University of Southampton

Faculty of Social and Human Sciences
School of Mathematics

Geometric actions of classical algebraic groups

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Ai miei genitori
che hanno sempre creduto in me
e sostenuto le mie scelte.

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ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES
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Doctor of Philosophy

GEOMETRIC ACTIONS OF CLASSICAL ALGEBRAIC GROUPS

by Raffaele Rainone

Let k be an algebraically closed field of arbitrary characteristic p. An affine algebraic group G is an affine algebraic variety over k with a group structure such that multiplication and inversion maps are morphisms of varieties. A special class of affine algebraic groups are the so called *classical groups* $Cl(V)$, groups of isometries of a finite dimensional k-vector space V with respect to a certain form on V – e.g. a zero form, a symplectic form or a non-degenerate quadratic form. These groups are: $\mathrm{GL}(V)$ the general linear group, $\mathrm{Sp}(V)$ the symplectic group and $\mathrm{O}(V)$ the orthogonal group.

Let $G = Cl(V)$. Various (closed) subgroups H of G can be defined naturally in terms of the geometry of V – H may be the stabiliser of a subspace of V, or a direct sum decomposition of V, or a non-degenerate form on V, for example. Let H be such a subgroup and let $\Omega = G/H$ be the corresponding coset space. Then Ω is a variety with a natural algebraic action of G. We define *geometric subgroups* of G to be the closed subgroups arising in this manner. Consequently, for H a geometric subgroup, we say that the natural action of G on $\Omega = G/H$ is a *geometric action*.

We define $C_\Omega(x)$ to be the set of points in Ω fixed by x. Then $C_\Omega(x)$ is a subvariety, and we can show that

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$$

The main aim of this thesis is to study the dimensions of fixed point spaces $C_\Omega(x)$ in the context of geometric actions of classical groups $G = Cl(V)$, with a focus on elements x in G of prime order. We define $f_\Omega(x)$ to be the ratio $\dim C_\Omega(x)/\dim \Omega$. Essentially, our aims are as follows:

(i) *Global bounds*: derive upper and lower bounds on $f_\Omega(x)$;
(ii) *Local bounds*: derive upper and lower bounds on $f_\Omega(x)$ in terms of the codimension of the largest eigenspace of x on V;
(iii) *Sharpness*: characterise the elements that realise such bounds.

Contents

Declaration of Authorship

I, Raffaele Rainone, declare that the thesis entitled *Geometric actions of classical algebraic groups* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of the work has been published before submission.

Signed:...

Date:..

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Introduction

Let k be an algebraically closed field of arbitrary characteristic p. An affine algebraic variety is a closed subset of k^n in the Zariski topology or, equivalently, the vanishing set of certain polynomials in n variables over k. Morphisms between affine varieties are maps that can be defined by polynomials in the coordinates. An affine algebraic group (or simply an *algebraic group*) is an affine variety over k with a group structure such that the multiplication and inversion maps are morphisms of varieties. For example the *special linear group* $\mathrm{SL}_n(k)$, the $n \times n$ matrices over k with determinant 1, is an affine variety and in fact an algebraic group.

Let G be an algebraic group with underlying field k and let $H \leq G$ be a closed subgroup. Then $\Omega = G/H$ is a variety over k and the natural action of G on Ω is an algebraic action, i.e. an action for which the map $G \times \Omega \to \Omega$ is a morphism of varieties. For $x \in G$ the *fixed point space*

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\} \subseteq \Omega$$

is a subvariety, so we can consider its dimension. This space will be of central interest in this thesis.

For any algebraic group G there exists a closed embedding $\iota\colon G \to \mathrm{GL}_n(k)$, for a suitable n. Fix an embedding ι. We say that $x \in G$ is unipotent (resp. semisimple) if $\iota(x) - 1$ is nilpotent (resp. $\iota(x)$ is diagonalisable). It turns out that these definitions are independent of the chosen embedding. Moreover, every element $x \in G$ has a unique Jordan-Chevalley decomposition $x = x_s x_u = x_u x_s$ where $x_s \in G$ is semisimple and $x_u \in G$ is unipotent. In particular, a prime order element is either semisimple or unipotent. In fact, in positive characteristic, a prime order element is unipotent if, and only if, it has order p.

Let V be a finite dimensional k-vector space equipped with the zero form f, a symplectic bilinear form f, or a non-degenerate quadratic form Q. In this thesis we say that $G = Cl(V)$ is a *classical group* if it is the isometry group of (V, κ) for $\kappa = f$ or Q. Formally,

$$\mathrm{I}(V, f) = \{A \in \mathrm{GL}(V) : f(A.v, A.u) = f(v, u) \text{ for all } u, v \in V\}$$
$$\mathrm{I}(V, Q) = \{A \in \mathrm{GL}(V) : Q(A.v) = Q(v) \text{ for all } v \in V\}$$

According to this definition the classical groups comprise the *general linear group* $\mathrm{GL}(V)$ in the case $f \equiv 0$, the *symplectic group* $\mathrm{Sp}(V)$ for f symplectic and the *orthogonal group* $\mathrm{O}(V)$ when Q is a non-degenerate quadratic form. If dim $V \geq 2$ (and also $\dim V \neq 4$ in the orthogonal case), then the derived subgroup $\mathrm{I}(V, \kappa)'$ is a simple algebraic groups, i.e. it has no non-trivial proper closed connected normal subgroups.

The geometry of (V, κ) gives rise to a number of natural spaces X on which the corresponding isometry group acts. An example is the space of linear subspaces of fixed dimension. In this thesis we investigate the actions of classical groups on the following spaces:

(i) the space of symplectic or non-degenerate quadratic forms on V;
(ii) the space of direct sum decompositions of V into maximal totally singular subspaces (subspaces on which the form is trivial), i.e.

$$X = \{(U, W) : V = U \oplus W,\ U, W \text{ totally singular}\}$$

(iii) the space of direct sum decompositions of V into orthogonal isometric subspaces.

In each case, the action of $G = Cl(V)$ on X is transitive. Thus, if H is a point stabiliser then this action is equivalent to the action on the coset variety $\Omega = G/H$. We call the actions arising from this construction *geometric actions*, and the stabilisers *geometric subgroups*. Incidentally, the subgroups arising in this manner are related to maximal subgroups, indeed their normalisers comprise the families $\mathscr{C}_6, \mathscr{C}_3$ and $\mathscr{C}_2$ in [**41**]. For each of the cases (i)–(iii) we say that H – a point stabiliser – lies, respectively, in the family $\mathscr{C}_6, \mathscr{C}_3$ or $\mathscr{C}_2$. Whenever we consider an action of a classical group G over $\Omega = G/H$ for $H \in \mathscr{C}_i$ we shall also refer to the action as a *$\mathscr{C}_i$-action*.

Each element of a classical group $G = Cl(V)$ is simultaneously acting on $\Omega = G/H$ and on V. Define $[V, x] = \langle x.v - v : v \in V\rangle \leq V$.

Definition 1. Let $G = Cl(V)$ and $x \in G$. Define

$$\nu(x) = \min\{\dim[V, \lambda x] : \lambda \in k^*\}$$

Notice that $\nu(x) > 0$ if x is not a scalar; in addition, $\nu(x)$ is the codimension of the largest eigenspace of x with respect to the natural action on V.

Let $G = Cl(V)$. Consider the action of G on $\Omega = G/H$, for some geometric subgroup $H \leq G$. Then we may expect a correspondence between $\dim C_\Omega(x)$ and $\nu(x)$. For example, if $\nu(x)$ is small then x has a large eigenspace and we might expect x to have a large fixed point space on Ω.

In this set up a number of questions arise naturally. One might ask how the dimension of $C_\Omega(x)$ depends on x and $\nu(x)$ for these geometric actions. A related problem, in the context of exceptional algebraic groups, was first studied by Lawther, Liebeck and Seitz [**39**], where upper bounds on $\dim C_\Omega(x)$ were obtained. We denote by x^G the *conjugacy class* of x. A key tool in their analysis is the following general formula (for $\Omega = G/H$):

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$$

In order to apply this formula we need a deep understanding of the G-conjugacy classes as well as the fusion of H-classes in G, as $x^G \cap H$ is a union of H-classes.

Notation. Let $G = Cl(V)$ be a classical algebraic group, $H \leq G$ be a closed geometric subgroup and $x \in G$ be an element of prime order r (or any unipotent element if the characteristic is zero). Set $\Omega = G/H$. The aim of this thesis is to study the ratio

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega}$$

In particular, we are interested in the following:

(a) *Global bounds*: derive close to best possible bounds on $f_\Omega(x)$ depending on G, H and r;
(b) *Local bounds*: derive close to best possible bounds on $f_\Omega(x)$ in terms of $\nu(x)$;
(c) *Sharpness*: justify why the bounds are close to best possible, by characterising elements that realise them, or by constructing elements with f_Ω-value sufficiently near the bounds.
(d) Focus on *involutions*: for $x \in G$ of order 2 find an explicit formula for $f_\Omega(x)$.

Some of these problems have already been addressed in the literature. An instance is the aforementioned paper [**39**] by Lawther, Liebeck and Seitz. Burness [**11**] with a systematic study of bounds on dimensions of conjugacy classes in classical algebraic groups derived upper bounds on the dimension of fixed point spaces in primitive actions. Hence (a) above extends the investigations in [**39, 11**] to upper and lower bounds on the dimension of fixed point spaces in actions of classical algebraic groups.

The only study on local bounds has been done by Frohardt and Magaard [**21**]; namely, they derived local upper bounds on the fixed point ratios (the proportion of points fixed by a group element) in actions of finite classical groups on the space of subspaces of fixed dimension. In this thesis we initiate the study of *local* upper and lower bounds for geometric actions of classical algebraic groups.

It is clear that problem (c) naturally arises when investigating global and local bounds. In the notation established, if $r = 2$ then $\dim C_\Omega(x)$ is essentially determined by $\nu(x)$: this motivates (d).

In this thesis we assume $H \leq G$ is a positive dimensional subgroup. Notice that if $H \leq G$ is a finite subgroup then $\dim C_\Omega(x) = \dim C_G(x)$. In particular, [**11**, Proposition 2.9] provides upper and lower local bounds on $\dim C_\Omega(x)$. Thus, our study on local bounds may be viewed as a generalisation of this result to the case where H is positive dimensional.

Further motivation arises from finite permutation groups. Let G be a transitive permutation group on a finite set Ω with point stabiliser H, i.e. $G \leq \mathrm{Sym}(\Omega)$. The *fixed point ratio* of x is defined to be

$$\mathrm{fpr}_\Omega(x) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap H|}{|x^G|}$$

Fixed point ratios have been widely studied since the 19th century, and bounds on them have found a wide range of applications [**40, 43, 28, 22, 23, 9, 16**]. For example, the key tool in [**22**] has been [**21**] whose main result implies that for a prime order element x in a finite classical group, defined over the field $\mathbb{F}_q$, acting on the set of subspaces of fixed dimension m, we have

$$\mathrm{fpr}_\Omega(x) \lesssim q^{-\nu(x)m}$$

Finite almost simple groups of Lie type can be defined by taking the fixed point groups of suitable morphisms of simple algebraic groups (see [**18**, Chapter 1] or [**45**, Chapter 22]). So it is not surprising that bounds on the dimension of fixed point spaces in actions of classical (or exceptional) groups can be translated to give bounds on the fixed point ratios in the corresponding actions of finite groups of Lie type. For example, the main tools to derive upper bounds on fixed point ratios in some primitive actions of finite exceptional simple groups in [**38**] are the bounds derived in [**39**] for the corresponding algebraic groups. Another instance is the series of papers [**12, 13, 14, 15**] by Burness; also here for the purpose of deducing upper bounds on fixed point ratios of finite simple classical groups in certain primitive actions the author uses bounds derived in [**11**] for primitive actions of classical algebraic groups.

Main results

Let k be an algebraically closed field of characteristic p; if the characteristic is zero we set $p = \infty$. Let G be a classical group $Cl(V)$. Then fixing a basis of V we gain an identification $V \cong k^n$ and also an identification of $Cl(V)$ with $Cl_n = \mathrm{GL}_n, \mathrm{Sp}_n, \mathrm{O}_n$. We will always assume $n > 4$ in order to simplify the statements of our results; however several of the following bounds hold in the case $n \leq 4$, as well. The main results of this thesis will deal with actions of classical groups $G = Cl_n$ on a coset variety $\Omega = G/H$ for $H \leq G$ a closed geometric subgroup in one of the families $\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_6$ defined in Table 1. For $\mathscr{C}_2$-subgroups we write $H = Cl_{n/t} \wr S_t < Cl_n$ where Cl ranges over the symbols $\{\mathrm{GL}, \mathrm{Sp}, \mathrm{O}\}$. Notice that we only consider positive dimensional subgroups, so for $\mathscr{C}_2$-subgroups we avoid the case $n = t$ if $G = \mathrm{O}_n$. For the reader's convenience in Appendix A we give tables where we provide references for all the cases studied.

$\mathscr{C}_i$	Rough description	Structure	Conditions
$\mathscr{C}_2$	Stabilisers of orthogonal decomposition $V = V_1 \oplus \ldots \oplus V_t,\ \dim V_i = n/t$	$Cl_{n/t} \wr S_t < Cl_n$	$n/t > 1$ if $Cl_n \neq \mathrm{GL}_n$
$\mathscr{C}_3$	Stabilisers of totally singular decomposition $V = U \oplus W$	$\mathrm{GL}_{n/2}.2 < \mathrm{Sp}_n$ or O_n	
$\mathscr{C}_6$	Classical subgroups	$\mathrm{Sp}_n, \mathrm{O}_n < \mathrm{GL}_n$	
		$\mathrm{O}_n < \mathrm{Sp}_n$	$p = 2$

Table 1

In the case $G = \mathrm{Sp}_n$ or O_n and $p = 2$ we label the conjugacy classes of involutions as in [**3**]. So if $x \in G$ is an involution with $\nu(x) = s$ odd then $x \in b_s^G$. In the case $\nu(x)$ is even we say $x \in a_s^G$ if $(v, x.v) = 0$ for all v in the natural module of G (where $(\cdot,\cdot)$ is the bilinear form on V fixed by G), otherwise $x \in c_s^G$. With a slight abuse of notation we shall also write $x = a_s$ or b_s or c_s.

For $x \in G$ we write $o(x)$ for the order of x. We define the following set:

$$\mathscr{R} = \mathscr{R}(G) = \{x \in G : o(x) \text{ prime or } x \text{ unipotent if } p = \infty\}$$

Theorems 4, 5, 6 give our strongest results on global bounds for $f_\Omega(x)$, where $\Omega = G/H$, $G = Cl_n$ and H is in $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_6$, respectively. (See Appendix A for a comprehensive list of references of all the results proved). However we first state the following theorem, which holds for any classical group and any subgroup $H \in \mathscr{C}_2 \cup \mathscr{C}_3 \cup \mathscr{C}_6$. Notice that if $x \in Z(G)$ then either $x \in H$, in which case $f_\Omega(x) = 1$, or $x \notin H$, in which case $C_\Omega(x) = \emptyset$ in particular $f_\Omega(x) = 0$.

In the following we set $t = 2$ if $H \in \mathscr{C}_3 \cup \mathscr{C}_6$ and t as in Table 1 if $H \in \mathscr{C}_2$.

Theorem 2. *Let $G = Cl_n$ and $H \leq G$ be a subgroup in $\mathscr{C}_i$ for $i = 2, 3, 6$. Set $\Omega = G/H$.*

(a) Let $x \in G \setminus Z(G)$. Then

$$f_\Omega(x) \leq 1 - \frac{1}{n} + \frac{1}{n^2}$$

(b) Let $x \in H \cap \mathscr{R}$ be unipotent.

(i) If $p > n/2$ then

$$f_\Omega(x) \geq \frac{1}{n}$$

(ii) If $p \leq n/2$ then

$$f_\Omega(x) \geq \frac{1}{p} - \frac{2}{n+2}$$

(c) Let $x \in H \cap \mathscr{R}$ be semisimple of order $r < n-1$. Assume $(G, H) \neq (\mathrm{Sp}_n, \mathrm{O}_n)$. Then

$$f_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{2n^2(t-1)} - \frac{2}{n}$$

Remark 3. Let us make some comments on the statement of Theorem 2.

(i) In (c), if $(G, H) = (\mathrm{Sp}_n, \mathrm{O}_n)$ or $r \geq n - 1$ then $f_\Omega(x) \geq 0$ and equality is possible. In fact, we determine all such elements in Theorems 6 and 12, below.

(ii) The lower bounds extend to any prime order element $x \in G$ whose G-conjugacy class meets H. In fact $x^G \cap H = \emptyset$ if, and only if, $C_\Omega(x) = \emptyset$.

(iii) The upper bound in (a) is not sharp, but best possible upper bounds are given in Theorems 4, 5, 6, which imply that the given bound is close to best possible.

(iv) The lower bound in (b)(i) is best possible. In fact, if $(G, H) = (\mathrm{GL}_n, \mathrm{O}_n)$, $p > n$ and $x \in H$ is a regular unipotent element, then $f_\Omega(x) = 1/n$.

(v) The lower bound in (b)(ii) is best possible when $p = 2$. For example, if $G = \mathrm{Sp}_n$ and $H \in \mathscr{C}_3$ then $f_\Omega(x) = 1/2 - 2/(n+2)$ for $x = b_{n/2}$ or $c_{n/2}$.

(vi) The lower bound in (c) for semisimple elements is not optimal; in fact it is often negative when $t \geq 2$ (but not if $r = 2$). However, in our later analysis we will give best possible bounds (which are more complicated to state) in almost all cases – see Remarks 8.1.2, 12.1.2 and 16.1.2 for details in the cases $H \in \mathscr{C}_6, \mathscr{C}_3, \mathscr{C}_2$, respectively.

Theorem 4. *Let $G = Cl_n$ and $H \leq G$ be a subgroup in $\mathscr{C}_2$. Set $\Omega = G/H$.*

(a) Let $x \in G \setminus Z(G)$. Then

$$f_\Omega(x) \leq 1 - \frac{2}{n} + \frac{2}{n(n-2)}$$

(b) Let $x \in H \cap \mathscr{R}$ be unipotent.

(i) *If* $p > n$ *then*

$$f_\Omega(x) \geq \frac{t}{n}$$

(ii) *If* $p \leq n$ *then*

$$f_\Omega(x) \geq \frac{1}{p}$$

(c) *Let* $x \in H \cap \mathscr{R}$ *be semisimple of order* $r < n-1$. *Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{2n^2(t-1)} - \frac{2}{n}$$

Theorem 5. *Let* $G = Cl_n$ *and* $H \leq G$ *be a subgroup in* $\mathscr{C}_3$. *Set* $\Omega = G/H$. *Then*

(a) *Let* $x \in G \setminus Z(G)$. *Then*

$$f_\Omega(x) \leq 1 - \frac{4}{n} + \left(\frac{4}{n}\right)^2$$

(b) *Let* $x \in H \cap \mathscr{R}$ *be unipotent.*

(i) *If* $p > n/2$ *then*

$$f_\Omega(x) \geq \frac{2}{n} - \frac{4}{n(n+2)}$$

(ii) *If* $p \leq n/2$ *then*

$$f_\Omega(x) \geq \frac{1}{p} - \epsilon$$

where $\epsilon = \frac{2}{n+2}$ *if* $p = 2$ *and* $\epsilon = 0$ *otherwise.*

(c) *Let* $x \in H \cap \mathscr{R}$ *be semisimple of order* $r < n$. *Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{r}{n(n-2)}$$

Theorem 6. *Let* $G = Cl_n$ *and* $H \leq G$ *be a subgroup in* $\mathscr{C}_6$. *Set* $\Omega = G/H$. *Then*

(a) *Let* $x \in G \setminus Z(G)$. *Then, for* $G = \mathrm{GL}_n$,

$$f_\Omega(x) \leq 1 - \frac{2}{n+1} + \frac{4}{n(n+1)}$$

If $G = \mathrm{Sp}_n$ *then* $f_\Omega(x) \leq 1 - 1/n$.

(b) *Let* $x \in H \cap \mathscr{R}$ *be unipotent.*

(i) *If* $p > n$ *then*

$$f_\Omega(x) \geq \frac{1}{n}$$

(ii) *If* $p \leq n$ *then*

$$f_\Omega(x) \geq \frac{1}{p}$$

(c) *Let* $x \in H \cap \mathscr{R}$ *be semisimple of order* r.

(i) *If* $G = \mathrm{GL}_n$ *then*

$$f_\Omega(x) \geq \begin{cases} \frac{1}{r} - \frac{1}{n(n-1)} & r < n \\ \frac{1}{n+1} & r \geq n \end{cases}$$

(ii) *Assume* $G = \mathrm{Sp}_n$. *Then* $f_\Omega(x) = 0$ *if, and only if,* $\dim C_V(x) = 0$.

Remark 7. Let us make some remarks on the statements of Theorems 4, 5 and 6. More detailed comments are given in Chapters 8, 12 and 16.

(i) As noted in Remark 3(vi), the lower bound in Theorem 4(c) is not best possible – see Remark 16.1.2 for optimal bounds in almost all cases.

(ii) We point out that, with the exception of the lower bounds for semisimple elements, the bounds are sharp. In fact, we can find elements $x \in G$ of order r such that $f_\Omega(x)$ realises the bound. Moreover, we can describe all such elements in most cases.

(iii) In Theorem 4(b) if we assumed $t < n$ then, instead of the two cases $p > n$ and $p \leq n$, we have the two cases $p > n/2$ (for (i)) and $p \leq n/2$ (for (ii)).

Let $G = Cl_n$ and $x \in G$. Recall the definition of $\nu(x)$ from Definition 1. In Theorem 8 below, we give upper and lower bounds on $f_\Omega(x)$, depending on $\nu(x)$, for $x \in H$ of odd prime order r. In order to state this result, we define

$$\mathcal{V}_s = \{x \in G : \nu(x) = s\}$$
$$\mathcal{V}_{s,r} = \{x \in \mathcal{V}_s : o(x) = r\}$$

Theorem 8. *Let $G = Cl_n$ and let $H \leq G$ be a subgroup in $\mathscr{C}_i$ for $i = 2, 3, 6$. Set $\Omega = G/H$. Let r be an odd prime. Let $x \in H \cap \mathcal{V}_{s,r}$.*

(i) We have

$$f_\Omega(x) \leq 1 - \frac{s}{n+2} + \frac{2}{n}$$

(ii) If $H \in \mathscr{C}_2$, assume $r \neq p$ and, in the orthogonal case, assume n/t is even. Then

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{5}{n}$$

Remark 9. We make some comments on Theorem 8 in the case $H = Cl_{n/t} \wr S_t$. Let H° be the connected component $Cl_{n/t} \times \cdots \times CL_{n/t}$. We shall compute an explicit formula for $\dim(x^G \cap H^\circ)$ for semisimple elements $x \in G$ of prime order (see Theorem 17.3.8) and we show that $\dim(x^G \cap H^\circ) \geq (1/t - \epsilon) \dim x^G$ for an explicit $\epsilon > 0$. These results are the key tools to derive local lower bounds. In particular, our method only applies if $x^G \cap H^\circ \neq \emptyset$. If $G = \mathrm{O}_n$ and n/t is odd then H does not contain a maximal torus; hence there are semisimple elements $x \in H$ for which $x^G \cap H^\circ = \emptyset$. Investigating local lower bounds for unipotent elements is difficult due to several reasons; for example, we do not have an explicit formula for $\dim(x^G \cap H^\circ)$ when $x \in H$ is unipotent (in general $x^G \cap H^\circ$ may be empty).

As expected, we see that there is a strong relationship between $\nu(x)$ and $f_\Omega(x)$. Excluding the cases in Theorem 8(ii), we immediately deduce the following result (with $r > 2$). See Remarks 8.1.11, 12.1.12 and 16.1.14 for references.

Corollary 10. *Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| \leq \frac{s(n-s)}{n^2} + \frac{2}{n}$$

In particular, if $s \leq \sqrt{n}$ or $s \geq n - \sqrt{n}$, then

$$|f_\Omega(x) - f_\Omega(y)| \leq \frac{1}{\sqrt{n}} + \frac{2}{n}$$

Assume $x, y \in G$ are involutions with $\nu(x) = \nu(y)$. Unless $p = 2$, $\nu(x)$ is even and $G \neq \mathrm{GL}_n$, we can show that $f_\Omega(x) = f_\Omega(y)$. In fact, in almost all cases, we can determine an explicit formula for $f_\Omega(x)$.

For the purpose of stating the next result we need some more notation. Let $G = Cl_n$ and $H = Cl_{n/t} \wr S_t$ be a $\mathscr{C}_2$-subgroup. Let $x \in G$ with $\nu(x) = s$. We define

$$h' = \max\Big\{0, \frac{s+t}{2} - \frac{n}{4}\Big\}$$

Theorem 11. *Let $G = Cl_n$ and $H \leq G$ be a subgroup in $\mathscr{C}_i$ for $i = 2, 3, 6$. Set $\Omega = G/H$. Let $x \in G$ be an involution with $\nu(x) = s$. Write $s = \lfloor s/t \rfloor t + b$ and $s/2 = \lfloor s/2t \rfloor t + b'$. Then*

$$g(n, t, s) \leq f_\Omega(x) \leq g(n, t, s) + \epsilon$$

where

(i) if $H \in \mathscr{C}_2$ then $g(n, t, s)$ and ϵ are recorded in Table 2;

(ii) if $H \in \mathscr{C}_3$ or $\mathscr{C}_6$ then $\epsilon = 0$ and $g(n, t, s)$ is recorded in Tables 3 and 4 for $H \in \mathscr{C}_3$ and $\mathscr{C}_6$, respectively.

G	Conditions	$g(n,t,s)$	ϵ
GL_n	n/t *even* or n/t *odd*, $s < \max\{\frac{n}{t}, \frac{n-t}{2}\}$	$1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$	0
Sp_n	$p = 2, x \in b_s^G$ *or* c_s^G		
O_n	$x \notin a_s^G$		
Sp_n	$p \neq 2$; $p = 2, x \in a_s^G$, $n/2t$ *even* or $h' = 0$	$1 - \frac{2s(n-s)}{n^2} - \frac{8b'(t-b')}{n^2(t-1)}$	$3/n$
O_n	$p = 2, x \in a_s^G, h' = 0$		
Sp_n	$p = 2, x \in a_s^G, n/2t$ *odd*, $h' > 0$	$1 - \frac{2s(n-s-1)}{n^2\left(1-\frac{1}{t}\right)} + \frac{n+2(s-t)}{4n(t-1)}$	$3/n$
O_n	$p = 2, x \in a_s^G, h' > 0$		
GL_n	n/t *odd*, $s \geq \max\{\frac{n}{t}, \frac{n-t}{2}\}$	$1 - \frac{2s(n-s-1)}{n^2\left(1-\frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$	0

Table 2. Involutions in $\mathscr{C}_2$-actions

An element $x \in G$ is regular if $\dim C_G(x) = \operatorname{rank} G$. In [**31**] Herb and O'Brian showed that $x \in G$ is regular if, and only if, $C_\Omega(x)$ is finite, where $\Omega = G/P$ and $P \leq G$ is parabolic (a particular type of $\mathscr{C}_1$-subgroup, in the terminology of Liebeck and Seitz). We establish a similar result for $\mathscr{C}_2$- and $\mathscr{C}_3$-actions.

Theorem 12. *Let $G = Cl_n$ and $H \leq G$ be a subgroup in $\mathscr{C}_i$ for $i = 2, 3$. Let $x \in G$ such that $x^G \cap H \neq \emptyset$. Then the following are equivalent*

(i) $C_\Omega(x)$ is finite;

(ii) $f_\Omega(x) = 0$;

(iii) x is regular, or $G = \mathrm{Sp}_n$, $H \in \mathscr{C}_2$ and $C_G(x) \cong \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1}$.

Let G be an algebraic group acting on a variety Ω. We define the *algebraic fixity*

$$M = M(G, \Omega) = \sup\{f_\Omega(x) : x \in G \setminus Z(G)\}$$

G	s	p	x	$g(n,t,s)$
Sp_n	$< n/2$	*any*		$1-\frac{2s(n-s)}{n(n+2)}$
	$n/2$	$\neq 2$	$[I_{n/2}, -I_{n/2}]$	$\frac{1}{2}+\frac{1}{n+2}$
	$n/2$	2	$a_{n/2}$	$\frac{1}{2}+\frac{2}{n+2}$
	$n/2$	2	$b_{n/2}, c_{n/2}$	$\frac{1}{2}-\frac{2}{n+2}$
O_n	$< n/2$	*any*		$1-\frac{2s(n-s)}{n(n-2)}+\frac{4s}{n(n-2)}\delta_{p,2}$
	$n/2$	$\neq 2$	$[I_{n/2}, -I_{n/2}]$	$\frac{1}{2}$
	$n/2$	2	$a_{n/2}$	$\frac{1}{2}+\frac{2}{n-2}$
	$n/2$	2	$b_{n/2}$	$\frac{1}{2}$
	$n/2$	2	$c_{n/2}$	$\frac{1}{2}-\frac{2}{n-2}$

Table 3. Involutions in $\mathscr{C}_3$-actions

G	H	$g(n,t,s)$
GL_n	Sp_n	$1-\frac{2s(n-s)}{n(n-1)}+\frac{2s}{n(n-1)}\delta_{p,2}$
GL_n	O_n	$1-\frac{2s(n-s)}{n(n+1)}+\frac{2s}{n(n+1)}\delta_{p,2}$
Sp_n	O_n	$1-\frac{s}{n}$

Table 4. Involutions in $\mathscr{C}_6$-actions

Let r be a prime, we define the *r-local algebraic fixity*

$$M_r = \sup\{f_\Omega(x) : o(x) = r\}$$

For any $x \in G$ we have $x = x_s x_u$ where $x_s \in G$ is semisimple and $x_u \in G$ is unipotent and $C_\Omega(x) = C_\Omega(x_s) \cap C_\Omega(x_u)$ (see Proposition 7.1.5). Clearly, for any integer m we have $C_\Omega(x) \subseteq C_\Omega(x^m)$. Therefore $M = \sup\{M_r : r\,\text{prime}\}$.

For finite permutation groups, M is defined to be the *fixity* of the group. Saxl and Shalev [**52**] showed that if a finite simple primitive permutation group has fixity f then either it is $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(q)$ (in their natural permutations actions), or the order of the group is bounded by some function of f.

Burness [**10**] provided a lower bound on the 2-local algebraic fixity for primitive actions of simple algebraic groups. The main result of [**10**] states that, apart from a short list of known exceptions,

$$M_2 \geq \frac{1}{2} + \frac{1}{2h+1}$$

where h is the *Coxeter number* of G: $h = -1 + \dim G/\mathrm{rank}\, G$. This bound is essentially the best possible (see [**10**, Remark 1]).

Theorem 13. *Let $G = Cl_n$ and $H \leq G$ be a subgroup in $\mathscr{C}_i$ for $i = 2,3,6$. Set $\Omega = G/H$. Then the algebraic fixity of G is listed in Table 5. In addition, if we assume $n > 10$ if $(G,H) = (\mathrm{O}_n, \mathrm{GL}_{n/2}.2)$, there exists $x \in G$ of prime order r such that $f_\Omega(x) = M$ if, and only if, r is as in the last column of the table.*

G	H	M	Conditions	r
GL_n	$\mathrm{GL}_{n/t} \wr S_t$	$1-\frac{2}{n}+\frac{1}{n(n-1)}$	$t=n$	2
		$1-\frac{2}{n}$	$t \neq n$	*any*
Sp_n	$\mathrm{Sp}_{n/t} \wr S_t$	$1-\frac{2}{n}$		p
O_n	$\mathrm{O}_{n/t} \wr S_t$	$1-\frac{2}{n}$		2
Sp_n	$\mathrm{GL}_{n/2}.2$	$1-\frac{4(n-2)}{n(n+2)}$		2
O_n	$\mathrm{GL}_{n/2}.2$	$1-\frac{4(n-4)}{n(n-2)}$		p
GL_n	Sp_n	$1-\frac{2(n-2)}{n(n-1)}$	$p \neq 2$	p
		$1-\frac{4(n-3)}{n(n-1)}$	$p=2$	2
GL_n	O_n	$1-\frac{2(n-1)}{n(n+1)}$	$p \neq 2$	2
		$1-\frac{2(n-2)}{n(n+1)}$	$p=2$	2
Sp_n	O_n	$1-\frac{1}{n}$	$p=2$	2

Table 5. Algebraic fixity

Remark 14. We could state a similar version of this theorem for $4 < n \leq 10$ in the case $G = \mathrm{O}_n$ and $H \in \mathscr{C}_3$. However, several cases arise depending if $n = 6, 8$, or 10. For these small rank cases, we refer the reader to Remark 12.1.4.

Corollary 15. *There exists $x \in G$, either an involution or an element of order p, such that $f_\Omega(x) = M$.*

During the analysis we will compute M_r for any r, see the references in Remarks 8.1.2(i), 12.1.2(ii) and 16.1.2(ii). The following shows that Theorem 2(a) provides close to the best possible upper bounds on $f_\Omega(x)$ for $x \in G \setminus Z(G)$.

Corollary 16. *For any prime r there exists $x \in G \setminus Z(G)$ of order r such that*

$$f_\Omega(x) \geq 1 - \frac{4}{n}$$

Part 1

Background

CHAPTER 1

Algebraic varieties

In this chapter we collect some basic results concerning classical algebraic geometry. The purpose is to introduce the main objects needed in order to define and discuss the basic properties of algebraic groups and their actions. We define affine and projective varieties, and we introduce important notions such as dimension and morphisms. We follow [**29**, Chapter 1]. Other references for algebraic geometry with a view towards algebraic groups are [**25**, Chapters 1-2] and [**45**, Chapter 1].

Throughout this thesis k is an algebraically closed field of characteristic p.

1.1. Zariski topology

The *affine n-space* over k, denoted $\mathbb{A}^n$, is the set of n-tuples of elements of k. We call the elements $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$ *points*. Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over k.

Each polynomial $f \in A$ can be naturally viewed as a function $f \colon \mathbb{A}^n \to k$ by defining $f(P) = f(a_1, \ldots, a_n)$.

Let $f \in A$ be a polynomial. We define the *zero set* of f to be

$$Z(f) = \{P \in \mathbb{A}^n : f(P) = 0\} \subseteq \mathbb{A}^n$$

More generally, if $T \subseteq A$ then $Z(T) = \{P \in \mathbb{A}^n : f(P) = 0 \text{ for all } f \in T\}$. It follows from the definition that if $T \subseteq T' \subseteq A$ then $Z(T') \subseteq Z(T) \subseteq \mathbb{A}^n$. Let $\mathfrak{a}$ be the ideal of A generated by $T \subseteq A$. It is rather easy to show that $Z(T) = Z(\mathfrak{a})$.

Recall that a ring R is said to be *Noetherian* if it satisfies the ascending chain condition on ideals, i.e. if given a chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ there exists m such that $I_m = I_{m+l}$ for all $l \geq 0$. Equivalently, R is Noetherian if every ideal is finitely generated. In particular, any field is a Noetherian ring. By Hilbert's Basis Theorem [**4**, Theorem 7.5], if R is a Noetherian ring then so is $R[x]$. In particular, A is Noetherian, then any ideal is finitely generated. Therefore, for any $T \subseteq A$, $Z(T)$ is defined by the zero set of a finite number of polynomials.

A subset $X \subseteq \mathbb{A}^n$ is called an *algebraic set* if there exists $T \subseteq A$ such that $X = Z(T)$. The following is [**29**, Proposition 1.1].

Proposition 1.1.1. *Algebraic sets satisfy the axioms for the closed subsets for a topology on $\mathbb{A}^n$.*

The *Zariski topology* is defined to be the topology on $\mathbb{A}^n$ whose closed sets are the algebraic sets.

Example 1.1.2. Let $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$. Then $\{P\} = Z(x_1 - a_1, \ldots, x_n - a_n)$. In particular, finite sets are algebraic sets.

Note that the closed sets of $\mathbb{A}^1$ in the Zariski topology are precisely the finite subsets. A topology whose collection of closed sets consists of finite subsets is also called cofinite topology.

Example 1.1.3. Let V be an n-dimensional k-vector space. Fixing a basis of V and recalling that $\mathbb{A}^n$ has a natural structure as vector space, we have an isomorphism $\varphi\colon V \to \mathbb{A}^n$. In particular, via φ, we can endow V with the Zariski topology (notice that φ depends on the choice of the basis hence the topology on V will depend on this choice). Let $T \subseteq A = k[x_1, \dots, x_n]$ be a set of linear polynomials with no constant term. Then $Z(T) \leq \mathbb{A}^n$ is a linear subspace. Conversely, any linear subspace $W \leq \mathbb{A}^n$ is the zero set of a system of some linear polynomials with no constant term.

Let us recall some general concepts in topology. Given a topological space X and a non-empty subset Y of X we say that Y is *irreducible* if Y cannot be expressed as a union of two proper subsets that are closed in X. A topological space X is called *Noetherian* if it satisfies the *descending chain condition* for closed subsets, i.e. if given a chain of closed subsets $Y_1 \supseteq Y_2 \supseteq \dots$ there exists an integer m such that $Y_m = Y_{m+l}$ for all $l \geq 0$. For the next result, see [**29**, Proposition 1.5].

Proposition 1.1.4. *Let X be a Noetherian topological space and let $Y \subseteq X$ be non-empty and closed. Then there exist finitely many closed irreducible subsets of X such that $Y = Y_1 \cup \dots \cup Y_n$. If we require $Y_i \not\subseteq Y_j$ for all $i \neq j$, then the Y_i are uniquely determined. In this situation, the Y_i are called the irreducible components of Y.*

In Remark 1.2.1 below, we prove that affine n-space $\mathbb{A}^n$, equipped with the Zariski topology, is a Noetherian space. Hence, as an immediate consequence of Proposition 1.1.4, any algebraic set can be expressed uniquely as a union of irreducible algebraic sets, no one containing another.

Set $\mathbf{0} = (0, \dots, 0) \in \mathbb{A}^{n+1}$. We say that $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ in $\mathbb{A}^{n+1} \backslash \{\mathbf{0}\}$ if, and only if, there exists $\lambda \in k^*$ such that $(a_0, \dots, a_n) = \lambda(b_0, \dots, b_n)$. It is straightforward to check that this is an equivalence relation. We define the *projective n-space* $\mathbb{P}^n = \mathbb{A}^{n+1}/_\sim$. For $P = (a_0, \dots, a_n) \in \mathbb{A}^{n+1}$ write $[a_0 : \dots : a_n]$ for the equivalence class containing P. A polynomial $f \in k[x_0, \dots, x_n]$ is said to be *homogeneous of degree d* if for every $\lambda \in k^*$ and for every $(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \setminus \{\mathbf{0}\}$ we have $f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$. We set $S \subseteq k[x_0, \dots, x_n]$ to be the collection of all homogeneous polynomials. And we observe that $S = \bigoplus_{d \geq 0} S_d$ where $S_d = \{f \in k[x_0, \dots, x_n] : f \text{ homogeneous of degree } d\}$. In particular, given $f \in S$, if $f(a_0, \dots, a_n) = 0$ then $f(\lambda a_0, \dots, \lambda a_n) = 0$ for all $\lambda \in k^*$. Therefore for any $f \in S$ the following set is well defined

$$Z(f) = \{P \in \mathbb{P}^n : f(P) = 0\}$$

As in the affine case, if $T \subseteq S$ and T consists of homogeneous polynomials then $Z(T) = \{P \in \mathbb{P}^n : f(P) = 0 \text{ for all } f \in T\}$.

We say that a subset $X \subseteq \mathbb{P}^n$ is an *algebraic set* if there exists $T \subseteq S$ such that $X = Z(T)$. We have the following result, which is the projective analogue of Proposition 1.1.1, see [**29**, Proposition 2.1].

Proposition 1.1.5. *The algebraic sets satisfy the axioms for the closed sets for a topology on* $\mathbb{P}^n$.

We define the *Zariski topology* on $\mathbb{P}^n$ to be the topology whose collection of closed sets consists of algebraic sets.

Also here we have that finite sets are algebraic sets.

Example 1.1.6. Let $P = [a_0 : \ldots : a_n] \in \mathbb{P}^n$, and assume $a_i \neq 0$. Let $f_j = a_i x_j - a_j x_i$. Then $f_j \in S$ and $Z(f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n) = \{P\}$.

Example 1.1.7. Let V be an $(n+1)$-dimensional vector space over k. We may define $\mathbb{P}(V) = \{\langle v \rangle : v \in V \setminus \{\mathbf{0}\}\}$ as the projective space associated to V. Here $v \sim u$ if, and only if, there exists $\lambda \in k^*$ such that $v = \lambda u$. In particular, fixing a basis of V and hence of $\mathbb{P}(V)$, we have an identification $\mathbb{P}(V) \cong \mathbb{P}^n$. Again, thanks to this identification we can endow $\mathbb{P}(V)$ with the Zariski topology.

1.2. Varieties

An *affine algebraic variety*, or affine variety, is an algebraic set in the affine space $\mathbb{A}^n$ together with the induced Zariski topology. An open subset of an affine variety is called a *quasi-affine variety*.

Let $Y \subseteq \mathbb{A}^n$ be a subset. The *ideal* of Y in A is

$$I(Y) = \{f \in A : f(P) = 0 \text{ for all } P \in Y\}$$

It is easy to check that $I(Y)$ is in fact an ideal of A. It is clear that if $Y \subseteq X$ then $I(X) \subseteq I(Y)$. Notice that given $Y \subseteq \mathbb{A}^n$ we have $Z(I(Y)) = \overline{Y}$, the closure of Y in $\mathbb{A}^n$ (see [**29**, Proposition 1.2]). Recall that in a topological space X we say that $Y \subseteq X$ is *dense* if $\overline{Y} = X$. We have that any open set in $\mathbb{A}^n$ is dense (see [**29**, Exercise 1.6]).

Remark 1.2.1. Let us observe that the affine space with the Zariski topology is Noetherian. Let $Y_1 \supseteq Y_2 \supseteq \ldots$ be a descending chain of closed subsets in $\mathbb{A}^n$. Then we have the ascending chain $I(Y_1) \subseteq I(Y_2) \subseteq \ldots$ of ideals in A. Since the polynomial ring is Noetherian, there exists m such that $I(Y_m) = I(Y_{m+l})$ for all $l \geq 0$. Since $Z(I(Y_i)) = Y_i$ for all i we deduce that $\mathbb{A}^n$ is Noetherian.

If $\mathfrak{a}$ is an ideal of a ring R then the *radical* of $\mathfrak{a}$ is defined to be

$$\sqrt{\mathfrak{a}} = \{f \in A : f^r \in \mathfrak{a} \text{ for some } r > 0\}$$

Consequently, an ideal $\mathfrak{a}$ is said to be *radical* if $\mathfrak{a} = \sqrt{\mathfrak{a}}$. For example, consider $a \in \mathbb{Z}_{>0}$ and $(x^a) \subseteq k[x]$ then $\sqrt{(x^a)} = (x)$. In particular (x^a) is radical if, and only if, $a = 1$.

The most important result in classical algebraic geometry is the following (for a proof see [**4**, Theorem 1, p.85]).

Theorem 1.2.2 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field. Let $\mathfrak{a}$ be an ideal of $A = k[x_1, \ldots, x_n]$. Then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

An immediate corollary of the Nullstellensatz is that there is a one-to-one, inclusion reversing, correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in A given by

$Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$, see [**29**, Corollary 1.4]. In particular, there is a one to one correspondence between points $\{P = (a_1, \ldots, a_n)\} \subseteq \mathbb{A}^n$ and maximal ideals $\mathfrak{m} \subseteq A$.

Let $Y \subseteq \mathbb{A}^n$ be an affine variety. We define the *affine coordinate ring* $k[Y]$ of Y to be $A/I(Y)$. We can regard $k[Y]$ as the ring of polynomial functions $Y \to k$.

In Section 1.1 we gave the definition of irreducibility for a topological space. We have the following characterization of an irreducible algebraic set in terms of its corresponding ideal, [**25**, Proposition 1.1.12 (b)].

Proposition 1.2.3. *Let $Y \subseteq \mathbb{A}^n$ be an algebraic set. Then Y is irreducible if, and only if, $I(Y)$ is a prime ideal.*

A topological space X is *connected* if it cannot be written as the disjoint union of two proper closed subsets. Clearly, if a space is irreducible then it is connected. The converse does not hold in general. For example, consider the algebraic set $Y = Z(xy)$ in $\mathbb{A}^2$. It is clear that

$$Y = \{(x, 0) : x \in k\} \cup \{(0, y) : y \in k\}$$

and these two sets are closed, hence Y is reducible. Meanwhile, it is possible to show directly that if $Y = X_1 \cup X_2$ where X_1, X_2 are disjoint and closed in Y (and hence closed in $\mathbb{A}^2$) then $X_1 = Y$ or $X_2 = Y$. In fact if such a sets exists then $X_i = Z(T_i)$ where $T_i \subseteq A$ is an ideal, for $i = 1, 2$. Thus $Y = X_1 \cup X_2 = Z(T_1 T_2) = Z(xy)$, which implies $T_1 T_2 = (xy)$. Since $X_1 \cap X_2 = \emptyset$, it is rather easy now to show that either T_1 or T_2 must be (xy), i.e. we cannot have $T_1 = (x)$ and $T_2 = (y)$ otherwise $(0, 0) \in X_1 \cap X_2$. Therefore Y is connected. For an algebraic group, these two concepts are equivalent (see Proposition 2.1.2).

We also have the related concept of a *projective variety*. This is an algebraic set in $\mathbb{P}^n$, together with the induced Zariski topology. An open subset of a projective variety is called a *quasi-projective variety*.

As in the affine case, we can prove that any algebraic set Y of $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. Again, we call these sets *irreducible components* of Y, see [**29**, Exercise 2.5 (b)]. Incidentally, we note that $\mathbb{P}^n$ can be covered by $(n+1)$-copies of $\mathbb{A}^n$, i.e. $\mathbb{P}^n = \bigcup_{i=0}^n \mathbb{A}^n_i$ where $\mathbb{A}^n_i = \{[a_0 : \ldots : a_n] : a_i = 0\}$, see [**29**, Proposition 2.2].

For $Y \subseteq \mathbb{P}^n$, the *homogeneous ideal* $I(Y)$ of Y in S is the ideal generated by $\{f \in S : f(P) = 0 \text{ for all } P \in Y\}$. And, as in the affine case, the *homogeneous coordinate ring* $S[Y]$ of Y is $S/I(Y)$.

In the following with the word *variety* we shall mean any affine, projective, quasi-affine, or quasi-projective variety as defined above, unless it is clear from the context which kind of variety we are considering.

1.3. Dimension

Let X be a topological space. The *dimension* of X, see [**29**, p.5], is defined to be

$$\dim X = \sup\{n : \exists\, \emptyset \neq X_0 \subset X_1 \subset \ldots \subset X_n = X,\ X_i \text{ irreducible}\}$$

Now, let $Y \subseteq \mathbb{A}^n$ be an affine (or quasi-affine) variety. We define the *dimension* of Y to be its dimension as a topological space with respect to the Zariski topology.

Let R be a ring. Given a prime ideal $\mathfrak{p}$ of R we define the *height* of $\mathfrak{p}$ to be

$$\mathrm{ht}(\mathfrak{p}) = \sup\{n : \exists\ 0 \neq \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{p},\ \mathfrak{p}_i \text{ prime ideal}\}$$

The *Krull dimension* of the ring R is the supremum of the heights of all its prime ideals. We denote the Krull dimension of the ring R by $\dim_{\mathrm{Kr}} R$, see [**29**, p.6].

If Y is an irreducible algebraic set then Proposition 1.2.3 implies that $k[Y]$ is an integral domain. Hence we may consider its field of fractions $k(Y)$. We denote by $\mathrm{Tr}_k k(Y)$ the transcendence degree of the field of fractions $k(Y)$ over the field k, this is the maximum number of elements of $k(Y)$ algebraically independent over k (we remind the reader that given fields $k \subset L$ then $\alpha_1, \ldots, \alpha_l \in L$ are said to be *algebraically independent* over k if for every non-zero polynomial $f \in k[x_1, \ldots, x_l]$ we have $f(\alpha_1, \ldots, \alpha_l) \neq 0$).

The following relates these concepts, see [**29**, Proposition 1.7] and [**4**, Chapter 11].

Theorem 1.3.1. *Let $Y \subseteq \mathbb{A}^n$ be an affine irreducible variety. Then*

$$\dim Y = \dim_{\mathrm{Kr}} k[Y] = \mathrm{Tr}_k k(Y)$$

We have that $\dim \mathbb{A}^n = n$. Indeed its coordinate ring is $A = k[x_1, \ldots, x_n]$ and its field of fractions, which is $k(x_1, \ldots, x_n)$, has transcendence degree equal to n over k. Let us give an additional example.

Example 1.3.2. Let $X = \{(x, y) \in k^2 : xy = 1\}$. Then $X = Z(xy - 1)$. The coordinate ring of X is $k[X] = k[x, y]/(xy - 1)$ and it is clear that $k[X] \cong k[x, x^{-1}]$ via $x \mapsto x$, $y \mapsto x^{-1}$. Since $k[x, x^{-1}]$ is an integral domain we deduce that X is an irreducible variety, by Proposition 1.2.3. The fraction field of $k[X]$ is given by $k(x)$, hence $\mathrm{Tr}_k k(x) = 1$ and thus $\dim X = 1$, by Theorem 1.3.1.

In order to compute the dimension of a quasi-affine variety we use the following result (see [**29**, Proposition 1.10]).

Proposition 1.3.3. *Let $Y \subseteq \mathbb{A}^n$ be a quasi-affine variety. Then* $\dim Y = \dim \overline{Y}$.

The projective analogue of Theorem 1.3.1 is the following, [**29**, Exercise 2.6, 2.7b].

Theorem 1.3.4. *Let Y be a projective variety. Then*

$$\dim Y = \dim_{\mathrm{Kr}} S[Y] - 1$$

If Y is a quasi-projective variety then $\dim Y = \dim \overline{Y}$.

Since $S[\mathbb{P}^n] = S \subseteq k[x_0, \ldots, x_n]$ and $\dim_{\mathrm{Kr}} S = n + 1$, by applying Theorem 1.3.4 we deduce $\dim \mathbb{P}^n = n$.

Let Y be any algebraic set in $\mathbb{A}^n$ or $\mathbb{P}^n$. By Proposition 1.1.4, write $Y = Y_1 \cup \cdots \cup Y_l$, where each Y_i is closed and irreducible and $Y_i \not\subseteq Y_j$ for all $i \neq j$. Then [**25**, Proposition 1.2.17] yields

$$\dim Y = \max\{\dim Y_i : 1 \leq i \leq l\} \qquad (1)$$

1.3.1. Subvarieties. Let X be topological space. We say that a subset is *locally closed* if it is the intersection of an open and a closed subset of X.

If X is an affine (resp. projective) variety and $Y \subseteq X$ is closed, we say that Y is a *subvariety* of X. If X is a quasi-affine (resp. quasi-projective) variety and Y is a locally closed subset then it is clear that Y is a quasi-affine (resp. quasi-projective) variety, and in this case we also say that Y is a *subvariety* of X.

Dimension behaves nicely with respect to subvarieties. Indeed if $Y \subseteq X$ is a subvariety then $\dim Y \leq \dim X$, see [**45**, Proposition 1.22]. Moreover if X and Y are irreducible then $\dim Y = \dim X$ if, and only if, $Y = X$, see [**29**, Exercise 1.10].

1.4. Morphisms

In the previous sections we have introduced the notion of affine and projective varieties. Now we define what maps are allowed between them.

Let Y be a variety. We say that a function $f\colon Y \to k$ is *regular at a point* $P \in Y$ if there exists an open neighbourhood $U \subseteq Y$ of P and polynomials $g, h \in A$ (homogeneous in S if Y is projective) with $h(Q) \neq 0$ for all $Q \in U$ such that $f = g/h$ on U. Consequently, we say that f is *regular* if it is regular at every point of Y.

The ring of regular function of Y is a k-algebra and, if Y is an irreducible variety, it is isomorphic to $k[Y]$, see [**29**, Theorem 3.2a].

Let X and Y be varieties. We say that $\varphi\colon X \to Y$ is a *morphism* if it is continuous and, for every open set $U \subseteq Y$ and for every regular function $f\colon U \to k$, the function $f \circ \varphi\colon \varphi^{-1}(U) \to k$ is regular.

A morphism $\varphi\colon X \to Y$ is defined to be an *isomorphism* if there exists a morphism $\psi\colon Y \to X$ such that $\varphi \circ \psi = id_Y$ and $\psi \circ \varphi = id_X$. In particular, an isomorphism is a bijection. In general, $\varphi\colon X \to Y$ does not need to be an isomorphism if it is a bijective morphism of varieties, we give an example after Proposition 1.4.1 below. We say that X and Y are isomorphic if there exists an isomorphism between them and we write $X \cong Y$.

Given X and Y, we write $\mathrm{Hom}(X, Y)$ for the set of morphisms $\varphi\colon X \to Y$. Similarly, $\mathrm{Hom}(k[X], k[Y])$ denotes the set of k-algebras homomorphisms $k[X] \to k[Y]$.

Let $\varphi\colon X \to Y$ be a morphism of affine varieties. We define its *comorphism* $\varphi^*\colon k[Y] \to k[X]$ as $f \mapsto f \circ \varphi$. The next result relates $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(k[X], k[Y])$, see [**25**, Proposition 1.3.4] for a proof.

Proposition 1.4.1. *Let X and Y be affine varieties. Then the map* $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(k[Y], k[X])$, $\varphi \mapsto \varphi^*$, *is bijective. Moreover, $\varphi\colon X \to Y$ is an isomorphism if, and only if, φ^* is an isomorphism of k-algebras.*

In particular given two varieties X, Y, one can check that the map $\varphi\colon X \to Y$ is a morphism of varieties by showing that $\varphi^*\colon k[Y] \to k[X]$ is a k-algebras homomorphism. Notice that this is true whenever the map $\varphi\colon X \to Y$ is defined by polynomial equations, i.e. $P = (a_1, \ldots, a_n) \mapsto (f_1(P), \ldots, f_m(P))$ and $f_i \in k(x_1, \ldots, x_n)$, here we are assuming $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$. We remark that this property coincides with the definition of morphism given by some authors, see [**30**, Definition 3.35].

As anticipated, we give an example of a morphism of varieties which is bijective but not an isomorphism. Recall that k is an algebraically closed field of characteristic p. Assume that the characteristic is non-zero. We consider the *Frobenius morphism*, $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t^p$. Then φ is a bijective map. It is clear that φ is a morphism of varieties since $\varphi^*\colon k[x] \to k[x]$, $f \mapsto f \circ \varphi$ is a k-algebra homomorphism (or, simply, because φ is a regular map). However φ is not an isomorphism since $\varphi^*\colon k[x] \to k[x]$ is not surjective; in fact $x^a \in \varphi^*(k[x])$ if, and only, $p \mid a$.

Let $\varphi\colon X \to Y$ be a morphism of varieties. The *fibres* of φ are defined to be the closed sets $\varphi^{-1}(y)$ for $y \in Y$. The following [**56**, Corollary 15.5.5(ii)] will be a useful tool in Proposition 7.1.8.

Proposition 1.4.2. *Let $\varphi\colon X \to Y$ be a morphism of varieties. Assume $\overline{\varphi(X)} = Y$ and $\dim \varphi^{-1}(y) = \dim \varphi^{-1}(z)$ for all $y, z \in \varphi(X)$. Then, for any $y \in \varphi(X)$,*

$$\dim X = \dim Y + \dim \varphi^{-1}(y)$$

1.5. Product of varieties

Let $X \subseteq \mathbb{A}^r$ and $Y \subseteq \mathbb{A}^s$ be affine varieties. Then $X \times Y \subseteq \mathbb{A}^{r+s}$ with the induced topology is a variety and it is called a *product variety*. Note that $X \times Y$ is not endowed with the usual product topology. As an example of this we note that $\mathbb{A}^1 \times \mathbb{A}^1$ with the product topology is not isomorphic to $\mathbb{A}^2$. For instance, $Z(y - x^2) = \{(x, y) : y = x^2\}$ is closed in $\mathbb{A}^2$ but not in $\mathbb{A}^1 \times \mathbb{A}^1$ (since $\mathbb{A}^1$ is endowed with the cofinite topology).

It is possible to prove that if X and Y are irreducible so is $X \times Y$. Moreover, it turns out that $k[X \times Y] \cong k[X] \otimes_k k[Y]$, so $\dim(X \times Y) = \dim X + \dim Y$, see [**25**, Proposition 1.3.8]. Moreover [**29**, Exercise 3.15(c)] implies that the natural projections $X \overset{\pi_1}{\leftarrow} X \times Y \overset{\pi_2}{\rightarrow} Y$ are morphisms.

For projective varieties $X \subseteq \mathbb{P}^r$ and $Y \subseteq \mathbb{P}^s$ the construction is more complicated. The variety structure of $X \times Y$ arises via the *Segre embedding*, [**29**, Exercise 2.14].

We conclude this chapter with the following observation. Let X be a variety. Then $X \times X$ is also a variety, and we define the *diagonal* subset of $X \times X$ to be

$$\Delta = \Delta(X) = \{(P, Q) \in X \times X : P = Q\}$$

It is not hard to show that Δ is closed in $X \times X$. In fact Δ is an algebraic subset of $\mathbb{A}^n \times \mathbb{A}^n$ (resp. $\mathbb{P}^n \times \mathbb{P}^n$). If $X \subseteq \mathbb{A}^n$ we have

$$\Delta = \{(P, Q) \in X \times X : f_i(P, Q) = 0, 1 \leq i \leq n\}$$

where $f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) = x_i - y_i \in k[x_i, y_i] \cong k[x_i] \otimes_k k[y_i]$.

Similarly if $X \subseteq \mathbb{P}^n$ is projective. Define $f_{i,j}(x_0, \ldots, x_n, y_0, \ldots, y_n) = x_i y_j - x_j y_i$. Then

$$\Delta = \{(P, Q) \in X \times X : f_{i,j}(P, Q) = 0, 0 \leq i \neq j \leq n\}$$

Proposition 1.5.1. *Let X be a variety. Then $\Delta \subseteq X \times X$ is closed.*

The following will be a useful tool for deriving a formula for the dimension of fixed point spaces, see Proposition 7.1.8.

Lemma 1.5.2. *Let $f, g\colon X \to Y$ be two surjective morphisms. Then $V = \{x \in X : f(x) = g(x)\} \subseteq X$ is closed.*

PROOF. Define $\varphi\colon X \to Y \times Y$, $x \mapsto (f(x), g(x))$. Then φ is a morphism, and $\Delta \subseteq Y \times Y$ is closed by Proposition 1.5.1. Hence $V = \varphi^{-1}(\Delta) \subseteq X$ is closed. *q.e.d.*

CHAPTER 2

Algebraic groups

In this chapter we define algebraic groups and present some basic examples (the classical algebraic groups will be introduced in Chapter 4).

2.1. Basic properties

An *affine algebraic group* G is an affine variety, defined over k, with a group structure such that the multiplication and inversion maps

$$\begin{array}{rcl} \mu\colon G\times G & \to & G \\ (g,h) & \mapsto & gh \end{array} \qquad \begin{array}{rcl} \iota\colon G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of varieties. We shall omit the word affine. We denote the identity element by 1. In some sense, algebraic groups via algebraic geometry are the analogues of *real Lie groups*, studied in differential geometry.

Example 2.1.1. The additive and the multiplicative group of the field k are affine algebraic groups. We denote these groups as $\mathbf{G}_a = (k,+)$ and $\mathbf{G}_m = (k^*,\cdot)$.

(1) Let us consider $\mathbf{G}_a$. It is clearly an affine variety with $I(\mathbf{G}_a) = (0)$ and so $k[\mathbf{G}_a] = k[x]$. In particular $\mathbf{G}_a$ is irreducible by Proposition 1.2.3 and has dimension 1. In addition, $\mu^*\colon k[x] \to k[x] \otimes_k k[y] \cong k[x,y]$ is defined by $x \mapsto x+y$ and $\iota^*\colon k[x] \to k[x]$, $x \mapsto -x$. Hence μ^* and ι^* are k-algebras homomorphisms. Thus $\mathbf{G}_a$ is an affine algebraic group.

(2) Consider $\mathbf{G}_m = k^*$. Notice that there is a one-to-one correspondence between $\mathbf{G}_m$ and $\{(x,y) \in k^2 : xy = 1\} \subseteq k^2$ via $x \mapsto (x, x^{-1})$. Therefore, $\mathbf{G}_m$ is an affine variety and $k[\mathbf{G}_m] = k[x,x^{-1}]$. By Proposition 1.2.3 and Example 1.3.2, $\mathbf{G}_m$ is irreducible with $\dim \mathbf{G}_m = 1$. We define the multiplication and inversion component-wise, i.e $(x,x^{-1})\cdot(y,y^{-1}) = (xy, x^{-1}y^{-1})$ and $(x,x^{-1})^{-1} = (x^{-1},x)$. So $\mu^*\colon k[x,x^{-1}] \to k[x,y,x^{-1},y^{-1}]$ is defined by $x \mapsto xy$ and $\iota^*\colon k[x,x^{-1}] \to k[x,x^{-1}]$, $x \mapsto x^{-1}$. In particular, μ^* and ι^* are k-algebras homomorpshims. Hence, $\mathbf{G}_m$ is an affine algebraic group.

We also give other examples of algebraic groups. Let V be an n-dimensional vector space over the field k. The group of bijective linear transformations of V is the *general linear group*, denoted by $\mathrm{GL}(V)$. Fixing a basis of V we gain an identification $V \cong k^n$. In this way, we identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(k) = \{A \in k^{n^2} : \det A \neq 0\}$. This isomorphism is not canonical; it does depend on the chosen basis. We shall also use the short notation GL_n.

We define $\mathrm{SL}_n = \{A \in \mathrm{GL}_n : \det(A) = 1\}$; clearly $\mathrm{SL}_n \leq \mathrm{GL}_n$. We call SL_n the *special linear group.* Then SL_n is an affine algebraic variety in $\mathbb{A}^{n^2}$, since $\mathrm{SL}_n = Z(\det(x_{i,j}) - 1)$. Furthermore it is clear that multiplication is a polynomial map and, by Cramer's rule, inversion is a map given by a quotient of polynomials. Therefore μ and ι are morphisms of varieties, according to the equivalent definition of morphisms given after Proposition 1.4.1. Therefore SL_n is an affine algebraic group.

It is not immediately clear that GL_n is an affine variety. As sets, we have

$$\mathrm{GL}_n \overset{1:1}{\longleftrightarrow} \{(A, a) \in k^{n^2} \times k : \det(A)a = 1\} \tag{2}$$

via $A \mapsto (A, \det(A)^{-1})$. Therefore GL_n is a closed subset in k^{n^2+1}. Hence it is a variety. We define multiplication and inversion component-wise in the set of couples $(A, \det(A)^{-1})$. Thus, in each component, μ and ι are polynomial maps. This, together with the observation given for SL_n, leads us to conclude that GL_n is an affine algebraic group. The coordinate ring of the general linear group is given by

$$k[\mathrm{GL}_n] = k[x_{i,j}, y : 1 \leq i, j \leq n]/(\det(x_{i,j})y - 1)$$

We have that $\det(x_{i,j})$ is an irreducible polynomial in $k[x_{i,j} : 1 \leq i, j \leq n]$, see [**7**, Section 61]. It is an easy calculation to deduce that $\det(x_{i,j})y - 1$ must be irreducible, too. Hence, by Proposition 1.2.3, GL_n is an irreducible variety. Then, [**29**, Proposition 1.13] yields

$$\dim \mathrm{GL}_n = n^2 \tag{3}$$

Let G be an affine algebraic group. Then, by Proposition 1.1.4, we may write $G = G_1 \cup \ldots \cup G_l$ where the G_i are irreducible affine subvarieties of G. With this notation we have the following result, see [**33**, Section 7.3].

Proposition 2.1.2. *Let G be an affine algebraic group and let $G = G_1 \cup \ldots \cup G_l$ be its decomposition into irreducible components. Then*

(i) there exists a unique irreducible component containing 1, we denote it by G°;
(ii) G° is a normal subgroup of finite index of G;
(iii) $G/G^\circ = \{G_i : 1 \leq i \leq l\}$;
(iv) let H be a closed subgroup of finite index in G. Then $G^\circ \leq H$.

Since the cosets of G° in G are pairwise disjoint, the irreducible components of G are also the connected components. Therefore, an algebraic group G is irreducible if, and only if, it is connected. Hence, G is irreducible if, and only if, $G = G^\circ$. If G is not irreducible, all the irreducible components of G are equidimensional. Thus, by (1), we have

$$\dim G = \dim G^\circ \tag{4}$$

We have several properties that relate the group structure with the topology. For example it is not difficult to see that the center $Z(G)$ of G is a closed subgroup. It is clear that we only consider closed subgroup in order to exploit algebraic geometry properties; however, if G is an algebraic group and $H \leq G$ then the algebraic closure $\overline{H}$ is again a subgroup of G, see [**33**, Proposition 7.4 A].

In an algebraic group G, a subgroup generated by closed and connected subgroups is itself closed and connected, [**33**, Proposition 7.5].

Proposition 2.1.3. *Let $\{X_i : i \in I\}$ be a collection of closed and connected subgroups of G. Then $\langle X_i : i \in I\rangle \leq G$ is closed and connected.*

This result is fundamental in the definition of a particular subgroup of the algebraic group G, see Definition 2.4.1.

2.2. Morphisms of algebraic groups

Let G and H be algebraic groups. A map $\varphi\colon G \to H$ is an *algebraic group morphism* if it is a homomorphism of groups and a morphism of varieties.

Let G and H be algebraic groups and $\varphi\colon G \to H$ be a morphism. In the following we relate group theoretic concepts, such as the kernel and the image of φ, with the affine variety structure, for a proof we refer to [**33**, Proposition 7.4 B].

Proposition 2.2.1. *Let $\varphi\colon G \to H$ be a morphism of algebraic groups. Then*

(i) $\ker\varphi \leq G$ *and* $\mathrm{Im}\varphi \leq H$ *are closed;*
(ii) $\varphi(G^\circ) = \varphi(G)^\circ$*;*
(iii) $\dim G = \dim\ker\varphi + \dim\mathrm{Im}\varphi$*.*

Example 2.2.2. It is clear that $\mathrm{GL}_1 = k^* = \mathbf{G}_m$. Let us consider the determinant map $\det\colon \mathrm{GL}_n \to \mathbf{G}_m$, being defined by a polynomial equation it is a morphism of varieties. Notice that $\ker(\det) = \mathrm{SL}_n$. In particular, using Proposition 2.2.1(iii) we deduce that $\dim \mathrm{SL}_n = n^2 - 1$.

The following is an easy consequence of Proposition 2.2.1(iii).

Corollary 2.2.3. *Let $1 \to G_1 \to G \to G_2 \to 1$ be a short exact sequence of algebraic groups. Then*

$$\dim G = \dim G_1 + \dim G_2$$

It is clear that any closed subgroup of GL_n is an algebraic group. The following result, known as the *Linearization theorem*, asserts that any algebraic group can be embedded in some GL_n as a closed subgroup, for a proof we refer the reader to [**33**, Theorem 8.6]

Theorem 2.2.4. *Let G be an affine algebraic group. Then there exists a closed embedding $G \hookrightarrow \mathrm{GL}_n$ for some n.*

This result will be fundamental for the definition of semisimple and unipotent elements in any algebraic group, see Section 2.3

We give further examples of algebraic groups and morphisms between them.

Example 2.2.5. Diagonal, upper triangular and upper uni-triangular matrices:

$$D_n = \{\mathrm{diag}(a_1, \dots, a_n) \in \mathrm{GL}_n : a_i \in k^* \text{ for } 1 \leq i \leq n\}$$
$$T_n = \{(a_{ij}) \in \mathrm{GL}_n : a_{ij} = 0 \text{ for } i > j\}$$
$$U_n = \{(a_{ij}) \in T_n : a_{ii} = 1 \text{ for } 1 \leq i \leq n\}$$

are algebraic groups, being defined by systems of polynomial equations. Note that we have a natural (split) exact sequence

$$1 \to U_n \to T_n \overset{\pi}{\to} D_n \to 1 \tag{5}$$

where π transforms $a_{i,j}$ to 0 whenever $i \neq j$ and leaves invariant the diagonal elements. It is clear that $\dim D_n = n$, in fact $D_n \cong (k^*)^n$. We may compute $\dim U_n = \frac{n}{2}(n-1)$. In particular, by Corollary 2.2.3 we deduce $\dim T_n = \frac{n}{2}(n+1)$.

Definition 2.2.6. Let T be an algebraic group. We say that T is a *torus* if $T \cong (k^*)^n$ for some $n \geq 1$. Furthermore, given an algebraic group G, we say that $T \leq G$ is a *maximal torus* of G if T is a torus and maximal with respect to inclusion amongst all the tori of G.

It turns out that all maximal tori in an algebraic group G are conjugate, [**45**, Corollary 6.5]. We define the *rank* of G to be the dimension of a maximal torus. This object is extremely important in the classification of (semi-) simple algebraic groups, see Section 2.4.

Let G be an algebraic group. We say that G is *solvable* if it is solvable as an abstract group. The following is a consequence of the Lie-Kolchin Theorem, [**33**, Theorem 17.6].

Proposition 2.2.7. *Let G be a closed connected solvable subgroup of GL_n. Then there exists $x \in \mathrm{GL}_n$ such that $x^{-1}Gx \leq T_n$.*

2.3. Jordan-Chevalley decomposition

In this section we first recall the Jordan decomposition for matrices in GL_n, then, via Theorem 2.2.4, we explain how such a decomposition holds in any algebraic group.

Let V be a finite dimensional k-vector space.

For any $f \in \mathrm{End}(V)$ there exist unique $s, n \in \mathrm{End}(V)$ such that $f = s + n$ and $sn = ns$, where s is diagonalizable and n is nilpotent, i.e. a power of n is the zero map, (see [**33**, Lemma 15.1A]). This property is known as *additive Jordan decomposition* of f.

We call $u \in \mathrm{GL}(V)$ *unipotent* if $1 - u \in \mathrm{End}(V)$ is nilpotent or, equivalently, if 1 is the only eigenvalue of u on V. If $p > 0$ an element $u \in \mathrm{GL}(V)$ is unipotent if, and only if, u has p-power order. We say that $s \in \mathrm{GL}(V)$ is *semisimple* if it is diagonalisable.

Remark 2.3.1. It is clear that if $x \in \mathrm{GL}(V)$ is unipotent (resp. semisimple) then x^g is unipotent (resp. semisimple) for all $g \in \mathrm{GL}(V)$.

The following is [**33**, Lemma 15.1B].

Proposition 2.3.2 (Jordan decomposition)**.** *Let $x \in \mathrm{GL}(V)$. Then there exist unique $s, u \in \mathrm{GL}(V)$ such that $x = su = us$, where s is semisimple and u is unipotent.*

Let G be an algebraic group. Thanks to Theorem 2.2.4, there exists an embedding $\rho\colon G \to \mathrm{GL}(V)$, for some V. For any $x \in G$, by Proposition 2.3.2, $\rho(x)$ has a Jordan decomposition $\rho(x) = su$. It is not clear a priori that s and u have a pre-image in G, and that the pre-images are independent of the chosen embedding. The following generalises the Jordan decomposition to any algebraic group, [**45**, Theorem 2.5].

Theorem 2.3.3. *Let G be an algebraic group.*

(a) For any embedding $\rho\colon G \to \mathrm{GL}(V)$ and for any $x \in G$, there exist unique $s, u \in G$ such that $x = su$ and $us = su$. Moreover, $\rho(s)$ is semisimple and $\rho(u)$ is unipotent.
(b) The decomposition $x = su$ is independent on the chosen embedding ρ.
(c) Let $\varphi\colon G \to H$ be a morphism of algebraic groups. Let $x = s_1u_1$ and $\varphi(x) = s_2u_2$, the respectively Jordan decompositions. Then $\varphi(s_1) = s_2$ and $\varphi(u_1) = u_2$.

An immediate consequence of Theorem 2.3.3 is the following definition of semisimple and unipotent elements in any algebraic group.

Let G be an algebraic group and ρ an isomorphism of G onto some closed subgroup of $\mathrm{GL}(V)$. We define an element $x \in G$ to be *semisimple* (resp. *unipotent*) if $\rho(x)$ is semisimple (resp. unipotent) in $\mathrm{GL}(V)$. We define

$$G_u = \{g \in G : g \text{ unipotent}\}$$
$$G_s = \{g \in G : g \text{ semisimple}\}$$

We say that an algebraic group G is a *unipotent group* if $G = G_u$. In the following result U_n is as defined in Example 2.2.5, see [**45**, Proposition 2.9].

Proposition 2.3.4. *Let $G \leq \mathrm{GL}_n$ be a unipotent group. Then there exists $x \in \mathrm{GL}_n$ such that $x^{-1}Gx \leq U_n$*

Let $G \leq T_n$ be a closed connected subgroup. The restriction of π, in (5), to G has kernel $G_u = G \cap U_n$. The image $T = \pi(G)$ is a closed subgroup of D_n, hence it is a torus, [**45**, Proposition 3.9]. So, we have the following short exact sequence

$$1 \to G_u \to G \to T \to 1$$

In the case G is solvable then [**45**, Theorem 4.4] shows that all the maximal tori are conjugate and, say T one of them, then $G = G_u \rtimes T$.

2.4. Simple algebraic groups

Let G be an algebraic group. Let $H_1, H_2 \leq G$ be closed connected (solvable) normal subgroups of G. Then $\langle H_1, H_2\rangle \leq G$ is closed and connected by Proposition 2.1.3, and by elementary group theory it is normal (and solvable).

Definition 2.4.1. Let G be an algebraic group. The unique maximal closed connected normal solvable subgroup of G is called the *radical* $R(G)$ of G.

We set $R_u(G) = (R(G))_u$, the set of unipotent elements of the radical. We call $R_u(G)$ the *unipotent radical* of G. The unipotent radical may be characterised as the maximal closed connected normal unipotent subgroup of G, [**33**, Section 19.5]. In the case $p > 0$, the unipotent radical is the algebraic analogue analogue of $\mathcal{O}_p(H)$, for a finite group H.

By Proposition 2.2.7, if $G \leq \mathrm{GL}_n$ is closed then, up to conjugacy, $R(G) \leq T_n$.

Definition 2.4.2. Let G be a connected algebraic group. We define G to be

(i) *reductive* if $R_u(G) = 1$;
(ii) *semisimple* if $R(G) = 1$;
(iii) *simple* if it contains no proper closed connected normal subgroups.

Notice that if a group G is simple then it is semisimple and hence reductive. None of these implications may be reversed. As we see in the following two examples. Recall that k is an algebraically closed field of characteristic p.

Example 2.4.3. Let $G = \mathrm{GL}_n$. Here we explicitly compute $R(G)$ and $R_u(G)$ and we deduce that G is reductive but not semisimple. Proposition 2.2.7 implies $R(G) \leq T_n$, recall that T_n is the subgroup of upper triangular matrices of GL_n. It is easy to check that T_n^-, the subgroup of lower triangular matrices, is conjugate to T_n. Therefore $R(G) \leq T_n \cap T_n^- = D_n$. By the definition of the radical we need to look for normal subgroups of D_n. By conjugating with matrices $I_n + E_{i,j}$ (here $E_{i,j}$ is the matrix with 1 in the position (i,j) and 0 elsewhere), we deduce that the subgroup of diagonal matrices $\{\lambda I_n : \lambda \in k^*\} \cong \mathbf{G}_m$ is the largest normal subgroup in D_n. Incidentally, observe that this is the centre of G. Therefore $R(G) = Z(G) \cong \mathbf{G}_m$. It is clear that the only unipotent element in $\mathbf{G}_m$ is the identity. Therefore $R_u(G) = 1$.

Example 2.4.4.

(i) Let $G = \mathrm{SL}_n$. Arguing in the same way of Example 2.4.3 we deduce that $R(G) \leq Z(G)$. It is clear that $Z(G)$ is always finite. Thus $R(G)$ is trivial. Therefore G is semisimple and, hence, reductive. In fact SL_n is simple, see [**45**, Example 9.8].
(ii) Similarly, we deduce that $\mathrm{SL}_n \times \mathrm{SL}_n$ is reductive and semisimple. However it is not simple since, for example, $1 \times \mathrm{SL}_n$ is a closed connected normal subgroup.

Remark 2.4.5. Note that an affine algebraic group may be simple in the algebraic sense without being simple in the abstract sense. For example, the special linear group SL_n is always simple as algebraic group, however it has non-trivial (finite) centre.

We have the following chain of normal subgroups

$$1 \leq R_u(G) \leq R(G) \leq G^\circ \leq G$$

In particular, G/G° is a finite group, by Proposition 2.1.2; $G^\circ/R(G)$ is semisimple and $G^\circ/R_u(G)$ is reductive, by definition.

The structure of reductive and semisimple connected groups is well known. For a proof of the following see [**45**, Proposition 6.20, Corollary 8.22] and [**33**, Theorem 27.5].

Theorem 2.4.6. *Let G be a connected algebraic group.*

(a) Assume G is reductive. Then $G = G'R(G)$. Furthermore, $G' = [G,G]$ is semisimple and $R(G) = Z(G)^\circ$ is a torus.
(b) Assume G is semisimple. Then there exist finitely many closed normal simple subgroups $G_1, \dots, G_n \leq G$ such that $G = G_1 \cdots G_n$ and
  (i) if $i \neq j$ then $[G_i, G_j] = 1$;
  (ii) $G_i \cap (G_1 \cdots G_{i-1} G_{i+1} \cdots G_n)$ is finite for all i.

Simple (and semisimple) algebraic groups are classified via combinatorial data. Now we give a brief overview of this, for a complete description we refer the reader to [**45**, Chapter 9] or [**33**, Chapter XI]. Let G be a connected algebraic group. We fix a maximal torus $T \leq G$ and we define the *character* and *cocharacter groups of* T to be $X(T) = \mathrm{Hom}_{\mathrm{alg}}(T, \mathbf{G}_m)$ and $Y(T) = \mathrm{Hom}_{\mathrm{alg}}(\mathbf{G}_m, T)$, respectively. Then, using the *Lie algebra* associated to G and a particular representation of G we define the *root system* $\Phi = \Phi(G)$ to be a certain subset of $X(T)$. It turns out that there is a link between $X(T)$ and $Y(T)$, which leads to the definition of the *dual root system* $\Phi^\vee \subseteq Y(T)$.

In general, one has the definition of *root datum* as a quadruple $\Psi = (X, \Phi, Y, \Phi^\vee)$ satisfying certain properties, see [**45**, Definition 9.10]. Also, we have the concept of homomorphisms and isomorphisms between root data, [**35**, Section II.1.13]. For G, T as above, $\Psi(T) = (X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum, see [**45**, Proposition 9.11].

The following fundamental result classifies semisimple algebraic groups in terms of their respective root data, (see, for example, [**45**, Theorem 9.13], [**54**, Theorems 9.6.2, 10.1.1] or [**33**, Section 32]). If Ψ is a root datum, we say that the algebraic group G realises Ψ if there exists a maximal torus $T \leq G$ such that $\Psi = \Psi(T)$.

Theorem 2.4.7 (Chevalley's Classification Theorem)**.** *Two semisimple algebraic groups are isomorphic if, and only if, they have isomorphic root data. In addition, for any root datum there exists a semisimple algebraic group that realises it.*

The classification of simple algebraic groups is given in terms of Dynkin diagrams, [**45**, Theorem 9.6, Exercise 10.33]:

$$\begin{aligned} \text{classical: } & A_n\ (n \geq 1),\ B_n\ (n \geq 2),\ C_n\ (n \geq 3),\ D_n\ (n \geq 4) \\ \text{exceptional: } & G_2,\ F_4,\ E_6,\ E_7,\ E_8 \end{aligned}$$

We shall define some of the classical groups in Chapter 4.

Remark 2.4.8. Chevalley's classification theorem is, in some way, the analogue for semisimple algebraic groups of the classification theorem of semisimple complex Lie algebras, see [**20**, Chapter 14]. Notice that there is a one-to-one correspondence between simple complex Lie algebras and Dynkin diagrams, whereas such correspondence fails to be one-to-one for algebraic groups. For example, both SL_n and $\mathrm{PGL_n} = \mathrm{GL}_n/Z(\mathrm{GL}_n)$ are simple with Dynkin diagram A_{n-1} and clearly they are not isomorphic.

The situation described in Remark 2.4.8 extends to all the other Dynkin diagrams (except for G_2, F_4, E_8). Without being formal – a detailed analysis would require further definitions and it is far from our purposes – we make some comments on this.

Let G be a simple algebraic group with root system Φ. Then, [**45**, Proposition 9.15] asserts that there exist algebraic groups G_{ad} and G_{sc} with root system Φ that are 'extreme' in the sense that there are *isogenies* (surjective morphisms with finite kernel):

$$G_{\mathrm{sc}} \to G \to G_{\mathrm{ad}}$$

The groups G_{sc} and G_{ad} are called *simply connected* and *adjoint type*, respectively. For example, SL_n is simply connected and PSL_n is the adjoint version of the special linear group.

CHAPTER 3

G-varieties

In group theory, the concept of an action of a group G on a set Ω is fundamental. In this chapter we define actions of algebraic groups on varieties. We define point stabilisers, orbits and fixed point sets, and we discuss algebraic properties of these sets.

3.1. Basic properties

Let G be an affine algebraic group and let Ω be a non-empty variety, both defined over k, so that $G \times \Omega$ is a variety. An *action* of G on Ω is a morphism of varieties $G \times \Omega \to \Omega$, $(g, \omega) \mapsto g.\omega$, which is also a group action (in the abstract sense), i.e. $1.\omega = \omega$ and $g.(h.\omega) = (gh).\omega$, for all $g, h \in G$ and $\omega \in \Omega$. If we have an action of G on Ω then we say that Ω is a G-variety.

Let Ω be a G-variety and $\omega \in \Omega$. We define the *stabiliser* of ω to be $G_\omega = \{g \in G : g.\omega = \omega\}$ which is a subgroup of G; the *orbit* of ω is $G.\omega = \{g.\omega : g \in G\} \subseteq \Omega$. The *fixed point space* of $x \in G$ is defined as follows:

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\} \subseteq \Omega \tag{6}$$

As already pointed out in the introduction, this set is of central interest in this thesis.

We say that the action is *transitive*, or *homogeneous*, if $G.\omega = \Omega$ for any $\omega \in \Omega$. Recall that in this situation, all the point stabilisers are conjugate. Moreover, say $H = G_\omega$ a point stabiliser, then the action of G on Ω is equivalent to the action of G on G/H (in the sense that there exists a one-to-one correspondence $G/H \cong \Omega$ preserved by G). A transitive action is called *primitive* if any point stabiliser is a maximal subgroup.

In fact we can relate these concepts with the variety structure, [**33**, Proposition 8.2]. First we have the following for stabilisers and fixed point spaces.

Proposition 3.1.1. *Let Ω be a G-variety. Then the following hold.*

(a) For every $\omega \in \Omega$, the stabiliser G_ω is a closed subgroup of G.
(b) For every $x \in G$, the fixed point space $C_\Omega(x)$ is closed in Ω.

Proof. Let $\omega \in \Omega$. Define $\varphi_\omega \colon G \to \Omega$, $g \mapsto g.\omega$. By definition of algebraic group action, φ_ω is a morphism of varieties. Then $\varphi^{-1}(\omega) = G_\omega$ is closed.

