * \longrightarrow Q$. As there are no more non-trivial differentials, the same is true of the E^∞ pages, and so $(\Omega(q \circ s'))_*$ is onto.

Finally, since $(\Omega(q \circ s'))_*$ is an algebra map and $(\Omega(q \circ s'))_*([u, v]) = 0$, there is a factorization

$$\begin{array}{ccc} T(u,v) & \xrightarrow{(\Omega(q\circ s'))_*} & H_*(\Omega Q) \cong \mathbb{Z}[a,b] \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle g} & \\ \mathbb{Z}[u,v] & & \end{array}$$

for some map g, where π is the abelianization map. Since $(\Omega(q \circ s'))_*$ is onto and both $\mathbb{Z}[u, v]$ and $\mathbb{Z}[a, b]$ have the same Poincaré series, g must be an isomorphism. The statement of the lemma now follows. □

By the Hilton–Milnor Theorem, the inclusion of the wedge into the product $S^m \vee S^{n-m} \xrightarrow{j} S^n \times S^{n-m}$ has a right homotopy inverse after looping. That is, there is a map

$$\phi : \Omega S^n \times \Omega S^{n-m} \longrightarrow \Omega\bigl(S^m \vee S^{n-m}\bigr)$$

which is a right homotopy inverse of Ωj.

Lemma 2.3. *The composite $\Omega S^m \times \Omega S^{n-m} \xrightarrow{\phi} \Omega(S^m \vee S^{n-m}) \xrightarrow{\Omega s'} \Omega P \xrightarrow{\Omega q} \Omega Q$ is a homotopy equivalence. Consequently, in the homotopy fibration sequence $\Omega Q \xrightarrow{\delta} F \xrightarrow{f} P \xrightarrow{q} Q$, the map δ is null homotopic, implying that there are homotopy equivalences*

$$\Omega P \simeq \Omega Q \times \Omega F \simeq \Omega S^m \times \Omega S^{n-m} \times \Omega F.$$

Proof. The fact that ϕ is a right homotopy inverse of Ωi implies that ϕ_* is a coalgebra map which maps onto the sub-coalgebra $\mathbb{Z}[u, v]$ of $T(u, v) \cong H_*(\Omega(S^m \vee S^{n-m}))$. By Lemma 2.2, $(\Omega(q \circ s'))_*$ maps this sub-coalgebra isomorphically onto $H_*(Q)$. Thus $\Omega q \circ \Omega s' \circ \phi$ induces an isomorphism in homology and so is a homotopy equivalence.

For the consequences, consider the homotopy fibration sequence $\Omega F \longrightarrow \Omega P \xrightarrow{\Omega q} \Omega Q \xrightarrow{\delta} F$. We have just shown that $\phi \circ \Omega s'$ is a right homotopy inverse for Ωq. Therefore, the map δ is null homotopic, and this immediately implies that there is a homotopy equivalence $\Omega P \simeq \Omega Q \times \Omega F$. □

Next, we wish to give an explicit homotopy decomposition of the space ΩF. The first step is to calculate its homology. By Observation (4), the composite $J \xrightarrow{s} P \xrightarrow{q} Q$ is a homotopy cofibration, so it is null homotopic. Therefore, s lifts through $F \xrightarrow{f} P$ to a map

$$\bar{s}\colon J \longrightarrow F.$$

By Observation (3), s_* induces an injection onto $\{a_2, \ldots, a_{\ell-1}\}$. So its lift $\bar{s}$ has the property that $(\bar{s})_*$ is an injection, and we will also label a basis for the image of $(\bar{s})_*$ by $\{a_2, \ldots, a_{\ell-1}\}$.

As the homotopy fibration

$$\Omega Q \xrightarrow{\delta} F \xrightarrow{f} P \tag{1}$$

is principal, there exists a left action

$$\theta : \Omega Q \times F \longrightarrow F$$

such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} \Omega Q \times \Omega Q & \xrightarrow{\mathbb{1}\times\delta} & \Omega Q \times F \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \theta} \\ \Omega Q & \xrightarrow{\delta} & F \end{array} \tag{2}$$

where $\mathbb{1}$ is the identity map and μ is the standard loop space multiplication.

Proposition 2.4. *There is an isomorphism of left $H_*(\Omega Q)$-modules*

$$H_*(F) \cong \mathbb{Z}\{a_2, \ldots, a_{\ell-1}\} \otimes H_*(\Omega Q),$$

where $\mathbb{Z}\{a_2, \ldots, a_{\ell-1}\}$ is the image of $\bar{s}_*$ and the left action of $H_*(\Omega Q)$ given by θ_*.

Proof. By a result of Moore [21], the homology Serre spectral sequence E for the principal homotopy fibration sequence $\Omega Q \xrightarrow{\delta} F \xrightarrow{f} P$ is a spectral sequence of left $H_*(\Omega Q)$-modules, with

$$E^2_{*,*} \cong H_*(P) \otimes H_*(\Omega Q). \tag{3}$$

Here, the left action is induced by θ_* and the differentials respect the left action of $H_*(\Omega Q)$. That is, up to sign, $d^n(f \otimes gh) = (1 \otimes g)d^n(f \otimes h)$ whenever the differential d^n is defined. We now proceed to calculate the spectral sequence. In doing so, it will be helpful to rewrite (3) as

$$E^2_{*,*} \cong \mathbb{Z}\{1, a_1, \ldots, a_\ell, z\} \otimes H_*(\Omega Q). \tag{4}$$

Initial information on the differentials. Consider the composite $S^m \vee S^{n-m} \xrightarrow{s'} P \xrightarrow{q} Q$. By Observation (4), $(q \circ s')_*$ is an injection in homology. The composite induces a homotopy fibration diagram

$$\begin{array}{ccccccc}
\Omega Q & \longrightarrow & Z & \longrightarrow & S^m \vee S^{n-m} & \xrightarrow{q\circ s'} & Q \\
\| & & \downarrow & & \downarrow{\scriptstyle s'} & & \| \\
\Omega Q & \xrightarrow{\delta} & F & \xrightarrow{f} & P & \xrightarrow{q} & Q
\end{array}$$

which defines the space Z. Since $(q \circ s')_*$ is an injection in homology and there is a coalgebra isomorphism $H_*(Q) \cong H_*(S^m \times S^{n-m})$, in the homology Serre spectral sequence for the fibration $\Omega Q \longrightarrow Z \longrightarrow S^m \vee S^{n-m}$ the generators $\iota_m, \iota_{n-m} \in H_*(S^m \vee S^{n-m})$ transgress to the elements $u, v \in H_*(\Omega Q)$ respectively, where u, v are as in Lemma 2.2. Now consider the homology Serre spectral sequence for the fibration $\Omega Q \xrightarrow{\delta} F \xrightarrow{f} P$. By Observation (3), we may assume that $(s')_*(\iota_m) = a_1$ and $(s')_*(\iota_{n-m}) = a_l$, so a comparison of spectral sequences implies that the elements a_1, a_l transgress to $u, v \in H_*(\Omega Q)$. That is, in terms of $E^2_{*,*}$, we have

$$d^m(a_1 \otimes 1) = 1 \otimes u, \qquad d^{n-m}(a_\ell \otimes 1) = 1 \otimes v.$$

Further, by Observation (3), the map $J \xrightarrow{s} P$ induces an injection in homology onto $\{a_2, \dots, a_{\ell-1}\}$, and it was observed before the statement of the proposition that the map s lifts through f to F. Therefore the elements $\{a_2, \dots, a_{\ell-1}\}$ survive the spectral sequence. Consequently,

$$d^t(a_i) = 0 \quad \text{for all } t \geq 2 \text{ and } 2 \leq i \leq \ell - 1. \tag{5}$$

Case 1: $m < n - m$. For degree reasons, the differentials $d^2, \dots, d^{m-1}$ are all zero on the elements $a_1, \dots, a_\ell$, so the left action of $H_*(\Omega Q)$ implies that these differentials are identially zero. Therefore

$$E^2_{*,*} \cong E^m_{*,*}.$$

For d^m we have $d^m(a_1 \otimes 1) = 1 \otimes u$. The left action of θ_* implies that for any element $g \in H_*(\Omega Q)$, we have (up to sign),

$$d^m(a_1 \otimes g) = (1 \otimes g)d^m(a_1 \otimes 1) = (1 \otimes g)(1 \otimes u) = 1 \otimes gu.$$

By (5), $d^m(a_i) = 0$ for $2 \leq i \leq \ell$. So the left action of θ_* implies that for $d^m(a_i \otimes g) = 0$ for any $2 \leq i \leq \ell$ and any $g \in H_*(\Omega Q)$. Next, consider the element $z \otimes 1$. Dualizing to the cohomology spectral sequence associated with E, we have for each i such that $|a_i| = n - m$,

$$d_m\big(a_i^*\otimes u^*\big)=\big(d_m\big(a_i^*\otimes 1\big)\big)\big(1\otimes u^*\big)+(-1)^{|a_i|}\big(a_i^*\otimes 1\big)d_m\big(1\otimes u^*\big)$$
$$=(-1)^{|a_i|}\big(a_i^*\otimes 1\big)\big(a_1^*\otimes 1\big)=(-1)^{|a_i|}c_{i1}\big(z^*\otimes 1\big).$$

This implies that in the homology Serre spectral sequence E we have

$$d^m(z\otimes 1)=\sum_{|a_i|=n-m}(-1)^{|a_i|}c_{i1}(a_i\otimes u).$$

The left action of θ_* therefore implies that

$$d^m(z\otimes g)=\sum_{|a_i|=n-m}(-1)^{|a_i|}c_{i1}(a_i\otimes gu)$$

for each $g\in H_*(\Omega Q)$. Therefore, as $c_{\ell 1}=1$ by Observation (6), $c_{\ell 1}(a_\ell\otimes gu)=(a_\ell\otimes gu)$ is identified in $E^{m+1}_{n-m,*}$ with a linear combination of elements $a_i\otimes gu$ for $|a_i|=n-m$. Note that a_1 is excluded here since $|a_1|=m$ and in this case we have assumed that $m<n-m$. Collectively, we have determined the differential d^m, and obtain an isomorphism of left $H_*(\Omega Q)$-modules

$$E^{m+1}_{*,*}\cong\mathbb{Z}\{a_2,\dots,a_\ell\}\otimes H_*(\Omega Q).$$

Continuing, by (5), $d^{m+1},\dots,d^{n-m-1}$ are all identically zero on the elements $a_2,\dots,a_{\ell-1}$ and for degree reasons, $d^{m+1},\dots,d^{n-m-1}$ are all identically zero on a_ℓ. So the left action of θ_* implies that these differentials are identically zero on all elements. Therefore there is an isomorphism

$$E^{m+1}_{*,*}\cong E^{n-m}_{*,*}.$$

For d^{n-m}, by (5), $d^{n-m}(a_i)=0$ for $2\le i\le\ell-1$, so the left action of θ_* implies that $d^{n-m}(a_i\otimes g)=0$ for any $2\le i\le\ell-1$ and for any $g\in H_*(\Omega Q)$. From the initial calculation of differentials, we obtained $d^{n-m}(a_\ell\otimes 1)=1\otimes v$. The left action of θ_* therefore implies that for any element $g\in H_*(\Omega Q)$ we have (up to sign),

$$d^{n-m}(a_\ell\otimes g)=(1\otimes g)d^{n-m}(a_\ell\otimes 1)=(1\otimes g)(1\otimes v)=1\otimes gv.$$

Thus we have determined the differential d^{n-m}, and obtain an isomorphism of left $H_*(\Omega Q)$-modules

$$E^{n-m+1}_{*,*}\cong\mathbb{Z}\{a_2,\dots,a_{\ell-1}\}\otimes H_*(\Omega Q).$$

Finally, by (5), the differentials d^t for $t>n-m$ are all identically zero on $a_2,\dots,a_{\ell-1}$, so the left acton of θ_* implies that these differentials are identically zero on all elements.

Hence

$$E^{\infty}_{*,*} \cong E^{n-m+1}_{*,*}.$$

Since there is no torsion in $E^{\infty}_{*,*}$, there is no extension problem, and we have

$$H_*(F) \cong \bigoplus_{i+j=*} E^{\infty}_{i,j} \cong \mathbb{Z}\{a_2, \dots, a_{\ell-1}\} \otimes H_*(\Omega Q). \tag{6}$$

To see that this is an isomorphism of left $H_*(\Omega Q)$-modules, recall that the left action $H_*(\Omega Q) \otimes E^{\infty}_{i,j} \longrightarrow E^{\infty}_{i,j+*}$ coincides with the left action of associated graded objects

$$H_*(\Omega Q) \otimes \frac{\mathcal{F}_{i,i+j}}{\mathcal{F}_{i-1,i+j}} \longrightarrow \frac{\mathcal{F}_{i,i+j+*}}{\mathcal{F}_{i-1,i+j+*}} \cong E^{\infty}_{i,j+*}$$

induced by the action $H_*(\Omega Q) \otimes H_{i+j}(F) \xrightarrow{\mu_*} H_{i+j+*}(F)$, where $\mathcal{F}_{i,j} = \mathcal{F}_i H_j(F) \subseteq H_j(F)$ is the increasing filtration associated with our spectral sequence. Observe from the calculations above that the action on the $E^{\infty}_{*,*}$ is free, so the action on the associated graded objects is free. Therefore the action μ_* must also be free, and so the isomorphism (6) is one of left $H_*(\Omega Q)$-modules.