Let $x \in G$. Define $\psi_x \colon \Omega \to \Omega \times \Omega$, $\omega \mapsto (x.\omega, \omega)$. Then, ψ is a morphism of varieties since the two projections are morphisms. By Proposition 1.5.1, the diagonal $\Delta \subseteq \Omega \times \Omega$ is closed. Therefore $\psi_x^{-1}(\Delta) = C_\Omega(x) \subseteq \Omega$ is closed. *q.e.d.*

Recall that a set is called *locally closed* if it is the intersection of an open and a closed set. Notice that if $X = A \cap Y$ where A is open and Y is closed, then $\overline{X} \subseteq Y$. In general, orbits are not closed, but they are locally closed, see [**33**, Proposition 8.3].

Proposition 3.1.2. *Let Ω be a G-variety and $\omega \in \Omega$. Then the orbit $G.\omega \subseteq \Omega$ is locally closed. Moreover the boundary of $G.\omega$ is a union of orbits of strictly lower dimension.*

Thanks to Proposition 3.1.2 we may define the dimension of an orbit as

$$\dim G.\omega = \dim \overline{G.\omega} \tag{7}$$

The following is the algebraic group analogue of the orbit-stabiliser theorem for finite group actions, see [**25**, Proposition 2.5.3].

Proposition 3.1.3. *Let Ω be a G-variety and $\omega \in \Omega$. Then*

$$\dim G = \dim G.\omega + \dim G_\omega$$

We give some examples of G-varieties.

Example 3.1.4. Let G be an affine algebraic group.

(1) Let us consider the transitive action of G on itself by left translation. So the action map is $\varphi\colon G \times G \to G$, $(x, g) \mapsto xg$. Note that any point stabiliser is trivial. Moreover $C_G(x) \neq \emptyset$ if, and only if, $x = 1$.
(2) Consider the action of G on itself by conjugation, i.e. $(x, g) \mapsto x^{-1}gx$. The fixed point space is given by the centraliser of the element: for $x \in G$ we have $C_G(x) = \{g \in G : x^{-1}gx = g\}$. Hence, by Proposition 3.1.1, the centraliser of any element of the group is a closed subgroup in G. In addition, orbits are given by conjugacy classes: for $x \in G$, $G.x = x^G$.

Let G be an algebraic group and $x \in G$. In Example 3.1.4(2) we have seen that conjugacy classes are orbits under the conjugation action. Thus, thanks to Proposition 3.1.2 and (7) we can define the dimension of a conjugacy class:

$$\dim x^G = \dim \overline{x^G} \tag{8}$$

In addition, by Proposition 3.1.3,

$$\dim G = \dim x^G + \dim C_G(x) \tag{9}$$

This formula will be very useful in Chapter 5, where we aim to describe conjugacy classes of classical groups and provide a formula for their dimensions.

3.2. Coset varieties

The main result of this section is Proposition 3.2.1 below, which asserts that the coset space is a variety. The key tool in order to show this fact is a theorem by Chevalley (see [**33**, Theorem 11.2] or [**45**, Theorem 5.5]) on the existence of a certain rational representation (a morphism of algebraic group $G \to \mathrm{GL}(V)$). Note that as special case this theorem implies Theorem 2.2.4.

For a proof of the following we refer to [**45**, Section 5.2] or [**54**, Section 5.5].

Proposition 3.2.1. *Let G be an algebraic group and $H \leq G$ be a closed subgroup. Then G/H is a quasi-projective variety. Moreover*

$$\dim(G/H) = \dim G - \dim H \tag{10}$$

This variety structure on G/H is independent of the chosen rational representation, see [**33**, Section 12.4]. With this structure G/H is called a *coset variety.*

If H is a closed normal subgroup of the affine algebraic group G then G/H is an abstract group. In fact, for closed normal subgroups the quotient space is an affine algebraic group, see [**8**, Theorem 6.8].

We remark that Richardson [**50**] proved that if G is a reductive affine algebraic group and H is a closed subgroup then G/H is an affine variety if, and only if, H is reductive.

CHAPTER 4

Classical groups

In this chapter we introduce the main object of this thesis: the classical algebraic groups. We start by giving some background on forms defined on vector spaces; here we follow [**36**, Chapter 2], and sometimes we refer to [**2**, Chapter 7]. Then, given a finite-dimensional k-vector space V, we define the classical groups $\mathrm{GL}(V), \mathrm{Sp}(V)$ and $\mathrm{O}(V)$ to be the isometry groups of certain forms defined on V.

Throughout this chapter V is an n-dimensional k-vector space.

4.1. Geometry and forms

A *bilinear form* on V is a map $f\colon V \times V \to k$ such that

$$f(\alpha_1 u_1 + \alpha_2 u_2, v_1) = \alpha_1 f(u_1, v_1) + \alpha_2 f(u_2, v_1)$$
$$f(u_1, \beta_1 v_1 + \beta_2 v_2) = \beta_1 f(u_1, v_1) + \beta_2 f(u_1, v_2)$$

for all $\alpha_i, \beta_i \in k$ and all $u_i, v_i \in V$, $i = 1, 2$. Let f be a bilinear form on V. Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis for V and define the matrix $A_{\mathcal{B}} = (a_{ij})$, where $a_{ij} = f(e_i, e_j)$. Then $A_{\mathcal{B}}$ is called the *Gram matrix* of f relative to $\mathcal{B}$. In the case $V = k^n$, and $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in k^n$ then

$$f(x, y) = x A_{\mathcal{B}} y^t \tag{11}$$

Let $\mathcal{B}' = \{w_1, \dots, w_n\}$ be another basis for V and write $w_i = \sum_j d_{ij} e_j$. Then $f(w_i, w_j)$ is the (i, j)-entry of the matrix $D^t A_{\mathcal{B}} D$, where $D = (d_{ij})$, so we have $A_{\mathcal{B}'} = D^t A_{\mathcal{B}} D$.

For a map $Q\colon V \to k$ we define $f_Q\colon V \times V \to k$ by

$$f_Q(u, v) = Q(u + v) - Q(u) - Q(v),$$

A *quadratic form* is a map $Q\colon V \to k$ such that f_Q is bilinear and

$$Q(\alpha u) = \alpha^2 Q(u)$$

for all $\alpha \in k$, $v \in V$. In this situation, we call f_Q the *bilinear form associated to Q*.

Example 4.1.1. Let $\{e_1, \dots, e_n\}$ be a basis for V.

(a) Define $Q_1\colon V \to k$ by $Q_1(x) = Q_1(\sum x_i e_i) = \sum_i x_i^2 + \sum_{i<j} x_i x_j$. Then

$$Q_1(x + y) = Q_1\Big(\sum (x_i + y_i) e_i\Big) = Q_1(x) + Q_1(y) + 2\sum_i x_i y_i + \sum_{i<j} (x_i y_j + x_j y_i)$$

It is clear that $Q_1(\lambda x) = \lambda^2 Q_1(x)$ for all $x \in V$ and all $\lambda \in k$. In addition, as defined above, $f_{Q_1}(x, y) = 2\sum_i x_i y_i + \sum_{i<j} (x_i y_j + x_j y_i)$ is clearly a bilinear map. Thus Q_1 is a quadratic form.

(b) Define $Q_2\colon V \to k$ by $Q_2(x) = Q_2(\sum x_i e_i) = \sum_{i<j} x_i x_j$. Then

$$Q_2(x+y) = Q_2\Big(\sum (x_i + y_i)e_i\Big) = Q_2(x) + Q_2(y) + \sum_{i<j}(x_i y_j + x_j y_i)$$

It is clear that $Q_2(\lambda x) = \lambda^2 Q_2(x)$ for all $x \in V$ and all $\lambda \in k$. In addition, as defined above, $f_{Q_2}(x,y) = \sum_{i<j}(x_i y_j + x_j y_i)$ is clearly a bilinear map. Thus Q_2 is a quadratic form.

For future reference, we also give the following.

Example 4.1.2. Let n be a positive even integer. Let $\{e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\}$ be a basis for V. Define $Q\colon V \to k$ as

$$Q\Big(\sum_i x_i e_i + y_i f_i\Big) = \sum_{i=1}^{n/2} x_i y_i$$

Clearly, for all $x \in V$ and $\lambda \in k$ we have $Q(\lambda x) = \lambda^2 Q(x)$. Let $x = \sum_i x_i e_i + y_i f_i$ and $y = \sum_i x_i' e_i + y_i' f_i$ be elements of V, then $Q(x+y) = Q(x) + Q(y) + \sum_i (x_i' x_i + x_i y_i')$. It is clear that $f_Q(x,y) = \sum_i (x_i' y_i + x_i y_i')$ is a bilinear form. Thus Q is a quadratic form.

Let f be a bilinear form on V. We say that f is degenerate if there exists $u \in V \setminus \{0\}$ such that $f(u,v) = 0$ for all $v \in V$, otherwise f is *non-degenerate*. Following [**36**], we say that the quadratic form Q on V is *non-degenerate* if the associated bilinear form f_Q is non-degenerate. A bilinear form f on V is *symmetric* if

$$f(u,v) = f(v,u)$$

for all u, v in V, and it is *alternating* if for all $v \in V$ we have

$$f(v,v) = 0$$

Note that f is alternating if, and only if, $f(u,v) = -f(v,u)$ for all $u, v \in V$. Moreover, if $p = 2$ then alternating and symmetric forms coincide.

Remark 4.1.3. The form f on V is non-degenerate if, and only if, the Gram matrix associated to f (in any basis) in non-singular, see [**26**, Proposition 2.1]. Similarly, f is symmetric (resp. alternating) if, and only if, the associated Gram matrix is symmetric (resp. skew-symmetric).

By definition, if Q is a quadratic form on V then f_Q is symmetric. Conversely, if $p \neq 2$ and f is a symmetric bilinear form on V then $Q(v) = \frac{1}{2} f(v,v)$ is a quadratic form with $f_Q = f$, so a quadratic form uniquely determines f_Q, and vice-versa. However, if $p = 2$ and f is a symmetric bilinear form there are many quadratic forms Q for which $f_Q = f$.

Example 4.1.4. If $p = 2$ then the two quadratic forms defined in Example 4.1.1 have associated bilinear form $f(x,y) = \sum_{i<j}(x_i y_j + x_j y_i)$.

We say that a bilinear form f is *symplectic* if it is non-degenerate and alternating. We say that (V, f) is a *symplectic space* if f is a symplectic form on V. Similarly, an *orthogonal space* is a pair (V, Q), where Q is a non-degenerate quadratic form on V.

Definition 4.1.5. Let V and V' be k-vector spaces equipped with bilinear forms f and f', respectively. We say that a map $\varphi\colon (V,f) \to (V',f')$ is an *isometry* if it is a bijective linear transformation $V \to V'$ such that $f(u,v) = f'(\varphi(u),\varphi(v))$ for all $u,v \in V$. Similarly, we define an isometry between (V,Q) and (V',Q') where Q,Q' are quadratic forms. If there exists an isometry between two spaces then we say that they are *isometric*, and in the case $V = V'$ the two forms are said to be *equivalent*.

Let $v \in V$ and let f be a bilinear form on V. We say that v is *singular* or *isotropic* if $f(v,v) = 0$. In the case V is equipped with a quadratic form Q we say that $v \in V$ is isotropic if $f_Q(v,v) = 0$ and singular if $Q(v) = 0$.

Following [**36**], we shall use κ to denote either a bilinear form or a quadratic form on V, hence $\kappa\colon V^\ell \to k$ where $\ell = 1,2$. We write $\kappa(\mathbf{v})$ for the image of $\mathbf{v} \in V^\ell$.

Let W be a subspace of V. We shall write (W,κ_W), or (W,κ) if there is no confusion, for the space obtained by restricting the form κ to W^ℓ, again $\ell = 1,2$ depending if κ is a quadratic or a bilinear form, respectively. We say that the subspace W is *non-degenerate* if the form κ_W is non-degenerate, and that W is *totally singular* if κ_W is the zero form (denoted $\kappa_W \equiv 0$). We define W to be *totally isotropic* if the restriction to $W \times W$ of the bilinear form f associated to $\kappa = Q$ is the zero form. In particular, in all the cases other than $(\kappa,p) = (Q,2)$ the terms totally singular and totally isotropic coincide. If $\kappa = Q$ is a quadratic form, $p = 2$ and W is a totally singular subspace then W is totally isotropic, but the converse does not hold in general. Consider, for example, $(V,Q) = (k^2,Q_2)$ where Q_2 is defined in Example 4.1.1, and $W = \langle(1,1)\rangle \le k^2$. Then W is totally isotropic but not totally singular.

The following is a well known result, see [**2**, Section 20] for a proof.

Lemma 4.1.6 (Witt's Lemma)**.** *Let (V_1,κ_1) and (V_2,κ_2) be symplectic or orthogonal isometric spaces and $W_i \le V_i$ for $i = 1$ and 2. Assume $\varphi\colon W_1 \to W_2$ is an isometry. Then φ extends to an isometry $\tilde{\varphi}\colon V_1 \to V_2$.*

An easy application of Witt's Lemma shows that maximal totally singular subspaces of a symplectic or orthogonal space are equidimensional.

Corollary 4.1.7. *Let (V,κ) be a symplectic or an orthogonal space. Let $W,W' \le V$ be maximal totally singular subspaces. Then $\dim W = \dim W'$.*

Proof. For convenience, we assume (V,f) is a symplectic space. Assume $\dim W \le \dim W'$. Then there exists $U \le W'$ such that $(U,f) \cong (W,f)$. Let $\varphi\colon U \to W$ be an isometry. By Witt's Lemma there exists an isometry $\psi\colon V \to V$ such that $\psi|_U = \varphi$. Now, for all $u,v \in W'$, since ψ is an isometry, we have

$$0 = f(u,v) = f(\psi(u),\psi(v))$$

So $\psi(W')$ is totally singular. In addition, $W \le \psi(W') \le V$, and this forces $W = \psi(W')$, since W is maximal totally singular. Thus $\dim W = \dim W'$. *q.e.d.*

Let (V,κ) be either a symplectic or an orthogonal space. Now we introduce the standard bases for V with respect to the form κ.

Proposition 4.1.8. *Let (V, f) be a symplectic space. Then n is even and there exists a basis $\mathcal{B} = \{e_1, \ldots, e_{n/2}, f_1, \ldots, f_{n/2}\}$ for V such that for all i, j*

$$f(e_i, e_j) = f(f_i, f_j) = 0, \; f(e_i, f_j) = \delta_{i,j}$$

where $\delta_{i,j}$ is the Kronecker delta.

Such a basis is called a *standard*, or *symplectic basis*. We refer the reader to [**36**, Proposition 2.4.1] for a proof.

Recall that two forms f, f' on V are equivalent if there exists an isometry $(V, f) \to (V, f')$. The following is a trivial consequence of Proposition 4.1.8.

Corollary 4.1.9. *Let (V, f) be a symplectic space. Then all symplectic forms on V are equivalent to f.*

In terms of the standard basis $\mathcal{B}$, the Gram matrix of a symplectic form is

$$A_{\mathcal{B}} = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \tag{12}$$

Remark 4.1.10. Let us consider the matrix $A'_n = \begin{pmatrix} 0 & K_{n/2} \\ -K_{n/2} & 0 \end{pmatrix}$, where $K_{n/2} = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ is the matrix with 1's along the minor diagonal and 0's elsewhere. Then A'_n is skew-symmetric, hence it represents a symplectic form. Therefore, by Corollary 4.1.9, $A_{\mathcal{B}}$ and A'_n are equivalent. Thus there exists $D \in \mathrm{GL}_n$ such that $D^t A_{\mathcal{B}} D = A'_n$.

Now, let (V, Q) be an orthogonal space, and let f_Q be the associated bilinear form. As already remarked, if $p = 2$ there is no difference between symplectic and non-degenerate symmetric bilinear forms. Hence, as an application of Proposition 4.1.8, we state the following.

Proposition 4.1.11. *Let (V, Q) be an orthogonal space and $p = 2$. Then $\dim V$ is even.*

In the orthogonal case we can also define standard bases. The following is a particular case of [**36**, Proposition 2.5.3].

Proposition 4.1.12. *Let (V, Q) be an orthogonal space.*

(i) If n is even there exists a basis $\mathcal{B} = \{e_1, \ldots, e_{n/2}, f_1, \ldots, f_{n/2}\}$ for V such that for all i, j

$$Q(e_i) = Q(f_i) = 0, \; f_Q(e_i, f_j) = \delta_{i,j}$$

(ii) If n is odd there exists a basis $\mathcal{B} = \{e_1, \ldots, e_{\lfloor n/2 \rfloor}, f_1, \ldots, f_{\lfloor n/2 \rfloor}, z\}$ for V such that for all i, j

$$Q(e_i) = Q(f_i) = 0$$

$$f_Q(e_i, f_j) = \delta_{i,j}, \; f_Q(e_i, z) = f_Q(f_i, z) = 0, \; f_Q(z, z) = 1$$

The bases for V as in (i) or (ii) of Proposition 4.1.12 will be called a *standard* or *orthogonal basis*.

Notice that all the previous definitions remain valid if k is a finite field. In the case k is finite the standard bases for a symplectic or an orthogonal space are as in Propositions

4.1.8 and 4.1.12 with a further case in the case (V, Q) is an even dimensional orthogonal space, see [**36**, Proposition 2.5.3(ii)].

The following is a trivial consequence of Proposition 4.1.12.

Corollary 4.1.13. *Let (V, Q) be an orthogonal space. Then all non-degenerate quadratic forms on V are equivalent to Q, and all non-degenerate bilinear forms are equivalent to f_Q.*

Notice that if k is a finite field then Corollary 4.1.13 does not hold. Indeed in this case there are two isometry classes of quadratic forms, see [**36**, Propositon 2.5.4].

4.2. Classical groups

The concept of *isometry* has been given in the previous section. In particular, an isometry $\varphi\colon V \to V$ is an automorphism of the vector space, so $\varphi \in \mathrm{GL}(V)$. It is clear that the set of isometries of V with respect to κ is a subgroup of $\mathrm{GL}(V)$. We call this subgroup the *isometry group* of V, denoted $\mathrm{I}(V, \kappa)$.

The following, stated in [**36**, Lemma 2.1.8], is an easy calculation.

Lemma 4.2.1. *Let f be a non-degenerate bilinear form and Q be a non-degenerate quadratic form on V. Let f_Q be the bilinear form associated to Q. Let $\mathcal{B}$ be a basis for V and identify $\mathrm{GL}(V) \cong \mathrm{GL}_n$. Let $A = A_{\mathcal{B}} \in \mathrm{GL}_n$ be the Gram matrix of f or f_Q. Then*

$$\mathrm{I}(V, f) = \{x \in \mathrm{GL}_n : x^t A x = A\}$$

and

$$\mathrm{I}(V, Q) = \{x \in \mathrm{I}(V, f_Q) : Q(xv) = Q(v), \text{ for all } v \in \mathcal{B}\}$$

Let us make some comments in the case $(V, \kappa) = (V, Q)$ in Lemma 4.2.1.

If $p \neq 2$ then, as already observed in Section 4.1, Q uniquely determines f_Q and vice-versa. Hence, $\mathrm{I}(V, Q) = \mathrm{I}(V, f_Q)$.

If $p = 2$ then $\mathrm{I}(V, Q) \leq \mathrm{I}(V, f_Q)$. Recall we denote $\dim V = n$. Then $\mathrm{I}(V, Q)$ is defined by n further relations. We can deduce explicit matrix relations that define $\mathrm{I}(V, Q)$. For example, in [**25**, Section 1.3.16] a particular non-degenerate symmetric bilinear form f with Gram matrix A in the basis $\mathcal{B}$ is fixed together with a matrix B such that $A = B + B^t$. Then it is deduced

$$\mathrm{I}(V, Q) = \{x \in \mathrm{I}(V, f_Q) : (x^t B x)_{i,i} = B_{i,i}, \text{ for } 1 \leq i \leq n\}$$

where subscript 'i, j' denotes the matrix entry in the position (i, j).

Remark 4.2.2. Let $x \in \mathrm{I}(V, f)$ then, taking the determinant of $x^t A x = A$ and using $\det(A) \neq 0$, we deduce $\det(x)^2 = 1$. Thus any element in $\mathrm{I}(V, \kappa)$ has determinant ± 1.

The classical groups are defined to be the isometry groups of particular forms. Thanks to Lemma 4.2.1 we see that $\mathrm{I}(V, \kappa)$ is defined by polynomials equations (recall that multiplication of matrices is given by polynomial maps). Hence we deduce that $\mathrm{I}(V, \kappa)$ is a closed subgroup of $\mathrm{GL}(V)$. Therefore $\mathrm{I}(V, \kappa)$ is an affine algebraic variety. Multiplication and inversion are morphisms of varieties (by the same argument of Section 2.1 for GL_n and SL_n). Therefore $\mathrm{I}(V, \kappa)$ is an affine algebraic group.

Thanks to the following we can choose a suitable form, up to equivalence.

Lemma 4.2.3. *Assume* κ *and* κ' *are equivalent forms on* V. *Then* $\mathrm{I}(V,\kappa), \mathrm{I}(V,\kappa')$ *are conjugate in* $\mathrm{GL}(V)$. *In particular,* $\mathrm{I}(V,\kappa) \cong \mathrm{I}(V,\kappa')$.

Proof. Let A, B be the Gram matrices associated to κ and κ' (with respect to a fixed basis of V). Assume $p \neq 2$ if κ is quadratic. Then there exists an invertible matrix D such that $B = D^t A D$. It is an easy computation to show that $x^t A x = A$ if, and only if, $(D^{-1}xD)^t B (D^{-1}xD) = B$. Hence, thanks to the description of the isometry group of Lemma 4.2.1, the map

$$\varphi\colon \mathrm{I}(V,\kappa) \to \mathrm{I}(V,\kappa'), \qquad x \mapsto D^{-1}xD$$

is an isomorphism. If κ, κ' are quadratic forms and $p = 2$ see [**25**, Section 1.3.16]. *q.e.d.*

In Table 4.2.1 we define the isometry groups we will work with in this thesis. See [**25**, Corollary 1.5.14] for $\dim G$ when $G = \mathrm{Sp}_n$ or O_n. In Sections 4.2.1–4.2.4 we give a more detailed description of these groups.

κ	$G = \mathrm{I}(V,\kappa)$	$\dim G$
zero-form	$\mathrm{GL}(V), \mathrm{GL}_n$	n^2
non-degenerate symplectic form	$\mathrm{Sp}(V), \mathrm{Sp}_n$	$\frac{n}{2}(n+1)$
non-degenerate quadratic form	$\mathrm{O}(V), \mathrm{O}_n$	$\frac{n}{2}(n-1)$

Table 4.2.1. The classical groups

We did not introduced the concept of Lie algebra associated to an algebraic group. However, we observe that the main strategy to compute the dimensions of the symplectic and orthogonal group is to show that their dimension is equal to the dimension of the associated Lie algebra, see [**25**, Proposition 1.5.2]. Then it is easy to compute such dimensions.

Remark 4.2.4. Assume we have an identification $V \cong k^n$. Then we write

$$\begin{aligned} Cl(V) &= \mathrm{GL}(V), \mathrm{Sp}(V) \text{ or } \mathrm{O}(V) \\ Cl_n &= \mathrm{GL}_n, \mathrm{Sp}_n \text{ or } \mathrm{O}_n \end{aligned}$$

4.2.1. General linear group. Here we define κ to be the zero form on V. Therefore $\mathrm{I}(V,\kappa) = \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of bijective linear transformations of V. As already observed, by fixing a basis of V we may identify $V \cong k^n$ and hence $\mathrm{GL}(V) \cong \mathrm{GL}_n$. As observed in (2), GL_n is an algebraic group. We have also observed that it is irreducible, hence, by Proposition 2.1.2, it is connected.

As we showed in Example 2.4.3, GL_n is reductive and its centre is

$$Z(\mathrm{GL}_n) = \{\lambda I_n : \lambda \in k^*\} \cong \mathbf{G}_m$$

and D_n, the subgroup of diagonal matrices, is a maximal torus. So $\operatorname{rank} \mathrm{GL}_n = n$.

We define the *projective general linear group* to be $\mathrm{PGL}_n = \mathrm{GL}_n / Z(\mathrm{GL}_n)$. It turns out that $G = \mathrm{PGL}_n$ is simple, see for example [**45**, Chapter 9].

4.2.2. Special linear group. Let $\det\colon \mathrm{GL}_n \to \mathbf{G}_m$ be the determinant map. Then det is a morphism of algebraic groups and the kernel is a closed normal subgroup of GL_n. We define the *special linear group* to be this kernel, i.e. $\mathrm{SL}_n = \{x \in \mathrm{GL}_n : \det(x) = 1\}$. It follows that SL_n is the derived group $(\mathrm{GL}_n)'$, see [**25**, Example 3.1.15]. As remarked in Example 2.2.2.

$$\dim \mathrm{SL}_n = n^2 - 1$$

A maximal torus of SL_n is given by $D_n \cap \mathrm{SL}_n$, hence $\operatorname{rank} \mathrm{SL}_n = n-1$. Moreover, as observed in Remark 2.4.8, SL_n is simple and it corresponds to a Dynkin diagram of type A_{n-1}.

Also here we define the *projective special linear group* to be $\mathrm{PSL}_n = \mathrm{SL}_n/Z(\mathrm{SL}_n)$. Notice that PSL_n is simple (in fact $\mathrm{PGL}_n = \mathrm{PSL}_n$) and it also corresponds to the Dynkin diagram A_{n-1}.

4.2.3. Symplectic group. Let (V, f) be a symplectic space (recall that n is even). We shall write $f(u, v) = (u, v)$. For this section, we denote $A \in \mathrm{GL}_n$ the Gram matrix associated to f. We define the *symplectic group* to be $\mathrm{I}(V, f)$ and we denote it by $\mathrm{Sp}(V)$. Thanks to Lemma 4.2.1 we have

$$\mathrm{Sp}_n = \{x \in \mathrm{GL}_n : x^t A x = A\}$$

where A is defined in (12). In this case it is not easy to give a description of the coordinate ring of Sp_n as an algebraic variety. By Remark 4.2.2 we deduce $\mathrm{Sp}_n \leq \mathrm{SL}_n$.

Remark 4.2.5. Assume V is 2-dimensional. So $A \in \mathrm{GL}_2$. By definition,

$$\mathrm{Sp}_2 = \{x \in \mathrm{GL}_2 : x^t A x = A\}$$

Let us denote $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and recall $A = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$. It is straightforward to see that $x \in \mathrm{Sp}_2$ if, and only if, $\det(x) = 1$. Therefore $\mathrm{Sp}_2 = \mathrm{SL}_2$.

Let $x \in \mathrm{Sp}_n \cap D_n$, where $D_n \leq \mathrm{GL}_n$ is the subgroup of diagonal matrices (in the standard basis $\{e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\}$). We write $x = [\lambda_1, \dots, \lambda_n]$ for a diagonal matrix with entries λ_i down the diagonal. Then $x.e_i = \lambda_i e_i$ and $x.f_i = \lambda_{n/2+i} f_i$. Since $x \in \mathrm{Sp}_n$ we have

$$1 = (e_i, f_i) = (x.e_i, x.f_i) = (\lambda_i e_i, \lambda_{n/2+i} f_i) = \lambda_i \lambda_{n/2+i}$$

Therefore

$$\mathrm{Sp}_n \cap D_n = \left\{[\lambda_1, \dots, \lambda_{n/2}, \lambda_1^{-1}, \dots, \lambda_{n/2}^{-1}] : \lambda_i \in k^*\right\} \cong (\mathbf{G}_m)^{n/2}$$

It is possible to show that $\mathrm{Sp}_n \cap D_n$ is a maximal torus in Sp_n, see for example [**45**, Example 6.7]. Hence

$$\operatorname{rank} \mathrm{Sp}_n = \frac{n}{2}$$

The symplectic group is connected, [**25**, Theorem 1.7.4(c)]. Furthermore Sp_n is simple and it corresponds to the Dynkin diagram of type $C_{n/2}$, see [**45**, Table 9.2].

4.2.4. Orthogonal group. Let (V, Q) be an orthogonal space. Denote by f_Q the corresponding symmetric bilinear form and write $f_Q(u,v) = (u,v)$. The *orthogonal group* is defined to be $\mathrm{I}(V,Q)$ and it is denoted by $\mathrm{O}(V)$.

We distinguish two cases, depending on whether or not $p = 2$. Notice that if $p = 2$ then n is even, by Proposition 4.1.11. Moreover, as already observed, symplectic and non-degenerate symmetric forms coincide, when $p = 2$.

Case 1. Let us assume $p \neq 2$. As explained after Lemma 4.2.1, $\mathrm{I}(V,Q) = \mathrm{I}(V, f_Q)$. In view of Lemma 4.2.3, it is enough to consider any non-degenerate bilinear form. Let $K_n \in \mathrm{GL}_n$ as in Remark 4.1.10. Then the orthogonal group is

$$\mathrm{O}_n = \{x \in \mathrm{GL}_n : x^t K_n x = K_n\}$$

In the case $n = 2$, a straightforward calculation leads to

$$\mathrm{O}_2 = \left\{ \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} & \lambda \\ \lambda^{-1} & \end{pmatrix} : \lambda \in k^* \right\} \tag{13}$$

Thus O_2 is an extension of GL_1 by an element of order 2 whose conjugation action on $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ gives $\begin{pmatrix} \lambda^{-1} & \\ & \lambda \end{pmatrix}$. And we write $\mathrm{O}_2 \cong \mathrm{GL}_1.2$.

Observe that $-I_n \in \mathrm{O}_n$, and for n odd $\det(-I_n) = -1$. In the case $n > 2$ is even we have that

$$\tau = \begin{pmatrix} & & 1 \\ & I_{n-2} & \\ 1 & & \end{pmatrix} \in \mathrm{O}_n$$

and $\det(\tau) = -1$. Thus, the determinant map $\det\colon \mathrm{O}_n \to \{\pm 1\}$ is surjective and the kernel is a normal closed subgroup of index 2 in O_n, which is called the *special orthogonal group*. It may also be defined as $\mathrm{SO}_n = \mathrm{O}_n \cap \mathrm{SL}_n$.

Therefore $\mathrm{O}_n = \mathrm{SO}_n \cup \mathrm{SO}_n\tau$ and $\mathrm{SO}_n \cap \mathrm{SO}_n\tau = \emptyset$, so the orthogonal group is not connected.

Standard bases for the vector space V are described in Proposition 4.1.12, they are: $\{e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\}$, for n even, and $\{e_1, \dots, e_{\lfloor n/2 \rfloor}, f_1, \dots, f_{\lfloor n/2 \rfloor}, z\}$ for n odd. Notice that $(z,z) = 1$ therefore if $g \in \mathrm{O}_n \cap D_n$ is such that $g.z = \lambda z$ then $\lambda = \pm 1$.

As for the symplectic group we have, for n odd

$$\mathrm{O}_n \cap D_n = \left\{ [\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor}, \lambda_1^{-1}, \dots, \lambda_{\lfloor n/2 \rfloor}^{-1}, \pm 1] : \lambda_i \in k^* \right\}$$

It is clear that $(\mathrm{O}_n \cap D_n)^\circ = \left\{ [\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor}, \lambda_1^{-1}, \dots, \lambda_{\lfloor n/2 \rfloor}^{-1}, 1] : \lambda_i \in k^* \right\} \cong (\mathbf{G}_m)^{\lfloor n/2 \rfloor}$.

If n is even then

$$\mathrm{O}_n \cap D_n = \left\{ [\lambda_1, \dots, \lambda_{n/2}, \lambda_1^{-1}, \dots, \lambda_{n/2}^{-1}] : \lambda_i \in k^* \right\} \cong (\mathbf{G}_m)^{n/2}$$

It is possible to show that $\mathrm{O}_n \cap D_n$ is a maximal torus in O_n, see for example [**45**, Example 6.7]. Hence

$$\operatorname{rank} \mathrm{O}_n = \left\lfloor \frac{n}{2} \right\rfloor$$

Case 2. When $p = 2$, symplectic and non-degenerate symmetric bilinear forms are one and the same. Also, $\dim V = n$ is even. Therefore $\mathrm{I}(V,Q) < \mathrm{I}(V, f_Q) = \mathrm{Sp}_n$.

Define $Q\colon V \to k$ as

$$Q(x_1, \dots, x_n) = \sum_{i=1}^{n/2} x_i x_{n/2+i}$$

Let us consider the standard basis $\mathcal{B} = \{e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\}$. Then for all $u \in V$ we can write $u = \sum_{i=1}^{n/2} (\lambda_i e_i + \mu_i f_i)$, for some $\lambda_i, \mu_i \in k$. Therefore

$$Q\Big(\sum_{i=1}^{n/2} (\lambda_i e_i + \mu_i f_i)\Big) = \sum_{i=1}^{n/2} \lambda_i \mu_i$$

Example 4.1.2 asserts that Q is a quadratic form. Therefore,

$$\mathrm{O}_n = \{x \in \mathrm{Sp}_n : Q(x.v) = Q(v), \text{ for all } v \in \mathcal{B}\}$$

4.2.5. Special orthogonal group. In any characteristic we have the following, see [**2**, 22.9].

Theorem 4.2.6. *The orthogonal group* O_n *has a unique normal subgroup of index* 2*, which is also its connected component.*

Therefore we define the *special orthogonal group* to be this subgroup, i.e.

$$\mathrm{SO}_n = \mathrm{O}_n^\circ$$

If the characteristic is $p \neq 2$ we have already given a description of SO_n as the kernel of the determinant map $\det\colon \mathrm{O}_n \to \{\pm 1\}$.

In the case $p = 2$ (here n is even) we give another description of the special orthogonal group, which also applies when $p \neq 2$. The following has been given in [**36**, p.42].

Assume n is even. Let $\mathcal{U}_{n/2}$ be the set of maximal totally singular subspaces of V. By Corollary 4.1.7, each of them is $n/2$-dimensional. Let $U, W \in \mathcal{U}_{n/2}$; we define the following relation: $U \sim W$ if, and only if, $n/2 - \dim(U \cap W)$ is even. Then, [**2**, 22.13] shows that $\sim$ is an equivalence relation with precisely two equivalence classes, say $\mathcal{U}^1_{n/2}$ and $\mathcal{U}^2_{n/2}$.

It is clear that O_n acts on $\mathcal{U}_{n/2}$. Moreover O_n preserves the equivalence relation defined above. Thus we have a group homomorphism $\gamma\colon \mathrm{O}_n \to \mathrm{Sym}\{\mathcal{U}^1_{n/2}, \mathcal{U}^1_{n/2}\}$. By Witt's Lemma 4.1.6, the action of O_n on $\mathcal{U}_{n/2}$ is transitive. In particular, γ is surjective. Therefore by Theorem 4.2.6 we have $\mathrm{SO}_n = \ker(\gamma)$. Furthermore, the action of SO_n on $\mathcal{U}_{n/2}$ has precisely two orbits, that are $\mathcal{U}^1_{n/2}$ and $\mathcal{U}^2_{n/2}$, see [**36**, Lemma 2.5.8].

A maximal torus of SO_n is given by $(\mathrm{O}_n \cap D_n)^\circ$. Therefore

$$\operatorname{rank} \mathrm{SO}_n = \left\lfloor \frac{n}{2} \right\rfloor$$

We have that SO_n is simple and the associated Dynkin diagram is $B_{\lfloor n/2 \rfloor}$ for n odd, and $D_{n/2}$ for n even, see [**45**, Table 9.2].

CHAPTER 5

Conjugacy classes

In this chapter we describe the conjugacy classes of prime order elements in the classical groups defined in Chapter 4. Let $G = Cl(V)$ denote one of the classical groups $\mathrm{GL}(V), \mathrm{Sp}(V)$ or $\mathrm{O}(V)$. We have three main aims:

(i) *Membership*: for $x \in \mathrm{GL}(V)$, give conditions for which $x \in G$.
(ii) *Conjugacy*: for $x, y \in G$ describe conditions for which x and y are G-conjugate.
(iii) *Dimension*: for $x \in G$ compute $\dim x^G$.

5.1. Preliminaries

Let V be an n-dimensional k-vector space. Fix a basis of V, then we have an identification $\mathrm{GL}(V) \cong \mathrm{GL}_n$.

In this first section we recall standard results from linear algebra for conjugacy of elements in GL_n, and we introduce the notation we shall use throughout the thesis.

For any $\lambda \in k^*$ we define the *Jordan block* of size i associated to the scalar λ the $(i \times i)$-matrix:

$$J_i(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \in \mathrm{GL}_i$$

and we write J_i for $J_i(1)$.

We denote by $[x_1, \dots, x_n]$ a block diagonal matrix with the matrices $x_1, \dots, x_n$ down the diagonal. We will also write $x = [x_1^{a_1}, \dots, x_n^{a_n}]$, where a_i represents the multiplicity of the block x_i, i.e. how many times it appears in x.

The following well known result characterise conjugacy classes in GL_n, see for example [**51**, Theorem 8.6].

Theorem 5.1.1 (The Jordan canonical form)**.** *Let $x \in \mathrm{GL}_n$. Assume the eigenvalues of x are $\{\lambda_1, \dots, \lambda_m\}$. Then there exists a basis of V in which*

$$x = [A_1, \dots, A_m], \text{ where } A_i = [J_n(\lambda_i)^{a_{i,n}}, \dots, J_1(\lambda_i)^{a_{i,1}}]$$

for some $a_{i,j}$ such that $\sum_{i,j} j a_{i,j} = n$.

We say that x is in *standard Jordan form* if x is a block diagonal matrix as in Theorem 5.1.1.

Let $x \in \mathrm{GL}_n$. Then, by Proposition 2.3.2, we have $x = x_s x_u = x_u x_s$, where x_s is semisimple and x_u is unipotent. Note that if x has prime order then x is either semisimple or unipotent. Now we discuss unipotent and semisimple elements in GL_n.

Unipotent elements. From Theorem 5.1.1 we immediately deduce that the standard Jordan form of any unipotent element $x \in \mathrm{GL}_n$ is

$$x = [J_n^{a_n}, \ldots, J_1^{a_1}] \tag{14}$$

where $n = \sum_i ia_i$. In particular, the vector space V under the action of x has a direct sum decomposition in indecomposable blocks:

$$V \downarrow x = (U_{n,1} \oplus \ldots \oplus U_{n,a_n}) \oplus \ldots \oplus (U_{1,1} \oplus \ldots \oplus U_{1,a_1}) \tag{15}$$

where $U_{i,j} \downarrow x$ has Jordan form J_i. Observe that there is a one-to-one correspondence between conjugacy classes of unipotent elements in GL_n and partitions of n:

$$(n^{a_n}, \ldots, 1^{a_1}) \vdash n \longleftrightarrow [J_n^{a_n}, \ldots, J_1^{a_1}]^{\mathrm{GL}_n} \tag{16}$$

Another consequence of Theorem 5.1.1 is the following.

Corollary 5.1.2. *Let $x = [J_n^{a_n}, \ldots, J_1^{a_1}]$ and $y = [J_n^{b_n}, \ldots, J_1^{b_1}] \in \mathrm{GL}_n$. Then x and y are GL_n-conjugate if, and only if, $a_i = b_i$ for all $1 \le i \le n$.*

Remark 5.1.3. Notice that if $p = 0$ then any nontrivial unipotent element has infinite order. If $p \neq 0$ then x has order a power of p; in addition, x has order p if, and only if, $a_i = 0$ for all $i > p$. In particular, if x is an involution, i.e. x has order 2, then we have, up to GL_n-conjugacy, $x = [J_2^s, J_1^{n-2s}]$, for some $0 < s \le n/2$.

In Section 5.2 we shall give a formula for $\dim x^{\mathrm{GL}_n}$ and discuss conjugacy of unipotent elements in the other classical groups.

We shall need the following observation; we denote x^{-t} the inverse transpose of x.

Lemma 5.1.4. *Let $G = \mathrm{GL}_n$. Then J_n and J_n^{-t} are G-conjugate.*

Proof. Let $\mathcal{B} = \{u_1, \ldots, u_n\}$ be a basis for $V = k^n$. Define $\varphi \in \mathrm{GL}_n$ as $u_i \mapsto u_i + u_{i-1}$ (we set $u_i = \mathbf{0}$ if $i < 1$ or $i > n$). Then φ has Jordan form J_n in the basis $\mathcal{B}$. Meanwhile J_n^t is given by $\varphi^t \colon u_i \mapsto u_i + u_{i+1}$. Define the linear map $\tau \colon V \to V$, $u_i \mapsto u_{n-i+1}$. Then we see $\varphi^t\tau = \tau\varphi$. Therefore J_n and J_n^t are G-conjugate.

In order to show that J_n and J_n^{-1} are conjugate, we show that they represent the same linear transformation in two different basis. Let φ as above. Define the basis $\mathcal{B}' = \{v_1, \ldots, v_n\}$ by recursion: $v_1 = u_1$ and $v_i = u_i - v_{i-1}$. Notice that $\varphi^{-1}(u_i) = v_i$. Again, the linear transformation $\psi \colon v_i \mapsto v_i + v_{i-1} = u_i$ is represented by J_n. The result follows. *q.e.d.*

Semisimple elements. Now assume $x \in \mathrm{GL}_n$ is a semisimple element. Then, by definition, x is GL_n-conjugate to a diagonal matrix $[\lambda_1 I_{a_1}, \ldots, \lambda_m I_{a_m}]$, where $\lambda_i \in k^*$, and a_i is the multiplicity of the eigenvalue λ_i, hence $n = \sum_i a_i$. Thus, we have

$$V \downarrow x = V_1 \oplus \ldots \oplus V_m$$

where $V_i \downarrow x$ has Jordan form $\lambda_i I_{a_i}$.

The following is an immediate consequence of Theorem 5.1.1.

Corollary 5.1.5. *Let* $x = [\lambda_1 I_{a_1}, \ldots, \lambda_m I_{a_m}]$ *and* $y = [\mu_1 I_{b_1}, \ldots, \mu_m I_{b_m}] \in \mathrm{GL}_n$*. Then* x *and* y *are* GL_n*-conjugate if, and only if, there exists a permutation* $\sigma \in S_m$ *such that* $\lambda_i = \mu_{\sigma(i)}$ *and* $a_i = b_{\sigma(i)}$ *for all* $1 \le i \le m$*.*

Assume x has prime order r, and let $\omega \in k$ be a primitive r-th root of unity. Then each eigenvalue of x is a power of ω. Hence, up to GL_n-conjugacy, we may write

$$x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \tag{17}$$

And, by Corollary 5.1.5, two semisimple elements $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ and $y = [I_{b_0}, \omega I_{b_1}, \ldots, \omega^{r-1} I_{b_{r-1}}] \in \mathrm{GL}_n$ of prime order r are GL_n-conjugate if, and only if, $a_i = b_i$ for all i. In particular, if x is a non-central involution then, up to GL_n-conjugacy, $x = [I_{n-s}, -I_s]$, for some $0 < s < n$.

In Section 5.3 we shall give the structure of $C_{\mathrm{GL}_n}(x)$, which leads to a formula for $\dim x^{\mathrm{GL}_n}$, and discuss conjugacy of semisimple elements in the other classical groups.

The ν parameter. We conclude this section by defining an important parameter.

Let $x \in \mathrm{GL}(V)$. Then, for $v \in V$ we define, see [**2**, Section 13.5],

$$[v, x] = x.v - v$$

Then, $[V, x] = \mathrm{span}\{[v, x] : v \in V\}$. Following [**43**, Section 3], we give the following.

Definition 5.1.6. Let $x \in \mathrm{GL}(V)$. We define

$$\nu(x) = \min\{\dim[V, \lambda x] : \lambda \in k^*\}$$

Notice that $\nu(x) > 0$ if $x \notin Z(\mathrm{GL}(V))$. Moreover $\nu(y) = \nu(x)$ for all $y \in x^{\mathrm{GL}(V)}$. The next result gives an alternative definition of $\nu(x)$.

Proposition 5.1.7. *Let* $x \in \mathrm{GL}(V)$*. Then* $\nu(x)$ *is the codimension of the largest, in terms of dimension, eigenspace of* x *with respect to the natural action on* V*.*

Proof. Under the action of x we have the decomposition in indecomposable blocks $V \downarrow x = U_1 \oplus \ldots \oplus U_\ell$.

Let $U \in \{U_1, \ldots, U_\ell\}$. Write $J_m(\lambda)$ for the Jordan form of $U \downarrow x$. Let $\{v_1, \ldots, v_m\}$ be a basis of V for which $x.v_i = \lambda v_i + v_{i-1}$ for all $1 \le i \le m$ (we write $v_0 = \mathbf{0}$). Then, for $\mu \in k^*$, we get $[v_i, \mu x] = (\lambda\mu - 1)v_i + \mu v_{i-1}$ for all $1 \le i \le m$. Thus $[U, \mu x] \le U$. It is clear that if $\mu = \lambda^{-1}$ then $[U, \mu x] < U$; in fact $v_m \notin [U, \mu x]$. Hence

$$\dim[U, \mu x] = \begin{cases} \dim U - 1 & \text{if } \mu = \lambda^{-1} \\ \dim U & \text{otherwise} \end{cases}$$

Since $V = \bigoplus_i U_i$ we have $\dim[V, \mu x] = \sum_{i=1}^{\ell} \dim[U_i, \mu x]$. Say $\mathcal{E}_\lambda$ the collection of U_i with eigenvalue λ. Notice that $|\mathcal{E}_\lambda|$ represents the dimension of $V_\lambda = \{v \in V : xv = \lambda v\}$, the eigenspace of λ. By the previous computation we get

$$\dim[V, \lambda^{-1} x] = \sum_{U \in \mathcal{E}_\lambda} (\dim U - 1) + \sum_{U \notin \mathcal{E}_\lambda} \dim U = \dim V - |\mathcal{E}_\lambda|$$

Thus $\dim[V, \mu x]$ is minimal when $\mu = \lambda^{-1}$ and λ is the eigenvalue with the largest eigenspace. *q.e.d.*

For example, if $x = [J_n^{a_n}, \dots, J_1^{a_1}] \in \mathrm{GL}_n$ then $\nu(x) = n - \sum_i a_i$.

5.2. Unipotent elements

Recall that k is an algebraically closed field of characteristic p. Let $G = Cl_n$. In the study of unipotent conjugacy classes there is a major dichotomy that arises from the characteristic of the field. We first deal with the case $p \neq 2$ if $G \neq \mathrm{GL}_n$.

Let $x \in G$ be a unipotent element. Then, x is GL_n-conjugate to $[J_n^{a_n}, \dots, J_1^{a_1}]$. In this section we shall give more information on conjugacy of unipotent elements in Sp_n and O_n. Recall, by Proposition 3.1.3 and Example 3.1.4(2), that

$$\dim x^G = \dim G - \dim C_G(x)$$

The following is a particular case of[**42**, Theorem 3.1] and it summarises all the informations we need on conjugacy of unipotent elements in the classical groups.

Theorem 5.2.1. *Let* $G = \mathrm{Sp}_n$ *or* O_n. *Assume* $p \neq 2$. *Let* $x \in \mathrm{GL}_n$ *be a unipotent element with Jordan form* $[J_p^{a_p}, \dots, J_1^{a_1}]$. *Then*

(i) A GL_n*-conjugate of* x *lies in* Sp_n *if, and only if,* a_i *is even for each odd* i*; and a* GL_n*-conjugate of* x *lies in* O_n *if, and only if,* a_i *is even for each even* i.

(ii) Two unipotent elements of G *are* G*-conjugate if, and only if, they are* GL_n*-conjugate.*

(iii) We have the following

$$\dim C_{\mathrm{GL}_n}(x) = 2\sum_{i<j} i a_i a_j + \sum_i i a_i^2$$

$$\dim C_{\mathrm{Sp}_n}(x) = \sum_{i<j} i a_i a_j + \frac{1}{2}\sum_i i a_i^2 + \frac{1}{2}\sum_{i \text{ odd}} a_i$$

$$\dim C_{\mathrm{O}_n}(x) = \sum_{i<j} i a_i a_j + \frac{1}{2}\sum_i i a_i^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i$$

Comments on the proof. *Parts (i) and (ii).* A unipotent element x in an algebraic group G is said to be *distinguished* if $C_G(x)^\circ$ is a unipotent group, see [**42**, Section 2.5]. The main strategy in order to prove (i) and (ii) relies on results by Bala and Carter [**5, 6**] thanks to which one can reduce to the study of distinguished unipotent classes in $Cl(V)$. (Notice that Bala and Carter were assuming $p = 0$ or p 'large' and odd if $G = \mathrm{Sp}_n$ or O_n, them results were extended to any odd p by Pommerening [**47, 48**]). After this task is completed, (i) and (ii) quickly follow.

Part (iii). This is [**42**, Proposition 3.7]. The proof relies on the following fact: $\dim C_G(x) = \dim C_{\mathrm{Lie}(G)}(x)$, where $\mathrm{Lie}(G)$ denotes the Lie algebra of G. By [**42**, Lemma 2.8], $\mathrm{Lie}(G) \cong V \otimes V, S^2V$ or $\bigwedge^2 V$ for $G = \mathrm{GL}(V), \mathrm{Sp}(V)$ or $\mathrm{O}(V)$, respectively. Then the result is a consequence of [**42**, Lemma 3.4]. *q.e.d.*

Remark 5.2.2. Indeed it is possible to give an explicit decomposition of V as an orthogonal sum of indecomposable blocks under the action of x. Then (i) is a trivial consequence. Given a unipotent element $x \in \mathrm{Sp}(V)$ we have ([**42**, Lemma 3.12]):

$$V \downarrow x = \Big(\bigoplus_{i\,\mathrm{odd}} (U_i \oplus U_i^*)\Big) \oplus \Big(\bigoplus_{i\,\mathrm{even}} W_i\Big) \qquad (18)$$

where the Jordan form of $(U_i \oplus U_i^*) \downarrow x$ is J_i^2, and the Jordan form of $W_i \downarrow x$ is J_i. In addition U_i, U_i^* are totally singular and $U_i \oplus U_i^*, W_i$ are non-degenerate. Notice that if $G = \mathrm{O}_n$ we get the same decomposition as (18) with the words 'odd' and 'even' transposed.

Let $\mathcal{P}_{\mathrm{GL}_n} = \mathcal{P}(n) = \{\lambda \vdash n\}$ the set of partitions of n. In (16) we saw that there is a one-to-one correspondence between GL_n-conjugacy classes of unipotent elements and $\mathcal{P}_{\mathrm{GL}_n}$. Let us define

$$\mathcal{P}_{\mathrm{Sp}_n} = \{\lambda = (n^{a_n}, \dots, 1^{a_1}) \vdash n : a_i \text{ even if } i \text{ is odd}\}$$
$$\mathcal{P}_{\mathrm{O}_n} = \{\lambda = (n^{a_n}, \dots, 1^{a_1}) \vdash n : a_i \text{ even if } i \text{ is even}\}$$

For $p \neq 2$ if $G \neq \mathrm{GL}_n$, Theorem 5.2.1 implies that the following correspondence is one-to-one

$$(n^{a_n}, \dots, 1^{a_1}) \in \mathcal{P}_{Cl_n} \longleftrightarrow [J_n^{a_n}, \dots, J_1^{a_1}]^{Cl_n}$$

Recall, a unipotent element x is an element for which a power of $1 - x$ vanishes; note that if x is unipotent then $(1-x)^n = 0$, where $\dim V = n$. Hence, we deduce that the set of all unipotent elements, say $\mathcal{U}$, in an algebraic group G, is closed. Therefore if $x \in \mathcal{U}$ then $\overline{x^G} \subseteq \mathcal{U}$. Notice that in general x^G is not closed; in fact in a reductive group a conjugacy class is closed if, and only if, it consists of semisimple elements (see [**34**, Corollary 1.7]). In particular, we can define a partial ordering on the set of unipotent classes. Let $x, y \in G$, then we write $x^G \leq y^G$ if, and only if, $x \in \overline{y^G}$. Notice that if $x^G \leq y^G$ then $\dim x^G \leq \dim y^G$.

Let n be an integer. For the purpose of defining an order on $\mathcal{P}(n)$ we use the following notation for partitions: $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$ where we allow some λ_i to be 0. On $\mathcal{P}(n)$ we define the *dominance ordering*, as follows:

$$\mu \preccurlyeq \lambda \Longleftrightarrow \sum_{i=1}^{l} \mu_i \leq \sum_{i=1}^{l} \lambda_i, \text{ for all } 1 \leq l \leq n$$

The following result (see [**34**, Theorem 7.19]) shows that the correspondence defined above between partitions in $\mathcal{P}_{Cl_n}$ and unipotent classes is, in fact, an order preserving bijection.

Proposition 5.2.3. *Let $G = Cl_n$. Assume $p \neq 2$ if $G \neq \mathrm{GL}_n$. Let x^G and y^G be two unipotent conjugacy classes in G with corresponding partitions $\lambda, \mu \vdash n$. Then*

$$x^G \leq y^G \Longleftrightarrow \lambda \preccurlyeq \mu$$

The previous result will be useful several times in the course of our analysis, for an application see for example Proposition 14.4.4. Moreover in Appendix B.1 minimal elements among the partitions with a fixed number of non-zero parts are determined.

We conclude this section by giving a remark on unipotent conjugacy classes in the special orthogonal group. Recall the description of SO_n given in Section 4.2.5.

Remark 5.2.4. Let $x = [J_n^{a_n}, \dots, J_1^{a_1}] \in \mathrm{O}_n$. Since $|\mathrm{O}_n : \mathrm{SO}_n| = 2$, $x^{\mathrm{O}_n} \cap \mathrm{SO}_n$ is union of at most two SO_n-conjugacy classes, say ℓ this number. Assume $a_i = 0$ for all i odd. Let $U_1, U_2 \in \mathcal{U}^1_{n/2}$ and $W_1, W_2 \in \mathcal{U}^2_{n/2}$ such that $V = U_1 \oplus U_2 = W_1 \oplus W_2$. Notice that, by the proof of [**36**, Lemma 4.1.9], if x acts as the matrix $A \in \mathrm{GL}_{n/2}$ on U_1 (or W_1) then the action of x on U_2 (resp. W_2) is given by A^{-t}. Recall, moreover, that a Jordan block and its inverse-transpose are GL_n-conjugate. Let y be O_n-conjugate to x for which

$$(U_1 \oplus U_2) \downarrow x = \left(J_n^{a_{n/2}} \oplus \ldots \oplus J_1^{a_1/2}\right) \oplus \left(J_n^{a_{n/2}} \oplus \ldots \oplus J_1^{a_1/2}\right)$$
$$(W_1 \oplus W_2) \downarrow y = \left(J_n^{a_{n/2}} \oplus \ldots \oplus J_1^{a_1/2}\right) \oplus \left(J_n^{a_{n/2}} \oplus \ldots \oplus J_1^{a_1/2}\right)$$

(here we have a slight abuse of notation: rather then writing the indecomposable blocks, as in (15), we write the Jordan form of x and y on these blocks.) Then, by the description of SO_n given in Section 4.2.5, x and y are not SO_n-conjugate. In fact, [**42**, Lemma 3.11(iii)] states that $\ell = 2$ if, and only if, $a_i = 0$ for all i odd.

5.2.1. Involutions. In this section we assume $p = 2$ and $G = \mathrm{Sp}_n$ or O_n. For more details we refer to [**3**, Section 7,8] and [**42**, Chapter 4,6].

Let $x \in G$ be an involution. Hence x has Jordan form $[J_2^s, J_1^{n-2s}]$ for some $1 \leq s \leq n/2$ (note that $\nu(x) = s$). However, the Jordan form of x does not uniquely determine the G-conjugacy class of x. More precisely, as described in [**3**, Sections 7, 8], if s is even there are precisely two different conjugacy classes of involutions, whose representative are denoted a_s and c_s. If s is odd there is a unique class with Jordan form $[J_2^s, J_1^{n-2s}]$, whose representative is denoted by b_s.

Let $(\cdot, \cdot)$ be the bilinear form defined on the natural module V of G. An involution x with $\nu(x) = s$ even is defined to be in a_s^G if, and only if, for all $v \in V$

$$(x.v, v) = 0 \qquad (19)$$

otherwise $x \in c_s^G$. We shall usually say that an involution $x \in G$ is a_s-type if $x \in a_s^G$, similarly in the other cases.

We write matrices in the standard (ordered) basis $\{e_1, f_1, \ldots, e_{n/2}, f_{n/2}\}$, see Propositions 4.1.8 and 4.1.12.

Now we give a property equivalent to the definition (19) of a_s-type involution. Define

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let $x \in G$ be an a_s-type involution. Then [**3**, 7.3 and 8.2] imply that

$$V \downarrow x = \left(\bigoplus_{i=1}^{s/2} V_i\right) \oplus W$$

where x acts trivially on W and each V_i is 4-dimensional with a symplectic (or orthogonal) basis $\{e_1^i, e_2^i, f_1^i, f_2^i\}$ such that in this basis $V_i \downarrow x$ is given by $[A, A]$. This fact is proved by constructing an explicit basis for which the required properties hold.

It is straightforward to verify that (19) holds for an involution x, with $\nu(x) = s$, that is G-conjugate to

$$[(A, A)^\diamond, \ldots, (A, A)^\diamond, I_{n-2s}]$$

where we denoted $(A, A)^\diamond$ the block diagonal matrix $[A, A]$ with respect to the ordered basis $\{e_i, e_{i+1}, f_i, f_{i+1}\}$ for odd $i \leq n - 2s - 2$.

Therefore an involution $x \in G$ with $\nu(x) = s$ is a_s-type if, and only if,

$$x \text{ is } G\text{-conjugate to } [(A,A)^\diamond, \dots, (A,A)^\diamond, I_{n-2s}]$$

In particular, this characterisation of a_s-type involution implies that an involution $x \in G$ (with $\nu(x) = s$ even) is of type c_s if, and only if, it is conjugate to a matrix which contains a pair $[A, A]$ in the basis $\{e_i, f_i, e_{i+1}, f_{i+1}\}$ for some i.

Proposition 5.2.5, below, may be deduced from [**3**, Sections 7, 8]; the precise formula for the dimension of the centraliser of involutions is also given in [**42**, (4.1) p.59], we remark that such a formula goes much further: in fact it gives the dimension of the centraliser of any unipotent element in even characteristic in Sp_n and O_n. Notice that [**42**, Chapter 4–6] relies on the work of Hesselink [**32**]. In [**32**] conjugacy classes of unipotent elements in classical groups in even characteristic are classified in terms of the indecomposable summands in $V \downarrow x$ and a function $\chi \colon \mathcal{I} \subseteq \mathbb{N} \to \mathbb{Z}$ where $\mathcal{I}$ is the set of dimensions of the indecomposable summands in $V \downarrow x$. It is not hard to show that this function χ is 0 for a-type involutions and $\chi(1) = 0, \chi(2) = 1$ otherwise.

Proposition 5.2.5. *Let $G = \mathrm{Sp}_n$ or O_n and $p = 2$. Let $s \leq n/2$ be a positive integer.*

(i) If s is odd there exists one G-conjugacy class of involutions with $\nu(x) = s$, whose representative is denoted by b_s;

(ii) Assume s is even. Then there are precisely two G-conjugacy classes of involutions with $\nu(x) = s$, with representatives a_s and c_s. Moreover x is a_s-type if, and only if, x is G-conjugate to

$$[(A,A)^\diamond, \dots, (A,A)^\diamond, I_{n-2s}]$$

otherwise x is c_s-type;

(iii) If $G = \mathrm{Sp}_n$ then

$$\dim a_s^G = s(n-s), \ \dim b_s^G = \dim c_s^G = s(n-s+1)$$

(iv) If $G = \mathrm{O}_n$ then

$$\dim a_s^G = s(n-s-1), \ \dim b_s^G = \dim c_s^G = s(n-s)$$

and $a_s, c_s \in \mathrm{SO}_n$, while $b_s \in \mathrm{O}_n \setminus \mathrm{SO}_n$.

Remark 5.2.6. The fact that $b_s \notin \mathrm{SO}_n$ is observed in [**3**, 8.10]. We prove this fact using the description of SO_n given in Section 4.2.5. Involutions of type b_s act on $\mathcal{U}_{n/2}$, the set of maximal isotropic subspaces of V. In particular, they interchange $\mathcal{U}^1_{n/2}$ and $\mathcal{U}^2_{n/2}$. Let $U = \langle e_1, \dots, e_{n/2} \rangle$ and $W = \langle f_1, \dots, f_{n/2} \rangle$. Define $x = [A, I_{n-2}]$, i.e. $x \colon e_1 \leftrightarrow f_1, e_i \mapsto e_i, f_i \mapsto f_i$ for $i > 1$. Notice that x is of b_1-type. Define $U' = x.U = \langle f_1, e_2, \dots, e_{n/2} \rangle$, similarly $W' = x.W$. We have $n/2 - \dim(U \cap U') = 1$ hence U, U' are not equivalent as well as W, W' (in the sense of the equivalence relation defined in Section 4.2.5). This is easily extended to any b_s. Thus we conclude that no b_s lies in SO_n.

Involutions with odd ν-value lie in Sp_n if, and only if, $p = 2$. Assume $x = [A, I_{n-2}]$ – in the ordered standard basis $\{e_1, f_1, \dots, e_{n/2}, f_{n/2}\}$ – then it is straightforward to check that x fixes a symplectic form if, and only if, $p = 2$. In the next section we provide more information on conjugacy classes of semisimple elements in the classical groups.

5.3. Semisimple elements

Let $G = Cl(V) = Cl_n$. Recall that G is the isometry group of the zero form or a non-degenerate form κ defined on V.

Let $x \in \mathrm{GL}_n$ be semisimple. Then

$$V \downarrow x = \bigoplus_{\lambda \in k^*} V_\lambda \tag{20}$$

where $V_\lambda = \{u \in V : x.u = \lambda u\}$ is the eigenspace of x relative to λ.

Recall the notation established in Section 5.1 for semisimple elements. If $r \neq p$ is an odd prime, we write $\omega \in k$ for a primitive r-th root of unity. Let $x \in \mathrm{GL}_n$ be a semisimple element of order r. Then, up to GL_n-conjugacy, we may write, for odd r

$$x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \tag{21}$$

for some $a_i \geq 0$ such that $\sum_i a_i = n$. If $r = 2$ then, up to GL_n-conjugacy, for some $0 < s < n$,

$$x = [I_s, -I_{n-s}] \tag{22}$$

Recall, Corollary 5.1.5 implies that x and $y = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}]$ (or $[I_t, -I_{n-t}]$ if $r = 2$), are GL_n-conjugate if, and only if, $a_i = b_i$ for all i, (or $s = t$ if $r = 2$).

This section is devoted to prove the following.

Theorem 5.3.1. *Assume $r \neq p$ is a prime. Let $G = Cl_n$ and $x \in \mathrm{GL}_n$ be of order r. Assume the Jordan form of x is as in (21) or (22).*

(i) *Two semisimple elements of G are G-conjugate if, and only if, they are GL_n-conjugate.*

(ii) *Assume $G \neq \mathrm{GL}_n$ and $r \neq 2$. Then $x^{\mathrm{GL}_n} \cap G \neq \emptyset$ if, and only if $a_0 \equiv n \pmod 2$ and $a_i = a_{r-i}$ for all $1 \leq i \leq \frac{r-1}{2}$.*

(iii) *Assume $r = 2$. Then $x^{\mathrm{GL}_n} \cap \mathrm{O}_n \neq \emptyset$, and $x^{\mathrm{GL}_n} \cap \mathrm{Sp}_n \neq \emptyset$ if, and only if, $s \equiv n \pmod 2$.*

(iv) *Assume $x \in G$. The structure and the dimension of $C_G(x)$ are recorded in Table 5.3.1.*

G	r	$C_G(x)$	$\dim C_G(x)$
GL_n	$= 2$	$\mathrm{GL}_s \times \mathrm{GL}_{n-s}$	$n^2 - 2s(n-s)$
GL_n	$\neq 2$	$\prod_{i=0}^{r-1} \mathrm{GL}_{a_i}$	$\sum_{i=0}^{r-1} a_i^2$
Sp_n	$= 2$	$\mathrm{Sp}_s \times \mathrm{Sp}_{n-s}$	$\frac{n}{2}(n+1) - s(n-s)$
Sp_n	$\neq 2$	$\mathrm{Sp}_{a_0} \times \prod_{i=1}^{(r-1)/2} \mathrm{GL}_{a_i}$	$\frac{a_0}{2}(a_0+1) + \frac{1}{2}\sum_{i=1}^{r-1} a_i^2$
O_n	$= 2$	$\mathrm{O}_s \times \mathrm{O}_{n-s}$	$\frac{n}{2}(n-1) - s(n-s)$
O_n	$\neq 2$	$\mathrm{O}_{a_0} \times \prod_{i=1}^{(r-1)/2} \mathrm{GL}_{a_i}$	$\frac{a_0}{2}(a_0-1) + \frac{1}{2}\sum_{i=1}^{r-1} a_i^2$

Table 5.3.1. Centralisers of prime order semisimple elements in $G = Cl(V)$

Let $G = \mathrm{Sp}_n$ or O_n and let $x \in G$ be a semisimple element (of any order). We have a decomposition of the natural module in eigenspaces as in (20). In Proposition 5.3.2, below, we establish the features of the eigenspaces. Then (i)–(iii) of Theorem 5.3.1 quickly follows.

Proposition 5.3.2. *Let $G = \mathrm{Sp}_n$ or O_n. Let $x \in G$ be a semisimple element. Then*

$$V \downarrow x = V_1 \oplus V_{-1} \oplus \big(\bigoplus_{\lambda \neq \pm 1} V_\lambda \oplus V_{\lambda^{-1}} \big)$$

and the following hold:

(i) $V_{\pm 1}$ *is either trivial or non-degenerate;*
(ii) $V_\lambda, V_{\lambda^{-1}}$ *are totally singular and* $\dim V_\lambda = \dim V_{\lambda^{-1}}$ *for all* $\lambda \neq \pm 1$*;*
(iii) $V_\lambda \oplus V_{\lambda^{-1}}$ *is either trivial or non-degenerate for all* $\lambda \neq \pm 1$*;*
(iv) If $\lambda \neq \mu^{-1}$ *then* $V_\lambda \oplus V_\mu$ *is an orthogonal sum.*

Proof. Let $f = (\cdot\,,\cdot)$ be the bilinear form defined on V. Since $x \in G$ is semisimple we have a decomposition of V in eigenspaces as (20).

Let $\lambda \in k^*$ such that V_λ is non-trivial. For $u, v \in V_\lambda$, we have $(u, v) = \lambda^2(u, v)$. Thus, if $\lambda \neq \pm 1$ we have $f|_{V_\lambda} \equiv 0$, so V_λ is totally singular. Let $u \in V_\lambda$ and $v \in V_\mu$. Then $(u, v) = \lambda\mu(u, v)$. So, $\lambda\mu \neq 1$ implies $(u, v) = 0$, hence $V_\lambda \oplus V_\mu$ is an orthogonal sum.

Assume V_1 is non trivial and, seeking a contradiction, let $u \in V_1$ such that $(u, V_1) = 0$. Since f is non-degenerate, there exists $v \in V$ such that $(u, v) \neq 0$. If $v \in V_{-1}$ we have $0 \neq (u, v) = (x.u, x.v) = -(u, v)$ which is absurd (note that if $p = 2$ then $V_1 = V_{-1}$). If $v \in V_\lambda$, for some $\lambda \neq \pm 1$ then, again, $0 \neq (u, v) = (x.u, x.v) = \lambda(u, v)$ is absurd. Therefore V_1 is non-degenerate. Similarly we can show that V_{-1} and $V_\lambda \oplus V_{\lambda^{-1}}$ are non-degenerate (whenever they are non-trivial).

Now, f induces a non-degenerate bilinear map $f_\lambda \colon V_\lambda \times V_{\lambda^{-1}} \to k$. Thus for all $u \in V_{\lambda^{-1}}$ we have the linear map $\varphi_u \colon V_\lambda \to k, v \mapsto (v, u)$. So we have the map $\Phi \colon V_{\lambda^{-1}} \to V_\lambda^*$, $u \mapsto \varphi_u$. It is straightforward to check that Φ is linear and $\ker \Phi = \{\mathbf{0}\}$. Thus $\dim V_{\lambda^{-1}} \leq \dim V_\lambda^*$. Similarly we show $\dim V_\lambda \leq \dim V_{\lambda^{-1}}^*$. Therefore we deduce $\dim V_\lambda = \dim V_{\lambda^{-1}}$. *q.e.d.*

Assume $G \neq \mathrm{GL}_n$. Let $x, y \in G$. Assume x and y are G-conjugate. Then $V \downarrow x$ and $V \downarrow y$ have isomorphic summands. In particular, they share the same eigenvalues with same multiplicities. Conversely, assume x, y have the same eigenvalues with same multiplicities, then we have the following.

Lemma 5.3.3. *Assume $G \neq \mathrm{GL}_n$. Let $x, y \in G$ be semisimple elements and write $V \downarrow x = \bigoplus V_\lambda$ and $V \downarrow y = \bigoplus U_\lambda$. Assume $\dim V_\lambda = \dim U_\lambda$ for all $\lambda \in k^*$. Then x, y are G-conjugate.*

Proof. By Proposition 5.3.2 we have the following orthogonal decompositions

$$V \downarrow x = V_1 \oplus V_{-1} \bigoplus (V_\lambda \oplus V_{\lambda^{-1}}), \;\; V \downarrow y = U_1 \oplus U_{-1} \bigoplus (U_\lambda \oplus U_{\lambda^{-1}})$$

where $V_{\pm 1}, U_{\pm 1}, V_\lambda \oplus V_{\lambda^{-1}}$ and $U_\lambda \oplus U_{\lambda^{-1}}$ are non-degenerate and $V_{\lambda^{\pm 1}}, U_{\lambda^{\pm 1}}$ are totally singular and equidimensional. Hence we can construct isometries $f_{\pm 1} \colon V_{\pm 1} \to U_{\pm 1}$.

Assume we have isometries $f_\lambda\colon V_\lambda \oplus V_{\lambda^{-1}} \to U_\lambda \oplus U_{\lambda^{-1}}$, for all $\lambda \neq \pm 1$, such that $f_\lambda(V_{\lambda^{\pm 1}}) = U_{\lambda^{\pm 1}}$. Then $f = f_1 \oplus f_{-1} \oplus (\bigoplus f_\lambda)\colon V \to V$ is an isometry and $fx = yf$, so the result follows. Notice that since V_λ and U_λ are totally singular any bijective linear map $\varphi\colon V_\lambda \to U_\lambda$ is an isometry (recall they are equidimensional). Now, Witt's Lemma 4.1.6 assures us the existence of the isometry f_λ. The result follows. *q.e.d.*

The following solves the membership problem.

Lemma 5.3.4. *Let $x \in \mathrm{GL}_n$ be semisimple. Assume $\dim V_\lambda = \dim V_{\lambda^{-1}}$.*

(i) If $\dim V_1, \dim V_{-1}$ are even then $x^{\mathrm{GL}_n} \cap \mathrm{Sp}_n \neq \emptyset$.

(ii) If $\dim V_1 + \dim V_{-1} \equiv \dim V \pmod 2$ then $x^{\mathrm{GL}_n} \cap \mathrm{O}_n \neq \emptyset$.

Proof. This is straightforward. Take a basis of eigenvectors and construct a one-to-one correspondence with the standard basis (with respect to the form considered) of V. *q.e.d.*

Combining together Proposition 5.3.2 and Lemmas 5.3.3 and 5.3.4 we deduce the first part of Theorem 5.3.1.

Corollary 5.3.5. *Conclusions (i)–(iii) in Theorem 5.3.1 hold.*

Let $x \in G$ be a semisimple element. Thanks to Corollary 5.3.5 we may assume that in the standard basis

$$x = [I_a, -I_b, \lambda_1 I_{a_1}, \lambda_1^{-1} I_{a_1}, \ldots, \lambda_m I_{a_m}, \lambda_m^{-1} I_{a_m}]$$

for some $a, b, a_i \geq 0$ and $1 \leq i \leq m$. We take as understood the order of the basis, for example in $G = \mathrm{Sp}_n$ the block $[\lambda_1 I_{a_1}, \lambda_1^{-1} I_{a_1}]$ is relative to

$$\{e_{a+b+1}, \ldots, e_{a+b+a_1}, f_{a+b+1}, \ldots, f_{a+b+a_1+1}\}$$

Proposition 5.3.6. *Let $G = Cl_n$ and $x \in G$ be semisimple. We record in Table 5.3.2 a representative of the G-conjugacy class of x, and its centraliser structure.*

G	x	$C_G(x)$
GL_n	$[I_a, -I_b, \lambda_1 I_{a_1}, \ldots, \lambda_m I_{a_m}]$	$\mathrm{GL}_a \times \mathrm{GL}_b \times \prod_{i=1}^m \mathrm{GL}_{a_i}$
Sp_n	$[I_a, -I_b, \lambda_1 I_{a_1}, \lambda_1^{-1} I_{a_1}, \ldots, \lambda_m I_{a_m}, \lambda_m^{-1} I_{a_m}]$	$\mathrm{Sp}_a \times \mathrm{Sp}_b \times \prod_{i=1}^m \mathrm{GL}_{a_i}$
O_n	$[I_a, -I_b, \lambda_1 I_{a_1}, \lambda_1^{-1} I_{a_1}, \ldots, \lambda_m I_{a_m}, \lambda_m^{-1} I_{a_m}]$	$\mathrm{O}_a \times \mathrm{O}_b \times \prod_{i=1}^m \mathrm{GL}_{a_i}$

Table 5.3.2. Centraliser of semisimple elements in $G = Cl(V)$

In the table, if $G \neq \mathrm{GL}_n$ then λ_i's are different and $\lambda_i\lambda_j \neq 1$ if $i \neq j$, and a, b are even in the case $G = \mathrm{Sp}_n$.

Proof. Representatives of the GL_n-conjugacy classes are given by Corollary 5.1.5. If $G = \mathrm{Sp}_n$ or O_n then we can write x as in the table thanks to Proposition 5.3.2 and Corollary 5.3.5.

Let us compute the centraliser of a semisimple element in G. An easy calculation shows that a matrix $y \in G$ that centralises x as in Table 5.3.2 must have block form

$y = [C_a, C_b, A_1, A_1', \ldots, A_m, A_m']$ (the A_i''s appear only in the symplectic and orthogonal case) where C_a, C_b have size a and b, respectively, and A_i, A_i' have size a_i.

First assume $G = \mathrm{GL}_n$; in this case we can write

$$x = [I_a, -I_b, \lambda_1 I_{a_1}, \ldots, \lambda_m I_{a_m}]$$

Then $y \in C_G(x)$ if, and only if, $yx = xy$ and $y = [C_a, C_b, A_1, \ldots, A_m]$. Thus $y \in C_G(x)$ if, and only if,

$$C_a I_a = I_a C_a,\ C_b(-I_b) = (-I_b)C_a,\ A_i(\lambda_i I_{a_i}) = (\lambda_i I_{a_i})A_i, \text{ for all } 1 \leq i \leq m$$

Thus there are no further conditions on C_a, C_b and A_i. Hence we deduce

$$C_G(x) = \mathrm{GL}_a \times \mathrm{GL}_b \times \prod_i \mathrm{GL}_{a_i}$$

Now assume $G = \mathrm{Sp}_n$ or O_n. Let $y = [C_a, C_b, A_1, A_1', \ldots, A_m, A_m'] \in C_G(x)$. Then the action $V_1 \downarrow y$ is given by C_a and since V_1 is non-degenerate and C_a is an isometry of V_1 we deduce $C_a \in \mathrm{Sp}_a$ or O_a; similarly we deduce $C_b \in \mathrm{Sp}_b$ or O_b. The action $(V_i \oplus V_i') \downarrow x$ is given by $[\lambda_i I_{a_i}, \lambda_i^{-1} I_{a_i}]$. By Proposition 5.3.2, V_i, V_i' are totally singular and $V_i \oplus V_i'$ is non-degenerate. We may assume $\{e_1, \ldots, e_i\}$ is a basis for V_i and $\{f_1, \ldots, f_i\}$ is a basis for V_i'. Then $[A_i, A_i']$ acts isometrically on $V_i \oplus V_i'$ and, by [**36**, Lemma 4.1.9], $A_i' = A_i^{-t}$. The result follows. *q.e.d.*

Notice that Proposition 5.3.6 implies part (iv) of Theorem 5.3.1.

5.3.1. Special orthogonal group. We conclude this section by giving some comments on conjugacy classes of semisimple elements in the special orthogonal group.

Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in \mathrm{O}_n$ of odd prime order r. Then $x^{\mathrm{O}_n} \cap \mathrm{SO}_n$ is a union of ℓ, at most 2 (as SO_n has index 2 in O_n), SO_n-conjugacy classes. In the following we roughly explain the conditions that determine the values $\ell = 1$ or 2.

Under the action of a semisimple element we have a decomposition of the vector space into eigenspaces, as proved in Proposition 5.3.2. We introduce the following notation. For $x \in \mathrm{O}_n$ semisimple we write

$$V \downarrow x = U \oplus (W \oplus W')$$

where $U, W \oplus W'$ are non-degenerate and W, W' are totally singular, and $U = V_1 \oplus V_{-1}$, $W = \bigoplus V_\lambda$ and $W' = \bigoplus V_{\lambda^{-1}}$, where the V_α are as in Proposition 5.3.2. Notice that W, W' are maximal totally singular subspaces if, and only if, $\dim U \leq 1$ (and $\dim U = 1$ if, and only if, $\dim V$ is odd).

Let W_1, W_2 and W_1', W_2' maximal totally singular subspaces which are not equivalent (with respect to the equivalence relation defined in Section 4.2.5). Define semisimple elements $x, y \in \mathrm{O}_n$ that are O_n-conjugate such that

$$V \downarrow x = U_1 \oplus (W_1 \oplus W_1'),\ V \downarrow y = U_1 \oplus (W_2 \oplus W_2')$$

It is clear that x and y are not SO_n-conjugate.

Indeed one can show that if $x \in \mathrm{O}_n$ is a semisimple element and $a_0 = \dim(V_1 \oplus V_{-1})$, then $\ell = 2$ if, and only if, $a_0 \leq 1$.

5.4. Dimensions

Let $G = Cl_n$. Let $x \in G$ be a prime order element. Recall that $\nu(x)$ is the codimension of the largest eigenspace of x on V, see Proposition 5.1.7. It is immediately clear, thanks to Theorems 5.2.1 and 5.3.1 and Proposition 5.2.5, that $\dim x^G$ is influenced by $\nu(x)$. Essentially a large eigenspace gives rise to a large centraliser. Thus we would expect $\dim x^G$ to be large for small value of $\nu(x)$ and vice-versa. We define

$$\delta_{a;b} = \begin{cases} 1 & b \mid a \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

The following is [**11**, Proposition 2.9].

Proposition 5.4.1. *Let $G = Cl_n$ and $x \in G$ be of prime order with $\nu(x) = s$. Then*

$$f(s) \leq \dim x^G \leq g(s)$$

where $f(s), g(s)$ are listed in Table 5.4.1.

G	$f(s)$	$g(s)$
GL_n	$\max\{2s(n-s), ns\}$	$s(2n-s-1)$
Sp_n	$\max\{s(n-s), \frac{ns}{2}\}$	$\frac{1}{2}(2ns - s^2 + 1)$
O_n	$\max\{s(n-s-1), \frac{n}{2}(s-1)\}$	$\frac{1}{2}(2ns - s^2 - 2s + \delta_{n-1;2})$

Table 5.4.1

This result will be one of the key tools in our analysis of global upper and local upper and lower bounds. Furthermore, these bounds may be interpreted as bounds on the dimension of the fixed point space of x for certain actions, see Remark 7.1.9.

CHAPTER 6

Subgroup structure of classical groups

The aim of this chapter is to give an overview of the subgroup structure of the classical groups. Let V be a finite dimensional k-vector space and let $G = Cl(V)$. Various closed subgroups $H \leq G$ can be defined naturally in terms of the geometry of V. A powerful theorem of Liebeck and Seitz provides a description of the maximal closed connected subgroups of G in terms of these geometric subgroups.

6.1. Preliminaries

Let V be a finite dimensional k-vector space equipped with a form κ. Let $G = Cl(V)$ be the corresponding classical group.

We define five natural collections of closed positive dimensional subgroups of G, labelled $\mathscr{C}_i$ for $i = 1, 2, 3, 4, 6$ (we will explain later the reason for this labelling) and we set $\mathscr{C} = \bigcup \mathscr{C}_i$. We call the subgroups in these families *geometric subgroups*.

We can describe these families of subgroups via their actions on V, which is either reducible (the $\mathscr{C}_1$-collection), irreducible and imprimitive (the $\mathscr{C}_2, \mathscr{C}_3$-collections), and finally irreducible and primitive ($\mathscr{C}_4, \mathscr{C}_6$-collections). In Table 6.1.1 we give a rough description of these families, we write $\mathscr{C}_4 = \mathscr{C}_4(i) \cup \mathscr{C}_4(ii)$.

$\mathscr{C}_i$	rough description
$\mathscr{C}_1$	stabilisers of $U \leq V$
$\mathscr{C}_2$	stabilisers of orthogonal isometric decompositions $V = \bigoplus_i V_i$
$\mathscr{C}_3$	stabilisers of totally singular decompositions $V = U \oplus W$
$\mathscr{C}_4(i)$	stabilisers of tensor product decompositions $V = U \otimes W$
$\mathscr{C}_4(ii)$	stabilisers of tensor product decompositions into isometric subspaces $V = \bigotimes_i V_i$
$\mathscr{C}_6$	stabilisers of a non degenerate form of V

Table 6.1.1. The $\mathscr{C}_i$-families

The motivation for these definitions comes from the study of the subgroup structure of finite and algebraic simple groups, in particular from the study of their maximal subgroups.

This problem, for finite simple groups, goes back to Galois, see his famous letter to Chevalier [**24**]. The main result for finite classical groups is due to Aschbacher [**1**], who provided a description of the maximal subgroups. In [**1**, Section 1] several families of subgroups of finite classical groups are defined, some of which have a description very similar to the $\mathscr{C}_i$ families in Table 6.1.1.

The problem of determining the maximal closed subgroups of a classical algebraic group G goes back to Dynkin [**19**] who first determined the maximal closed connected subgroups of $G = \mathrm{SL}(V), \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ if $k = \mathbb{C}$. Then, in the '80s his analysis was essentially redone, using completely different techniques (mainly group-theoretic whereas Dynkin used to work with the Lie algebras of the groups), and extended to arbitrary characteristic by Seitz [**53**]. The key tool in their analysis is the following observation: let $\varphi\colon G \to \mathrm{SL}(V)$ be an irreducible rational representation, then (under some conditions) $\varphi(G)$ is maximal in one of the classical groups. In fact Dynkin, and then Seitz, determined all the triples (G, H, V) where $G \leq \mathrm{SL}(V)$ is a (simply connected for $p > 0$) simple closed classical group and $H \leq G$ is closed and connected, both acting irreducibly on V.

The main result on the subgroup structure of classical algebraic groups is due to Liebeck and Seitz [**41**], where also non-connected subgroups are considered. They define six families of subgroups in terms of the geometry of V: $\mathscr{C}_i$, $i = 1, \ldots, 6$. Again $\mathscr{C} = \bigcup_i \mathscr{C}_i$. Note that our collections $\mathscr{C}_i$ are very similar to these and we miss the family $\mathscr{C}_5$ because it consists of certain finite groups. We state the main result of [**41**] as Theorem 6.1.1, below. For a sketch of the proof we refer the reader to [**45**, Theorem 18.6]. For more details we refer the reader to the original paper [**41**].

Recall that a group H is said to be *almost simple* if $T \leq H \leq \mathrm{Aut}(T)$, where T is a non-abelian simple group. In the case H is an algebraic group we require $H^\circ \leq H \leq \mathrm{Aut}(H^\circ)$, where H° is simple and $\mathrm{Aut}(H^\circ)$ denotes the algebraic group automorphisms of H°; e.g. $A_n.2, D_n.2, D_4.3, D_4.S_3, E_6.2$.

Theorem 6.1.1. *Let $G \in \{\mathrm{SL}(V), \mathrm{Sp}(V), \mathrm{SO}(V)\}$ and let $H \leq G$ be a closed subgroup of positive dimension. Then either*

(i) H is contained in a member of $\mathscr{C}$; or,
(ii) modulo scalars, H is almost simple and H° is irreducible and tensor indecomposable on V and, if $G = \mathrm{SL}(V)$, fixes no non-degenerate form on V.

An important consequence of Theorem 6.1.1 is a new proof of Aschbacher's Theorem, obtained by considering the fixed points under the action of a Frobenius morphism of G, and using a theorem by Lang and Steinberg. For details we refer to [**41**, Section 4] or [**45**, Chapter 27]

Remark 6.1.2. The conditions in (ii) of Theorem 6.1.1 are added to ensure that such a subgroup is not contained in any element of the $\mathscr{C}_i$ collections. In the literature, the collection comprising groups in (ii) is usually denoted by $\mathscr{S}$. In order to know all the groups in $\mathscr{S}$ one should know all the possible irreducible representations $\rho\colon H \to \mathrm{SL}(V)$ of almost simple groups H. As explained above, this has been achieved by Dynkin and Seitz with some restrictive hypothesis (e.g. the groups are connected).

In the next sections we give some more details on the geometric subgroups.

6.2. Collection $\mathscr{C}_1$: subspace stabilisers

In order to motivate our definition of the $\mathscr{C}_1$-collection, we state the following [**45**, Proposition 18.4(1)].

Lemma 6.2.1. *Let $G = Cl(V)$ and let $H \leq G$ be a closed subgroup acting reducibly on V. Then there exists a non-zero subspace $U < V$ stabilised by H such that one of the following holds:*

(i) U is totally singular;
(ii) U is non-degenerate;
(iii) $(G,p) = (\mathrm{O}(V),2)$ and U is a 1-dimensional non-singular subspace.

For $G = Cl(V)$, we say that $H \leq G$ is a $\mathscr{C}_1$-subgroup if $H = \mathrm{Stab}_G(U)$ where U is a proper non-zero subspace of V, that is either totally singular or non-degenerate, or a non-singular 1-space if $(G,p) = (\mathrm{O}(V),2)$.

Notice that subgroups in the $\mathscr{C}_1$-collection are not reductive, [**33**, Theorem 30.4(a)]. We give an example in the case $G = \mathrm{GL}_n$. Assume $m < n/2$, and define P_m to be the stabiliser of an m-dimensional subspace of the natural module of G. The subgroup P_m is a *parabolic subgroup* and, in a suitable basis,

$$P_m = \begin{pmatrix} \mathrm{GL}_m & M_{m,n-m} \\ 0 & \mathrm{GL}_{n-m} \end{pmatrix} \tag{24}$$

where $M_{m,n-m}$ denotes the set of $m \times (n-m)$ matrices over k. We have $P_m = U \rtimes L$ where U is the unipotent radical of P_m and L is called a *Levi factor*:

$$U = \begin{pmatrix} I_m & M_{m,n-m} \\ 0 & I_{n-m} \end{pmatrix}, \ L = \begin{pmatrix} \mathrm{GL}_m & 0 \\ 0 & \mathrm{GL}_{n-m} \end{pmatrix}$$

For details see [**33**, Section 30].

6.3. Collection $\mathscr{C}_2$: imprimitive subgroups

A subgroup H lies in the $\mathscr{C}_2$ family if there exists $t > 1$ dividing n and isometric, pairwise orthogonal subspaces $V_1, \dots, V_t$ for which $V = V_1 \oplus \dots \oplus V_t$ such that $H = \mathrm{Stab}_G\{V_1, \dots, V_t\}$. We write $\mathrm{Stab}_G(V_1, \dots, V_t)$ for the pointwise stabiliser, i.e. if $x \in \mathrm{Stab}_G(V_1, \dots, V_t)$ then $x.V_i = V_i$ for all i.

Let $V = V_1 \oplus \dots \oplus V_t$ be such a decomposition, then V_i and V_j are isometric (see Definition 4.1.5), so $Cl(V_i) \cong Cl(V_j)$. In Section 17 we shall prove that $H = Cl(V_1) \wr S_t$ and $H^\circ = \mathrm{Stab}_G(V_1, \dots, V_t)$, see Proposition 17.1.6. For example, if $G = \mathrm{GL}_n$ and $t > 1$ is a divisor of n, then

$$H = \mathrm{GL}_{n/t} \wr S_t$$

where the action of S_t on the summands $\{V_1, \dots, V_t\}$ is given by the standard permutation action. For more details we refer to Section 17. Notice that the action of H on V is irreducible, due to the presence of S_t, but imprimitive, in fact $\{V_1, \dots, V_t\}$ is a system of imprimitivity [**17**, Section 1.9].

In the decomposition $V = V_1 \oplus \dots \oplus V_t$ we require the V_i's to be isometric otherwise the stabiliser of $\{V_1, \dots, V_t\}$ would be contained in a member of the $\mathscr{C}_1$-family (H has to transitively permute the V_i's in order to act irreducibly on V). For example, let $G = \mathrm{GL}(V)$ and $V = V_1 \oplus V_2$ with $\dim V_1 = a, \dim V_2 = b$ with $a < b$. Then $\mathrm{Stab}_G\{V_1, V_2\} = \mathrm{Stab}_G(V_1, V_2)$ and, in a suitable basis, is given by $H = \begin{pmatrix} \mathrm{GL}_a & 0 \\ 0 & \mathrm{GL}_b \end{pmatrix} < P_a$, where P_a is as in (24).

6.4. Collection $\mathscr{C}_3$: stabilisers of totally singular decompositions

Let $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$. Then H is a $\mathscr{C}_3$-subgroup if there exist maximal totally singular subspaces U and W of V with $V = U \oplus W$ such that $H = \mathrm{Stab}_G\{U, W\}$. In this situation, Corollary 4.1.7 implies $\dim U = \dim W = n/2$. Observe that if $G = \mathrm{GL}_n$ then H is a $\mathscr{C}_2$-subgroup. In Section 13.1, namely in Proposition 13.1.3, we show $H \cong \mathrm{GL}_{n/2}.\langle\tau\rangle = \mathrm{GL}_{n/2}.2$ and

$$H^\circ = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\} \cong \mathrm{GL}_{n/2}$$

and $\tau = \begin{pmatrix} & I_{n/2} \\ \epsilon I_{n/2} & \end{pmatrix}$, where $\epsilon = 1$ if $G = \mathrm{O}_n$ and -1 if $G = \mathrm{Sp}_n$.

Notice that H acts irreducibly on V due to the presence of τ; but the action is imprimitive. In fact $\{U, W\}$ is a system of imprimitivity.

6.5. Collection $\mathscr{C}_4$: tensor product subgroups

We start by recalling the following, [**41**, Proposition 2.2], which is an important tool in the definition of $\mathscr{C}_4$-subgroups. Recall that the *central product* of two groups H_1 and H_2 with same centre Z is $H_1 \circ H_2 = (H_1 \times H_2)/\{(z, z^{-1}) : z \in Z\}$.

Proposition 6.5.1. *Let f_i be a non-degenerate bilinear form on V_i, $i = 1, 2$ and set $V = V_1 \otimes V_2$. Then*

(i) There is a unique bilinear form $f = f_1 \otimes f_2$ on V such that

$$f(u_1 \otimes u_2, v_1 \otimes v_2) = f_1(u_1, v_1) f_2(u_2, v_2)$$

for all $u_i, v_i \in V_i$. Moreover f is preserved by $Cl(V_1) \circ Cl(V_2)$.

(ii) The form f is symmetric if, and only if, f_1, f_2 are either both symmetric or both alternating, and f is alternating otherwise.

(iii) If $p = 2$ then there is a unique quadratic form Q on V such that $f_Q = f$ and $Q(u_1 \otimes u_2) = 0$ for all $u_i \in V_i$, and Q is preserved by $\mathrm{Sp}(V_1) \circ \mathrm{Sp}(V_2)$.

Now we can define the $\mathscr{C}_4$-subgroups. There are two cases:

Case 1. We say that H is a $\mathscr{C}_4$-subgroup if there exists a decomposition $V = V_1 \otimes V_2$ with $\dim V_1 = a$, $\dim V_2 = b$ and $1 < a < b$, that is stabilised by H. Then $H = Cl(V_1) \circ Cl(V_2)$ acts naturally on the tensor product. We use the notation $H = Cl(V_1) \otimes Cl(V_2)$. By Proposition 6.5.1 we only have the following possibilities

$$\mathrm{GL}_a \otimes \mathrm{GL}_b < \mathrm{GL}_{ab},\ \mathrm{Sp}_a \otimes \mathrm{O}_b < \mathrm{Sp}_{ab}\ (p \neq 2)$$

$$\mathrm{Sp}_a \otimes \mathrm{Sp}_b < \mathrm{O}_{ab},\ \mathrm{O}_a \otimes \mathrm{O}_b < \mathrm{O}_{ab}\ (p \neq 2)$$

Notice that the central product assures the action of H on V to be faithful.

Case 2. We say that H is a $\mathscr{C}_4$-subgroup if there exists a decomposition $V = \bigotimes_{i=1}^t V_i$, with $t > 1$ and V_i, V_j isometric for all i, j, stabilised by H. In particular $H = N_G(\prod_i Cl(V_i))$ acting naturally on the tensor product. The possibilities for $\prod_i Cl(V_i)$

are given by

$$\bigotimes \mathrm{GL}_a < \mathrm{GL}_{a^t}, \ \bigotimes \mathrm{Sp}_a < \mathrm{Sp}_{a^t} \ (t \text{ odd}, p \neq 2)$$
$$\bigotimes \mathrm{Sp}_a < \mathrm{O}_{a^t} \ (t \text{ even or } p = 2), \ \bigotimes \mathrm{O}_a < \mathrm{O}_{a^t} \ (p \neq 2, \ \dim V_i \neq 2, 4)$$

The conditions are given in order to assure that the subgroups are maximal, for example in the last case if $p = 2$ then $\bigotimes \mathrm{O}_a < \bigotimes \mathrm{Sp}_a < \mathrm{O}_{a^t}$. We refer to the beginning of [**41**, Section 1] for more details on the conditions.

6.6. Collection $\mathscr{C}_6$: classical subgroups

Let $G = Cl(V)$. Here we define $H \leq G$ to be in the $\mathscr{C}_6$-family if H fixes a non-degenerate form on V. Therefore either $G = \mathrm{GL}_n$ and $H = \mathrm{Sp}_n$ or O_n or, $(G, p) = (\mathrm{Sp}_n, 2)$ and $H = \mathrm{O}_n$. We shall give more details on these subgroups in Chapter 9.

CHAPTER 7

Fixed point spaces

In this final background chapter we study the fixed point spaces that arise in the action of algebraic groups on coset varieties. We start in full generality relating fixed point spaces of abstract group actions with conjugacy classes. Then we concentrate on actions of algebraic groups. We give an explicit formula to compute the dimension of a fixed point space, which will be the key tool in our subsequent analysis.

7.1. Preliminaries

Let G be a group and $H \leq G$ be a subgroup. Consider the standard action of G on the coset space $\Omega = G/H$. For $x \in G$ we have defined in (6) the fixed point set of x:

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\}$$

First we observe that $C_\Omega(x)$ is non-empty if, and only if, x^G meets H.

Lemma 7.1.1. *Let $x \in G$. Then $C_\Omega(x) \neq \emptyset$ if, and only if, $x^G \cap H \neq \emptyset$.*

Proof. Let $g \in G$. Then $x^g \in x^G \cap H \Leftrightarrow x^g \in H \Leftrightarrow xgH = gH \Leftrightarrow gH \in C_\Omega(x)$. The result follows. *q.e.d.*

Moreover conjugates of any element have isomorphic (as sets) fixed point sets.

Lemma 7.1.2. *Let $x \in G$. Then for all $g \in G$ we have $C_\Omega(x) \cong C_\Omega(x^g)$.*

Proof. Let $\omega = yH \in \Omega$ and $g \in G$. Then $\omega \in C_\Omega(x)$ if, and only if, $x.\omega = xyH = yH$. Therefore $\omega \in C_\Omega(x)$ if, and only if, $y^{-1}xy = (g^{-1}y)^{-1}x^g(g^{-1}y) \in H$, which is equivalent to $g^{-1}yH \in C_\Omega(x^g)$. Thus $\omega \in C_\Omega(x)$ if, and only if, $g^{-1}\omega \in C_\Omega(x^g)$. Therefore $\varphi\colon C_\Omega(x) \to C_\Omega(x^g)$, $\omega \mapsto g^{-1}\omega$ is a bijective map. *q.e.d.*

It is clear that for any integer n we have

$$C_\Omega(x) \subseteq C_\Omega(x^n) \tag{25}$$

In particular given any $x \in G$ of finite order there exists a prime order element $y \in G$ for which $C_\Omega(x) \subseteq C_\Omega(y)$. For example if x has order $m = qn$ and q is prime then we can choose $y = x^n$. We shall state a similar property, when G is an algebraic group, in the case x is semisimple of infinite order in Lemma 7.1.7.

In the following we show that multiplying by a central element of H does not change the fixed point set.

Lemma 7.1.3. *Let $x \in H$. Then for all $\lambda \in Z(H)$, $C_\Omega(x) = C_\Omega(\lambda x)$.*

Proof. Let $\lambda \in Z(H)$ and $\omega = gH \in \Omega$. Then $\lambda\omega = \lambda gH = g\lambda H = gH = \omega$.

Let $\lambda \in Z(H)$. Then $\omega \in C_\Omega(x)$ if, and only if, $\omega = x.\omega = x.\lambda\omega = \lambda x.\omega$, which is equivalent to $\omega \in C_\Omega(\lambda x)$. The result follows. *q.e.d.*

We are interested in studying fixed point spaces in actions of algebraic groups.

7.1.1. Algebraic groups. For the reminder of the chapter the notation is as follows.

> Let G be an affine algebraic group defined over an algebraically closed field k of characteristic p and let H be a closed subgroup of G. Let $\Omega = G/H$ be the corresponding coset variety.

Remark 7.1.4. Notice that, for algebraic groups, Lemma 7.1.2 gives an isomorphism as varieties since $\varphi\colon \omega \mapsto g^{-1}w$ and $\varphi^{-1}\colon \omega \mapsto\ g\omega$ are morphisms of varieties.

Let $x \in G$. Then, by Theorem 2.3.3, $x = x_s x_u = x_u x_s$, where x_s is semisimple and x_u is unipotent. The next result relates the fixed point space of x with the fixed point spaces of its semisimple and unipotent parts.

Proposition 7.1.5. *Let $x \in G$ with Jordan decomposition $x = x_s x_u$. Then $C_\Omega(x) = C_\Omega(x_s) \cap C_\Omega(x_u)$.*

Proof. Assume $\omega \in C_\Omega(x_s) \cap C_\Omega(x_u)$ then $x.\omega = x_s.(x_u.\omega) = \omega$. Hence $\omega \in C_\Omega(x)$.

Conversely, let $\omega \in C_\Omega(x)$, so that $x.\omega = \omega$. For some $g \in G$ we have $\omega = gH$. Then, as in Lemma 7.1.1, $x.\omega = \omega$ is equivalent to $x^g \in H$. In particular $g^{-1}x_s x_u g = x_s^g x_u^g \in H$. Since x_u and x_s commute we also have $x_u^g x_s^g \in H$. Therefore $x_u^g \in H$ if, and only if, $x_s^g \in H$. Recall that x_s^g is semisimple and x_u^g is unipotent, see Remark 2.3.1. Since H is an algebraic group, by Theorem 2.3.3, there exist $s, u \in H$ such that s is semisimple, u is unipotent and $x^g = su = us$. If $s \neq x_s^g$ or $u \neq x_u^g$ we would have two different Jordan decompositions of x^g which contradicts the uniqueness of Theorem 2.3.3. Therefore $x_u^g, x_s^g \in H$ and so $\omega \in C_\Omega(x_s) \cap C_\Omega(x_u)$. *q.e.d.*

Recall that Proposition 3.1.1 implies that fixed point spaces are subvarieties of Ω.

The following is an immediate consequence of Proposition 7.1.5.

Corollary 7.1.6. *Let $x \in G$ with Jordan decomposition $x = x_s x_u$. Then*

$$\dim C_\Omega(x) \leq \min\{\dim C_\Omega(x_s), \dim C_\Omega(x_u)\}$$

In particular, from Corollary 7.1.6 we deduce that in order to derive upper bounds on $\dim C_\Omega(x)$ we may always assume that x is either semisimple or unipotent.

In Section 7.1 we observed that if $x \in G$ has finite order then there exists a prime order element $y \in G$ such that $C_\Omega(x) \subseteq C_\Omega(y)$. The following result [**11**, Section 8] shows that a similar property holds for semisimple elements of infinite order, provided H is reductive.

Lemma 7.1.7. *Assume $H \leq G$ is a reductive closed subgroup. Let $x \in G$ be a semisimple element of infinite order. Then there exists $y \in H$ semisimple of finite order such that $C_\Omega(x) \subseteq C_\Omega(y)$.*

As seen in Lemmas 7.1.1 and 7.1.2 there is a link between the conjugacy class of x and $C_\Omega(x)$. Recall Lemma 7.1.1, if $x \in G$ is such that $x^G \cap H = \emptyset$ then $C_\Omega(x) = \emptyset$; in particular, $\dim C_\Omega(x) = 0$. The following [**39**, Proposition 1.14] is a key result we shall use several times in this thesis.

Proposition 7.1.8. *Let $x \in G$. Assume $x^G \cap H \neq \emptyset$. Then*

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$$

Proof. Let us define

$$V = \{(g, \omega) \in G \times \Omega : g.\omega = \omega\}$$

We claim that V is a closed subset of $G \times \Omega$. In order to prove this we define $\pi\colon G \times \Omega \to \Omega$, the projection $\pi(g, \omega) = \omega$; and $\varphi\colon G \times \Omega \to \Omega$, the action map $\varphi(g, \omega) = g.\omega$. Then π and φ are morphisms of varieties. Notice that both π and φ are surjective. Therefore $V = \{(g, \omega) \in G \times \Omega : \varphi(g, \omega) = \pi(g, \omega)\}$ is closed in $G \times \Omega$ by Lemma 1.5.2.

Since $x^G \cap H \neq \emptyset$, thanks to Lemma 7.1.2, we may assume $x \in H$. Define

$$V_x = \{(x^g, \omega) : g \in G \text{ and } x^g.\omega = \omega\} \subseteq x^G \times \Omega$$

Then V_x is a subvariety of V, by the same argument given above: $V_x = \{(x^g, \omega) : \varphi(x^g, \omega) = \pi(x^g, \omega)\}$.

Consider the projection on the first coordinate $\pi_1\colon V_x \to x^G$, $(x^g, \omega) \mapsto x^g$. Notice that π_1 is surjective since $(x^g, g^{-1}H) \in V_x$ for all $g \in G$, as $x^g g^{-1}H = g^{-1}xH = g^{-1}H$ (recall $x \in H$). Moreover

$$\pi_1^{-1}(x^g) = \{(x^g, \omega) : x^g.\omega = \omega\} \cong C_\Omega(x^g)$$

In particular $\dim \pi_1^{-1}(x^g) = \dim C_\Omega(x^g)$. Hence, by Lemma 7.1.2, $\dim \pi_1^{-1}(x^g) = \dim C_\Omega(x)$ for all $g \in G$. Hence, Proposition 1.4.2 yields

$$\dim V_x = \dim x^G + \dim C_\Omega(x)$$

Now consider the projection on the second coordinate $\pi_2\colon V_x \to \Omega$, $(x^g, \omega) \mapsto \omega$, which is surjective since $(x^{g^{-1}}, gH) \mapsto gH$ for all $g \in G$. Let $yH \in \Omega$. Then

$$f^{-1}(yH) = \{(x^g, yH) : x^g yH = yH\}$$

Notice that $x^g yH = yH$ if, and only if, $x^{gy} \in H$. Hence there exists a one to one correspondence $f^{-1}(yH) \cong x^G \cap H$. Again, Proposition 1.4.2 implies

$$\dim V_x = \dim \Omega + \dim(x^G \cap H)$$

The result follows. *q.e.d.*

Remark 7.1.9. Assume $H \leq G$ is a finite subgroup. Set $\Omega = G/H$. Then $\dim \Omega = \dim G$. In particular, thanks to Proposition 7.1.8, for any $x \in H$, $\dim C_\Omega(x) = \dim G - \dim x^G$. Therefore the bounds on $\dim x^G$ stated in Proposition 5.4.1 may be viewed as bounds on the fixed point space $C_\Omega(x)$.

As mentioned in the Introduction, a motivation of this project arises from finite groups. Let $G \leq \mathrm{Sym}(\Omega)$ be a finite permutation group on the finite set Ω. Then for

$x \in G$ the *fixed point ratio* is defined to be the proportion of points in Ω fixed by x:

$$\mathrm{fpr}_\Omega(x) = \frac{|C_\Omega(x)|}{|\Omega|} \tag{26}$$

Assume the action of G on Ω is transitive then all the point stabiliser are G-conjugate, say $H = G_\omega$ one of them. There is a one to one correspondence $\Omega \cong G/H$ preserved by G. Moreover, considering the set $V = \{(g, \omega) \in x^G \times \Omega : g.\omega = \omega\}$ and counting its order in two different ways we deduce

$$\mathrm{fpr}_\Omega(x) = \frac{|x^G \cap H|}{|x^G|}$$

Notice the similarity with the formula proved in Proposition 7.1.8.

Remark 7.1.10. Often results for algebraic groups can be translated for finite groups of Lie type (an example is the Aschbacher theorem deduced from the Liebeck–Seitz theorem on the subgroup structure of classical algebraic groups, cf. Section 6.1). We list some more examples. The bounds on the fixed point ratios in actions of finite exceptional groups established by Lawther, Liebeck and Seitz in [**38**] used bounds on the fixed point spaces in actions of simple exceptional algebraic groups [**39**]. Also, the work of Burness [**12, 13, 14, 15**] used results for algebraic groups, for example the main theorem of [**11**].

The following is a consequence of Lemma 7.1.7.

Corollary 7.1.11. *Assume $p > 0$. Let $H \leq G$ be reductive. Assume G acts on $\Omega = G/H$. Let $x \in G$. Then there exists $y \in G$ of prime order such that* $\dim C_\Omega(x) \leq \dim C_\Omega(y)$.

We will be interested in the case $G = Cl(V)$, $H \in \mathscr{C}_i$ and $x \in H$ a prime order element. The aim will be to derive bounds on $\dim C_\Omega(x)$ using Proposition 7.1.8 as the main tool. By the definitions of G and H given in Chapters 4 and 6, $\dim G$ and $\dim H$ are easy to compute, so $\dim \Omega = \dim G - \dim H$ quickly follows. By results in Chapter 5, we can also compute $\dim x^G$. The main challenge is estimating $\dim(x^G \cap H)$. In the next section we shall give properties on this intersection.

7.2. Intersection of conjugacy classes with subgroups

Let $x \in G$. Then there exists a family $\{x_i\}_{i \in \mathcal{I}} \subseteq H$, where $x_i^H \neq x_j^H$ for $i \neq j$, such that $x^G \cap H = \bigcup_i x_i^H$. In general the union is not finite, as we show in the following.

Example 7.2.1. Recall the notation of Example 2.2.5, $T_n, U_n \leq \mathrm{GL}_n$ are, respectively, the subgroups of upper triangular and upper uni-triangular matrices of GL_n. Let $G = T_3$ and $H = U_3 \leq G$. Notice that G is a unipotent group, i.e. $G = G_u$. For any $\alpha \in k$ we define

$$x_\alpha = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H$$

We compute

$$\begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix}^{-1} x_\alpha \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} = x_{f\alpha/a}$$

Therefore $x_1^G = \{x_\alpha : \alpha \in k^*\}$. In addition, for all $h \in H$ we have $h^{-1}x_\alpha h = x_\alpha$. Thus $x_\alpha^H = \{x_\alpha\}$. Therefore

$$x_1^G \cap H = \bigcup_{\alpha\in k^*} x_\alpha^H = \bigcup_{\alpha\in k^*} \{x_\alpha\}$$

In particular the union is not finite.

In the case $G = \mathrm{GL}_n$ and H is a parabolic subgroup (defined in (24)) then the union may be not finite, for example see [**46**, Corollary 7.1].

The following is [**27**, Theorem 1.2].

Theorem 7.2.2. *Let $x \in G$. Assume $H \leq G$ are both reductive. Then $\mathcal{I}$ is finite.*

Remark 7.2.3. In Theorem 7.2.2 the groups are not necessary connected. In fact in [**27**] the author defines a group to be reductive if its connected component is reductive.

The following is an immediate consequence of Theorem 7.2.2.

Corollary 7.2.4. *Let $x \in G$ and $H \leq G$ both reductive. Then* $\dim(x^G \cap H) = \max_i\{\dim x_i^H\}$.

The key tool for the proof of Theorem 7.2.2 is the finiteness of number of conjugacy classes of unipotent elements in a reductive connected algebraic group. This was a long standing problem: Steinberg [**55**] asked whether the number of such classes is always finite. An affirmative answer was given by the works of Dynkin and Konstant [**37**] (for the characteristic 0 case), Richardson [**49**] (for the case $p \geq 0$ apart few special cases, e.g. $p = 2$ if $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$) and, finally, Lusztig [**44**].

Now assume $G = \mathrm{SL}(V), \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ is a simple classical algebraic group. Let $H \leq G$ be a maximal closed subgroup which is not a subspace subgroup, e.g. $H \notin \mathscr{C}_1$. Burness [**11**, Theorem 1] gave upper bounds on $\dim(x^G \cap H)$ for elements $x \in G$ of prime order or any unipotent element in the characteristic 0 case. The next result is a particular case of [**11**, Theorem 1]; in fact in [**11**] the statement is more general: it also considers subgroups in $\mathscr{S}$. In the following the definition of the $\mathscr{C}_i$ families is the same as in [**41**].

Theorem 7.2.5. *Let G be a simple classical algebraic group. Let $x \in G \setminus Z(G)$ be an element of prime order, or an arbitrary unipotent element in the characteristic zero case. Let $H \in \bigcup_{i>1} \mathscr{C}_i$ and assume $(G, H, p) \neq (\mathrm{Sp}_n, \mathrm{O}_n, 2)$. Then*

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leq \frac{1}{2} + \epsilon$$

where $\epsilon = 0$, or (G, H°, ϵ) lies in Table 7.2.1.

Notice it is possible to show that Theorem 7.2.5 holds in the case $G = \mathrm{GL}_n$ or O_n, as well (with the definition of the $\mathscr{C}_i$ families as in Chapter 6).

G	Type of H	Collection	ϵ
Sp_n	$\mathrm{Sp}_{n/2} \wr S_2$	$\mathscr{C}_2$	$1/n$
SO_n	$\mathrm{GL}_{n/2}.2$	$\mathscr{C}_3$	$1/(n-2)$
SL_n	Sp_n	$\mathscr{C}_6$	$1/n$

Table 7.2.1

7.3. Aim

In this last section we state the main aims of the thesis. We define the following set

(27) $$\mathscr{R} = \mathscr{R}(G) = \{x \in G : o(x) \text{ is prime, or } x \text{ is unipotent in characteristic } 0\}$$

General aim. Let G be an affine algebraic group and $H \leq G$ be a closed subgroup. Set $\Omega = G/H$. For $x \in \mathscr{R}$ of order r we aim to study the following ratio (which provides a natural analogue of the fixed point ratio (26) for finite permutation groups):

(28) $$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega}$$

Fixed point spaces in actions of exceptional groups are widely studied. In [**39**], Lawther, Liebeck and Seitz derived upper bounds on the dimension of these spaces in primitive actions of exceptional algebraic groups. As already mentioned in Remark 7.1.10, the work done in [**39**] has been an important tool in order to derive upper bounds on the fixed point ratio for finite simple groups of exceptional type, see [**38**]. For more motivations we refer to the Introduction.

As a matter of notation, we set $p = \infty$ in the characteristic zero case.

With the notation introduced in the previous chapters we can state the precise aims of this thesis. For the remainder of the chapter we fix the following notation.

> Let $G = Cl(V)$ and $H \in \mathscr{C}_i$ (as defined in Chapter 6). Let $\Omega = G/H$ be the corresponding coset variety. Let $x \in G \setminus Z(G)$ be an element of prime order r or any unipotent element if $p = \infty$.

7.3.1. Aim 1. Global bounds.

The first aim is to derive *global bounds* on $f_\Omega(x)$.

Aim 1.1. Find functions $f(G, H, r), g(G, H, r)$ such that

(29) $$g(G, H, r) \leq f_\Omega(x) \leq f(G, H, r)$$

This would extend the main results of [**39**] to classical groups. In addition this would be the first study on lower bounds on the dimension of fixed point spaces.

The goal is to derive the best possible bounds on $f_\Omega(x)$. Clearly either the bounds in (29) are sharp or they are not.

If the bounds in (29) are sharp we have the following.

Aim 1.2. Classify (or provide example of) the elements $z \in G$ of order r such that

$$f_\Omega(z) = \iota(G, H, r)$$

where ι ranges in the symbols $\{f, g\}$.

In the case the bounds in (29) are not sharp:

Aim 1.3. Show the existence of an element $z \in G$ of order r such that

$$|f_\Omega(z) - \iota(G, H, r)| < \epsilon$$

for a small $\epsilon > 0$, also here ι ranges in the symbols $\{f, g\}$.

In order to state the next aim we introduce some new notations.

Definition 7.3.1. For G and H as above and $\Omega = G/H$, we define the following numbers

$$M = \sup\{f_\Omega(x) : x \in G \setminus Z(G)\}$$
$$M_r = \sup\{f_\Omega(x) : o(x) = r\}$$
$$M_{r'} = \sup\{f_\Omega(x) : x \in \mathscr{R}, o(x) \neq r\}$$

As said in the Introduction, we call M the *algebraic fixity* and M_r the *r-local algebraic fixity.*

Thanks to Corollary 7.1.6 and Lemma 7.1.7 we have $M = \sup_r\{M_r\}$.

Aim 1.4. Find the best possible $\epsilon \geq 0$ such that for all prime r there exists $z \in \mathscr{R}$ for which

$$f_\Omega(z) \geq M - \epsilon$$

7.3.2. Aim 2. Local bounds. Recall that, for $x \in Cl(V)$, $\nu(x)$ is the codimension of the largest eigenspace of x on V, see Proposition 5.1.7. So $\nu(x)$ captures the *fixity* of the natural action of x on V. Therefore we expect the value of $f_\Omega(x)$ to be influenced by $\nu(x)$. Let r be a prime. We define the following sets

$$\mathcal{V}_s = \{x \in G : \nu(x) = s\}$$
$$\mathcal{V}_{s,r} = \{x \in \mathcal{V}_s : o(x) = r\}$$

Given a prime r, for $x \in \mathcal{V}_{s,r}$, the aims are as in Aim 1.2, 1.3 and 1.3. In this case we aim to find the best possible functions $f_\nu(G, H, s)$ and $g_\nu(G, H, s)$.

As anticipated in the introduction, the only similar study has been done by Frohardt and Magaard [**21**] for certain actions of finite classical groups.

7.3.3. Aim 3. Involutions. Let $x \in G$ be an involution with $\nu(x) = s$. The main aim is the following.

Aim 3. Find an explicit function $f_2(G, H, s)$ and the best possible $\epsilon \geq 0$ such that

$$f_2(G, H, s) \leq f_\Omega(x) \leq f_2(G, H, s) + \epsilon$$

Part 2

$\mathscr{C}_6$-actions of classical groups

CHAPTER 8

Introduction

Let $G = \mathrm{GL}_n$ or Sp_n, $n > 2$, defined over an algebraically closed field of characteristic p. A closed subgroup $H \leq G$ is a $\mathscr{C}_6$-subgroup if it fixes a certain form on the natural module of G: we have the following possibilities $(G, H) = (\mathrm{GL}_n, \mathrm{Sp}_n)$ or $(\mathrm{GL}_n, \mathrm{O}_n)$, or $(\mathrm{Sp}_n, \mathrm{O}_n)$ when $p = 2$. Set $\Omega = G/H$. In this chapter we state the main results of this part, in which we derive bounds on $f_\Omega(x)$ for $x \in H$ of prime order or any unipotent element in the characteristic zero case.

8.1. Main results

Recall from (27) the definition of $\mathscr{R}$ as the set of elements of G of prime order, or any unipotent element in characteristic zero. Recall that we denote $p = \infty$ in the characteristic zero case. Theorem 8.1.1 below provides upper and lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order.

Theorem 8.1.1. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_6$-subgroup. Set $\Omega = G/H$.*

(a) Let $x \in G \setminus Z(G)$. Then, for $G = \mathrm{GL}_n$,

$$f_\Omega(x) \leq 1 - \frac{2}{n+1} + \frac{4}{n(n+1)}$$

If $G = \mathrm{Sp}_n$ then $f_\Omega(x) \leq 1 - 1/n$.

(b) Let $x \in H \cap \mathscr{R}$ be an element of prime order r.

(i) If $r = p$ then

$$f_\Omega(x) \geq \begin{cases} \frac{1}{p} & p < n \\ \frac{1}{n} & p \geq n \end{cases}$$

(ii) If $G = \mathrm{GL}_n$ and $r \neq p$ then

$$f_\Omega(x) \geq \begin{cases} \frac{1}{r} - \frac{1}{n(n-1)} & r < n \\ \frac{1}{n+1} & r \geq n \end{cases}$$

(iii) If $G = \mathrm{Sp}_n$ and $r \neq 2$ then $f_\Omega(x) = 0$ if, and only if, $\dim C_V(x) = 0$.

Remark 8.1.2. Let us make some comments on the statement of Theorem 8.1.1.

(i) Referring to part (a), notice that the bound does not depend on r and p. Moreover this bound is the best possible: indeed for $(G, H, p) = (\mathrm{GL}_n, \mathrm{O}_n, 2)$ and $x = [J_2, J_1^{n-2}]$ or $x = [J_2^2]$ we have $f_\Omega(x) = 1 - \frac{2}{n+1} + \frac{4}{n(n+1)}$. Studying separately the different cases we derive the best possible bounds, characterising elements that realise them. For $(G, H) = (\mathrm{GL}_n, \mathrm{Sp}_n)$ see Proposition 10.1.1, for $(\mathrm{GL}_n, \mathrm{O}_n)$ see Proposition 11.1.1, finally for $(\mathrm{Sp}_n, \mathrm{O}_n)$ we refer to Proposition 9.4.1. In general, elements that realise the bound are $x \in H$ such that

$\nu(x)$ is minimal, i.e. $\nu(x) = 1$ or 2. Notice that elements with this property always exist, for any r and p.

(ii) The bounds in (b) extend to any prime order element x for which $x^G \cap H \neq \emptyset$.

(iii) Regarding the lower bounds in (b)(i): studying separately the different cases for $G = \mathrm{GL}_n$ or Sp_n we can give slightly better bounds, we refer to Proposition 10.2.1 for $(G, H) = (\mathrm{GL}_n, \mathrm{Sp}_n)$, Proposition 11.2.1 if $(G, H) = (\mathrm{GL}_n, \mathrm{O}_n)$ and Proposition 9.4.3 if $(G, H) = (\mathrm{Sp}_n, \mathrm{O}_n)$. The bounds stated in the aforementioned results are sharp and we also characterise elements which realise them; often the existence of these elements is subjected to a divisibility condition on n and p. In the general case we show that the bounds are close to best possible by constructing an explicit element whose f_Ω-value is close to the bound.

(iv) The lower bounds in (b)(ii) are close to best possible. For example if $(G, H) = (\mathrm{GL}_n, \mathrm{O}_n)$, with n even, and $x \in H$ is such that $\nu(x) = n - 1$ then $f_\Omega(x) = \frac{1}{n+1}$. The best possible bounds are derived in Corollaries 10.3.7 and 11.3.5 for $H = \mathrm{Sp}_n$ and O_n, respectively. We shall construct a family of elements that we call *special*, see Definition 10.3.2, and we prove that lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order $r \neq p$ are realised by the f_Ω-value of any special element.

Recall Definition 7.3.1 of the algebraic fixity M, the r-local algebraic fixity M_r and of $M_{r'}$.

Theorem 8.1.3. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_6$-subgroup.*

(i) If $(G, H) = (\mathrm{GL}_n, \mathrm{Sp}_n)$ then

$$M = 1 - \frac{2(n-2)}{n(n-1)}, \ \textit{if } p \neq 2, \ M = 1 - \frac{4(n-3)}{n(n-1)}, \ \textit{if } p = 2$$

(i) If $(G, H) = (\mathrm{GL}_n, \mathrm{O}_n)$ then

$$M = 1 - \frac{2(n-1)}{n(n+1)}, \ \textit{if } p \neq 2, \ M = 1 - \frac{2(n-2)}{n(n+1)}, \ \textit{if } p = 2$$

(iii) If $(G, H) = (\mathrm{Sp}_n, \mathrm{O}_n)$ then

$$M = 1 - \frac{1}{n}$$

Theorem 8.1.4. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_6$-subgroup.*

(i) Assume $(G, H) = (\mathrm{GL}_n, \mathrm{Sp}_n)$. Then $M = M_p$ and $M > M_{p'}$;

(ii) Assume $(G, H) = (\mathrm{GL}_n, \mathrm{O}_n)$. Then $M = M_2$ and $M > M_{2'}$;

(iii) Assume $(G, H, p) = (\mathrm{Sp}_n, \mathrm{O}_n, 2)$. Then $M = M_2$ and $M > M_{2'}$.

Moreover in any case $M_{p'}, M_{2'} \geq 1 - 2/n$.

Remark 8.1.5. In the analysis we explicitly compute M_r for any r. We refer to the results mentioned in Remark 8.1.2(i).

Corollary 8.1.6. *For any prime r, there exists $x \in H$ of order r such that*

$$f_\Omega(x) \geq 1 - \frac{2}{n}$$

Let $x \in G$, recall the definition of $\nu(x)$ to be the codimension of the largest eigenspace in the action of x on V, the natural module of G, see Proposition 5.1.7. In the proofs of the results that lead to Theorem 8.1.1 and in the result on which we characterise the elements that satisfy the bounds we see a strong relationship between $f_\Omega(x)$ and $\nu(x)$. Roughly, $f_\Omega(x)$ is maximal when $\nu(x)$ is minimal and vice-versa. Recall that $\mathcal{V}_s = \{x \in G : \nu(x) = s\}$, in addition we denote $\mathcal{V}_{s,r} = \{x \in \mathcal{V}_s : o(x) = r\}$.

Theorems 8.1.7 and 8.1.9 provide upper and lower bounds on $f_\Omega(x)$ depending on $\nu(x)$ for $x \in \mathcal{V}_s$. We deduce the following from the main results of Sections 10.4 and 11.4.

Theorem 8.1.7. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_6$-subgroup. Assume $r \neq 2$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n} + \frac{1}{n}$$

Remark 8.1.8. We refer to Sections 10.4 and 11.4 for more details. We shall study separately unipotent and semisimple elements for $H = \mathrm{Sp}_n$ or O_n in GL_n. In the case x has prime order $r \neq p$ we shall define elements z, z' in (41) and (42), for $H = \mathrm{Sp}_n$, or (61) and (62) for $H = \mathrm{O}_n$. Then in Propositions 10.4.10 and 11.4.4 we prove $f_\Omega(x) \leq \max\{f_\Omega(z), f_\Omega(z')\}$ for any $x \in H \cap \mathcal{V}_{s,r}$. See Proposition 9.4.5 for $G = \mathrm{Sp}_n$.

The following, instead, follows from the main results of Sections 10.5 and 11.5.

Theorem 8.1.9. *Let $G = \mathrm{GL}_n$ and $H \leq G$ be a $\mathscr{C}_6$-subgroup. Assume $r \neq 2$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{2s(n-s)}{n(n-1)} - \frac{2}{n}$$

It is interesting to note that the f_Ω-values of elements in $H \cap \mathcal{V}_s$ are close to each other.

Corollary 8.1.10. *Let $G = \mathrm{GL}_n$. Let $x, y \in H \cap \mathcal{V}_{s,r}$, with $r \neq 2$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{2s(n-s)}{n(n-1)} + \frac{4}{n-1}$$

Remark 8.1.11. Corollary 8.1.10 is a straightforward consequence of the local upper and lower bounds, see Sections 10.6 and 11.6

In particular, we deduce the following.

Corollary 8.1.12. *Let $G = \mathrm{GL}_n$. Let $x, y \in H \cap \mathcal{V}_{s,r}$ and assume $s \leq \sqrt{n}$ or $s \geq n - \sqrt{n}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{2}{\sqrt{n}} + \frac{4}{n-1}$$

Remark 8.1.13. Observe that for lower bounds on $f_\Omega(x)$ we do not consider the case $G = \mathrm{Sp}_n$ with $p = 2$. In fact in this case Propositions 9.4.4 and 9.4.5 provide all the information we need. Let $x \in H \cap \mathcal{V}_{s,r}$. If $s < n/2$ then $f_\Omega(x) = 1 - \frac{s}{n}$, if $s \geq n/2$ then $f_\Omega(x) = 0$ if, and only if, $\dim C_V(x) = 0$.

For $\mathscr{C}_6$-actions it is very easy to compute an explicit formula for $f_\Omega(x)$, for $x \in G$ of any prime order r. If $x \in G$ is an involution Theorem 8.1.14 provides an explicit formula for $f_\Omega(x)$. For the purpose of the next result, we introduce the following notation

$$\epsilon = \begin{cases} -1 & H = \mathrm{Sp}_n \\ 1 & H = \mathrm{O}_n \end{cases}$$

Theorem 8.1.14. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_6$-subgroup. Let $x \in G$ be an involution with $\nu(x) = s \leq n/2$.*

(i) If $G = \mathrm{GL}_n$ then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n+\epsilon)} + \frac{2}{n(n+\epsilon)}\delta_{p,2}$$

(ii) If $G = \mathrm{Sp}_n$ then $p = 2$ and

$$f_\Omega(x) = 1 - \frac{s}{n}$$

CHAPTER 9

The $\mathscr{C}_6$-collection

9.1. Structure

Let V be a finite n-dimensional k-vector space and $G = \mathrm{GL}(V)$ or $\mathrm{Sp}(V)$. The elements of the $\mathscr{C}_6$-family are subgroups that fix an appropriate non-degenerate form on V.

$\mathscr{C}_6$ We say that $H \in \mathscr{C}_6$ if there exists a symplectic form or a non-degenerate quadratic form on V stabilised by H. Thus, here $G = \mathrm{GL}(V)$ and $H = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$; or $G = \mathrm{Sp}(V)$ with $p = 2$ and $H = \mathrm{O}(V)$.

We give a geometric interpretation of how these subgroups arise. We define

$$X_S = \{f \colon V \times V \to k : f \text{ is a symplectic form}\}$$
$$X_O = \{Q \colon V \to k : Q \text{ is a non-degenerate quadratic form}\}$$

Let $G = \mathrm{GL}(V)$ and $x \in G$. For $f \in X_S$ we define $x.f$ to be the bilinear form $V \times V \to k$ defined as $(x.f)(u,v) = f(x.u, x.v)$. Similarly, for $Q \in X_O$ we define $x.Q$ to be the map $V \to k$ defined as $(x.Q)(u) = Q(x.u)$. With these definitions, G acts on X_κ, where κ ranges in the symbols $\{S, O\}$. In Proposition 9.1.1 we show that these actions are transitive. Therefore, given a point stabiliser H, the action of G on X_κ is equivalent to the action of G on $\Omega = G/H$. In particular, given the one-to-one correspondence $X_\kappa \cong \Omega$, we can endow X_κ of the structure of a variety. It is clear that the stabiliser of an element in X_S, resp. X_O, is $\mathrm{Sp}(V)$, resp. $\mathrm{O}(V)$.

Proposition 9.1.1. *Let V be an n-dimensional k-vector space and define X_κ as above, where κ ranges in the symbols $\{S, O\}$.*

(i) Let $x \in \mathrm{GL}(V)$, $f \in X_S$ and $Q \in X_O$. Then $(x, f) \mapsto x.f$ and $(x, Q) \mapsto x.Q$ define transitive actions of $\mathrm{GL}(V)$ on X_κ.

(ii) If $p = 2$ then $\mathrm{Sp}(V) \times X_O \to X_O, (x, Q) \mapsto x.Q$ is a well defined transitive action.

Proof. Let $x \in \mathrm{GL}(V)$ and $f \in X_S$. It is clear that $x.f$ is a non-degenerate bilinear form, moreover, for all $u, v \in V$, $(x.f)(u,v) = f(x.u, x.v) = -f(x.v, x.u) = -(x.f)(v,u)$. Thus $x.f \in X_S$. Similarly one can show that, for $Q \in X_O$, $x.Q \in X_O$. In the case $p = 2$ then $\mathrm{Sp}(V)$ acts on X_O.

Moreover, these actions are transitive thanks to Corollaries 4.1.9 and 4.1.13. *q.e.d.*

Before giving more information on conjugacy classes in $\mathscr{C}_6$-subgroups we list the dimension of the variety $\Omega = G/H$ when H lies in the $\mathscr{C}_6$-family in Table 9.1.1.

G	H	$\dim \Omega$
GL_n	Sp_n	$\frac{n(n-1)}{2}$
GL_n	O_n	$\frac{n(n+1)}{2}$
Sp_n	O_n	n

Table 9.1.1. Dimensions of $\Omega = G/H$ for $H \in \mathscr{C}_6$

9.2. Conjugacy classes in $\mathscr{C}_6$-subgroups

Let $x \in G$ be of prime order r. If $r = p$ then x is unipotent and, up to G-conjugacy,

(30) $$x = [J_p^{a_p}, \dots, J_1^{a_1}]$$

The conditions on the a_i's such that $x \in H$ are listed in Theorem 5.2.1, together with the formula for $\dim x^H$. For involutions in the symplectic and orthogonal groups we refer the reader to Section 5.2.1.

If $r \neq p$, then x is semisimple and, up to G-conjugacy,

(31) $$x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$$

The conditions for which $x \in H$ are given in Theorem 5.3.1. Let us recall them here, we have $a_0 \equiv n \pmod 2$ and $a_i = a_{r-i}$ for all $i \leq \frac{r-1}{2}$. We also find the structure of $C_G(x)$ for these elements in the aforementioned result.

Proposition 9.2.1. *Let $x \in G$ be an element of prime order r.*

(i) If $(p,r) \neq (2,2)$ then $x^G \cap H = x^H$;
(ii) If $p = r = 2$ say $\nu(x) = s$. Then, $x^G \cap H = b_s^H$, for s odd, and $x^G \cap H = a_s^H \cup c_s^H$ if s is even.

Proof. Part (i) is straightforward since two elements are conjugate in Sp_n, or O_n if, and only if, they are conjugate in GL_n, see Theorems 5.2.1 and 5.3.1. Part (ii) follows from Section 5.2.1. *q.e.d.*

The following is a clear consequence of Proposition 9.2.1.

Corollary 9.2.2. *Let $x \in G$ be an element of prime order r.*

(i) If $(p,r) \neq (2,2)$ then $\dim(x^G \cap H) = \dim x^H$;
(ii) If $p = r = 2$ say $\nu(x) = s$. Then, $\dim(x^G \cap H) = \dim b_s^H$, for s odd, and $\dim(x^G \cap H) = \dim c_s^H$ if s is even.

9.3. Explicit f_Ω-formulae

In this case it is easy to establish explicit formulae for $f_\Omega(x)$ for elements $x \in H$ of prime order r. In the following we will assume $x \in H$. However notice that the same results holds for any $x \in G$ such that $x^G \cap H \neq \emptyset$.

9.3.1. $(G,H) = (\mathrm{GL}_n, \mathrm{Sp}_n)$**.** We have the following.

Proposition 9.3.1. *Let $x \in H$ be of prime order r.*

(i) If $r \neq p$ is odd and x is as in (31) then

$$f_\Omega(x) = \frac{\sum_{i\geq 0} a_i^2 - a_0}{n(n-1)}$$

(ii) If $r = 2 \neq p$ and $x = [I_{n-s}, -I_s]$ then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n-1)}$$

(iii) If $r = p \neq 2$ and x is as in (30) then

$$f_\Omega(x) = \frac{2\sum_{i<j} ia_ia_j + \sum_i ia_i^2 - \sum_{i \text{ odd}} a_i}{n(n-1)}$$

(iv) If $r = p = 2$ and $x = [J_2^s, J_1^{n-2s}]$ then

$$f_\Omega(x) = 1 - \frac{2s(n-s-1)}{n(n-1)}$$

Proof. We use Theorems 5.2.1, 5.3.1 and Proposition 5.2.5 for computing the dimension of conjugacy classes of the element considered. Thus the formulae listed in (i)–(iv) follow by easy calculations. *q.e.d.*

9.3.2. $(G, H) = (\mathrm{GL}_n, \mathrm{O}_n)$**.** We have the following.

Proposition 9.3.2. *Let $x \in H$ be of prime order r.*

(i) If $r \neq p$ is odd and x is as in (31) then

$$f_\Omega(x) = \frac{\sum_{i\geq 0} a_i^2 + a_0}{n(n+1)}$$

(ii) If $r = 2 \neq p$ and $x = [I_{n-s}, -I_s]$ then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n+1)}$$

(iii) If $r = p \neq 2$ and x is as in (30) then

$$f_\Omega(x) = \frac{2\sum_{i<j} ia_ia_j + \sum_i ia_i^2 + \sum_{i \text{ odd}} a_i}{n(n+1)}$$

(iv) If $r = p = 2$ and $x = [J_2^s, J_1^{n-2s}]$ then

$$f_\Omega(x) = 1 - \frac{2s(n-s-1)}{n(n+1)}$$

9.3.3. $(G, H) = (\mathrm{Sp}_n, \mathrm{O}_n)$**.** Here we assume $p = 2$. We have the following.

Proposition 9.3.3. *Let $x \in H$ be of prime order r.*

(i) If $r \neq 2$ and x is as in (31) then

$$f_\Omega(x) = \frac{a_0}{n}$$

(ii) If $r = 2$ then

$$f_\Omega(a_s) = f_\Omega(b_s) = f_\Omega(c_s) = 1 - \frac{s}{n}$$

From the formulae computed above we see that the case $(G, H) = (\mathrm{Sp}_n, \mathrm{O}_n)$ is the easiest to deal with. We give all the information in the next section.

9.4. Symplectic group

9.4.1. Global bounds. Assume $p = 2$. Let $G = \mathrm{Sp}_n$ and $H = \mathrm{O}_n$ be a $\mathscr{C}_6$-subgroup of G. Set $\Omega = G/H$. The following results are trivial consequences of the formulae given in Proposition 9.3.3.

Proposition 9.4.1. *Let $x \in G$ be of prime order r. Then*

$$f_\Omega(x) \leq 1 - \frac{\iota}{n}$$

where $\iota = 1$ if $r = 2$ and $\iota = 2$ otherwise. Moreover equality holds if, and only if, $\nu(x) = \iota$.

Remark 9.4.2. Notice that there is no semisimple element x in H with $\nu(x) = 1$ as it would have Jordan form $[\lambda I_{n-1}, \mu]$ for some $\lambda, \mu \in k^*$ and, according to the conditions of Theorem 5.3.1, this is not possible.

Proposition 9.4.3. *Let $x \in H$ be of prime order r. Then*

$$f_\Omega(x) \geq \frac{\delta_{r,2}}{n}$$

Furthermore, if $r = 2$ equality holds if, and only if, $\nu(x) = \frac{n}{2}$. If $r \neq 2$ equality holds if, and only if, $a_0 = 0$.

9.4.2. Local bounds. Assume $r \neq 2$. Let $x \in H \cap \mathcal{V}_{s,r}$. For $s < n/2$, up to G-conjugation, we have

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$

with $a_i \leq n - s$ for all i. Therefore $f_\Omega(x) = 1 - \frac{s}{n}$.

If $s \geq n/2$ then, up to G-conjugation and up to relabelling the eigenvalues, we have

$$x = \begin{cases} [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \\ [I_{a_0}, \omega I_{n-s}, \ldots, \omega^{r-2} I_{a_{r-2}}, \omega^{r-1} I_{n-s}] \end{cases}$$

with $a_i \leq n - s$ for all i, and for s odd $a_0 \leq n - s - 1$ being a_0 even. Notice that in the first case s must be even.

Therefore we deduce the following.

Proposition 9.4.4. *Let $x \in H \cap \mathcal{V}_{s,r}$.*

(i) If $s < n/2$ then

$$f_\Omega(x) = 1 - \frac{s}{n}$$

(ii) If $s \geq n/2$ then

$$f_\Omega(x) \leq 1 - \frac{s + 1 - \delta_{s;2}}{n}$$

with equality if, and only if, $a_0 = n - s - 1 + \delta_{s;2}$.

In the same way it is easy to deduce a lower bound.

Proposition 9.4.5. *Let $x \in H \cap \mathcal{V}_{s,r}$. Assume $s \geq n/2$. Then $f_\Omega(x) \geq 0$ with equality if, and only if $a_0 = 0$.*

CHAPTER 10

General linear group, $H = \mathrm{Sp}_n$

We consider the case $G = \mathrm{GL}_n$, $H = \mathrm{Sp}_n$ a $\mathscr{C}_6$-subgroup of G, with $n > 2$. Denote $\Omega = G/H$. The aim of this chapter is to derive bounds on $f_\Omega(x)$ for $x \in G$ of prime order, proving the results stated in Chapter 8.

10.1. Upper bound

In this section we shall prove the following result.

Proposition 10.1.1. *Let $x \in G$ be an element of prime order r.*

(i) If $r = p$ is odd then
$$f_\Omega(x) \leq 1 - \frac{2(n-2)}{n(n-1)}$$
with equality if, and only if, $\nu(x) = 1$ and $(n, \nu(x)) = (4, 2)$.

(ii) If $r \neq p$ is odd then either $n \neq 6$ if $r \neq 2$ and
$$f_\Omega(x) \leq 1 - \frac{4(n-2)}{n(n-1)}$$
with equality if, and only if, $\nu(x) = 2$ or one of the following holds:
- *$n = 4$ and $\nu(x) = 2$ or 3;*
- *$n = 8$ and either $\nu(x) = 2$ or $x = [\omega I_4, \omega^{-1} I_4]$.*

Or, $n = 6$ and $f_\Omega(x) \leq \frac{3}{5}$, with equality if, and only if $\nu(x) = 3$.

(iii) If $r = 2 \neq p$ then
$$f_\Omega(x) \leq 1 - \frac{4(n-2)}{n(n-1)}$$
with equality if, and only if, $\nu(x) = 2$.

(iv) If $r = p = 2$ then
$$f_\Omega(x) \leq 1 - \frac{4(n-3)}{n(n-1)}$$
with equality if, and only if, $\nu(x) = 2$.

Remark 10.1.2.

(i) Proposition 10.1.1 provides upper bounds on $f_\Omega(x)$ for $x \in G$ of prime order r. Notice that thanks to Lemma 7.1.1 and Corollary 7.1.11 these bounds extend to bounds on $f_\Omega(y)$ for any element $y \in G \setminus Z(G)$, in the sense that $f_\Omega(x)$ is at most the maximum of the bounds stated in Proposition 10.1.1.

(ii) Notice that Proposition 10.1.1 immediately implies Theorems 8.1.1, 8.1.3, 8.1.4 and Corollary 8.1.6.

Remark 10.1.3. Recall that for any $x \in G$, $x^G \cap H \neq \emptyset$ if, and only if, $C_\Omega(x) \neq \emptyset$, see Lemma 7.1.1. Moreover, by Lemma 7.1.2 fixed point spaces of conjugate elements are equidimensional. Hence in the following we will always assume $x \in H$ has prime order and we will work only with elements in Jordan normal form.

We prove Proposition 10.1.1 in a sequence of lemmas. We start with unipotent elements.

Lemma 10.1.4. *Let $x \in H$ be an element of order $p \neq 2$.*

(i) If $\nu(x) = 1$ then

$$f_\Omega(x) = 1 - \frac{2(n-2)}{n(n-1)}$$

(ii) If $\nu(x) = 2$ then

$$f_\Omega(x) = 1 - \frac{4(n-3)}{n(n-1)}$$

(iii) If $\nu(x) \geq 3$ then

$$f_\Omega(x) < 1 - \frac{2(n-2)}{n(n-1)}$$

In particular, the conclusions of Proposition 10.1.1 hold.

Proof. If $\nu(x) = 1$ then $x = [J_2, J_1^{n-2}]$ and the result follows by an easy calculation using the formula in Proposition 9.3.1(iii). Similarly if $\nu(x) = 2$ then either $x = [J_2^2, J_1^{n-4}]$ or $x = [J_3, J_1^{n-3}]$. Notice that the latter element does not belong to Sp_n. Again, with a straightforward calculation we deduce the result. Thus (i) and (ii) follow.

Let $x \in H$ with $\nu(x) \geq 3$. Then, by Proposition 5.4.1, if $n \geq 6$ we have $\dim x^G \geq 6(n-3) > n$. Using Theorem 7.2.5 we have

$$f_\Omega(x) \leq 1 - \frac{\left(\frac{1}{2} - \frac{1}{n}\right)\dim x^G}{\dim \Omega} = 1 - \frac{(n-2)\dim x^G}{n^2(n-1)} < 1 - \frac{2(n-2)}{n(n-1)}$$

If $n = 4$ and $\nu(x) = 3$ then $x = [J_4]$ and $f_\Omega(x) = \frac{1}{3} < \frac{2}{3} = 1 - \frac{2(n-2)}{n(n-2)}$. *q.e.d.*

In the following we prove Proposition 10.1.1 in the case $p = 2$ and $x \in H$ is an involution.

Lemma 10.1.5. *Let $x \in H$ be an involution and assume $p = 2$. Then*

$$f_\Omega(x) \leq 1 - \frac{2(n-2)}{n(n-1)}$$

with equality if, and only if, $x = [J_2, J_1^{n-2}]$ or, $n = 4$ and $x = [J_2, J_1^2], [J_2^2]$.

Proof. Proposition 9.3.1(iii) yields an explicit formula for $f_\Omega(x)$ for any involution x. Let $g(s) = 1 - \frac{2s(n-s-1)}{n(n-1)}$, differentiating $g(s)$ we see that $g(s) \leq g(1)$ for all $s \leq n/2$ and $g(s) = g(1)$ if, and only if $s = 1$ and $n > 4$ or, $n = 4$ and $s = 1, 2$. The result follows. *q.e.d.*

We continue the analysis with semisimple elements.

Lemma 10.1.6. *Let $x \in H$ be an element of odd prime order $r \neq p$.*

(i) If $\nu(x) = 2$ then

$$f_\Omega(x) = 1 - \frac{4(n-2)}{n(n-1)}$$

(ii) If $\nu(x) = s \in \{3, 4, n-1\}$ then $(n, C_G(x), f_\Omega(x))$ are listed in Table 10.1.1;

(iii) If $4 < \nu(x) < n-1$ *then*

$$f_\Omega(x) < 1 - \frac{4(n-2)}{n(n-1)}$$

In particular, the conclusions of Proposition 10.1.1 hold.

s	n	$C_G(x)$	$f_\Omega(x)$
3	4	$(\mathrm{GL}_1)^4$	1/3
	6	$(\mathrm{GL}_3)^2$	3/5
4	6	$(\mathrm{GL}_2)^2 \times \mathrm{GL}_1^2$	1/3
	8	$(\mathrm{GL}_4)^2$	4/7
	> 4	$\mathrm{GL}_{n-4} \times (\mathrm{GL}_1)^4$	$1 - \frac{8(n-3)}{n(n-1)}$
	> 4	$\mathrm{GL}_{n-4} \times (\mathrm{GL}_2)^2$	$1 - \frac{4(2n-7)}{n(n-1)}$
$n-1$	–	(GL_1^n)	$\frac{1}{n-1}$

Table 10.1.1

Proof. If $\nu(x) = 2$ then $x = [I_{n-2}, \omega, \omega^{-1}]$. An easy calculation, using Proposition 9.3.1(i), leads to the result.

If $\nu(x) = 3, 4, n-1$ an easy application of the formula in Proposition 9.3.1(i) leads to the values listed in Table 10.1.1.

Now assume $4 < \nu(x) < n-1$. Then, by Proposition 9.3.1(i), the result is equivalent to $\sum_i a_i^2 - a_0 < n^2 - 5n + 8$. Using $n = \sum_i a_i$ so that $n^2 = \sum_i a_i^2 + 2\sum_{i<j} a_i a_j$, we get

$$2\sum_{i<j} a_i a_j - 5n + a_0 + 8 > 0$$

For $\nu(x) = s$ we have two cases.

Case 1. We have $a_0 = n - s$ and $a_i \le n - s$ for all i. Thus $\sum_{i>0} a_i = s$ and

$$\begin{aligned} 2\sum_{i<j} a_i a_j - 5n + a_0 + 8 &= 2s(n-s) + 2\sum_{0<i<j} a_i a_j - 4n - s + 8 \\ &> 2ns - 2s^2 - 4n + 8 = g(s) \end{aligned}$$

where we used the trivial inequality $2\sum_{i<j} a_i a_j > \sum_i a_i = s$. We see that for $s \le \frac{n}{2}$, $g(s) \ge g(5) = 6n - 42$, which is positive if $n \ge 8$. In the case $\frac{n}{2} < s < n-1$ we have $g(s) \ge g(n-2) = 0$. We need to check the case $s = n-2$, for which the following argument applies: by Proposition 5.4.1 we have $\dim x^G \ge \frac{n(n-2)}{2} > 4n$, if $n > 10$. Using Theorem 7.2.5 we have

$$f_\Omega(x) \le 1 - \frac{\left(\frac{1}{2} - \frac{1}{n}\right)\dim x^G}{\dim \Omega} = 1 - \frac{(n-2)\dim x^G}{n^2(n-1)} < 1 - \frac{4(n-2)}{n(n-1)}$$

Notice the conditions $n \le 8$ and $s \le n/2$ imply $s \le 4$. The case $(n, s) = (10, 8)$ follows with an easy computation.

Case 2. We have $a_j = a_{r-j} = n - s$ for some $j \le \frac{r-1}{2}$, we may assume $j = 1$ (up to relabelling the eigenvalues) so that $a_1 = a_{r-1} = n - s$ and $a_i \le n - s$ for all i. Hence we have

$$\begin{aligned}2\sum_{i<j} a_i a_j - 5n + a_0 + 8 &= 4s(n-s) + 2(n-s)^2 + 2\sum_{\substack{i<j,\\ i,j\neq 1,r-1}} a_i a_j - 5n + a_0 + 8\\ &\geq 4s(n-s) + 2(n-s)^2 - 5n + 8 = 2n^2 - 2s^2 - 5n + 8\\ &\geq 2n^2 - 2(n-2)^2 - 5n + 8 = 3n\end{aligned}$$

where the last inequality follows from $s \leq n-2$. The result follows. *q.e.d.*

The last step in order to have a complete proof of Proposition 10.1.1 is to establish an upper bound on $f_\Omega(x)$ when x is an involution and $p \neq 2$.

Lemma 10.1.7. *Assume $p \neq 2$ and let $x \in H$ be an involution. Then*

$$f_\Omega(x) \leq 1 - \frac{4(n-2)}{n(n-1)}$$

with equality if, and only if, $\nu(x) = 2$.

Proof. By Proposition 9.3.1(ii) and (iv), for any involution $x \in H$ with $\nu(x) = s \leq n/2$ we have $f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n-1)}$. By a straightforward computation we see that, as a function of s, $f_\Omega(x)$ is monotonically decreasing in $[2, n/2]$. Therefore, if $p \neq 2$ for any involution $x \in H$ we have $f_\Omega(x) \leq f_\Omega([I_2, -I_{n-2}]) = 1 - \frac{4(n-2)}{n(n-1)}$. *q.e.d.*

10.2. Unipotent elements: lower bound

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H$ of order p. Recall that we set $p = \infty$ for the characteristic zero case. We shall prove the following.

Proposition 10.2.1. *Let $x \in H$ be an element of order p.*

(i) If $2 \neq p < n$ then

$$f_\Omega(x) \geq \frac{1}{p}$$

with equality if, and only if, $n = ap$ and $x \in [J_p^a]^G$.

(ii) If $p \geq n$ then

$$f_\Omega(x) \geq \frac{1}{n-1}$$

with equality if, and only if $x \in [J_n]^G$.

(iii) If $p = 2$ then

$$f_\Omega(x) \geq \frac{1}{2} + \frac{1}{2(n-1)}$$

with equality if, and only if, $\nu(x) \in \left\{\frac{n}{2}, \frac{n}{2} - 1\right\}$.

Remark 10.2.2. If $x \in H$ is an element of order p then the largest Jordan block allowed in x is either J_p for $p < n$ or J_n for $p \geq n$. In the characteristic zero case, any unipotent element has infinite order and can have Jordan blocks of any size $i \leq n$. So the characteristic zero case falls in part (ii) of Proposition 10.2.1, this motivates the choice of the notation $p = \infty$ for this case.

10.2.1. Odd prime order elements. Let us start from the case $2 \neq p < n$.

Lemma 10.2.3. *Assume $p < n$ is odd. Let $x \in H$ be an element of order p. Then*

$$f_\Omega(x) \geq \frac{1}{p}$$

PROOF. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in \mathrm{Sp}_n$. Using Proposition 9.3.1(iii), the result is equivalent to the following

$$2\sum_{i<j} ia_i a_j + \sum_i ia_i^2 - \sum_{i \text{ odd}} a_i \geq \frac{n^2-n}{p} \tag{32}$$

using $n = \sum_i ia_i$ a straightforward calculation shows that (32) is equivalent to

$$2\sum_{i<j} ia_i a_j \Big(1 - \frac{j}{p}\Big) + \sum_{i \text{ odd}} a_i(ia_i - 1)\Big(1 - \frac{i}{p}\Big) + \sum_{i \text{ even}} \Big(ia_i^2\Big(1 - \frac{i}{p}\Big) + \frac{ia_i}{p}\Big) \geq 0 \tag{33}$$

It is clear that in (33) the first and last summands are non-negative. We prove that also the second summand is non-negative. Let i be odd and assume $a_i > 0$. Then $a_i \geq 2$ is even, since $x \in \mathrm{Sp}_n$, so $ia_i - 1 > 0$. The result follows. *q.e.d.*

Lemma 10.2.4. *Assume $p < n$ is odd. Let $x \in H$ be an element of order p. Then $f_\Omega(x) = \frac{1}{p}$ if, and only if $n = ap$ and $x \in [J_p^a]^G$.*

PROOF. If $p \mid n$ and $x = [J_p^{n/p}]$ it is straightforward, using Proposition 9.3.1(iii), to compute $f_\Omega(x) = \frac{1}{p}$.

Conversely, let $x \in H$ of order p such that $f_\Omega(x) = \frac{1}{p}$. By the proof of Lemma 10.2.3 we have that each summand in (33) vanishes. If i is even then $a_i = 0$, otherwise the last summand would be positive. Assume $i < p$ is odd and $a_i \neq 0$. This forces $ia_i - 1 = 0$ which is never possible (recall that a_i is even if i is odd), hence $a_i = 0$. Notice that if $i = p$ then $\big(1 - \frac{i}{p}\big) = 0$. Therefore $a_p \neq 0$ and, $p \mid n$ since $n = \sum_i ia_i = pa_p$. *q.e.d.*

In the case $p \mid n$, Lemma 10.2.4 provides all the elements that realise the lower bound. The following deals with the case $p \nmid n$.

Proposition 10.2.5. *Assume $p < n$ is odd and $n = ap + b$ with $0 < b < p$. Then there exists $x \in H$ of order p such that*

$$f_\Omega(x) < \frac{1}{p} + \frac{1}{2n} + \frac{3}{n(n-1)}$$

PROOF. First notice that $a \equiv b \pmod 2$, since p is odd and n even. Consider the element $x_1 = [J_p^a, J_b] \in \mathrm{Sp}_n$ if a is even, otherwise consider $x_2 = [J_p^{a-1}, J_{p-1}, J_{b+1}] \in \mathrm{Sp}_n$. Then, using Proposition 9.3.1(ii) and $a = \frac{n-b}{p}$, we find

$$f_\Omega(x_1) = \frac{1}{p} + \frac{b(p+1-b)}{pn(n-1)} \leq \frac{1}{p} + \frac{(p+1)^2}{4pn(n-1)}$$

where the inequality follows from the fact that the function $g(b) = b(p+1-b)$ for $b \in (0,p)$ is maximal when $b = (p+1)/2$ hence $g(b) \leq g\big(\frac{p+1}{2}\big)$. Now, we see that $\frac{(p+1)^2}{4pn(n-1)}$ is increasing in $p > 2$. Thus, using $p < n$, we deduce

$$f_\Omega(x_1) < \frac{1}{p} + \frac{(n+1)^2}{4n^2(n-1)} < \frac{1}{p} + \frac{1}{2n}$$

Notice that we used $\frac{(n+1)^2}{4n^2(n-1)} < \frac{1}{2n}$, which is true for $n > 4$. In the case $n = 4$ we have that $b = 1$ is odd.

Now assume a, b are odd, we compute

$$f_\Omega(x_2) = \frac{1}{p} + \frac{b(p+1-b)}{pn(n-1)} + \frac{3}{n(n-1)} \leq \frac{1}{p} + \frac{(p+1)^2}{4pn(n-1)} + \frac{3}{n(n-1)}$$

the inequality follows from the same argument as above. Moreover, using $p < n$,

$$f_\Omega(x_2) < \frac{1}{p} + \frac{(n+1)^2}{4n^2(n-1)} + \frac{3}{n(n-1)} < \frac{1}{p} + \frac{1}{2n} + \frac{3}{n(n-1)}$$

Again the last inequality it is true for $n > 4$.

In the case $n = 4$ we necessarily have $p = 3$. We consider the element $[J_2^2] \in \mathrm{Sp}_n$. A straightforward computation leads to $f_\Omega([J_2^2]) < \frac{1}{p} + \frac{1}{2n} + \frac{3}{n(n-1)}$. *q.e.d.*

We now study the case $p > n$.

Lemma 10.2.6. *Assume $p > n$. Let $x \in H$ be an element of order p. Then*

$$f_\Omega(x) \geq \frac{1}{n-1}$$

Proof. As in Lemma 10.2.3 we see that the result is equivalent to

$$2\sum_{i<j} ia_ia_j + \sum_i ia_i^2 - \sum_{i \text{ odd}} a_i \geq n = \sum_i ia_i$$

which is equivalent to

$$2\sum_{i<j} ia_ia_j + \sum_{i \text{ even}} ia_i(a_i-1) + \sum_{i \text{ odd}} a_i\big(i(a_i-1)-1\big) \geq 0 \tag{34}$$

and it is clear that each summand is non-negative, recall that for i odd if $a_i \neq 0$ then $a_i \geq 2$ is even. *q.e.d.*

Lemma 10.2.7. *Assume $p > n$. Let $x \in H$ be an element of order p. Then $f_\Omega(x) = \frac{1}{n-1}$ if, and only if, $x \in [J_n]^G$.*

Proof. If $x = [J_n] \in \mathrm{Sp}_n$ then an easy calculation shows that $f_\Omega(x) = \frac{1}{n-1}$, using Proposition 9.3.1(ii).

Conversely, let $x \in H$ be an element of order p such that $f_\Omega(x) = \frac{1}{n-1}$. Thanks to the proof of Lemma 10.2.6, each of the summand in (34) vanishes. Since $\sum_{i<j} ia_ia_j = 0$ we deduce that there is only one index l such that $a_l \neq 0$, hence $n = \sum_i ia_i = la_l$. For l even we have $la_l(a_l-1) = 0$ if, and only if, $a_l = 0$ or 1. For l odd if $a_l \neq 0$ then $l(a_l-1)-1 = 0$ if, and only if, $(l, a_l) = (1, 2)$. Therefore, from $n = la_l$ we deduce $l = n$ and $a_n = 1$, the case $n = 2$ and $x = I_2$ can be excluded since $n \geq 4$. The result follows. *q.e.d.*

10.2.2. Involution.

Now assume $p = 2$.

Lemma 10.2.8. *Assume $p = 2$. Let $x \in H$ be an involution with $\nu(x) = s$. Then*

$$f_\Omega(x) \geq \frac{1}{2} + \frac{1}{2(n-1)}$$

with equality if, and only if, $\nu(x) = \frac{n}{2}$ or $\frac{n}{2} - 1$.

PROOF. For $x \in H$ an involution we have, by Proposition 9.3.1(iv), $f_\Omega(x) = 1 - \frac{2s(n-s-1)}{n(n-1)}$, differentiating this expression with respect to s we see that it is minimal for $s = \frac{n-1}{2}$. Hence the minimal f_Ω-value is realised when $s = \frac{n}{2} - 1$ or $\frac{n}{2}$. A direct check leads to the result. *q.e.d.*

10.3. Semisimple elements: lower bound

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H$ an element of prime order $r \neq p$. The main result of this section is Proposition 10.3.1, below. Recall the definition of $\delta_{a;b}$

$$\delta_{a;b} = \begin{cases} 1 & b \mid a \\ 0 & \text{otherwise} \end{cases}$$

Proposition 10.3.1. *Let $x \in H$ be an element of prime order $r \neq p$.*

(i) If $2 \neq r < n$ then

$$f_\Omega(x) \geq \frac{1}{r} - \frac{1 - \delta_{\lfloor n/r \rfloor;2}}{n(n-1)}$$

(ii) If $r \geq n$ then

$$f_\Omega(x) \geq \frac{1}{n-1}$$

with equality if, and only if, $C_G(x) \cong \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1}$ or $(\mathrm{GL}_1)^{n/2}$.

(iii) If $r = 2$ then

$$f_\Omega(x) \geq \frac{1}{2} - \frac{1}{2(n-1)} + \frac{2(1-\delta_{n;4})}{n(n-1)}$$

with equality if, and only if, $\nu(x) \in \{n/2, n/2-1\}$.

We first study elements of odd prime order. In the case $r < n$ we shall also compute the best possible lower bounds, see Remark 10.3.6. For involutions the result quickly follows using Proposition 9.3.1, see Section 10.3.2.

10.3.1. Odd prime order elements. We start the analysis giving the following definition. Recall that for $x \in G$ semisimple, up to G-conjugacy, $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. Moreover $x^G \cap H \neq \emptyset$ if, and only if, a_0 is even and $a_i = a_{r-i}$ for all $0 < i \leq \frac{r-1}{2}$.

Definition 10.3.2. Let $x \in H$ be an element of odd prime order $r \neq p$. We define x to be *special* if $|a_i - a_j| \leq 1$ for all i, j.

Observe that if x is not special then there exist $i, j \leq \frac{r-1}{2}$ such that $a_i - a_j \geq 2$. Notice, also, that if $r < n$ and x is special then $a_i \neq 0$ for all i.

Claim. Let $x \in H$ be an element of odd prime order $r \neq p$. Then $f_\Omega(x) \geq f_\Omega(y)$ for any special element $y \in H$ of order r.

Lemma 10.3.3. *Let $x \in H$ be an element of odd prime order $r \neq 2$. Then either x is special or there exists $y = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}]$ with $f_\Omega(x) = f_\Omega(y)$ such that one of the following hold*

(i) $|b_0 - b_1| \geq 2$;

(ii) $b_1 - b_2 \geq 2$

Proof. Up to G-conjugacy, we may assume $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H$ is not special. Then $|a_i - a_j| \geq 2$ for some $i, j \leq \frac{r-1}{2}$. If $i = 0$, let $y = [I_{a_0}, \omega I_{a_j}, \ldots, \omega^j I_{a_1}, \ldots]$. Then, by Proposition 9.3.1, we have $f_\Omega(x) = f_\Omega(y)$ and (i) holds. Now assume $|a_i - a_j| = a_i - a_j \geq 2$ and $i, j \neq 0$. Then we define $y = [I_{a_0}, \omega I_{a_i}, \omega^2 I_{a_j}, \ldots, \omega^i I_{a_1}, \omega^j I_{a_2}, \ldots]$. As in the previous case we see that $f_\Omega(x) = f_\Omega(y)$ and (ii) holds for y. *q.e.d.*

Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H$, and assume x is not special. Set $a_i = \max_l\{a_l\}$, $a_j = \min_l\{a_l\}$, and $i, j \leq \frac{r-1}{2}$. Since x is not special we have $a_i - a_j \geq 2$. In view of Lemma 10.3.3, we may assume (i, j) is one of the following $(0, 1), (1, 0), (1, 2)$.

Case 1. Assume $a_0 - a_1 \geq 2$. Then we define

$$y = [I_{a_0-2}, \omega I_{a_1+1}, \omega^2 I_{a_2}, \ldots, \omega^{r-1} I_{a_{r-1}+1}] \tag{35}$$

Case 2. If $a_1 - a_0 \geq 2$ then we define

$$y = [I_{a_0+2}, \omega I_{a_1-1}, \omega^2 I_{a_2}, \ldots, \omega^{r-1} I_{a_{r-1}-1}] \tag{36}$$

Case 3. Finally, assume $a_1 - a_2 \geq 2$. Then we define

$$y = [I_{a_0}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \ldots, \omega^{r-2} I_{a_{r-2}+1}, \omega^{r-1} I_{a_{r-1}-1}] \tag{37}$$

Lemma 10.3.4. *In the above notation, let $x \in H$ be a semisimple element of prime order r. Assume x is not special. Then*

$$f_\Omega(x) \geq f_\Omega(y)$$

where y is defined in (35), (36) or (37).

Proof. Using the formula stated in Proposition 9.3.1(i) the result reduces to a straightforward computation. Let us write $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$.

Case 1. Assume $a_0 - a_1 \geq 2$. Then $f_\Omega(x) - f_\Omega(y) = \frac{4(a_0 - a_1 - 2)}{n(n-1)} \geq 0$. Notice that $f_\Omega(x) = f_\Omega(y)$ if, and only if $a_0 - a_1 = 2$.

Case 2. Assume $a_1 - a_0 \geq 2$. Then $f_\Omega(x) - f_\Omega(y) = \frac{4(a_1 - a_0 - 1)}{n(n-1)} > 0$.

Case 3. Assume $a_1 - a_2 \geq 2$. Then $f_\Omega(x) - f_\Omega(y) = \frac{4(a_1 - a_2 - 1)}{n(n-1)} > 0$. *q.e.d.*

Proposition 10.3.5. *Let $x \in H$ be an element of odd prime order r. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

where z is a special element.

Proof. If x is special the result trivially holds. Hence, we may assume x is not special. Let $a_i = \max_l\{a_l\}$ and $a_j = \min_l\{a_l\}$ in x. If $(i, j) \notin \{(0, 1), (1, 0), (1, 2)\}$ then, by Lemma 10.3.3, there exists x' with $f_\Omega(x) = f_\Omega(x')$ and the eigenvalues of x' satisfy (i) or (ii) in the statement of the lemma. So we may assume $(i, j) \in \{(0, 1), (1, 0), (1, 2)\}$. In particular, Lemma 10.3.4 yields $f_\Omega(x) = f_\Omega(x') \geq f_\Omega(y)$, for y as in (35), (36) or (37). Then we apply the same argument to y, substituting y with a suitable y' if necessary. In a finite number of steps we get $f_\Omega(x) = f_\Omega(x') \geq f_\Omega(y) \geq \ldots \geq f_\Omega(z)$ for a special element z. *q.e.d.*

The last step in order to have a complete proof of Proposition 10.3.1 is to compute $f_\Omega(x)$ for $x \in H$ a special element. Write $n = ar + b$ with $0 \leq b < r$ and $a \equiv b \pmod 2$. For the purpose of giving the structure of the general special element let us define

$$A = [(\omega, \omega^{-1})I_{a+1}, \ldots, (\omega, \omega^{-1})^{\lfloor \frac{b}{2} \rfloor} I_{a+1}, (\omega, \omega^{-1})^{\lfloor \frac{b}{2} \rfloor + 1} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a]$$

Then if z is any special element we have $C_G(z) \cong C_G(x)$ where

$$x = \begin{cases} [I_a, A] & a \text{ even} \\ [I_{a+1}, A] & a \text{ odd} \end{cases} \tag{38}$$

Using Proposition 9.3.1(i) we compute $f_\Omega(x)$ for x a special element as in (38). We assume $r < n$ (the case $r \geq n$ is dealt with in Lemma 10.3.8, below).

Case 1. Assume a is even. Let x as in (38). Then, using *Mathematica*, we compute

$$f_\Omega(x) = \frac{1}{r} + \frac{b(r+1-b)}{rn(n-1)} \geq \frac{1}{r} \tag{39}$$

where the inequality follows from the fact that $g(b) := b(r+1-b) \geq g(0) = 0$ for $b \in [0, r-1]$, observe that $f_\Omega(x) = \frac{1}{r}$ if, and only if $b = 0$.

Moreover we see that $g(b) \leq g\big(\frac{r+1}{2}\big)$. Hence, assisted by *Mathematica*, we get

$$f_\Omega(x) \leq \frac{1}{r} + \frac{(r+1)^2}{4rn(n-1)} < \frac{1}{r} + \frac{(n+1)^2}{4n^2(n-1)} < \frac{1}{r} + \frac{1}{2n}$$

where the second inequality follows from the fact that $\frac{(r-1)^2}{4rn(n-1)}$ is monotonically increasing in r and $r < n$. The last inequality is true for $n > 4$, notice that if $n = 4$ then $r = 3$ and $a = b = 1$ are odd.

Case 2. Assume a is odd. Let x as in (38). Then, using *Mathematica*, we compute

$$f_\Omega(x) = \frac{1}{r} + \frac{b(r+1-b)}{rn(n-1)} - \frac{1}{n(n-1)} \geq \frac{1}{r} - \frac{1}{n(n-1)} \tag{40}$$

as above, we use $g(b) = b(r+1-b) \geq g(0) = 0$ for $b \in [0, r-1]$. Notice that $f_\Omega(x) = \frac{1}{r} - \frac{1}{n(n-1)}$ if, and only if, $b = 0$.

Again, since $g(b) \leq g\big(\frac{r+1}{2}\big)$, we get

$$f_\Omega(x) \leq \frac{1}{r} + \frac{(r-1)^2}{4rn(n-1)} - \frac{1}{n(n-1)} < \frac{1}{r} + \frac{1}{2n} - \frac{1}{n(n-1)}$$

again, the second inequality is due the fact that $\frac{(r-1)^2}{4rn(n-1)}$ is monotonically increasing in $r < n$.

Remark 10.3.6. Let $r \neq p$ be an odd prime and x be an element of order r. Then, thanks to Proposition 10.3.5, the best possible lower bound are computed in (39) and (40). In particular, the bounds in Proposition 10.3.1 hold. Notice that if $r \mid n$ then any special element realises the lower bound in Proposition 10.3.1.

Here we state the best possible lower bounds for semisimple elements of odd prime order r. We write $n = ar + b$ with $0 \leq b < r$.

Corollary 10.3.7. *Let $x \in H$ be a semisimple element of odd prime order $r < n$. Then*

$$f_\Omega(x) \geq \frac{1}{r} + \frac{b(r+1-b)}{rn(n-1)} - \frac{1-\delta_{a;2}}{n(n-1)}$$

Moreover, equality holds if x is special.