Case 2: $m = n - m$. This case is simpler. We have $n = 2m$ and $|u| = |v| = m - 1$. So the only differential which comes into play is d^m. This time $d^m(z \otimes g)$ is the sum of linear combinations of the elements $c_{i1}(a_i \otimes gu)$ and $c_{i\ell}(a_i \otimes gv)$ for all i, where $c_{\ell 1} = 1$, $c_{1\ell} = (-1)^{m(n-m)} = -(-1)^{|u||v|}$ and $c_{11} = c_{\ell\ell} = 0$. Therefore the elements $a_\ell \otimes gu - (-1)^{|u||v|} a_1 \otimes gv$ are identified in $E^{m+1}_{m,*}$ with a linear combination of elements of the form $a_i \otimes gu$ or $a_i \otimes gv$ for $2 \leq i \leq \ell - 1$, and the calculation goes through as before. □

Now we refine the homotopy decomposition $\Omega P \simeq \Omega Q \times \Omega F$ of Lemma 2.3 by identifying the homotopy type of F. For spaces X and Y, the *left half-smash* of X and Y is defined by

$$X \ltimes Y = (X \times Y)/(* \times Y).$$

It is well-known that if Y is a suspension then there is a homotopy equivalence

$$X \ltimes Y \simeq Y \vee (X \wedge Y).$$

Proposition 2.5. *There is a homotopy equivalence*

$$F \simeq \Omega Q \ltimes J.$$

Proof. Using the lift $J \xrightarrow{\bar{s}} F$ of $J \xrightarrow{s} P$ and the homotopy action $\Omega Q \times F \xrightarrow{\theta} F$, define λ as the composite

$$\lambda : \Omega Q \times J \xrightarrow{\mathbb{1}\times\bar{s}} \Omega Q \times F \xrightarrow{\theta} F.$$

By (2), the restriction of θ to ΩQ is homotopic to δ, which by Lemma 2.3 is null homotopic. Therefore the composite

$$\Omega Q \times * \xrightarrow{\mathbb{1}\times *} \Omega Q \times J \xrightarrow{\lambda} F$$

is null homotopic. Since the homotopy cofibre of $\mathbb{1} \times *$ is $\Omega Q \ltimes F$, the map λ extends to a map $\hat{\lambda}$ that makes the following diagram homotopy commute

$$\begin{array}{ccccc} \Omega Q \times * & \xrightarrow{\mathbb{1}\times *} & \Omega Q \times J & \longrightarrow & \Omega Q \ltimes J \\ & & \downarrow{\scriptstyle \lambda} & \swarrow{\scriptstyle \hat{\lambda}} & \\ & & F. & & \end{array}$$

By definition, $\lambda = \theta \circ (\mathbb{1} \times \bar{s})$, so Proposition 2.4 implies that $\hat{\lambda}_*$ is an isomorphism. Thus $\hat{\lambda}$ is a homotopy equivalence. □

Theorem 2.6. *Let $P \in \mathcal{P}$ and suppose that P is $(m-1)$-connected and n-dimensional. Then the following hold:*

(i) *there is a homotopy equivalence*

$$\Omega P \simeq \Omega\big(S^m \times S^{n-m}\big) \times \Omega\big(\Omega\big(S^m \times S^{n-m}\big) \ltimes J\big),$$

which, if J is a suspension, refines to a homotopy equivalence

$$\Omega P \simeq \Omega\big(S^m \times S^{n-m}\big) \times \Omega\big(J \vee \big(J \wedge \Omega\big(S^m \times S^{n-m}\big)\big)\big);$$

(ii) *the map $\Omega\bar{P} \xrightarrow{\Omega i} \Omega P$ has a right homotopy inverse.*

Proof. For part (i), by Lemma 2.3, $\Omega P \simeq \Omega Q \times \Omega F$ and $\Omega Q \simeq \Omega S^m \times \Omega S^{n-m}$, and by Proposition 2.5, $F \simeq \Omega Q \ltimes J$. Thus

$$\Omega P \simeq \Omega S^m \times \Omega S^{n-m} \times \Omega\big(\big(\Omega S^m \times \Omega S^{n-m}\big) \ltimes J\big).$$

If J is a suspension, this decomposition refines due to the fact that $\Omega Q \ltimes J \simeq J \vee (J \wedge \Omega Q)$.

For part (ii), define $\bar{q}$ as the composite

$$\bar{q} : \bar{P} \xrightarrow{i} P \xrightarrow{q} Q.$$

From this composite we obtain a homotopy pullback diagram

$$\begin{array}{ccccccc} \Omega Q & \xrightarrow{\bar{\delta}} & \bar{F} & \xrightarrow{\bar{f}} & \bar{P} & \xrightarrow{\bar{q}} & Q \\ \| & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle i} & & \| \\ \Omega Q & \xrightarrow{\delta} & F & \xrightarrow{f} & P & \xrightarrow{q} & Q \end{array} \tag{7}$$

which defines the space $\bar{F}$ and the maps $\bar{f}$, $\bar{\delta}$ and τ. In particular, this is a homotopy commutative diagram of principal fibration sequences, so if $\bar{\theta} : \Omega Q \times \bar{F} \longrightarrow \bar{F}$ is the homotopy action for the top fibration sequence, then there is a homotopy commutative diagram of actions

$$\begin{array}{ccc} \Omega Q \times \bar{F} & \xrightarrow{\bar{\theta}} & \bar{F} \\ \downarrow{\scriptstyle \mathbb{1}\times\tau} & & \downarrow{\scriptstyle \tau} \\ \Omega Q \times F & \xrightarrow{\theta} & F. \end{array}$$

By definition, the map $J \xrightarrow{s} P$ factors as the composite $J \xrightarrow{r} \bar{P} \xrightarrow{i} P$, where r is the inclusion of the wedge summand in $\bar{P} \simeq J \vee (S^m \vee S^{n-m})$. Since s lifts through f to the map $J \xrightarrow{\bar{s}} F$, the definition of $\bar{F}$ as a homotopy pullback in (7) implies that there is a pullback map $\bar{r} : J \longrightarrow \bar{F}$ such that $\bar{f} \circ \bar{r} \simeq r$ and $\tau \circ \bar{r} \simeq \bar{s}$. Combining this with the preceding diagram, we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega Q \times J & \xrightarrow{\mathbb{1}\times\bar{r}} & \Omega Q \times \bar{F} & \xrightarrow{\bar{\theta}} & \bar{F} \\ \| & & \downarrow{\scriptstyle \mathbb{1}\times\tau} & & \downarrow{\scriptstyle \tau} \\ \Omega Q \times J & \xrightarrow{\mathbb{1}\times\bar{s}} & \Omega Q \times F & \xrightarrow{\theta} & F. \end{array} \tag{8}$$

By definition, the map $S^m \vee S^{n-m} \xrightarrow{s'} P$ factors as the composite $S^m \vee S^{n-m} \xrightarrow{j} \bar{P} \xrightarrow{i} P$, where j is the inclusion of the wedge summand in $\bar{P} \simeq J \vee (S^m \vee S^{n-m})$. By Lemma 2.3, $\Omega(q \circ s')$ has a right homotopy inverse. As $\bar{q} = q \circ i$, we have $q \circ s' = q \circ i \circ j = \bar{q} \circ j$, so $\Omega(\bar{q} \circ j)$ has a right homotopy inverse. Consequently, $\Omega\bar{q}$ has a right homotopy inverse, which implies that in the homotopy fibration $\Omega P \xrightarrow{\Omega\bar{q}} \Omega Q \xrightarrow{\bar{\delta}} \bar{F}$, the map $\bar{\delta}$ is null homotopic.