Now assume $r > n$. Then $x = [\omega, \omega^{-1}, \ldots, \omega^{\frac{n}{2}}, \omega^{-\frac{n}{2}}] \in \mathrm{Sp}_n$ is a special element. The proof of Lemma 10.3.4, in particular **Case 1**, suggests to consider also $y = [I_2, \omega, \omega^{-1}, \ldots, \omega^{\frac{n}{2}-1}, \omega^{-\frac{n}{2}+1}] \in \mathrm{Sp}_n$, since in the proof we observed that $f_\Omega(x) = f_\Omega(y)$ (with the notation of the lemma) if, and only if, $a_0 - a_1 = 2$. Notice that y is a special element if $r = n - 1$. In this case, using Proposition 9.3.1, we can also characterise elements that realise the lower bound.

Lemma 10.3.8. *Let $r > n$ be a prime. Let $x \in H$ be an element of order r. Then*

$$f_\Omega(x) \geq \frac{1}{n-1}$$

with equality if, and only if, $C_G(x) \cong \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1}$ or $(\mathrm{GL}_1)^{n/2}$.

Proof. Thanks to Proposition 9.3.1 we have $f_\Omega(x) \geq 1/(n-1)$ if, and only if, $\sum_i a_i^2 - a_0 \geq n = \sum_i a_i$. This last inequality is equivalent to $\sum_{i>0} a_i(a_i - 1) + a_0(a_0 - 2) \geq 0$. Notice that each of the summands is non-negative.

It is a straightforward computation to see that if $C_G(x) \cong \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1}$ or $(\mathrm{GL}_1)^{n/2}$ then $f_\Omega(x) = 1/(n-1)$. Conversely, let $x \in G$ such that $f_\Omega(x) = 1/(n-1)$. Then by the previous argument, $a_0(a_0 - 2) = 0$ and $a_i(a_i - 1) = 0$ for all $i \neq 0$. Therefore $a_0 = 0, 2$ and $a_i = 0, 1$. The result follows. *q.e.d.*

10.3.2. Involutions. Thanks to Proposition 9.3.1(ii), it is straightforward to derive lower bounds on $f_\Omega(x)$ for an involution $x \in H$. As already seen in the proof of Lemma 10.1.7, $f_\Omega(x)$, as a function in s, is monotonically decreasing in $[2, n/2]$. Therefore $f_\Omega(x) \geq f_\Omega([I_{n/2}, -I_{n/2}])$ or $f_\Omega([I_{n/2-1}, -I_{n/2+1}])$. This observation with an easy computation leads to the following.

Lemma 10.3.9. *Assume $p \neq 2$. Let $x \in H$ be an involution. Then the conclusion of Proposition 10.3.1 holds.*

10.4. Local upper bound

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$, where r is an odd prime. Recall that $\mathcal{V}_s = \{x \in H : \nu(x) = s\}$ and, for r a prime $\mathcal{V}_{s,r} = \{x \in \mathcal{V}_s : o(x) = r\}$.

The main result of this section is the following.

Proposition 10.4.1. *Let $x \in H \cap \mathcal{V}_{s,r}$, where $r \neq 2$. Then*

$$f_\Omega(x) \leq 1 - \frac{s-1}{n-1}$$

In order to prove Proposition 10.4.1 we study separately unipotent and semisimple elements.

10.4.1. Unipotent elements. We start with the case $r = p$. Recall that we are assuming $r \neq 2$.

Lemma 10.4.2. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then the conclusion of Proposition 10.4.1 holds.*

PROOF. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in H \cap \mathcal{V}_{s,r}$, so that $n - s = \sum_i a_i$. Using the argument of [**11**, Proposition 2.9] we have

$$\begin{aligned} n(n-s) = \Big(\sum_i i a_i\Big)\Big(\sum_i a_i\Big) &= 2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 + \sum_{i<j}(j-i)a_i a_j \\ &\geq 2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 - \sum_{i \text{ odd}} a_i \end{aligned}$$

Proposition 9.3.1 provides an explicit formula for $f_\Omega(x)$. Using also the previous inequality, we have

$$f_\Omega(x) \leq \frac{n(n-s)}{n(n-1)} = 1 - \frac{s-1}{n-1}$$

q.e.d.

In general the bound is not sharp. The following example shows that the bound is close to best possible.

Example 10.4.3. Assume $n - s$ divides n and $\frac{n}{n-s} \leq p$. Thus $x = \left[J_{\frac{n}{n-s}}^{n-s}\right] \in H \cap \mathcal{V}_{s,p}$. Applying Proposition 9.3.1(iii), we compute

$$f_\Omega(x) = 1 - \frac{s-1}{n-1} - \frac{n-s}{n(n-1)} > 1 - \frac{s-1}{n-1} - \frac{1}{n}$$

10.4.2. Semisimple elements. Assume $r \neq p$ is an odd prime. Then we have the following.

Proposition 10.4.4. *Let $x \in H \cap \mathcal{V}_{s,r}$. Say $m = \max\{s(n-s-1), \frac{n}{2}(s-1)\}$. Then*

$$f_\Omega(x) \leq 1 - \frac{2m}{n(n-1)}$$

In particular, the conclusion of Proposition 10.4.1 holds.

PROOF. Observe that for any $x \in H$ of prime order $r \neq p$ we have that $x^G \cap \mathrm{O}_n \neq \emptyset$. Moreover, by Proposition 9.3.1(i), we have

$$f_\Omega(x) = \frac{\dim C_{\mathrm{O}_n}(x)}{\dim \Omega} \leq 1 - \frac{2m}{n(n-1)}$$

where the last inequality follows from Proposition 5.4.1.

If $m = s(n-s-1)$ we have $1 - \frac{s-1}{n-1} - \left(1 - \frac{2s(n-s-1)}{n(n-1)}\right) = \frac{2ns-2s^2-2s+n}{n(n-1)}$, we see that $g(s) = 2ns - 2s^2 - 2s + n$ is minimal for $s = 0$ of $s = \frac{n}{2}$. Thus we deduce $1 - \frac{2m}{n(n-1)} \leq 1 - \frac{s-1}{n-1}$. If $m = \frac{n}{2}(s-1)$ the conclusion of Proposition 10.4.1 is immediate. *q.e.d.*

In this case, $r \neq p$, we can give more information. In fact we can construct elements that give the best possible upper bound. Let $n = a(n-s) + b$, where $0 \leq b < n - s$.

For $s \geq n/2$ we define

$$B = [(\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\lfloor\frac{a-1}{2}\rfloor}I_{n-s}]$$
$$B' = [(\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\lfloor\frac{a}{2}\rfloor}I_{n-s}]$$

We define the following elements

$$z = \begin{cases} \left[I_{n-s}, B, (\omega,\omega^{-1})^{\frac{a+1}{2}} I_{\frac{b}{2}}\right] & s \text{ even}, a \text{ odd} \\ \left[I_{n-s}, B, (\omega,\omega^{-1})^{\frac{a}{2}} I_{\frac{n-s+b}{2}}\right] & s \text{ even}, a \text{ even} \\ \left[I_{n-s-1}, B, (\omega,\omega^{-1})^{\frac{a+1}{2}} I_{\frac{b+1}{2}}\right] & s \text{ odd}, a \text{ odd} \\ \left[I_{n-s-1}, B, (\omega,\omega^{-1})^{\frac{a}{2}} I_{\frac{n-s+b+1}{2}}\right] & s \text{ odd}, a \text{ even} \end{cases} \tag{41}$$

and

$$z' = \begin{cases} \left[B', (\omega,\omega^{-1})^{\frac{a}{2}+1} I_{\frac{b}{2}}\right] & a \text{ even} \\ \left[B', (\omega,\omega^{-1})^{\frac{a+1}{2}} I_{\frac{n-s+b}{2}}\right] & a \text{ odd} \end{cases} \tag{42}$$

Claim. Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega(x) \leq \max\{f_\Omega(z), f_\Omega(z')\}$.

In order to prove the claim we need some preliminary results.

First assume $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. In the next result we show that we can always assume $a_1 = \max\{a_i : a_i < n-s, 0 \leq i \leq r-1\}$ and $a_2 = \min\{a_i : a_i > 0, 0 \leq i \leq r-1\}$.

Lemma 10.4.5. *Let $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H \cap \mathcal{V}_{s,r}$. Then either $C_G(x) \cong C_G(z)$ or there exists $y = [I_{n-s}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}] \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega(x) = f_\Omega(y)$, $b_1 = \max\{b_i : b_i < n-s, 0 \leq i \leq r-1\}$ and $b_2 = \min\{b_i : b_i > 0, 0 \leq i \leq r-1\}$.*

Proof. Assume $C_G(x) \not\cong C_G(z)$. Let $a_l = \max\{a_i : a_i < n-s, 0 \leq i \leq r-1\}$ and $a_m = \min\{a_i : a_i > 0, 0 \leq i \leq r-1\}$, for $l, m \leq \frac{r-1}{2}$. Then we define $y = [I_{n-s}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}]$ where $b_i = a_i$ for all $i \notin \{1, 2, l, m\}$ and $i \leq \frac{r-1}{2}$, and $b_1 = a_l, b_2 = a_m$. The result follows thanks to Proposition 9.3.1(i). *q.e.d.*

Let $x \in H \cap \mathcal{V}_{s,r}$. Assume $C_G(x) \not\cong C_G(z)$ and $a_0 = n-s, a_1 = \max\{a_i : a_i < n-s, 0 \leq i \leq r-1\}$ and $a_2 = \min\{a_i : a_i > 0, 0 \leq i \leq r-1\}$. We define

$$y = [I_{n-s}, \omega I_{a_1+1}, \omega^2 I_{a_2-1}, \omega^3 I_{a_3}, \dots] \tag{43}$$

And, for elements whose 1-eigenspace is $(n-s)$-dimensional, we have the following.

Lemma 10.4.6. *Let $x \in H \cap \mathcal{V}_{s,r}$. Assume $C_G(x) \not\cong C_G(z)$. Then, for y as in (43),*

$$f_\Omega(x) < f_\Omega(y)$$

Proof. Using Proposition 9.3.1(i) we have $f_\Omega(x) - f_\Omega(y) = \frac{4(a_2-a_1+1)}{n(n-1)} > 0$. *q.e.d.*

Now assume $x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \dots, \omega^{r-1} I_{n-s}] \in H \cap \mathcal{V}_{s,r}$. The following result is similar to Lemma 10.4.5.

Lemma 10.4.7. *Let $x \in H \cap \mathcal{V}_{s,r}$ with $a_1 = n-s$. Then either $C_G(x) \cong C_G(z)$ or $C_G(z')$ or there exists $y = [I_{b_0}, \omega I_{n-s}, \omega^2 I_{b_2}, \dots, \omega^{r-1} I_{n-s}] \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega(x) = f_\Omega(y)$ and at least one of the following hold* *(i)* $b_0 = \max\{b_i : b_i < n-s, 0 \le i \le r-1\}, b_2 = \min\{b_i : b_i > 0, 0 \le i \le r-1\}$, *and only one of the following holds*

(i') $b_0 = n-s-1$ *and* $b_3 = \max\{b_i : b_i < n-s, 0 < i \le r-1\}$;

(i") $b_0 \le n-s-2$.

(ii) $b_0 = \min\{b_i : b_i > 0, 0 \le i \le r-1\}, b_2 = \max\{b_i : b_i < n-s, 0 \le i \le r-1\}$;

(iii) $b_2 = \min\{b_i : b_i > 0, 0 \le i \le r-1\}, b_3 = \max\{b_i : b_i < n-s, 0 \le i \le r-1\}$.

Proof. The same argument of Lemma 10.4.5 applies. *q.e.d.*

Let $x \in H \cap \mathcal{V}_{s,r}$ with $a_1 = n-s$ and assume that x satisfies one of the four possibilities in Lemma 10.4.7. If (i') or (iii) holds we define

(44) $$y = [I_{a_0}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3+1}, \ldots]$$

If x satisfies (i") we define

(45) $$y = [I_{a_0+2}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3}, \ldots]$$

Finally, for x as in (ii) we define

(46) $$y = [I_{a_0-2}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2+1}, (\omega, \omega^{-1})^3 I_{a_3}, \ldots]$$

Remark 10.4.8. If, for some $h > 0$, $a_0 = a_h = \min_i\{a_i : 0 < a_i < n-s\}$ then we can construct two different new elements, namely (44) or (46) (assuming $h = 2$).

With this notation we can show the following.

Lemma 10.4.9. *Let $x \in H \cap \mathcal{V}_{s,r}$. Assume $a_1 = n-s$ and one of the conditions of Lemma 10.4.7 holds. For y as in (44)–(46) we have*

$$f_\Omega(y) > f_\Omega(x)$$

Proof. The result easily follows using the formula in Proposition 9.3.1(i).

If y is as in (44) we compute $f_\Omega(y) - f_\Omega(x) = \frac{4(a_3-a_2+1)}{n(n-1)} > 0$.

If y is as in (45) we compute $f_\Omega(y) - f_\Omega(x) = \frac{4(a_0-a_2+\frac{1}{2})}{n(n-1)} > 0$.

If y is as in (46) we compute $f_\Omega(y) - f_\Omega(x) = \frac{4(a_2-a_0+2)}{n(n-1)} > 0$. *q.e.d.*

Now we can prove the claim.

Proposition 10.4.10. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \le \max\{f_\Omega(z), f_\Omega(z')\}$$

Proof. If $C_G(x) \cong C_G(z)$ or $C_G(z')$ the result immediately follows. Hence we may assume there are no such isomorphisms.

Assume $a_0 = n-s$. Then, thanks to Lemma 10.4.6, we can construct an element y such that $f_\Omega(x) < f_\Omega(y)$ (if the eigenvalues of x does not satisfy the hypothesis of the lemma, by Lemma 10.4.5 we may replace x with suitable x' for which the hypothesis holds). If $C_G(y) \ncong C_G(z), C_G(z')$, we apply the same argument to y finding $f_\Omega(y) < f_\Omega(y')$. Eventually, in a finite number of steps, iterating this construction, we have $f_\Omega(x) < f_\Omega(y) < \ldots < f_\Omega(z)$.

If $a_0 < n-s$, hence $a_0 < n-s-1$, we may assume $a_1 = n-s$ (up to relabelling the eigenvalues). Applying the argument of the previous case, using the construction given for Lemma 10.4.9, substituting the element with a suitable one (Lemma 10.4.7), if necessary, we will find $f_\Omega(x) < \ldots < f_\Omega(z)$ or $f_\Omega(x) < \ldots < f_\Omega(z')$, in a finite number of steps. Notice that the chain terminates either with $f_\Omega(z)$ or $f_\Omega(z')$ in this case. The result follows. *q.e.d.*

Thanks to Proposition 10.4.10, it is easy to derive upper bound on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$.

Proposition 10.4.11. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s-1}{n-1}$$

Proof. Thanks to Proposition 10.4.10 it is enough to compute $f_\Omega(z)$ and $f_\Omega(z')$, which is a straightforward computation thanks to Proposition 9.3.1. Write $n = a(n-s)+b$, where $0 \leq b < n-s$.

Using *Mathematica* we get, for a odd,

$$f_\Omega(z) = 1 - \frac{s}{n} - \frac{b(n-s-b/2)}{n(n-1)} \leq 1 - \frac{s}{n} \tag{47}$$

where the inequality follows from the fact that $g(b) := b(n-s-\frac{b}{2}) \geq g(0) = 0$. If a is even we compute

$$f_\Omega(z) = 1 - \frac{s}{n} - \frac{(n-s+b)(n-s-b)}{2n(n-1)} < 1 - \frac{s}{n} \tag{48}$$

and the inequality follows from the fact that $(n-s+b)(n-s-b) > 0$ for $0 \leq b < n-s$.

Now assume s is odd so that $a_0 = n-s-1$, we may assume $s < n-1$ otherwise $a_0 = 0$ and we shall compute $f_\Omega(z')$ below. If a is even we compute

$$f_\Omega(z) = 1 - \frac{s}{n} - \frac{2b(n-s-1) - b^2 + 4(n-s) - 5}{2n(n-1)} < 1 - \frac{s}{n}$$

where the inequality follows from the fact that $2b(n-s-1) - b^2$ is minimal for $b = 0$ and $4(n-s) - 5 > 0$.

If a is odd we compute

$$f_\Omega(x) = 1 - \frac{s}{n} - \frac{(n-s)(n-s+2) - b(b+2) - 5}{n(n-1)} < 1 - \frac{s}{n} - \frac{4(n-s)-5}{2n(n-1)} < 1 - \frac{s}{n}$$

where the inequality follow since $b(b+2)$ is maximal for b maximal, and we use $b \leq n-s-2$ (recall that b is odd since a is odd). The last inequality follows from the assumption $s < n-1$.

Now, let us compute $f_\Omega(z')$. If a is even we have

$$f_\Omega(z') = 1 - \frac{s-1}{n-1} - \frac{b(n-s-\frac{b}{2})}{n(n-1)} \leq 1 - \frac{s-1}{n-1} \tag{49}$$

For a odd

$$f_\Omega(z') = \frac{n^2 - s^2 + b^2}{2n(n-1)} < \frac{n^2 - s^2 + (n-s)^2}{2n(n-1)} = 1 - \frac{s-1}{n-1} \tag{50}$$

q.e.d.

Remark 10.4.12. The best possible upper bound on f_Ω for elements in $H \cap \mathcal{V}_{s,r}$ is given by the maximum of (47) and (50) for $a = \lfloor \frac{n}{n-s} \rfloor$ odd, and the maximum of (48) and (49) for a even. Notice, moreover, (49) and (50) may be considered only if $s \geq n/2$ (if $s < n/2$, z' does not exist).

The bound given in Proposition 10.4.11 is close to the best possible in a variety of examples. For this purpose, in the following, we shall bound the differences $1 - \frac{s-1}{n-1} - f_\Omega(z)$ and $1 - \frac{s-1}{n-1} - f_\Omega(z')$.

Let us denote $U = 1 - \frac{s-1}{n-1}$. For a even we have

$$U - f_\Omega(z) = \frac{(n-s)(n-s+2) - b^2}{2n(n-1)} < \frac{(n-s+2)^2}{2n(n-1)}$$

For a odd

$$U - f_\Omega(z) = \frac{b(n-s-\frac{b}{2})}{n(n-1)} + \frac{n-s}{n(n-1)} < \frac{(n-s)^2}{2n(n-1)} + \frac{1}{n-1}$$

For a even

$$U - f_\Omega(z') = \frac{b(n-s-\frac{b}{2})}{n(n-1)} < \frac{(n-s)^2}{2n(n-1)}$$

For a odd

$$U - f_\Omega(z') = \frac{(n-s)^2 - b^2}{2n(n-1)} < \frac{(n-s)^2}{2n(n-1)}$$

In particular, in all cases if $s > n - \sqrt{n}$ we have

$$U - f_\Omega(z), U - f_\Omega(z') < \frac{3}{2(n-1)}$$

Furthermore, notice that if $n = a(n-s)$ and a is even then $f_\Omega(z') = 1 - \frac{s-1}{n-1}$. Hence the upper bound in Proposition 10.4.1 is the best possible.

10.5. Local lower bound

The main result of this section is Proposition 10.5.1, in which we give a lower bound on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$ where r is an odd prime.

Proposition 10.5.1. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n(n-1)}$$

As for the study of local upper bounds we shall first study unipotent elements and next semisimple elements.

10.5.1. Unipotent elements.

We have the following.

Proposition 10.5.2. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s-1)}{n(n-1)} + \frac{1}{n-1}$$

In particular the conclusion of Proposition 10.5.1 holds.

PROOF. Thanks to Proposition 9.3.1(iii) we have

$$f_\Omega(x) \geq \frac{2\sum_{i<j} ia_ia_j + \sum_i ia_i^2}{n(n-1)} = \frac{\dim C_G(x)}{n(n-1)}$$

By Proposition 5.4.1 we have $\dim C_G(x) \geq n^2 - 2ns + s^2 + s$. The result follows. *q.e.d.*

Example 10.5.3. For $s \leq \frac{n}{2}$ let $x = [J_2^s, J_1^{n-2s}]$. Then, using Proposition 9.3.1(iii) we compute

$$f_\Omega(x) = 1 - \frac{s(2n-s)}{n(n-1)} + \frac{s(s+2)}{n(n-1)}$$

Write ℓ for the lower bound in Proposition 10.5.2. Then

$$f_\Omega(x) - \ell = \frac{s(s+1)}{n(n-1)} - \frac{1}{n-1}$$

In particular, for $s \leq \sqrt{n}$ we deduce $f_\Omega(x) - \ell < \frac{1}{n-1}$.

10.5.2. Semisimple elements. Here we assume $r \neq p$ is an odd prime and $x \in H \cap \mathcal{V}_{s,r}$. We prove the following.

Proposition 10.5.4. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n(n-1)} + \frac{s^2}{n(n-1)(r-1)}$$

In particular, the conclusion of Proposition 10.5.2 holds.

PROOF. First, assume $x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H \cap \mathcal{V}_{s,r}$, so that $a_i \leq n-s$ and $\sum_{i>0} a_i = s$. Then, using Proposition 9.3.1(i),

$$\begin{aligned} f_\Omega(x) &= \frac{(n-s)^2 + \sum_{i>0} a_i^2 - (n-s)}{n(n-1)} \geq \frac{(n-s)(n-s-1) + \frac{1}{r-1}\left(\sum_{i>0} a_i\right)^2}{n(n-1)} \\ &= 1 - \frac{s(2n-s)}{n(n-1)} + \frac{s^2}{n(n-1)(r-1)} \end{aligned}$$

where the inequality follows from Proposition B.2.1.

Now assume $x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \ldots] \in H \cap \mathcal{V}_{s,r}$, so $a_i \leq n-s$ for all i. As in the previous case, we have

$$\begin{aligned} f_\Omega(x) &= \frac{(n-s)^2 + \sum_{i\neq 1} a_i^2 - a_0}{n(n-1)} \geq \frac{(n-s)^2 + \frac{s^2}{r-1} - a_0}{n(n-1)} \\ &= 1 - \frac{s(2n-s)}{n(n-1)} + \frac{s^2}{n(n-1)(r-1)} + \frac{s}{n(n-1)} \end{aligned}$$

The result follows. *q.e.d.*

Let $x \in H \cap \mathcal{V}_{s,r}$. If $s < \frac{n}{2}$ then s is even and the 1-eigenspace is $(n-s)$-dimensional. We give the following.

Definition 10.5.5. Let $x \in H \cap \mathcal{V}_{s,r}$ with $s < n/2$. We call x *special* if $|a_i - a_j| \leq 1$ for all $i, j > 0$.

Claim. Let $x \in H \cap \mathcal{V}_{s,r}$ with $s < \frac{n}{2}$. Then $f_\Omega(x) \geq f_\Omega(y)$ for any special element $y \in H \cap \mathcal{V}_{s,r}$.

In order to prove the claim we may use the same argument given in Section 10.3. In fact a version of Lemma 10.3.4 holds in this case as well. And this is the main tool to prove the claim. We point out that the argument given for global lower bounds for semisimple elements does not work here in full generality. In fact we need to assume $s < n/2$. In this situation, for $x \in H \cap \mathcal{V}_{s,r}$ not special with $a_1 - a_2 \geq 2$, we define

$$y = [I_{n-s}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \omega^3 I_{a_3}, \dots,] \in H \cap \mathcal{V}_{s,r} \tag{51}$$

Then, as in Lemma 10.3.4 we have $f_\Omega(x) > f_\Omega(y)$ (the strict inequality leads to the characterisation). Iterating these constructions we, eventually, deduce the claim.

Proposition 10.5.6. *Let $x \in H \cap \mathcal{V}_{s,r}$ with $s < n/2$. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

where z is a special element. Moreover equality holds if, and only if, x is special.

Proof. The usual argument applies: thanks to the above construction (and the version of Lemma 10.3.4 applied to x and y as in (51)) we have a finite chain of inequalities $f_\Omega(x) > f_\Omega(y) > \dots > f_\Omega(z)$. The characterisation follows from the strict inequalities. *q.e.d.*

Let $s < n/2$ be a positive even integer. Write $s = a(r-1) + b$ where $0 \leq b < r-1$. Then b is even, as $(r-1)$ and s are even. It is clear that

$$z = [I_{n-s}, (\omega, \omega^{-1}) I_{a+1}, \dots, (\omega, \omega^{-1})^{b/2} I_{a+1}, (\omega, \omega^{-1})^{b/2+1} I_a, \dots, (\omega, \omega^{-1})^{(r-1)/2} I_a]$$

is a special element in $H \cap \mathcal{V}_{s,r}$. Using Proposition 9.3.1(i) and *Mathematica* we compute

$$\begin{aligned} f_\Omega(z) &= \frac{b^2 - br + b + n^2(r-1) - n(r-1)(2s+1) + s(rs + r - 2s - 1)}{n(n-1)(r-1)} \\ &\geq \frac{b^2 - 2b + 2n^2 - 2n(2s+1) + s(s+2)}{2n(n-1)} \\ &\geq \frac{2n^2 - 2n(2s+1) + s(s+2) - 1}{2n(n-1)} = 1 - \frac{s(2n-s)}{n(n-1)} - \frac{s(s-2)+1}{n(n-1)} \end{aligned} \tag{52}$$

the first inequality follows from the fact that $f_\Omega(z)$ is minimal when r is minimal, so $r = 3$; the second inequality follows from $b^2 - 2b \geq -1$.

Notice that if $r - 1 > s$ then $z = [I_{n-s}, (\omega, \omega^{-1}), \dots, (\omega, \omega^{-1})^{s/2}]$ and

$$f_\Omega(z) = 1 - \frac{s(2n-s)}{n(n-1)} + \frac{2s}{n(n-1)} \tag{53}$$

In the case $s < \frac{n}{2}$, the best possible lower bounds are computed in (52) and (53), moreover Proposition 10.5.6 characterises elements which realise it.

When $s \geq n/2$, the largest eigenspace may be different from the 1-eigenspace. However, given $x \in H \cap \mathcal{V}_{s,r}$, if we mimic the construction of y as in (51) then $f_\Omega(x) > f_\Omega(y)$ fails to be true in general.

10.6. Further comments on local bounds

In Propositions 10.4.1 and 10.5.1, we have established upper and lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$. Combining these bounds together, we easily deduce the following. Here r is an odd prime.

Proposition 10.6.1. *Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| \leq \frac{s(n-s)}{n(n-1)} + \frac{1}{n+1}$$

Remark 10.6.2. If $s \leq \sqrt{n}$ or $s \geq n - \sqrt{n}$ we have $\frac{s(n-s)}{n(n-1)} \leq \frac{2}{\sqrt{n}}$. Therefore the f_Ω-values of elements in $H \cap \mathcal{V}_{s,r}$ are close to each other.

CHAPTER 11

General linear group, $H = \mathrm{O}_n$

We consider the case $G = \mathrm{GL}_n$, $H = \mathrm{O}_n$ a $\mathscr{C}_6$-subgroup of G, with $n > 2$. Denote $\Omega = G/H$. The aim of this chapter is to derive bounds on $f_\Omega(x)$ for $x \in G$ of prime order, proving the results stated in Chapter 8.

11.1. Upper bound

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H$ of prime order r. We shall prove the following.

Proposition 11.1.1. *Let $x \in G \setminus Z(G)$ be an element of prime order r.*

(i) If $r \neq 2$ then
$$f_\Omega(x) \le 1 - \frac{4(n-1)}{n(n+1)}$$
with equality if, and only if, $\nu(x) = 2$.

(ii) If $r = 2 \neq p$ then
$$f_\Omega(x) \le 1 - \frac{2(n-1)}{n(n+1)}$$
with equality if, and only if, $\nu(x) = 1$.

(iii) If $r = p = 2$ then
$$f_\Omega(x) \le 1 - \frac{2(n-2)}{n(n+1)}$$
with equality if, and only if, $\nu(x) = 1$ or $(n, \nu(x)) = (4, 2)$.

Remark 11.1.2. The observations made in Remark 10.1.2 hold in this case, as well.

We prove Proposition 11.1.1 in a sequence of lemmas; by Remark 10.1.3 we will prove the next results for $x \in H$.

Lemma 11.1.3. *Let $x \in H$ be an element of odd prime order r and assume $\nu(x) = 2$. Then*
$$f_\Omega(x) = 1 - \frac{4(n-1)}{n(n+1)}$$

Proof. If $\nu(x) = 2$ then either $x = [I_{n-2}, \omega, \omega^{-1}]$, if $r \neq p$, or $x = [J_2^2, J^{n-4}]$ or $[J_3, J_1^{n-3}]$ if $r = p$. In all cases the result follows from applying formulae in Proposition 9.3.2(i) and (iii). *q.e.d.*

In the following two results we prove that for $x \in H$, with $\nu(x) > 2$, $f_\Omega(x) < 1 - \frac{4(n-1)}{n(n+1)}$. Notice that using Theorem 7.2.5 and Proposition 5.4.1 we could prove Proposition 11.1.1 assuming that $\nu(x)$ is bounded below and then proving the remaining cases by inspection. In order to avoid these computations we deduce the result from the explicit formulae for $f_\Omega(x)$ given in Proposition 9.3.2.

Lemma 11.1.4. *Let $x \in H$ be of odd order p. If $\nu(x) > 2$ then*

$$f_\Omega(x) < 1 - \frac{4(n-1)}{n(n+1)}$$

Proof. Notice that $n > 5$, as we require $\nu(x) > 2$. By Proposition 9.3.2(iii) we have an explicit formula for $f_\Omega(x)$. By the proof of [**43**, Lemma 3.4] we have $f_\Omega(x) \le 1 - \frac{2s(n-s)}{n(n+1)}$ if $s \le n/2$ and $f_\Omega(x) \le 1 - \frac{s}{n+1}$ for $s > n/2$.

Assume $s \le n/2$. Then $1 - \frac{2s(n-s)}{n(n+1)} < 1 - \frac{4(n-1)}{n(n+1)}$ if, and only if, $n - \sqrt{n^2 - 8n + 8} < 2s < n + \sqrt{n^2 - 8n + 8}$. It is straightforward to check that for $n > 7$, $n - \sqrt{n^2 - 8n + 8} < 6$ and $n + \sqrt{n^2 - 8n + 8} > n$. Therefore, we need to check the cases $n = 6, 7$. In the case $n = 6$ the possible choices are $x = [J_2^3], [J_3, J_2, J_1], [J_4, J_1^2]$ and none of them lies in H. If $n = 7$ then $x = [J_2^3, J_1], [J_3, J_2, J_1^2]$, again none of these elements lies in H.

Assume $s > n/2$. Then $1 - \frac{s}{n+1} < 1 - \frac{4(n-1)}{n(n+1)}$ if, and only if, $s > 4\left(1 - \frac{1}{n}\right)$, which is always true for $s \ge 4$. In the case $s = 3$ we deduce $n < 6$, as $s > n/2$; hence we need to check the cases $n = 4, 5$ and $s = 3$. In the case $n = 4$ we must have $x = [J_4] \notin H$. Similarly, if $n = 5$, $x = [J_3, J_2]$ or $[J_4, J_1]$ and none of them lies in H. *q.e.d.*

Lemma 11.1.5. *Let $x \in H$ be of odd order $r \neq p$. If $\nu(x) > 2$ then*

$$f_\Omega(x) < 1 - \frac{4(n-1)}{n(n+1)}$$

Proof. Also here $n > 5$, otherwise $\nu(x) \le 2$. Using the explicit formula in Proposition 9.3.2(i) and $n = \sum_i i a_i$ (from which $n^2 = \sum_i a_i^2 + 2\sum_{i<j} a_i a_j$) we deduce that the result is equivalent to

$$2\sum_{i<j} a_i a_j + 4 - a_0 - 3n > 0 \tag{54}$$

First assume the largest eigenspace of x is the 1-eigenspace. Then the left hand side of (54) is $2s(n-s) + 2\sum_{0<i<j} a_i a_j + 4 - 4n + s > 2s(n-s) + 4 - 4n + s$. In order to get the result it is enough to show $2s(n-s) + 4 - 4n + s \ge 0$, which is true if, and only if, $2n + 1 - \sqrt{4n^2 - 28n + 33} \le 4s \le 2n + 1 + \sqrt{4n^2 - 28n + 33}$. It is straightforward to check that, for $n > 6$, $2n + 1 - \sqrt{4n^2 - 28n + 33} \le 12$ and $2n + 1 + \sqrt{4n^2 - 28n + 33} \ge n - 2$. Therefore for $3 \le s \le n-2$ if $n \ge 6$ the inequality (54) holds. We need to check the case $s = n-1$ for any n. Assume $s = n-1$ then the 1-eigenspace is 1-dimensional, therefore n is odd and $x = [1, \omega, \omega^{-1}, \ldots, \omega^{\frac{n-1}{2}}, \omega^{-\frac{n-1}{2}}]$, using Proposition 9.3.2(i) we compute $f_\Omega(x) = \frac{1}{n} < 1 - \frac{4(n-1)}{n(n+1)}$.

Now assume that the largest eigenspace of x is the ω^i-eigenspace, for simplicity of notation we may assume $i = 1$. Then the left hand side of (54) is

$$4s(n-s) + 2a_0 \sum_{j>1} a_j + 2\sum_{1<i<j<r-1} a_i a_j + 4 - a_0 - 3n \ge 4s(n-s) + 4 - 4n + s$$

Again, it is enough to show $4s(n-s) + 4 - 4n + s > 0$. This is equivalent to $4n + 1 - \sqrt{16n^2 - 56n + 65} < 8s < 4n + 1 + \sqrt{16n^2 - 56n + 65}$. It is easy to check that $4n + 1 - \sqrt{16n^2 - 56n + 65} < 24$ for $n > 3$; similarly $4n + 1 + \sqrt{16n^2 - 56n + 65} > 8(n-2)$ if $n > 2$. Therefore we need to check only the case $s = n-1$. Here, the only possibility is $x = [\omega, \omega^{-1}, \ldots, \omega^{\frac{n}{2}}, \omega^{-\frac{n}{2}}]$, and we compute $f_\Omega(x) = \frac{1}{n+1} < 1 - \frac{4(n-1)}{n(n+1)}$. *q.e.d.*

Proposition 11.1.1(i) follows from Lemmas 11.1.3, 11.1.4 and 11.1.5. We prove parts (ii) and (iii) of Proposition 11.1.1 in the following.

Lemma 11.1.6. *Let $x \in H$ be an involution. Then*

$$f_\Omega(x) \leq 1 - \frac{2(n-1-\delta_{p,2})}{n(n+1)}$$

with equality if, and only if, $\nu(x) = 2$ in the case $p \neq 2$ or, $\nu(x) = 1$ or $(n, \nu(x)) = (4,2)$ in the case $p = 2$.

Proof. In any characteristic, we see that the f_Ω-value of an involution decreases in $\nu(x)$, for $1 \leq \nu(x) \leq \lfloor n/2 \rfloor$, by the explicit formulae in Proposition 9.3.2(ii) and (iv). The result follows by a straightforward computation. *q.e.d.*

11.2. Unipotent elements: lower bound

In this section we establish lower bounds on $f_\Omega(x)$ for $x \in H$ of order p, or any unipotent element in characteristic zero (recall, in this case we set $p = \infty$). We prove the following.

Proposition 11.2.1. *Let $x \in H$ be an element of order p.*

(i) *If $2 \neq p < n$ then*

$$f_\Omega(x) \geq \frac{1}{p}$$

with equality if, and only if, $n = ap$ and $x \in [J_p^a]^G$.

(ii) *If $p \geq n$ then*

$$f_\Omega(x) \geq \frac{1}{n} + \frac{3\delta_{n;2}}{n(n+1)}$$

with equality if, and only if, $x \in [J_n]^G, [J_{n-1}, J_1]^G$ or $[J_2^2]^G$.

(iii) *If $p = 2$ then*

$$f_\Omega(x) \geq \frac{1}{2} + \frac{3}{2(n+1)} - \frac{3(1-\delta_{n;2})}{2n(n+1)}$$

with equality if, and only if, $\nu(x) \in \{\frac{n-1}{2}, \frac{n}{2}, \frac{n}{2} - 1\}$.

Remark 11.2.2. As in Remark 10.2.2, also here we notice that in the characteristic zero case the unipotent elements have infinite order and the largest Jordan block allowed is J_n. Therefore the characteristic zero case falls in case (ii) of Proposition 11.2.1, this motivates the choice to denote $p = \infty$.

11.2.1. Odd prime order elements. Here we assume $p \neq 2$ and show parts (i) and (ii) of Proposition 11.2.1. **First we assume p < n.**

Lemma 11.2.3. *Assume $p < n$ is odd. Let $x \in H$ be an element of order p. Then*

$$f_\Omega(x) \geq \frac{1}{p}$$

Proof. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in \mathrm{O}_n$. Using Proposition 9.3.2(iii) and $n = \sum_i i a_i$, we see that the result is equivalent to

$$2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 + \sum_{i \text{ odd}} a_i \geq \frac{n^2+n}{p} = \frac{1}{p}\sum_i i^2 a_i^2 + \frac{2}{p}\sum_{i<j} ij a_i a_j + \frac{1}{p}\sum_i i a_i$$

which is equivalent to

$$(55)\quad 2\sum_{i<j} ia_ia_j\Big(1-\frac{j}{p}\Big)+\sum_{i \text{ odd}} ia_i^2\Big(1-\frac{i}{p}\Big)+\frac{1}{p}\sum_{i \text{ even}} ia_i\Big(a_i(p-i)-1\Big)+\sum_{i \text{ odd}} a_i\Big(1-\frac{i}{p}\Big)\geq 0$$

It is straightforward to check that each summand in (55) is non-negative. For example, for i even and $a_i \neq 0$ we have $a_i(p-i)-1 \geq a_i(p-(p-1))-1 \geq 0$, since $i \leq p-1$. The result follows. *q.e.d.*

Lemma 11.2.4. *Assume $p < n$ is odd. Let $x \in H$ be an element of order p. Then $f_\Omega(x) = \frac{1}{p}$ if, and only if, $n = ap$ and $x \in [J_p^a]^G$.*

Proof. For $x = [J_p^a]$, the result follows by an easy calculation using Proposition 9.3.2(iii).

Conversely, let $x \in H$ such that $f_\Omega(x) = \frac{1}{p}$. Then, by the proof of Lemma 11.2.3, each of the summands in (55) vanishes. Let $i < p$ be odd, then $a_i(p-i) = 0$ hence $a_i = 0$. For i even we have $ia_i(a_i(p-i)-1) = 0$ if, and only if $a_i = 0$ or $i = p-1$ and $a_{p-1} = 1$. Therefore, using also $n = \sum_i ia_i$, we deduce that either $n = ap$ and $x = [J_p^a]$ or $n = ap + (p-1)$ and $x = [J_p^a, J_{p-1}]$. Notice that the latter element does not lie in O_n. The result follows. *q.e.d.*

In the case $p \mid n$, Lemma 11.2.4 provides all the elements of order p that realise the lower bound. So we may assume $p \nmid n$. The following shows that the bound $\frac{1}{p}$ is close to the best possible.

Proposition 11.2.5. *Assume $p < n$ is odd and $n = ap+b$ with $0 < b < p$. Then there exists $x \in H$ of order p such that*

$$f_\Omega(x) < \frac{1}{p} + \frac{1}{4(n+1)} + \frac{4}{n(n+1)}$$

Proof. Let $x_1 = [J_p^a, J_b]$ if b is odd and $x_2 = [J_p^a, J_{b-1}, J_1]$ if b is even. Then x_1, x_2 lie in O_n. Using Proposition 9.3.2(iii) and $a = \frac{n-b}{p}$, we compute

$$f_\Omega(x_1) = \frac{1}{p} + \frac{b(p-1-b)}{pn(n+1)} + \frac{1}{n(n+1)} \leq \frac{1}{p} + \frac{(p-1)^2}{4pn(n+1)} + \frac{1}{n(n+1)}$$

where the inequality follows from $g(b) := b(p-1-b) \leq g(\frac{p-1}{2})$. We also observe that $\frac{(p-1)^2}{4pn(n+1)}$ is monotonically increasing in p for $2 < p < n$. Thus

$$f_\Omega(x_1) < \frac{1}{p} + \frac{(n-1)^2}{4n^2(n+1)} + \frac{1}{n(n+1)} < \frac{1}{p} + \frac{1}{4(n+1)} + \frac{1}{n(n+1)}$$

Similarly,

$$f_\Omega(x_2) = \frac{1}{p} + \frac{b(p-1-b)}{pn(n+1)} + \frac{4}{n(n+1)} < \frac{1}{p} + \frac{1}{4(n+1)} + \frac{4}{n(n+1)}$$

where the inequality follows by the same argument as above. *q.e.d.*

Now assume $\mathbf{p \geq n}$. In this case the largest Jordan block allowed in an element of order p is J_n. Recall (23), the definition of $\delta_{a;b}$

Lemma 11.2.6. *Assume $p \geq n$. Let $x \in H$ be an element of order p. Then*
$$f_\Omega(x) \geq \frac{1}{n} + \frac{3}{n(n+1)}\delta_{n;2}$$

Proof. Assume n is odd. By Proposition 9.3.2(iii), $f_\Omega(x) \geq \frac{1}{n}$ if, and only if,

$$2\sum_{i<j} ia_ia_j + \sum_i ia_i(a_i-1) + \sum_{i\text{ odd}} a_i - 1 \geq 0 \tag{56}$$

clearly the first two summands are non-negative. Since n is odd, there exists i odd such that $a_i \neq 0$. In particular $a_i \geq 1$. So we deduce that the last summand is non-negative.

Now assume n is even. The result is equivalent to

$$2\sum_{i<j} ia_ia_j + \sum_i ia_i(a_i-1) - 4 + \sum_{i\text{ odd}} a_i \geq 0 \tag{57}$$

Assume there exists ℓ odd such that $a_\ell \neq 0$. Since n is even either there exists ℓ' odd such that $a_{\ell'} \neq 0$ or $a_i = 0$ for all $i \neq \ell$ odd and a_ℓ is even. If, instead, for all i odd $a_i = 0$ then there exists ℓ even such that $a_\ell \neq 0$, so $a_\ell \geq 2$ (since $x \in \mathrm{O}_n$).

Case 1. There exist $\ell < \ell'$ odd such that $a_\ell, a_{\ell'} \neq 0$. Then (57) is satisfied since

$$2\sum_{i<j} ia_ia_j - 2 \geq 2\ell a_\ell a_{\ell'} - 2 \geq 0, \quad \sum_i ia_i(a_i-1) \geq 0, \quad \sum_{i\text{ odd}} a_i - 2 \geq a_\ell + a_{\ell'} - 2 \geq 0$$

Case 2. There exists ℓ odd such that $a_\ell \neq 0$ and for all $i \neq \ell$ odd $a_i = 0$. This forces a_ℓ to be even. Again (57) is satisfied as

$$2\sum_{i<j} ia_ia_j \geq 0, \quad \sum_{i\text{ odd}} a_i - 2 \geq a_\ell - 2 \geq 0, \quad \sum_i ia_i(a_i-1) - 2 \geq \ell a_\ell(a_\ell - 1) - 2 \geq 0$$

Case 3. For all i odd, $a_i = 0$. Hence there exists ℓ even with $a_\ell > 0$ (recall that $x \in \mathrm{O}_n$, so a_i is even for i even by Theorem 5.2.1). So

$$2\sum_{i<j} ia_ia_j \geq 0, \quad \sum_{i\text{ odd}} a_i = 0, \quad \sum_i ia_i(a_i-1) - 4 \geq \ell a_\ell(a_\ell - 1) - 4 \geq 0$$

and (57) is satisfied. The result follows. *q.e.d.*

Lemma 11.2.7. *Assume $p > n$. Let $x \in H$ be of order p.*

(i) If n is odd then $f_\Omega(x) = \frac{1}{n}$ if, and only if, $x \in [J_n]^G$.
(ii) If n is even then $f_\Omega(x) = \frac{1}{n} + \frac{3}{n(n+1)}$ if, and only if, $x \in [J_{n-1}, J_1]^G$ or $[J_2^2]^G$.

Proof. For $x = [J_n], [J_{n-1}, J_1]$ or $[J_2^2]$ the result quickly follows, using Proposition 9.3.2(iii).

Assume n is odd and $f_\Omega(x) = \frac{1}{n}$. Then, by the proof of Lemma 11.2.6, each of the summands in (56) vanishes. Since $\sum_{i<j} ia_ia_j = 0$ we deduce that there exists only one index i such that $a_i \neq 0$. Thus $n = ia_i$. The second summand vanishes if, and only if, $a_i = 0$ or 1. Hence $a_n = 1$ and $a_i = 0$ for all $i < n$.

Now assume n is even and $f_\Omega(x) = \frac{1}{n} + \frac{3}{n(n+1)}$. Again, by the proof of Lemma 11.2.6, each of the summands (as organized in the three cases) in (57) vanishes. With the same notation as in Lemma 11.2.6 we have the following.

Case 1. We have $0 = \sum_{i \text{ odd}} a_i - 2 = a_\ell + a_{\ell'} + \sum_{i \neq \ell, \ell'} a_i - 2$ hence $a_\ell = a_{\ell'} = 1$ and $a_i = 0$ for all $i \neq \ell, \ell'$ odd. In addition, $0 = 2\sum_{i<j} i a_i a_j - 2 \geq 2\ell a_\ell a_{\ell'} - 2$, so $\ell = 1$ and $a_i = 0$ if $i \neq \ell$ or ℓ'. We have $n = \sum_i a_i = \ell a_\ell + \ell' a_{\ell'} = 1 + \ell'$, hence $x = [J_{n-1}, J_1]$.

Case 2. From $2\sum_{i<j} i a_i a_j = 0$ we deduce that there exists a unique ℓ (which is odd by hypothesis) such that $a_\ell \neq 0$. Thus $0 = \sum_{i \text{ odd}} a_i - 2 = a_\ell - 2$ so $a_\ell = 2$. Then we have $\ell a_\ell (a_\ell - 1) - 2 = 0$ which implies $\ell = 1$. So $n = 2$. This case is excluded since $n > 2$.

Case 3. By hypothesis $a_i = 0$ if i is odd. From $2\sum_{i<j} i a_i a_j = 0$ we deduce that there exists only one even index ℓ such that $a_\ell \neq 0$. Moreover since $0 = \sum_i i a_i (a_i - 1) - 4 = \ell a_\ell (a_\ell - 1) - 4$ we deduce $\ell = 2,\ a_\ell = 2$. Therefore $n = 4$ and $x = [J_2^2]$. *q.e.d.*

11.2.2. Involutions. Here we assume $p = 2$ and we derive lower bounds on $f_\Omega(x)$ for $x \in H$ an involution.

Lemma 11.2.8. *Assume $p = 2$. Let $x \in H$ be an involution.*

(i) If n is odd then

$$f_\Omega(x) \geq \frac{1}{2} + \frac{3}{2(n+1)} - \frac{3}{2n(n+1)}$$

with equality if, and only if $\nu(x) = \frac{n-1}{2}$.

(ii) If n is even then

$$f_\Omega(x) \geq \frac{1}{2} + \frac{3}{2(n+1)}$$

with equality if, and only if $\nu(x) = \frac{n}{2}$ or $\frac{n}{2} - 1$.

Proof. Thanks to Proposition 9.3.2(iv) we know that given an involution x with $\nu(x) = s$ then $f_\Omega(x) = 1 - \frac{2s(n-1-s)}{n(n+1)}$. With a straightforward computation we see that $f_\Omega(x)$ is minimal if, and only if, $\nu(x) = \frac{n-1}{2}$ if n is odd and $\nu(x) = \frac{n}{2}, \frac{n}{2} - 1$ if n is even. The result follows. *q.e.d.*

11.3. Semisimple elements: lower bound

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H$ an element of order $r \neq p$. We shall prove the following.

Proposition 11.3.1. *Let $x \in H$ be an element of order $r \neq p$.*

(i) If $r \leq n$ then

$$f_\Omega(x) \geq \frac{1}{r}$$

(ii) If $r > n$ then

$$f_\Omega(x) \geq \frac{1}{n+1}$$

Remark 11.3.2. Notice that we will derive the best possible lower bounds, see Corollary 11.3.5 and Lemma 11.3.7.

11.3.1. Odd prime order elements. The conditions defining semisimple elements in O_n are very similar to the ones defining semisimple elements in Sp_n. Moreover the formulae of f_{G/Sp_n} and f_Ω for semisimple elements are very similar. In fact, the same arguments given in Section 10.3 hold in the case $H = \mathrm{O}_n$, as well.

Recall Definition 10.3.2. An element $x \in H$ of order r is said to be *special* if $|a_i - a_j| \leq 1$ for all i, j.

Notice that Lemmas 10.3.3 and 10.3.4 hold in the case $H = \mathrm{O}_n$, as well. Therefore with the same argument used for Proposition 10.3.5 we have the following. We assume r is an odd prime.

Proposition 11.3.3. *Let $x \in H$ be of odd order r. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

where z is any special element.

The last step in order to have a complete proof of Proposition 11.3.1 is to compute $f_\Omega(x)$ for $x \in H$ a special element. Write $n = ar + b$ with $0 \leq b < r$. Notice that if n is even then $a \equiv b \pmod 2$; if n is odd then $a \equiv b + 1 \pmod 2$. For the purpose of giving the structure of the general special element we define

$$A = [(\omega, \omega^{-1})I_{a+1}, \ldots, (\omega, \omega^{-1})^{\lfloor \frac{b}{2} \rfloor} I_{a+1}, (\omega, \omega^{-1})^{\lfloor \frac{b}{2} \rfloor + 1} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a]$$

Then if z is a special element we have $C_G(z) \cong C_G(x)$ where

$$x = \begin{cases} [I_a, A] & n \text{ even} \quad a \text{ even} \\ [I_{a+1}, A] & n \text{ even} \quad a \text{ odd} \\ [I_{a+1}, A] & n \text{ odd} \quad a \text{ even} \\ [I_a, A] & n \text{ odd} \quad a \text{ odd} \end{cases} \tag{58}$$

Using Proposition 9.3.2(i) we compute $f_\Omega(x)$ for $x \in H$ a special element as in (58). **First assume r < n.**

Case 1. Let x as in (58). Using *Mathematica*, we compute

$$f_\Omega(x) = \frac{1}{r} + \frac{b(r-1-b)}{rn(n+1)} \geq \frac{1}{r} \tag{59}$$

where the inequality follows from the fact that $g(b) := b(r-1-b) \geq g(0) = g(r-1) = 0$ for $b \in [0, r-1]$, observe that $f_\Omega(x) = \frac{1}{r}$ if, and only if, $b = 0$ or $r - 1$.

Moreover we see that $g(b) \leq g\left(\frac{r-1}{2}\right)$. Hence, assisted by *Mathematica*, we get

$$f_\Omega(x) \leq \frac{1}{r} + \frac{(r-1)^2}{4rn(n+1)} < \frac{1}{r} + \frac{(n-1)^2}{4n^2(n+1)} < \frac{1}{r} + \frac{1}{4(n+1)}$$

where the second inequality follows from the fact that $\frac{(r-1)^2}{4rn(n+1)}$ is monotonically increasing in r and $r < n$.

Case 2. Let $x = [I_{a+1}, A]$ as in (58). Using *Mathematica*, we compute

$$f_\Omega(x) = \frac{1}{r} + \frac{b(r-1-b)}{rn(n+1)} + \frac{1}{n(n+1)} \geq \frac{1}{r} \tag{60}$$

as above, we use $g(b) = b(r-1-b) \geq g(0) = g(r-1) = 0$ for $b \in [0, r-1]$. Notice that $f_\Omega(x) = \frac{1}{r} + \frac{1}{n(n+1)}$ if, and only if, $b = 0$ or $r - 1$.

Again, since $g(b) \leq g\big(\frac{r-1}{2}\big)$, we get

$$f_\Omega(x) \leq \frac{1}{r} + \frac{(r-1)^2}{4rn(n+1)} + \frac{1}{n(n+1)} < \frac{1}{r} + \frac{1}{4(n+1)} + \frac{1}{n(n-1)}$$

Remark 11.3.4. Let $r \neq p$ be an odd prime and x be an element of order $r < n$. Then, thanks to Proposition 11.3.3, the best possible lower bounds are computed in (59) and (60). In particular, the conclusion of Proposition 11.3.1 holds.

We summarise the above computation of the best possible lower bounds in the following.

Corollary 11.3.5. *Let $x \in H$ be an element of odd order $r < n$. Then*

$$f_\Omega(x) \geq \frac{1}{r} + \frac{b(r-1-b)}{rn(n+1)} + \frac{\iota}{n(n+1)}$$

where $\iota = 1$ if $n \not\equiv \lfloor n/r \rfloor \pmod 2$ and 0 otherwise.

Now assume r > n. Then any special element has centraliser isomorphic to the one of $x = [I_{\delta_{n-1;2}}, (\omega, \omega^{-1}), \dots, (\omega, \omega^{-1})^{\lfloor n/2 \rfloor}]$. In the following we use Proposition 9.3.2 and we also characterise elements that realise the bound.

Lemma 11.3.6. *Assume $r \geq n$. Let $x \in H$ be of order r. Then*

$$f_\Omega(x) \geq \frac{1}{n + \delta_{n;2}}$$

with equality if, and only if, $C_G(x) \cong \mathrm{O}_{\delta_{n-1;2}} \times (\mathrm{GL}_1)^{\lfloor n/2 \rfloor}$

Proof. First assume n is odd. Then $a_0 \geq 1$ is odd. Using Proposition 9.3.2, we have $f_\Omega(x) \geq 1/n$ if, and only if, $\sum_i a_i^2 + a_0 \geq n + 1 = \sum_i a_i + 1$. This last inequality is equivalent to $\sum_{i \neq 0} a_i(a_i - 1) + a_0^2 - 1 \geq 0$, which is clearly satisfied since $a_0^2 - 1 \geq 0$ and $a_i(a_i - 1) \geq 0$ for all $i > 0$.

It is clear that if $a_i = 0, 1$ and $a_0 = 1$ then $f_\Omega(x) = 1/n$. Conversely, assume $f_\Omega(x) = 1/n$. Then, by the proof of the inequality, $a_0 = 1$ and $a_i = 0, 1$. The result follows.

Similarly if n is even. *q.e.d.*

11.3.2. Involutions. We assume $p \neq 2$. Thanks to Proposition 9.3.2(ii), it is straightforward to derive lower bounds on $f_\Omega(x)$ for $x \in H$ an involution. As already seen in the proof of Lemma 11.1.6, $f_\Omega(x)$, as function in $\nu(x)$, is monotonically decreasing in $[2, n/2]$. We have the following, which concludes the proof of Proposition 11.3.1.

Lemma 11.3.7. *Assume $p \neq 2$. Let $x \in H$ be an involution. Then*

$$f_\Omega(x) \geq \frac{1}{2} + \frac{1}{2(n + \delta_{n;2})}$$

with equality if, and only if, $\nu(x) = \lfloor n/2 \rfloor$.

11.4. Local upper bound

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$.

Proposition 11.4.1. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then.*

$$f_\Omega(x) \leq 1 - \frac{s}{n+1}$$

We start by deriving local upper bounds for unipotent elements, then we move on to semisimple elements. The analysis for semisimple elements will mirror that used in the case $H = \mathrm{Sp}_n$, see Section 10.4.2.

11.4.1. Unipotent elements.

Let $x \in H \cap \mathcal{V}_{s,p}$. Then, up to G-conjugacy, $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ and $n - s = \sum_i a_i$. We have the following.

Proposition 11.4.2. *Let $x \in H \cap \mathcal{V}_{s,p}$.*

(i) If $s \leq \frac{n}{2}$ then

$$f_\Omega(x) \leq 1 - \frac{2s(n-s)}{n(n+1)}$$

(ii) If $s > \frac{n}{2}$ then

$$f_\Omega(x) \leq 1 - \frac{s}{n+1}$$

In particular, the conclusion of Proposition 11.4.1 holds.

Proof. By Proposition 9.3.2(iii) we have, for $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$,

$$f_\Omega(x) = \frac{2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 + \sum_{i \text{ odd}} a_i}{n(n+1)}$$

The result follows applying

$$n(n+1) f_\Omega(x) \leq \begin{cases} (n-s)^2 + s^2 + n & s \leq \frac{n}{2} \\ n(n-s+1) & s > \frac{n}{2} \end{cases}$$

which follows by the proof of [**43**, Lemma 3.4]. *q.e.d.*

The following example is meant to show the existence of $x \in H \cap \mathcal{V}_{s,r}$ (for $s \leq n/2$), such that $f_\Omega(x)$ is close to the bound established.

Example 11.4.3. If $s \leq \frac{n}{2}$ is even then $x = [J_2^s, J_1^{n-2s}] \in H \cap \mathcal{V}_{s,p}$. So, using Proposition 9.3.2(iii) we compute $f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n+1)} - \frac{2s}{n(n+1)} \geq 1 - \frac{2s(n-s)}{n(n+1)} - \frac{1}{n+1}$.

11.4.2. Semisimple elements.

Here we assume $x \in H \cap \mathcal{V}_{s,r}$ and $r \neq p$ is an odd prime. So, we may assume that either $x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ with s even and $a_i \leq n - s$, or $x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \ldots, \omega^{r-1} I_{n-s}]$ with $s \geq \frac{n}{2}$.

Then, write $n = a(n-s) + b$, with $0 \leq b < n - s$. As in Section 10.4.2, let

$$B = [(\omega, \omega^{-1}) I_{n-s}, \ldots, (\omega, \omega^{-1})^{\lfloor \frac{a-1}{2} \rfloor} I_{n-s}]$$
$$B' = [(\omega, \omega^{-1}) I_{n-s}, \ldots, (\omega, \omega^{-1})^{\lfloor \frac{a}{2} \rfloor} I_{n-s}]$$

We define

$$z = \begin{cases} \left[I_{n-s}, B, (\omega,\omega^{-1})^{\frac{a+1}{2}} I_{\frac{b}{2}}\right] & s \text{ even}, a \text{ odd} \\ \left[I_{n-s}, B, (\omega,\omega^{-1})^{\frac{a}{2}} I_{\frac{n-s+b}{2}}\right] & s \text{ even}, a \text{ even} \\ \left[I_{n-s-1}, B, (\omega,\omega^{-1})^{\frac{a+1}{2}} I_{\frac{b+1}{2}}\right] & s \text{ odd}, a \text{ odd} \\ \left[I_{n-s-1}, B, (\omega,\omega^{-1})^{\frac{a}{2}} I_{\frac{n-s+b+1}{2}}\right] & s \text{ odd}, a \text{ even} \end{cases} \tag{61}$$

and

$$z' = \begin{cases} [B', (\omega,\omega^{-1})^{\frac{a+1}{2}} I_{\frac{n-s+b}{2}}] & a \text{ odd},\ n \text{ even} \\ [B', (\omega,\omega^{-1})^{\frac{a}{2}+1} I_{\frac{b}{2}}] & a \text{ even},\ n \text{ even} \\ [1, B', (\omega,\omega^{-1})^{\frac{a}{2}+1} I_{\frac{b-1}{2}}] & a \text{ even},\ n \text{ odd} \\ [1, B', (\omega,\omega^{-1})^{\frac{a+1}{2}} I_{\frac{n-s+b-1}{2}}] & a \text{ odd},\ n \text{ odd} \end{cases} \tag{62}$$

The same arguments used in Section 10.4.2 provide the following.

Proposition 11.4.4. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq \max\{f_\Omega(z), f_\Omega(z')\}$$

where z and z' are defined in (61) and (62).

Now, we can derive upper bounds on $f_\Omega(x)$ for any $x \in H \cap \mathcal{V}_{s,r}$.

Proposition 11.4.5. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n+1}$$

Proof. In view of Proposition 11.4.4, it is enough to compute $f_\Omega(z)$ and $f_\Omega(z')$ to deduce upper bounds on $f_\Omega(x)$ for any $x \in H \cap \mathcal{V}_{s,r}$. Recall that z and z' are defined in (61) and (62). Thanks to Proposition 9.3.2(i) we can easily compute $f_\Omega(z)$ and $f_\Omega(z')$. Let us write $n = a(n-s) + b$, where $0 \leq b < n-s$. We use *Mathematica* for the following calculations.

If a is odd we have

$$f_\Omega(z) = 1 - \frac{s}{n} - \frac{b(n-s-\frac{b}{2})}{n(n+1)} \leq 1 - \frac{s}{n} \tag{63}$$

If a is even we compute

$$f_\Omega(z) = 1 - \frac{s}{n} - \frac{(n-s)^2 - b^2}{2n(n+1)} < 1 - \frac{s}{n} \tag{64}$$

Assume s is odd and $a_0 = n - s - 1$. If a is even then

$$f_\Omega(z) = 1 - \frac{s}{n} - \frac{2b(b-s-1) - b^2 + 4(n-s) - 1}{2n(n+1)} \leq 1 - \frac{s}{n} - \frac{4(n-s)-1}{2n(n+1)} \tag{65}$$

where the inequality follows from $b^2 - 2b(n-s-1) \leq 0$.

If a is odd then we compute

$$f_\Omega(z) = 1 - \frac{s}{n+1} - \frac{(n-s)^2 - b(b+2) + 2n - 1}{2n(n+1)} \leq 1 - \frac{s}{n+1} - \frac{1}{n+1} \tag{66}$$

where the inequality follows from the fact that $g(b) := (n-s)^2 - b(b+2) + 2n - 1 \geq g(n-s-1) = 2n$.

Now we compute $f_\Omega(z')$. First, let us assume n even. If a is odd then

$$f_\Omega(z') = \frac{1}{2} - \frac{s^2 - b^2 + n}{2n(n+1)} < 1 - \frac{s}{n} \tag{67}$$

where the inequality follows from the following computation (recall $b < n - s$)

$$1 - \frac{s}{n} - \frac{1}{2} + \frac{s^2 - b^2 + n}{2n(n+1)} = \frac{(n-s)^2 - b^2 + 2(n-s)}{2n(n+1)} > 0$$

If a is even we compute

$$f_\Omega(z') = 1 - \frac{s+1}{n+1} - \frac{b(n-s-\frac{b}{2})}{n(n+1)} \leq 1 - \frac{s+1}{n+1} \tag{68}$$

Now assume n is odd. If a is even we have

$$f_\Omega(z') = 1 - \frac{s+1}{n+1} - \frac{b(n-s+1-\frac{b}{2})}{n(n+1)} + \frac{5}{2n(n+1)} \leq 1 - \frac{s}{n+1} \tag{69}$$

where the last inequality follows from $b(n-s+1-\frac{b}{2}) \geq 0$ and $1-\frac{s+1}{n+1}+\frac{5}{2n(n+1)} \leq 1-\frac{s}{n+1}$ for $n \geq 3$.

If a is even we compute

$$f_\Omega(z') = \frac{n^2 - 2n - s^2 + 2s + 5 + b(b-2)}{2n(n+1)} \tag{70}$$

and we have $1 - \frac{s}{n+1} - f_\Omega(z') = \frac{(n-s)^2 - b(b-2) + 2(n-s) + 2n - 5}{2n(n+1)} > 0$, since $b < n - s$. *q.e.d.*

Remark 11.4.6. Thanks to Proposition 11.4.4 the best possible upper bound on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$ is given by $\max\{f_\Omega(z), f_\Omega(z')\}$. The values of $f_\Omega(z)$ and $f_\Omega(z')$ have been computed in (63)–(70).

Let us denote by U the bound of Proposition 11.4.5. We bound the differences $U - f_\Omega(z)$ and $U - f_\Omega(z')$ in several cases. We omit the computations of the following.

For $f_\Omega(z)$ as in (63) and (64) we have $U - f_\Omega(z) < \frac{2s+(n-s)^2}{2n(n+1)}$.

For $f_\Omega(z')$ as in (67), (68), (69) and (70) we have $U - f_\Omega(z') < \frac{2}{n+1} + \frac{(n-s)^2}{n(n+1)}$.

In particular, in all the cases listed above if $s > n - \sqrt{n}$ we have

$$U - \max\{f_\Omega(z), f_\Omega(z')\} < \frac{5}{n+1}$$

11.5. Local lower bound

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$. We shall prove the following.

Proposition 11.5.1. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s+1)}{n(n+1)} - \frac{1}{n+1}$$

We study separately the cases $r = p$ and $r \neq p$.

11.5.1. Unipotent elements. Let $x \in H \cap \mathcal{V}_{s,p}$. Up to G-conjugacy, $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ and $n - s = \sum_i a_i$. We have the following.

Proposition 11.5.2. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s+1)}{n(n+1)}$$

In particular, the conclusion of Proposition 11.5.1 holds.

Proof. By the proof of [**43**, Lemma 3.4] we have, using Proposition 9.3.2,

$$f_\Omega(x) = \frac{\sum_{i<j} ia_ia_j + \sum_i ia_i^2 + \sum_{i \text{ odd}} a_i}{n(n+1)} \geq \frac{(n-s)(n-s+1)}{n(n+1)}$$

The result follows. *q.e.d.*

In the following we show that the bound is close to best possible.

Example 11.5.3. Assume $s < p$ is odd. Let $x = [J_{s+1}, J_1^{n-s-1}]$. Then $x \in H \cap \mathcal{V}_{s,p}$. Using Proposition 9.3.2(iii) we compute

$$f_\Omega(x) = 1 - \frac{s(2n-s)}{n(n+1)} - \frac{1}{n(n+1)}$$

Let ℓ be the bound in Proposition 11.5.2. Then $f_\Omega(x) - \ell \leq \frac{s-1}{n(n-1)} < \frac{1}{n}$.

11.5.2. Semisimple elements. Here we assume $x \in H \cap \mathcal{V}_{s,r}$ with $r \neq p$. So, up to G-conjugacy, we can write

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \tag{71}$$

with s even, or

$$x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \ldots, \omega^{r-2} I_{a_{r-2}}, \omega^{r-1}, \omega^{r-1} I_{n-s}] \tag{72}$$

with $s \geq \frac{n}{2}$.

We prove the following.

Proposition 11.5.4. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s+1)}{n(n+1)} + \frac{s^2}{n(n+1)(r-1)} - \frac{1}{n+1}$$

In particular, the conclusion of Proposition 11.5.1 holds.

Proof. Assume x is as in (71) then, by Proposition 9.3.2(i) we have

$$\begin{aligned} f_\Omega(x) &= \frac{(n-s)(n-s+1) + \sum_{i>0} a_i^2}{n(n+1)} \geq \frac{(n-s)(n-s+1) + \frac{s^2}{r-1}}{n(n+1)} \\ &= 1 - \frac{s(2n-s+1)}{n(n+1)} + \frac{s^2}{n(n+1)(r-1)} \end{aligned}$$

where we used $\sum_{i>0} a_i = s$ and Proposition B.2.1.

If x is as in (72) we get

$$f_\Omega(x) = \frac{(n-s)^2 + \sum_{i\neq 1} a_i^2 + a_0}{n(n+1)} \geq \frac{(n-s)^2 + \frac{s^2}{r-1}}{n(n+1)}$$
$$> 1 - \frac{s(2n-s+1)}{n(n+1)} + \frac{s^2}{n(n+1)(r-1)} - \frac{1}{n+1}$$

q.e.d.

As done in Section 10.5.2 we have the following.

Proposition 11.5.5. *Let $x \in H \cap \mathcal{V}_{s,r}$ with $s < \frac{n}{2}$. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

where z is a special element. Moreover equality holds if, and only if, x is special.

Write $s = a(r-1) + b$ with $0 \leq b < r-1$. Let

$$z = [I_{n-s}, \omega I_{a+1}, \ldots, \omega^{\frac{b}{2}} I_{a+1}, \omega^{\frac{b}{2}+1} I_a, \ldots, \omega^{r-1} I_a] \tag{73}$$

Then z is special. In addition, if x is special then $C_G(x) \cong C_G(z)$.

Now we compute $f_\Omega(z)$ for z as in (73). Using Proposition 9.3.2 we have

$$f_\Omega(z) = 1 - \frac{s(2n-s+1)}{n(n+1)} + \frac{b(r-1-b)}{n(n+1)(r-1)} + \frac{s^2}{n(n+1)r-1}$$

Observe that $b(r-1-b) < \frac{(r-1)^2}{4}$. Hence, if $r-1 \leq s$ we have $0 \leq b < r-1$ and

$$f_\Omega(z) \leq 1 - \frac{s(2n-s+1)}{n(n+1)} + \frac{s^2}{n(n+1)r-1} + \frac{s}{4n(n+1)}$$

In the case $r-1 > s$ we have $z = [I_{n-s}, (\omega, \omega^{-1}), \ldots, (\omega, \omega^{-1})^{\frac{s}{2}}]$ and

$$f_\Omega(z) = 1 - \frac{s(2n-s)}{n(n+1)}$$

Denote by ℓ the bound of Proposition 11.5.1. Then

$$f_\Omega(z) - \ell = \frac{s+n}{n(n+1)} < \frac{2}{n+1}$$

Therefore the bound is close to best possible.

11.6. Further comments on local bounds

We easily deduce the following from Propositions 11.4.1 and 11.5.1.

Proposition 11.6.1. *Assume $r \neq p$ is an odd prime. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| \leq \frac{s(n-s)}{n(n+1)} + \frac{2}{n+1}$$

Proof. Using the bounds in Propositions 11.4.1 and 11.5.1 we have

$$|f_\Omega(x) - f_\Omega(y)| \leq \frac{s(n-s)}{n(n+1)} + \frac{n+s}{n(n+1)} < \frac{s(n-s)}{n(n+1)} + \frac{2}{n+1}$$

q.e.d.

Remark 11.6.2. If $s \leq \sqrt{n}$ or $s \geq n - \sqrt{n}$ we have $\frac{s(n-s)}{n(n-1)} \leq \frac{2}{\sqrt{n}}$. Therefore the f_Ω-values of elements in $H \cap \mathcal{V}_{s,r}$ are close to each other. In fact, with this assumption on x, $|f_\Omega(x) - f_\Omega(y)| < \frac{2}{\sqrt{n}} + \frac{2}{n+1}$.

Part 3

$\mathscr{C}_3$-actions of classical groups

CHAPTER 12

Introduction

Let $G = \mathrm{Sp}_n$ or O_n, with $n > 4$ even, defined over an algebraically closed field of characteristic p. A closed subgroup $H \leq G$ is a $\mathscr{C}_3$-subgroup if there exist $U, W \leq V$ maximal totally singular subspaces such that $V = U \oplus W$ and $H = \mathrm{Stab}_G\{U, W\}$. So

$$H^\circ = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\} \cong \mathrm{GL}_{n/2}$$

and $H = \mathrm{GL}_{n/2}.2$, see Proposition 13.1.3. Set $\Omega = G/H$ the coset variety.

In this chapter we state the main results of this part, in which we derive bounds on $f_\Omega(x)$ for $x \in H$ of prime order or any unipotent element in the characteristic zero case.

12.1. Main results

Recall the definition of $\mathscr{R}$ as the set of elements of G of prime order, including any unipotent elements in characteristic zero; in this last case recall that we set $p = \infty$. Theorem 12.1.1 below provides global upper and lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order.

Theorem 12.1.1. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_3$-subgroup. Set $\Omega = G/H$.*

(a) Let $x \in G \setminus Z(G)$. Then

$$f_\Omega(x) \leq 1 - \frac{4}{n} + \left(\frac{4}{n}\right)^2$$

(b) Let $x \in H \cap \mathscr{R}$ be unipotent.

(i) If $p \neq 2$ then

$$f_\Omega(x) \geq \begin{cases} \frac{1}{p} & \text{if } p < \frac{n}{2} \\ \frac{2}{n} - \frac{4}{n(n+2)} & \text{if } p \geq \frac{n}{2} \end{cases}$$

(ii) If $p = 2$ then

$$f_\Omega(x) \geq \frac{1}{2} - \frac{2}{n+2}$$

(c) Let $x \in H \cap \mathscr{R}$ be semisimple of order $r < n$. Then

$$f_\Omega(x) \geq \frac{1}{r} - \frac{r}{n(n-2)}$$

Remark 12.1.2. Let us make some comments on the statement of Theorem 12.1.1.

(i) Note that the cases $r \neq p$ and $r > n$ are omitted. In this case we classify elements with vanishing f_Ω-value in Theorem 12.1.6.

(ii) Referring to part (a), notice that the bound does not depend on r and p. Studying separately the cases $G = \mathrm{Sp}_n$ and O_n, and taking into account the order of the element, we will derive sharp upper bounds, see Proposition 14.1.1

for odd order elements and Corollary 14.7.2 for involutions ($G = \mathrm{Sp}_n$), and Proposition 15.1.1 and Corollary 15.7.2 ($G = \mathrm{O}_n$).

(iii) The lower bounds extend to any element $x \in \mathscr{R}$ for which $x^G \cap H \neq \emptyset$.

(iv) Let $p < n/2$ be odd. The lower bound in (b)(i) is proved in Lemma 14.2.3 ($G = \mathrm{Sp}_n$) and Lemma 15.2.3 ($G = \mathrm{O}_n$). Moreover this bound is the best possible. Indeed, we characterise the elements that realise it; see Lemmas 14.2.6 and 15.2.4 for $G = \mathrm{Sp}_n$ and O_n, respectively. In the case $p \nmid n$, in Propositions 14.2.8 and 15.2.5 (for $G = \mathrm{Sp}_n$ and O_n, respectively) we show the existence of an element $y \in H$ of order p for which $f_\Omega(y) - \frac{1}{p} \leq \frac{1}{n}$.

(v) For $p \geq n/2$ we deduce the lower bound in (b)(i) from Lemmas 14.2.5 and 15.2.7. The bounds stated in the aforementioned two results are sharp. Indeed if $G = \mathrm{Sp}_n$ we characterise elements that realise it, (see Lemma 14.2.7). If $G = \mathrm{O}_n$ we present two elements that realise the lower bound of Lemma 15.2.7 and we conjecture that those elements are the only ones with this property (see Proposition 15.2.8 and Conjecture 15.2.9)

(vi) The lower bound in (c) is not sharp, in general. However, we shall define a collection of so-called *special* elements, (see Definitions 14.3.10 and 15.3.9, for $G = \mathrm{Sp}_n$ and O_n, respectively), and we prove that special elements realise the best possible lower bound on f_Ω, (for details see Lemmas 14.3.13 and 15.3.12). Moreover the best possible lower bounds are computed and we list them in Table 14.3.1 ($G = \mathrm{Sp}_n$) and Table 15.3.1 ($G = \mathrm{O}_n$). The stated bound follows from Propositions 14.3.5 and 15.3.5.

(vii) For involutions, we shall give an explicit formula for $f_\Omega(x)$, see Theorem 12.1.13. In particular, the best possible lower bounds, with characterisations of elements that realise them, are stated in Corollaries 14.7.2 and 15.7.2 for $G = \mathrm{Sp}_n$ and O_n, respectively.

Recall Definition 7.3.1 of the algebraic fixity M, the r-local algebraic fixity M_r, and $M_{r'}$.

Theorem 12.1.3. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_3$-subgroup. Set $\Omega = G/H$.*

(i) If $G = \mathrm{Sp}_n$ then

$$M = 1 - \frac{4}{n} + \frac{16}{n(n+2)}$$

Moreover $M = M_2$ and $M > M_{2'}$.

(ii) If $G = \mathrm{O}_n$ assume $n > 10$. Then

$$M = 1 - \frac{4}{n} + \frac{8}{n(n-2)}$$

Moreover $M = M_p$ and $M > M_{p'}$.

In addition, for any r, $M_r \geq 1 - 4/n$.

Remark 12.1.4. Theorem 12.1.3 easily follows by combining the upper bounds and the characterisations proved in Propositions 14.1.1, 15.1.1 and Corollaries 14.7.2, 15.7.2. We could state a version of this result also for O_n when $n = 6, 8, 10$. However in this situation there are several cases. If $n = 10$ then $M = M_r$ for any r. If $n = 6, 8$ then $M = 2/3$ and $M = M_r$ for any odd prime $r \neq 2$.

Corollary 12.1.5. *For any prime r, there exists $x \in H$ of order r such that*

$$f_\Omega(x) \geq 1 - \frac{4}{n}$$

Let G be a connected reductive algebraic group and let P be a parabolic subgroup, set $\mathcal{P} = G/P$. In [**31**, Theorem 1(b)] it is stated that an element $x \in G$ is regular if, and only if, $f_{\mathcal{P}}(x) = 0$. We prove a similar version of this result for $\mathscr{C}_3$-actions.

Theorem 12.1.6. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_3$-subgroup. Set $\Omega = G/H$. Let $x \in G$. Assume $x^G \cap H \neq \emptyset$. Then the following are equivalent*

(i) *$C_\Omega(x)$ is finite;*
(ii) *$f_\Omega(x) = 0$;*
(iii) *x is regular.*

Let $x \in G$. Recall from Proposition 5.1.7, that $\nu(x)$ is the codimension of the largest eigenspace of x. In the proof of various results which lead to the bounds stated in Theorem 12.1.1, we see a quite strong relation between $f_\Omega(x)$ and $\nu(x)$. Roughly speaking, $f_\Omega(x)$ is maximal for elements x such that $\nu(x)$ is minimal and vice-versa.

Thus, we proceed with the analysis of bounds on $f_\Omega(x)$ by studying elements in $\mathcal{V}_s = \{x \in G : \nu(x) = s\}$ and $\mathcal{V}_{s,r} = \{x \in \mathcal{V}_s : o(x) = r\}$.

In Theorems 12.1.7 and 12.1.9, below, we assume x is not an involution. Indeed if x is an involution then, apart a few exceptions, $\nu(x)$ precisely determines the value of $f_\Omega(x)$, see Theorem 12.1.13 in which we state explicit formulae for $f_\Omega(x)$ when $x \in G$ is an involution.

We deduce the following from the main results of Sections 14.4 and 15.4.

Theorem 12.1.7. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_3$-subgroup. Set $\Omega = G/H$. Assume $r \neq 2$. Let $x \in \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n+2} + \frac{2}{n}$$

Remark 12.1.8. We shall compute close to best possible upper bounds on $f_\Omega(x)$ for each of the two cases $G = \mathrm{Sp}_n$ and O_n when x is either a unipotent or a semisimple element of prime order. For example, if $G = \mathrm{Sp}_n$ we shall prove $f_\Omega(x) \leq 1 - \frac{s}{n}$ for all $x \in \mathcal{V}_s$.

(i) For unipotent elements better upper bounds are computed in Propositions 14.4.4 and 15.4.3 for $G = \mathrm{Sp}_n$ and O_n, respectively. Then in Propositions 14.4.6 ($G = \mathrm{Sp}_n$) and 15.4.6 ($G = \mathrm{O}_n$) we prove that these bounds are close to best possible, constructing an element in $H \cap \mathcal{V}_{s,p}$ whose f_Ω-value is close to the bound computed.

(ii) For semisimple elements we provide more information. We shall construct elements $z_1, z_2 \in H \cap \mathcal{V}_{s,r}$, see (102) and (103) for $G = \mathrm{Sp}_n$ and (137) and (138) for $G = \mathrm{O}_n$. Then we prove that the optimal upper bound is $\max\{f_\Omega(z_1), f_\Omega(z_2)\}$, see Propositions 14.4.13 and 14.4.13, for $G = \mathrm{Sp}_n$ and O_n, respectively. Then we derive bounds by studying the value of $f_\Omega(z_1)$ and $f_\Omega(z_2)$.

The following provides local lower bounds.

Theorem 12.1.9. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_3$-subgroup. Set $\Omega = G/H$. Assume $r \neq 2$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n(n-2)} - \iota$$

where $\iota = \frac{1}{n}$ if $(G, r) = (\mathrm{Sp}_n, p)$, and $\iota = 0$ otherwise.

Remark 12.1.10. Also here, we observe that studying separately the two cases $G = \mathrm{Sp}_n$ or O_n we can derive more accurate lower bounds, see Sections 14.5 and 15.5. For semisimple elements, as in the previous cases, we have more information. We shall define a family of so-called *special* elements, see Definition 14.5.8 ($G = \mathrm{Sp}_n$) and 15.5.6 ($G = \mathrm{O}_n$), and we prove that the optimal lower bound is given by the f_Ω-value of some special element, see Lemmas 14.5.11 and 15.5.7.

It is interesting to note that the f_Ω-values of elements in $H \cap \mathcal{V}_{s,r}$ are close to each other.

Theorem 12.1.11. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_3$-subgroup. Set $\Omega = G/H$. Assume $r \neq 2$. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{s(n-s)}{n(n-2)} + \frac{2}{n}$$

Remark 12.1.12. The bound follows from looking separately at the upper and lower bounds in the case $G = \mathrm{Sp}_n$ or O_n. It may be slightly improved in both cases, see Propositions 14.6.1 and 15.6.1. Notice that if $s \leq \sqrt{n}$ or $s \geq n - \sqrt{n}$ then $|f_\Omega(x) - f_\Omega(y)| < \frac{1}{\sqrt{n}} + \frac{2}{n}$.

Given $x \in G$ an involution, we establish an explicit formula for $f_\Omega(x)$. In order to state the following result we introduce the following notation

$$\epsilon = \begin{cases} 1 & G = \mathrm{Sp}_n \\ -1 & G = \mathrm{O}_n \end{cases}$$

Recall the Kronecker delta symbol: $\delta_{a,b} = 1$ if $a = b$ and 0 otherwise.

Theorem 12.1.13. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_3$-subgroup. Set $\Omega = G/H$. Let $x \in G$ be an involution with $\nu(x) = s$. Assume $x^G \cap H \neq \emptyset$.*

(i) If $s < n/2$ then s is even and

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n+2\epsilon)} + \iota$$

where $\iota = 0$ if $G = \mathrm{Sp}_n$ and $\iota = \frac{4s}{n(n-2)}\delta_{p,2}$ if $G = \mathrm{O}_n$.

(ii) If $s = n/2$ then $n/2$ is even if $G = \mathrm{Sp}_n$ and $p > 2$. In addition, we list $(x, f_\Omega(x))$ in Table 12.1.1.

Remark 12.1.14.

(i) These formulae are proved in Propositions 14.7.1 and 15.7.1, for $G = \mathrm{Sp}_n$ and O_n, respectively.

(ii) If $p = 2$ and $x \in G$ is an involution with $\nu(x) = s < n/2$ and $x^G \cap H \neq \emptyset$ then x is a_s-type and $x^G \cap H = x^G \cap H^\circ$, see Lemma 13.2.11.

G	p	x	$f_\Omega(x)$
Sp_n	$\neq 2$	$[I_{n/2}, -I_{n/2}]$	$\frac{1}{2} + \frac{1}{n+2}$
	$= 2$	$a_{n/2}$	$\frac{1}{2} + \frac{2}{n+2}$
		$b_{n/2}$	$\frac{1}{2} - \frac{2}{n+2}$
		$c_{n/2}$	$\frac{1}{2} - \frac{2}{n+2}$
O_n	$\neq 2$	$[I_{n/2}, -I_{n/2}]$	$\frac{1}{2}$
	$= 2$	$a_{n/2}$	$\frac{1}{2} + \frac{2}{n-2}$
		$b_{n/2}$	$\frac{1}{2}$
		$c_{n/2}$	$\frac{1}{2} - \frac{2}{n-2}$

Table 12.1.1. Involutions $x \in H$ with $\nu(x) = n/2$

CHAPTER 13

The $\mathscr{C}_3$-collection

13.1. Structure

Let V be a finite dimensional k-vector space and $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$. Following the description given in [**41**, Section 1] we define the $\mathscr{C}_3$-family as follows:

$\mathscr{C}_3$ For $G = \mathrm{Sp}(V), \mathrm{O}(V)$ we say that $H \in \mathscr{C}_3$ if there exists a decomposition $V = U \oplus W$ where U and W are maximal totally singular subspaces and $H = \mathrm{Stab}_G\{U, W\}$.