Let $\bar{\lambda}$ be the composite along the top row of (8),

$$\bar{\lambda} : \Omega Q \times J \xrightarrow{\mathbb{1}\times\bar{r}} \Omega Q \times F \xrightarrow{\bar{\theta}} \bar{F}.$$

Since $\bar{\theta}$ is a homotopy action, its restriction to ΩQ is $\bar{\delta}$. Therefore the restriction of $\bar{\lambda}$ to ΩQ is $\bar{\delta}$, which is null homotopic. Thus there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega Q \times * & \xrightarrow{\mathbb{1}\times *} & \Omega Q \times J & \longrightarrow & \Omega Q \ltimes J \\ & & \downarrow{\scriptstyle \bar{\lambda}} & \swarrow{\scriptstyle \tilde{\lambda}} & \\ & & F & & \end{array}$$

where the top row is a homotopy cofibration and $\tilde{\lambda}$ is an extension of $\bar{\lambda}$. Now let γ be the composite

$$\gamma : \Omega Q \ltimes \bar{F} \xrightarrow{\tilde{\lambda}} \bar{F} \xrightarrow{\tau} F.$$

Observe that γ is a choice of the extension $\hat{\lambda}$ in the proof of Proposition 2.5. Thus γ induces an isomorphism in homology and so is a homotopy equivalence. Consequently, the map τ has a right homotopy inverse $\sigma : F \longrightarrow \bar{F}$.

Finally, consider the diagram

$$\begin{array}{ccccccc} (\Omega S^m \times \Omega S^{n-m}) \times \Omega F & \xrightarrow{\mathbb{1}\times \Omega\sigma} & (\Omega S^m \times \Omega S^{n-m}) \times \Omega \bar{F} & \xrightarrow{\Omega j \times \Omega \bar{f}} & \Omega \bar{P} \times \Omega \bar{P} & \xrightarrow{\mu} & \Omega \bar{P} \\ & \searrow\!\!= & \downarrow{\scriptstyle \mathbb{1}\times \Omega\tau} & & \downarrow{\scriptstyle \Omega i \times \Omega i} & & \downarrow{\scriptstyle \Omega i} \\ & & (\Omega S^m \times \Omega S^{n-m}) \times \Omega F & \xrightarrow{\Omega s' \times \Omega f} & \Omega P \times \Omega P & \xrightarrow{\mu} & \Omega P \end{array}$$

where μ is the standard loop multiplication. The left triangle homotopy commutes since σ is a right homotopy inverse of τ. The middle square homotopy commutes since, by definition, $s' = i \circ j$, and by (7), $f \simeq i \circ \bar{f}$. The right square homotopy commutes since Ωi is a loop map. By part (i), the bottom row is a homotopy equivalence, so the homotopy commutativity of the diagram implies that Ωi has a right homotopy inverse. □

3. Consequences

In this section we apply Theorem 2.6 to two classes of examples, first to certain connected sums, and then to certain Poincaré duality complexes, and prove Theorems 1.4.

If M is a closed oriented n-dimensional manifold, let $\bar{M}$ be the $(n-1)$-skeleton of M. In particular, M is homotopy equivalent to $M - *$, and is obtained from $\bar{M}$ by attaching a single n-cell. Observe that if N is a closed oriented n-dimensional manifold and there is a ring isomorphism $H^*(N) \cong H^*(S^m \times S^{n-m})$, then $\bar{N} \simeq S^m \vee S^{n-m}$. Denote the connected sum of two closed oriented n-dimensional manifolds M and N by $M\#N$. Observe that the $(n-1)$-skeleton of $M\#N$ is homotopy equivalent to $\bar{M} \vee \bar{N}$. Let

$$i : \bar{M} \vee \bar{N} \longrightarrow M\#N$$

be the skeletal inclusion.

Proof of Theorem 1.5. We will show that $M\#N \in \mathcal{P}$. Let $P = M\#N$, let $\bar{P}$ be the $(n-1)$-skeleton of P, and let $\bar{P} \xrightarrow{i} P$ be the skeletal inclusion. By the definitions of M and N, P is an $(m-1)$-connected, n-dimensional CW-complex. Since $P = M\#N$ is a closed oriented manifold, it satisfies Poincaré duality, which implies that $\bar{P} \simeq \bar{M} \vee \bar{N}$ is actually $(n-m)$-dimensional. Note that as $m > 1$ we have $n - m < n - 1$, so $H_*(P)$ is torsion-free if and only if $H_*(\bar{P})$ is torsion-free. But as $H_*(M)$ is torsion-free, so is $H_*(\bar{M})$, which implies that $\bar{P} \simeq \bar{M} \vee \bar{N} \simeq \bar{M} \vee (S^m \vee S^{n-m})$ also has $H_*(\bar{P})$ torsion-free. Thus $H_*(P)$ is torsion-free.

Now if $J = \bar{M}$ then as $\bar{N} \simeq S^m \vee S^{n-m}$, we have $\bar{P} \simeq J \vee (S^m \vee S^{n-m})$. Let Q be the cofibre of the composite $J \longrightarrow \bar{P} \xrightarrow{i} P$, that is, Q is the cofibre of the composite $\bar{M} \longrightarrow \bar{M} \vee \bar{N} \xrightarrow{i} M\#N$. Then $Q \simeq N$, which implies that $H^*(Q) \cong H^*(N) \cong H^*(S^m \times S^{n-m})$. Thus $P = M\#N$ satisfies all the conditions of Definition 2.1, so $P \in \mathcal{P}$. The assertions of the proposition are now all direct applications of Theorem 2.6. □

Example 3.1. As an example of Theorem 1.5 in action, recall that an n-dimensional manifold M is a connected sum of sphere products if

$$M \cong \left(S^{m_1} \times S^{n-m_1}\right)\# \cdots \#\left(S^{m_k} \times S^{n-m_k}\right)$$

for some integers $m_1, \ldots, m_k$. Let $M_1 = (S^{m_1} \times S^{n-m_1})\# \cdots \#(S^{m_{k-1}} \times S^{n-m_{k-1}})$ and $N = S^{m_k} \times S^{n-m_k}$ so that $M = M_1\#N$. Observe that $\bar{M}_1 = \bigvee_{i=1}^{k-1}(S^{m_i} \vee S^{n-m_i})$. So by Theorem 1.5, there is a homotopy equivalence

$$\Omega M \simeq \Omega(M_1\#N) \simeq \Omega\left(S^{m_k} \times S^{n-m_k}\right) \times \Omega\left(\bar{M}_1 \vee \left(\bar{M}_1 \wedge \Omega\left(S^{m_k} \times S^{n-m_k}\right)\right)\right).$$

Recall that P is a *Poincaré duality complex* if it has the homotopy type of a finite CW-complex and its cohomology ring $H^*(P; R)$ satisfies Poincaré duality for all coefficient rings R. In particular every oriented simply-connected manifold is a Poincaré duality complex.