By Corollary 4.1.7, U and W are equidimensional. Hence V is even-dimensional.

Remark 13.1.1. The $\mathscr{C}_3$-collection is only defined for $G = \mathrm{Sp}(V)$ and $\mathrm{O}(V)$. Indeed, in the case $G = \mathrm{GL}(V)$ any subspace is totally singular, since the form is the zero form. Hence, if we had $V = U \oplus W$ where U, W are equidimensional H would be of type $\mathrm{GL}_{n/2} \wr S_2$, which is a $\mathscr{C}_2$-subgroup, see Proposition 17.1.6.

Let (V, κ) be a symplectic or an orthogonal space. We define the following set

$$X = \big\{\{U, W\} : V = U \oplus W,\ U, W \text{ maximal totally singular}\big\} \tag{74}$$

Let $G = \mathrm{I}(V, \kappa)$. Then G acts naturally on X, as $g.\{U, W\} = \{g.U, g.W\}$, where $g.U = \{g.u : u \in U\}$. Witt's Lemma implies the following.

Lemma 13.1.2. *Let (V, κ) be a symplectic or an orthogonal space. Let $G = \mathrm{I}(V, \kappa)$. Then the action of G on X is transitive.*

Proof. Let $\{U, W\}$ and $\{U', W'\}$ be in X. We need to show that there exists $g \in G$ such that $g.\{U, W\} = \{U', W'\}$. Since the subspaces taken are all totally singular, there exists an isometry $\varphi \colon U \to U'$. By Witt's Lemma 4.1.6 there exists an isometry $\psi \colon G \to G$ such that $\psi|_U = \varphi$. In particular, $\varphi(U) = U'$ and $\varphi(W) = W'$ and $\psi \in G$. The result follows. *q.e.d.*

Fixing a standard basis as given in Proposition 4.1.8 and 4.1.12, we may identify $G = \mathrm{Sp}_n$ or O_n, with n even.

By Lemma 13.1.2, we may assume $U = \langle e_1, \dots, e_{n/2}\rangle$ and $W = \langle f_1, \dots, f_{n/2}\rangle$.

Proposition 13.1.3. *Let $G = Cl_n$ and $H \le G$ be a $\mathscr{C}_3$-subgroup. Then*

$$H^\circ = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\} \cong \mathrm{GL}_{n/2} \tag{75}$$

Moreover $H \cong \mathrm{GL}_{n/2}.2$.

Proof. First, we notice that the pointwise stabiliser in GL_n of the set $\{U, W\}$ is given by $\mathrm{Stab}_{\mathrm{GL}_n}(U, W) = \{g \in \mathrm{GL}_n : g.U = U, g.W = W\} = \{[A, B] : A, B \in \mathrm{GL}_{n/2}\}$.

In particular, in all the cases other than $(G,p) = (\mathrm{O}_n, 2)$ it is clear that $[A,B] \in \mathrm{Stab}_G(U,W)$ if, and only if, $B = A^{-t}$, see also [**36**, Lemma 4.1.9].

In the case $(G,p) = (\mathrm{O}_n, 2)$ we claim $\mathrm{Stab}_G(U,W) = \mathrm{Stab}_{\mathrm{Sp}_n}(U,W)$ (recall that $\mathrm{O}_n < \mathrm{Sp}_n$ when $p = 2$). It is clear that $\mathrm{Stab}_G(U,W) \subseteq \mathrm{Stab}_{\mathrm{Sp}_n}(U,W)$. Thus we need to show that any element $[A, A^{-t}]$, with $A \in \mathrm{GL}_{n/2}$, fixes the quadratic form Q on V. We follow [**36**, p.101]. For every $v = \sum_i (\alpha_i e_i + \beta_i f_i) \in V$ let $Q(\sum_i \alpha_i e_i + \beta_i f_i) = \sum_i \alpha_i \beta_i$. Then Q is a non-degenerate quadratic form as noted in Example 4.1.2. Let $x = [A, A^{-t}]$, say $A = (a_{ij})$ and $A^{-t} = (b_{ij})$. We have $Q(x.e_i) = Q(\sum_j a_{ij} e_j) = 0$ and $Q(x.f_i) = Q(\sum_j b_{ij} f_j) = 0$. Therefore, any element in $\mathrm{Stab}_{\mathrm{Sp}_n}(U,W)$ fixes Q. Thus $\mathrm{Stab}_G(U,W) = \{[A, A^{-t}] : A \in \mathrm{GL}_{n/2}\} \cong \mathrm{GL}_{n/2}$.

The elements of $\mathrm{Stab}_G\{U,W\} \setminus \mathrm{Stab}_G(U,W)$ interchange the two spaces. Therefore $H = \mathrm{Stab}_G(U,W).\langle\tau\rangle$. The element $\tau \in \mathrm{Stab}_G\{U,W\}$ is defined as $\tau.U = W$ and $\tau.W = U$ and, in the standard basis $\{e_1, \ldots, e_{n/2}, f_1, \ldots, f_{n/2}\}$, it is represented by

$$\tau = \begin{pmatrix} & I_{n/2} \\ \epsilon I_{n/2} & \end{pmatrix} \tag{76}$$

where $\epsilon = +$ if $G = \mathrm{O}_n$, and $\epsilon = -$ when $G = \mathrm{Sp}_n$. Therefore $H \cong \mathrm{GL}_{n/2}.2$ and since $\mathrm{GL}_{n/2}$ is connected we have $H^\circ \cong \mathrm{GL}_{n/2}$. *q.e.d.*

Remark 13.1.4. The corresponding subgroup in the case $G = \mathrm{SO}_n$ may be slightly different. In fact, if $m = n/2$ is odd $H = H^\circ$. Recall, from Section 4.2.5, that the action of O_n on $\mathcal{U}_m$, the set of maximal totally singular subspaces, is transitive and preserves the sets $\mathcal{U}_m^1, \mathcal{U}_m^2$. It is clear that a representative of $\mathcal{U}_m^1$ is $\langle e_1, \ldots, e_m\rangle$ and a representative of $\mathcal{U}_m^2$ is $\langle f_1, \ldots, f_m\rangle$. By definition SO_n is the kernel of the corresponding homomorphism $\gamma \colon \mathrm{O}_n \to \mathrm{Sym}\{\mathcal{U}_m^1, \mathcal{U}_m^2\}$. In addition, SO_n has two orbits on $\mathcal{U}_m$ and γ is surjective. In particular, no element in SO_n interchanges $\langle e_1, \ldots, e_m\rangle$ and $\langle f_1, \ldots, f_m\rangle$. Notice, also, that $\mathrm{GL}_m < \mathrm{SO}(V)$ is not maximal since $\mathrm{GL}_m < P_m < \mathrm{SO}(V)$, where P_m is a parabolic subgroup fixing U.

Remark 13.1.5. Notice that if $(G,p) = (\mathrm{Sp}_n, 2)$ then O_n is a subgroup of G, since, as observed in Section 4.1, symplectic and symmetric bilinear forms coincide when $p = 2$. Hence, $\mathrm{GL}_{n/2}.2 < \mathrm{O}_n < \mathrm{Sp}_n$.

Recall the dimensions of Sp_n and O_n from Table 4.2.1. We have $\dim H = \dim H^\circ = (n/2)^2$. In Table 13.1.1 we record the dimension of $\Omega = G/H$ for $H \in \mathscr{C}_3$.

G	H	$\dim \Omega$
Sp_n	$\mathrm{GL}_{n/2}.2$	$\frac{n}{4}(n+2)$
O_n	$\mathrm{GL}_{n/2}.2$	$\frac{n}{4}(n-2)$

Table 13.1.1. Dimension of $\Omega = G/H$ for $H \in \mathscr{C}_3$

13.2. Conjugacy classes in $\mathscr{C}_3$-subgroups

In this section we shall describe the conjugacy classes of prime order elements in a $\mathscr{C}_3$-subgroup $H \leq G$. In particular, we shall compute $\dim(x^G \cap H)$.

13.2.1. Unipotent elements. Assume $p \neq 2$, see Section 13.2.3 for involutions. Let $x \in G$ be an element of order p. Then, up to G-conjugacy, we write $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$. Notice that for any unipotent element $x \in G$ we have $x^G \cap H = x^G \cap H^\circ$, because $H \setminus H^\circ = H^\circ\tau$ and τ has order either 2 or 4, in particular there are no odd order elements in $H \setminus H^\circ$. Thus we may assume $x \in H^\circ$ is an element of order p. Then $x = [x_1, x_2]$ and $x_2 = x_1^{-t}$. Lemma 5.1.4 shows that J_i and J_i^{-t} are GL_i-conjugate. Therefore a_i is even for all i and, up to conjugation,

$$x = \left(\begin{array}{c|c} [J_p^{a_p/2}, \ldots, J_1^{a_1/2}] & \\ \hline & [J_p^{a_p/2}, \ldots, J_1^{a_1/2}] \end{array}\right) \tag{77}$$

Remark 13.2.1. Let $x \in H^\circ$ be of order p, possibly even. Then $\nu(x)$ is even.

Now, we shall give a formula for $\dim(x^G \cap H)$ for a unipotent element $x \in H$. We have $x^G \cap H^\circ = x^{H^\circ}$, because, as said above, there is only one possible block decomposition. Hence $\dim(x^G \cap H) = \dim x^{H^\circ}$, where $x = [x_1, x_2]$ is as in (77). Using Theorem 5.2.1, we easily compute $\dim(x^G \cap H)$.

For future reference we state the following.

Proposition 13.2.2. *Assume $p \neq 2$. Let $x \in G$ be of order p. Assume $x^G \cap H^\circ \neq \emptyset$. Then $[x_1, x_2]$ as in (77) lies in $x^G \cap H^\circ$ and*

$$\dim(x^G \cap H) = \frac{n^2}{4} - \frac{1}{2}\sum_{i<j} i a_i a_j - \frac{1}{4}\sum_i i a_i^2 \tag{78}$$

13.2.2. Semisimple elements. Here we assume $r \neq p$ is an odd prime (again, see Section 13.2.3 for the case $r = 2$). Let $x \in G$ be of order r. Then, by Theorem 5.3.1, up to conjugation,

$$x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \tag{79}$$

with a_0 even and $a_i = a_{r-i}$ for $1 \leq i \leq \frac{r-1}{2}$. **In the following we denote $t = \frac{r-1}{2}$.**

As for unipotent elements, since $r > 2$ and τ has order either 2 or 4, we deduce that $x^G \cap H = x^G \cap H^\circ$.

Assume $x = [x_1, x_2]$ with $x_1 = x_2^{-t} \in \mathrm{GL}_{n/2}$. Then, for some $b_1, \ldots, b_t$, with $0 \leq b_i \leq a_i$, we have, up to conjugation,

$$x = \left(\begin{array}{c|c} [I_{a_0/2}, \omega I_{a_1 - b_1}, \omega^{-1} I_{b_1}, \ldots] & \\ \hline & [I_{a_0/2}, \omega^{-1} I_{a_1 - b_1}, \omega I_{b_1}, \ldots] \end{array}\right) \tag{80}$$

Thus a block decomposition $x = [x_1, x_2]$ is uniquely determined by the t-tuple, $\mathbf{b} = (b_1, \ldots, b_t)$. We define

$$\mathcal{T}_x = \left\{\mathbf{b} = (b_1, \ldots, b_t) : 0 \leq b_i \leq a_i,\ i = 1, \ldots, t\right\}$$

Given $\mathbf{b} \in \mathcal{T}_x$ we denote by $x_{\mathbf{b}}$ the block decomposition $[x_1, x_2]$ as in (80). Using Theorem 5.3.1, we have

$$\dim x_{\mathbf{b}}^{H^\circ} = \dim x_1^{\mathrm{GL}_{n/2}} = \frac{n^2}{4} - \frac{a_0^2}{4} - \sum_{i=1}^{t}(a_i - b_i)^2 - \sum_{i=1}^{t} b_i^2 \tag{81}$$

Notice that for all $\mathbf{b} \in \mathcal{T}_x$ we have $x_{\mathbf{b}} \in x^G \cap H$. Indeed, we have the following.

Proposition 13.2.3. *Let $x \in G$ be of order r. Then*

$$x^G \cap H = \bigcup_{\mathbf{b} \in \mathcal{T}_x} x_{\mathbf{b}}^{H^\circ}$$

In particular, $\dim(x^G \cap H) = \max_{\mathbf{b} \in \mathcal{T}_x}\{\dim x_{\mathbf{b}}^{H^\circ}\}$.

We now classify the t-tuples $\mathbf{b}$ for which $\dim(x^G \cap H) = \dim x_{\mathbf{b}}^{H^\circ}$. In the following, a necessary condition for a t-tuple, $\mathbf{b} \in \mathcal{T}_x$, such that $\dim(x^G \cap H) = \dim x_{\mathbf{b}}^{H^\circ}$, is given.

Lemma 13.2.4. *Let $x \in G$ be of order r. Assume* $\dim(x^G \cap H) = \dim x_{\mathbf{b}}^{H^\circ}$. *Then, for all $i = 1, \ldots, t$,*

$$|a_i - 2b_i| \leq 1$$

Proof. Seeking a contradiction, assume there exists i such that $|a_i - 2b_i| \geq 2$. Without loss of generality, we may assume $i = 1$. Rewriting (81) we have

$$\dim x_{\mathbf{b}}^{H^\circ} = \frac{n^2}{4} - \frac{a_0^2}{4} - (a_1 - b_1)^2 - b_1^2 - \sum_{i=2}^{t}(a_i - b_i)^2 - \sum_{i=2}^{t} b_i^2$$

We claim that there exists $\mathbf{a} \in \mathcal{T}_x$ such that $\dim x_{\mathbf{a}}^{H^\circ} > \dim x_{\mathbf{b}}^{H^\circ}$, which would contradict the maximality of $\dim x_{\mathbf{b}}^{H^\circ}$ in $\{\dim x_{\mathbf{c}}^{H^\circ} : \mathbf{c} \in \mathcal{T}_x\}$.

Case 1. Assume $a_1 - 2b_1 \geq 2$. Define $\mathbf{a} = (b_1 + 1, b_2, \ldots, b_t)$. Then we compute $\dim x_{\mathbf{a}}^{H^\circ} - \dim x_{\mathbf{b}}^{H^\circ} = 2(a_1 - 2b_1 - 1) \geq 2 > 0$.

Case 2. Assume $2b_1 - a_1 \geq 2$. Define $\mathbf{a} = (b_1 - 1, b_2, \ldots, b_t)$. Then we compute $\dim x_{\mathbf{a}}^{H^\circ} - \dim x_{\mathbf{b}}^{H^\circ} = 2(2b_1 - a_1 - 1) \geq 2 > 0$. The result follows. *q.e.d.*

We state the following number-theoretic result in order to give a complete classification of the tuples $\mathbf{b} \in \mathcal{T}_x$ such that $\dim x_{\mathbf{b}}^{H^\circ} = \dim(x^G \cap H^\circ)$.

Lemma 13.2.5. *Let l be a positive integer and let $a_1, b_1, \ldots, a_l, b_l \in \mathbb{Z}_{\geq 0}$ with $b_i \leq a_i$ and $|a_i - 2b_i| \leq 1$ for all $i = 1 \ldots, l$. Then, for all $i = 1, \ldots, l$, $b_i \in \left\{\left\lfloor \frac{a_i}{2} \right\rfloor, \left\lfloor \frac{a_i}{2} \right\rfloor + 1\right\}$.*

Proof. By the assumptions we deduce $\frac{a_i - 1}{2} \leq b_i \leq \frac{a_i + 1}{2}$ for all i. If a_i is even then $b_i = \frac{a_i}{2}$. Instead, for a_i odd $\left\lfloor \frac{a_i}{2} \right\rfloor = \frac{a_i - 1}{2}$ and $\left\lfloor \frac{a_i}{2} \right\rfloor + 1 = \frac{a_i + 1}{2}$. The result follows. *q.e.d.*

Proposition 13.2.6. *Let $x \in G$ be of order r. Let $\mathbf{b} = (b_1, \ldots, b_t) \in \mathcal{T}_x$. Then* $\dim(x^G \cap H) = \dim x_{\mathbf{b}}^{H^\circ}$ *if, and only if, $|a_i - 2b_i| \leq 1$ for all $i = 1, \ldots, t$.*

Proof. If $\dim(x^G \cap H) = \dim x_{\mathbf{b}}^{H^\circ}$ then the result follows from Lemma 13.2.4.

Now, assume $|a_i - 2b_i| \leq 1$ for all i. There exists an element $\mathbf{c} = (c_1, \ldots, c_t) \in \mathcal{T}_x$ such that $\dim(x^G \cap H^\circ) = \dim x_{\mathbf{c}}^{H^\circ}$. By Lemma 13.2.4 we have $|a_i - 2c_i| \leq 1$ for all i. And, by Lemma 13.2.5, $b_i, c_i \in \left\{\left\lfloor \frac{a_i}{2} \right\rfloor, \left\lfloor \frac{a_i}{2} \right\rfloor + 1\right\}$. If a_i is even, by the proof of Lemma 13.2.5 $b_i = c_i = \frac{a_i}{2}$. Now assume a_i is odd and $b_i \neq c_i$. Without loss of generality, we may assume $b_i = \left\lfloor \frac{a_i}{2} \right\rfloor, c_i = \left\lfloor \frac{a_i}{2} \right\rfloor + 1$. Then the factors in the centralizer in H° of $x_{\mathbf{b}}$ arising from the eigenvalues ω^i, ω^{-i} are $\mathrm{GL}_{a_i - b_i} \times \mathrm{GL}_{b_i}$, and, for $x_{\mathbf{c}}$, they are $\mathrm{GL}_{a_i - c_i} \times \mathrm{GL}_{c_i}$. Let us observe that $a_i - b_i = a_i - \frac{a_i - 1}{2} = \frac{a_i + 1}{2} = c_i$ and, similarly $b_i = a_i - c_i$. Thus $\mathrm{GL}_{a_i - b_i} \times \mathrm{GL}_{b_i} \cong \mathrm{GL}_{a_i - c_i} \times \mathrm{GL}_{c_i}$. Therefore $\dim x_{\mathbf{b}}^{H^\circ} = \dim x_{\mathbf{c}}^{H^\circ}$. The result follows. *q.e.d.*

It is now easy to compute $\dim(x^G \cap H)$ for any semisimple element of odd prime order. Notice that the following formula holds in both cases $G = \mathrm{Sp}_n$ and O_n.

We define

$$d(x) = \left|\left\{i \in \{1, \ldots, r-1\} : a_i \text{ odd}\right\}\right| \tag{82}$$

Proposition 13.2.7. *Let $x \in H$ be of order r. Then*

$$\dim(x^G \cap H) = \frac{n^2}{4} - \frac{1}{4}\sum_{i=0}^{r-1} a_i^2 - \frac{d(x)}{4}$$

where $d(x)$ is defined in (82).

PROOF. By Proposition 13.2.6, $\dim(x^G \cap H) = \dim x_{\mathbf{b}}^{H^\circ}$, where $\mathbf{b} = (b_1, \ldots, b_t)$ and $b_i \in \left\{\lfloor \frac{a_i}{2} \rfloor, \lfloor \frac{a_i}{2} \rfloor + 1\right\}$. Thus, we have the block decomposition $x_{\mathbf{b}} = [x_1, x_2]$, and for $i = 1, \ldots, t$ the eigenvalue ω^i in x_1 has multiplicity $a_i - \lfloor \frac{a_i}{2} \rfloor - \epsilon_i$, similarly ω^{-i} has multiplicity $\lfloor \frac{a_i}{2} \rfloor + \epsilon_i$, with $\epsilon_i \in \{0, 1\}$, in particular $\epsilon_i = 0$ if a_i is even and $\epsilon_i = 1$ if a_i is odd, by the proof of Proposition 13.2.6. Note that $a_i = 2\lfloor \frac{a_i}{2} \rfloor + \epsilon_i$. Using (81),

$$\begin{aligned}\dim(x^G \cap H) = \dim x_{\mathbf{b}}^{H^\circ} &= \frac{n^2}{4} - \frac{a_0^2}{4} - \sum_{i=1}^{\frac{r-1}{2}} \left(a_i - \left\lfloor \frac{a_i}{2} \right\rfloor - \epsilon_i\right)^2 - \sum_{i=1}^{\frac{r-1}{2}} \left(\left\lfloor \frac{a_i}{2} \right\rfloor + \epsilon_i\right)^2 \\ &= \frac{n^2}{4} - \frac{a_0^2}{4} - \sum_{i=1}^{\frac{r-1}{2}} a_i^2 - 2\sum_{i=1}^{\frac{r-1}{2}} \left\lfloor \frac{a_i}{2} \right\rfloor \left(\left\lfloor \frac{a_i}{2} \right\rfloor - a_i\right)\end{aligned}$$

Recall the definition of $d(x)$ from (82). For the purpose of this proof, let $d'(x) = \left|\left\{i \in \{1, \ldots, t\} : a_i \text{ odd}\right\}\right|$, notice that $d(x) = 2d'(x)$. Note that $\lfloor \frac{a_i}{2} \rfloor = \frac{a_i}{2}$ if a_i is even, and $\lfloor \frac{a_i}{2} \rfloor = \frac{a_i - 1}{2}$ if a_i is odd. Therefore

$$\begin{aligned}\dim(x^G \cap H) &= \frac{n^2}{4} - \frac{a_0^2}{4} - \sum_{i=1}^{\frac{r-1}{2}} a_i^2 - \sum_{\substack{i \le t, \\ a_i \text{ odd}}} (a_i - 1)\left(\frac{a_i - 1}{2} - a_i\right) - \sum_{\substack{i \le t, \\ a_i \text{ even}}} a_i\left(\frac{a_i}{2} - a_i\right) \\ &= \frac{n^2}{4} - \frac{a_0^2}{4} - \frac{1}{2}\sum_{i=1}^{t} a_i^2 - \frac{d'(x)}{2} = \frac{n^2}{4} - \frac{a_0^2}{4} - \frac{1}{4}\sum_{i=1}^{r-1} a_i^2 - \frac{d(x)}{4}\end{aligned}$$

q.e.d.

13.2.3. Involutions. Let $x \in G$ be an involution. In the case $p \neq 2$, up to conjugation, we may write $x = [I_{n-s}, -I_s]$ with s even if $G = \mathrm{Sp}_n$.

In Proposition 13.2.8 we shall study conjugacy classes of involutions in $H \setminus H^\circ$. Then we study classes of involutions in the connected component H°.

For the purpose of the next result define, when $n/2$ is even, $z = [B, \ldots, B] \in H^\circ$ where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{83}$$

Proposition 13.2.8. *The following hold.*

(i) Assume $n \equiv 2 \pmod 4$. Then

(a) *If* $G = \mathrm{Sp}_n$*, there are no involutions in* $H \setminus H^\circ$ *for* $p \neq 2$*; if* $p = 2$ *then all the involutions in* $H \setminus H^\circ$ *are* G*-conjugate to* τ*, which is* $b_{n/2}$*-type.*

(b) *If* $G = \mathrm{O}_n$*, all the involutions in* $H \setminus H^\circ$ *are* G*-conjugate to* τ*, which is a* $b_{n/2}$*-involution when* $p = 2$*.*

(ii) *Assume* $n \equiv 0 \pmod 4$ *and let* $x \in H \setminus H^\circ$ *be an involution.*

(a) *If* $p \neq 2$*, then* x *is* G*-conjugate to* $[I_{n/2}, -I_{n/2}]$*.*

(b) *If* $p = 2$*, then* x *has Jordan form* $[J_2^{n/2}]$*. In addition,* $z\tau$ *is* $a_{n/2}$*-type and* τ *is* $c_{n/2}$*-type.*

Proof. Let $x \in H^\circ\tau$. Say

$$x = [A, A^{-t}]\tau = \begin{pmatrix} & A \\ \epsilon A^{-t} & \end{pmatrix}$$

for some $A \in \mathrm{GL}_{n/2}$, where $\epsilon = 1$ if $G = \mathrm{O}_n$ and $\epsilon = -1$ if $G = \mathrm{Sp}_n$. Thus x is an involution if, and only if,

$$A = \begin{cases} A^t & \text{if } G = \mathrm{O}_n \\ -A^t & \text{if } G = \mathrm{Sp}_n \end{cases} \tag{84}$$

Assume x is an involution. Since $A = \pm A^t$, we also have $A^{-1} = \pm A^{-t}$. Thus

$$x = \begin{pmatrix} & A \\ A^{-1} & \end{pmatrix}$$

We defined H to be the stabiliser of the set $\{U, W\}$ where $U = \langle e_1, \dots, e_{n/2}\rangle$ and $W = \langle f_1, \dots, f_{n/2}\rangle$ and $U \oplus W = V$, see Section 13.1. Since $x \notin H^\circ$, x transposes U and W. Thus $W = \langle x.e_1, \dots, x.e_{n/2}\rangle$.

Assume n/2 is even. In the basis

$$\{e_1, x.e_1, \dots, e_{n/4}, x.e_{n/4}, e_{n/4+1}, -x.e_{n/4+1}, \dots, e_{n/2}, -x.e_{n/2}\}$$

the element x is represented by $\tilde{x} = [B, \dots, B, -B, \dots, -B]$ with B as in (83). It is clear that $\nu(\tilde{x}) = n/2$ and that $\tilde{x}$ has Jordan form $[I_{n/2}, -I_{n/2}]$ if $p \neq 2$ and $[J_2^{n/2}]$ if $p = 2$, since the characteristic polynomial of B is $p_B(\lambda) = \lambda^2 - 1$.

For $p = 2$, it is clear that τ is a $c_{n/2}$-type involution, since $(e_i, x.e_i) = (e_i, f_i) = 1$. Let $z = [B, \dots, B]$ and let us consider $z\tau$. The action of $z\tau$ on V is given by the following

$$\begin{array}{lll} e_i \mapsto f_{i+1}, & f_i \mapsto e_{i+1}, & i \text{ odd} \\ e_i \mapsto f_{i-1}, & f_i \mapsto e_{i-1}, & i \text{ even} \end{array}$$

Moreover we have $(e_i, z\tau.e_i) = (e_i, f_{i\pm1}) = 0$ and, similarly $(f_i, z\tau.f_i) = (f_i, e_{i\pm1}) = 0$. Therefore $z\tau$ is an $a_{n/2}$-type involution.

Assume n/2 is odd. If $G = \mathrm{Sp}_n$ and $p \neq 2$, we have $\det(x) = \det([A, A^{-t}])\det(\tau) = -1$, since $\det([A, A^{-t}]) = \det(-AA^{-1}) = -1$ (recall $A = -A^{-t}$) and $\det(\tau) = 1$. Therefore, we conclude that there are no involutions in $H \setminus H^\circ$. If $G = \mathrm{Sp}_n$ and $p = 2$, or $G = \mathrm{O}_n$, we use an argument similar to the previous one. Indeed, we see that x, in the basis $\{e_1, x.e_1, \dots, e_{n/2}, x.e_{n/2}\}$, is represented by $z = [B, \dots, B]$, which is represented by τ in the basis $\{e_1, \dots, e_{n/2}, x.e_1, \dots, x.e_{n/2}\}$. Notice that $\nu(z) = n/2$.

In the case $p = 2$ and $G = \mathrm{Sp}_n$ or O_n we have proved that all the involutions in $H \setminus H^\circ$ have Jordan form $[J_2^{n/2}]$. In the case $n/2$ is odd $[J_2^{n/2}]$ is a $b_{n/2}$-type involution. If $p \neq 2$, all involutions have Jordan form $[I_{n/2}, -I_{n/2}]$, since they are all conjugate to z and $\nu(z) = n/2$. *q.e.d.*

Remark 13.2.9. Notice that $[B, \ldots, B, -B, \ldots, -B]$ and $K = [K_{n/2}, K_{n/2}^{-t}]\tau$ are G-conjugate, where

$$K_{n/2} = \begin{pmatrix} & I_{n/4} \\ \pm I_{n/4} & \end{pmatrix} \tag{85}$$

is the standard symplectic, respectively orthogonal, form on $k^{n/2}$. Similarly one defines $K_{n/2}$ when $n/2$ is odd, so $K_{n/2}$ will be the Gram matrix of the bilinear form associated to a non-degenerate quadratic form (provided $p \neq 2$).

Lemma 13.2.10. *Assume $p \neq 2$. Any involution in $H \setminus H^\circ$ is H°-conjugate to K.*

Proof. Let $x \in H \setminus H^\circ$ be an involution. Then, up to G-conjugation, $x = [A, A^{-t}]\tau$, for some $A \in \mathrm{GL}_{n/2}$. Notice, (84) implies that A and A^{-t} are Gram matrices associated to a symplectic or orthogonal form on $k^{n/2}$. Hence they are congruent to $K_{n/2}$, see [**26**, p.14]. Thus there exists $P \in \mathrm{GL}_{n/2}$ such that $P^t A P = K_{n/2}$. Define $g = [P^{-t}, P]$. Then it is straightforward to check $g^{-1}xg = K$. *q.e.d.*

In the case $p = 2$ we can show that any involution in H° is of a_s-type.

Lemma 13.2.11. *Assume $p = 2$. Let $x \in H^\circ$ be an involution with $\nu(x) = s$. Then x is an a_s-type involution.*

Proof. Observe that $x.U = U$ and $x.W = W$, where $U = \langle e_1, \ldots, e_{n/2}\rangle$ and $W = \langle f_1, \ldots, f_{n/2}\rangle$. Then $x.e_i \in U$ and $x.f_i \in W$ for all i. So

$$(x.e_i, e_i) = (x.f_i, f_i) = 0$$

Therefore x is a_s-type. *q.e.d.*

Let $x \in H^\circ$ be an involution. Then $x = [x_1, x_2]$. In particular, x_1 and x_2 have the same Jordan form. Thus x has Jordan form $[I_{n-s}, -I_s]$ or $[J_2^s, J_1^{n-2s}]$, with s even. And $x_1 = \left[I_{(n-s)/2}, -I_{s/2}\right]$ if $p \neq 2$, or $\left[J_2^{s/2}, J_1^{n/2-s}\right]$ if $p = 2$. We deduce the following.

Lemma 13.2.12. *Let $x \in G$ be an involution. Assume $\nu(x)$ is odd. Then $x^G \cap H^\circ = \emptyset$.*

Proposition 13.2.13 below is a direct consequence of Proposition 13.2.8. In the following we provide representatives of the G- and H-conjugacy classes of involutions in $H \setminus H^\circ$.

Proposition 13.2.13. *Let $x \in G$ be an involution. Assume $x^G \cap H \neq \emptyset$. Then the following hold.*

(i) Assume $\nu(x) < n/2$. Then $\nu(x)$ is even and $x^G \cap H = x^G \cap H^\circ$.
(ii) If $\nu(x) = n/2$, then

$$x^G \cap H = \bigcup_{y \in \mathcal{A}} y^{H^\circ} \cup \bigcup_{\tau_i \in \mathcal{B}} \tau_i^{H^\circ} \tag{86}$$

where $\mathcal{A} \subseteq H^\circ$, $\mathcal{B} \subseteq H \setminus H^\circ$ and $|\mathcal{A}|, |\mathcal{B}| \in \{0,1\}$. In Table 13.2.1 we give representatives of the distinct H°-classes in (86). The sizes of $\mathcal{A}$ and $\mathcal{B}$ can be deduced from the last two columns.

G	n	p	G-class repr.	τ_i	H°-class repr.
Sp_n	$4m+2$	$p \neq 2$	–	–	–
		$p = 2$	$b_{n/2}$	τ	–
	$4m$	$p \neq 2$	$[I_{n/2}, -I_{n/2}]$	$[B, \dots, B]\tau$	$[I_m, -I_m]$
		$p = 2$	$c_{n/2}$	τ	–
			$a_{n/2}$	$[B, \dots, B]\tau$	$[J_2^m]$
O_n	$4m+2$	$p \neq 2$	$[I_{n/2}, -I_{n/2}]$	τ	–
		$p = 2$	$b_{n/2}$	τ	–
	$4m$	$p \neq 2$	$[I_{n/2}, -I_{n/2}]$	τ	$[I_m, -I_m]$
	$4m$	$p = 2$	$c_{n/2}$	τ	–
			$a_{n/2}$	$[B, \dots, B]\tau$	$[J_2^m]$

Table 13.2.1. Representatives of H°-classes of involutions $x \in H$ with $\nu(x) = n/2$, $H \in \mathscr{C}_3$

Proof. Proposition 13.2.8 implies that if $x^G \cap (H \setminus H^\circ)$ is non-empty then $\nu(x) = n/2$. Thus (i) follows. Furthermore, for an involution x there exists a unique block decomposition $[x_1, x_2]$ with $x_1 = x_2^{-t}$. Thus $\nu(x)$ is even and $x^G \cap H = x^G \cap H^\circ$.

Assume $\mathbf{G} = \mathbf{Sp_n}$ and $n/2$ odd. Then by Proposition 13.2.8(i)(a), there are no involutions in $H \setminus H^\circ$ if $p \neq 2$; for $p = 2$ there is only one G-class of involutions represented by τ, which is a $b_{n/2}$-type, hence $\tau^G \cap H^\circ = \emptyset$, by Remark 13.2.1 and Lemma 13.2.11. If $n/2$ is even and $p \neq 2$ then by Proposition 13.2.8(ii) there is only one G-class of involutions whose representative is $[B, \dots, B]\tau$, which is the only H°-class by Lemma 13.2.10. Moreover the Jordan form of $[B, \dots, B]\tau$ is $[I_{n/2}, -I_{n/2}]$ hence a representative of the H°-class is given by $[I_{n/4}, -I_{n/4}]$. In the case $n/2$ is even and $p = 2$ by Proposition 13.2.8 we know that there are two distinct G-conjugacy classes of involutions, an $a_{n/2}$-class and a $c_{n/2}$-class, whose representatives are, respectively, $[B, \dots, B]\tau$ and τ. Observe that $\tau^G \cap H^\circ = \emptyset$ by Lemma 13.2.11.

Assume $\mathbf{G} = \mathbf{O_n}$ and $n/2$ is odd. Thanks to Proposition 13.2.8(i) we see that there is only one G-class of involutions, whose representative is τ. In the case $p \neq 2$, we have, by Lemma 13.2.10, that τ is H°-conjugate to $[I_{n/2}, -I_{n/2}]$. If $p = 2$ then Proposition 13.2.8 implies that τ is a $b_{n/2}$-type involution, in addition $\tau^G \cap H^\circ = \emptyset$, by Remark 13.2.1. If $n/2$ is even and $p = 2$, Proposition 13.2.8 shows that there are two G-classes of involutions with representative τ, with Jordan form $[I_{n/2}, -I_{n/2}]$. In addition, all the involutions in $H^\circ\tau$ are H°-conjugate by Lemma 13.2.10. If $p \neq 2$ Proposition 13.2.8 implies that τ is a $c_{n/2}$-type involution, while $[B, \dots, B]\tau$ is a $a_{n/2}$-type involution. Since τ is a $c_{n/2}$-involution, Lemma 13.2.11 implies $\tau^G \cap H^\circ = \emptyset$. *q.e.d.*

In the case $\nu(x) < n/2$ we easily compute $\dim(x^G \cap H)$. Assume $p \neq 2$. Let $G = \mathrm{Sp}_n, \mathrm{O}_n$, and $x = [I_{n-s}, -I_s]$, in order to have $x \in H^\circ$ we need $x = [x_1, x_2]$ with

$x_1 = x_2^{-t}$. Thus, the only possibility to get such a block decomposition is given by $x_1 = x_2 = [I_{(n-s)/2}, I_{s/2}]$. Hence, by Theorem 5.3.1,

$$\dim(x^G \cap H^\circ) = \dim x_1^{\mathrm{GL}_{n/2}} = \frac{s}{2}(n-s) \tag{87}$$

Assume $p = 2$ and let $x \in H^\circ$ be an involution with $\nu(x) < n/2$. By Proposition 13.2.13(i) we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Hence, by Theorem 5.2.1,

$$\dim(x^G \cap H) = \dim x^{H^\circ} = \dim[J_2^{s/2}, J_1^{n/2-s}]^{\mathrm{GL}_{n/2}} = \frac{s}{2}(n-s) \tag{88}$$

Assume, now $s = n/2$ and $G = \mathrm{Sp}_n$ or O_n. The next result gives the centraliser in H° of involutions in $H \setminus H^\circ$. This was originally stated in [**39**, Proposition 1.4] using the isogeny types of the groups, we essentially re-state it in our notation.

Proposition 13.2.14. *Let $G = \mathrm{Sp}_n$ or O_n, $H^\circ = \mathrm{GL}_{n/2} < G$, τ as in (76) and B as in (83).*

(i) Assume $n/2$ is odd. Then $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$.
(ii) If $n/2$ is even, then $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$ and $C_{H^\circ}([B,\dots,B]\tau) = \mathrm{Sp}_{n/2}$ if $p \neq 2$; and $C_{H^\circ}(\tau) = \mathrm{Sp}_{n/2}$, $C_{H^\circ}([B,\dots,B]\tau) = C_{\mathrm{Sp}_{n/2}}(b_1)$ if $p = 2$.

Proof. The centraliser in H° of τ is given by

$$C_{H^\circ}(\tau) = \{[A, A^{-t}] \in H^\circ : A = A^{-t}\}$$

and for $n/2$ odd, or $n/2$ even and $p \neq 2$ we have $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$. If $n/2$ is even and $p = 2$ then $C_{H^\circ}(\tau) = \mathrm{Sp}_{n/2}$.

Assume $n/2$ is even. Recall that $[B,\dots,B]\tau$ is H°-conjugate to $K = [K_{n/2}, K_{n/2}]\tau$, see Remark 13.2.9 and Lemma 13.2.10. So $C_{H^\circ}([B,\dots,B]\tau) \cong C_{H^\circ}(K)$ and

$$C_{H^\circ}(K) = \{[A, A^{-t}] \in H^\circ : AK_{n/2}A^t = K_{n/2}\}$$

and if $p \neq 2$ we have $C_{H^\circ}(K) \cong \mathrm{Sp}_{n/2}$.

Finally, for $n/2$ even and $p = 2$ the equality $C_{H^\circ}([B,\dots,B]\tau) = C_{\mathrm{Sp}_{n/2}}(t)$ follows from [**39**, Proposition 1.4(ii)]. *q.e.d.*

In Section 14.7 and 15.7 we shall explicitly compute $\dim(x^G \cap H)$ for involutions $x \in G$ with $\nu(x) = n/2$.

Let $x \in G$ be an involution. Thanks to the previous discussions we know when $x^G \cap H \neq \emptyset$. We conclude this section with the following result.

Proposition 13.2.15. *Let $x \in G$ be an involution with $\nu(x) = s$. Then $x^G \cap H \neq \emptyset$ if, and only if,*

(i) $s < n/2$ and s is even;
(ii) $s = n/2$ with $n/2$ even, when $G = \mathrm{Sp}_n$ and $p \neq 2$.

CHAPTER 14

Symplectic group

Throughout this chapter the notation is as follows. Let $G = \mathrm{Sp}_n$, with $n \geq 4$, and $H = \mathrm{GL}_{n/2}.2$ a $\mathscr{C}_3$-subgroup of G. Set $\Omega = G/H$. The aim of this chapter is to derive bounds on $f_\Omega(x)$ for $x \in G$ of prime order.

14.1. Upper bounds

In this section we shall prove the following, only for odd order elements. For convenience we postpone involutions to Section 14.7.

Proposition 14.1.1. *Let $x \in G$ be an element of order $r > 2$. Then*

$$f_\Omega(x) \leq 1 - \frac{4}{n+2}$$

Furthermore equality holds if, and only if, $\nu(x) = 2$.

Remark 14.1.2. Notice that, after we have derived lower bound for involutions, thanks to Lemma 7.1.1 and Corollary 7.1.11, the upper bound will extend to any element in $G \setminus Z(G)$.

The same observation made in Remark 10.1.3 holds. Hence we will prove the next results for elements in H.

In order to prove Proposition 14.1.1 we shall study separately the cases $\nu(x) \leq 4$ and then $\nu(x) > 4$. Observe that in $H^\circ < \mathrm{Sp}_n$ there are no elements y such that $\nu(y) = 1$.

Lemma 14.1.3. *Let $x \in H$ be of order r. Assume $\nu(x) = 2$. Then $f_\Omega(x) = 1 - \frac{4}{n+2}$.*

Proof. **Assume $\mathbf{r = p}$.** Then, up to conjugation, $x = [J_2^2, J_1^{n-4}]$. Indeed the only other unipotent element $y \in \mathrm{GL}_n$ with $\nu(y) = 2$ is, up to G-conjugacy, $y = [J_3, J_1^{n-3}] \notin \mathrm{Sp}_n$. Hence, by Theorem 5.2.1 and formula (78) we have $\dim x^G = 2(n-1), \dim(x^G \cap H) = n-2$. Therefore, using the value of $\dim \Omega$ recorded in Table 13.1.1 and Proposition 7.1.8 we get $f_\Omega(x) = 1 - \frac{4}{n+2}$.

Assume $\mathbf{r \neq p}$. Then $x = [I_{n-2}, \omega, \omega^{-1}]$ where $\omega \in k$ is a primitive r-th root of unity. The only other semisimple elements in GL_n with ν-value equal to 2 is $[I_{n-2}, \omega I_2] \notin \mathrm{Sp}_n$; in the case $n = 4$ we also have $y = [\omega I_2, \omega^{-1} I_2] \in \mathrm{Sp}_4$. Again, using Theorem 5.3.1 and the formula of $\dim(x^G \cap H)$ given in Proposition 13.2.7, the result follows. Similarly for $y \in \mathrm{Sp}_4$. *q.e.d.*

Lemma 14.1.4. *Let $x \in H$ be of order r. Assume $\nu(x) = 3$ or 4. Then*

$$f_\Omega(x) < 1 - \frac{4}{n+2}$$

Proof. We record in Table 14.1.1 the elements $x \in H$ with $\nu(x) = 3, 4$ (up to conjugacy and up to centraliser structure) and the corresponding value of $f_\Omega(x)$. The bound $f_\Omega(x) < 1 - \frac{4}{n+2}$ quickly follows. *q.e.d.*

x	$\dim x^G$	$\dim(x^G \cap H)$	$f_\Omega(x)$
$[\lambda I_3, \lambda^{-1} I_3]$	12	4	$\frac{1}{3}$
$[\lambda, \lambda^{-1}, \mu, \mu^{-1}]$	8	2	0
$[I_{n-4}, \lambda, \lambda^{-1}, \mu, \mu^{-1}]$	$4n - 8$	$2n - 6$	$1 - \frac{8(n-1)}{n(n+2)}$
$[I_{n-4}, \lambda I_2, \lambda^{-1} I_2]$	$4n - 10$	$2n - 6$	$1 - \frac{8(n-2)}{n(n+2)}$
$[\lambda I_4, \lambda^{-1} I_4]$	20	8	$\frac{2}{5}$
$[J_2^4, J_1^{n-8}]$	$4n - 12$	$2n - 8$	$1 - \frac{8(n-2)}{n(n+2)}$
$[J_3^2, J_1^{n-6}]$	$4n - 10$	$2n - 6$	$1 - \frac{8(n-2)}{n(n+2)}$

Table 14.1.1. Elements $x \in H$ with $\nu(x) = 3$ or 4

The final step in the proof of Proposition 14.1.1 is the following.

Lemma 14.1.5. *Let $x \in H$ be of order r. Assume $\nu(x) > 4$. Then* $f_\Omega(x) < 1 - \frac{4}{n+2}$

Proof. By Proposition 5.4.1 we have $\dim x^G > 2n$ for $n \geq 8$. Moreover by Theorem 7.2.5 we have $\dim(x^G \cap H) \leq \frac{1}{2} \dim x^G$. Therefore

$$f_\Omega(x) \leq 1 - \frac{\dim x^G}{2 \dim \Omega} < 1 - \frac{n}{\dim \Omega} = 1 - \frac{4}{n+2}$$

The only case left is $n = 6$ and $\nu(x) = 5$. Here $x = [\omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}]$ and $f_\Omega(x) = 0$. If $n = 4$ then any element has $\nu(x) < 4$. The result follows. *q.e.d.*

Collecting the results given in Lemmas 14.1.3, 14.1.4 and 14.1.5 we have a complete proof of Proposition 14.1.1.

Remark 14.1.6. The above results, Lemma 14.1.3, 14.1.4, 14.1.5, hold also in the case x is unipotent and the characteristic is zero. Indeed, in the characteristic zero case any unipotent element has infinite orde and $x^G \cap H = x^G \cap H^\circ$, see Lemma B.3.1. Moreover the analysis done makes no assumption on the order of the elements in the unipotent case, and Proposition 5.4.1, Theorem 7.2.5 hold in this case.

14.2. Unipotent elements: lower bounds

Let $x \in H$ be an element of order p (or any unipotent element in characteristic 0). In this section we shall derive a lower bound on $f_\Omega(x)$. Recall that we denote $p = \infty$ in the characteristic zero case. Up to G-conjugation, for $x \in H$, we write $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ with a_i even for all i.

For integers a, b, recall the definition given in (23): $\delta_{a;b} = 1$ if $b \mid a$ and 0 otherwise.

Proposition 14.2.1. *Let $x \in H$ be an element order p.*

(i) If $p \leq n/2$ then $f_\Omega(x) \geq 1/p$. Moreover, equality holds if, and only if,

(a) $n/2 = ap$ and $x \in [J_p^{2a}]^G$; or

(b) $n/2 = ap + (p-1)$ *and* $x \in [J_p^{2a}, J_{p-1}^2]^G$.

(ii) If $p > n/2$ *then* $f_\Omega(x) \geq \frac{2}{n+2\delta_{n;4}}$. *Moreover, equality holds if, and only if,* $x \in [J_{n/2}^2]^G$.

We shall prove Proposition 14.2.1 in Lemmas 14.2.3 – 14.2.7, below. For future reference we state the following, which follows immediately from Proposition 7.1.8, Theorem 5.2.1 and the formula (78).

Proposition 14.2.2. *Let* $x \in G$ *be of order* p. *Assume* $x^G \cap H \neq \emptyset$. *Then*

$$f_\Omega(x) = \frac{2\sum_{i<j} ia_ia_j + \sum_i ia_i^2 + 2\sum_{i \text{ odd}} a_i}{n(n+2)} \tag{89}$$

Lemma 14.2.3. *Assume* $p \leq n/2$ *is odd. Let* $x \in H$ *be an element of order* p. *Then* $f_\Omega(x) \geq 1/p$.

Proof. Write $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$. Let $\alpha(x) = n(n+2)\big(f_\Omega(x) - \frac{1}{p}\big)$. Thus we need to show $\alpha(x) \geq 0$. Using $n = \sum_i ia_i$, and the formula (89), we have

$$\alpha(x) = 2\sum_{i<j} ia_ia_j\Big(1-\frac{j}{p}\Big) + \sum_{i \text{ odd}} ia_i^2\Big(1-\frac{i}{p}\Big) + 2\sum_{i \text{ odd}} a_i\Big(1-\frac{i}{p}\Big) + \sum_{i \text{ even}} ia_i\Big(a_i - \frac{ia_i}{p} - \frac{2}{p}\Big)$$

Since $i \leq p$, it is clear that the first three summands in $\alpha(x)$ are non-negative. We claim that the last summand is non-negative.

Let $i < p$ be even. Assume $a_i > 0$. Since a_i is even we have $a_i \geq 2$. Thus

$$ia_i\Big(a_i - \frac{ia_i}{p} - \frac{2}{p}\Big) \geq 2i\Big(2 - \frac{2i}{p} - \frac{2}{p}\Big) = \frac{4i}{p}(p-i-1) \geq 0$$

q.e.d.

Now we assume $p > n/2$.

Remark 14.2.4. The characteristic zero case is part of this analysis. In this case, the largest Jordan block allowed in any unipotent element $x \in H$ is $J_{n/2}$, as $x^G \cap H = x^G \cap H^\circ$, see Lemma B.3.1. This motivates the choice to denote $p = \infty$.

Lemma 14.2.5. *Assume* $p > n/2$. *Let* $x \in H$ *be an element of order* p. *Then* $f_\Omega(x) \geq \frac{2}{n+2\delta_{n;4}}$.

Proof. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in H$. Recall that each a_i is even.

Assume n/2 is even. Using (89) we compute $n(n+2)\big(f_\Omega(x) - \frac{2}{n+2}\big)$:

$$n(n+2)\Big(f_\Omega(x) - \frac{2}{n+2}\Big) = 2\sum_{i<j} ia_ia_j + \sum_i ia_i(a_i-2) + 2\sum_{i \text{ odd}} a_i$$

It is clear that all summands in this expression are non-negative. So $f_\Omega(x) \geq \frac{2}{n+2}$.

Assume n/2 is odd. Here we have

$$n(n+2)\Big(f_\Omega(x) - \frac{2}{n}\Big) = 2\sum_{i<j} ia_ia_j + \sum_i ia_i(a_i-2) + 2\sum_{i \text{ odd}} a_i - 4$$

Since $n/2$ is odd there exists an odd i with $a_i \neq 0$ even. Thus, $2\sum_{i \text{ odd}} a_i \geq 4$, so the above expression is non-negative. Hence $f_\Omega(x) \geq \frac{2}{n}$, as required. *q.e.d.*

The next step is to characterise elements which realise the bounds. Once again we split the analysis into the cases $p \leq n/2$ and $p > n/2$.

Lemma 14.2.6. *Assume $p \leq n/2$. Let $x \in H$ be of order p. Then $f_\Omega(x) = 1/p$ if, and only if, one of the following holds*

(i) $n/2 = ap$ and $x \in [J_p^{2a}]^G$; or
(ii) $n/2 = ap + (p-1)$ and $x \in [J_p^{2a}, J_{p-1}^2]^G$.

Proof. It is straightforward, using (89), to see that if x is one of the elements given in the statement then $f_\Omega(x) = 1/p$.

Conversely, suppose $x \in H$ has order p and $f_\Omega(x) = 1/p$. Then, by the proof of Lemma 14.2.3, we have

$$\sum_{i \text{ odd}} a_i\Big(1 - \frac{i}{p}\Big) = 0, \qquad \sum_{i \text{ even}} \frac{ia_i}{p}(a_i(p-i) - 2) = 0$$

Therefore for all $i \neq p$ odd we deduce that $a_i = 0$.

For i even we have $a_i(p-i) = 2$ if, and only if, $(i, a_i) = (p-1, 2)$. Therefore $a_{p-1} \in \{0, 2\}$ and $a_i = 0$ for all even $i \neq p-1$. Recall that $n = \sum_i ia_i$. Hence either $n = pa_p$, in the case $a_{p-1} = 0$, or, $n = pa_p + 2(p-1)$ in the case $a_{p-1} = 2$. The result follows. *q.e.d.*

The next lemma completes the proof of Proposition 14.2.1.

Lemma 14.2.7. *Assume $p > n/2$. Let $x \in H$ be of order p. Then $f_\Omega(x) = \frac{2}{n+2\delta_{n;4}}$ if, and only if, $x \in [J_{n/2}^2]^G$.*

Proof. Up to G-conjugacy, $x = [J_{n/2}^{a_{n/2}}, \ldots, J_1^{a_1}]$, where $\sum_i ia_i = n$ and each a_i is even. Using (89) we compute $f_\Omega([J_{n/2}^2]) = \frac{2}{n+2\delta_{n;4}}$.

Let us prove the converse. **Assume n/2 is even**. Then, by the proof of Lemma 14.2.5 we have $a_i = 0$ for all i odd, and $ia_i(a_i - 2) = 0$ for all i. In particular for i even either $a_i = 0$ or $a_i = 2$. We claim that there exists only one i even for which $a_i \neq 0$. Assume there exists $i_1 < i_2$ even such that $a_{i_1}, a_{i_2} \neq 0$, this contradicts the hypothesis $\sum_{i<j} ia_ia_j = 0$, in fact $i_1a_{i_1}a_{i_2} \neq 0$. Therefore $a_i \neq 0$ for only one even index i. Thus, since $n = \sum_i ia_i$, we deduce $a_{n/2} = 2$ and $a_i = 0$ for all $i < n/2$.

Assume n/2 is odd. Then, by the proof of Lemma 14.2.5 we have $ia_i(a_i - 2) = 0$ for all i and $\sum_{i \text{ odd}} a_i = 2$. From the second equality we deduce that there exists only one odd i for which $a_i = 2$. As above, we can show that there exists only one i for which $a_i \neq 0$. Therefore x is G-conjugate to $[J_{n/2}^2]$. *q.e.d.*

The following result shows that the bound given in the case $p \leq n/2$ is, in fact, close to best possible.

Proposition 14.2.8. *Assume $p \leq n/2$ and $n/2 \not\equiv 0, p-1 \pmod p$. Then there exists $x \in H$ of order p such that*

$$f_\Omega(x) \leq \frac{1}{p} + \frac{1}{2(n+2)} + \frac{4}{n(n-2)}$$

PROOF. Write $n/2 = ap + b$ with $0 < b < p-1$. Define $x = [J_p^{2a}, J_b^2]$. Then, using (89), we have

$$f_\Omega(x) = \frac{1}{p} + \frac{4b(p-b-1) + 4p\delta_{b-1;2}}{pn(n+2)} \leq \frac{1}{p} + \frac{4b(p-b) + 4(p-b)}{pn(n+2)} < \frac{1}{p} + \frac{p}{n(n+2)} + \frac{4}{n(n-2)}$$

where the last inequality follows from the observation that, as function of b, $b(p-b)$ is maximal when $b = p/2$. Then, since $p < n/2$ we have $\frac{p}{n(n+2)} < \frac{1}{2(n+2)}$. *q.e.d.*

14.3. Semisimple elements: lower bounds

First notice that H contains a maximal torus of G. Therefore given $x \in G$ semisimple we have $x^G \cap H \neq \emptyset$. Hence it is enough to derive lower bounds on $f_\Omega(x)$ for $x \in H$ semisimple. Let $x \in H$ be of prime order $r > 2$. Up to G-conjugation, we have

$$x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \tag{90}$$

with a_0 even and $a_i = a_{r-i}$ for all $i \leq \frac{r-1}{2}$. For convenience, we postpone the analysis of involutions to Section 14.7. In this section we prove the following result.

Proposition 14.3.1. *Let $x \in H$ be a semisimple element of order $r > 2$.*

(i) If $r > n$, then $f_\Omega(x) \geq 0$. Moreover $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$;
(ii) If $r \leq n$, then $f_\Omega(x) \geq \frac{1}{r} - \frac{r^2-1}{rn(n+2)}$

Remark 14.3.2. For $r \leq n$, the bound given in part (ii) is sharp. In fact, if $\lfloor n/r \rfloor = \frac{n-r+1}{r}$ and z is as in (95) then $f_\Omega(z)$ realises it. We shall construct a family of *special* elements, with the property that each of them realises the best possible lower bound on f_Ω. Notice that $\frac{1}{r} - \frac{r^2-1}{rn(n+2)} \geq \frac{1}{r} - \frac{r}{n(n+2)}$ and $\frac{1}{r} - \frac{1}{n}$. In Remark 14.3.15, we give comments on the best possible lower bounds, indeed in Table 14.3.1 we list the best possible bounds.

Notice that H contains a maximal torus of G, hence $x^G \cap H^\circ \neq \emptyset$ for any element of order r. In addition, as already observed, since r is odd $x^G \cap H = x^G \cap H^\circ$.

We have already given a formula for $\dim(x^G \cap H)$ in Proposition 13.2.7. Using Theorem 5.3.1 and Proposition 7.1.8, it is straightforward to compute $f_\Omega(x)$. For convenience of the reader we recall the following definition, given in (82):

$$d(x) = |\{i \in \{1, \ldots, r-1\} : a_i \text{ odd}\}|$$

Recall that we listed the values of $\dim \Omega$ in Table 13.1.1.

Proposition 14.3.3. *Let $x \in G$ be of order r. Then $x^G \cap H = x^G \cap H^\circ \neq \emptyset$ and*

$$f_\Omega(x) = \frac{2a_0 + \sum_{i=0}^{r-1} a_i^2 - d(x)}{n(n+2)} \tag{91}$$

Remark 14.3.4. For $x \in G$ semisimple of (prime) order $r > 2$ we have $\dim C_\Omega(x) = \frac{1}{4}\left(2a_0 + \sum_i a_i^2 - d(x)\right)$. We see that $\dim C_\Omega(x)$ is an even integer. In fact we can rewrite

$$\dim C_\Omega(x) = \frac{a_0(a_0+2)}{4} + \frac{1}{4} \sum_{i>0:\, a_i \text{ even}} a_i^2 + \frac{1}{4} \sum_{i>0:\, a_i \text{ odd}} (a_i^2 - 1)$$

Since a_0 is even then $8 \mid a_0(a_0+2)$; recall that $a_i = a_{r-i}$ for all $0 < i \leq \frac{r-1}{2}$ then $8 \mid (a_i^2 + a_{r-i}^2)$, when a_i is even, similarly $8 \mid (a_i^2 - 1)(a_{r-i}^2 - 1)$, when a_i is odd.

Now, using Proposition 14.3.3 it is not hard to derive a lower bound on $f_\Omega(x)$.

Proposition 14.3.5. *Let $x \in H$ be of order r. Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{r^2-1}{rn(n+2)}$$

Proof. Since $d(x) \leq r-1$, Proposition 14.3.3 implies

$$f_\Omega(x) = \frac{2a_0 + a_0^2 + \sum_{i=1}^{r-1} a_i^2 - d(x)}{n(n+2)} \geq \frac{2a_0 + a_0^2 + \sum_{i=1}^{r-1} a_i^2 - (r-1)}{n(n+2)}$$

$$\geq \frac{2a_0 + a_0^2 + \frac{(n-a_0)^2}{r-1} - (r-1)}{n(n+2)} \geq \frac{\frac{n(n+2)}{r} - \frac{r^2-1}{r}}{n(n+2)} = \frac{1}{r} - \frac{r^2-1}{rn(n+2)}$$

The first inequality in the second line is given by Proposition B.2.1. The second inequality, in the second line, is obtained by observing that the one-variable function $g(a) = 2a + a^2 + \frac{(n-a)^2}{r-1}$ has a minimum at $a = \frac{n}{r} + \frac{1}{r} - 1$. *q.e.d.*

Remark 14.3.6. It is clear that the bound $\frac{1}{r} - \frac{r^2-1}{rn(n+2)}$ is a monotonically decreasing function in r, let us refer to it as $g(r)$. Then $g(n) = \frac{3}{n(n+2)} > 0$ and $g(n+1) = 0$.

In fact, for $r > n$ there exist semisimple elements $x \in H$ of order r with $f_\Omega(x) = 0$.

Proposition 14.3.7. *Let $x \in H$ be of order $r > n$. Then $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$.*

Proof. If $\nu(x) = n-1$, then all the eigenvalues of x have multiplicity 1, and $a_0 = 0$. Hence $d(x) = n$ and using Proposition 14.3.3 we have $f_\Omega(x) = 0$.

Conversely, assume $f_\Omega(x) = 0$ and $\nu(x) < n-1$. Then there exists an eigenvalue with multiplicity at least 2. There are two cases to consider.

Case 1. Assume $a_0 \geq 2$. Then using the formula (91) we have

$$f_\Omega(x) = \frac{2a_0 + a_0^2 + \sum_{i\geq 1} a_i^2 - d(x)}{n(n+2)} \geq \frac{2a_0 + a_0^2 + (n-a_0) - n}{n(n+2)} > 0$$

where the first inequality follows from the fact that $d(x) \leq n$, for $r > n$, and $\sum_{i\geq 1} a_i^2 \geq \sum_{i\geq 1} a_i = n - a_0$. Therefore $a_0 = 0$.

Case 2. Assume $a_0 = 0$ and $a_i \geq 2$, for some $i \geq 1$. We may assume $i = 1$. Again, using the formula (91) we have

$$f_\Omega(x) = \frac{\sum_{i\geq 1} a_i^2 - d(x)}{n(n+2)} \geq \frac{2a_1^2 + \sum_{i=2}^{r-2} a_i^2 - n}{n(n+2)} \geq \frac{2a_1^2 + n - 2a_1 - n}{n(n+2)} > 0$$

As in the previous case the second inequality follows from the fact that $d(x) \leq n$ and $\sum_{i=2}^{r-2} a_i^2 \geq n - 2a_1$. This is a contradiction. The result follows. *q.e.d.*

We give a geometric interpretation of Proposition 14.3.7. Recall that the action of G on Ω is equivalent to the action of G on X, the set of direct sum decompositions of V into maximal totally singular subspaces, as defined in (74). In particular, there is a one to one correspondence $C_\Omega(x) \leftrightarrow C_X(x)$.

In the following two results we use elementary arguments based only on linear algebra.

Proposition 14.3.8. *Let $x \in H$ be an odd prime order element. Assume $\nu(x) < n-1$. Then $C_\Omega(x)$ is not finite. In particular* $\dim C_\Omega(x) > 0$.

PROOF. Up to conjugation we may write $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Since $\nu(x) < n-1$ there exists i such that $a_i \geq 2$. Without loss of generality we may assume $a_0 \geq 2$; the argument is the same if $i \neq 0$.

Let $\{e_1, f_1, \ldots, e_{n/2}, f_{n/2}\}$ be the standard basis of the natural module V of G. Then, under the action of V, we may assume $e_1 \mapsto e_1$ and $f_1 \mapsto f_1$. For $\lambda, \mu \in k$ we define

$$U(\lambda, \mu) = \langle \lambda e_1 + \mu f_1, e_2, \ldots, e_{n/2} \rangle, \ W(\lambda, \mu) = \langle \lambda e_1 + \mu f_1, f_2, \ldots, f_{n/2} \rangle$$

The following facts are clear:

(i) If (λ, μ) and (λ', μ') are linearly independent $\ell(\lambda, \mu) \neq \ell(\lambda', \mu')$, for $\ell \in \{U, W\}$.
(ii) If $(\lambda, \mu) \neq (0, 0)$ then $U(\lambda, \mu)$ and $W(\lambda, \mu)$ are maximal totally singular.
(iii) If (λ, μ) and (λ', μ') are linearly independent then $V = U(\lambda, \mu) \oplus W(\lambda', \mu')$.
(iv) If $\{U(\lambda, \mu), W(\lambda', \mu')\} \in X$ then $\{U(\lambda, \mu), W(\lambda', \mu')\} \in C_X(x)$.

It is clear that $\{U(\lambda, 1), W(1, \mu)\} \in X$ for all $\lambda, \mu \in k \setminus \{1\}$. Therefore $C_X(x)$ contains an infinite set. The result follows. *q.e.d.*

With the notation introduced in the proof of Proposition 14.3.8 we prove that semisimple elements $x \in H$ with $\nu(x) = n-1$ have finite fixed point spaces.

Proposition 14.3.9. *Let $x \in G$ be of order r. Assume $\nu(x) = n-1$. Then $C_\Omega(x)$ is finite.*

PROOF. Up to G-conjugacy and up to centraliser structure, we may write $x = [\omega, \omega^{-1}, \ldots, \omega^{n/2}, \omega^{-n/2}]$. Under the action of x we have $e_i \mapsto \omega^i e_i$ and $f_i \mapsto \omega^{-i} f_i$. It is clear that $x.U(0, \mu) = U(0, \mu) = U(0, 1)$ and $x.U(\lambda, 0) = U(\lambda, 0) = U(1, 0)$; the same is true if we read the previous equalities with W instead of U. Let $\mathcal{I} = \{(a_1, \ldots, a_{n/2}) \in \{0, 1\}^{n/2}\}$. For $\mathbf{a} \in \mathcal{I}$, we define $U(\mathbf{a})$ to be generated by $e_1, \ldots, e_{n/2}$ with f_i instead of e_i in the position i such that $a_i = 1$, similarly for $W(\mathbf{a})$. For example, $U((1, 0, 1)) = \langle f_1, e_2, f_3 \rangle$ and $W((1, 0, 1)) = \langle e_1, f_2, e_3 \rangle$. It is clear that for all $\mathbf{a} \in \mathcal{I}$ we have $\{U(\mathbf{a}), W(\mathbf{a})\} \in X$ and that x fixes this decomposition.

For all $\lambda, \mu \in k^*$, we have $x.U(\lambda, \mu) = U(\lambda\omega, \mu\omega^{-1}) \neq U(\lambda, \mu)$, by (i) in the proof of Proposition 14.3.8. Similarly, one may define maximal totally singular subspaces using different linear combinations of the e_i's and f_i's (in the same spirit as the definition of $U(\lambda, \mu)$); we can show that these spaces are not fixed by x. For example, assume $U \neq U(\mathbf{a})$, for all $\mathbf{a} \in \mathcal{I}$. Let $u = \sum_i (\lambda_i e_i + \mu f_i) \in U$ with $\lambda_i \mu_j \neq 0$ for some i, j. Then one sees that if $x.u \in U$ then $x^i.u \in U$ for all $0 < i \leq r-1$; but those vectors are linearly independent, which is absurd as $\dim U = n/2 < r$.

Therefore only the decompositions $\{U(\mathbf{a}), W(\mathbf{a})\} \in X$ are fixed by x. *q.e.d.*

When $r < n$ we construct a family of elements of order r that realise the best possible lower bound on f_Ω. We start with the following.

Definition 14.3.10. Let $x \in G$ be of order r. We say that x is *special* if $|a_i - a_j| \leq 1$ for all $i, j = 0, \ldots, r-1$.

In particular, if x is special and $r < n$ then $a_i \neq 0$ for all $i \geq 0$.

Claim: Let $x \in H$ be of order $r < n$. Then $f_\Omega(x) \geq f_\Omega(z)$ for any special element z of order r.

Lemma 14.3.11. *Let $x \in G$ be of order $r < n$. Then either x is special or there exists $y = [I_{b_0}, \omega I_{b_1}, \ldots, \omega^{r-1} I_{b_{r-1}}] \in G$ such that $f_\Omega(x) = f_\Omega(y)$ and one of the following holds*

(i) $|b_0 - b_1| \geq 2$;
(ii) $b_1 - b_2 \geq 2$.

Proof. As in Lemma 10.3.3 is suffices to relabel the eigenvalues. *q.e.d.*

Assume x is not special. Then $|a_i - a_j| \geq 2$ for some $i, j \geq 0$. Thanks to Lemma 14.3.11, we may assume $(i, j) \in \{(0,1), (1,2)\}$.

Assume $a_0 - a_1 \geq 2$. We define

$$y = [I_{a_0-2}, (\omega, \omega^{-1}) I_{a_1+1}, (\omega, \omega^{-1})^2 I_{a_2}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{92}$$

Similarly, if $a_1 - a_0 \geq 2$ then set

$$y = [I_{a_0+2}, (\omega, \omega^{-1}) I_{a_1-1}, (\omega, \omega^{-1})^2 I_{a_2}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{93}$$

Finally, if $a_1 - a_2 \geq 2$, we define

$$y = [I_{a_0}, (\omega, \omega^{-1}) I_{a_1-1}, (\omega, \omega^{-1})^2 I_{a_2+1}, (\omega, \omega^{-1})^3 I_{a_3} \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{94}$$

With this notation we shall prove the following.

Lemma 14.3.12. *Let $x \in H$ be a non-special semisimple element of odd prime order $r < n$. Then, for y as in (92), (93) or (94)*

$$f_\Omega(x) \geq f_\Omega(y)$$

Proof. Clearly it is enough to show that $\dim C_\Omega(x) - \dim C_\Omega(y) \geq 0$, where $\dim C_\Omega(x)$ is given in Remark 14.3.4. We prove this inequality for each of the three different cases.

Case 1. Let $a_0 - a_1 \geq 2$. Then with a straightforward computation, using y as in (92),

$$\dim C_\Omega(x) - \dim C_\Omega(y) = a_0 - a_1 - \frac{d(x) - d(y) + 2}{4}$$

Note that $d(x) = d(y) \pm 2$ depending on whether a_1 is even or odd. Hence

$$\dim C_\Omega(x) - \dim C_\Omega(y) \geq a_0 - a_1 - 1 > 0$$

Case 2. Assume $a_1 - a_0 \geq 2$. Using y as in (93),we have

$$\dim C_\Omega(x) - \dim C_\Omega(y) = a_1 - a_0 - 2 - \frac{d(x) - d(y) + 2}{4}$$

Again, $d(x) = d(y) \pm 2$ according to the parity of a_1. In particular, $d(x) - d(y) \leq 2$. Hence

$$\dim C_\Omega(x) - \dim C_\Omega(y) \geq a_1 - a_0 - 2 - 1 \geq -1$$

Recall that $\dim C_\Omega(x), \dim C_\Omega(y)$ are even, by Remark 14.3.4. Hence the difference is non-negative.

Case 3. Let $a_1 - a_2 \geq 2$. Using y as in (94),

$$\dim C_\Omega(x) - \dim C_\Omega(y) = a_1 - a_2 - 1 - \frac{d(x) - d(y)}{4}$$

Here, we have: $d(x) = d(y) + 4$ if a_1, a_2 are odd; $d(x) = d(y)$ if $a_1 \not\equiv a_2 \pmod 2$; and $d(x) = d(y) - 4$ if they are both even. Namely $d(x) - d(y) \leq 4$. Therefore

$$\dim C_\Omega(x) - \dim C_\Omega(y) \geq a_1 - a_2 - 2 \geq 0$$

q.e.d.

Iterating Lemma 14.3.12, we prove the claim.

Lemma 14.3.13. *Let $x \in H$ be a semisimple element of odd prime order $r < n$. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

for some special element $z \in H$.

PROOF. The same argument as in Proposition 10.3.5 applies, using Lemma 14.3.12.
q.e.d.

Let us give an explicit description of special elements. Let $n = ar + b$ where $0 \leq b < r$ and $a = \left\lfloor \frac{n}{r} \right\rfloor$. Notice that $\frac{n-r+1}{r} \leq a \leq \frac{n}{r}$, also $a \equiv b \pmod 2$. Then it is clear that every special element has G-centraliser isomorphic to the centraliser of z, where

$$z = \begin{cases} [I_a, A] & a \text{ even} \\ [I_{a+1}, A] & a \text{ odd} \end{cases} \tag{95}$$

for $A = [(\omega, \omega^{-1}) I_{a+1}, \ldots, (\omega, \omega^{-1})^{\lfloor \frac{b}{2} \rfloor} I_{a+1}, (\omega, \omega^{-1})^{\lfloor \frac{b}{2} \rfloor + 1} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a]$.

Remark 14.3.14. Let $x, y \in H$ be semisimple elements of prime order $r < n$. Assume both x and y are special. Then $f_\Omega(x) = f_\Omega(y)$. This is clear by Proposition 14.3.3.

Assume a is even. With the aid of *Mathematica*, using Proposition 14.3.3, we compute

$$f_\Omega(z) = \frac{2a + (a+1)^2 b + (r-b)a^2 - b}{n(n+2)} = \frac{2an + 2a - a^2 r}{n(n+2)} \tag{96}$$

The numerator, as a function of a, is increasing for $\frac{n-r+1}{r} \leq a \leq \frac{n}{r}$. Therefore it is minimal for $a = \frac{n-r+1}{r}$. Hence, using *Mathematica*, we get

$$f_\Omega(z) \geq \frac{1}{r} - \frac{r^2 - 1}{rn(n+2)} \tag{97}$$

And it is maximal when $a = n/r$. Thus $f_\Omega(z) \leq 1/r$. Observe that this is precisely the same bound as that given in Proposition 14.3.5.

Now assume a is odd. Using Proposition 14.3.3, assisted by *Mathematica*, we compute:

$$f_\Omega(z) = \frac{2an + 2n + 2a - 2ar - a^2 r - r + 2}{n(n+2)} \tag{98}$$

As in the previous case we see that the numerator is minimal when $a = \frac{n}{r}$ since it is monotonically decreasing for $\frac{n-r+1}{r} \leq a \leq \frac{n}{r}$. Hence,

$$f_\Omega(z) \geq \frac{1}{r} - \frac{r-2}{n(n+2)} \tag{99}$$

Since $f_\Omega(z)$ is maximal when $a = \frac{n-r+1}{r}$ we also have $f_\Omega(z) \leq \frac{1}{r} + \frac{1}{rn(n+2)}$.

Remark 14.3.15. As proved in Lemma 14.3.13, $f_\Omega(x)$ is minimal when x is special, namely for x as in (95). The best possible lower bounds are given in (96) and (98), good approximations are given in (97) and (99).

We have also computed upper bounds on $f_\Omega(x)$ for a special element x, as in (95), and it is clear that, in any case, $f_\Omega(x)$ is close to the lower bound given in Proposition 14.3.5. For the convenience of the reader we record the best possible lower bounds, given in (96) and (98), in Table 14.3.1. We write $n = ar + b$ with $0 \leq b < r$.

$\lfloor \frac{n}{r} \rfloor$	$f_\Omega(x) \geq$
even	$\frac{1}{r} - \frac{b(b+2)}{rn(n+2)}$
odd	$\frac{1}{r} - \frac{(r-b)(r-b-2)}{rn(n+2)}$

Table 14.3.1. Lower bounds on $f_\Omega(x)$ for $x \in H$ of odd prime order $r \neq p$

14.4. Local upper bounds

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$. Here we assume r is an odd prime, see Section 14.7 for involutions. For convenience of the reader we recall that

$$\mathcal{V}_s = \{x \in G : \nu(x) = s\}$$

In addition, we denote by $\mathcal{V}_{s,r}$ the set of elements of $\mathcal{V}_s$ of order r.

The main result of this section is Proposition 14.4.1, below. This is a straightforward consequence of Proposition 14.4.4 for unipotent elements and Proposition 14.4.14 for semisimple elements. In order to justify the reason why these bounds are close to the best possible we shall produce explicit elements whose f_Ω-values are close to the upper bound stated; see Proposition 14.4.6 for the unipotent case, and Proposition 14.4.14 for semisimple elements.

Proposition 14.4.1. *Assume $r \neq 2$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n+2}$$

Remark 14.4.2. We can easily derive a bound using Theorem 7.2.5 and Proposition 5.4.1. Let $x \in H \cap \mathcal{V}_{s,r}$. Then

$$f_\Omega(x) \leq 1 - \frac{\dim x^G}{2 \dim \Omega} \leq 1 - \frac{2 \max\{s(n-s), \frac{ns}{2}\}}{n(n+2)}$$

using Theorem 7.2.5 and Proposition 5.4.1, thanks to which $\dim x^G \geq \max\{s(n-s), \frac{ns}{2}\}$. Observe that $\max\{s(n-s), \frac{ns}{2}\} = s(n-s)$ for $s \leq \frac{n}{2}$ and $\frac{ns}{2}$ for $s \geq \frac{n}{2}$.

However, in the following we shall use different methods to obtain bounds and prove that they are close to best possible.

We record here some conditions on $\nu(x)$, for $x \in H$ of prime order.

Proposition 14.4.3. *Let $x \in H$ be of order r. If $r = p$, then $\nu(x)$ is even and*

$$2 \leq \nu(x) \leq n - 2\left\lfloor \frac{n}{2p} \right\rfloor$$

If $r \neq p$ then

$$2 \leq \nu(x) \leq n - \left\lfloor \frac{n}{r} \right\rfloor - 1$$

Moreover, if $\nu(x)$ is odd then $\nu(x) \geq n/2$.

Proof. If $x \in H$ is unipotent we have already remarked in Section 13.2.1 that $\nu(x)$ is even. The bounds on $\nu(x)$ follow by arguing that the prime order elements with extreme ν-value are $[J_2^2, J_1^{n-4}], [J_p^{2a}, J_b^2] \in H$, where $n/2 = ap + b$.

Similarly, we deduce bounds on $\nu(x)$ for x semisimple of prime order r. Observe that if $\nu(x)$ is odd then the eigenvalue with the largest eigenspace is $\omega^i \neq 1$ for some $i > 0$. Hence $\nu(x) \leq n/2$. *q.e.d.*

14.4.1. Unipotent elements. Let $s < n$ even. We write $n = a(n-s) + b$, where $0 \leq b < n - s$ and b is even. Here we assume $p \neq 2$, we study involutions in Section 14.7.

Proposition 14.4.4. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \leq 1 - \frac{(n-b)(s+b)+s}{n(n+2)}$$

Proof. Up to G-conjugation we have $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$, where each a_i is even. Using Theorem 5.2.1, we compute

$$\dim x^G = 2\dim y^{\mathrm{GL}_{n/2}} + \frac{n - \sum_{i \text{ odd}} a_i}{2} \geq 2\dim y^{\mathrm{GL}_{n/2}} + \frac{s}{2} \tag{100}$$

where $y = [J_p^{a_p/2}, \ldots, J_1^{a_1/2}] \in \mathrm{GL}_{n/2}$. Note that $\nu(y) = s/2$. For all $y \in \mathrm{GL}_{n/2}$ of order p,

$$\dim y^{\mathrm{GL}_{n/2}} \geq \dim[J_{a+1}^{\frac{b}{2}}, J_a^{\frac{n-s-b}{2}}]^{\mathrm{GL}_{n/2}}$$

by Lemma B.1.1 and Example B.1.2. Again, by Theorem 5.2.1, we have

$$\dim[J_{a+1}^{\frac{b}{2}}, J_a^{\frac{n-s-b}{2}}]^{\mathrm{GL}_{n/2}} = \frac{(n-b)(s+b)}{4}$$

Therefore, by Proposition 7.1.8 and Theorem 7.2.5, we have

$$\begin{aligned} f_\Omega(x) &\leq 1 - \frac{\dim x^G}{2\dim\Omega} \leq 1 - \frac{2\dim y^{\mathrm{GL}_{n/2}} + \frac{s}{2}}{2\dim\Omega} \\ &\leq 1 - \frac{4\dim[J_{a+1}^{\frac{b}{2}}, J_a^{\frac{n-s-b}{2}}]^{\mathrm{GL}_{n/2}} + s}{4\dim\Omega} = 1 - \frac{(n-b)(s+b)+s}{n(n+2)} \end{aligned}$$

q.e.d.

Corollary 14.4.5. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n+2}$$

Proof. It is enough to observe that the upper bound in Proposition 14.4.4 is maximal for $b \in \{0, n-s\}$. *q.e.d.*

In the following we show that the bound given in Proposition 14.4.4 is accurate.

Proposition 14.4.6. *Assume $p \neq 2$. There exists $x \in H \cap \mathcal{V}_{s,p}$ such that*

$$f_\Omega(x) \geq 1 - \frac{(n-b)(s+b)+s}{n(n+2)} - \frac{2}{n+2}$$

Proof. Write $n = a(n-s)+b$, where $0 \leq b < n-s$ even. Let $x = [J_{a+1}^b, J_a^{n-s-b}] \in H$. Notice that $s < n\frac{p-2}{p-1}$. Hence $a+1 = \lfloor \frac{n}{n-s} \rfloor + 1 \leq \frac{n}{n-s} + 1 \leq p$. Therefore $x \in \mathcal{V}_{s,p}$. Notice that $x = [J_2^s, J_1^{n-2s}]$ if $s \leq \frac{n}{2}$. Write U for the bound in Proposition 14.4.4, then by that proof,

$$U = 1 - \frac{4\dim x^{H^\circ} + s}{4\dim\Omega}$$

Therefore, since $\dim(x^G \cap H) = \dim x^{H^\circ}$ (see Proposition 13.2.2), we have $U - f_\Omega(x) = \frac{\dim x^G - 2\dim x^{H^\circ}}{\dim\Omega} - \frac{s}{4\dim\Omega}$. An easy computation shows that $\dim x^G - 2\dim x^{H^\circ} = \frac{n-\delta}{2} \leq \frac{n}{2}$, where $\delta = b$ if a is even and $\delta = n-s-b$ if a is odd. In particular,

$$U - f_\Omega(x) \leq \frac{2}{n+2} - \frac{s}{n(n+2)} < \frac{2}{n+2}$$

q.e.d.

Remark 14.4.7. The bound deduced in Corollary 14.4.5 is close to best possible. Indeed, as a function of b, the upper bound $1 - \frac{(n-b)(s+b)+s}{n(n+2)}$ is maximal when $b = \frac{n-s}{2}$, and it is symmetric in b. For example, if $b = 0$ then $n = (n-s)a$ and $f_\Omega([J_a^{n-s}]) = 1 - \frac{s}{n+2} - \frac{s}{n(n+2)} \approx 1 - \frac{s}{n+2}$.

14.4.2. Semisimple elements. Assume $r \neq 2$. We shall study involutions in Section 14.7. Then, up to the centraliser structure, $x \in G$ of order r is G-conjugate to one of the following

$$(101)\qquad \begin{array}{lll} (a) & [I_{n-s}, (\omega,\omega^{-1})I_{a_1}, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] & s \text{ even} \\ (b) & [I_{a_0}, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_{a_2}, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] & s \geq \frac{n}{2} \end{array}$$

where $a_i \leq n-s$ for all i.

In this case we explicitly construct elements in $H \cap \mathcal{V}_{s,r}$ with maximal f_Ω-value.

Assume s is even if $s < \frac{n}{2}$, see Proposition 14.4.3. Let $n = (n-s)l + m$ with $0 \leq m < n-s$. We define elements $z_1, z_2 \in H \cap \mathcal{V}_{s,r}$ as follows:

$$(102)\qquad z_1 = \begin{cases} [I_{n-s}, A, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{\frac{m}{2}}] & s \text{ even} \quad l \text{ odd} \\ [I_{n-s}, A, (\omega,\omega^{-1})^{\frac{l}{2}} I_{\frac{n-s+m}{2}}] & s \text{ even} \quad l \text{ even} \\ [I_{n-s-1}, A, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{\frac{m+1}{2}}] & s \text{ odd} \quad l \text{ odd} \\ [I_{n-s-1}, A, (\omega,\omega^{-1})^{\frac{l}{2}} I_{\frac{n-s+m+1}{2}}] & s \text{ odd} \quad l \text{ even} \end{cases}$$

where $A = [(\omega,\omega^{-1})I_{n-s},\dots,(\omega,\omega^{-1})^{\lfloor\frac{l-1}{2}\rfloor}I_{n-s}]$. And

$$z_2 = \begin{cases} [B,(\omega,\omega^{-1})^{\frac{l}{2}+1}I_{\frac{m}{2}}] & l \text{ even} \\ [B,(\omega,\omega^{-1})^{\frac{l+1}{2}}I_{\frac{n-s+m}{2}}] & l \text{ odd} \end{cases} \tag{103}$$

where $B = [(\omega,\omega^{-1})I_{n-s},\dots,(\omega,\omega^{-1})^{\lfloor\frac{l}{2}\rfloor}I_{n-s}]$.

It is straightforward, with the upper bound on $\nu(x)$ given in Proposition 14.4.3, to check that $\frac{l+1}{2},\frac{l}{2},\frac{l}{2}+1 \leq \frac{r-1}{2}$.

Claim. Let $y \in H\cap\mathcal{V}_{s,r}$. Then $f_\Omega(y) \leq \max\{f_\Omega(z_1), f_\Omega(z_2)\}$.

We shall prove this claim in Proposition 14.4.13. For convenience we delegate the technical part of the proof to Lemmas 14.4.9–14.4.12.