Proposition 3.2. *Fix $1 < m \leq n$. If $m = n - m$, assume that $m \notin \{2, 4, 8\}$. Let P be an $(m-1)$-connected n-dimensional Poincaré duality complex such that $(n-1)$-skeleton $\bar{P}$ of P has the homotopy type of a wedge of spheres. Let $i : \bar{P} \longrightarrow P$ be the skeletal inclusion. Then the following hold:*

(i) *there is a homotopy equivalence*

$$\Omega P \simeq \Omega\left(S^m \times S^{n-m}\right) \times \Omega\left(J \vee \left(J \wedge \Omega\left(S^m \times S^{n-m}\right)\right)\right)$$

where J is obtained from $\bar{P}$ by quotienting out a copy of $S^m \vee S^{n-m}$;

(ii) *the map $\Omega\bar{P} \xrightarrow{\Omega i} \Omega P$ has a right homotopy inverse.*

Consequently, the homotopy type of ΩP depends only on the homotopy type of $\bar{P}$.

We will need a preliminary lemma about the cohomology ring of Poincaré duality complexes before we can prove this.

Lemma 3.3. *Let P be an n-dimensional Poincaré duality complex such that $H_*(P)$ is torsion-free, and let e^* be a generator of $H^n(P) \cong \mathbb{Z}$. Then for any positive integer $i \leq n$ and basis element x^* in $H^i(P)$, there exists a choice of basis for $H^{n-i}(P)$ such that $x^*y^* = e^*$ for some y^* in this basis.*

Proof. Let x and e be the homology duals of x^* and e^*. Since $H^*(P)$ satisfies Poincaré duality, the cap product homomorphism

$$e \cap H^i(P) \longrightarrow H_{n-i}(P)$$

is an isomorphism, so it maps a basis of $H^i(P)$ to a basis of $H_{n-i}(P)$. Therefore

$$y = e \cap x^*$$

is an element in a basis for $H_{n-i}(P)$.

Since $H_*(P)$ is torsion-free, the cup product is dual to the cap product. That is, there is a commutative diagram

$$\begin{array}{ccc} H^{n-i}(P) & \xrightarrow{\cong} & Hom(H_{n-i}(P), Z) \\ \downarrow{\scriptstyle \cup x^*} & & \downarrow{\scriptstyle (\cap x^*)^*} \\ H^n(P) & \xrightarrow{\cong} & Hom(H_n(P), Z). \end{array}$$

In particular, since the homomorphism $(\cap x^*)$ sends e to y and e generates $H^n(P)$, its dual $(\cap x^*)^* = (\cup x^*)$ sends y^* to e^*, so we have

$$y^* \cup x^* = e^*.$$

Since y is an element in a basis for $H_{n-i}(P)$, y^* is an element in the dual basis for $H^{n-i}(P)$, and we are done. □

Note that if $m = n$ and $m \notin \{2, 4, 8\}$ then the element y^* in Lemma 3.3 is *not* equal to $\pm x^*$. But if $m \in \{2, 4, 8\}$ then we may have $y^* = \pm x^*$. This is the reason for the exclusion of this case in the statement of Proposition 3.2.

Proof of Proposition 3.2. We will check that $P \in \mathcal{P}$. By Poincaré duality, $\bar{P}$ is $(n - m)$-dimensional. So as $m > 1$, we have $n - m < n - 1$, implying that $H_*(P)$ is torsion-free if and only if $H_*(\bar{P})$ is torsion-free. But as $\bar{P}$ is homotopy equivalent to a wedge of spheres, $H_*(\bar{P})$ is torsion-free and therefore $H_*(P)$ is torsion-free.

Fix e^* as a generator of $H^n(P) \cong \mathbb{Z}$. Let $x^* \in H^m(P)$ be a basis element. By Lemma 3.3 there exists a basis element $y^* \in H^{n-i}(P)$ such that $x^*y^* = e^*$. Since $\bar{P}$ is homotopy equivalent to a wedge of spheres, x^* and y^* are spherical classes represented by maps $S^m \xrightarrow{\alpha} \bar{P}$ and $S^{n-m} \xrightarrow{\beta} \bar{P}$ and the wedge sum $S^m \vee S^{n-m} \xrightarrow{\alpha+\beta} \bar{P}$ has a left homotopy inverse. Thus $\bar{P} \simeq J \vee (S^m \vee S^{n-m})$ where J is the homotopy cofibre of $\alpha + \beta$. Let Q be the homotopy cofibre of the composite $J \longrightarrow \bar{P} \xrightarrow{i} P$. The homotopy equivalence for $\bar{P}$ and the fact that, as a CW-complex, $P = \bar{P} \cup e^n$ implies that Q is a three-cell complex, $Q = (S^m \vee S^{n-m}) \cup e^n$, and the map to the cofibre, $P \xrightarrow{q} Q$, is onto in homology. Dualizing, q^* is an injection. Suppose that $u* \in H^m(Q)$, $v^* \in H^{n-m}(Q)$ and $z^* \in H^n(Q)$ satisfy $q^*(u^*) = x^*$, $q^*(v^*) = y^*$ and $q^*(z^*) = e^*$. Then the fact that $x^*y^* = e^*$ implies that $u^*v^* = z^*$. Thus there is a ring isomorphism $H^*(Q) \cong H^*(S^m \times S^{n-m})$. Thus P satisfies all the conditions of Definition 2.1, so $P \in \mathcal{P}$. The assertions of the proposition are now all direct applications of Theorem 2.6. □

As an example of Proposition 3.3 in action, we prove Theorem 1.4. Let M be an $(n-1)$-connected $2n$-dimensional manifold. Observe that the $(2n-1)$-skeleton of M is homotopy equivalent to $\bigvee_{i=1}^k S^n$, where $k = \dim H^n(M)$. We aim to decompose ΩM.