Lemma 14.4.8. *Let $x \in H\cap\mathcal{V}_{s,r}$. Then either $C_G(x) \cong C_G(z_1), C_G(z_2)$ or there exists $y = [I_{b_0},\omega I_{b_1},\dots,\omega^{r-1}I_{b_{r-1}}] \in H\cap\mathcal{V}_{s,r}$ such that $f_\Omega(x) = f_\Omega(y)$ and one of the following holds*

(i) $b_0 = n-s, b_1 = \min\{b_i : b_i \neq 0, i>0\}$ *and* $b_2 = \max\{b_i : b_i < n-s, i>0\}$;
(ii) $b_1 = n-s, b_2 = \min_i\{b_i : b_i \neq 0\}$ *and* $b_0 = \max_i\{b_i : b_i < n-s\} < n-s-1$;
(iii) $b_1 = n-s, b_2 = \min_i\{b_i : b_i \neq 0\}, b_0 = n-s-1$ *and* $b_3 = \max\{b_i : b_i < n-s, i>0\}$;
(iv) $b_2 = n-s, b_0 = \min_i\{b_i : b_i \neq 0\}$ *and* $b_1 = \max_i\{b_i : b_i < n-s\}$;
(v) $b_1 = n-s, b_2 = \min_i\{b_i : b_i \neq 0\}$ *and* $b_3 = \max_i\{b_i : b_i < n-s\}$.

Proof. If the centraliser in G of x is not isomorphic to that of z_1 or z_2 then it is enough to relabel the eigenvalues of x, with a procedure analogous to that used in the proof of Lemma 14.3.11. *q.e.d.*

For all the possible cases arising from Lemma 14.4.8 we shall define a suitable element $y \in H\cap\mathcal{V}_{s,r}$ for which we prove $f_\Omega(x) \leq f_\Omega(y)$.

For the first technical lemma we need the following terminology. Let $x \in H\cap\mathcal{V}_{s,r}$ as in (101) and assume $a_0 = n-s$. In view of Lemma 14.4.8, we may assume $a_1 = \min_{i>0}\{a_i : a_i \neq 0\}$ and $a_2 = \max_{i>0}\{a_i : a_i < n-s\}$. Observe that if $C_G(x)$ is not isomorphic to $C_G(z_1)$ or $C_G(z_2)$ then $a_2 - a_1 \geq 1$. Let us define

$$y = [I_{n-s},(\omega,\omega^{-1})I_{a_1-1},(\omega,\omega^{-1})^2I_{a_2+1},\dots,(\omega,\omega^{-1})^{\frac{r-1}{2}}I_{a_{\frac{r-1}{2}}}] \in H\cap\mathcal{V}_{s,r} \tag{104}$$

Lemma 14.4.9. *Let $x \in H\cap\mathcal{V}_{s,r}$ with $a_0 = n-s$, $a_1 = \min\{a_i : a_i \neq 0\}$ and $a_2 = \max\{a_i : a_i < n-s\}$. Assume $a_1 < a_2$ and define $y \in H\cap\mathcal{V}_{s,r}$ as in (104). Then*

$$f_\Omega(x) \leq f_\Omega(y)$$

Proof. Using Proposition 14.3.3 we compute $f_\Omega(x), f_\Omega(y)$. Thus we have $f_\Omega(x) \leq f_\Omega(y)$ if, and only if, $4(a_2-a_1+1)+d(x)-d(y) \geq 0$. Recall that $d(x) = |\{i \in \{1,\dots,r-1\} : a_i \text{ odd}\}|$. Then $d(x) \in \{d(y)+2, d(y), d(y)-2\}$, depending on the parity of a_1 and a_2. In particular, $4(a_2-a_1+1)+d(x)-d(y) \geq 4(a_2-a_1+1)-2 > 0$, since $a_2 - a_1 \geq 1$. *q.e.d.*

Remark 14.4.10. Notice that the case (v) in Lemma 14.4.8 is very similar to (i). Here, we would define

$$y = [I_{a_0}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3+1}, \ldots] \in H \cap \mathcal{V}_{s,r} \tag{105}$$

and the proof that $f_\Omega(x) \leq f_\Omega(y)$ is entirely similar to Lemma 14.4.9

Let $x \in \mathcal{V}_{s,r}$ as in (101). Assume, now, that ω has the largest eigenspace, so that $s \geq n/2$. Thanks to Lemma 14.4.8 we may assume $a_0 = \max_i\{a_i : a_i < n-s\}$ and $a_2 = \min_i\{a_i : a_i \neq 0\}$.

If $a_0 < n-s-1$ then we define

$$y = [I_{a_0+2}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{106}$$

If, instead, $a_0 = n-s-1$ we may assume $a_3 = \max_{i>0}\{a_i : a_i < n-s\}$. We define

$$y = [I_{a_0}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3+1}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{107}$$

Lemma 14.4.11. *Let $x \in H \cap \mathcal{V}_{s,r}$ with $a_0 = \max_i\{a_i : a_i < n-s\}, a_1 = n-s$ and $a_2 = \min_i\{a_i : a_i \neq 0\}$. Define y as in (106) if $a_0 < n-s-1$ or (107) if $a_0 = n-s-1$. Then*

$$f_\Omega(x) \leq f_\Omega(y)$$

Proof. First, assume y is as in (106). Then using Proposition 14.3.3 we compute $f_\Omega(x), f_\Omega(y)$. Thus we have $f_\Omega(x) \leq f_\Omega(y)$ if, and only if, $4(a_0-a_2+3)+d(x)-d(y) \geq 0$. We see that $d(x) = d(y) \pm 1$, depending on the parity of a_2. Thus $4(a_0 - a_2 + 3) + d(x) - d(y) \geq 4(a_0 - a_2 + 3) - 1 > 0$, since $a_0 \geq a_2$.

Now, assume y is as in (107). Again, by Proposition 14.3.3, we compute $f_\Omega(x), f_\Omega(y)$. We see $f_\Omega(x) \leq f_\Omega(y)$ if, and only if, $4(a_3 - a_2 + 1) + d(x) - d(y) \geq 0$. We have $d(x) \in \{d(y) + 2, d(y), d(x) - 2\}$ depending on the parity of a_2, a_3. In particular, $4(a_3 - a_2 + 1) + d(x) - d(y) \geq 4(a_3 - a_2 + 1) - 2 > 0$, since $a_3 \geq a_2$. *q.e.d.*

Now assume $a_0 = \min\{a_i : a \neq 0\}$. Without loss of generality, thanks to Lemma 14.4.8, let $a_1 = \max\{a_i : a_i < n-s\}, a_2 = n-s$. We define

$$y = [I_{a_0-2}, (\omega, \omega^{-1})I_{a_1+1}, (\omega, \omega^{-1})^2 I_{n-s}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{108}$$

Lemma 14.4.12. *Let $x \in H \cap \mathcal{V}_{s,r}$ with $a_0 = \min\{a_i : a_i > 0\}$, $a_1 = \max\{a_i : a_i < n-s\}$ and $a_0 < a_1$. Define y as in (108). Then $f_\Omega(x) \leq f_\Omega(y)$.*

Proof. Using formula (91) we compute $f_\Omega(x), f_\Omega(y)$. We have $f_\Omega(x) \leq f_\Omega(y)$ if, and only if, $4a_0 - 4a_1 + 2 + d(x) - d(y) \geq 0$. Since $d(x) = d(y) \pm 1$ depending on the parity of a_1, we have $4a_0 - 4a_1 + 2 + d(x) - d(y) \geq 4a_0 - 4a_1 + 2 - 1 > 0$, being $a_1 \geq a_0$. *q.e.d.*

Proposition 14.4.13. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega(x) \leq \max\{f_\Omega(z_1), f_\Omega(z_2)\}$ where z_1 is defined in (102) and z_2 in (103).*

Proof. With the same argument as in Lemma 14.3.13 we see that, depending on the conditions on the multiplicities of the eigenvalues of x, we have $f_\Omega(x) \leq f_\Omega(z_1)$ or $f_\Omega(z_2)$. Indeed, assume that the centraliser in G of x is not isomorphic to the centraliser in G of z_1 or z_2, then there exists y, as in (104)–(108) such that $f_\Omega(x) \leq f_\Omega(y)$, by Lemma 14.4.9, 14.4.11 or 14.4.12. If $C_G(y) \cong C_G(z_1)$ or $C_G(z_2)$ then we are done. Otherwise, we construct y' such that $f_\Omega(y) \leq f_\Omega(y')$. Eventually, in a finite number of steps we have $f_\Omega(x) \leq f_\Omega(y) \leq \ldots \leq f_\Omega(z_1)$ or $f_\Omega(x) \leq f_\Omega(y) \leq \ldots \leq f_\Omega(z_2)$. Notice that z_2 arises from repeatedly applying Lemma 14.4.12. *q.e.d.*

Recall that $\delta_{a;b} = 1$ if $b \mid a$ and $\delta_{a;b} = 0$ otherwise.

Proposition 14.4.14. *Let $x \in H \cap \mathcal{V}_{s,r}$. Write $n = (n-s)l + m$ with $0 \leq m < n-s$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n} - \frac{m(n-s) - m^2}{2n(n+2)} \tag{109}$$

In particular, the conclusion of Proposition 14.4.1 holds.

Proof. By Proposition 14.4.13, we know that $f_\Omega(x) \leq \max\{f_\Omega(z_1), f_\Omega(z_2)\}$ for all $x \in H \cap \mathcal{V}_{s,r}$. Hence, we need to compute $f_\Omega(z_1)$ and $f_\Omega(z_2)$ using the formula in Proposition 14.3.3. We use *Mathematica* for the following calculations.

Case 1. Assume s even and l odd. Then

$$f_\Omega(z_1) = 1 - \frac{s}{n} - \frac{m(n-s-\frac{m}{2}) + 2(1-\delta_{m;4})}{n(n+2)} \tag{110}$$

and $\frac{m(n-s-\frac{m}{2})+2(1-\delta_{m;4})}{n(n+2)} > \frac{m(n-s)-m^2}{2n(n+2)} > 0$, since $m < n-s$.

Case 2. Assume s even and l even. Then

$$f_\Omega(z_1) = 1 - \frac{s}{n} - \frac{(n-s)^2 - m^2 + 4(1-\delta_{n-s+m;4})}{2n(n+2)} \tag{111}$$

and $\frac{(n-s)^2-m^2+4(1-\delta_{n-s+m;4})}{2n(n+2)} > \frac{m(n-s)-m^2}{2n(n+2)} > 0$.

Case 3. Assume s odd and l odd. Then

(112)

$$\begin{aligned} f_\Omega(z_1) &= \frac{n^2 - ns - m(n-s) + \frac{m^2+1}{2} + m - \frac{n-m}{n-s} - 2(1-\delta_{m+1;4})}{n(n+2)} \\ &= 1 - \frac{s}{n} - \frac{(n-s)(m+2) - \frac{m^2+1}{2} - m + \frac{n-m}{n-s} + 2(1-\delta_{m+1;4})}{n(n+2)} = 1 - \frac{s}{n} - \epsilon' \end{aligned}$$

and we see that

$$\epsilon' \geq \frac{2m(n-s)^2 - m^2(n-s) + (n-m)}{2n(n+2)(n-s)} > \frac{m(n-s)^2 - m^2(n-s)}{2n(n+2)(n-s)} = \frac{m(n-s) - m^2}{2n(n+2)}$$

the first inequality follows from an easy computation and the fact that $\delta_{m+1;4} \geq 0$, for the second one we use $n - m > 0$.

Case 4. Assume s odd and l even. Then

$$\begin{aligned} f_\Omega(z_1) &= \frac{n^2 + 2n + m^2 + 2m - s^2 - 2s + 3 - 2\frac{n-m}{n-s} - 4(1-\delta_{n-s+m+1})}{2n(n+2)} \\ &= 1 - \frac{s}{n} - \frac{(n-s-1)(n-s+3) - m(m+3)}{2n(n+2)} = 1 - \frac{s}{n} - \epsilon' \end{aligned} \tag{113}$$

And we compute

$$\epsilon' = \frac{m(n-s)-m^2}{2n(n+2)} + \frac{(n-s)^2+2(n-s)-m(n-s)-3(m+1)}{2n(n+2)}$$

and it is easy to see that $\frac{(n-s)^2+2(n-s)-m(n-s)-3(m+1)}{2n(n+2)} > 0$, one way is to observe that it is monotonically decreasing in m and for $m=0$ it is positive. In particular, we have

$$f_\Omega(z_1) \le 1 - \frac{s}{n} - \frac{m(n-s)-m^2}{2n(n+2)}$$

Finally, let us compute $f_\Omega(z_2)$ for z_2 defined in (103).

Case 1′. Assume l odd. Then

$$f_\Omega(z_2) = 1 - \frac{s}{n} - \frac{(n-s)(n-s+4)-m^2+l(1-\delta_{s;2})+2(1-\delta_{m;4})}{2n(n+2)} \tag{114}$$

Clearly, (109) holds in this case, as well.

Case 2′. Assume l even. Then

$$f_\Omega(z_2) = 1 - \frac{s}{n} - \frac{2(n-s)(m+2)-m^2+(l-1)(1-\delta_{s;2})+2(1-\delta_{n-s+m;4})}{2n(n+2)} \tag{115}$$

Clearly, (109) holds in this case, as well. *q.e.d.*

Remark 14.4.15. The best possible bounds, thanks to Proposition 14.4.13, are given in (110)–(115).

Let us denote by U the bound given in Proposition 14.4.14. Let us bound the difference $U - f_\Omega(z_i)$, for $i=1,2$, in some cases.

For $f_\Omega(z_1)$ as (110) we have

$$U - f_\Omega(z_1) \le \frac{m(n-s)}{2n(n+2)} + \frac{2}{n(n+2)}$$

In particular, for small values of m the difference $U - f_\Omega(z_1)$ is very small. In general we can write $U - f_\Omega(z_1) < \frac{(n-s)^2}{2n(n+2)} + \frac{2}{n(n+2)}$. If, for example, $s \ge n - \sqrt{n}$ then $U - f_\Omega(z_1) < \frac{1}{2(n+1)} + \frac{2}{n(n+2)}$.

For $f_\Omega(z_1)$ as in (111) we have

$$U - f_\Omega(z_1) \le \frac{(n-s-m)(n-s)}{2n(n+2)} + \frac{2}{n(n+2)}$$

In this case the difference is small for large values of m, for example if $m = n-s-1$ we see $U - f_\Omega(z_1) \le \frac{n-s}{2n(n+2)} + \frac{2}{n(n+2)} < \frac{1}{2(n+2)} + \frac{2}{n(n+2)}$. In general, as before, we have $U - f_\Omega(z_1) \le \frac{(n-s)^2}{2n(n+2)} + \frac{2}{n(n+2)}$. So, if $s \ge n - \sqrt{n}$ then $U - f_\Omega(z_1) \le \frac{1}{2(n+2)} + \frac{2}{n(n+2)}$.

Now, assume $f_\Omega(z_2)$ is as in (114). So

$$U - f_\Omega(z_2) \le \frac{(n-s)(n-s+4-m)}{2n(n+2)} + \frac{l+2}{2n(n+2)}$$

as the previous case, also here we see that for large values of m the difference is very small. In general, $U - f_\Omega(z_2) \le \frac{(n-s)^2}{2n(n+2)} + \frac{2(n-s)}{n(n+2)} + \frac{l+2}{2n(n+2)} \le \frac{(n-s)^2}{2n(n+2)} + \frac{2(n-s)}{n(n+2)} + \frac{1}{2n}$. Again, if $s \ge n - \sqrt{n}$ we easily compute $U - f_\Omega(z_2) \le \frac{3}{n}$.

Further let us observe that the upper bound $1-\frac{s}{n}$ is a good bound in several cases. In fact, if $m \leq \sqrt{n}$ or $m \geq n-s-\sqrt{n-s}$ we have

$$\frac{m(n-s-m)}{2n(n+2)} \leq \frac{\sqrt{n-s}(n-s-\sqrt{n-s})}{2n(n+2)} < \frac{n(\sqrt{n}-1)}{2n(n+2)} < \frac{1}{2\sqrt{n}}$$

In general,

$$\frac{m(n-s-m)}{2n(n+2)} \leq \frac{(n-s)^2}{8n(n+2)} \leq \frac{1}{8(n+2)}$$

where the last inequality follows if we assume $s \geq n-\sqrt{n}$.

14.5. Local lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$. We assume $r \neq 2$, see Section 13.2.3 for involutions. The main result of this section is the following.

Proposition 14.5.1. *Let $x \in H \cap \mathcal{V}_{s,r}$ with $r \neq 2$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)+\epsilon}{n(n+2)}$$

where $\epsilon = 2s$ if $r \neq p$ and $\epsilon = n+1$ otherwise.

We first study elements of order p. We derive a lower bound in Proposition 14.5.2. Then, in Proposition 14.5.3 we show that the bound is close to the best possible, constructing an element $x \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega(x)$ is close to the lower bound.

For elements of order $r \neq p$ we use the same strategy as in Section 14.3: we define a class of special elements in $\mathcal{V}_{s,r}$, see Definition 14.5.8, then we show that any special element realises the best possible lower bound.

14.5.1. Unipotent elements. Let $x \in H \cap \mathcal{V}_{s,p}$. Then up to G-conjugacy we have $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ where each a_i is even, $\sum_i i a_i = n$ and $s = n - \sum_i a_i$.

Proposition 14.5.2. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \geq 1 - \frac{n+1}{n(n+2)} - \frac{s(2n-s)}{n(n+2)}$$

PROOF. According to Proposition 14.2.2 we have

$$\begin{aligned} f_\Omega(x) &= \frac{2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 + 2\sum_{i \text{ odd}} a_i}{n(n+2)} = \frac{n(n+1) - 2\dim x^G + \sum_{i \text{ odd}} a_i}{n(n+2)} \\ &\geq \frac{n(n+1) - 2\dim x^G}{n(n+2)} \geq \frac{n(n+1) - (2ns - s^2 + 1)}{n(n+2)} \\ &= 1 - \frac{n+1}{n(n+2)} - \frac{s(2n-s)}{n(n+2)} \end{aligned}$$

where the final inequality follows from Proposition 5.4.1. *q.e.d.*

In fact, the bound of Proposition 14.5.2 is close to the best possible.

Proposition 14.5.3. *Let s be an even integer such that $H \cap \mathcal{V}_{s,p} \neq \emptyset$. Then there exists $x \in H \cap \mathcal{V}_{s,p}$ such that*

$$f_\Omega(x) \leq 1 + \frac{1}{n(n+2)} - \frac{s(2n-s)}{n(n+2)} + \epsilon$$

where $\epsilon = 0$, *if* $s \le 2p-2$, *and* $\epsilon = \frac{s^2}{n(n+2)(p-1)}$, *otherwise.*

Proof. Let $x = [J^2_{\frac{s+2}{2}}, J_1^{n-s-2}]$, thus $\nu(x) = s$. Clearly $x \in H$. Using Proposition 14.2.2, we have

$$f_\Omega(x) = \frac{n^2+2n-2ns+s^2-4+4(1-\delta_{s+2;4})}{n(n+2)} = 1 - \frac{s(2n-s)+4\delta_{s+2;4}}{n(n+2)}$$

Let ℓ is the bound in Proposition 14.5.2. A straightforward computation leads to

$$f_\Omega(x) - \ell \le \frac{n+1}{n(n+2)} < \frac{1}{n}$$

Note that x has prime order if, and only if, $s+2 \le 2p$.

For the remainder of the proof let us assume $s+2 > 2p$. Define $h = \min\{i \ge 0 : \frac{s+2}{2} + i - ip \le p\}$. It is clear that

$$h = \left\lfloor \frac{s}{2(p-1)} - 1 \right\rfloor \le \frac{s}{2(p-1)} - 1$$

Hence, x has prime order if, and only if, $h = 0$. In the case $h > 0$ we define

$$z = \left[J_p^{2h}, J^2_{\frac{s}{2}+1+h-hp}, J_1^{n-s-2h-2}\right] \in H \cap \mathcal{V}_{s,p}$$

For convenience we denote $\alpha = \frac{s}{2}+1+h-hp$. Notice that $\alpha \le p$, with equality if, and only if $h = \frac{s}{2(p-1)} - 1$. Using Proposition 14.2.2 and *Mathematica*, we compute

$$f_\Omega(z) = 1 - \frac{s(2n-s)}{n(n+2)} - \frac{4\delta_{\alpha;2}}{n(n+2)} + \frac{4g(h)}{n(n+2)}$$

where $g(h) := hs - h(p-1) - h^2(p-1)$. Observe that in the case $p = \alpha$ we get the element $z' = [J_p^{2h+2}, J_1^{n-s-2-2h}]$ and using Proposition 14.2.2 we see that $f_\Omega(z) = f_\Omega(z')$. Therefore we have

$$f_\Omega(z) - \ell \le \frac{n+1+4g(h)}{n(n+2)}$$

We need to bound $g(h)$. Observe that, in the case $h > 0$, we have $h = \left\lfloor \frac{s}{2(p-1)} - 1 \right\rfloor = \left\lfloor \frac{s}{2(p-1)} \right\rfloor - 1$. Hence, for $a = \left\lfloor \frac{s}{2(p-1)} \right\rfloor$ we have $g(h) = g(a-1) = a(s+p-1) - s - a^2(p-1) \le g\left(\frac{s}{2(p-1)} - 1\right) = \frac{s^2}{4(p-1)}$. Therefore

$$f_\Omega(x) - \ell \le \frac{1}{n} + \frac{4g(h)}{n(n+2)} \le \frac{1}{n} + \frac{s^2}{n(n+2)(p-1)}$$

q.e.d.

Remark 14.5.4. If $x \in H \cap \mathcal{V}_{s,r}$ and $s > 2p-2$, then $p < n/2$ and, by Proposition 14.4.3, $s \le n-2$. So by Proposition 14.5.3, $\epsilon = \frac{s^2}{n(n+2)(p-1)} < \frac{1}{p-1}$. Hence, for large values of p the bound in Proposition 14.5.2 is accurate.

14.5.2. Semisimple elements. Let $x \in H \cap \mathcal{V}_{s,r}$ as in (101). Notice that we may assume $s \le n-2$, because if $\nu(x) = n-1$ by Proposition 14.3.7 we have $f_\Omega(x) = 0$. We shall prove the following result.

Proposition 14.5.5. *Assume* $s < n-1$. *Let* $x \in H \cap \mathcal{V}_{s,r}$. *Then*

$$f_\Omega(x) \ge 1 - \frac{s(2n-s)+2s}{n(n+2)}$$

Remark 14.5.6. Assume $r = 3$ and let $x \in \mathcal{V}_{s,3}$. Then, up to conjugation, $x = [I_{n-s}, \omega I_{\frac{s}{2}}, \omega^{-1} I_{\frac{s}{2}}]$ or $[I_{2s-n}, \omega I_{n-s}, \omega^{-1} I_{n-s}]$. Notice that the second element may occur only for $s \geq n/2$ and, if $s \geq n/2$ is odd, only the second element occurs. Moreover, in all cases $s \leq \frac{2}{3}n$.

The following is an easy computation using the observation in Remark 14.5.6.

Proposition 14.5.7. *Let $x \in H \cap \mathcal{V}_{s,3}$.*

(i) If $s < n/2$ is even, then

$$f_\Omega(x) = 1 - \frac{s(2n-s)}{n(n+2)} - \frac{4s - s^2 + 4(1 - \delta_{s;4})}{2n(n+2)}$$

(ii) If $s \geq n/2$ is odd, then

$$f_\Omega(x) = \frac{3n^2 - 2n + 6s^2 - 8ns + 4s - 2(1 - \delta_{s;2})}{n(n+2)}$$

(iii) If $s \geq n/2$ is even, then

$$f_\Omega(x) \geq \min \left\{ \begin{array}{l} 1 - \frac{s(2n-s)}{n(n+2)} - \frac{4s-s^2+4(1-\delta_{s;4})}{2n(n+2)} \\ \frac{3n^2-2n+6s^2-8ns+4s}{n(n+2)} \end{array} \right\}$$

In particular, the conclusion of Proposition 14.5.5 holds.

For the remainder of this section we may assume $r > 3$. In order to find sharp lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$, we use the same argument as that given in Section 14.3.

Let $x \in H \cap \mathcal{V}_{s,r}$. Assume, for some $h \in \{0, \ldots, \frac{r-1}{2}\}$, $\dim V_{\omega^h} = \dim V_{\omega^{-h}} = n - s$.

Definition 14.5.8. We define $x \in \mathcal{V}_{s,r}$ to be *special* if $|a_i - a_j| \leq 1$ for all $i, j \neq h, r-h$.

Claim. Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega(x) \geq f_\Omega(z)$ for some special element $z \in H \cap \mathcal{V}_{s,r}$.

If $x \in \mathcal{V}_{s,r}$ is not special, then $|a_i - a_j| \geq 2$ for some i, j.

Lemma 14.5.9. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then either x is special or there exists an element $y = [I_{b_0}, \omega I_{b_1}, \ldots, \omega^{r-1} I_{b_{r-1}}] \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega(x) = f_\Omega(y)$ and one of the following holds*

(i) $b_0 = n - s, b_1 - b_2 \geq 2$;
(ii) $b_0 < n - s, b_1 = n - s, b_0 - b_2 \geq 2$;
(iii) $b_0 < n - s, b_1 = n - s, b_2 - b_0 \geq 2$;
(iv) $b_0 < n - s, b_1 = n - s, b_2 - b_3 \geq 2$.

Proof. As in Lemma 14.3.11 and 14.4.8 if x is non-special we define y by relabelling the eigenvalues of x. Let us give the details for (ii) and (iii), the other cases are very similar. Assume x is non-special, $a_0 < n - s, a_i = n - s$ and $|a_0 - a_j| \geq 2$ for some $j \neq i$ and $i, j \leq \frac{r-1}{2}$. We define $y = [I_{a_0}, \omega I_{a_i}, \omega^2 I_{a_j}, \ldots, \omega^i I_{a_1}, \omega^j I_{a_2}, \ldots] \in H \cap \mathcal{V}_{s,r}$. Notice that, by Proposition 14.3.3, $f_\Omega(x) = f_\Omega(y)$. *q.e.d.*

If $x \in H \cap \mathcal{V}_{s,r}$ is non-special, then, in view of Lemma 14.5.9, we may substitute x with an element of $H \cap \mathcal{V}_{s,r}$ for which one of the four different cases (i)–(iv) occurs.

Case 1. $a_0 = n - s$ and $a_1 - a_2 \geq 2$. We define

$$y = [I_{n-s}, (\omega, \omega^{-1})I_{a_1-1}, (\omega, \omega^{-1})^2 I_{a_2+1}, (\omega, \omega^{-1})^3 I_{a_3}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{116}$$

Case 2. $a_0 < n - s$ and $a_1 = n - s$. Here we have two sub-cases, either $|a_0 - a_2| \geq 2$ or $|a_2 - a_3| \geq 2$.

Case 2a. If $a_0 - a_2 \geq 2$ we define

$$y = [I_{a_0-2}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2+1}, (\omega, \omega^{-1})^3 I_{a_3}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{117}$$

Case 2b. If $a_2 - a_0 \geq 2$ we define

$$y = [I_{a_0+2}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{118}$$

Case 2c. Assume $|a_2 - a_3| \geq 2$, without loss of generality we may assume $a_2 - a_3 \geq 2$. Define

$$y = [I_{a_0}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3+1}, (\omega, \omega^{-1})^4 I_{a_3}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{119}$$

Using this notation, with an easy computation similar to Lemma 14.3.12 we can prove the following.

Lemma 14.5.10. *Let $x \in H \cap \mathcal{V}_{s,r}$ be non-special. Define $y \in H \cap \mathcal{V}_{s,r}$ as in (116), (117), (118) or (119). Then*

$$f_\Omega(x) \geq f_\Omega(y)$$

Now, iterating the construction of Lemma 14.5.10 we deduce the claim.

Lemma 14.5.11. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

for some special element $z \in H \cap \mathcal{V}_{s,r}$.

PROOF. The same argument as in Lemma 14.3.13 applies. *q.e.d.*

Let us give a general description of the special elements in $\mathcal{V}_{s,r}$.

For any s even we write $s = a(r-1) + b$ with $0 \leq b < r - 1$, notice that b is even since $r - 1$ is even. The following is a special element

$$\bar{x} = [I_{n-s}, (\omega, \omega^{-1})I_{a+1}, \ldots, (\omega, \omega^{-1})^{\frac{b}{2}} I_{a+1}, (\omega, \omega^{-1})^{\frac{b}{2}+1} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a] \tag{120}$$

For $s \geq \frac{n}{2}$, even or odd, let $2s - n = a(r-2) + b$, notice that $a \equiv b \pmod 2$. If a is odd then a special element is given by

$$\bar{x} = [I_{a+1}, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a+1}, \ldots, (\omega, \omega^{-1})^{\frac{b+1}{2}} I_{a+1}, (\omega, \omega^{-1})^{\frac{b+3}{2}} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a] \tag{121}$$

Similarly, if a is even then

$$\bar{x} = [I_a, (\omega, \omega^{-1})I_{n-s}, (\omega, \omega^{-1})^2 I_{a+1}, \ldots, (\omega, \omega^{-1})^{\frac{b}{2}+1} I_{a+1}, (\omega, \omega^{-1})^{\frac{b}{2}+2} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a] \tag{122}$$

The following is clear.

Proposition 14.5.12. *Let* $x \in H \cap \mathcal{V}_{s,r}$ *be a special element. Then* $C_G(x) \cong C_G(\bar{x})$*, where* $\bar{x}$ *is as in (120)–(122), above.*

Remark 14.5.13. Let $s \geq n/2$ odd. Then all the elements in $\mathcal{V}_{s,r}$ have the ω^i-eigenspace of dimension $(n-s)$, for some $i > 0$. Therefore, up to the centraliser structure a special element z is as in (121) or (122). However for $s \geq n/2$ even in $\mathcal{V}_{s,r}$ there are special elements of any type as (120), (121) or (122) according to the right divisibility conditions.

In the latter case of Remark 14.5.13, Lemma 14.5.11 does not say which of the elements in (120), (121) or (122) realises the lower bound. However, it is not hard to compute $f_\Omega(\bar{x})$, using Proposition 14.3.3.

For simplicity we divide the analysis depending on the value of a. First we study the case $a = 0$, then the case $a > 0$.

Assume $a = 0$ so that $r - 1 \geq s, 2s - n$, respectively for (120) or (122). Hence, for x as in (120) we have, using Proposition 14.3.3,

$$f_\Omega([I_{n-s}, (\omega, \omega^{-1}), \dots, (\omega, \omega^{-1})^{\frac{s}{2}}]) = 1 - \frac{2s(n+1) - s^2}{n(n+2)}$$

For x as in (122) we have

$$f_\Omega([(\omega, \omega^{-1}) I_{n-s}, (\omega, \omega^{-1})^2, \dots, (\omega, \omega^{-1})^{s - \frac{n}{2} + 1}]) = \frac{2(n-s)^2 - 2(1 - \delta_{s;2})}{n(n+2)}$$

Lemma 14.5.14. *Assume* $r - 1 \geq s$ *if* $s < n/2$ *and* $r - 1 \geq 2s - n$ *otherwise. Let* $x \in H \cap \mathcal{V}_{s,r}$*. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n - s) + 2s}{n(n+2)}$$

Proof. By Lemma 14.5.11, $f_\Omega(x)$ is minimal when x is special. By the above computation if x is a special then $f_\Omega(x) \in \left\{1 - \frac{2s(n+1)-s^2}{n(n+2)}, \frac{2(n-s)^2 - 2(1-\delta_{s;2})}{n(n+2)}\right\}$. The result follows by computing the minimal element of the previous set. *q.e.d.*

In the case $a > 0$ it is still possible to compute $f_\Omega(\bar{x})$ in the three cases we are interested in. Set $\ell = 1 - \frac{2s(n+1)-s^2}{n(n+2)}$.

Case 1. Assume x is as in (120) and $r - 1 < s$, so that $a = \left\lfloor \frac{s}{r-1} \right\rfloor \geq 1$. Using Proposition 14.3.3 we have

$$f_\Omega(x) = \frac{2(n-s) + (n-s)^2 + (a+1)^2 b + a^2(r - 1 - b) - d(x)}{n(n+2)}$$

and $d(x) = b$ if a is even or, $r - 1 - b$ for a odd. Using *Mathematica*, we simplify

$$f_\Omega(x) = 1 - \frac{2s(n+1)}{n(n+2)} + \frac{rs^2 - b^2 - \epsilon(r-1)}{n(n+2)(r-1)} \tag{123}$$

where $\epsilon = 0$ if a is even and $\epsilon = r - 1 - 2b$ for a odd. This expression is minimal when either $b = 0$ if a is odd, or $b = r - 1$ for a even; in both cases

$$f_\Omega(x) \geq 1 - \frac{2s(n+1)}{n(n+2)} + \frac{rs^2 - (r-1)^2}{n(n+2)(r-1)} \geq 1 - \frac{2s(n+1) - s^2}{n(n+2)}$$

where the final inequality follows by observing that the second expression is minimal when $r-1=s$.

Case 2. Now assume x is as in (121) with $a=\lfloor\frac{2s-n}{r-2}\rfloor\geq 1$, i.e. $r-2\leq 2s-n$. Then, using Proposition 14.3.3 we have

(124)
$$\begin{aligned} f_\Omega(x) &= \frac{-b^2+2b(r-3)+n^2(2r-3)+n(4s-4rs-2)+2rs^2-(r-s)^2+4s-2(1-\delta_{s;2})}{n(n+2)(r-2)} \\ &\geq \frac{n^2(2r-3)+n(4s-4rs-2)+2rs^2-(r-s)^2+4s-2}{n(n+2)(r-2)} \end{aligned}$$

($f_\Omega(x)$ is minimal when $b=0$). Moreover, differentiating with respect to $r-2$ we see that it is minimal when $r-2$ is maximal, i.e. $r-2=2s-n$. Thus

$$f_\Omega(x)\geq \frac{2(n^3-4n^2s+5ns^2+n-2s^3-2s+1)}{n(n+2)(n-2s)}=\ell_1$$

At this point it is not hard to see that $\ell_1\geq\ell=1-\frac{2s(n+1)-s^2}{n(n+2)}$. In the particular case $s=n/2$ so that $x=[\omega I_{n/2},\omega^{-1}I_{n/2}]$ the claimed inequality quickly follows. Then we assume $s>n/2$ and we proceed as follows. First we compute $\ell_1-\ell$, then we differentiate this difference with respect to s and we see that the first derivative is always positive and $(\ell_1-\ell)|_{s=\frac{n}{2}+1}\geq 0$. Using *Mathematica*, we get

$$(\ell_1-\ell)(s)=\frac{n^3-4n^2s-2n^2+5ns^2+6ns+2n-2s^3-4s^2-4s+2}{n(n+2)(n-2s)}$$

Differentiating with respect to s we have

$$(\ell_1-\ell)'(s)=\frac{2}{n(n+2)(n-2s)^2}\big(-n^3+n^2(5s+1)-4ns(2s+1)+4s^3+4s^2+2\big)$$

It is not hard to see that $(\ell_1-\ell)'(s)$ is always positive. Finally it is a straightforward computation to check that $(\ell_1-\ell)|_{s=\frac{n}{2}}>0$.

Case 3. Take x as in (122) and assume $a=\lfloor\frac{2s-n}{r-2}\rfloor\geq 1$, i.e. $r-2\leq 2s-n$. Then, using Proposition 14.3.3, with the fact $d(x)=b+2(1-\delta_{s;2})$ and $a=\frac{2s-n-b}{r-2}$, we have

$$\begin{aligned} f_\Omega(x) &= \frac{2a+2(n-s)^2+b(a+1)^2+(r-b-2)a^2-d(x)}{n(n+2)} \\ &= \frac{-2b-2n-2bn-3n^2+br+2n^2r+4s+4bs+4ns-4nrs+2rs^2}{n(n+2)(r-2)} \qquad (125) \end{aligned}$$

In particular,

(126)
$$\begin{aligned} f_\Omega(x) &\geq \frac{-2n-3n^2+2n^2r+4s+4ns-4nrs+2rs^2}{n(n+2)(r-2)} \\ &\geq \frac{-2n-3n^2+2n^2(2s-n+2)+4s+4ns-4ns(2s-n+2)+2s^2(2s-n+2)}{n(n+2)(2s-n-2)} \\ &= \frac{2n^2-4ns-n+2s^2+2s+2}{n(n+2)} \end{aligned}$$

where the first inequality is due to the fact that $f_\Omega(x)$ is minimal when b is minimal. The second inequality follows from the fact that (126) is minimal when r is maximal, (differentiating (126) with respect to r, we get $\frac{-n^2+n(4s+2)-4s(s+1)}{n(n+2)(r-2)^2} < 0$). Then, using *Mathematica* we see that

$$\frac{2n^2-4ns-n+2s^2+2s+2}{n(n+2)} - \left(1-\frac{2s(n+1)-s^2}{n(n+2)}\right) = \frac{(n-s)^2-2n+4s+2}{n(n+2)} \geq 0$$

We have just proved the following.

Lemma 14.5.15. *Assume $r-1 < s$ or $r-1 < 2s-n$ in the above cases. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{2s(n+1)-s^2}{n(n+2)}$$

We deduce Proposition 14.5.5 from Lemma 14.5.14 and Lemma 14.5.15. The best possible bounds are given in (123), (124) and (125).

Remark 14.5.16. The bound is the best possible. In fact as shown in Lemma 14.5.14 special elements realise the bound.

14.6. Further comments on local bounds

Using Propositions 14.4.1 and 14.5.1, we have the following.

Proposition 14.6.1. *Assume $r \neq 2$. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{s(n-s)}{n(n+2)} + \frac{1}{n+2}$$

Remark 14.6.2. Notice that if $s < \sqrt{n}$ or $s > n-\sqrt{n}$ then $\frac{s(n-s)}{n(n+2)} < \frac{1}{\sqrt{n}}$. In particular, by Proposition 14.6.1 for all $x, y \in H \cap \mathcal{V}_{s,r}$ we have $|f_\Omega(x) - f_\Omega(y)| < \frac{2}{\sqrt{n}}$.

14.7. Involutions

The main result of this section is the following.

Proposition 14.7.1. *Let $x \in G$ be an involution with $\nu(x) = s$. Assume $x^G \cap H \neq \emptyset$. Then for $(s,p) \neq (n/2, 2)$*

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n+2)}$$

In addition, if $p = 2$ then x is an a_s-involution.

If $(\nu(x), p) = (n/2, 2)$ then

$$f_\Omega(a_{n/2}) = \frac{1}{2} + \frac{2}{n+2}, \quad f_\Omega(b_{n/2}) = f_\Omega(c_{n/2}) = \frac{1}{2} - \frac{2}{n+2}$$

We prove Proposition 14.7.1 in Section 14.7.1 for the case $p \neq 2$ and Section 14.7.2 when $p = 2$. We quickly deduce bounds on $f_\Omega(x)$ for $x \in H$ an involution.

Corollary 14.7.2. *Let $x \in H$ be an involution. Then either $n > 4$ if $p = 2$, and*

$$f_\Omega(x) \leq 1 - \frac{4}{n} + \frac{16}{n(n+2)}$$

or $(n,p) = (4,2)$ *and* $f_\Omega(x) \le 5/6$. *Furthermore, equality holds if, and only if,* $(n,p,x) = (4,2,a_2)$ *or if* $\nu(x) = 2$ *in the remaining cases. In addition,*

$$f_\Omega(x) \ge \frac{1}{2} + \epsilon_p$$

where $\epsilon = \frac{1}{n+2\delta_{n;4}}$ *if* $p \neq 2$ *and* $\epsilon = -\frac{2}{n+2}$ *if* $p = 2$. *Furthermore, equality holds if, and only if,* $\nu(x) \in \{n/2, n/2-1\}$.

Proof. We have that $f_\Omega(x)$ is decreasing for s in $[1, n/2]$. Therefore $f_\Omega(x) \ge f_\Omega([I_{n/2}, -I_{n/2}])$ when $n/2$ is even, and $f_\Omega(x) \ge f_\Omega([I_{n/2+1}, -I_{n/2-1}])$ for $n/2$ odd. Using Proposition 14.7.1 we compute

$$f_\Omega([I_{n/2}, -I_{n/2}]) = 1 - \frac{n}{2(n+2)} = \frac{1}{2} + \frac{1}{n+2}, \quad f_\Omega([I_{n/2+1}, -I_{n/2-1}]) = \frac{1}{2} + \frac{1}{n}$$

In the case $p = 2$ a direct check shows that $f_\Omega(x) \ge \frac{1}{2} - \frac{2}{n+2}$ with equality if, and only if, $x = b_{n/2}$ or $c_{n/2}$. Similarly for the upper bound. *q.e.d.*

Remark 14.7.3.

(i) Recall that $f_\Omega(x) \le 1 - \frac{4}{n+2}$ for all $x \in G$ of odd prime order (see Proposition 14.1.1). In view of Corollary 14.7.2 we have that for any $x \in H$ (of any order), $f_\Omega(x) \le 1 - \frac{4}{n} + \frac{16}{n(n+2)}$, with equality if, and only if, $[I_{n-2}, -I_2]$ or a_2. By Corollary 7.1.11, Theorem 12.1.1 follows.

(ii) For any element $x \in H$ of odd prime order $r \neq p$ we have, by Proposition 14.3.1, $f_\Omega(x) \ge \frac{1}{r} - \epsilon$. This bound still holds for involutions, when $p \neq 2$. However, the best possible lower bound, for $r = 2$, is given in Corollary 14.7.2, with a characterisation of elements that realise it.

(iii) If $p = 2$ then the bound given for $p \neq 2$ in Proposition 14.2.1 does not hold. In fact for involutions we have $f_\Omega(x) \ge \frac{1}{2} - \frac{2}{n+2}$.

14.7.1. Semisimple involutions. Assume $p \neq 2$. Let $x \in H$ be an involution. Up to conjugation, $x = [I_{n-s}, -I_s]$. Then, by Theorem 5.3.1, $\dim x^G = s(n-s)$. Assume $x \in H^\circ$, so $x = [x_1, x_2]$, with $x_1 = x_2^{-t}$. Therefore $x_1 = x_2 = \left[I_{\frac{n-s}{2}}, -I_{\frac{s}{2}}\right]$. Further, if $s < n/2$ then $\dim(x^G \cap H) = \dim x^{H^\circ} = \frac{s}{2}(n-s)$, see Proposition 13.2.13(i).

Now suppose $s = n/2$. If $n/2$ odd then there are no involution in $H \setminus H^\circ$, by Proposition 13.2.8. Let $n/2$ be even. The same result shows that there is only one conjugacy class of involutions, whose representative is $\tau_2 = [K_{n/2}, K_{n/2}]\tau$, defined in (85).

Proof of Proposition 14.7.1. As said above for $x = [I_{n-s}, -I_s]$ with $s \le n/2$ we have $\dim x^G = s(n-s)$. Moreover, if $s < n/2$, thanks to Proposition 13.2.13 we have $x^G \cap H = x^{H^\circ}$. So $\dim(x^G \cap H) = \frac{s}{2}(n-s)$. Using Proposition 7.1.8, we compute $f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n+2)}$ (♣).

If $n/2$ is even and $s = n/2$, we may write $x = [I_{n/2}, -I_{n/2}]$. Then, by Proposition 13.2.13, $x^G \cap H = x^{H^\circ} \cup \tau_2^{H^\circ}$ where $\tau_2 = [B, \dots, B]\tau$ and $C_{H^\circ}(\tau_2) = \mathrm{Sp}_{n/2}$, by Proposition 13.2.14. Therefore

$$\dim x^{H^\circ} = \frac{n^2}{8}, \qquad \dim \tau_2^{H^\circ} = \dim H^\circ - \dim C_{H^\circ}(\tau_2) = \frac{n^2}{8} - \frac{n}{4}$$

Thus, we conclude that $\dim(x^G \cap H) = \dim x^{H^\circ} = \frac{n^2}{8}$. In particular, (♣) computed above holds in this case, as well. *q.e.d.*

14.7.2. Unipotent involutions. Here we assume $p = 2$. Let $x \in H$ be an involution. If $\nu(x) = s < n/2$ then $x^G \cap H = x^G \cap H^\circ$, by Proposition 13.2.13(i). And, by Lemma 13.2.11, x is an a_s-type.

If $s = n/2$ there are two cases depending on the parity of $n/2$. If $n/2$ is odd x is $b_{n/2}$-type and $x^G \cap H = \tau^{H^\circ}$ (Proposition 13.2.13(ii)). If $n/2$ is even there are two different G-classes of involutions: $a_{n/2}^G$ and $c_{n/2}^G$. And, Proposition 13.2.13(ii) yields $a_{n/2}^G \cap H = [J_2^{n/4}]^{H^\circ} \cup ([B, \dots, B]\tau)^{H^\circ}$ and $c_{n/2}^G \cap H = \tau^{H^\circ}$, where B is as in (83).

PROOF OF PROPOSITION 14.7.1. Assume $s < n/2$. Then, as remarked above x is a_s-type. So, $\dim x^G = s(n-s)$ and $\dim(x^G \cap H) = \dim[J_2^{s/2}, J_1^{n/2-s}]^{H^\circ} = \frac{s}{2}(n-s)$, by Theorem 5.2.1. Therefore, using Proposition 7.1.8 and the formula for $\dim \Omega$ given in Table 13.1.1 we get

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n+2)}$$

If $s = n/2$ is odd, then $x \in b_{n/2}^G$ and $x^G \cap H = \tau^{H^\circ}$ moreover $C_{H^\circ}(\tau) = \mathrm{Sp}_{n/2}$, by Proposition 13.2.14. Therefore, by Proposition 5.2.5, $\dim x^G = n(n+2)/4$ and $\dim \tau^{H^\circ} = \dim H^\circ - \dim C_{H^\circ}(\tau) = n(n-2)/8$. Thus $f_\Omega(x) = 1/2 - 2/(n+2)$.

Now assume $s = n/2$ is even. As stated above there are two G-classes of involutions. If x is an $a_{n/2}$-involution we have $\dim x^G = \frac{n^2}{4}$ and $\dim(x^G \cap H) = \max\{\dim[J_2^{n/4}]^{H^\circ}, \dim([B, \dots, B]\tau)^{H^\circ}\}$. We compute $\dim[J_2^{n/4}]^{H^\circ} = n^2/8$. By Proposition 13.2.14, $C_{\mathrm{GL}_{n/2}}([B, \dots, B]\tau) = C_{\mathrm{Sp}_{n/2}}(b_1)$. So $\dim([B, \dots, B]\tau)^{H^\circ} = \dim \mathrm{GL}_{n/2} - \dim C_{\mathrm{GL}_{n/2}}([B, \dots, B]\tau) = \frac{n}{8}(n+2)$. Therefore $\dim(x^G \cap H) = n(n+2)/8$. Using Proposition 7.1.8 and the value of $\dim \Omega$ in Table 13.1.1 we get $f_\Omega(x) = 1/2 + 2/(n+2)$.

If x is a $c_{n/2}$-involution we have $\dim x^G = n(n+2)/4$ and $\dim(x^G \cap H) = \dim \tau^{H^\circ} = \dim H^\circ - \dim C_{H^\circ}(\tau) = n(n-2)/8$, where $C_{H^\circ}(\tau) = \mathrm{Sp}_{n/2}$, by Proposition 13.2.14. So $f_\Omega(x) = 1/2 - 2/(n+2)$. *q.e.d.*

CHAPTER 15

Orthogonal group

Throughout this chapter the notation is as follows. Let $G = \mathrm{O}_n$, $n > 4$ even, $H = \mathrm{GL}_{n/2}.2$ be a $\mathscr{C}_3$-subgroup of G. Set $\Omega = G/H$. The aim of this chapter is to derive bounds on $f_\Omega(x)$ for $x \in G$ of prime order.

15.1. Upper bounds

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H$ an element of odd prime order r. The main result of this section is Proposition 15.1.1, below. For convenience, the study of involutions is postponed to Section 15.7. As usual, $\omega \in k$ denotes a primitive r-th root of unity.

Proposition 15.1.1. *Let $x \in G$ be of order $r > 2$. Then either*

$$f_\Omega(x) \leq 1 - \frac{4}{n} + \frac{8\delta_{r,p}}{n(n-2)} \tag{127}$$

or, $r \neq p$, $n = 6, 8, f_\Omega(x) \leq 2/3$ and $f_\Omega(x) = 2/3$ if, and only if $C_G(x) \cong \mathrm{GL}_{n/2}$.

Furthermore, in (127) equality holds if, and only if, $\nu(x) = 2$ or $C_G(x) \cong C_G(\bar{x})$, where $\bar{x}$ is listed in Table 15.1.1.

n	p	r	$\bar{x}$
8	> 2	$= p$	$[J_2^4]$
8	> 3	$= p$	$[J_4^2]$
10		$\neq p$	$[\omega I_5, \omega^{-1} I_5]$

Table 15.1.1

Remark 15.1.2. The same observations made in Remarks 10.1.3 and 14.1.2 hold here. Hence, in the next results we will only consider elements $x \in H$ of odd prime order r.

Remark 15.1.3. If $n = 4$ then we see that $f_\Omega([\omega I_2, \omega^{-1} I_2]) = 1$, This motivates the choice to assume $n > 4$.

We spread the proof of Proposition 15.1.1 in a sequence of lemmas. In the following we will assume r is an odd prime.

Recall that if $x \in H$ is a unipotent element then $\nu(x)$ is even, see Section 13.2.1.

Lemma 15.1.4. *Let $x \in H$ be of order r. Assume $\nu(x) = 2$. Then $f_\Omega(x) = 1 - \frac{4}{n} + \frac{8\delta_{r,p}}{n(n-2)}$.*

Proof. First assume x is unipotent. Up to G-conjugacy, $x = [J_2^2, J_1^{n-4}]$, notice that $[J_3, J_1^{n-3}]^G \cap H = \emptyset$. Hence, by Theorem 5.2.1 and (78) we have $\dim x^G = 2n - 6, \dim(x^G \cap H) = n - 2$. So Proposition 7.1.8 yields $f_\Omega(x) = 1 - \frac{4}{n} + \frac{8}{n(n-2)}$.

Now, consider semisimple elements. Up to G-centraliser, $x = [I_{n-2}, \omega, \omega^{-1}]$. Using Theorem 5.3.1 and Proposition 13.2.7, we get $\dim x^G = 2n-4, \dim(x^G \cap H) = n-2$. Thus, $f_\Omega(x) = 1 - \frac{4}{n}$. *q.e.d.*

Lemma 15.1.5. *Let $x \in H$ be of order $p > 2$. Assume $\nu(x) = 4$. Then*

$$f_\Omega(x) \leq 1 - \frac{4}{n} + \frac{8}{n(n-2)}$$

Furthermore equality holds if, and only if, $n = 8$ and $x \in [J_2^4]^G$.

Proof. Up to G-conjugacy, x is either $[J_3^2, J_1^{n-6}]$ or $[J_2^4, J_1^{n-8}]$. We denote these two elements by x_1, x_2, respectively. Then, we compute

$$f_\Omega(x_1) = 1 - \frac{8}{n} + \frac{16}{n(n-2)}, \qquad f_\Omega(x_2) = 1 - \frac{8}{n} + \frac{32}{n(n-2)}$$

At this point it is straightforward to check that $f_\Omega(x_i) \leq 1 - \frac{4}{n} - \frac{8}{n(n-2)}$ for $i = 1, 2$ and equality holds if, and only if $x = x_2$ and $n = 8$. *q.e.d.*

We now give a similar result for semisimple elements.

Lemma 15.1.6. *Let $x \in H$ be of order $r \neq p$. Assume $\nu(x) \in \{3, 4, 5\}$.*

(i) If $n \geq 10$ then

$$f_\Omega(x) \leq 1 - \frac{4}{n}$$

with equality if, and only if, $n = 10$ and $C_G(x) \cong \mathrm{GL}_5$.

(ii) If $n = 6$ or 8 then $f_\Omega(x) \leq 2/3$, with equality if, and only if, $C_G(x) \cong \mathrm{GL}_{n/2}$.

Proof. We list the elements $x \in H$ for which $\nu(x) \in \{3, 4, 5\}$, together with $\dim x^G, \dim(x^G \cap H)$ and $f_\Omega(x)$ in Table 15.1.2. *q.e.d.*

$\nu(x)$	x	$\dim x^G$	$\dim(x^G \cap H)$	$f_\Omega(x)$
3	$[\omega I_3, \omega^{-1} I_3]$	6	4	2/3
4	$[I_{n-4}, \omega I_2, \omega^{-1} I_2]$	$4n-14$	$2n-6$	$1 - \frac{8}{n} + \frac{16}{n(n-2)}$
	$[I_{n-4}, \omega, \omega^{-1}, \omega^2, \omega^{-2}]$	$4n-12$	$2n-6$	$1 - \frac{8}{n} + \frac{8}{n(n-2)}$
	$[\omega I_4, \omega^{-1} I_4]$	12	8	2/3
	$[\omega I_2, \omega^{-2} I_2, \omega^2, \omega^{-2}]$	10	6	1/3
5	$[\omega I_5, \omega^{-1} I_5]$	20	12	3/5
	$[\omega I_3, \omega^{-1} I_3, \omega^2, \omega^{-2}]$	18	10	1/3
	$[I_2, \omega I_3, \omega^{-1} I_3]$	18	10	1/3
	$[\omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}]$	12	6	0

Table 15.1.2. Elements $x \in H$ of prime order $r \neq p$ with $\nu(x) \in \{3, 4, 5\}$

Now, we deal with the general case. Recall that if $\nu(x) \geq a$ then $n > a$.

Lemma 15.1.7. *Let $x \in H$ be of order r. Assume $\nu(x) \geq 6$. Then*

$$f_\Omega(x) \leq 1 - \frac{4}{n} + \frac{8\delta_{r,p}}{n(n-2)}$$

with equality if, and only if, $n = 8$ and $x \in [J_4^2]^G$ or $C_G(x) \cong (\mathrm{GL}_2)^2$.

Proof. By Proposition 5.4.1 we have $\dim x^G \geq \frac{5}{2}n$ for $n \geq 12$, and $\dim x^G \geq 6(n-7)$ if $n < 12$. Moreover Theorem 7.2.5 implies $\dim(x^G \cap H) \leq \left(\frac{1}{2} + \frac{1}{n-2}\right) \dim x^G$.

Therefore, for $n \geq 12$,

$$f_\Omega(x) \leq 1 - \frac{\frac{1}{2}-\epsilon}{\dim \Omega}\dim x^G \leq 1 - \frac{\frac{1}{2}-\frac{1}{n-2}}{\dim \Omega}\Big(\frac{5}{2}n\Big) = 1 - \frac{5}{n-2} + \frac{10}{(n-2)^2}$$

Using *Mathematica* we compute, for $n \geq 12$

$$1 - \frac{4}{n} - \Big(1 - \frac{5}{n-2} + \frac{10}{(n-2)^2}\Big) = \frac{n^2-4n-16}{n(n-2)^2} > 0$$

If $n < 12$ we have $f_\Omega(x) \leq 1 - \frac{12(n-7)}{n(n-2)} + \frac{6(n-7)}{n(n-2)^2}$. Again, using *Mathematica*, we have, for $n \geq 10$,

$$1 - \frac{4}{n} - \Big(1 - \frac{12(n-7)}{n(n-2)} + \frac{6(n-7)}{n(n-2)^2}\Big) = \frac{2(4n^2-49n+97)}{n(n-2)^2} > 0$$

In the case $n = 8$, the only unipotent elements in H (up to G-conjugacy) are $[J_4^2], [J_3^2, J_1^2], [J_2^4], [J_2^2, J_1^4]$. The only x with $\nu(x) \geq 6$ is $[J_4^2]$. We compute

$$f_\Omega([J_4^2]) = \frac{1}{3} = 1 - \frac{4}{n} + \frac{8}{n(n-2)}$$

Similarly, up to centraliser structure, the elements $x \in H$ with $\nu(x) = 6, 7$ are

$$x_1 = [I_2, \omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}],\ x_2 = [\omega I_2, \omega^{-1} I_2, \omega^2 I_2, \omega^{-2} I_2]$$
$$x_3 = [\omega I_2, \omega^{-1} I_2, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}],\ x_4 = [\omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \omega^{-3}, \omega^4, \omega^{-4}]$$

We compute $f_\Omega(x_1) = f_\Omega(x_4) = 0$, $f_\Omega(x_2) = 1/3$ and $f_\Omega(x_3) = 1/6$. *q.e.d.*

Remark 15.1.8. Lemmas 15.1.4, 15.1.5, 15.1.7, hold in characteristic zero for unipotent elements. Indeed, in this case any unipotent element has infinite order and $x^G \cap H = x^G \cap H^\circ$, see Lemma B.3.1. Moreover the analysis done makes no assumption on the order of the elements in the unipotent case.

This completes the proof of Proposition 15.1.1. Once we have derived upper bounds for involutions, thanks to Corollary 7.1.11, we deduce upper bounds for any $x \in G \setminus Z(G)$.

15.2. Unipotent elements: lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H$ of order $p \neq 2$. For convenience we study involutions in Section 13.2.3. Recall that $x^G \cap H = x^G \cap H^\circ$. Moreover, for $x \in H^\circ$, $x = [x_1, x_1^{-t}]$, by Proposition 13.2.2. Up to G-conjugacy we may write $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$, so $x_1 = [J_p^{a_p/2}, \ldots, J_1^{a_1/2}]$.

Proposition 15.2.1. *Let $x \in H$ be of order $p \neq 2$.*

(i) If $p \leq n/2$ then $f_\Omega(x) \geq 1/p$. Moreover, equality holds if, and only if, $n = ap$ and $x \in [J_p^a]^G$.
(ii) If $p > n/2$ then $f_\Omega(x) \geq \frac{2}{n-2\delta_{n;4}}$.

We shall prove Proposition 15.2.1 in a sequence of lemmas. First, for $x \in H$ of order p, we give an explicit formula for $f_\Omega(x)$, that quickly follows from Theorem 5.2.1 and Propositions 13.2.2 and 7.1.8.

Proposition 15.2.2. *Let $x \in G$ be of odd order p. Assume $x^G \cap H \neq \emptyset$. Then $x \in [J_p^{a_p}, \ldots, J_1^{a_1}]^G$ with a_i even for all i and*

$$f_\Omega(x) = \frac{2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 - 2\sum_{i \text{ odd}} a_i}{n(n-2)} \tag{128}$$

Lemma 15.2.3. *Let $x \in H$ be of odd order p. Assume $p \leq n/2$. Then $f_\Omega(x) \geq 1/p$.*

Proof. Let $\alpha(x) = n(n-2)\big(f_\Omega(x) - \frac{1}{p}\big)$. We need to show $\alpha(x) \geq 0$, which leads to the result. Using Proposition 15.2.2 and $n = \sum_i i a_i$, we get

$$\begin{aligned}
\alpha(a_i) &= 2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 - \sum_{i \text{ odd}} a_i - \frac{n^2 - 2n}{p} \\
&= 2\sum_{i<j} i a_i a_j \Big(1 - \frac{j}{p}\Big) + \sum_{i \text{ even}} i a_i \Big(a_i\Big(1 - \frac{i}{p}\Big) + \frac{2a_i}{p}\Big) + \sum_{i \text{ odd}} a_i \Big(i a_i \Big(1 - \frac{i}{p}\Big) + \frac{2i}{p} - 2\Big)
\end{aligned}$$

The first two summands are clearly non-negative. Assume i is odd and $a_i > 0$. Then $i a_i\big(1 - \frac{i}{p}\big) + \frac{2i}{p} - 2 = \frac{1}{p}(i a_i - 2)(p - i) \geq 0$. Therefore, the last summand is also non-negative. The result follows. *q.e.d.*

In the case $p < n/2$ we can characterise elements which realise the bound.

Lemma 15.2.4. *Let $x \in G$ be of odd order p. Assume $p \leq n/2$. Then $f_\Omega(x) = 1/p$ if, and only if, $n = ap$ and $x \in [J_p^a]^G$.*

Proof. Assume $n = ap$ and $x = [J_p^a]$ (note that a is even since n is even). Then, using Proposition 15.2.2 it is straightforward to compute $f_\Omega(x) = 1/p$.

Now assume $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in H$ and $f_\Omega(x) = 1/p$. Then, by the proof of Lemma 15.2.3, we have:

(i) $i a_i a_j = 0$ for all $i < j$,
(ii) $i a_i\big(a_i\big(1 - \frac{i}{p}\big) + 2\frac{a_i}{p}\big) = 0$ for all i even,
(iii) $a_i\big(i a_i\big(1 - \frac{i}{p}\big) + 2\frac{i}{p} - 2\big) = 0$ for all i odd.

From (ii), we deduce that $a_i = 0$ for all i even. Similarly, from (iii) we have $a_i = 0$ for all $1 < i < p$ odd. Assume $a_1, a_p \neq 0$. Then $a_1 a_p \neq 0$, this is a contradiction by (i). Therefore either $a_1 \neq 0$ or $a_p \neq 0$. Since $x \neq I_n$, we deduce $a_p \neq 0$. The result follows. *q.e.d.*

Lemma 15.2.4 provides all the informations for the case $p \mid n$. In the case $p \nmid n$, the next result shows that the bound $1/p$ is close to best possible.

Proposition 15.2.5. *Assume $p < n/2$ and $p \nmid n$. Then there exists $x \in H$ of order p such that*

$$f_\Omega(x) \leq \frac{1}{p} + \frac{1}{n}$$

Proof. Write $n/2 = ap + b$ with $0 < b < p$. Let $x = [J_p^{2a}, J_b^2]$. Then, using Proposition 15.2.2 and the aid of *Mathematica*, we compute

$$f_\Omega(x) = \frac{1}{p} + \frac{4b(p-b+1)}{pn(n+2)} - \frac{4(1-\delta_{b;2})}{n(n+2)} \leq \frac{1}{p} + \frac{(p+1)^2}{pn(n+2)}$$

where the inequality follows from $4b(p-b+1) \leq (p+1)^2$. Since $p \leq n/2 - 1$ we have $\frac{(p+1)^2}{p} \leq \frac{n^2}{2(n-2)}$, so

$$f_\Omega(x) \leq \frac{1}{p} + \frac{n^2}{2n(n-2)(n+2)} \leq \frac{1}{p} + \frac{1}{n}$$

where the second inequality follows from the assumption $n \geq 6$. *q.e.d.*

Now we assume $p > n/2$.

Remark 15.2.6. This analysis also includes the characteristic zero case. In fact if x is a unipotent element then x has infinite order and $x^G \cap H = x^G \cap H^\circ$, see Remark 15.1.8. In particular, the largest Jordan block that may appear in x is $J_{n/2}$.

Lemma 15.2.7. *Assume $p > n/2$. Let $x \in H$ be of order p. Then*

$$f_\Omega(x) \geq \frac{2}{n - 2\delta_{n;4}}$$

Proof. First, assume $n/2$ even, i.e. $\delta_{n;4} = 1$. Using Proposition 15.2.2, we have $f_\Omega(x) \geq \frac{2}{n-2}$ if, and only if,

$$2\sum_{i<j} ia_ia_j + \sum_i ia_i^2 - 2\sum_{i \text{ odd}} a_i \geq 2n = 2\sum_i ia_i$$

which is equivalent to

$$2\sum_{i<j} ia_ia_j + \sum_{i \text{ even}} ia_i(a_i-2) + \sum_{i \text{ odd}} a_i\big(i(a_i-2)-2\big) \geq 0$$

The first summand is clearly non-negative. Since a_i is even we have that $ia_i(a_i-2) \geq 0$ for all i even. Also, for i odd, unless $a_i = 2$, we have $a_i(i(a_i-2)-2) \geq 0$. Assume there exists i odd such that $a_i = 2$. Say $i_1 \leq i_2 \leq \ldots \leq i_\ell$ all the odd multiplicities for which $a_{i_j} = 2$. If $\ell = 1$, we have two cases.

Case 1. There exists $j \neq i$ such that $a_j \neq 0$. If $i < j$, we have $2ia_ia_j + a_i(i(a_i-2)-2) = 4ia_j - 4 > 0$. If, instead, $j < i$ then $2ja_ia_j + a_i(i(a_i-2)-2) = 4ja_j - 4 > 0$.

Case 2. For all $j \neq i$ we have $a_j = 0$. Since $\sum_i ia_i = n$ we have $i = n/2$ so $x = [J_{n/2}^2]$ and $f_\Omega(x) = \frac{2}{n-2}$.

In the case $\ell > 1$ we have

$$2i_1a_{i_1}a_{i_2} + \cdots + 2i_1a_{i_1}a_{i_{\ell-1}} - 2a_{i_1} - 2a_{i_\ell} \geq 0,$$
$$2i_2a_{i_2}a_{i_3} + \cdots + 2i_2a_{i_2}a_{i_\ell} - 2a_{i_2} > 0,\ \ldots, 2i_{\ell-1}a_{i_{\ell-1l}}a_{i_\ell} - 2a_{i_{\ell-1}} > 0.$$

We conclude that $f_\Omega(x) \geq \frac{2}{n-2}$ if $n/2$ is even.

Now assume $n/2$ is odd, so $\delta_{n;4} = 0$. Again, we have $f_\Omega(x) \geq 2/n$ if, and only if,

$$2\sum_{i<j} ia_ia_j + \sum_i ia_i^2 - 2\sum_{i \text{ odd}} a_i \geq 2(n-2) = 2\sum_i ia_i - 4$$

which is equivalent to

$$2\sum_{i<j} ia_ia_j + \sum_{i \text{ even}} ia_i(a_i-2) + \sum_{i \text{ odd}} a_i\big(i(a_i-2)-2\big) + 4 \geq 0$$

With the same argument as above, we can prove that this inequality is always satisfied. The result follows. *q.e.d.*

In the next result we show that the bounds are sharp.

Proposition 15.2.8. *Assume $p > n/2$. Then there exists $x \in H$ of order p such that $f_\Omega(x) = \frac{2}{n-2\delta_{n;4}}$.*

Proof. Let $x = [J_{n/2}^2]$ and $y = [J_{n/2-1}^2, J_1^2]$. Applying Proposition 15.2.2 we get

$$f_\Omega(x) = f_\Omega(y) = \frac{2}{n-2\delta_{n;4}}$$

q.e.d.

The proof of Lemma 15.2.7 is very similar to that of Lemma 15.2.4. However, the major difference between the cases $p \leq n/2$ and $p > n/2$ is that in the latter case we do not prove that each summand, in the expression of $f_\Omega(x)$, is non-negative. Hence, for the purpose of getting a characterisation when $p > n/2$ it is not possible to argue as in Lemma 15.2.4. We conjecture that the only elements that realise the bounds are those given in the proof of Proposition 15.2.8.

Conjecture 15.2.9. *Let $x \in G$ be of order $p > n/2$. Then $f_\Omega(x) = \frac{2}{n-2\delta_{n;4}}$ if, and only if, $x \in [J_{n/2}^2]^G$ or $[J_{n/2-1}^2, J_1^2]^G$.*

15.3. Semisimple elements: lower bounds

First notice that H contains a maximal torus of G. Hence, if $x \in G$ is semisimple then $x^G \cap H \neq \emptyset$. Hence it is enough to derive lower bounds on $f_\Omega(x)$ for $x \in H$ semisimple. For convenience, we postpone the analysis of involutions to Section 15.7. In the following r denotes an odd prime other than p. Up to G-conjugacy, we write

$$x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \tag{129}$$

where a_0 is even and for all $1 \leq i \leq \frac{r-1}{2}$ we have $a_i = a_{r-i}$. The main result of the section is the following.

Proposition 15.3.1. *Let $x \in H$ be of order r.*

(i) If $r > n$ then $f_\Omega(x) \geq 0$, with equality if, and only if, $\nu(x) = n-1$ or $C_G(x) \cong \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}$.
(ii) If $r < n$ then $f_\Omega(x) \geq \frac{1}{r} - \frac{r^2-1}{rn(n-2)}$.

Remark 15.3.2. The bound given in the case $r < n$ is sharp. In fact if $n = \lfloor n/r \rfloor r + 1$ then there exists $z \in H$ that realises the bound, see Remark 15.3.14. We prove the bound in Proposition 15.3.5. Then we construct a family of special elements, see Definition 15.3.9, and we prove that special elements realise the best possible lower bound. The best possible lower bounds are given in Table 15.3.1. Notice that $\frac{1}{r} - \frac{r^2-1}{rn(n-2)} \geq \frac{1}{r} - \frac{1}{n}$.

Recall that $d(x) = \{i \in \{1, \dots, r-1\} : a_i \text{ odd}\}$. In Proposition 13.2.7 we gave an explicit formula for $\dim(x^G \cap H)$. By Theorem 5.3.1 we have $\dim x^G = \frac{1}{2}\big(n(n-1) + a_0 - \sum_{i=0}^{r-1} a_i^2\big)$. Using Proposition 7.1.8, we get the following.

Proposition 15.3.3. *Let $x \in G$ be of order r. Then*

$$f_\Omega(x) = \frac{-2a_0 + \sum_{i=0}^{r-1} a_i^2 - d(x)}{n(n-2)}$$

Remark 15.3.4. We have $\dim C_\Omega(x) = \frac{1}{4}\big(-2a_0 + \sum_i a_i^2 - d(x)\big)$. As in Remark 14.3.4 it is easily verified that $\dim C_\Omega(x)$ is an even integer.

We use Proposition 15.3.3 to derive a lower bound on $f_\Omega(x)$. The analysis is similar to that done in Section 14.3, when $G = \mathrm{Sp}_n$. The following shows Proposition 15.3.1(ii).

Proposition 15.3.5. *Let $x \in H$ be of order $r < n$. Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{r^2-1}{rn(n-2)}$$

Proof. Since $d(x) \leq r-1$, Proposition 15.3.3 implies that

$$f_\Omega(x) = \frac{-2a_0 + a_0^2 + \sum_{i>0} a_i^2 - d(x)}{n(n-2)} \geq \frac{-2a_0 + a_0^2 + \frac{(n-a_0)^2}{r-1} - (r-1)}{n(n-2)}$$
$$\geq \frac{\frac{n^2-2n-r+1}{r} - r + 1}{n(n-2)} = \frac{1}{r} - \frac{r^2-1}{rn(n-2)}$$

where the first inequality follows from Proposition B.2.1, while the second inequality follows from the fact that $-2a_0 + a_0^2 + \frac{(n-a_0)^2}{r-1}$, as a function in the variable a_0, is minimal when $a_0 = \frac{n+r-1}{r}$. *q.e.d.*

Remark 15.3.6. As observed in Remark 14.3.6, in the case $G = \mathrm{Sp}_n$, also here we notice that the lower bound $\frac{1}{r} - \frac{r^2-1}{rn(n-2)}$ is monotonically decreasing in r, say $g(r)$. In particular, $g(n-1) = 0$ while $g(n) = -\frac{2n-1}{n^2(n-2)} < 0$.

In fact, for $r > n$ we can classify the semisimple elements $x \in H$ of order r such that $f_\Omega(x) = 0$. With the following we complete the proof of Proposition 15.3.1.

Lemma 15.3.7. *Let $x \in H$ be of order $r > n$. Then $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$ or $C_G(x) \cong \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}$.*

Proof. First, assume $\nu(x) = n-1$. Up to centraliser structure, we may write $x = [\omega, \omega^{-1}, \ldots, \omega^{n/2}, \omega^{-n/2}]$. Then, using Proposition 15.3.3 we get $f_\Omega(x) = 0$. In the same way we see that for $x = [I_2, \omega, \omega^{-1}, \ldots, \omega^{n/2-1}, \omega^{-n/2+1}]$ we get $f_\Omega(x) = 0$.

Now suppose $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H$ and $f_\Omega(x) = 0$. We assume $\nu(x) < n-1$. We consider three cases.

Case 1. Assume $a_0 = 0$. Then, by Proposition 15.3.3, $f_\Omega(x) = \frac{\sum_{i>0} a_i^2 - d(x)}{n(n-2)}$. Observe that $d(x) \leq n$. Assume there exists i such that $a_i \geq 2$, without loss of generality say $i = 1$. Then $\sum_i a_i^2 \geq 2^2 + 2^2 + \sum_{i=2}^{r-2} a_i^2 \geq 8 + \frac{1}{r-3}(n-2a_1)^2$, the last inequality follows from Proposition B.2.1. Therefore $f_\Omega(x) > 0$.

Case 2. Assume $a_0 = 2$. Here $f_\Omega(x) = \frac{\sum_{i\geq 0} a_i^2 - d(x)}{n(n-2)}$. Assume there exists $i > 0$ such that $a_i \geq 2$, say $i = 1$. Notice $d(x) \leq n-2$ and $\sum_{i\geq 1} a_i^2 \geq 8 + \sum_{i=2}^{r-2} a_i^2 \geq 8 + \frac{1}{r-3}(n-2-2a_1)^2$. Therefore $f_\Omega(x) > 0$. So, if $f_\Omega(x) = 0$ and $a_0 = 2$, we deduce $x = [I_2, \omega, \omega^{-1}, \ldots, \omega^{n/2-1}, \omega^{-n/2+1}]$.

Case 3. Assume $a_0 \geq 4$. So $f_\Omega(x) = \frac{-2a_0 + a_0^2 + \sum_{i\geq 1} a_i^2 - d(x)}{n(n-2)}$. Using $\sum_i a_i^2 \geq n - a_0$ and $d(x) \leq n - a_0$ we deduce $f_\Omega(x) \geq \frac{-2a_0 + a_0^2 + n - a - (n-a_0)}{n(n-2)} = \frac{a_0^2 - 2a_0}{n(n-2)}$. We see that $a_0^2 - 2a_0$ is minimal when a_0 is minimal, i.e. $a_0 = 4$. Thus $f_\Omega(x) \geq \frac{8}{n(n-2)} > 0$. *q.e.d.*

Remark 15.3.8. Recall that Propositions 14.3.8 and 14.3.9 give a geometric proof of the fact that $C_\Omega(x)$ is finite if, and only if, $\nu(x) = n-1$ for $G = \mathrm{Sp}_n$. Those results, with minor adjustments, hold in this case as well.

We, now, proceed as in Section 14.3, in order to prove that the lower bound in Proposition 15.3.1(ii) is close to best possible.

Definition 15.3.9. Let $x \in G$ be of order r. We say that x is *special* if $|a_i - a_j| \leq 1$ for all $i, j = 0, \dots, r-1$.

Observe that, if $r > n$ then x is special if, and only if, $\nu(x) = n-1$. Also, when $r < n$, if x is special then $a_i \neq 0$ for all $i \geq 0$.

Lemma 15.3.10. *Let $x \in G$ be of order r. Then either x is special or there exists $y = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H$ such that $f_\Omega(x) = f_\Omega(y)$ and one of the following holds*

(i) $|b_0 - b_1| \geq 2$;
(ii) $b_1 - b_2 \geq 2$.

Proof. The same argument of Lemma 14.3.11 applies. *q.e.d.*

If x is not special we have $a_i - a_j \geq 2$ for some i, j. In view of Lemma 15.3.10 we may assume $(i,j) \in \{(0,1),(1,0),(1,2)\}$. For these couples we define y as in (92), (93) and (94), respectively, see Section 14.3.

As for the symplectic case we have the following. Notice that the proof is similar to that of Lemma 14.3.12, hence we omit it.

Lemma 15.3.11. *Assume $r < n$. Let $x \in H$ be of order r. Assume x is not special. Then*

$$f_\Omega(x) \geq f_\Omega(y)$$

for y defined in (92), (93) and (94).

Iterating Lemma 15.3.11 we have the following, whose proof is totally equal to that of Lemma 14.3.13.

Lemma 15.3.12. *Let $x \in H$ be of order r. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

where z is a special element.

As for the symplectic case (Section 14.3), we see that any special element is given by z, as in (95). For the convenience of the reader we recall the structure of special elements also here. Let $n = ar + b$, where $0 \leq b < r$ and $a = \lfloor n/r \rfloor$. Notice that $\frac{n-r+1}{r} \leq a \leq \frac{n}{r}$. Then every special element has G-centraliser isomorphic to $C_G(z)$, where

$$z = \begin{cases} [I_a, A] & a \text{ even} \\ [I_{a+1}, A] & a \text{ odd} \end{cases} \tag{130}$$

for $A = [(\omega,\omega^{-1})I_{a+1}, \dots, (\omega,\omega^{-1})^{\lfloor b/2 \rfloor} I_{a+1}, (\omega,\omega^{-1})^{\lfloor b/2 \rfloor + 1} I_a, \dots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_a]$.

The following is the analogue of Remark 14.3.14.

Remark 15.3.13. Let $x, y \in H$ be of prime order $r < n$. Assume both x and y are special. Then $f_\Omega(x) = f_\Omega(y)$.

If a is even, we see that $d(z) = b$ and, using Proposition 15.3.3, we compute

$$f_\Omega(z) = \frac{(n-b)(n-2+b)}{rn(n-2)} = \frac{1}{r} - \frac{b(b-2)}{rn(n-2)} \tag{131}$$

As a function of b, $f_\Omega(z)$ is minimal when $b = r-1$, and maximal when $b = 1$. Thus

$$\frac{1}{r} - \frac{(r-1)(r-3)}{rn(n-2)} \leq f_\Omega(z) \leq \frac{1}{r} + \frac{1}{r(n-2)} \tag{132}$$

If a is odd then b is odd and we have

$$f_\Omega(z) = \frac{(n+r-b)(n-2-r+b)}{rn(n-2)} = \frac{1}{r} - \frac{(r-b)(r-b+2)}{rn(n-2)} \tag{133}$$

As a function of b, $f_\Omega(z)$ is minimal for $b = 1$ and maximal for $b = r$, so

$$\frac{1}{r} - \frac{r^2-1}{rn(n-2)} \leq f_\Omega(z) \leq \frac{1}{r} \tag{134}$$

Remark 15.3.14. From the previous computation we see that if $\lfloor n/r \rfloor$ is odd and $n = \lfloor n/r \rfloor r + 1$ then, for $z \in H$ special, $f_\Omega(z) = \frac{1}{r} - \frac{r^2-1}{rn(n-2)}$, which is the lower bound stated in Proposition 15.3.1. Thus such a bound is the best possible.

The optimal lower bounds on $f_\Omega(x)$ have been computed in (131) and (133). However, good approximations of these bounds are given in (132) and (134). We record in Table 15.3.1 the best possible lower bound on $f_\Omega(x)$, for $x \in H$ of order r. We denote $b = n - \lfloor n/r \rfloor r$.

$\lfloor n/r \rfloor$	$f_\Omega(x) \geq$
even	$\frac{1}{r} - \frac{b(b-2)}{rn(n-2)}$
odd	$\frac{1}{r} - \frac{(r-b)(r-b+2)}{rn(n-2)}$

Table 15.3.1. Lower bounds on $f_\Omega(x)$ for $x \in H$ of odd prime order $r \neq p$

15.4. Local upper bounds

In this section we establish upper bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$, where r is an odd prime, see Section 15.7 for involutions. For convenience of the reader we recall that

$$\mathcal{V}_s = \{x \in G : \nu(x) = s\}$$

Also, $\mathcal{V}_{s,r}$ denotes the set of elements of $\mathcal{V}_s$ of order r.

The main result of this section is Proposition 15.4.1, below. Note that, studying separately unipotent and semisimple elements, we shall derive better upper bounds, see Proposition 15.4.3 and Proposition 15.4.14.

Proposition 15.4.1. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n-2} + \frac{2}{n}$$

Remark 15.4.2. Also in this case, as in Remark 14.4.2, using 7.2.5 and Proposition 5.4.1, we deduce the following less accurate bounds. Let $x \in H \cap \mathcal{V}_{s,r}$. Then

$$f_\Omega(x) \leq 1 - \frac{\dim x^G\left(\frac{1}{2} - \frac{1}{n-2}\right)}{\dim \Omega} \leq 1 - \frac{4\left(\frac{1}{2} - \frac{1}{n-2}\right)\max\{s(n-s-1), \frac{n}{2}(s-1)\}}{n(n-2)}$$

In what follows we shall compute bounds, which we prove to be close to best possible, using different methods.

Recall Proposition 14.4.3 for bounds on $\nu(x)$ depending on the prime order r of x. We start with local upper bounds for unipotent elements.

15.4.1. Unipotent elements. The next result is the analogue of Proposition 14.4.4.

Proposition 15.4.3. *Let $x \in H \cap \mathcal{V}_{s,p}$. Write $n = a(n-s) + b$, where $0 \leq b < n-s$. Then*

$$f_\Omega(x) \leq 1 - \frac{(n-b)(s+b)}{n(n-2)} + \frac{2(n-b)(s+b)}{n(n-2)^2}$$

Proof. Up to conjugation we write $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in H \cap \mathcal{V}_{s,r}$. Let $y = [J_p^{a_p/2}, \ldots, J_1^{a_1/2}] \in \mathrm{GL}_{n/2}$. Using Theorem 5.2.1, we compute

$$\dim x^G = 2\dim y^{\mathrm{GL}_{n/2}} - \frac{n - \sum_{i \text{ odd}} a_i}{2} \geq 2\dim y^{\mathrm{GL}_{n/2}} \tag{135}$$

Then, by Proposition 5.2.3, Lemma B.1.1 and Example B.1.2 we have, for all $y \in \mathrm{GL}_{n/2} \cap \mathcal{V}_{s,p}$

$$\dim y^{\mathrm{GL}_{n/2}} \geq \dim[J_{a+1}^{\frac{b}{2}}, J_a^{\frac{n-s-b}{2}}]^{\mathrm{GL}_{n/2}} = \frac{(n-b)(s+b)}{4}$$

Therefore, using Theorem 7.2.5, we have

$$\begin{aligned} f_\Omega(x) &\leq 1 - \frac{\left(\frac{1}{2} - \frac{1}{n-2}\right)\dim x^G}{\dim \Omega} \leq 1 - \frac{2\left(\frac{1}{2} - \frac{1}{n-2}\right)\dim y^{\mathrm{GL}_{n/2}}}{\dim \Omega} \\ &\leq 1 - \frac{(n-b)(s+b)}{n(n-2)} + \frac{2(n-b)(s+b)}{n(n-2)^2} \end{aligned}$$

q.e.d.

Remark 15.4.4. We see that $(n-b)(s+b)$ is maximal for $b = \frac{n-s}{2}$. In particular, $\frac{(n-b)(s+b)}{n(n-2)^2} \leq \frac{(n+s)^2}{4n(n-2)^2}$. Recall that $s \leq n-2$ is even. Hence $\frac{(n-b)(s+b)}{n(n-2)^2} \leq \frac{(n-1)^2}{n(n-2)^2} \leq \frac{2}{n}$, where the last inequality holds for $n \geq 6$.

With the same argument of Corollary 14.4.5, we deduce the following upper bound.

Corollary 15.4.5. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n-2} + \frac{2}{n}$$

In the next result we show that the upper bound given in Proposition 15.4.3 is close to the best possible. The following is the analogue of Proposition 14.4.6.

Proposition 15.4.6. *There exists $x \in H \cap \mathcal{V}_{s,p}$ such that*

$$f_\Omega(x) \geq 1 - \frac{(n-b)(s+b)}{n(n-2)} + \frac{2(n-b)(s+b)}{n(n-2)^2} - \frac{3}{n}$$

PROOF. Write $n = a(n-s) + b$, where $0 \le b < n-s$ is even. We consider $x = [J_{a+1}^b, J_a^{n-s-b}] \in H$. It is straightforward to check that $a+1 = \lfloor \frac{n}{n-s} \rfloor + 1 \le p$ hence $x \in \mathcal{V}_{s,p}$. Say U the bound of Proposition 15.4.3, by that proof we know

$$U = 1 - \frac{2\left(\frac{1}{2} - \frac{1}{n-2}\right)\dim x^{H^\circ}}{\dim \Omega}$$

In particular,

$$f_\Omega(x) \ge U - \frac{\dim x^G}{\dim \Omega} + \frac{\dim x^{H^\circ}}{\dim \Omega} - \frac{2\dim x^{H^\circ}}{(n-2)\dim \Omega} > U - \frac{3}{n} - \frac{\dim x^G - 2\dim x^{H^\circ}}{\dim \Omega}$$

We see that $\dim x^G - 2\dim x^{H^\circ} = \frac{1}{2}\left(\sum_{i \text{ odd}} a_i - n\right) < -\frac{s}{2}$, since $\sum_{i \text{ odd}} a_i = b$ if a is even or $n-s-b$ if a is odd. Therefore

$$f_\Omega(x) > U - \frac{3}{n} + \frac{s/2}{\dim \Omega} = U - \frac{3}{n} + \frac{2s}{n(n-2)} > U - \frac{3}{n}$$

q.e.d.