Proof of Theorem 1.4. If $n \notin \{2, 4, 8\}$ and $k \geq 2$, then by Proposition 3.3 there is a homotopy equivalence

$$\Omega M \simeq \Omega(S^n \times S^n) \times \Omega\bigl(J \vee \bigl(J \wedge \Omega(S^n \times S^n)\bigr)\bigr)$$

where $J = \bigvee_{i=1}^{k-2} S^n$. □

4. The case of simply-connected 4-manifolds

Proposition 3.2 does not cover the cases of simply-connected 4-manifolds, 3-connected 8-manifolds, or 7-connected 16-manifolds, due to the potential presence of nonzero cup product squares. To handle the case of simply-connected 4-manifolds and prove Theorem 1.3, we use the fact that such spaces appear as the base space in a certain S^1 homotopy fibration whose total space is a Poincaré duality complex. These homotopy fibrations generalize the fiber bundle $S^1 \longrightarrow S^5 \longrightarrow \mathbb{C}P^2$.

Let M be a simply-connected oriented 4-manifold. If $H^2(M) = 0$ then M is homotopy equivalent to S^4, and the homotopy type of ΩS^4 is well known to be $S^3 \times \Omega S^7$. So we will assume from now on that $H^2(M) \neq 0$. Then, up to homotopy equivalence, there is a homotopy cofibration

$$S^3 \xrightarrow{\alpha} \bigvee_{i=1}^{k} S^2 \longrightarrow M$$

for some map α. Suppose that there is an isomorphism of $\mathbb{Z}$-modules

$$H^*(M) \cong \mathbb{Z}\{x_1, \dots, x_k, z\}$$

where $|x_i| = 2$ and $|z| = 4$. Let c_{ij} be such that $x_i x_j = c_{ij} z$. Let C be the $k \times k$ matrix

$$C = [c_{ij}].$$

The anti-commutativity of the cup product implies that $c_{ij} = c_{ji}$, so C is symmetric, and Poincaré duality implies that C is nonsingular.

Focus on the class $x_k \in H^2(M)$. By Lemma 3.3, we may assume the basis of $H^2(M)$ has been chosen so that $c_{k\bar{k}} = 1$ for some $\bar{k}$. That is,

$$x_k x_{\bar{k}} = z.$$

The cohomology class x_k is represented by a map

$$q: M \longrightarrow K(\mathbb{Z}, 2).$$

Note that $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$, and $\Omega \mathbb{C}P^\infty \simeq S^1$. Define the space Z by the homotopy fibration sequence

$$S^1 \longrightarrow Z \longrightarrow M \xrightarrow{q} \mathbb{C}P^\infty.$$

A theorem of Quinn [24] says that in a fibration of spaces having the homotopy type of finite CW-complexes, the total space is a Poincaré duality complex if and only if the fiber and base space are Poincaré duality complexes. This, of course, also holds for homotopy fibrations. Therefore, as we have a homotopy fibration $S^1 \longrightarrow Z \longrightarrow M$ and both S^1 and M are Poincaré duality complexes, then so is Z.

Lemma 4.1. *The Poincaré duality complex Z satisfies the following:*

(i) *there is a homotopy cofibration*

$$S^4 \xrightarrow{\gamma} \bigvee_{i=1}^{k} \left(S^2 \vee S^3\right) \longrightarrow Z$$

for some map γ;

(ii) *$H^*(Z)$ is torsion-free.*

Proof. Consider the homotopy fibration $S^1 \longrightarrow Z \longrightarrow M$. We will use a Serre spectral sequence to calculate $H^*(Z)$. We have $E_2^{*,*} \cong H^*(S^1) \otimes H^*(M)$. Let $a \in H^1(S^1)$ represent a generator and recall that, as a $\mathbb{Z}$-module, $H^*(M) \cong \mathbb{Z}\{x_1, \dots, x_k, z\}$. Thus a $\mathbb{Z}$-module basis for $E_2^{*,*}$ is given by

$$\{1, 1 \otimes x_1, \dots, 1 \otimes x_k, 1 \otimes z, a \otimes 1, a \otimes x_1, \dots, a \otimes x_k, a \otimes z\}.$$

The fibration in question is induced by the map $M \xrightarrow{q} \mathbb{C}P^\infty$ which represents the cohomology class x_k. Therefore $d_2(a) = \pm x_k$. Changing the basis of $H^1(S^1)$ if need be, assume that $d_2(a) = x_k$. As d_2 is a differential, the fact that $x_k x_{\bar{k}} = z$ implies that $d_2(a \otimes x_{\bar{k}}) = x_k x_{\bar{k}} = z$, while $d_2(a \otimes x_j) = x_k x_j = c_{kj} z$. Thus a $\mathbb{Z}$-module basis for $E_3^{*,*}$ is given by

$$\begin{aligned}&\{1, 1 \otimes x_1, \dots, 1 \otimes x_{k-1}, (a \otimes x_1 - a \otimes c_{k1} x_{\bar{k}}), \dots, (a \otimes x_{\bar{k}-1} - a \otimes c_{k(\bar{k}-1)} x_{\bar{k}}),\\ &(a \otimes x_{\bar{k}+1} - a \otimes c_{k(\bar{k}+1)} x_{\bar{k}}), (a \otimes x_k - a \otimes c_{kk} x_{\bar{k}}), a \otimes z\}.\end{aligned}$$

All other differentials are trivial for degree reasons, so we have $H^*(Z) \cong E_\infty^{*,*} \cong E_3^{*,*}$.

Notice that the calculation for the rational cohomology Serre spectral is exactly the same. Thus the rationalization map $H^*(Z;\mathbb{Z}) \longrightarrow H^*(Z;\mathbb{Q})$ preserves the number of basis elements in each dimension. Thus $H^*(Z)$ is torsion-free, proving part (ii).

Notice that the description of $H^*(Z)$ implies that Z has $k-1$ cells in dimension 2 and $k-1$ cells in dimension 3. The fact that $H^*(Z)$ is torsion-free therefore implies that the 3-skeleton of Z is homotopy equivalent to $\bigvee_{i=1}^k (S^2 \vee S^3)$. The one remaining nontrivial cell of Z occurs in dimension 5, so Z is the homotopy cofibre of a map $S^4 \longrightarrow \bigvee_{i=1}^k (S^2 \vee S^3)$, proving part (i). $\square$

Remark 4.2. The space Z is in fact a manifold, not just a Poincaré duality complex, which is diffeomorphic to the connected sum of k copies of $S^2 \times S^3$ [10]. As we only use the much simpler properties of Z listed in Lemma 4.1, it is clarifying to leave the analysis of Z as it stands in the statement and proof of the lemma.

Before proceeding to decompose the loop space of a simply-connected 4-manifold, we first decompose the loop space of the associated Poincaré duality space Z. Let

$$i : \bigvee_{i=1}^{k-1} (S^2 \vee S^3) \longrightarrow Z$$

be the skeletal inclusion.