Remark 15.4.7. The bound deduced in Corollary 15.4.5 is close to best possible when b is either very small or very large, by the observation made in Remark 15.4.4. For example, if $b = 0$ then $n = (n-s)a$ and $f_\Omega([J_a^{n-s}]) = 1 - \frac{s}{n-2} + \frac{2s}{(n-2)^2} \approx 1 - \frac{s}{n-2}$.

15.4.2. Semisimple elements. Now assume $x \in H$ is a semisimple element of prime order $r > 2$ with $\nu(x) = s$. Then, up to the centraliser structure, x is conjugate to one of the following

$$(136) \qquad \begin{array}{llr} (a) & [I_{n-s}, (\omega,\omega^{-1})I_{a_1}, \dots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] & s \text{ even} \\ (b) & [I_{a_0}, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_{a_2}, \dots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] & s \ge \frac{n}{2} \end{array}$$

where $a_i \le n-s$, for all i.

Similarly to Section 14.4.2 also here we shall construct elements $z \in H \cap \mathcal{V}_{s,r}$ and we show that $f_\Omega(z)$ is maximal.

Assume s even if $s < n/2$. Let $n = (n-s)l + m$ with $0 \le m < n-s$. We define

$$(137) \qquad z_1 = \begin{cases} [I_{n-s}, A, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{\frac{m}{2}}] & s \text{ even} \quad l \text{ odd} \\ [I_{n-s}, A, (\omega,\omega^{-1})^{\frac{l}{2}} I_{\frac{n-s+m}{2}}] & s \text{ even} \quad l \text{ even} \\ [I_{n-s-1}, A, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{\frac{m+1}{2}}] & s \text{ odd} \quad l \text{ odd} \\ [I_{n-s-1}, A, (\omega,\omega^{-1})^{\frac{l}{2}} I_{\frac{n-s+m+1}{2}}] & s \text{ odd} \quad l \text{ even} \end{cases}$$

where $A = [(\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\lfloor \frac{l-1}{2} \rfloor} I_{n-s}]$. And

$$(138) \qquad z_2 = \begin{cases} [B, (\omega,\omega^{-1})^{\frac{l}{2}+1} I_{\frac{m}{2}}] & l \text{ even} \\ [B, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{\frac{n-s+m}{2}}] & l \text{ odd} \end{cases}$$

where $B = [(\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\lfloor \frac{l}{2} \rfloor} I_{n-s}]$.

So $z_1, z_2 \in H \cap \mathcal{V}_{s,r}$. It is straightforward, using the upper bound on $\nu(x)$ given in Proposition 14.4.3, to check that $\frac{l+1}{2}, \frac{l}{2}, \frac{l}{2} - 1 \le \frac{r-1}{2}$.

Claim. Let $y \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega(y) \le \max\{f_\Omega(z_1), f_\Omega(z_2)\}$.

We will prove this claim in a sequence of lemmas (see Lemma 15.4.9–15.4.12), with final conclusion established in Proposition 15.4.13.

Lemma 15.4.8. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then either $C_G(x) \cong C_G(z_1)$ or $C_G(z_2)$, or there exists $y = [I_{b_0}, \omega I_{b_1}, \ldots, \omega^{r-1} I_{b_{r-1}}] \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega(x) = f_\Omega(y)$ and one of the following holds*

(i) $b_0 = n - s,\ b_1 = \min_{i>0}\{b_i : b_i \neq 0\}$ *and* $b_2 = \max_{i>0}\{b_i : b_i < n - s\}$;
(ii) $b_1 = n - s,\ b_2 = \min_{i\geq 0}\{b_i : b_i \neq 0\}$ *and* $b_0 = \max_{i\geq 0}\{b_i : b_i < n - s\} < n - s - 1$;
(iii) $b_1 = n-s,\ b_2 = \min_{i\geq 0}\{b_i : b_i \neq 0\},\ b_0 = n-s-1$ *and* $b_3 = \max_{i>0}\{b_i : b_i < n-s\}$;
(iv) $b_2 = n - s,\ b_0 = \min_{i\geq 0}\{b_i : b_i \neq 0\}$ *and* $b_1 = \max_{i\geq 0}\{b_i : b_i < n - s\}$;
(v) $b_1 = n - s, b_2 = \min_i\{b_i : b_i \neq 0\}$ *and* $b_3 = \max_i\{b_i : b_i < n - s\}$.

Proof. If the centraliser in G of x is not isomorphic to that of z_1 or z_2 then it is enough to relabel the eigenvalues of x. *q.e.d.*

As in Section 14.4.2, for every possible case arising from Lemma 15.4.8 we define a suitable element $y \in H \cap \mathcal{V}_{s,r}$ for which we will prove $f_\Omega(x) \leq f_\Omega(y)$. Here we shall omit the proofs as the arguments comprise easy computations similar to those of Lemmas 14.4.9, 14.4.11 and 14.4.12.

Let $x \in H \cap \mathcal{V}_{s,r}$ as in (136) assume that $a_0 = n - s$. In view of Lemma 15.4.8, we may assume $a_1 = \min\{a_i : a_i \neq 0\}$ and $a_2 = \max_i\{a_i : a_i < n - s\}$. Then, we define

$$y = [I_{n-s}, (\omega, \omega^{-1}) I_{a_1-1}, (\omega, \omega^{-1})^2 I_{a_2+1}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{139}$$

Observe that if $C_G(y)$ is not isomorphic to $C_G(z_1)$ or $C_G(z_2)$ then $a_2 - a_1 > 1$.

The following is the analogue of Lemma 14.4.9.

Lemma 15.4.9. *Let $x \in H \cap \mathcal{V}_{s,r}$ with $a_0 = n - s$, $a_1 = \min\{a_i : a_i \neq 0\}$ and $a_2 = \max\{a_i : a_i < n - s\}$. Assume $a_1 < a_2$ and define y as in (139). Then*

$$f_\Omega(x) \leq f_\Omega(y)$$

Remark 15.4.10. Notice that the case (v) in Lemma 15.4.8 is very similar to (i). In fact in case (v) we would define

$$y = [I_{a_0}, (\omega, \omega^{-1}) I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3+1}, \ldots] \in H \cap \mathcal{V}_{s,r} \tag{140}$$

and the proof that $f_\Omega(x) \leq f_\Omega(y)$ is entirely similar to Lemma 15.4.9

Let $x \in H \cap \mathcal{V}_{s,r}$ as in (136). Assume $a_i = n - s$ for some $i \neq 0$, so that $s \geq n/2$. Then, by Lemma 15.4.8 we may assume $a_0 = \max_i\{a_i : a_i < n - s\}$, $a_1 = n - s$ and $a_2 = \min_i\{a_i : a_i \neq 0\}$. Also here we remark that if $C_G(x)$ is not isomorphic to $C_G(z_1)$ or $C_G(z_2)$ then $a_0 > a_2$.

If $a_0 < n - s - 1$ then we define

$$y = [I_{a_0+2}, (\omega, \omega^{-1}) I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{141}$$

If, instead, $a_0 \geq n - s - 1$ we may assume $a_3 = \max_{i>0}\{a_i : a_i < n - s\}$ and we define

$$y = [I_{a_0}, (\omega, \omega^{-1}) I_{n-s}, (\omega, \omega^{-1})^2 I_{a_2-1}, (\omega, \omega^{-1})^3 I_{a_3+1}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{142}$$

The following is the analogue of Lemma 14.4.11.

Lemma 15.4.11. *Let* $x \in H \cap \mathcal{V}_{s,r}$ *with* $a_1 = n - s, a_2 = \min_i\{a_i : a_i \neq 0\}$ *and* $a_0 = \max_i\{a_i : a_i < n - s\}$. *Assume* $a_0 > a_2$ *and define* y *as in (141) or (142) depending on the above conditions on* a_0. *Then*

$$f_\Omega(x) \leq f_\Omega(y)$$

Now assume $a_0 = \min\{a_i : a_i \neq 0\}$. Without loss of generality, by Lemma 15.4.8, let $a_1 = \max\{a_i : a_i < n - s\}$ and $a_2 = n - s$. We define

$$y = [I_{a_0-2}, (\omega, \omega^{-1})I_{a_1+1}, (\omega, \omega^{-1})^2 I_{n-s}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{143}$$

The following is the analogue of Lemma 14.4.12.

Lemma 15.4.12. *Let* $x \in H \cap \mathcal{V}_{s,r}$. *Assume* $a_0 = \min\{a_i : a_i \neq 0\}, a_1 = \max\{a_i : a_i < n - s\}, a_2 = n - s$ *and* $a_1 > a_0$. *Then, for* y *as in (143),*

$$f_\Omega(x) \leq f_\Omega(y)$$

Now Lemmas 15.4.9, 15.4.11 and 15.4.12 provide the technical tools to prove the claim.

Proposition 15.4.13. *Let* $x \in H \cap \mathcal{V}_{s,r}$. *Then* $f_\Omega(x) \leq \max\{f_\Omega(z_1), f_\Omega(z_2)\}$, *where* z_1 *and* z_2 *are defined in (137) and (138).*

Proof. With the same argument of Proposition 14.4.13 we see that, depending on the conditions on the multiplicities of the eigenvalues of x, we have $f_\Omega(x) \leq f_\Omega(z_1)$ or $f_\Omega(z_2)$. *q.e.d.*

Write $n = (n - s)l + m$, for $0 \leq m < n - s$. Recall that $\delta_{a;b} = 1$ if $b \mid a$ and 0 otherwise.

Proposition 15.4.14. *Let* $x \in H \cap \mathcal{V}_{s,r}$. *Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n} - \frac{m(n-s) - m^2}{2n(n-2)} \tag{144}$$

Proof. By Proposition 15.4.13, we know that $f_\Omega(x) \leq \max\{f_\Omega(z_1), f_\Omega(z_2)\}$. Hence we need to compute $f_\Omega(z_1), f_\Omega(z_2)$. We use Proposition 15.3.3 and *Mathematica* for the calculations.

Case 1. Assume s even and l odd. Then

$$f_\Omega(z_1) = 1 - \frac{s}{n} - \frac{2m(n-s) - m^2 + 2(1 - \delta_{m;4})}{2n(n-2)} \tag{145}$$

since $m < n - s$ we deduce $\frac{2m(n-s)-m^2+2(1-\delta_{m;4})}{2n(n-2)} > \frac{m(n-s)-m^2}{2n(n-2)} > 0$.

Case 2. Assume s even and l even. Then

$$f_\Omega(z_1) = 1 - \frac{s}{n} - \frac{(n-s)^2 - m^2 + 2(1 - \delta_{n-s+m;4})}{2n(n-2)} \tag{146}$$

again, using $m < n - s$ we have $\frac{(n-s)^2-m^2+2(1-\delta_{n-s+m;4})}{2n(n-2)} > \frac{m(n-s)-m^2}{2n(n-2)} > 0$.

Case 3. Assume s odd and l odd. Then

(147)
$$\begin{aligned} f_\Omega(z_1) &= 1 - \frac{s}{n} - \frac{2(n-s)^2(m+2) - m(n-s)(m+2) - 2m - 7n + 9s + 2(1-\delta_{m+1;4})}{2n(n-2)(n-s)} \\ &= 1 - \frac{s}{n} - \epsilon' \end{aligned}$$

with the aid of *Mathematica* we see

$$\epsilon' - \frac{m(n-s) - m^2}{2n(n-2)} = \frac{(n-s)^2(m+4) - 2m(n-s+1) - 7(n-s) + 2s}{2n(n-2)(n-s)} > 0$$

Case 4. Assume s odd and l even. Then

(148)
$$\begin{aligned} f_\Omega(z_1) &= 1 - \frac{s}{n} - \frac{(n-s)^3 - m(n-s)(m+2) + 2(n-s)^2 - 2m - 9(n-s) + 2s + 2(1-\delta_{n-s+m;4})}{2n(n-2)(n-s)} \\ &= 1 - \frac{s}{n} - \epsilon' \end{aligned}$$

Again, assisted by *Mathematica* we compute

$$\epsilon' - \frac{m(n-s) - m^2}{2n(n-2)} = \frac{(n-s)^3 + (n-s)^2(2-m) - 2m(n-s) - 9(n-s) + 2s}{2n(n-2)(n-s)} > 0$$

Thanks to the explicit formula given in Proposition 15.3.3, we see that, $f_\Omega(z_2)$ is the same as $f_\Omega(z_2)$ computed in Proposition 14.4.14 for $G = \mathrm{Sp}_n$. Hence we refer the reader to the proof of Proposition 14.4.14, in particular (114) and (115). *q.e.d.*

Remark 15.4.15. The best possible upper bound is given by $\max\{f_\Omega(z_1), f_\Omega(z_2)\}$ and $f_\Omega(z_1)$ is computed in (145), (146), (147), while the explicit value of $f_\Omega(z_2)$ is computed in (114) and (115).

Let U be the upper bound given in Proposition 15.4.14. We conclude this section with some examples in which we notice that this bound is close to best possible.

For $f_\Omega(z_1)$ as in (145) we have

$$U - f_\Omega(z_1) = \frac{m(n-s)}{2n(n-2)} + \frac{1-\delta_{m;4}}{n(n-2)}$$

Thus, for small values of m the upper bound U is optimal. Notice that if $m = 0$ then $f_\Omega(z_1) = U$. If $s \geq n - \sqrt{n}$ then $U - f_\Omega(z_1) < \frac{1}{n-2}$.

For $f_\Omega(z_1)$ as in (148) we have

(149)
$$U - f_\Omega(z_1) = \frac{(n-s)^3 + (n-s)^2(2-m) - 2m(n-s) - 9(n-s) + 2s}{2n(n-2)(n-s)}$$

Also, $U - f_\Omega(z_1) < \frac{n-s}{n(n-2)} + \frac{(n-s)^2}{2n(n-2)} - \frac{9}{2n(n-2)} + \frac{s}{n(n-2)(n-s)}$, which is very small for large values of s. For example, if $s \geq n - \sqrt{n}$ we have $U - f_\Omega(z_1) < \frac{4}{n}$.

15.5. Local lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$, with $r > 2$. We deal with involutions in Section 15.7. The main result of this section is the following.

Proposition 15.5.1. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s-1)}{n(n-2)}$$

In the case $r = p$ a slightly better bound is given in Proposition 15.5.2. Then in Proposition 15.5.3 we show that the bound is, essentially, the best possible by constructing $x \in H \cap \mathcal{V}_{s,p}$ such that $f_\Omega(x)$ is close to the bound.

If $r \neq p$ we proceed as in Section 14.5. We define a class of special elements in Definition 15.5.6. Then we show that $f_\Omega(x) \geq f_\Omega(z)$ for all $x \in H \cap \mathcal{V}_{s,r}$ where z is special. We compute close to the best possible lower bounds in Proposition 15.5.8 and 15.5.9.

15.5.1. Unipotent elements. Assume $p \neq 2$. Let $x \in H \cap \mathcal{V}_{s,p}$. Up to G-conjugacy, we write $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$. By Proposition 5.1.7, $s = n - \sum_i a_i$. Recall from Section 13.2 that s is always even. In particular, $s \leq n-2$.

Here, we prove the following result, which is the analogue of Proposition 14.5.2.

Proposition 15.5.2. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s-3)}{n(n-2)}$$

Proof. We proceed as in Proposition 14.5.2. Let $x \in H \cap \mathcal{V}_{s,p}$. Then, by Proposition 15.2.2, Theorem 5.2.1 and Proposition 5.4.1 we have

$$\begin{aligned} f_\Omega(x) &= \frac{2\sum_{i<j} i a_i a_j + \sum_i i a_i^2 - 2\sum_{i \text{ odd}} a_i}{n(n-2)} = \frac{n^2 - \dim x^{\mathrm{GL}_n} - 2\sum_{i \text{ odd}} a_i}{n(n-2)} \\ &= \frac{n(n-2) - \dim x^{\mathrm{GL}_n} + 2(n - \sum_{i \text{ odd}} a_i)}{n(n-2)} \geq 1 - \frac{\dim x^{\mathrm{GL}_n} - 2s}{n(n-2)} \\ &\geq 1 - \frac{s(2n-s-3)}{n(n-2)} \end{aligned}$$

q.e.d.

The following shows that the bound given in Proposition 15.5.2 is close to best possible.

Proposition 15.5.3. *Let s such that $H \cap \mathcal{V}_{s,p} \neq \emptyset$. Then there exists $x \in H \cap \mathcal{V}_{s,p}$ such that*

$$f_\Omega(x) \leq 1 - \frac{s(2n-s-3)}{n(n-2)} + \frac{3}{n} + \epsilon$$

where $\epsilon = 0$ if $s \leq 2p-2$ and $\epsilon = \frac{s^2}{n(n-2)(p-1)}$ otherwise.

Proof. This is essentially the same proof as that of Proposition 14.5.3.

In the case $s \leq 2p-2$ we consider $x = [J_{\frac{s+2}{2}}^2, J_1^{n-s-2}] \in H \cap \mathcal{V}_{s,p}$. Using Proposition 15.2.2 we have

$$f_\Omega(x) = 1 - \frac{s(2n-s-3)}{n(n-2)} + \frac{s + 4\delta_{s+2;4}}{n(n-2)} \leq 1 - \frac{s(2n-s-3)}{n(n-2)} + \frac{3}{n}$$

If $s > 2p-2$. We define $h = \left\lfloor \frac{s}{2(p-1)} - 1 \right\rfloor$. And we consider

$$z = \left[J_p^{2h}, J_{\frac{s}{2}+1+h-hp}^2, J_1^{n-s-2h-2}\right] \in H \cap \mathcal{V}_{s,p}$$

We compute
$$f_\Omega(z) = 1 - \frac{s(2n-s-3)}{n(n-2)} + \frac{s+4\delta_{\alpha;2}}{n(n-2)} + \frac{4g(h)}{n(n-2)}$$
where $g(h) := h(s-p+1) - h^2(p-1)$. Eventually, we have
$$f_\Omega(x) \leq 1 - \frac{s(2n-s-3)}{n(n-2)} + \frac{3}{n} + \frac{4g(h)}{n(n-2)} \leq 1 - \frac{s(2n-s-3)}{n(n-2)} + \frac{3}{n} + \frac{s^2}{n(n-2)(p-1)}$$

q.e.d.

Remark 15.5.4. If $x \in H \cap \mathcal{V}_{s,p}$ and $s > 2p-2$ then $p < n/2$ and, by Proposition 14.4.3, $s \leq n-2$. So by Proposition 14.5.3, $\epsilon = \frac{s^2}{n(n+2)(p-1)} < \frac{1}{p-1}$. Hence, for large values of p the bound in Proposition 14.5.2 is accurate.

15.5.2. Semisimple elements. Assume $r \neq p$ odd. In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$. The case $r = 2$ is studied in Section 15.7. By Proposition 15.3.1, we may assume $\nu(x) < n-2$. The main result of this section is the following.

Proposition 15.5.5. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*
$$f_\Omega(x) \geq 1 - \frac{s(2n-s-1)}{n(n-2)}$$

All the arguments are the same as in Section 14.5.2. Let $x \in H \cap \mathcal{V}_{s,r}$ be semisimple of prime order $r > 2$. Up to G-conjugacy, $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Assume, for some $h \leq \frac{r-1}{2}$, $\dim V_{\omega^h} = n - s$.

Definition 15.5.6. Let $x \in H \cap \mathcal{V}_{s,r}$. We define x to be *special* if $|a_i - a_j| \leq 1$ for all $i, j \neq h, r-h$.

Claim. Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega(x) \geq f_\Omega(z)$, where $z \in H \cap \mathcal{V}_{s,r}$ is special.

As in Section 14.5.2, four cases arise. In each of these cases, given $x \in H \cap \mathcal{V}_{s,r}$ non-special, we define a new element y, see (116), (117), (118), (119). Using a result analogous to Lemma 14.5.10 we eventually can show the claim.

Lemma 15.5.7. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then, for some special element $z \in H \cap \mathcal{V}_{s,r}$,*
$$f_\Omega(x) \geq f_\Omega(z)$$

As for the symplectic group, any special element has centraliser isomorphic to that of the elements defined in (120), (121), (122). For the convenience of the reader we recall the definitions of these elements here.

Let s be even, and write $s = a(r-1) + b$ where $0 \leq b < r-1$, in particular, b is even. Thus, a special element is given by

(150) $$z = [I_{n-s}, (\omega, \omega^{-1}) I_{a+1}, \ldots, (\omega, \omega^{-1})^{\frac{b}{2}} I_{a+1}, (\omega, \omega^{-1})^{\frac{b}{2}+1} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a]$$

For $s \geq n/2$, let $2s - n = a(r-2) + b$ with $0 \leq b < r-2$, notice $a \equiv b \pmod 2$. We define two elements, according to whether or not a is even. If a is odd then

(151)
$$z = [I_{a+1}, (\omega, \omega^{-1}) I_{n-s}, (\omega, \omega^{-1})^2 I_{a+1}, \ldots, (\omega, \omega^{-1})^{\frac{b+1}{2}} I_{a+1}, (\omega, \omega^{-1})^{\frac{b+3}{2}} I_a, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_a]$$

Similarly, if a is even then

(152)

$$z = [I_a, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_{a+1}, \ldots, (\omega,\omega^{-1})^{\frac{b}{2}+1} I_{a+1}, (\omega,\omega^{-1})^{\frac{b}{2}+2} I_a, \ldots, (\omega,\omega^{-1})^{\frac{r-1}{2}} I_a]$$

As in Section 14.5.2 we first consider the case $a = 0$.

Proposition 15.5.8. *Assume $r-1 \geq s, 2s-n$, respectively for the above cases. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s-2)}{n(n-2)}$$

PROOF. For z as in (150) we have, using Proposition 15.3.3

$$f_\Omega(z) = \frac{(n-s)(n-s-2)}{n(n-2)} = 1 - \frac{s(2n-s-2)}{n(n-2)}$$

For z as in (152) we have $f_\Omega(z) = \frac{2(n-s)^2 - 2(1-\delta_{s;2})}{n(n-2)}$.

A straightforward computation shows that $(n-s)(n-s-2) \leq 2(n-s)^2 - 2(1-\delta_{s;2})$ in the case $s \geq n/2$. *q.e.d.*

In the case $s > r-1$, it is still possible to compute $f_\Omega(z)$ where z is as in (150), (151), (152). We have the following.

Proposition 15.5.9. *Assume $r-1 < s$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s-1)}{n(n-2)}$$

PROOF. In order to show the result it is enough to show that the f_Ω-value of a special element satisfies the lower bound, thanks to Lemma 15.5.7.

Case 1. Assume z is as in (150). Using Proposition 15.3.3, we compute

(153)
$$f_\Omega(z) = 1 - \frac{2s(n-1)}{n(n-2)} + \frac{rs^2 - \epsilon^2}{n(n-2)(r-1)}$$

where $\epsilon = b$ if a is even or, $\epsilon = r-1$ if a is odd. It is clear, since $s > \epsilon \geq 0$, that

$$f_\Omega(z) \geq 1 - \frac{2s(n-1) - s^2}{n(n-2)} = 1 - \frac{s(2n-s-2)}{n(n-2)}$$

Case 2. Assume z is as in (151) and $r-2 \leq 2s-n$, so that $a \geq 1$ is odd. Then, by Proposition 15.3.3, we have

$$f_\Omega(z) = \frac{-b^2 + 2br - 2b + 2n^2 r - 3n^2 - 4nrs + 4ns + 2n - 4s - r^2 + 2rs^2 + 2r - 2(1-\delta_{s;2})(r-2)}{n(n-2)(r-2)}$$

(154)
$$\geq \frac{1 - r^2 + n^2(2r-3) - 4s + 2n - 4ns(r-1) + 2r(s^2+1)}{n(n-2)(r-2)}$$

where the inequality follows from the fact that the numerator is minimal for $b = 1$ (here b is odd). Then, differentiating with respect to r we see that (154) is minimal for r maximal, i.e. $r = 2s-n+2$. Then, using *Mathematica*, we get

$$f_\Omega(z) \geq \frac{-1 + 2n^3 + 8s - 4s^3 - 8n^2 s + 2n(5s^2-2)}{n(n-2)(n-2s)}$$

And we see that

(155)

$$f_\Omega(z)-\Big(1-\frac{s(2n-s-1)}{n(n-2)}\Big) \geq \frac{1-n^3-n^2(2-4s)-n(-4+5(s-1)s)-2s(4+s-s^2)}{n(n-2)(2s-n)}$$

differentiating with respect to s the right hand side of (155) we get $-\frac{2+(n-2s)^2(2n-2s+1)}{n(n-2)(2s-n)^2} < 0$. Therefore the right hand side of (155) is minimal for s maximal. Substituting the value $s = n-1$ we get $\frac{n(n-4)+5}{n(n-2)^2} > 0$.

Case 3. This is totally similar to **Case 2** studied above. Assume z is as in (152) and $r-2 \leq 2s-n$, so that $a \geq 2$ is even. Then, by Proposition 15.3.3, we have

$$f_\Omega(z) = \frac{2b-b^2+n^2(2r-3)+4sn+2n-4rsn+2rs^2-4s-2(1-\delta_{s;2})(r-2)}{n(n-2)(r-2)}$$

$$\geq \frac{4+2n^2(r-2)-s(s+6)+4n-2ns(2r-3)+2r(s^2-1)}{n(n-2)(r-2)} \tag{156}$$

where the inequality follows from the fact that the expression in b is minimal for b maximal, i.e. $b = n-s$. Differentiating with respect to r we get $\frac{(2n-3s)(s-2)}{(r-2)^2}$ whose sign depends whether or not $s \leq 2n/3$. In the case the first derivative in r is always negative, i.e. $s > 2n/3$, we have, substituting $r = 3$,

$$f_\Omega(z) \geq \frac{2n^2+4n-6n^2-6s+5s^2-2}{n(n-2)}$$

and

$$f_\Omega(z) - \Big(1-\frac{s(2n-s-2)}{n(n-2)}\Big) \geq \frac{n^2-2-2n(2s-3)+4s(s-2)}{n(n-2)} \geq 0$$

Now, let us assume $s \leq \frac{2}{3}n$ so that the expression in (156) is minimal for r maximal, i.e. $r = 2s-n+2$. Thus

$$f_\Omega(z) \geq \frac{2n^3-8n^2s-s(2+s)(-5+4s)+2n(-3+s+5s^2)}{n(n-2)(n-2s)}$$

Again, with the aid of *Mathematica* we deduce the result. *q.e.d.*

Remark 15.5.10. In **Case 1** we can take as lower bound, $1-\frac{s(2n-s-2)}{n(n-2)}$. In **Case 2** and the second part **Case 3** we need the slightly worse lower bound $1-\frac{s(2n-s-1)}{n(n-2)}$.

The best possible lower bounds are given by $\min\{f_\Omega(x) : x \text{ special}\}$, by Proposition 15.5.7. The best possible bound is the minimum of the values computed in (153), (154) and (156).

Say $\ell = 1-\frac{s(2n-s-1)}{n(n-2)}$. We conclude this section with an example of a bound on $f_\Omega(z)-\ell$, where $z \in H \cap \mathcal{V}_{s,r}$ is a special element.

For $f_\Omega(z)$ as in (153) we have

$$f_\Omega(z)-\ell = \frac{s}{n(n-2)} + \frac{s^2-\epsilon^2}{n(n-2)(r-1)} \leq \frac{s}{n(n-2)} + \frac{s^2}{n(n-2)(r-1)}$$

In particular, we see that, for small s, $f_\Omega(z)-\ell$ is small. In general, $f_\Omega(z)-\ell < \frac{1}{n-2}+\frac{25}{24(r-1)}$. However, if we assume $s \leq \sqrt{n}$ then $f_\Omega(z)-\ell < \frac{2}{n-2}$.

15.6. Further comments on local bounds

Using the bounds in Propositions 15.4.1 and 15.5.1 we have the following. We omit the proof because it is an easy calculation.

Proposition 15.6.1. *Assume $r \neq 2$. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{s(n-s)}{n(n-2)} + \frac{2}{n}$$

Remark 15.6.2. Notice that if $s \leq \sqrt{n}$ or $s \geq n - \sqrt{n}$ then $\frac{s(n-s)}{n(n-2)} \leq \frac{2}{\sqrt{n}}$. By Proposition 15.6.1 for all $x, y \in H \cap \mathcal{V}_{s,r}$ we have $|f_\Omega(x) - f_\Omega(y)| \leq \frac{2}{\sqrt{n}} + \frac{2}{n}$.

15.7. Involutions

In this section we study involutions in H. The main result of this section in Proposition 15.7.1, below, in which we give explicit formulae for $f_\Omega(x)$, for any involution $x \in H$. Notice that there are involutions $x \in G$ such that $x^G \cap H \neq \emptyset$ (e.g. all those x such that $\nu(x) < n/2$ is odd).

Let $x \in H$ and write $\nu(x) = s$. By Proposition 13.2.13, if $s < n/2$ then $x^G \cap H = x^G \cap H^\circ$ and if $p = 2$ then x is a_s-type, by Lemma 13.2.11. If $\nu(x) = n/2$ then $x^G \cap H^\circ\tau \neq \emptyset$ and Proposition 13.2.13 provides all the informations needed. Notice that if $(p, s) = (2, n/2)$ and $x \in H$ then x may be of $a_{n/2}, b_{n/2}$ or $c_{n/2}$-type.

Proposition 15.7.1. *Let $x \in H$ be an involution. Assume $\nu(x) = s$.*

(i) Suppose $p \neq 2$. If $s < n/2$ then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n-2)}$$

If $s = n/2$ then $f_\Omega(x) = \frac{1}{2}$.

(ii) Suppose $p = 2$. If $s < n/2$ then s is even, x is a_s-type and

$$f_\Omega(x) = 1 - \frac{2s(n-s-2)}{n(n-2)}$$

If $s = n/2$ then

$$f_\Omega(a_{n/2}) = \frac{1}{2} + \frac{2}{n-2}, \quad f_\Omega(c_{n/2}) = \frac{1}{2} - \frac{2}{n-2}, \quad f_\Omega(b_{n/2}) = \frac{1}{2}$$

The following is an immediate consequence.

Corollary 15.7.2. *Let $x \in H$ be an involution.*

(a) We have

$$f_\Omega(x) \leq 1 - \frac{4}{n} + \frac{8\delta_{p,2}}{n(n-2)}$$

with equality if, and only if $\nu(x) = 2$.

(b) For $p \neq 2$

$$f_\Omega(x) \geq \frac{1}{2}$$

with equality if, and only if, $\nu(x) = n/2$.

(c) For $p = 2$

$$f_\Omega(x) \geq \frac{1}{2} - \frac{\delta_{n;4}}{n-2}$$

with equality if, and only if $x \in \{b_{n/2}, c_{n/2}\}$.

PROOF. First, assume $p \neq 2$. We see that $f_\Omega(x)$ in decreasing for $1 \leq s \leq n/2$. Therefore $f_\Omega(x) \leq f_\Omega([I_{n-2}, -I_2]) = 1 - \frac{4}{n}$. And $f_\Omega(x) \geq f_\Omega(z) = \frac{1}{2}$, where $\nu(z) = n/2$.

Assume now $p = 2$. Again, we see that the formula for $f_\Omega(a_s)$ with $s < n/2$ is monotonically decreasing in $[2, \frac{n}{2} - 1]$. A direct check shows that $f_\Omega(x)$ is minimal for $x \in \{b_{n/2}, c_{n/2}\}$. The result follows *q.e.d.*

Remark 15.7.3. Let us make some comments and relate these bounds with the global bounds given so far for odd prime order elements.

(i) The upper bound given in Corollary 15.7.2 is the same as the upper bound stated for odd prime order elements in Proposition 15.1.1. Moreover, also for involutions, the bound is realised by $x \in H$ if, and only if, $\nu(x) = 2$.
(ii) Notice that thanks to Corollary 7.1.11 we can extend the upper bounds of Proposition 15.1.1 and Corollary 15.7.2 to any non-central element of G.
(iii) Assume $p = 2$. The lower bound given in Proposition 15.2.1 in the case $p \leq n/2$ is $1/p$. In the case $p = 2$ we clearly have $2 \leq n/2$ and the lower bound stated in Corollary 15.7.2 agrees with the bound of Proposition 15.2.1. When $n/2$ is odd, Corollary 15.7.2 slightly improves the bound in Proposition 15.2.1.
(iv) Assume $p \neq 2$. The lower bound stated in Proposition 15.3.1(ii) is still valid for involutions. However, Corollary 15.7.2 provides a slightly better bound. Notice that the elements that realise the lower bound are special, as in Definition 15.3.9.

15.7.1. Semisimple involutions. Here we prove Proposition 15.7.1 for $p \neq 2$. Recall the definition of B in (83).

Lemma 15.7.4. *Assume* $p \neq 2$. *Then conclusions of Proposition 15.7.1 hold.*

PROOF. Up to conjugation, write $x = [I_{n-s}, -I_s] \in H$. Then, by Proposition 13.2.13, unless $s = n/2$ we have $x^G \cap H = x^G \cap H^\circ$. In the case $s = n/2$, we have $x^G \cap H = x^{H^\circ} \cup \tau^{H^\circ}$, see Table 13.2.1. Observe that $x^{H^\circ} = \emptyset$ if $n/2$ is odd.

First assume $\mathbf{s < n/2}$. Let $x = [I_{n-s}, -I_s]$. Then, by Theorem 5.3.1 and (87), we have $\dim x^G = s(n-s)$ and $\dim(x^G \cap H^\circ) = \frac{s}{2}(n-s)$. Therefore, using $\dim \Omega = \frac{n}{4}(n-2)$, given in Table 13.1.1, we get

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n(n-2)} \tag{157}$$

Now assume $\mathbf{s = n/2}$.

Case 1. If $n/2$ is odd then $x \in \tau^G$, by Proposition 13.2.13, and $\dim(x^G \cap H) = \dim \tau^{H^\circ}$. Then, by Proposition 13.2.14, $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$. In particular, $\dim(x^G \cap H) = \frac{n(n+2)}{8}$, recall that $\dim \tau^{H^\circ} = \dim H^\circ - \dim C_{H^\circ}(\tau)$. Thus $f_\Omega(x) = 1/2$.

Case 2. Assume now $n/2$ is even. Then $x \in [I_{n/2}, -I_{n/2}]^G$ and, as reminded above, $x^G \cap H = x^{H^\circ} \cup \tau^{H^\circ}$. Proposition 13.2.14 yields $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$. Therefore we compute

$$\dim x^{H^\circ} = \frac{n^2}{8}, \quad \dim \tau_1^{H^\circ} = \frac{n(n+2)}{8}$$

Therefore $\dim(x^G \cap H) = \dim \tau^{H^\circ} = \frac{n(n+2)}{8}$. Using the above formula we have $\dim x^G = n^2/4$. Thus, for $x \in H$ with $\nu(x) = n/2$ we get $f_\Omega(x) = 1/2$. *q.e.d.*

15.7.2. Unipotent involutions. Here, we complete the proof of Proposition 15.7.1.

Lemma 15.7.5. *Assume $p = 2$. Then conclusions of Proposition 15.7.1 hold.*

PROOF. Let $x \in H$ be an involution with $\nu(x) = s$.

First assume $\mathbf{s < n/2}$. Proposition 13.2.13 gives $x^G \cap H = x^{H^\circ}$. In addition Lemma 13.2.11 implies that x is of a_s-type. Using Proposition 5.2.5 and (88), we have

$$\dim x^G = s(n-s-1), \ \dim(x^G \cap H) = \frac{s}{2}(n-s)$$

Therefore, using $\dim \Omega = \frac{n}{4}(n-2)$, as in Table 13.1.1, we compute

$$f_\Omega(x) = 1 - \frac{2s(n-s-2)}{n(n-2)}$$

Now assume $\mathbf{s = n/2}$.

Case 1. If $n/2$ is odd then x is $b_{n/2}$-type. By Proposition 13.2.8, $x \in \tau^{H^\circ}$. Moreover $x^G \cap H = \tau^{H^\circ}$, by Proposition 13.2.13. Therefore, using Proposition 5.2.5 and (88), we compute,

$$\dim x^G = \frac{n^2}{4}, \quad \dim(x^G \cap H) = \frac{n(n+2)}{8}$$

where $\dim(x^G \cap H) = \dim \tau^{H^\circ} = \dim H^\circ - \dim C_{H^\circ}(\tau)$ and $C_{H^\circ}(\tau) = \mathrm{O}_{n/2}$, by Proposition 13.2.14. Therefore, using Proposition 7.1.8, we compute

$$f_\Omega(x) = \frac{1}{2}$$

Case 2. If $n/2$ is even then, as proved in Proposition 13.2.13 there are two different conjugacy classes of involutions with ν-value equal to $n/2$. The representatives of these classes are given in Table 13.2.1. First, assume x is a $c_{n/2}$-type involution so $x^G \cap H^\circ = \tau^{H^\circ}$. We compute, using Proposition 5.2.5, $\dim x^G = n^2/4$ and

$$\dim \tau^{H^\circ} = \frac{n(n-2)}{8}$$

where $\dim(x^G \cap H) = \dim \tau^{H^\circ} = \dim H^\circ - \dim C_{H^\circ}(\tau)$ and $C_{H^\circ}(\tau) = \mathrm{Sp}_{n/2}$, by Proposition 13.2.14. Therefore

$$f_\Omega(x) = 1 - \frac{n+2}{2(n-2)} = \frac{1}{2} - \frac{2}{n-2}$$

Now, assume x is an $a_{n/2}$-type involution. Then, following the notation in Table 13.2.1, we have $x^G \cap H = x^{H^\circ} \cup \tau_2^{H^\circ}$ for $\tau_2 = [B, \ldots, B]\tau$, where B is defined in (83). By Proposition 13.2.14, $C_{H^\circ}(\tau_2) = C_{\mathrm{Sp}_{n/2}}(b_1)$. Proposition 5.2.5 implies $\dim C_{\mathrm{Sp}_{n/2}}(b_1) = \frac{n(n-2)}{8}$. Thus $\dim x^G = \frac{n(n-2)}{4}$ and

$$\dim x^{H^\circ} = \frac{n^2}{8}, \quad \dim \tau_2^{H^\circ} = \frac{n(n+2)}{8}$$

Thus $\dim(x^G \cap H) = \frac{n(n+2)}{8}$. Therefore, using Proposition 7.1.8, we compute

$$f_\Omega(x) = \frac{1}{2} + \frac{2}{n-2}$$

q.e.d.

Part 4

$\mathscr{C}_2$-actions of classical groups

CHAPTER 16

Introduction

Let $G = Cl_n$ be defined over an algebraically closed field k of arbitrary characteristic p. As usual, V stands for the natural module of G. A closed subgroup $H \leq G$ is a $\mathscr{C}_2$-subgroup if there exists a decomposition $V = V_1 \oplus \ldots \oplus V_t$ into isometric subspaces such that $H = \mathrm{Stab}_G\{V_1, \ldots, V_t\}$. In Proposition 17.1.6 we show that $H = \mathrm{GL}_{n/t} \wr S_t$ or $\mathrm{Sp}_{n/t} \wr S_t$ or $\mathrm{O}_{n/t} \wr S_t$ and $H^\circ = (\mathrm{GL}_{n/t})^t, (\mathrm{Sp}_{n/t})^t$ or $(\mathrm{O}_{n/t})^t$, respectively. Notice that we will always assume $n/t > 1$ in the orthogonal case. Set $\Omega = G/H$ for the coset variety.

Notice that n/t is even if $p = 2$ and $G \neq \mathrm{GL}_n$.

In this chapter we state the main results of this part, in which we derive bounds on $f_\Omega(x)$ for $x \in H$ of prime order or any unipotent element in the characteristic zero case.

Let $x \in G$; we define the following related ratio

$$f^\circ_\Omega(x) = \frac{\dim \Omega - \dim x^G + \dim(x^G \cap H^\circ)}{\dim \Omega} \tag{158}$$

We conclude this chapter giving a brief overview of the strategy used to derive bounds for $\mathscr{C}_2$-actions.

16.1. Main results

Recall from (27) the definition of the set $\mathscr{R}$ as the subset of G consisting of prime order elements or any unipotent element in the characteristic zero case. Recall, moreover, that we set $p = \infty$ in the characteristic zero case.

Theorem 16.1.1 below provides global upper and lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order.

Theorem 16.1.1. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_2$-subgroup. Set $\Omega = G/H$.*

(a) Let $x \in G \setminus Z(G)$. Then

$$f_\Omega(x) \leq 1 - \frac{2}{n} + \frac{2}{n(n-2)}$$

(b) Let $x \in H \cap \mathscr{R}$ be unipotent.

(i) If $p > n$ then

$$f_\Omega(x) \geq \frac{t}{n}$$

(ii) If $p \leq n$ then

$$f_\Omega(x) \geq \frac{1}{p}$$

(c) *Let $x \in H \cap \mathscr{R}$ be semisimple of order $r < n-1$. Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{2n^2(t-1)} - \frac{2}{n}$$

Remark 16.1.2. Let us make some comments on the statement of Theorem 16.1.1.

(i) Note that the case $r \geq n$ is a prime other than p is omitted. In this situation, we characterise elements $x \in H$ of order r for which $f_\Omega(x) = 0$, see Theorem 16.1.7.

(ii) The upper bound does not depend on the order of the element. Moreover the upper bound is sharp: in fact if $p = 2$ and $(G, H) = (\mathrm{Sp}_n, \mathrm{Sp}_2 \wr S_{n/2})$ and x is a b_1-involution then $f_\Omega(x) = 1 - \frac{2}{n} + \frac{2}{n(n-2)}$ (also if x is an a_2-involution and $n = 4$). Indeed, for each classical group we shall give the best possible upper bound and we characterise elements which realise the bound, see Sections 18.1, 19.1 and 20.1.

(iii) The lower bounds extend to any (prime order) element $x \in G$ for which $x^G \cap H \neq \emptyset$.

(iv) Part (b) is proved in Propositions 18.2.1, 19.2.1 and 20.2.1. Notice that the lower bound in Proposition 20.2.1 for involutions is improved in Section 20.7, see Remark 20.7.4. In the case $p > n$ we characterise elements that realise the bound. In the case $p \leq n$ we will give a characterisation of the elements realising the bound only when $p \mid n$. In the case $p \nmid n$, we prove that there exists $x \in H^\circ$ such that $f_\Omega(x) \leq 1/p + \epsilon$, for a small ϵ.

(v) The characteristic zero case for unipotent elements is included in the case $p > n$.

(vi) Part (c) is proved in Section 18.3, 19.3 and 20.3. For $r < n$ the bound given is not the best possible, in fact it is often negative. However, close to best possible lower bounds are given in Propositions 18.3.19, 19.3.16 and 20.3.16 (if $G = \mathrm{O}_n$ only in the case n/t is even). In fact these bounds are the best possible on f_Ω°, as in (158). In this case we do not have a classification of the elements that realise the bound. However, we shall define a class of *special* elements in H° (see Definition 18.3.9, for example) and we show that these elements realise the best possible lower bound on f_Ω°.

(vii) Assume $G = \mathrm{O}_n$ and n/t is odd. Then H° does not contain a maximal torus of G. Therefore there are semisimple elements $x \in G$ for which $x^G \cap H^\circ = \emptyset$. In fact we characterise semisimple elements $x \in G$ for which $x^G \cap H \neq \emptyset$, see Proposition 20.3.18, and we deduce a characterisation for which $x^G \cap H^\circ \neq \emptyset$. In this case, in Section 20.3.2.2 we will derive lower bounds on f_Ω° when some restrictive hypothesis holds (e.g. special elements lie in H°), see Proposition 20.3.22.

Recall Definition 7.3.1 of the algebraic fixity M, the r-local algebraic fixity M_r and $M_{r'}$.

Theorem 16.1.3. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_2$-subgroup. Set $\Omega = G/H$.*

(a) *Assume $G = \mathrm{GL}_n$. Then*

$$M = 1 - \frac{2}{n} + \frac{\delta_{t,n}}{n(n-1)}$$

(b) *Assume $G = \mathrm{Sp}_n$. Then*

$$M = 1 - \frac{2}{n} + \frac{2\delta_{t,n/2}}{n(n-2)}$$

(c) Assume $G = \mathrm{O}_n$. *Then*

$$M = 1 - \frac{2}{n}$$

Theorem 16.1.4. *Let* $G = Cl_n$ *and* $H \leq G$ *be a* $\mathscr{C}_2$*-subgroup. Set* $\Omega = G/H$.

(a) If $G = \mathrm{GL}_n$ *then*

(i) Assume $t = n$. *Then* $M = M_2$, $M > M_r$ *for any odd prime* r *and*

$$M_r = 1 - \frac{2}{n}$$

(ii) Assume $t < n$. *Then for every prime* r, $M = M_r$.

(b) If $G = \mathrm{Sp}_n$ *and* $n > 4$ *then* $M = M_p$ *and* $M_p > M_{p'}$

(i) Assume $t = n/2$. *Then for* $r \neq p$ *odd*

$$M_2 = 1 - \frac{2}{n} + \frac{2\delta_{p,2}}{n(n-2)}, \quad M_r = 1 - \frac{4}{n}$$

(ii) Assume $t < n/2$. *Then for* $r \neq p$

$$M_r = 1 - \frac{4}{n} + \frac{6\delta_{r,2}}{n(n-2)}$$

(c) Assume $G = \mathrm{O}_n$ *and* $n > 4$. *Then* $M = M_2$ *and* $M > M_r$ *for any odd prime* r. *In addition*

$$M_r = 1 - \frac{4}{n}$$

Remark 16.1.5.

(i) In the case $G = \mathrm{Sp}_4$ we have $M = M_2 = 3/4$ and $M_r = 0$ while $M_p = 1/2$.
(ii) If $G = \mathrm{O}_4$ we refer the reader to Lemma 20.1.3.
(iii) Theorem 16.1.3 easily follows by combining the upper bounds and the characterisations provided by Propositions 18.1.1, 19.1.1, 20.1.1.

The following is a straightforward consequence of Theorems 16.1.3 and 16.1.4.

Corollary 16.1.6. *For any prime* r *there exists* $x \in H$ *of order* r *such that*

$$f_\Omega(x) \geq 1 - \frac{4}{n}$$

Let G be a connected reductive algebraic group and let P be a parabolic subgroup, set $\mathcal{P} = G/P$. In [**31**, Theorem 1(c)] it is stated that an element $x \in G$ is regular if, and only if, $f_{\mathcal{P}}(x) = 0$. We prove a similar version of this result for $\mathscr{C}_2$-actions. Recall that $x \in Cl_n$ is regular if, and only if, $\dim C_G(x) = \operatorname{rank} G$.

Theorem 16.1.7. *Let* $G = Cl_n$ *and* $H \leq G$ *be a* $\mathscr{C}_2$*-subgroup. Set* $\Omega = G/H$. *Let* $x \in G$. *Assume* $x^G \cap H \neq \emptyset$. *Then the following are equivalent*

(i) $C_\Omega(x)$ *is finite;*
(ii) $f_\Omega(x) = 0$;
(iii) x *is regular, or* $G = \mathrm{Sp}_n$ *and* $C_{\mathrm{Sp}_n}(x) \cong \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1}$.

Remark 16.1.8.

(i) Notice that if $G = \mathrm{O}_n$ and n/t is odd then conjugacy classes of regular elements do not meet H. In this case, let $x \in G$ be semisimple of order $r \geq n-1$. Then we shall show in Proposition 20.3.1 that $f_\Omega(x) \geq (t/n)^2$.

(ii) Notice that if x is a regular unipotent element then $x^G \cap H = \emptyset$. Thus this result holds only for semisimple elements, see Propositions 18.3.1, 19.3.1 and 20.3.1

Let $x \in G$. Recall from Proposition 5.1.7 that $\nu(x)$ is the codimension of the largest eigenspace of x. In the proof of various results which lead to the bounds stated in Theorem 16.1.1, we see a quite strong relation between $f_\Omega(x)$ and $\nu(x)$. For example, we prove that for a prime order element $x \in H$, $f_\Omega(x)$ is maximal if, and only if, $\nu(x)$ has the lowest possible value, apart from finitely many known exceptions; see the main results of the sections mentioned in Remark 16.1.2(ii).

In Theorems 16.1.9 and 16.1.11, below, we assume x is not an involution. Indeed if x is an involution $\nu(x)$ precisely determines the value of $f_\Omega(x)$, unless $p = 2$ (see Section 5.2.1). For involutions in some cases we will determine an explicit formula for $f_\Omega(x)$, see Theorem 16.1.15.

We deduce the following from Sections 18.4, 19.4 and 20.4.

Theorem 16.1.9. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_2$-subgroup. Set $\Omega = G/H$. Assume $r \neq 2$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1 - \frac{s}{n+1} + \frac{1}{n}$$

Remark 16.1.10. First, notice that this upper bound extends to any element $x \in \mathcal{V}_s$ (if $x \in \mathcal{V}_s$ and $x^G \cap H = \emptyset$ then $f_\Omega(x) = 0$, see Lemma 7.1.1). Let us consider separately the two cases $r \neq p$ and $r = p$.

(i) Assume $r \neq p$. In Sections 18.4.1. 19.4.2 and 20.4.2 we prove that the bound is close to best possible. We shall define elements $\bar{x}$, for example (212) if $G = \mathrm{GL}_n$, such that the difference between $f^\circ_\Omega(\bar{x})$ and the upper bound stated is bounded. Then in Propositions 18.4.8, 19.4.8 and 20.4.4 we prove that $f^\circ_\Omega(\bar{x})$ is the best possible upper bound on f°_Ω.

(ii) We study the case $r = p$ in Sections 18.4.2, 19.4.1 and 20.4.1. For $s \leq n/2$ we show in Proposition 18.4.11, 19.4.2 and 20.4.3 that there exists $x \in H^\circ \cap \mathcal{V}_{s,p}$ for which the difference between the bound and $f^\circ_\Omega(x)$ is small. In the case $s > n/2$ we do not have a similar result in general; however if $G = \mathrm{GL}_n$ and $t = 2$ we will provide such information, see Proposition 18.4.12. Furthermore, if $G = \mathrm{GL}_n$ and $t = n$ we give explicit formulae for $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,p}$, see Proposition 18.4.13.

Notice that, unless $G = \mathrm{O}_n$ and n/t is odd, for any semisimple $x \in G$ we have $x^G \cap H^\circ \neq \emptyset$, since H° contains a maximal torus. Therefore it is enough to derive lower bounds on $f_\Omega(x)$ for $x \in H^\circ \cap \mathcal{V}_{s,r}$. The following provides lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$, for $r \neq p$ an odd prime.

Theorem 16.1.11. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_2$-subgroup. Set $\Omega = G/H$. Assume $r \neq 2, p$. Let $x \in H^\circ \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{5}{n}$$

Remark 16.1.12. As in Remark 16.1.2(iii) we point out that this bound extends to any element $x \in \mathcal{V}_{s,r}$ such that $x^G \cap H^\circ \neq \emptyset$. Let us make some observations on Theorem 16.1.11.

(i) If $G = \mathrm{O}_n$ and $H = \mathrm{O}_{n/t} \wr S_t$ with n/t odd, the case $x \in (H \setminus H^\circ) \cap \mathcal{V}_{s,r}$ is omitted. In this case there are, in general, semisimple elements $x \in H$ whose G-class does not meet H°. Therefore our strategy to derive lower bounds on f_Ω° is not effective in this case.

(ii) In the case $r = p$ we do not have any general result. The condition $\nu(x) = s$ for an element of prime order p given in Jordan form $x = [J_p^{a_p}, \dots, J_1^{a_1}]$ translates into $s = n - \sum_i a_i$; this condition together with $x^G \cap H^\circ \neq \emptyset$ is not easy to deal with. However, in the case $t = 2$ for $G = \mathrm{GL}_n$ we give close to best possible lower bounds on f_Ω in Proposition 18.5.10.

(iii) In the case $r \neq p$ we shall compute the best possible lower bound on f_Ω°. We shall define a class of *special* elements $z \in H \cap \mathcal{V}_{s,r}$ in Definitions 18.5.2 and 18.5.2, and we prove $f_\Omega^\circ(x) \geq f_\Omega^\circ(z)$ for any $x \in H \cap \mathcal{V}_{s,r}$, see Lemmas 18.5.4 and 19.5.4.

It is interesting to note that the f_Ω-values of elements in $H \cap \mathcal{V}_{s,r}$ are approximately the same; more formally we state the following.

Theorem 16.1.13. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_2$-subgroup. Set $\Omega = G/H$. Assume $r \neq 2, p$. Let $x, y \in H^\circ \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{s(n-s)}{n^2} + \frac{5}{n}$$

Remark 16.1.14. This is proved in Propositions 18.6.1, 19.6.1 and 20.6.1. Notice that for $s \leq \sqrt{n}$ and $s \geq n - \sqrt{n}$ we have $|f_\Omega(x) - f_\Omega(y)| \leq \frac{2}{\sqrt{n}} + \frac{5}{n}$.

Let $x \in G$ be an involution. In most of the cases we can give an explicit formula for $f_\Omega(x)$. Notice that for any involution $x \in G$ we have $x^G \cap H \neq \emptyset$. For more details and proofs we refer the reader to Sections 18.7, 19.7 and 20.7.

For the purpose of stating the next result we define

$$h' = \max\left\{0, \frac{s+t}{2} - \frac{n}{4}\right\}$$

Theorem 16.1.15. *Let $G = Cl_n$ and $H \leq G$ be a $\mathscr{C}_2$-subgroup. Let $x \in H$ be an involution with $\nu(x) = s$. Write $s = \lfloor s/t \rfloor t + b$ and $s/2 = \lfloor s/2t \rfloor t + b'$. Then*

$$g(n,t,s) \leq f_\Omega(x) \leq g(n,t,s) + \epsilon$$

where the values of $g(n,t,s)$ and ϵ are listed in Table 16.1.1.

Remark 16.1.16.

(i) Notice that, for any involution $x \in G$ with $\nu(x) = s$, we may always assume $s \leq n/2$. For unipotent involutions this is clear. If $p \neq 2$ then $f_\Omega([I_{n-s}, -I_s]) = f_\Omega([I_s, -I_{n-s}])$, thanks to Lemma 7.1.3.

(ii) We will define h'' to be the smallest integer such that $x^G \cap H^\circ \pi_i = \emptyset$ for all $i < h''$. Then we shall show that $h' = h''$.

(iii) In the main theorems of Sections 18.7, 19.7 and 20.7 we will give slightly better values for ϵ. Notice that if $s - h'\frac{n}{t} < \frac{n}{t}$ then $\epsilon = 0$, as in this case $x^G \cap H = x^G \cap H^\circ \pi_{h'}$.

G	Conditions	$g(n,t,s)$	ϵ
GL_n Sp_n O_n	*n/t even* *or n/t odd and $s < \max\{\frac{n}{t}, \frac{n-t}{2}\}$* $p=2, x \in b_s^G$ *or* c_s^G $x \notin a_s^G$	$1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$	0
Sp_n O_n	$p \neq 2$ $p=2, x \in a_s^G$ *and* *$n/2t$ even or $h'=0$* $p=2, x \in a_s^G, h'=0$	$1 - \frac{2s(n-s)}{n^2} - \frac{8b'(t-b')}{n^2(t-1)}$	$3/n$
Sp_n O_n	$p=2, x \in a_s^G, n/2t$ *odd and* $h'>0$ $p=2, x \in a_s^G, h'>0$	$1 - \frac{2s(n-s-1)}{n^2\left(1-\frac{1}{t}\right)} + \frac{n+2(s-t)}{4n(t-1)}$	$3/n$
GL_n	*n/t odd and $s \geq \max\{\frac{n}{t}, \frac{n-t}{2}\}$*	$1 - \frac{2s(n-s)-s}{n^2\left(1-\frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$	0

Table 16.1.1. Involutions in $\mathscr{C}_2$-actions

16.2. Strategy and layout

In Chapter 17 we define the $\mathscr{C}_2$-collection and prove that subgroups in this family have the following structure:

$$\mathrm{GL}_{n/t} \wr S_t < \mathrm{GL}_n,\ \mathrm{Sp}_{n/t} \wr S_t < \mathrm{Sp}_n,\ \mathrm{O}_{n/t} \wr S_t < \mathrm{O}_n$$

for some divisor $t > 1$ of n. Then we study conjugacy classes, first in H° then in $H \setminus H^\circ$ in Sections 17.3 and 17.4, respectively. The main results of Section 17.3 are Theorem 17.3.8 which provides an explicit formula for $\dim(x^G \cap H^\circ)$ for semisimple elements $x \in G$ and Proposition 17.3.17 that shows $\dim(x^G \cap H^\circ) \geq (1/t - \epsilon)\dim x^G$, for semisimple element $x \in G$ whose G-class meets H° with an explicit $\epsilon \geq 0$. Notice that Proposition 17.3.17 is, in some sense, the dual of [**11**, Theorem 1] and [**14**, Proposition 2.1]. In fact combining these results together we will deduce $f_\Omega(x) - f_\Omega^\circ(x) \leq 4/n$. In Section 17.4 we shall study the Jordan form of prime order elements $x \in G$ whose G-class meets $H \setminus H^\circ$.

Let $G = Cl_n$, $H \leq G$ be a $\mathscr{C}_2$-subgroup and $\Omega = G/H$. Let $x \in H \cap \mathscr{R}$ of order r.

In Chapters 18, 19 and 20 we study actions of G on $\Omega = G/H$, deriving bounds on $f_\Omega(x)$. Although the methods used and the main arguments are very similar to each other, we do not try to combine the analysis in order to avoid over-complicated notations. Often we will omit the proofs that are similar to arguments already given; however we will record such proofs in Appendix B. All these three chapters will have the same structure. We study: global upper bounds, global lower bounds for unipotent and then semisimple elements, local upper and lower bounds and we conclude each chapter with a more detailed study of involutions. We start each section stating the main result of that section which immediately implies the theorems stated in this introduction.

In Sections 18.1, 19.1 and 20.1 we shall prove part (a) of Theorem 16.1.1 and Theorems 16.1.3 and 16.1.4. The main strategy is to compute explicit values of $f_\Omega(x)$ for $x \in H$ when $\nu(x)$ is small and then use [**14**, Proposition 2.1], for the general case.

In Sections 18.2, 19.2 and 20.2 we prove part (b) of Theorem 16.1.1. First we study the case $p > n$ (which also includes the characteristic zero case), here we have $x^G \cap H = x^G \cap H^\circ$, see Proposition 17.4.11, hence the largest Jordan block allowed in x has size n/t. Then we move onto the case $p \leq n$, where the largest Jordan block allowed is J_p.

In the case $p > n$ we claim that the lower bound is realised by elements with at least $(t-1)$ Jordan blocks of size n/t. The main tool (in the case $G = \mathrm{GL}_n$, but the analysis will be similar for the other groups) is given by Lemma 18.2.5, in which, given an element x with $h < t-1$ Jordan blocks of size n/t, we construct a suitable new element y with $h+1$ Jordan blocks of size n/t and we prove $f_\Omega(x) > f_\Omega(y)$. Then iterating this construction we prove the claim.

In the case $p \leq n$ we directly show that for any unipotent element $x \in H$ we have $f_\Omega(x) \geq 1/p$. Then proving a few technical lemmas we characterise elements of order p that realise the bound (for $G = \mathrm{GL}_n$ see Proposition 18.2.16), only in the case $p \mid n$. If $p \nmid n$ we construct an element $x \in H$ such that $f_\Omega(x) - 1/p$ is universally bounded.

In Sections 18.3, 19.3 and 20.3 we prove part (c) of Theorem 16.1.1 and Theorem 16.1.7. Also here we shall divide the analysis according to whether $r > n$ or $r \leq n$ (or $n-1$ for the symplectic and orthogonal cases).

For $r > n$ we explicitly compute a formula for $\dim C_\Omega(x)$, see (202) and (233) and (258), for $G = \mathrm{GL}_n, \mathrm{Sp}_n$ and O_n, respectively. Then we prove that $f_\Omega(x) = 0$ if, and only if $\nu(x) = n-1$ (with a further exception in the symplectic and orthogonal cases), proving Theorem 16.1.7, as well.

In the case $r \leq n$ we can easily deduce the lower bound stated in Theorem 16.1.1. However this bound is often negative. With a more detailed study, we will deduce the best possible lower bounds on f_Ω° (notice that $x^G \cap H^\circ \neq \emptyset$, unless $G = \mathrm{O}_n$ and n/t is odd). We define a class of *special* elements, roughly an element of prime order r is special if all the powers of a primitive r-th root of 1 occur as eigenvalues and all the eigenvalues have almost equal multiplicities, see Definitions 18.3.9, 19.3.9 and 20.3.14. We shall prove that for any semisimple element $x \in H^\circ$, $f_\Omega^\circ(x) \geq f_\Omega^\circ(z)$, where z is a special element. The main strategy is to show a "swapping lemma". Roughly, given $x \in H$ a non-special element of order r we define a new element $y \in H$ of order r with $\nu(x) \leq \nu(y) \leq \nu(x) + 1$, and we prove $f_\Omega^\circ(x) \geq f_\Omega^\circ(y)$. For example, assume $G = \mathrm{GL}_n$ and $x = [I_5, \omega I_3, \omega^2] \in H$ is a semisimple element of order 3 then we define $y = [I_4, \omega I_2, \omega^2 I_2]$, increasing by 1 the multiplicity of the eigenvalue with smallest eigenspace and decreasing by 1 the multiplicity of the eigenvalue of largest eigenspace. Notice that we can continue this process getting $z = [I_3, \omega I_3, \omega^2 I_2]$, from y. And, according to our definition, z is a special element. Note that the best possible lower bound for the orthogonal group is given only in the case n/t is even (when n/t is odd we only show that special elements realise it whenever they lie in H°). For $G = \mathrm{O}_n$ and n/t even we observe that if $x \in H \leq G$ then a suitable G-conjugate of x lies in the corresponding $\mathscr{C}_2$-subgroup of Sp_n, say K, and $f_\Omega(x) = f_{\mathrm{Sp}_n/K}(x)$, therefore the results will be quickly deduced from the analogous symplectic results.

In Sections 18.4, 19.4 and 20.4 we prove Theorem 16.1.9, in fact we give slightly better upper bounds. The main tool again is [**14**, Proposition 2.1].

For $r = p$, in order to show that the bounds are close to best possible we will construct a unipotent element $x \in H$ with $\nu(x) = s$ (with some restriction on s if needed) such that $f_\Omega(x)$ is sufficiently near the upper bound. In the case $G = \mathrm{GL}_n$ we shall give slightly more information. For example, if $H = \mathrm{GL}_1 \wr S_t$ we will compute an explicit formula for $f_\Omega(x)$ when $x \in H$ is a unipotent element with $\nu(x) = s$, cf. Proposition 18.4.13.

For semisimple elements we will have a "reversed" version of the swapping lemma. In fact, for any $x \in H \cap \mathcal{V}_{s,r}$ we prove $f_\Omega^\circ(x) \leq f_\Omega^\circ(\bar{x})$, where $\bar{x}$ is defined in (212), (243) or (270). Notice that the argument given does not hold in full generality: as pointed out before for $G = \mathrm{O}_n$, if n/t is odd we may have semisimple elements $x \in G$ whose G-class does not meet H°.

In Sections 18.5, 19.5 and 20.5, we prove Theorem 16.1.11. Also here we shall define special elements in $H \cap \mathcal{V}_{s,r}$. And, again, the key result will be a similar version of the swapping lemma given for global lower bounds on semisimple elements.

In general, we do not give lower bounds on unipotent elements in $H \cap \mathcal{V}_s$. However, for $G = \mathrm{GL}_n$ in the case $t = 2$, we shall derive such lower bounds in Section 18.5.1.

We conclude each chapter with a detailed study of involutions where we prove Theorem 16.1.15.

CHAPTER 17

The $\mathscr{C}_2$-collection

17.1. Structure

Let V be a finite dimensional k-vector space and $G = Cl(V)$, i.e. $\mathrm{GL}(V), \mathrm{Sp}(V)$ or $\mathrm{O}(V)$. Recall that given a vector space V with a form κ we say that two subspaces $U, W \leq V$ are orthogonal and we write $U \perp W$ if, and only if, $(u, w) = 0$ for all $u \in U$ and $w \in W$, where $(\cdot, \cdot)$ is the bilinear form associated to κ. Following the description given in [**41**, Section 1] we define the $\mathscr{C}_2$-family as follows:

$\mathscr{C}_2$ We say that $H \in \mathscr{C}_2$ if there exists a decomposition $V = V_1 \oplus \ldots \oplus V_t$, where the V_i's are mutually orthogonal and isometric and $H = \mathrm{Stab}_G\{V_1, \ldots, V_t\}$.

Remark 17.1.1. If $G = \mathrm{GL}(V)$ then the bilinear form defined on V is the zero form. Hence, any two subspaces of V are orthogonal and, given $U, W \leq V$, they are isometric if, and only if, they are equidimensional.

Remark 17.1.2. In the definition of $\mathscr{C}_2$-subgroups we consider a direct sum decomposition of the vector space V into orthogonal isometric subspaces. Saying that $V_i, V_j \leq V$ are isometric implies that $\mathrm{I}(V_i, \kappa_i) \cong \mathrm{I}(V_j, \kappa_j)$, where κ is the form defined on V and $\kappa_i = \kappa_{|V_i}$ is the restriction of κ on the subspace V_i. Given $G = Cl(V)$ we denote $Cl(V_i) = \mathrm{I}(V_i, \kappa_i)$. In particular, for all i, j, we have $Cl(V_i) \cong Cl(V_j)$.

Remark 17.1.3. Notice that if $G \neq \mathrm{GL}_n$ and $p = 2$ then n/t must be even, cf. Proposition 4.1.11 for orthogonal the case.

Let κ be the zero form or a symplectic form or a non-degenerate quadratic form on V and denote $G = \mathrm{I}(V, \kappa)$. If $V_1, V_2 \leq V$ we write κ_1, κ_2 for the restriction of κ to V_1 and V_2, respectively. We define the following set

$$X = \left\{\{V_1, \ldots, V_t\} : V = \bigoplus_{i=1}^{t} V_i,\ V_i \perp V_j,\ \mathrm{I}(V_i, \kappa_i) \cong \mathrm{I}(V_j, \kappa_j) \text{ for } i \neq j\right\} \tag{159}$$

Then G acts naturally on X, for $g \in G$, $g.\{V_1, \ldots, V_t\} = \{g.V_1, \ldots, g.V_t\}$.

Lemma 17.1.4. *Let $G = Cl(V)$. Then the action of G on X is transitive.*

Proof. First assume $G = \mathrm{GL}(V)$. Then $\{V_1, \ldots, V_t\} \in X$ if, and only if, $V = \bigoplus_i V_i$ and $\dim V_i = \dim V_j$. Let $\{V_1, \ldots, V_t\}$ and $\{V_1', \ldots, V_t'\}$ be in X. Then, given a basis for each V_i and V_i', there exists $g \in G$ that sends the first basis to the second one, hence $g.V_i = V_i'$. The result follows.

Assume now, $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$, i.e. $G = \mathrm{I}(V, \kappa)$ where κ is a symplectic or non-degenerate quadratic form. Let $\{V_1, \ldots, V_t\}$ and $\{V_1', \ldots, V_t'\}$ be in X, denote $\kappa_i = \kappa_{|V_i}$ and $\kappa_i' = \kappa_{|V_i'}$. Then κ_i and κ_i' are symplectic (if κ is symplectic) or non-degenerate quadratic forms (if κ is quadratic). Thus (V_i, κ_i) and (V_i', κ_i') are isometric.

For $1 \leq i \leq t$ let $g_i \colon V_i \to V_i'$ be an isometry. The map $g \colon V \to V$ defined to be $g_{|V_i} = g_i$ is clearly an isometry of V. Hence $g \in \mathrm{I}(V, \kappa)$ and $g.\{V_1, \ldots, V_t\} = \{V_1', \ldots, V_t'\}$. The result follows. *q.e.d.*

Fixing a standard basis for V we gain an identification $G = \mathrm{GL}_n$ or Sp_n or O_n. Let $t > 1$ be a divisor of n. Since the action on X is transitive, we may fix a decomposition, for the purpose of understanding the structure of H.

For $G = \mathrm{GL}_n$, let $\{e_1, \ldots, e_n\}$ be the standard basis for k^n. Then we consider $V_1 = \langle e_1, \ldots, e_{n/t}\rangle, \ldots, V_t = \langle e_{n-n/t+1}, \ldots, e_n\rangle$.

For $G = \mathrm{Sp}_n$, let $\{e_1, \ldots, e_{n/2}, f_1, \ldots, f_{n/2}\}$ be the standard basis for k^n. Then n/t is even and we consider $V_i = \langle e_{(i-1)n/t+1}, \ldots, e_{in/2t}, f_{(i-1)n/t+1}, \ldots, f_{in/2t}, \rangle$ for $1 \leq i \leq t$.

Let $G = \mathrm{O}_n$. If $t = n$ then $H = \mathrm{O}_1 \wr S_t$ and since $\mathrm{O}_1 = \{\pm 1\}$ we have that H is finite. Hence we assume $1 < t < n$. If n is odd then so is n/t. If n is even then n/t may be either even or odd. For example, let V be a 6-dimensional vector spaces endowed with a non-degenerate quadratic form Q and assume $p \neq 2$. Let $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ be the standard basis for V. Define $V_1 = \langle e_1, f_1, e_3 + f_3\rangle$ and $V_1 = \langle e_2, f_2, e_3 - f_3\rangle$. We have that V_1, V_2 are orthogonal and isometric and $V = V_1 \oplus V_2$.

In order to give the explicit structure of $\mathscr{C}_2$-subgroups we need the following result.

Lemma 17.1.5. *Let $G = Cl(V)$. Assume $V = V_1 \oplus \ldots \oplus V_t$ is a decomposition into orthogonal and isometric subspaces, $t > 1$. Then there exists $J \leq G$ with $J \cong S_t$ acting naturally on $\{V_1, \ldots, V_t\}$.*

Proof. If $G = \mathrm{GL}_n$ the conclusion is clear. Assume $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$. It is enough to show that the transpositions $\tau = (i, j)$ lie in G since these transpositions generate S_t. For example, assume $G = \mathrm{Sp}(V) = \mathrm{Sp}_n$ and $\tau = (1, 2)$. Then, in the basis $\{e_1, \ldots, e_{n/2t}, f_1, \ldots, f_{n/2t}, e_{n/2t+1}, \ldots, e_{n/t}, \ldots, \}$, τ is represented by the block matrix

$$\tau = \begin{pmatrix} & I_{n/t} & & & \\ I_{n/t} & & & & \\ & & I_{n/t} & & \\ & & & \ddots & \\ & & & & I_{n/t} \end{pmatrix}$$

Therefore we have $\tau e_i = e_{n/2t+i}$ and $\tau f_i = f_{n/2t+i}$ for $1 \leq i \leq n/2t$, $\tau e_i = e_i$ and $\tau f_i = f_i$ for $i > n/2t$. Hence it is clear that $\tau \in \mathrm{Sp}(V)$. The same argument applies for any other transposition (i, j) and in the case V is endowed with a non-degenerate quadratic form. The result follows. *q.e.d.*

In the next result we describe the structure of $\mathscr{C}_2$-subgroups. Assume $V = V_1 \oplus \ldots \oplus V_t$ is a decomposition into orthogonal and isometric spaces. We define $Cl_i = \mathrm{I}(V_i, \kappa_{|V_i})$. Since $Cl_i \cong Cl_j$ for all $1 \leq i, j \leq t$ and $\dim V_i = n/t$; we denote $Cl_{n/t} = Cl_1$.

Proposition 17.1.6. *Let $G = Cl(V)$ and $V = V_1 \oplus \ldots \oplus V_t$ be a decomposition into orthogonal isometric subspaces. Let $H = \mathrm{Stab}_G\{V_1, \ldots, V_t\}$ be a $\mathscr{C}_2$-subgroup of G. Then $H = Cl_{n/t} \wr S_t$ and $H^\circ = \left(Cl_{n/t}\right)^t$.*

Proof. By Lemma 17.1.4 we can fix the decomposition $\{V_1,\dots,V_t\}$, where the V_i's are defined above, in term of the standard basis for V. First, we consider the pointwise stabiliser $\mathrm{Stab}_G(V_1,\dots,V_t)$. So, $x \in \mathrm{Stab}_G(V_1,\dots,V_t)$ if, and only if, $x.V_i = V_i$. From $x.V_1 = V_1$ we deduce that x is given by

$$x = \begin{pmatrix} x_1 & * \\ 0 & B \end{pmatrix}$$

for some $x_1 \in Cl_{n/t}$. Iterating this argument, we deduce that x is a block diagonal matrix $x = [x_1,\dots,x_t]$ where $x_i \in Cl_{n/t}$. Hence, $\mathrm{Stab}_G(V_1,\dots,V_t) = (Cl_{n/t})^t$.

By the proof of Lemma 17.1.4, for any $(i,j) \in S_t$ we have that the permutation matrix associated to (i,j), that acts on $\{V_1,\dots,V_t\}$ transposing V_i and V_j and fixing all the other subspaces, lies in G, and hence in H. Therefore $S_t \leq \mathrm{Stab}_G\{V_1,\dots,V_t\}$. In particular, $Cl_{n/t} \wr S_t \leq \mathrm{Stab}_G\{V_1,\dots,V_t\}$. Let $x \in \mathrm{Stab}_G\{V_1,\dots,V_t\}$. Then $x.V_{i_1} = V_{j_1},\dots,x.V_{i_l} = V_{j_l}$ and $x.V_i = V_i$ for all $i \notin \{i_1,\dots,i_l\}$. Consequently define $\tau \in S_t$ to be the permutation $(i_1,j_1)\cdots(i_l,j_l)$. Thus $x = [x_1,\dots,x_t]\tau$ hence $x \in Cl_{n/t} \wr S_t$. Therefore $\mathrm{Stab}_G\{V_1,\dots,V_t\} = Cl_{n/t} \wr S_t$.

Observe that we have $Cl_{n/t} \wr S_t = \bigcup_{\tau\in S_t}(Cl_{n/t})^t\tau$ and, for $\tau \neq \tau'$, $(Cl_{n/t})^t\tau \cap (Cl_{n/t})^t\tau' = \emptyset$. Therefore, the connected component H° of H (see Proposition 2.1.2) is given by the direct product $(Cl_{n/t})^t$. *q.e.d.*

We summarise in Table 17.1.1 the dimension of the coset variety $\Omega = G/H$ for H a $\mathscr{C}_2$-subgroup of G. Recall, by (10), $\dim\Omega = \dim G - \dim H$.

G	H	$\dim\Omega$
GL_n	$\mathrm{GL}_{n/t} \wr S_t$	$n^2(1-\frac{1}{t})$
Sp_n	$\mathrm{Sp}_{n/t} \wr S_t$	$\frac{n^2}{2}(1-\frac{1}{t})$
O_n	$\mathrm{O}_{n/t} \wr S_t$	$\frac{n^2}{2}(1-\frac{1}{t})$

Table 17.1.1. Dimension of $\Omega = G/H$ for $H \leq G$ a $\mathscr{C}_2$-subgroup

17.2. Conjugacy classes in $\mathscr{C}_2$-subgroups

Let $G = Cl(V)$ and let H be a $\mathscr{C}_2$-subgroup of G. In this section we shall describe conjugacy classes of prime order elements in H, in order to deduce information on $x^G \cap H$. Recall that any prime order element is either unipotent (if it has order p), or semisimple (if it has order $r \neq p$).

Notice that both G and H are reductive. Hence, for $x \in H$, by Theorem 7.2.2,

$$x^G \cap H = x_1^H \cup \ldots \cup x_l^H \tag{160}$$

for some $l < \infty$. In particular $\dim(x^G \cap H) = \max_i\{\dim x^H\}$.

Furthermore, notice that $H = \bigcup_{\tau\in S_t} H^\circ\tau$, hence

$$x^G \cap H = \bigcup_{\tau\in S_t} (x^G \cap H^\circ\tau) \tag{161}$$

In particular, $x^G \cap H = (x^G \cap H^\circ) \cup (x^G \cap H \setminus H^\circ)$. In view of this last equality we shall divide the analysis into two cases starting with the study of $x^G \cap H^\circ$.

The following is [**14**, Proposition 2.1], for the case $G = \mathrm{Sp}_n$ we record the upper bounds proved in the proof.

Proposition 17.2.1. *Let $G = Cl_n$ and $H = Cl_{n/t} \wr S_t$ be a $\mathscr{C}_2$-subgroup of G. Let $x \in G$ be of prime order r. Then*

$$\dim(x^G \cap H) \leq \left(\frac{1}{t} + \zeta\right) \dim x^G$$

where $\zeta = 0$ if $G = \mathrm{GL}_n$ or O_n, for $G = \mathrm{Sp}_n$ we record ζ in Table 17.2.1, with a representative of x^G.

p	r	x^G-representative	ζ
$\neq 2$	$= 2$	$[I_{n-s}, -I_s]$	$\frac{1}{n}$
$= 2$	$= 2$	$[J_2^s, J_1^{n-2s}]$	$\frac{s}{\dim x^G}\left(1 - \frac{1}{t}\right)$
any	$\neq p, > 2$	$[I_{a_0}, \ldots, \omega^{r-1} I_{a_{r-1}}]$	$\frac{n-a_0}{2\dim x^G}\left(1 - \frac{1}{t}\right)$
$\neq 2$	$= p$	$[J_p^{a_p}, \ldots, J_1^{a_1}]$	$\frac{n - \sum_{i \text{odd}} a_i}{2 \dim x^G}\left(1 - \frac{1}{t}\right)$

Table 17.2.1

17.3. Conjugacy classes in H°

Let $x \in G$ be of prime order r. Assume $x^G \cap H^\circ \neq \emptyset$. Up to G-conjugacy we may assume $x \in H^\circ$. For infinitely many primes we prove that $x^G \cap H = x^G \cap H^\circ$.

Lemma 17.3.1. *Let $x \in H$ of prime order r. If $r > t$ then $x^G \cap H = x^G \cap H^\circ$.*

Proof. Assume $x^G \cap (H \setminus H^\circ) \neq \emptyset$; then there exist $y \in H^\circ$ and $\tau \in S_t$ such that $y\tau \in x^G \cap (H \setminus H^\circ)$. Then τ has order r, since x has order r. But there is no permutation in S_t of order r. The result follows. *q.e.d.*

Remark 17.3.2. We shall prove in Proposition 17.4.11 that in the characteristic zero case, given $x \in G$ unipotent, we have $x^G \cap H = x^G \cap H^\circ$.

Now, we describe conjugacy classes in H°. We start with elements of prime order p, i.e. unipotent elements, and then elements of prime order $r \neq p$.

Recall that $H = Cl_{n/t} \wr S_t$ where $Cl_{n/t}$ is either $\mathrm{GL}_{n/t}$ or $\mathrm{Sp}_{n/t}$ or $\mathrm{O}_{n/t}$ depending on whether $G = \mathrm{GL}_n$ or Sp_n or O_n, respectively. Moreover $H^\circ = (Cl_{n/t})^t$.

Notation. For $x \in H^\circ$ we shall write $x = [x_1, \ldots, x_t]$ with the implicit assumption that each x_i lies in a factor of the direct product defining H°.

17.3.1. Unipotent elements.

Here we assume $x \in H^\circ$ has prime order p.

Fix an integer n and let $\mathcal{P} = \mathcal{P}(n) = \{\lambda : \lambda \vdash n\}$ be the set of partitions of n, write $\lambda = (n^{a_n}, \ldots, 1^{a_1})$, so that $n = \sum_{i=1}^n i a_i$.

Recall, from Section 5.2, there exists a one-to-one correspondence between conjugacy classes of unipotent elements in GL_n and $\mathcal{P}$. If $G = \mathrm{Sp}_n$ or O_n we have a one-to-one correspondence with a proper subset, say $\mathcal{P}_G$, of $\mathcal{P}$. The additional properties defining $\mathcal{P}_G$ can be deduced from Theorem 5.2.1: $\mathcal{P}_{\mathrm{GL}_n} = \mathcal{P}$ and

$$\mathcal{P}_{\mathrm{Sp}_n} = \{\lambda \in \mathcal{P} : a_i \text{ even for all } i \text{ odd}\}$$
$$\mathcal{P}_{\mathrm{O}_n} = \{\lambda \in \mathcal{P} : a_i \text{ even for all } i \text{ even}\}$$

We define the following notation for partitions. Let $\lambda = (n^{a_n}, \dots, 1^{a_1}) \vdash n$. Let $\mu = (n^{b_n}, \dots, 1^{b_1})$ and $\eta = (n^{c_n}, \dots, 1^{c_1})$. We write $\lambda = \mu \oplus \eta$ if $a_i = b_i + c_i$ for all i.

Assume $x \in H^\circ$ has order p. Then, up to G-conjugation, we may assume $x = [J_p^{a_p}, \dots, J_1^{a_1}]$. Moreover, there exists a block decomposition $x = [z_1, \dots, z_t]$ such that $z_i \in Cl_{n/t}$ for all i. Notice that if $x = [y_1, \dots, y_t]$ and there exists i such that y_i and z_i are not $Cl_{n/t}$-conjugate then $[z_1, \dots, z_t]$ and $[y_1, \dots, y_t]$ are not H°-conjugate. For example, $[x_1, x_2]$ and $[x_2, x_1]$ in $H^\circ = (Cl_{n/2})^2$, with x_1 and x_2 not $Cl_{n/t}$-conjugate, are not H°-conjugate, but they are H-conjugate. In Example 17.3.3 we shall observe that, for the purpose of computing dimensions, we don't need to consider the order of the blocks.

In general, $x^G \cap H^\circ$ is a finite union of H°-classes. We have

$$x^G \cap H^\circ = \bigcup_{i=1}^{l} A_i^{H^\circ} \tag{162}$$

for some $l < \infty$, where $A_i \in H^\circ$. Thanks to the previous discussion it is clear that there is a one to one correspondence

$$\{A_1, \dots, A_l\} \longleftrightarrow \{\lambda_1 \oplus \dots \oplus \lambda_t : \lambda_i \vdash n/t \text{ and } \lambda_i \in \mathcal{P}_{Cl_{n/t}} \text{ for all } i\} \tag{163}$$

In order to compute $\dim(x^G \cap H^\circ)$ we need to know all the possible block decompositions $x = [x_1, \dots, x_t]$ where $x_i \in Cl_{n/t}$.

In the following example we illustrate how (162) and (163) arise for unipotent elements.