Proposition 4.3. *If $k = 1$ then $Z \simeq S^5$, so $\Omega Z \simeq \Omega S^5$. If $k \geq 2$ then the following hold:*

(i) *there is a homotopy equivalence*

$$\Omega Z \simeq \Omega(S^2 \times S^3) \times \Omega\big(J \vee \big(J \wedge \Omega(S^2 \times S^3)\big)\big)$$

where $J = \bigvee_{i=1}^{k-1} (S^2 \vee S^3)$ if $k > 2$ and $J = *$ if $k = 2$;

(ii) *the map $\Omega(\bigvee_{i=1}^{k-1} (S^2 \vee S^3)) \xrightarrow{\Omega i} \Omega Z$ has a right homotopy inverse.*

Proof. Notice that Proposition 4.1(i) implies that if $k = 1$ then $Z \simeq S^5$. Assume from now on that $k \geq 2$. We will show that the conditions of Proposition 3.2 hold. The result of Quinn already cited implies that Z is a Poincaré duality space, and by Proposition 4.1(i), Z is 1-connected and 5-dimensional. So with $m = 2$ and $n = 5$ we have $m = 2 < n - m = 3$. By Proposition 4.1, the 4-skeleton of Z is homotopy equivalent to $\bigvee_{i=1}^{k-1}(S^2 \vee S^3)$. Thus Z satisfies the hypotheses of Proposition 3.2, and applying the proposition immediately gives the statements of the proposition. □

We now prove Theorem 1.3, restated as follows.

Theorem 4.4. *Let M be a simply-connected 4-manifold and suppose* $\dim H^2(M) = k$ *for* $k > 0$. *If $k = 1$ then there is a homotopy equivalence*

$$\Omega M \simeq S^1 \times \Omega S^5$$

and if $k \geq 2$ then there is a homotopy equivalence

$$\Omega M \simeq S^1 \times \Omega Z \simeq S^1 \times \Omega\big(S^2 \times S^3\big) \times \Omega\big(J \vee \big(J \wedge \Omega\big(S^2 \times S^3\big)\big)\big)$$

where $J = \bigvee_{i=1}^{k-1}(S^2 \vee S^3)$ if $k > 2$ and $J = *$ if $k = 2$. Consequently, the homotopy type of ΩM depends only on the integer $k = \dim H^2(M)$.

Proof. Consider the map $M \xrightarrow{q} \mathbb{C}P^\infty$ representing the cohomology class x_k. Since M is simply-connected, any generator of $H_2(M)$ is in the image of the Hurewicz homomorphism. In our case, the homology class dual to x_k is the Hurewicz image of a map $t : S^2 \longrightarrow M$. Dualizing, $t^*(x_k) = \iota_2^*$, where ι_2^* is a generator of $H^2(S^2)$. Therefore, the composite $S^2 \xrightarrow{t} M \xrightarrow{q} \mathbb{C}P^\infty$ is degree one in cohomology. Let $\bar{t} : S^1 \longrightarrow \Omega M$ be the adjoint of t. Then the composite $S^1 \xrightarrow{\bar{t}} \Omega M \xrightarrow{\Omega q} S^1$ is degree one in cohomology, implying that it is a homotopy equivalence. Therefore, in the homotopy fibration $\Omega Z \longrightarrow \Omega M \xrightarrow{\Omega q} S^1$, the map Ωq has a right homotopy inverse, implying that there is a homotopy equivalence

$$\Omega M \simeq S^1 \times \Omega Z.$$

The theorem now follows from the decomposition of ΩZ in Proposition 4.3. □

An analogue of Theorem 4.4 holds for 3-connected 8-manifolds M, provided that there is a map $M \longrightarrow \mathbb{H}P^2$ that induces a surjection onto $H^4(\mathbb{H}P^2) \cong \mathbb{Z}$. In such a case, composing this map with the inclusion $\mathbb{H}P^2 \longrightarrow \mathbb{H}P^\infty$ and then using the fact that $\mathbb{H}P^\infty \simeq S^3$, one obtains a principal homotopy fibration $S^3 \longrightarrow Z \longrightarrow M$ with total space Z an 11-dimensional Poincaré duality complex. The only nonzero homology groups of Z are in degrees 4, 7, and 11, and using the associated action of S^3 on Z, it is not difficult to show that the 10-skeleton of Z is homotopy equivalent to a wedge of 4-spheres and

11-spheres. It is not really clear what may happen in the case of 7-connected 16-manifolds, as S^7 does not have a classifying space.

We now re-organize the information appearing in the decomposition in Theorem 4.4 when $k \geq 2$ to make it more clear how the decomposition depends on the 2-skeleton of the 4-manifold. Let $i\colon \bigvee_{i=1}^{k} S^2 \longrightarrow M$ be the skeletal inclusion.

Theorem 4.5. *Let M be a simply-connected 4-manifold and suppose $\dim H^2(M) = k$ for $k \geq 2$. Then the map $\Omega(\bigvee_{i=1}^{k} S^2) \xrightarrow{\Omega i} \Omega M$ has a right homotopy inverse.*

Proof. Recall that there is a homotopy fibration $Z \xrightarrow{r} M \xrightarrow{q} \mathbb{C}P^\infty$. In Theorem 4.4 it was shown that Ωq has a right homotopy inverse, $f : S^1 \longrightarrow \Omega M$. Thus the composite

$$S^1 \times \Omega Z \xrightarrow{f \times \Omega r} \Omega M \times \Omega M \xrightarrow{\mu} \Omega M$$

is a homotopy equivalence, where μ is the loop multiplication.

By Proposition 4.3, the map $\Omega(\bigvee_{s=1}^{k-1}(S^2 \vee S^3)) \xrightarrow{\Omega j} \Omega Z$ has a right homotopy inverse, where j is the inclusion of the 4-skeleton into the 5-dimensional space Z. Let $g : \Omega Z \longrightarrow \Omega(\bigvee_{s=1}^{k-1}(S^2 \vee S^3))$ be a right homotopy inverse of Ωj. Let h be the composite

$$h : \bigvee_{s=1}^{k-1} (S^2 \vee S^3) \xrightarrow{j} Z \xrightarrow{r} M.$$

Then $\Omega h \circ g$ is homotopic to Ωr. Therefore, by the previous paragraph, the composite

$$S^1 \times \Omega Z \xrightarrow{f \times g} \Omega M \times \Omega\left(\bigvee_{s=1}^{k-1} (S^2 \vee S^3)\right) \xrightarrow{\mathbb{1} \times \Omega h} \Omega M \times \Omega M \xrightarrow{\mu} \Omega M$$

is a homotopy equivalence.

Since $\bigvee_{s=1}^{k-1}(S^2 \vee S^3)$ is 3-dimensional, the map h factors through the 3-skeleton of M, which is homotopy equivalent to $\bigvee_{i=1}^{k} S^2$. Thus h factors as a composite $\bigvee_{s=1}^{k-1}(S^2 \vee S^3) \xrightarrow{h'} \bigvee_{i=1}^{k} S^2 \xrightarrow{i} M$ for some map h'. Also, for connectivity and dimension reasons, the map $S^1 \xrightarrow{f} M$ factors as a composite $S^1 \xrightarrow{f'} \Omega(\bigvee_{i=1}^{k} S^2) \xrightarrow{\Omega i} \Omega M$ for some map f'. Therefore, inserting these factorizations into the homotopy equivalence $\mu \circ (\mathbb{1} \times \Omega h) \circ (f \times g)$, we obtain a homotopy equivalence

$$S^1 \times \Omega Z \xrightarrow{f' \times g} \Omega\left(\bigvee_{s=1}^{k-1} S^2\right) \times \Omega\left(\bigvee_{s=1}^{k-1} (S^2 \vee S^3)\right) \xrightarrow{\mathbb{1} \times \Omega h'} \Omega\left(\bigvee_{i=1}^{k} S^2\right) \times \Omega\left(\bigvee_{i=1}^{k} S^2\right)$$
$$\xrightarrow{\Omega i \times \Omega i} \Omega M \times \Omega M \xrightarrow{\mu} \Omega M.$$

Finally, since Ωi is a loop map, it commutes with the loop multiplication, so we obtain a homotopy equivalence

$$S^1 \times \Omega Z \xrightarrow{f' \times (\Omega h' \circ g)} \Omega\Big(\bigvee_{i=1}^{k} S^2\Big) \times \Omega\Big(\bigvee_{i=1}^{k} S^2\Big) \xrightarrow{\mu} \Omega\Big(\bigvee_{i=1}^{k} S^2\Big) \xrightarrow{\Omega i} \Omega M.$$

Consequently, the map Ωi has a right homotopy inverse. □

Theorem 4.5 is useful. For example, we apply it to determine the homotopy type of the loops on certain principle G-bundles.

Corollary 4.6. *Let G be a simply-connected, simple compact Lie group. Let M be a simply-connected 4-manifold with $\dim H^2(M) \geq 2$. Let $P \xrightarrow{\pi} M$ be a principle G-bundle. Then $\Omega\pi$ has a right homotopy inverse, implying that there is a homotopy equivalence*

$$\Omega P \simeq \Omega M \times \Omega G.$$

Proof. Any principle G-bundle $P \xrightarrow{\pi} M$ is classified by a map $M \xrightarrow{g} BG$, where BG is the classifying space of G and P is the homotopy fibre of g. In our case, since G is a simply-connected, compact simple Lie group, BG is 2-connected (in fact, it is 3-connected). Thus the composite $\bigvee_{i=1}^{k} S^2 \xrightarrow{i} M \xrightarrow{g} BG$ is null homotopic by connectivity. By Theorem 4.5, Ωi has a right homotopy inverse. Therefore Ωg is null homotopic. Hence in the homotopy fibration sequence $\Omega G \longrightarrow \Omega P \xrightarrow{\Omega\pi} \Omega M \xrightarrow{\Omega g} G$ the null homotopy for Ωg implies that $\Omega\pi$ has a right homotopy inverse, and therefore $\Omega P \simeq \Omega M \times \Omega G$. □

Corollary 4.6 says something interesting. While there are $[M, BG] \cong \mathbb{Z}$ distinct principle G-bundles over M, after looping all those bundles become homotopy equivalent. Further, the decomposition of ΩP can be refined by inserting the decomposition of ΩM in Theorem 4.4, and — after localizing at a prime p — by the decompositions of ΩG that arise from the p-local decompositions of G due to Mimura, Nishida and Toda [20].

5. Looped Configuration Spaces

We end with a quick application that is in the spirit of our previous results. Let

$$F_k(X) = \big\{(x_1, \ldots, x_k) \in X^{\times k} \mid x_i \neq x_j \text{ if } i \neq j\big\}$$

be the *ordered configuration space* of k distinct points in X. The literature on these spaces is substantial, but many basic questions remain unanswered. For example, their integral homology is not clearly understood in most cases, and it is now known that their homotopy type generally does not depend only on the homotopy type of X, even after restricting the input space to compact manifolds [17].

Things do simplify after looping however. If we were to take M to be a smooth manifold with a nonvanishing tangent vector field, then the projection map $F_k(M) \longrightarrow M$ onto the first coordinate has a section. By [11,7] there is a homotopy decomposition

$$\Omega F_k(M) \simeq \Omega M \times \Omega(M - Q_1) \times \cdots \times \Omega(M - Q_k) \tag{9}$$

for any choice of distinct points $q_1, \ldots, q_k$ in M, with $Q_i = \{q_1, \ldots, q_i\}$. Thus, not only are the Betti numbers $\Omega F_k(M)$ relatively easy to compute, but the homotopy type of $\Omega F_k(M)$ depends only on the homotopy type of the input manifold M when M is simply connected. The following takes this a step further:

Corollary 5.1. *Let $1 < m \leq n - m$, n be odd, and let M be a closed oriented $(m-1)$-connected n-dimensional smooth manifold with torsion-free homology. Then the homotopy type of the looped configuration space $\Omega F_k(M\#(S^m \times S^{n-m}))$ depends only on the homotopy type of $M - *$ for each $k \geq 1$.*

Proof. Recall that the connected sum of smooth manifolds can be constructed so that the resulting manifold also has a smooth structure. Then $M\#(S^m \times S^{n-m})$ is a smooth manifold, and moreover it is odd dimensional, so it has a nonvanishing tangent vector field. Thus, the decomposition (9) specializes to

$$\Omega F_k\big(M\#\big(S^m \times S^{n-m}\big)\big) \simeq \Omega\big(M\#\big(S^m \times S^{n-m}\big)\big) \times \Omega\big(M\#\big(S^m \times S^{n-m}\big) - Q_1\big) \times \cdots \\ \times \Omega\big(M\#\big(S^m \times S^{n-m}\big) - Q_k\big)$$

for any choice of k distinct points $q_1, \ldots, q_k$ in $M\#(S^m \times S^{n-m})$.

Notice that $M\#(S^m \times S^{n-m}) - Q_i$ is homotopy equivalent to the wedge sum of $(M - *) \vee S^m \vee S^{n-m}$ with $i - 1$ copies of the $(n-1)$-sphere. Thus, the homotopy type of each factor $\Omega(M\#(S^m \times S^{n-m}) - Q_i)$ in the decomposition above depends only on the homotopy type of $M - *$. Likewise, the homotopy type of the remaining factor $\Omega(M\#(S^m \times S^{n-m}))$ depends only on that of $M - *$ by Theorem 1.5. The result follows. □

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