Example 17.3.3. Let $p \geq 3$, $G = \mathrm{GL}_{21}$ and let $H = \mathrm{GL}_7 \wr S_3$. We consider $x = [J_3^2, J_2^4, J_1^7]$. In order to get all the possible A_i's in (162) we need to find all the possible ways to write x in block form $x = [x_1, x_2, x_3]$. Notice

$$\dim x^{H^\circ} = \dim x_1^{\mathrm{GL}_{n/t}} + \dim x_2^{\mathrm{GL}_{n/t}} + \dim x_3^{\mathrm{GL}_{n/t}}$$

Observe that $[x_1, x_2, x_3]$ and $[x_1, x_3, x_2]$ are not H°-conjugate (whenever $x_2 \notin x_3^{\mathrm{GL}_7}$). However, by the above formula, they have the same H°-dimension. We list the possible decompositions $x = [x_1, x_2, x_3]$ (by the above formula we do not consider permutations of the blocks) together with $\dim x^{H^\circ}$ in Table 17.3.1. We use Theorem 5.2.1 to compute $\dim x_i^{\mathrm{GL}_{n/t}}$.

Thus we get

$$\dim(x^G \cap H^\circ) = \dim[J_3, J_2, J_1^2]^{\mathrm{GL}_7} + \dim[J_3, J_2, J_1^2]^{\mathrm{GL}_7} + \dim[J_2^2, J_1^3]^{\mathrm{GL}_7} = 76$$

Assume $G = \mathrm{GL}_n$. For some unipotent elements of H° we can easily compute $\dim(x^G \cap H^\circ)$ and, in fact, $\dim(x^G \cap H)$.

x_1	x_2	x_3	$\dim x^{H^\circ}$
$[J_3^2, J_1]$	$[J_2^3, J_1]$	$[J_2, J_1^5]$	68
$[J_3^2, J_1]$	$[J_2^2, J_1^3]$	$[J_2^2, J_1^3]$	72
$[J_3, J_2^2]$	$[J_3, J_2^2]$	$[J_1^7]$	60
$[J_3, J_2^2]$	$[J_3, J_2, J_1^2]$	$[J_2, J_1^5]$	70
$[J_3, J_2^2]$	$[J_3, J_1^4]$	$[J_2^2, J_1^3]$	72
$[J_3, J_2, J_1^2]$	$[J_3, J_2, J_1^2]$	$[J_2^2, J_1^3]$	76
$[J_3, J_2, J_1^2]$	$[J_3, J_1^4]$	$[J_2^3, J_1]$	74

Table 17.3.1

Lemma 17.3.4. *Let $G = \mathrm{GL}_n$. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in G$. Assume t divides a_i for all i. Then $x^G \cap H^\circ \neq \emptyset$ and there exists a decomposition $x = [x_1, \ldots, x_t]$ such that $\dim x^{H^\circ} = (1/t) \dim x^G$. In particular,*

$$\dim(x^G \cap H) = \frac{1}{t} \dim x^G$$

Proof. Let $x_i = [J_p^{a_p/t}, \ldots, J_1^{a_1/t}]$ for all i. Then, by Theorem 5.2.1 and Proposition 17.2.1 the result follows. *q.e.d.*

A similar result holds when $G = \mathrm{Sp}_n$ if we assume stronger hypotheses.

Lemma 17.3.5. *Let $G = \mathrm{Sp}_n$. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in G$. Assume t divides a_i for all i and a_i/t is even whenever i is odd. Then $x^G \cap H^\circ \neq \emptyset$ and there exists a decomposition $x = [x_1, \ldots, x_t]$ such that $\dim x^{H^\circ} = (1/t) \dim x^G + (n - \sum_{i \text{ odd}} a_i)(1 - \frac{1}{t})/2$. In particular,*

$$\dim(x^G \cap H) = \frac{1}{t} \dim x^G + \frac{n - \sum_{i \text{ odd}} a_i}{2} \Big(1 - \frac{1}{t}\Big)$$

Proof. Let $x_1 = \ldots = x_t = [J_p^{a_p/t}, \ldots, J_1^{a_1/t}]$. Then $x_i \in \mathrm{Sp}_{n/t}$ since a_i/t is even for i odd. Thus the result follows by an easy calculation using Theorem 5.2.1. Then, applying Proposition 17.2.1 we deduce $\dim(x^G \cap H) = \dim x^{H^\circ}$. *q.e.d.*

We do not have the same result for the orthogonal group.

Remark 17.3.6. Let $G = \mathrm{O}_n$. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in G$. Assume t divides a_i for all i and a_i/t is even whenever i is even. Let $x_1 = \ldots = x_t = [J_p^{a_p/t}, \ldots, J_1^{a_1/t}] \in \mathrm{O}_{n/t}$. Then $[x_1, \ldots, x_t] \in x^G \cap H^\circ$ and an easy calculation leads to

$$\dim x^{H^\circ} = \frac{1}{t} \dim x^G - \frac{n + \sum_{i \text{ odd}} a_i}{2} \Big(1 - \frac{1}{t}\Big)$$

17.3.2. Semisimple elements. Assume $x \in G$ has prime order $r \neq p$. Then, up to G-conjugation, $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ where $\omega \in k$ is a primitive r-th root of unity. Therefore given x in G of prime order $r \neq p$ we have $x^G \cap H^\circ \neq \emptyset$, whenever H° contains a maximal torus. The main aim of this section is to give an explicit formula for $\dim(x^G \cap H^\circ)$ and, in the same spirit as Proposition 17.2.1, derive a lower bound on this dimension of the form $(1/t - \epsilon) \dim x^G$.

Remark 17.3.7. We remind the reader of the conditions on the multiplicities of the eigenvalues of $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$, for $r > 2$, see Theorem 5.3.1.

- If $G = \mathrm{GL}_n$, there are no conditions.
- If $G = \mathrm{Sp}_n$ or O_n then $a_0 \equiv n \pmod 2$ and $a_i = a_{r-i}$ for all $0 < i \leq \frac{r-1}{2}$.

For semisimple elements we can give an explicit formula for $\dim(x^G \cap H^\circ)$. The following result will be the essential tool when we deal with bounds on $f_\Omega(x)$ for x semisimple, see for example Lemma 18.3.11.

Theorem 17.3.8. *Let $x \in G$ be of primer order r. Assume $x^G \cap H^\circ \neq \emptyset$. Write $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ when $r > 2$ and $x = [I_s, -I_{n-s}]$ with $s \leq n/2$ for $r = 2$.*

(i) If $G = \mathrm{GL}_n$ then

$$\dim(x^G \cap H^\circ) = \frac{n^2}{t} - n + \sum_{i=0}^{r-1} \Big(\Big\lfloor \frac{a_i}{t} \Big\rfloor^2 t + (t - 2a_i) \Big\lfloor \frac{a_i}{t} \Big\rfloor \Big)$$

(ii) If $G = \mathrm{Sp}_n$ and $r \neq 2$, then

$$\dim(x^G \cap H^\circ) = \frac{n^2}{2t} - a_0 + 2\Big(\Big\lfloor \frac{a_0}{2t} \Big\rfloor^2 t + (t - a_0) \Big\lfloor \frac{a_0}{2t} \Big\rfloor \Big) + \frac{1}{2} \sum_{i=1}^{r-1} \Big(\Big\lfloor \frac{a_i}{t} \Big\rfloor^2 t + (t - 2a_i) \Big\lfloor \frac{a_i}{t} \Big\rfloor \Big)$$

Assume $r = 2$. Write $s/2 = at + b$ where $0 \leq b < t$. Then

$$\dim(x^G \cap H^\circ) = \frac{s(n-s)}{t} - \frac{4b(t-b)}{t}$$

(iii) Assume $G = \mathrm{O}_n$ and $r \neq 2$. If n/t is even, then

$$\dim(x^G \cap H^\circ) = \frac{n^2}{2t} - n + 2\Big(\Big\lfloor \frac{a_0}{2t} \Big\rfloor^2 t + (t - a_0) \Big\lfloor \frac{a_0}{2t} \Big\rfloor \Big) + \frac{1}{2} \sum_{i=1}^{r-1} \Big(\Big\lfloor \frac{a_i}{t} \Big\rfloor^2 t + (t - 2a_i) \Big\lfloor \frac{a_i}{t} \Big\rfloor \Big)$$

If n/t is odd then

$$\begin{aligned} \dim(x^G \cap H^\circ) &= \frac{n^2}{2t} - n + \frac{a_0}{2} + \frac{1}{2} \sum_{i=1}^{r-1} \Big(\Big\lfloor \frac{a_i}{t} \Big\rfloor^2 t + (t - 2a_i) \Big\lfloor \frac{a_i}{t} \Big\rfloor \Big) \\ &+ 2\Big(\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor^2 t + (2t - a_0) \Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor \Big) - \frac{3}{2}(a_0 - t) \end{aligned}$$

Assume $r = 2$ and write $s = ct + b$ where $0 \leq b < t$. Then

$$\dim(x^G \cap H^\circ) = \frac{s(n-s)}{t} - \frac{b(t-b)}{t}$$

In order to prove Theorem 17.3.8 we need some general lemmas that establish a link between the decomposition $x = [x_1, \ldots, x_t]$ and $\dim(x^G \cap H^\circ)$.

For $x = [x_1, \ldots, x_t] \in H^\circ$ of order r we establish the following notation

$$x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \ldots, \omega^{r-1} I_{a_{i,r-1}}] \tag{164}$$

As observed in Example 17.3.3 the order of the x_i's is not relevant for the purpose of computing the dimension of x^{H°.

We introduce the following claim, which will be the essential tool to show Theorem 17.3.8. Let $G = Cl_n$ and r be a prime. We define $\iota = 1$ if $G = \mathrm{GL}_n$ or $(G, r) = (\mathrm{O}_n, 2)$, and $\iota = 2$ otherwise.

Claim. Let $x = [x_1, \ldots, x_t] \in H^\circ$ be of order r. We have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ if, and only if, $|a_{i,l} - a_{j,l}| \leq 1$ and $|a_{i,0} - a_{j,0}| \leq \iota$ for all $1 \leq l \leq r-1$ and $1 \leq i, j \leq t$.

After proving the claim we can easily write down a block decomposition of x for which $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ and the proof of Theorem 17.3.8 will reduce to an easy calculation. The following technical lemma will be a useful tool in order to prove the claim.

Lemma 17.3.9. *Assume $r > 2$ if $G \neq \mathrm{GL}_n$. Let $x = [x_1, \ldots, x_t] \in H^\circ$ as in (164) be of order r. Assume there exists $m \in \{0, \ldots, r-1\}$ and blocks x_i, x_j such that $a_{i,m} = a_{j,m} + 2 + h$, for $h \geq 0$ or, $h \geq 2$ if $m = 0$ and $G \neq \mathrm{GL}_n$.*

(i) If $G = \mathrm{GL}_n$ then there exists $l \in \{0, 1, \ldots, r-1\} \setminus \{m\}$ such that $a_{j,l} \neq 0$ and $a_{i,l} - a_{j,l} < h$.

(ii) Let $G = \mathrm{Sp}_n$ or O_n.

  (a) If $m = 0$, there exists $l \in \{1, \ldots, r-1\}$ such that $a_{j,l} \neq 0$ and $a_{i,l} - a_{j,l} < h - 2$.

  (b) If $m \neq 0$, one of the following holds.

  * *$a_{j,0} \neq 0$ and $a_{i,0} - a_{j,0} < -1$; or,*
  * *$a_{j,0} = 0$ and there exists $l \in \{1, \ldots, r-1\} \setminus \{m\}$ such that $a_{i,l} - a_{j,l} < h$.*

Proof. In this proof we shall use the following notation: $a_{i,h} = b_h$ and $a_{j,h} = c_h$ for $0 \leq h \leq r-1$.

Case (i). If $G = \mathrm{GL}_n$, we may assume, without loss of generality, $m = 0$. By the hypothesis, $b_0 = c_0 + 2 + h$, in particular, $c_0 < n/t$. Hence, there exists $i > 0$ such that $c_i > 0$. Up to relabelling the eigenvalues, and not considering c_0 (that may be 0), we may assume that $c_1, \ldots, c_l$ are the only non-zero multiplicities in the block x_j. Suppose that $b_i - c_i \geq h$ for all $1 \leq i \leq l$, then summing over i we have

$$(b_0 - c_0) + \sum_{i=1}^{l} (b_i - c_i) > (h+1) + lh \geq 1 \tag{165}$$

Notice that $c_0 + \sum_i c_i = n/t$ and $b_0 + \sum_i b_i \leq n/t$. Hence $(b_0 - c_0) + \sum_{i=1}^{l}(b_i - c_i) \leq 0$; this, together with (165), is absurd. The result follows.

Case (ii). Assume $G = \mathrm{Sp}_n$ or O_n. First assume $m = 0$, so that $b_0 - c_0 = 4 + h$ for some $h \geq 0$. We use the same argument as above. We have $n/t = c_0 + c_1 + \ldots + c_l$ where $c_1, \ldots, c_l$ are the only non-zero multiplicities (possibly together with c_0). Thus $\sum_i c_i = n/t$ and $n/t \geq b_0 + \sum_{i=1}^{l} b_i$. Assume $b_i - c_i \geq h$ for all i. Hence,

$$0 \geq (b_0 - c_0) + \sum_{i=1}^{l} (b_i - c_i) \geq 4 + h + lh > 0$$

So, there exists i such that $c_i \neq 0$ and $b_i - c_i < h$ (where $h \geq 0$).

In the case $m \neq 0$ we have $b_m - c_m \geq h + 2$. We distinguish two cases depending on whether or not $c_0 = 0$.

If $c_0 = 0$ then $b_0 - c_0 \geq 0$. We use the same argument as above. Assume $b_i - c_i \geq h$ for all $i \neq 0, m$. (Notice that we may always assume $m > l$). We have $n/t = c_0 + c_m + \sum_{i=1}^{l} c_i$, where $c_i \neq 0$, and $n/t \geq b_0 + b_m + \sum_{i=1}^{l} b_i$. Thus

$$0 \geq (b_0 - c_0) + (b_m - c_m) + \sum_{i=1}^{l} (b_i - c_i) \geq h + 2 + lh > 0$$

which is absurd. Hence $b_i - c_i < h$ for some i.

Assume $c_0 \neq 0$. Then either there exists i such that $b_i - c_i < h$, in which case the result follows, or $b_i - c_i \geq h$ for all i. In the latter case, with the usual argument, we have $n/t = c_0 + c_m + \sum_{i=1}^{l} c_i$ and $n/t \geq b_0 + b_m + \sum_{i=1}^{l} b_i$ hence

$$0 \geq (b_0 - c_0) + (b_m - c_m) + \sum_{i=1}^{l}(b_i - c_i) \geq (b_0 - c_0) + h + 2 + lh \geq (b_0 - c_0) + 2$$

therefore $b_0 - c_0 \leq -2 < -1$. *q.e.d.*

Here we prove one implication of the claim. Recall, if $x = [x_1, \ldots, x_t] \in H^\circ$ then $\dim x^{H^\circ} = \sum_i \dim x_i^{Cl_{n/t}}$.

Proposition 17.3.10. *Let $x = [x_1, \ldots, x_t] \in H^\circ$ be of order r. Assume $\dim x^{H^\circ} = \dim(x^G \cap H^\circ)$ and say $a_{i,h}$ the multiplicity of ω^h in x_i.*

(i) If $r \neq 2$ then

$$|a_{h,i} - a_{l,i}| \leq 1$$

for all $1 \leq h, l \leq t$ and $1 \leq i \leq r-1$. In addition, $|a_{h,0} - a_{l,0}| \leq \iota$ for all h, l where $\iota = 1$ if $G = \mathrm{GL}_n$ and $\iota = 2$ otherwise.

(ii) If $r = 2$ then $x_i \in [I_{n/t-s_i}, -I_{s_i}]^{Cl_{n/t}}$ and

$$|s_i - s_j| \leq 1, \text{ if } G = \mathrm{GL}_n, \mathrm{O}_n$$
$$|s_i - s_j| \leq 2, \text{ if } G = \mathrm{Sp}_n$$

Proof. We prove the contrapositive. Assume there exist blocks $x_l, x_{l'}$ in which the multiplicities of the eigenvalue ω^i satisfy $a_{l,i} - a_{l',i} \geq 2$, i.e. $a_{l,i} = a_{l',i} + 2 + h$ for some $h \geq 0$. We shall construct $y = [y_1, \ldots, y_t] \in x^G \cap H^\circ$ such that $\dim y^{H^\circ} > \dim x^{H^\circ}$, which is absurd, as $\dim x^{H^\circ} = \dim(x^G \cap H^\circ)$.

If $(l, l') \neq (1, 2)$ we permute the blocks. So we assume $l = 1, l' = 2$. For $r \neq 2$ we denote the multiplicities $a_{1,j} = b_j$ and $a_{2,j} = c_j$ for $0 \leq j \leq r-1$. Therefore $b_i = c_i + 2 + h$ or $b_0 = c_0 + 4 + h$ if $G = \mathrm{Sp}_n, \mathrm{O}_n$ (recall that $b_0, c_0 \equiv n/t \pmod 2$) for some $h \geq 0$. We split the analysis into different cases.

Case 1. Assume $G = \mathrm{GL}_n$.
Up to relabelling the eigenvalues we may assume $i = 0$. Hence $b_0 = c_0 + 2 + h$. Up to conjugation,

$$x_1 = [I_{c_0+2+h}, \omega I_{b_1}, \ldots, \omega^{r-1} I_{b_{r-1}}]$$
$$x_2 = [I_{c_0}, \omega I_{c_1}, \ldots, \omega^{r-1} I_{c_{r-1}}]$$

By Lemma 17.3.9, there exists $i > 0$ such that $c_i \neq 0$ and $b_i - c_i < h$. Up to relabelling the eigenvalue we may assume $i = 1$. Thereby we define $y = [y_1, y_2, x_3, \ldots, x_t]$, where

$$y_1 = [I_{c_0+1+h}, \omega I_{b_1+1}, \omega^2 I_{b_2}, \ldots, \omega^{r-1} I_{b_{r-1}}]$$
$$y_2 = [I_{c_0+1}, \omega I_{c_1-1}, \omega^2 I_{c_2}, \ldots, \omega^{r-1} I_{c_{r-1}}]$$

Clearly x and y are G-conjugate but not H°-conjugate, i.e. $x^{H^\circ} \cap y^{H^\circ} = \emptyset$. After a straightforward computation, using Theorem 5.3.1, we see that $\dim y^{H^\circ} - \dim x^{H^\circ} =$

$2(c_1 - b_1 + h) > 0$. This implies $\dim y^{H^\circ} > \dim(x^G \cap H^\circ)$ which is absurd. The result follows (even if $r = 2$).

Case 2. Assume $G = \mathrm{Sp}_n$ or O_n, and $r \neq 2$.
Recall we are assuming $b_i = c_i + 2 + h$ for some $i \in \{1, \dots, r-1\}$ and $h \geq 0$ or $b_0 = c_0 + 4 + h$ for some $h \geq 0$. We have two cases depending whether $i = 0$ or $i \neq 0$.

Assume $b_0 - c_0 = 4 + h$ for some $h \geq 0$. By Lemma 17.3.9 (note that here we assume $h \geq 0$) there exists $j > 0$ such that $c_j \neq 0$ and $b_j - c_j < h$. Also here, we may assume $j = 1$. We define $y = [y_1, y_2, x_3, \dots, x_t]$, where

$$y_1 = [I_{b_0-2}, \omega I_{b_1+1}, \omega^2 I_{b_2}, \dots, \omega^{r-1} I_{b_1+1}]$$
$$y_2 = [I_{c_0+2}, \omega I_{c_1-1}, \omega^2 I_{c_2}, \dots, \omega^{r-1} I_{c_1-1}]$$

Again: $y \in x^G$ and $y^{H^\circ} \cap x^{H^\circ} = \emptyset$. In both cases $G = \mathrm{Sp}_n$ and O_n we get:

$$\dim y^{H^\circ} - \dim x^{H^\circ} = 2(b_0 - c_0 - (b_1 - c_1) - 3) = 2(h + 1 - (b_1 - c_1)) > 2$$

where the last inequality follows from $b_1 - c_1 < h$. As above, the result follows.

Assume $b_i - c_i = 2 + h$ for some $i \neq 0$. Up to relabel the eigenvalues, we assume $i = 1$. Then, by Lemma 17.3.9, either there exists $j > 0$ such that $c_j \neq 0$ and $b_j - c_j < h$ or $c_0 \neq 0$ and $b_0 - c_0 < -1$. In the former case we assume $j = 2$. Thereby we define $y = [y_1, y_2, x_3, \dots, x_t]$, where

$$y_1 = [I_{b_0}, \omega I_{b_1-1}, \omega^2 I_{b_2+1}, \omega^3 I_{b_3}, \dots, \omega^{r-1} I_{b_1-1}]$$
$$y_2 = [I_{c_0}, \omega I_{c_1+1}, \omega^2 I_{c_2-1}, \omega^3 I_{c_3}, \dots, \omega^{r-1} I_{c_1+1}]$$

As above we have $y \in x^G$ and $y^{H^\circ} \cap x^{H^\circ} = \emptyset$. In both cases $G = \mathrm{Sp}_n, \mathrm{O}_n$ we compute

$$\dim y^{H^\circ} - \dim x^{H^\circ} = 2(b_1 - c_1 - 1) - 2(b_2 - c_2 + 1) = 2(h - b_2 + c_2) > 0$$

where the inequality follows from $b_2 - c_2 < h$. Again, the result follows.

Now assume $b_1 - c_1 = 2 + h$ $(h \geq 0)$, $c_0 \neq 0$ and $b_0 - c_0 < -1$. Then we define $y = [y_1, y_2, x_3 \dots, x_t]$, where

$$y_1 = [I_{b_0+2}, \omega I_{b_1-1}, \omega^2 I_{b_2}, \dots, \omega^{r-1} I_{b_1-1}]$$
$$y_2 = [I_{c_0-2}, \omega I_{c_1+1}, \omega^2 I_{c_2}, \dots, \omega^{r-1} I_{c_1+1}]$$

Again, $y \in x^G$ and $y^{H^\circ} \cap x^{H^\circ} = \emptyset$. For $G = \mathrm{Sp}_n$ and O_n we get

$$\dim y^{H^\circ} - \dim x^{H^\circ} = 2(b_1 - c_1 - (b_0 - c_0) - 3) = 2(h - 1 - (b_0 - c_0)) > 2h \geq 0$$

where the inequality follows from $b_0 - c_0 < -1$. Again, the result follows.

Case 3(i). Now assume $G = \mathrm{Sp}_n$ and x is an involution.
Up to conjugation, $x = [I_{n-s}, -I_s]$ where $s \equiv n \pmod 2$. Let $x = [x_1, \dots, x_t]$ with $x_i = [I_{n/t-s_i}, -I_{s_i}]$. Assume $s_1 - s_2 > 2$. Define $y = [y_1, y_2, x_3, \dots, x_t]$ where

$$y_1 = [I_{n/t-s_1+2}, -I_{s_1-2}]$$
$$y_2 = [I_{n/t-s_2-2}, -I_{s_2+2}]$$

So $y \in x^G$ and $y^{H^\circ} \cap x^{H^\circ} = \emptyset$. We compute

$$\dim y^{H^\circ} - \dim x^{H^\circ} = 4(s_1 - s_2 - 2) > 0$$

The result follows.

Case 3(ii). If $G = \mathrm{O}_n$ and $x = [I_{n-s}, -I_s]$ is an involution then s does not need to be even or odd (cf. Section 5.3). As in the previous case write $x = [x_1, \ldots, x_t] \in H^\circ$ and assume $s_1 - s_2 \geq 2$. We define $y = [y_1, y_2, x_3, \ldots, x_t] \in H^\circ$ where

$$y_1 = [I_{n/t-s_1+1}, -I_{s_1-1}]$$
$$y_2 = [I_{n/t-s_2-1}, -I_{s_2+1}]$$

So $y \in x^G$ and $y^{H^\circ} \cap x^{H^\circ} = \emptyset$. Again, we compute

$$\dim y^{H^\circ} - \dim x^{H^\circ} = 2(s_1 - s_2 - 1) \geq 2 > 0$$

The result follows. *q.e.d.*

Remark 17.3.11. Let $G = \mathrm{Sp}_n\ \mathrm{O}_n$. Let $x = [x_1, \ldots, x_t] \in H^\circ$ be a semisimple element of odd prime order or a semisimple involution if $G = \mathrm{Sp}_n$. Assume $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. By Proposition 17.3.10 we have $|a_{i,0} - a_{j,0}| \leq 2$ for $1 \leq i, j \leq t$. Since $a_{i,0} \equiv n/t \pmod 2$ we have $a_{i,0} - a_{j,0} \in \{0, \pm 2\}$.

Only for the purpose of uniform notation in the following (Lemma 17.3.12) if x is an involution we write $x = [I_{a_0}, -I_{a_1}]$. Notice that $a_1 = n - a_0$. If $x = [x_1, \ldots, x_t] \in H^\circ$ we denote each x_i as in (164).

Lemma 17.3.12. *Let $x \in G$ be of order r. Assume $x = [x_1, \ldots, x_t] \in H^\circ$ is such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.*

(i) If $G = \mathrm{GL}_n$ then $a_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$ for $1 \leq i \leq t$ and $0 \leq h \leq r-1$;
(ii) If $G = \mathrm{Sp}_n$ or O_n (assume $r \neq 2$ if $G = \mathrm{O}_n$) then $a_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$ for $1 \leq i \leq t$ and $1 \leq h \leq r-1$; $a_{i,0} \in \{2\lfloor a_0/2t \rfloor, 2\lfloor a_0/2t \rfloor + 2\}$;
(iii) If $G = \mathrm{O}_n$ and $r = 2$ then $a_{i,0} \in \{\lfloor a_0/t \rfloor, \lfloor a_0/t \rfloor + 1\}$ for $1 \leq i \leq t$.

Proof. Let us recall Proposition 17.3.10. If $G = \mathrm{GL}_n$ then $|a_{i,h} - a_{j,h}| \leq 1$ for all $1 \leq i, j \leq t$ and $0 \leq h \leq r-1$. The same holds if $(G, r) = (\mathrm{O}_n, 2)$. If $G = \mathrm{Sp}_n, \mathrm{O}_n$ then $|a_{i,h} - a_{j,h}| \leq 1$ for $h \neq 0$ and $|a_{i,0} - a_{j,0}| \leq 2$, provided $(G, r) \neq (\mathrm{O}_n, 2)$. Recall, in the notation (164), that $a_h = \sum_{i=1}^t a_{i,h}$.

Assume $G = \mathrm{GL}_n$. For simplicity let us fix $h = 0$. We may assume $a_{1,0} = \max_i\{a_{i,0}\}$ and $a_{2,0} = \min_i\{a_{i,0}\}$. Then $a_{1,0} - a_{2,0} \in \{0, 1\}$. If $a_{1,0} - a_{2,0} = 0$ then for all i we have $a_{i,0} = a_{1,0}$ and so $a_{i,0} = a_0/t$. Let us assume $a_{1,0} - a_{2,0} = 1$. So $a_{2,0} = a_{1,0} - 1$. For all i, $a_{1,0} - 1 \leq a_{i,0} \leq a_{1,0}$. Thus summing over $i = 1, \ldots, t$ we get

$$(a_{1,0} - 1)t + 1 \leq a_0 \leq a_{1,0}t - 1 = (a_{1,0} - 1)t + (t - 1)$$

Thus $a_{1,0} - 1 = \lfloor a_0/t \rfloor$. The same argument holds for $h \neq 0$. The result follows.

Assume $G = \mathrm{Sp}_n$ or O_n and $r \neq 2$. The previous argument applies for $h \neq 0$. Hence, assume $h = 0$. Then, for all $1 \leq i, j \leq t$, we have $a_{i,0} - a_{j,0} \in \{0, \pm 2\}$, by Remark 17.3.11. Let $a_{1,0} = \max_i\{a_{i,0}\}$ and $a_{2,0} = \min\{a_{i,0}\}$. If $a_{1,0} = a_{2,0}$ then $a_{i,0} = a_0/t$ for all i. If $a_{1,0} - a_{2,0} = 2$ then for all i we have $a_{1,0}/2 - 1 \leq a_{i,0}/2 \leq a_{i,0}/2$. Thus,

$$\Big(\frac{a_{1,0}}{2} - 1\Big)t + 1 \leq \frac{a_0}{2} \leq \frac{a_{1,0}}{2}t - 1 = \Big(\frac{a_{1,0}}{2} - 1\Big)t + (t-1)$$

and $a_{1,0}/2 - 1 = \lfloor a_0/2t \rfloor$. Notice that this argument applies also in the case $(G, r) = (\mathrm{Sp}_n, 2)$.

In the case $(G, r) = (\mathrm{O}_n, 2)$ the same argument as that used for GL_n applies. The result follows. *q.e.d.*

Now we can prove the claim.

Proposition 17.3.13. *Let $x \in G$ be of order r. Assume $x = [x_1, \ldots, x_t] \in H^\circ$. Then $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ if, and only if,*

$$|a_{i,h} - a_{j,h}| \leq 1, \quad |a_{i,0} - a_{j,0}| \leq \iota$$

for all $1 \leq i, j \leq t$ and all $1 \leq h \leq r-1$; and $\iota = 1$ if $G = \mathrm{GL}_n$ or $(G, r) = (\mathrm{O}_n, 2)$, and $\iota = 2$ otherwise.

Proof. If $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ then, by Proposition 17.3.10, the result follows.

Conversely, let us assume $G = \mathrm{GL}_n$ and $|a_{i,h} - a_{j,h}| \leq 1$ for all i, j, h. Then by Lemma 17.3.12 we have $a_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$. Notice that $\dim(x^G \cap H^\circ) \geq \dim x^{H^\circ}$. There exists $z = [z_1, \ldots, z_t] \in x^G \cap H^\circ$ such that $\dim(x^G \cap H^\circ) = \dim z^{H^\circ}$. Say $b_{i,h}$ is the multiplicity of ω^h in the block z_i. Then, by Proposition 17.3.10, $|b_{i,h} - b_{j,h}| \leq 1$ for all i, j, h and by Lemma 17.3.12 we have $b_{i,h} \in \{\lfloor a_h/t \rfloor, \lfloor a_h/t \rfloor + 1\}$. It follows that, up to a permutation of the blocks, $C_{\mathrm{GL}_{n/t}}(x_i) \cong C_{\mathrm{GL}_{n/t}}(z_i)$, thus $\dim x^{H^\circ} = \dim z^{H^\circ}$. Therefore $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Similarly if $G \neq \mathrm{GL}_n$. *q.e.d.*

Now, we can prove Theorem 17.3.8.

Proof of Theorem 17.3.8. Thanks to Proposition 17.3.13, given $x \in H^\circ$ we can find a suitable decomposition $x = [x_1, \ldots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. We divide the analysis according if $G = \mathrm{GL}_n, \mathrm{Sp}_n$ or O_n.

Case 1. $G = \mathrm{GL}_n$.
Let $x = [I_{a_0}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in G$ be of order r. Write $a_i = c_i t + b_i$ where $0 \leq b_i < t$. Let $x = [x_1, \ldots, x_t]$, where we define

$$x_i = [I_{c_0 + \epsilon_{i,0}}, \omega I_{c_1 + \epsilon_{i,1}}, \ldots, \omega^{r-1} I_{c_{r-1} + \epsilon_{i,r-1}}] \tag{166}$$

where for every $j \in \{0, \ldots, r-1\}$ we have $\sum_{i=1}^t \epsilon_{i,j} = b_j$ and $\epsilon_{i,j} \in \{0, 1\}$ for all i and j. Then, by Proposition 17.3.13, we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Using Theorem 5.3.1, we compute

$$\begin{aligned}
\dim x^{H^\circ} &= \sum_{i=1}^{t} \Big(\big(\frac{n}{t}\big)^2 - \sum_{j=0}^{r-1} (c_j + \epsilon_{i,j})^2 \Big) = \frac{n^2}{t} - \sum_i \sum_j (c_j^2 + 2c_j \epsilon_{i,j} + \epsilon_{i,j}^2) \\
&= \frac{n^2}{t} - \frac{1}{t} \sum_j \Big(c_j^2 t^2 + 2 c_j b_j t + b_j t \Big) = \frac{n^2}{t} - \frac{1}{t} \sum_j \Big(a_j^2 + t b_j - b_j^2 \Big)
\end{aligned}$$

$$= \frac{n^2}{t} - n + \sum_{j=0}^{r-1} \Big(t \Big\lfloor \frac{a_j}{t} \Big\rfloor^2 + (t - 2a_j) \Big\lfloor \frac{a_j}{t} \Big\rfloor \Big) \tag{167}$$

where we used $\epsilon_{i,j}^2 = \epsilon_{i,j}$, $\sum_i \epsilon_{i,j} = b_j$, $b_j = a_j - c_j t$ and $c_j = \lfloor a_j/t \rfloor$.

Case 2. $G = \mathrm{Sp}_n$.
Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1}] \in G$. First assume $r > 2$. Write $a_0/2 = c_0 t + b_0$ and $a_i = c_i t + b_i$, for $i > 0$, where $0 \leq b_i < t$. Let $x = [x_1, \ldots, x_t]$, where we define

$$x_i = [I_{2c_0+2\epsilon_{i,0}}, \omega I_{c_1+\epsilon_{i,1}}, \ldots, \omega^{r-1} I_{c_{r-1}+\epsilon_{i,r-1}}] \tag{168}$$

with the same conditions given for (166). Then, by Proposition 17.3.13, $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Thus

$$\begin{aligned}
\dim x^{H^\circ} &= \sum_{i=1}^{t}\Big(\frac{n}{2t}\Big(\frac{n}{t}+1\Big) - (c_0+\epsilon_{i,0})(2c_0+2\epsilon_{i,0}+1) - \frac{1}{2}\sum_{j=1}^{r-1}(c_j+\epsilon_{i,j})^2\Big)\\
&= \frac{n}{2}\Big(\frac{n}{t}+1\Big) - \frac{a_0^2}{2t} - \frac{a_0}{2} - \frac{2b_0}{t}(t-b_0) - \frac{1}{2t}\sum_j\big(a_j^2 + b_j(t-b_j)\big)\\
&= \frac{n^2}{2t} - a_0 + 2\Big(\Big\lfloor\frac{a_0}{2t}\Big\rfloor^2 t + (t-a_0)\Big\lfloor\frac{a_0}{2t}\Big\rfloor\Big) + \frac{1}{2}\sum_{j=1}^{r-1}\Big(\Big\lfloor\frac{a_j}{t}\Big\rfloor^2 t + (t-2a_j)\Big\lfloor\frac{a_j}{t}\Big\rfloor\Big)
\end{aligned}$$

Now assume $r = 2$. Up to conjugation, $x = [I_{n-s}, -I_s]$ with s even. Write $s/2 = ct+b$ where $0 \leq b < t$. Let $x = [x_1, \ldots, x_t]$, where we define

$$x_i = [I_{n/t-2c-2\epsilon_i}, -I_{2c+2\epsilon_i}] \tag{169}$$

where $\epsilon_i \in \{0,1\}$ and $\sum_i \epsilon_i = b$. Then by Proposition 17.3.13 we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Using Theorem 5.3.1, we compute

$$\dim x^{H^\circ} = \sum_{i=1}^{t}\Big(\frac{n}{t} - 2c_i - 2\epsilon_i\Big)(2c_i+2\epsilon_i) = \frac{s(n-s)}{t} - \frac{4b(t-b)}{t}$$

Case 3. $G = \mathrm{O}_n$.
We use the same notation for the case $G = \mathrm{Sp}_n$. First assume $r > 2$. Assume n/t is even, so n is even and hence a_0 is even. Let $a_0/2 = c_0 t + b_0$ and $a_j = c_j t + b_j$ for $j \geq 1$. We define x_i as in (168) and we compute

$$\begin{aligned}
\dim x^{H^\circ} &= \sum_{i=1}^{t}\Big(\frac{n}{2t}\Big(\frac{n}{t}-1\Big) - (c_0+\epsilon_{i,0})(2c_0+2\epsilon_{i,0}-1) - \frac{1}{2}\sum_{j=1}^{r-1}(c_j+\epsilon_{i,j})^2\Big)\\
&= \frac{n^2}{2t} - n + 2\Big(\Big\lfloor\frac{a_0}{2t}\Big\rfloor^2 t + (t-a_0)\Big\lfloor\frac{a_0}{2t}\Big\rfloor\Big) + \frac{1}{2}\sum_{j=1}^{r-1}\Big(\Big\lfloor\frac{a_j}{t}\Big\rfloor^2 t + (t-2a_j)\Big\lfloor\frac{a_j}{t}\Big\rfloor\Big)
\end{aligned}$$

Now assume n/t is odd. Recall that n is even if, and only if, a_0 is even. Then, for n even we deduce that t is even; if n is odd then t is odd. In particular $a_0 - t$ is even.

Let $[y_1, \ldots, y_t] \in x^G \cap H^\circ$ be any decomposition. Then $y_i \in \mathrm{O}_{n/t}$, and since n/t is odd we deduce that the multiplicity of the 1-eigenvalue in each y_i is odd, in particular it is always positive. Therefore $a_0 \geq t$. Let $\frac{a_0-t}{2} = c_0 t + b_0$ where $0 \leq b_0 < t$ and $c_0 = \lfloor \frac{a_0-t}{2t} \rfloor$. For $1 \leq i \leq t$, define

$$x_i = [I_{2c_0+2\epsilon_{i,0}+1}, \omega I_{c_1+\epsilon_{i,1}}, \ldots, \omega^{r-1} I_{c_{r-1}+\epsilon_{i,r-1}}]$$

where $\sum_i \epsilon_{i,0} = b_0$ and $\epsilon_{i,j} = \epsilon_{i,r-j}$ for $j \neq 0$. Then $x_i \in \mathrm{O}_{n/t}$ for all i and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, by Proposition 17.3.13. We compute

$$\begin{aligned}\dim x^{H^\circ} &= \sum_{i=1}^{t}\Big(\frac{n}{2t}\Big(\frac{n}{t}-1\Big) - (c_0+\epsilon_{i,0})(2c_0+2\epsilon_{i,0}+1) - \frac{1}{2}\sum_{j=1}^{r-1}(c_j+\epsilon_{i,j})^2\Big)\\ &= \frac{n^2}{2t} - n + \frac{a_0}{2} + 2\Big(\Big\lfloor\frac{a_0-t}{2t}\Big\rfloor^2 t + (2t-a_0)\Big\lfloor\frac{a_0-t}{2t}\Big\rfloor\Big) - \frac{3}{2}(a_0-t)\\ &+ \frac{1}{2}\sum_{j=1}^{r-1}\Big(\Big\lfloor\frac{a_j}{t}\Big\rfloor^2 t + (t-2a_j)\Big\lfloor\frac{a_j}{t}\Big\rfloor\Big)\end{aligned}$$

Now assume $x = [I_{n-s}, -I_s]$ is an involution. Let $s = ct + b$ where $0 \leq b < t$. Define

$$x_i = [I_{n/t-c-\epsilon_i}, -I_{c+\epsilon_i}] \tag{170}$$

where $\epsilon_i \in \{0,1\}$ and $\sum_i \epsilon_i = b$. Then $x = [x_1, \ldots, x_t] \in H^\circ$ and, again, by Proposition 17.3.13 we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Thanks to Theorem 5.3.1, we compute

$$\dim x^{H^\circ} = \sum_{i=1}^{t} \dim x_i^{\mathrm{O}_{n/t}} = \sum_i \Big(\frac{n}{t} - c - \epsilon_i\Big)(c+\epsilon_i) = \frac{s(n-s)}{t} - \frac{b(t-b)}{t}$$

q.e.d.

Remark 17.3.14. From the computations done in the proof of Theorem 17.3.8 we deduce also the following formulae.

- Assume $G = \mathrm{GL}_n$. Let $x \in G$ be a semisimple element of prime order r. Say $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$ and write $a_i = c_i t + b_i$, where $0 \leq b_i < t$. Then, by the proof of Theorem 17.3.8 we have

$$\dim(x^G \cap H^\circ) = \frac{1}{t}\dim x^G - \frac{1}{t}\sum_{i=0}^{r-1} b_i(t-b_i) \tag{171}$$

- Assume $G = \mathrm{Sp}_n$. For x of order $r > 2$ we write $a_0/2 = c_0 t + b_0$ and $a_i = c_i t + b_i$, for $i > 0$. Then

$$\dim(x^G \cap H^\circ) = \frac{\dim x^G}{t} + \frac{n-a_0}{2}\Big(1-\frac{1}{t}\Big) - \frac{1}{t}\Big(2b_0(t-b_0) + \frac{1}{2}\sum_{j=1}^{r-1} b_j(t-b_j)\Big) \tag{172}$$

- Assume $G = \mathrm{O}_n$. Let x be of order $r > 2$. Here we have several cases. Write $a_j = c_j t + b_j$ where $0 \leq b_j < t$.
  - Assume n/t is even and write $a_0/2 = c_0 t + b_0$. Then

$$\dim(x^G \cap H^\circ) = \frac{\dim x^G}{t} - \frac{n-a_0}{2}\Big(1-\frac{1}{t}\Big) - \frac{1}{t}\Big(2b_0(t-b_0) + \frac{1}{2}\sum_{j=1}^{r-1} b_j(t-b_j)\Big) \tag{173}$$

  - Assume n/t is odd. If $a_0 - t > 0$, write $\frac{a_0-t}{2} = c_0 t + b_0$. Define $\iota = 0$ if $a_0 \leq t$ and $\iota = 1$ otherwise. Then

$$\dim(x^G \cap H^\circ) = \frac{\dim x^G}{t} - \frac{n-a_0}{2}\Big(1-\frac{1}{t}\Big) - \iota\Big(\frac{t}{2} + 2\frac{b_0}{t}(t-b_0)\Big) - \frac{1}{2t}\sum_{j=1}^{r-1} b_j(t-b_j) \tag{174}$$

Corollary 17.3.15. *Let $x, y \in G$ be of order r. Assume $C_G(x) \cong C_G(y)$. Then $\dim(x^G \cap H^\circ) = \dim(y^G \cap H^\circ)$.*

PROOF. Up to G-conjugacy we may write $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$ and $y = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}]$. The centralisers of x and y can be read off from Theorem 5.3.1. Notice that if $G = \mathrm{Sp}_n$ or O_n then $a_0 = b_0$. The result is a straightforward consequence of Theorem 17.3.8, it is enough to observe that $C_G(x) \cong C_G(y)$ if, and only if, there exists a permutation σ in S_t for which $x = [I_{\sigma(b_0)}, \omega I_{\sigma(b_1)}, \dots, \omega^{r-1} I_{\sigma(b_{r-1})}]$. Thus, by the formulae given in Theorem 17.3.8, the result follows. *q.e.d.*

By analogy with Lemmas 17.3.4 and 17.3.5 given for elements of order p we prove the following. Recall the number ζ from Proposition 17.2.1.

Lemma 17.3.16. *Assume $G = \mathrm{GL}_n$ or Sp_n. Let $x \in G$ be of prime order r. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ) = \Big(\frac{1}{t} + \zeta\Big) \dim x^G$$

if, and only if, $t \mid a_i$ for all $i > 0$ and $t \mid a_0/\iota$ where $\iota = 1$ if $G = \mathrm{GL}_n$ and $\iota = 2$ if $G = \mathrm{Sp}_n$.

PROOF. This is clear thanks to Remark 17.3.14 and Proposition 17.2.1. *q.e.d.*

Thanks to Lemma 17.3.12, for $x \in H^\circ$ of prime order $r \neq p$, we know how to construct a block decomposition $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. The following is analogous to Proposition 17.2.1. Notice that the main tool is the construction of a block decomposition as explained above.

Proposition 17.3.17. *Let $x \in G$ be an element of odd prime order $r \neq p$. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$\dim(x^G \cap H^\circ) \geq \Big(\frac{1}{t} - \frac{1}{n} \pm \xi\Big) \dim x^G \tag{175}$$

where $\xi = 0$ if $G = \mathrm{GL}_n$ otherwise

$$\xi = \frac{n - a_0}{2 \dim x^G}\Big(1 - \frac{1}{t} + \frac{1}{n}\Big)$$

where $+$ occurs if $G = \mathrm{Sp}_n$ and $-$ for $G = \mathrm{O}_n$.

PROOF. For $t = n$ the inequality is trivially satisfied. Therefore we may assume $t < n$.

Case 1. Assume $G = \mathrm{GL}_n$. (Notice that the following argument holds for $r = 2$, as well).
Write $x = [x_1, \dots, x_t] \in H^\circ$, such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Up to conjugation,

$$x_i = [I_{a_{i,0}}, \dots, \omega^{r-1} I_{a_{i,r-1}}]$$

where $\sum_{i=1}^t a_{i,j} = a_j$ for all j. Using Theorem 5.3.1, we see that (175) is equivalent to

$$\frac{n^2}{t} - \sum_{i=1}^{t} \sum_{j=0}^{r-1} a_{i,j}^2 \geq \Big(\frac{1}{t} - \frac{1}{n}\Big)\Big(n^2 - \sum_{j=0}^{r-1} a_j^2\Big) \tag{176}$$

Since $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, by Lemma 17.3.12 we have $a_{i,j} \in \{\lfloor a_j/t \rfloor, \lfloor a_j/t \rfloor + 1\}$. Therefore, writing $a_j = q_j t + r_j$ where $0 \leq r_j < t$, we have, for all $j \in \{0, \dots, r-1\}$,

$$\sum_{i=1}^{t} a_{i,j}^2 = (q_j + 1)^2 r_j + q_j^2 (t - r_j) = q_j^2 t + 2 q_j r_j + r_j$$

Using $n = \sum_i a_i$ and $a_j = q_j t + r_j$, we have

$$n + \Big(\frac{1}{t} - \frac{1}{n}\Big)\sum_{j=0}^{r-1} a_j^2 - \sum_{i=1}^{t}\sum_{j=0}^{r-1} a_{i,j}^2 = \sum_{j=0}^{r-1}\Big(a_j + \Big(\frac{1}{t} - \frac{1}{n}\Big)a_j^2 - \sum_i a_{i,j}^2\Big)$$
$$= \sum_{j=0}^{r-1}\Big(a_j\Big(1 - \frac{a_j}{n}\Big) - r_j\Big(1 - \frac{r_j}{t}\Big)\Big)$$

Therefore inequality (176) is equivalent to

$$\sum_{j=0}^{r-1}\Big(a_j\Big(1 - \frac{a_j}{n}\Big) - r_j\Big(1 - \frac{r_j}{t}\Big)\Big) \geq 0$$

We claim that, for every $j \in \{0, \dots, r-1\}$, each summand is non-negative. Hence, let us fix j and write $a_j = a, r_j = s$. We claim

$$a\Big(1 - \frac{a}{n}\Big) - s\Big(1 - \frac{s}{t}\Big) > 0 \tag{177}$$

If $a < t$ we have $a = s$ and the inequality (177) is clearly true since $n > t$.

Assume $a \geq t$ then $a = qt + s$ with $q \geq 1$ and $0 \leq s < t$. Thus (177) is equivalent to

$$g(s) = \frac{n(qt^2 + s^2) - t(qt + s)^2}{nt} > 0 \tag{178}$$

We see that $g(s)$ is minimal when $s = \frac{qt^2}{n-t}$. Therefore we distinguish two cases. Assume $\frac{qt^2}{n-t} < t$. Then we can actually have $s = \frac{qt^2}{n-t}$ and, using *Mathematica*, we compute

$$g(s) = \frac{n(qt^2 + s^2) - t(qt + s)^2}{nt} \geq g\Big(\frac{qt^2}{n-t}\Big) = \frac{nqt^2(n - (q+1)t)}{nt(n-t)}$$

In particular, (178) is satisfied, since, in this case, $qt + t < n$.

Now assume $\frac{qt^2}{n-t} \geq t$. Then the left hand side of the inequality (178) is minimal when $s = t - 1$. Therefore, since $g(s) \geq g(t-1)$, it is sufficient to prove the inequality for $s = t - 1$, that is

$$g(t-1) = a\Big(1 - \frac{a}{n}\Big) - \Big(1 - \frac{1}{t}\Big) \geq 0 \tag{179}$$

which is true if, and only if,

$$\Big(1 - \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \leq a \leq \Big(1 + \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2}$$

We claim that

$$\Big(1 + \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \geq n - 1$$

which is equivalent to $\sqrt{1 - (4/n)(1 - 1/t)} \geq (1 - 2/n)^2$, we have

$$1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big) - \Big(1 - \frac{2}{n}\Big)^2 = \frac{4}{n}\Big(\frac{1}{t} - \frac{1}{n}\Big) > 0$$

Furthermore, we claim that

$$\Big(1 - \sqrt{1 - \frac{4}{n}\Big(1 - \frac{1}{t}\Big)}\Big)\frac{n}{2} \leq t$$

which is equivalent to

$$t\Big(1-\frac{t}{n}\Big)-\Big(1-\frac{1}{t}\Big)\geq 0 \tag{180}$$

These two claims, together with the fact that $t\leq a\leq n-1$, will lead to the conclusion that (179) is verified.

As a function of t, the left hand side of (180) is monotonically increasing in $[2,\frac{n}{2}]$. Therefore it is minimal for $t=2$ and in this case the inequality (180) is satisfied if $n>3$. The case $n=2,3$ necessarily implies $n=t$ and we can never be in this case since $n>t$.

Case 2. Assume $G=\mathrm{Sp}_n$.
As for the previous case we write $x=[x_1,\dots,x_t]$ with $\dim(x^G\cap H^\circ)=\dim x^{H^\circ}$ and we write

$$x_i=[I_{a_{i,0}},\dots,\omega^{r-1}I_{a_{i,r-1}}]$$

where $\sum_{i=1}^t a_{i,j}=a_j$ for all $j\in\{1,\dots,t\}$.Using Theorem 5.3.1, we see that inequality (175) is equivalent to

$$\frac{n}{2}\Big(\frac{n}{t}+1\Big)-\sum_{i=1}^{t}\Big(\frac{a_{i,0}}{2}+\frac{1}{2}\sum_{j=0}^{r-1}a_{i,j}^2\Big)\geq\Big(\frac{1}{t}-\frac{1}{n}+\xi\Big)\Big(\frac{n}{2}(n+1)-\frac{a_0}{2}-\frac{1}{2}\sum_{j=0}^{r-1}a_j^2\Big) \tag{181}$$

By the computation done in the previous case we know that

$$\frac{n^2}{2t}-\frac{1}{2}\sum_{i=1}^{t}\Big(\sum_{j=0}^{r-1}a_{i,j}^2\Big)\geq\Big(\frac{1}{t}-\frac{1}{n}\Big)\Big(\frac{n^2}{2}-\frac{1}{2}\sum_{j=0}^{r-1}a_j^2\Big)$$

In addition substituting the value of $\xi=\frac{n-a_0}{2\dim x^G}\big(1-\frac{1}{t}+\frac{1}{n}\big)$

$$\frac{n-a_0}{2}\geq\Big(\frac{1}{t}-\frac{1}{n}\Big)\frac{n-a_0}{2}+\xi\dim x^G=\frac{n-a_0}{2}$$

Thus (181) is satisfied.

Case 3. Assume $G=\mathrm{O}_n$.
With the same notation as above, (175) is equivalent to

$$\frac{n}{2}\Big(\frac{n}{t}-1\Big)+\frac{a_0}{2}-\sum_{i=1}^{t}\Big(\frac{1}{2}\sum_{j=0}^{r-1}a_{i,j}^2\Big)\geq\Big(\frac{1}{t}-\frac{1}{n}-\xi\Big)\Big(\frac{n}{2}(n-1)+\frac{a_0}{2}-\frac{1}{2}\sum_{j=0}^{r-1}a_j^2\Big) \tag{182}$$

As for the symplectic case, the analysis done for $G=\mathrm{GL}_n$ allows us to conclude

$$\frac{n^2}{2t}-\frac{1}{2}\sum_{i=1}^{t}\Big(\sum_{j=0}^{r-1}a_{i,j}^2\Big)\geq\Big(\frac{1}{t}-\frac{1}{n}\Big)\Big(\frac{n^2}{2}-\frac{1}{2}\sum_{j=0}^{r-1}a_j^2\Big)$$

In addition, substituting the value $\xi=\frac{n-a_0}{2\dim x^G}\big(1-\frac{1}{t}+\frac{1}{n}\big)$

$$-\frac{n-a_0}{2}\geq-\frac{n-a_0}{2}\Big(\frac{1}{t}-\frac{1}{n}\Big)-\xi\dim x^G=-\frac{n-a_0}{2}$$

Therefore (182) is satisfied. *q.e.d.*

Remark 17.3.18. We do not have the same result for elements of order p. One reason is that given $x\in G$ of order p such that $x^G\cap H\neq\emptyset$ we may have $x^G\cap H^\circ=\emptyset$, see Example 17.4.10. Similarly, if $x \in \mathrm{O}_{n/t} \wr S_t < \mathrm{O}_n$ and n/t is odd, we may have $x^G \cap (\mathrm{O}_{n/t})^t = \emptyset$, see Corollary 20.3.19.

Combining Propositions 17.2.1 and 17.3.17 we have the following two corollaries.

Corollary 17.3.19. *Let $x \in G$ be of odd order $r \neq p$. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$0 \leq \frac{\dim(x^G \cap H) - \dim(x^G \cap H^\circ)}{\dim x^G} \leq \frac{1}{n} + \frac{\iota}{\dim x^G}$$

where $\iota = 0$ if $G \neq \mathrm{O}_n$ and $\iota = n/2$ otherwise.

Proof. For $x \in H$ we have, in general, $\dim(x^G \cap H) \geq \dim(x^G \cap H^\circ)$. Moreover, by Propositions 17.2.1 and 17.3.17 we have

$$\dim(x^G \cap H) - \dim(x^G \cap H^\circ) \leq \Big(\zeta \mp \xi + \frac{1}{n}\Big) \dim x^G$$

where $-$ occurs if $G = \mathrm{Sp}_n$ and $+$ if $G = \mathrm{O}_n$.

In the case $G = \mathrm{GL}_n$ we have $\zeta = \xi = 0$, hence the result follows.

If $G = \mathrm{Sp}_n$ then $\zeta - \xi = -\frac{n-a_0}{2n \dim x^G}$, and $\big(\zeta - \xi + \frac{1}{n}\big) \dim x^G = \frac{\dim x^G}{n} - \frac{n-a_0}{2n} < \frac{\dim x^G}{n}$. The result follows.

Assume $G = \mathrm{O}_n$. Then $\zeta + \xi = \frac{n-a_0}{2 \dim x^G}\big(1 - \frac{1}{t} + \frac{1}{n}\big)$. Therefore $\big(\zeta + \xi + \frac{1}{n}\big) \dim x^G = \frac{n-a_0}{2}\big(1 - \frac{1}{t} + \frac{1}{n}\big) + \frac{\dim x^G}{n}$. Thus, using $t \leq n/2$ and $a_0 \geq 0$ we deduce the result. *q.e.d.*

Recall that, for $x \in G$, we defined

$$f_\Omega^\circ(x) = \frac{\dim \Omega - \dim x^G + \dim(x^G \cap H^\circ)}{\dim \Omega}$$

Corollary 17.3.20. *Let $x \in G$ be of odd order $r \neq p$. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$f_\Omega^\circ(x) \leq f_\Omega(x) < f_\Omega^\circ(x) + \frac{2t}{n(t-1)}$$

Proof. Using Corollary 17.3.19, we have

$$f_\Omega(x) - f_\Omega^\circ(x) = \frac{\dim(x^G \cap H) - \dim(x^G \cap H^\circ)}{\dim \Omega} \leq \frac{\dim x^G + n\iota}{n \dim \Omega} < \frac{n}{\dim \Omega} \leq \frac{2t}{n(t-1)}$$

where we used $\dim x^G < n^2$ for $G = \mathrm{GL}_n$ or Sp_n and $\dim x^{\mathrm{O}_n} < n^2/2$ (this easily follows from Proposition 5.4.1). *q.e.d.*

Notice that $\frac{t}{t-1}$ is maximal (for $t > 1$) when t is minimal. Therefore $f_\Omega(x) - f_\Omega^\circ(x) \leq 4/n$.

17.4. Conjugacy classes in $H \setminus H^\circ$

Let $x \in H$ be an element of prime order r. Denote by $\pi_i \in S_t$ any permutation with cycle shape $(r^i, 1^{t-ir})$. The aim of this section is to give conditions on x in order to have $x^G \cap H^\circ \pi_i \neq \emptyset$, see Proposition 17.4.8, and so deduce a formula for $\dim(x^G \cap H^\circ \pi_i)$, see (184).

Let $x \in H$. Then $x = [x_1, \ldots, x_t]\pi$ where $[x_1, \ldots, x_t] \in H^\circ$ and $\pi \in S_t$. Since π acts on $[x_1, \ldots, x_t]$ by permuting the blocks we have that π is a block matrix of G. For example if $t = 3$ and $\pi = (1, 2)$, then, in a suitable basis,

$$\pi = \begin{pmatrix} 0 & I_{n/3} & 0 \\ I_{n/3} & 0 & 0 \\ 0 & 0 & I_{n/3} \end{pmatrix}$$

Let $x \in H$ be an element of prime order r. As remarked above we have $\dim(x^G \cap H) = \max_{\sigma^r=1}\{\dim(x^G \cap H^\circ\sigma)\}$, see (161).

Let $\pi_i \in S_t$ be any permutation with cycle shape $(r^i, 1^{t-ir})$. Then, thanks to the following result we have $\dim(x^G \cap H) = \max_i\{\dim(x^G \cap H^\circ\pi_i)\}$.

Lemma 17.4.1. *Let $x \in H$ be of order r. Let $\tau, \sigma \in S_t$ be conjugate permutations of order r. Then*

$$\dim(x^G \cap H^\circ\tau) = \dim(x^G \cap H^\circ\sigma)$$

Proof. Let V be the natural module for G. The subgroup H stabilises a direct sum decomposition of V into orthogonal and isometric subspaces, say $V = V_1 \oplus \ldots \oplus V_t$.

Since τ and σ are conjugate in S_t they have the same cycle shape, say $(r^h, 1^{t-hr})$, for some $h > 0$. Thus we have

$$\begin{aligned} \tau &= (1, \ldots, r)(r+1, \ldots, 2r)(2r+1, \ldots, 3r) \cdots ((h-1)r+1, \ldots, hr) \\ \sigma &= (j_1, \ldots, j_r)(j_{r+1}, \ldots, j_{2r})(j_{2r+1}, \ldots, j_{3r}) \cdots (j_{(h-1)r+1}, \ldots, j_{hr}) \end{aligned}$$

The set of points fixed by τ is $\{hr+1, \ldots, t\}$, while the set of points fixed by σ is given by $\{j_{hr+1}, \ldots, j_t\}$. Let $\pi = \sigma^{-1}$ in S_t.

Assume, seeking a contradiction, that $\dim(x^G \cap H^\circ\tau) > \dim(x^G \cap H^\circ\sigma)$. Then there exists $y \in x^G \cap H^\circ\tau$ such that $\dim(x^G \cap H^\circ\tau) = \dim y^{H^\circ}$. Let us denote $y = [A_1, \ldots, A_t]\tau$. We consider the following element of $H^\circ\sigma$

$$y' = [A_{\pi(1)}, \ldots, A_{\pi(t)}]\sigma$$

Then $y, y' \in x^G$. Moreover y' stabilises the decomposition $V_{\pi(1)} \oplus V_{\pi(2)} \oplus \ldots \oplus V_{\pi(t)}$ of V. Therefore, by construction of π and y', we have $y = y'$. Hence $\dim y^{H^\circ} = \dim(y')^{H^\circ}$, which is a contradiction since

$$\dim y^{H^\circ} = \dim(x^G \cap H^\circ\tau) > \dim(x^G \cap H^\circ\sigma) \geq \dim(y')^{H^\circ}$$

q.e.d.

Let $V = V_1 \oplus \ldots \oplus V_t$ be the direct sum decomposition stabilised by H. Assume $x^G \cap H^\circ\tau \neq \emptyset$. Then we may always assume $x = [x_1, \ldots, x_t]\tau$.

First, we observe that the action of x on the permuted linear spaces V_i is trivial. The following result is in the proof of [**43**, Lemma 4.5].

Lemma 17.4.2. *Let $x = [x_1, \ldots, x_t]\tau \in H \setminus H^\circ$ be of order r. Let $(r^h, 1^{t-hr})$ be the cycle shape of τ. Then x is H°-conjugate to $[I_{n/t}, \ldots, I_{n/t}, x_{hr+1}, \ldots, x_t]\tau$.*

Proof. Since x has prime order r then so ha τ. So, by Lemma 17.4.1, we may assume $\tau = \prod_{i=1}^{h}\Big((i-1)r+1, \ldots, ir\Big)$, say $f = t - hr$ is the number of fixed points of

τ. Therefore, in a suitable basis we have $\tau = [\tau_1, \ldots, \tau_h, I_{nf/t}]$, where

$$\tau_i = \begin{pmatrix} 0 & I_{n/t} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & I_{n/t} \\ I_{n/t} & \cdots & \cdots & 0 \end{pmatrix} \tag{183}$$

where $\tau_i \in Cl_{rn/t}$ by Lemma 17.1.5(ii).

Since x has order r we have $x_{(i-1)r+1} \cdots x_{ir} = I_{n/t}$ for all i. Define

$$\begin{aligned} g =& [I_{n/t}, x_1, x_1x_2, \ldots, x_1 \cdots x_{r-1}, I_{n/t}, x_{r+1}, x_{r+1}x_{r+2}, \ldots, x_{r+1} \cdots x_{2r-1}, \\ & I_{n/t} \ldots, x_{(h-1)r+1}, x_{(h-1)r+1}x_{(h-1)r+2}, \ldots, x_{(h-1)r+1} \cdots x_{hr-1}, I_{nf/t}] \in H^\circ \end{aligned}$$

We see that $g^{-1}\tau g = [x_1, x_2, \ldots, x_{hr}, I_{n/t}, \ldots, I_{n/t}]\tau$. Thus, defining

$$b = [I_{n/t}, \ldots, I_{n/t}, x_{hr+1}, \ldots, x_t]$$

we have $g^{-1}(b\tau)g = b(g^{-1}\tau g) = x$. The result follows. *q.e.d.*

It is not possible to extend Lemma 17.4.2 to elements of non-prime finite order as we see in the following example.

Remark 17.4.3. It is straightforward to check that the argument in Lemma 17.4.2 holds in the case where $x = [x_1, \ldots, x_t]\tau$ and $\tau = \tau_1 \cdots \tau_l$ is a product of prime order permutations (say of orders $r_1, \ldots, r_l$) and $x_1, \ldots x_{r_1}$ have prime order r_1 and similarly for the other blocks. If the blocks do not have prime order the argument fails in the equality regarding $g^{-1}\tau g$. It is straightforward to check this if $x = [A_1, \ldots, A_6]\tau$, where $\tau = (123)(45)$ and each A_i has order 6.

A straightforward consequence of Lemma 17.4.2 is that we obtain a formula for $\dim(x^G \cap H^\circ\tau)$. Let $x = [x_1, \ldots, x_t]\pi_h \in H^\circ\pi_h$ of prime order r. Notice that given $[A_1, \ldots, A_t] \in H^\circ$ we have – by a straightforward computation, using the matrix form of π_1 given in (183) – $[A_1, \ldots, A_t]\pi_1 = \pi_1[A_1, \ldots, A_t]$ if, and only if $A_1 = A_2 = \ldots = A_r$. Therefore $C_{H^\circ}(\pi_1) \cong Cl_{n/t}$. In the same way, we see that $C_{H^\circ}(\pi_h) = (Cl_{n/t})^h$. Thus, we have the following.

Proposition 17.4.4. *Let $x \in G$ be of order r. Assume $x^G \cap H^\circ\pi_h \neq \emptyset$. Then*

$$\begin{aligned} \dim(x^G \cap H^\circ\pi_h) &= \dim H^\circ - \dim C_{H^\circ}([I_{n/t}, \ldots, I_{n/t}, B_{hr+1}, \ldots, B_t]\pi_h) \\ &= h(r-1)\dim Cl_{n/t} + \sum_{i \geq hr+1} \dim B_i^{Cl_{n/t}} \end{aligned} \tag{184}$$

for suitable B_i's in $Cl_{n/t}$.

The following two lemmas provide the Jordan form in GL_n of $\pi_h \in S_t$ of prime order.

Lemma 17.4.5. *Let $\tau \in S_t$ be an element of order p. Let $(p^h, 1^f)$ be the cycle shape of τ. Then τ is GL_n-conjugate to $[J_p^{nh/t}, J_1^{nf/t}]$.*

Proof. Without loss of generality we may assume

$$\tau = \big(1, 2, \ldots, p\big)\big(p+1, \ldots, 2p\big) \cdots \big((h-1)p+1, \ldots, hp\big)$$

In terms of matrices we have $\tau = [\tau_1, \ldots, \tau_h, I_{nf/t}]$, where $\tau_i \in \mathrm{GL}_{np/t}$ is defined in (183). Each τ_i is $\mathrm{GL}_{np/t}$-conjugate to the matrix $[g, \ldots, g] \in \mathrm{GL}_{np/t}$, where

$$g = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 1 & & \ldots & 0 \end{pmatrix} \in \mathrm{GL}_p \tag{185}$$

The characteristic polynomial of g is $cp_g(\lambda) = 1 - \lambda^p = (1-\lambda)^p$. So g is GL_p-conjugate to J_p. Therefore τ is conjugate to $[J_p^{nh/t}, J_1^{nf/t}]$. *q.e.d.*

Lemma 17.4.6. *Let $\tau \in S_t$ be of order $r \neq p$. Let $(r^h, 1^f)$ be the cycle shape of τ. Then τ is GL_n-conjugate to $[I_{n(h+f)/t}, \omega I_{nh/t}, \ldots, \omega^{r-1} I_{nh/t}]$.*

Proof. As seen in the proof of Lemma 17.4.5 we have $\tau = [\tau_1, \ldots, \tau_h, I_{nf/t}]$, where $t = hr + f$. Again, each τ_i is conjugate to the matrix $[g, \ldots, g] \in \mathrm{GL}_{nr/t}$, where $g \in \mathrm{GL}_r$ is given in (185). When $p \neq r$, the characteristic polynomial of g is $cp(\lambda) = 1 - \lambda^r$. Therefore each $r \times r$ block is GL_r-conjugate to $[I_1, \omega I_1, \ldots, \omega^{r-1} I_1]$. Therefore τ is GL_n-conjugate to $[I_{n(h+f)/t}, \omega I_{nh/t}, \ldots, \omega^{r-1} I_{nh/t}]$. *q.e.d.*

Remark 17.4.7. Observe that the Jordan form of $x \in \mathrm{Sp}_n$ or O_n is given by the Jordan form in GL_n thanks to Theorems 5.2.1 and 5.3.1. Thus if $G = \mathrm{Sp}_n$ or O_n and $\pi_h \in S_t$ is a prime order element then π_h is G-conjugate to the elements given in Lemmas 17.4.5 and 17.4.6. Moreover notice that if $G = \mathrm{Sp}_n$ or O_n and $p = 2$ then π_h is an $a_{hn/t}$-type involution.

Let $x \in H$ be of order r. For future reference, we define

$$h = \max\{l : x^G \cap H^\circ \pi_l \neq \emptyset, \pi_l \in S_t\} \tag{186}$$

where π_l is a permutation of cycle shape $(r^l, 1^{t-rl})$, and we set $\pi_0 = 1$. By replacing x with a suitable G-conjugate, we may assume $x \in H^\circ \pi_h$ and, thanks to Lemma 17.4.2, x is H°-conjugate to $[I_{n/t}, \ldots, I_{n/t}, x_{hr+1}, \ldots, x_t]\pi_h$.

We conclude this section by giving the Jordan form of a prime order element $x \in H$ such that $x^G \cap H^\circ \pi_l \neq \emptyset$. Recall that if $p = 2$ and $G = \mathrm{Sp}_n$ or O_n then n/t is even.

Proposition 17.4.8. *Let $x \in H$ be of prime order r (possibly even). Then $x^G \cap H^\circ \pi_h \neq \emptyset$ if, and only if, one of the following two conditions holds.*

(i) If $r \neq p$, then for some b_i's

$$x \in \left[I_{\frac{n}{t}h+b_0}, \omega I_{\frac{n}{t}h+b_1}, \ldots, \omega^{r-1} I_{\frac{n}{t}h+b_{r-1}}\right]^G$$

(ii) Let $r = p$. Assume $p \neq 2$ if $G \neq \mathrm{GL}_n$. Then, for some b_i's,

$$x \in \left[J_p^{\frac{n}{t}h+b_p}, J_{p-1}^{b_{p-1}}, \ldots, J_1^{b_1}\right]^G$$

If $p = 2$ and $G = \mathrm{Sp}_n$ or O_n then $x \in a_{hn/t}^G$.

Proof. This follows directly from Lemmas 17.4.2, 17.4.5 and 17.4.6, and Remark 17.4.7. *q.e.d.*

The following are trivial consequences of Proposition 17.4.8.

Corollary 17.4.9. *Let $x \in H$ be of prime order r (possibly even).*

(i) Assume $r \neq p$ and $x \in [I_{a_0}, \dots, \omega^{r-1} I_{a_{r-1}}]^G$. If there exists i such that $a_i < n/t$ then $x^G \cap H = x^G \cap H^\circ$.

(ii) Assume $r = p$ and $x \in [J_p^{a_p}, \dots, J_1^{a_1}]^G$. If $a_p < n/t$ then $x^G \cap H = x^G \cap H^\circ$.

Using the previous results on conjugacy classes in $H \setminus H^\circ$, we give the following.

Example 17.4.10. Let $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_1 \wr S_n$. Assume $p \leq n$ and write $n = ap + b$. Define $x_h = [J_p^h, J_1^{n-hp}]$ for any $1 \leq h \leq a$. It is clear that $x_h^G \cap H^\circ = \emptyset$. Then, using Lemma 17.4.5, we have that x_h is G-conjugate to π_h. In fact, combining this result with Lemma 17.4.2 we find $x_h^G \cap H = x_h^G \cap H^\circ \pi_h = \pi_h^{H^\circ}$.

The last observation is devoted to unipotent elements in the case where the field has characteristic zero. Recall that in this case we set $p = \infty$. For $x \in G$ unipotent, up to G-conjugation, $x = [J_n^{a_n}, \dots, J_1^{a_1}]$.

Proposition 17.4.11. *Assume $p = \infty$. Let $x \in G$ be unipotent. Assume $x^G \cap H \neq \emptyset$. Then $x^G \cap H = x^G \cap H^\circ$.*

PROOF. This is Lemma B.3.1. *q.e.d.*

In particular, in characteristic zero, unipotent elements in H must have no Jordan block of size larger than n/t.

Corollary 17.4.12. *Assume $p = \infty$. Let $x \in H$ be unipotent. Then, up to G-conjugacy, $x = [J_{n/t}^{a_{n/t}}, \dots, J_1^{a_1}]$.*

CHAPTER 18

General linear group

Throughout this chapter the notation is as follows. Let $G = \mathrm{GL}_n$, $n > 1$, $H = \mathrm{GL}_{n/t} \wr S_t$ for $t > 1$. Set $\Omega = G/H$. The aim of this chapter is to derive bounds on $f_\Omega(x)$ for $x \in G$ of prime order.

18.1. Upper bounds

In this section we shall derive upper bound on $f_\Omega(x)$ for $x \in H$ of any prime order r. We prove the following.

Proposition 18.1.1. *Let $x \in G$ be of order r.*

(i) Assume $t \neq n$ if $r = 2$. Then

$$f_\Omega(x) \leq 1 - \frac{2}{n}$$

with equality if, and only if, $\nu(x) = 1$ or one of the following holds

(a) $r = p$ and $(t, x^G) = (2, [J_2^2]^G)$ or $(3, [J_3]^G)$;

(b) $r \neq p$ and $(r, t, C_G(x)) = (3, 4, \mathrm{GL}_3 \times (\mathrm{GL}_1)^2)$ or $(t, C_G(x)) = (2, (\mathrm{GL}_2)^2)$.

(ii) Assume $r = 2$ and $t = n$. Then

$$f_\Omega(x) \leq 1 - \frac{2}{n} + \frac{1}{n(n-1)}$$

with equality if, and only if, $\nu(x) = 1$.

Remark 18.1.2. Thanks to Lemma 7.1.1 and Corollary 7.1.11, Proposition 18.1.1 extends to any non-central element in $G \setminus Z(G)$. In particular all the results that follow hold for unipotent elements in characteristic zero, too. Therefore, Theorems 16.1.1, 16.1.3 and 16.1.4 follow.

We divide the proof of Proposition 18.1.1 into several technical lemmas. First, we study elements with $\nu(x) = 1$. As observed in Remark 10.1.3 we shall prove the following results for $x \in H$ of prime order.

Lemma 18.1.3. *Let $x \in H$ be of order r. Assume $\nu(x) = 1$. Then either $(r, t) \neq (2, n)$ and*

$$f_\Omega(x) = 1 - \frac{2}{n}$$

or $(r, t) = (2, n)$ and $f_\Omega(x) = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$.

Proof. Let us study separately the cases $r = p$ and $r \neq p$.

Case 1. Assume $r = p$. Hence, up to G-conjugacy, $x = [J_2, J_1^{n-2}]$. In the case $(r, t) \neq (2, n)$ we have, by Corollary 17.4.9, $x^G \cap H = x^G \cap H^\circ$. Up to conjugation, $x = [x_1, \ldots, x_t]$ with $x_1 = [J_2, J_1^{n/t-2}]$ and $x_i = I_{n/t}$ for $i > 1$. Using Theorem 5.2.1, we compute $\dim x^G = 2n-2$ and

$$\dim(x^G \cap H^\circ) = \dim x^{H^\circ} = \dim x_1^{\mathrm{GL}_{n/t}} = 2\frac{n}{t} - 2$$

Therefore, thanks to Proposition 7.1.8 and the value of $\dim \Omega$ in Table 17.1.1, we compute $f_\Omega(x) = 1 - 2/n$.

In the case $(r,t) = (2,n)$ we have, $x^G \cap H^\circ = \emptyset$ and $x^G \cap H = x^G \cap H^\circ \pi_1$. In fact, by Lemma 17.4.5, we have that x is G-conjugate to $\pi_1 = (1,2)$. Then, using (184), we compute $\dim(x^G \cap H) = \dim \pi_1^{H^\circ} = (n/t)^2 = 1$. Therefore $f_\Omega(x) = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$.

Case 2. Assume $r \neq p$. Let $\omega \in k$ be a primitive r-th root of unity. Then, up to centraliser structure, $x = [I_{n-1}, \omega]$. Again, if $(r,t) \neq (2,n)$ we have $x^G \cap H = x^G \cap H^\circ$, by Corollary 17.4.9. The only block decomposition, up to a permutation of the blocks, is given by $x = [x_1, \ldots, x_t]$ where $x_1 = [I_{n/t-1}, \omega]$ and $x_i = I_{n/t}$ for $i > 1$. Using Theorem 5.3.1, we compute $\dim x^G = 2n-2$ and

$$\dim(x^G \cap H^\circ) = \dim x^{H^\circ} = \dim x_1^{\mathrm{GL}_{n/t}} = 2\frac{n}{t} - 2$$

Therefore, thanks to Proposition 7.1.8 and the value of $\dim \Omega$ in Table 17.1.1, we compute $f_\Omega(x) = 1 - 2/n$.

In the case $(r,t) = (2,n)$ the previous argument holds. We have that x is G-conjugate to $\pi_1 = (1,2)$ and $\dim(x^G \cap H) = 1$. Therefore $f_\Omega(x) = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$. *q.e.d.*

Now we study prime order elements $x \in H$ with $\nu(x) = 2$. First we study elements of odd order and then involutions.

Lemma 18.1.4. *Let $x \in H$ be of order $r > 2$. Assume $\nu(x) = 2$. Then either*

$$f_\Omega(x) < 1 - \frac{2}{n}$$

or one of the following holds

(i) $r = p$, $(t, x^G) = (2, [J_2^2]^G)$ or $(3, [J_3]^G)$, and $f_\Omega(x) = 1 - 2/n$.
(ii) $r \neq p$, $(r, t, C_G(x)) = (3, 4, \mathrm{GL}_2 \times (\mathrm{GL}_1)^2)$, or $(t, x) = (2, (\mathrm{GL}_2)^2)$, and $f_\Omega(x) = 1 - 2/n$.

Proof. Also here we study first the case $r = p$ and then $r \neq p$.

Case 1. Assume $r = p$. Hence, up to G-conjugacy, $x = [J_2^2, J_1^{n-4}]$ or $[J_3, J_1^{n-3}]$.

Let $x = [J_2^2, J_1^{n-4}]$. For $(r,t) \neq (2,n)$ or $(2, n/2)$, thanks to Corollary 17.4.9, $x^G \cap H = x^G \cap H^\circ$. The possible decompositions of x are given by $[x_1, \ldots, x_t]$ or $[x'_1, \ldots, x'_t]$, where $x_1 = [J_2^2, J_1^{n/t-4}]$ and $x_i = I_{n/t}$ for $i > 1$; or $x'_1 = x'_2 = [J_2, J_1^{n/t-2}]$ and $x'_i = I_{n/t}$ for $i > 2$. A direct computation shows $\dim x^G = 4n - 8$ and

$$\dim(x^G \cap H) = 4\frac{n}{t} - 4$$

Therefore $f_\Omega(x) = 1 - \frac{4}{n} + \frac{4t}{n^2(t-1)}$. Here we have $f_\Omega(x) = 1 - 2/n$ if, and only if, $\frac{n}{t}(t-1) = 2$. Hence $f_\Omega(x) = 1 - 2/n$ if, and only if, $(n,t) = (4,2)$ or $(3,3)$. Observe that if $(n,t) = (3,3)$ then $x^G \cap H = \emptyset$.

Now, let $x = [J_3, J_1^{n-3}]$. Then, unless $(r,t) = (3,n)$, we have, by Corollary 17.4.9, $x^G \cap H = x^G \cap H^\circ$. The only block decomposition, up to a permutation of the blocks, is given by $x = [x_1, \ldots, x_t]$ where $x_1 = [J_3, J_1^{n/t-3}]$ and $x_i = I_{n/t}$ for $i > 1$. Therefore

$\dim(x^G \cap H^\circ) = \dim x_1^{\mathrm{GL}_{n/t}}$. Using Theorem 5.2.1 we compute $\dim x^G = 4n - 6$ and $\dim(x^G \cap H) = 4n/t - 6$. Therefore $f_\Omega(x) = 1 - 4/n$. The result follows.

Now assume $x = [J_3, J_1^{n-3}]$ and $(r,t) = (3,n)$. Here, we have $x^G \cap H = x^G \cap H^\circ\pi_1$ and, by Lemma 17.4.5, $\pi_1 \in x^G$. So, using (184), $\dim(x^G \cap H) = \dim \pi_1^{H^\circ} = 2$. Therefore $f_\Omega(x) = 1 - \frac{4}{n} + \frac{4}{n(n-1)}$. Furthermore $f_\Omega(x) = 1 - 2/n$ if, and only if, $n = 3$.

Case 2. Assume $r \neq p$. Then, up to G-conjugacy, $x = [I_{n-2}, \omega, \omega^2]$ or $[I_{n-2}, \omega I_2]$. Notice that if y has order r and $\nu(y) = 2$ then $C_G(y) \cong C_G(x)$, for x defined before. The result follows with an easy computation. *q.e.d.*

Lemma 18.1.5. *Let $x \in H$ be an involution. Assume $\nu(x) = 2$. Then*

$$f_\Omega(x) < 1 - \frac{2}{n} + \frac{\delta_{t,n}}{n(n-1)}$$

Furthermore equality holds if and only if $(n,t) = (4,2)$.

PROOF. As usual, we divide the computations according to whether or not $p = 2$.

Case 1. Assume $p = 2$. In the case $x = [J_2^2, J_1^{n-4}]$ and $t \neq n, n/2$ the same argument as in Lemma 18.1.4 applies and the result follows. Recall that we have already computed $\dim x^G = 4n - 8$.

Let $x = [J_2^2, J_1^{n-4}]$ and assume $t = n$. By the same argument of Lemma 18.1.3, x is G-conjugate to π_2 and $x^G \cap H = \pi_2^{H^\circ}$. Then, using (184), we compute

$$\dim(x^G \cap H) = \dim \pi_2^{H^\circ} = 2(n/t)^2 = 2$$

Hence $f_\Omega(x) = 1 - \frac{4}{n} + \frac{6}{n(n-1)}$. A straightforward computation shows that, for $n \geq 4$, $f_\Omega(x) < 1 - \frac{2}{n} + \frac{1}{n(n-1)}$.

Let $x = [J_2^2, J_1^{n-4}]$ and assume $t = n/2$. Then $x^G \cap H = (x^G \cap H^\circ) \cup (x^G \cap H^\circ\pi_1)$. With an easy computation we get $\dim(x^G \cap H^\circ) = 4 = \dim(x^G \cap H^\circ\pi_1)$. Therefore, $f_\Omega(x) = 1 - \frac{4}{n} + \frac{4}{n(n-2)}$. Again, it is straightforward to check that $f_\Omega(x) \leq 1 - \frac{2}{n}$ with equality if, and only if, $n = 4$.

Case 2. Assume $p \neq 2$. Then, up to a multiplication by the scalar -1, $x = [I_{n-2}, -I_2]$ (here $n \geq 4$, otherwise $\nu(x) \neq 2$). By Corollary 17.4.9, unless $t = n$ or $n/2$ we have $x^G \cap H = x^G \cap H^\circ$. Therefore, $f_\Omega(x) = 1 - \frac{4}{n} + \frac{4t}{n^2(t-1)}$. We see that $f_\Omega(x) = 1 - 2/n$ if, and only if, $\frac{n}{t}(t-1) = 2$, i.e. $(n,t) = (4,2)$ or $(3,3)$. However, we cannot consider either of the cases since we are assuming $t < n/2$. Hence $f_\Omega(x) < 1 - 2/n$.

Assume $t = n$. Then $x^G \cap H = (x^G \cap H^\circ) \cup (x^G \cap H^\circ\pi_1) \cup (x^G \cap H^\circ\pi_2)$. And we have, using (184) and Theorem 17.3.8,

$$\dim(x^G \cap H^\circ) = 0,\ \dim(x^G \cap H^\circ\pi_1) = 1,\ \dim(x^G \cap H^\circ\pi_2) = 2$$

Therefore $\dim(x^G \cap H) = 2$ and $f_\Omega(x) = 1 - \frac{4}{n} + \frac{6}{n(n-1)}$. It is straightforward to check that $f_\Omega(x) < 1 - \frac{2}{n} + \frac{1}{n(n-1)}$.

The last case is $x = [I_{n-2}, -I_2]$, with $n \geq 4$ and $t = n/2$. Here we have $x^G \cap H = (x^G \cap H^\circ) \cup (x^G \cap H^\circ\pi_1)$. Again, we compute $\dim(x^G \cap H^\circ) = 4$ and $\dim(x^G \cap H^\circ\pi_1) = 4$. Hence, $f_\Omega(x) = 1 - \frac{4}{n} + \frac{4}{n(n-2)}$. In addition, $f_\Omega(x) = 1 - 2/n$ if, and only if, $n = 4$. *q.e.d.*

Lemma 18.1.6. *Let $x \in H$ be an element of prime order r. Assume $\nu(x) > 2$. Then*

$$f_\Omega(x) < 1 - \frac{2}{n}$$

Proof. Thanks to Proposition 5.4.1 we have, for $n \geq 4$, $\dim x^G \geq \max\{2s(n-s), ns\} > 2n$. Hence, using Proposition 17.2.1, we have

$$f_\Omega(x) \leq 1 - \frac{\dim x^G}{n^2} < 1 - \frac{2}{n}$$

If $n < 4$ then for all $x \in G$ we have $\nu(x) = 1$ or 2. *q.e.d.*

18.2. Unipotent elements: lower bounds

In this section we establish lower bounds on $f_\Omega(x)$ for $x \in H$ of order p (or any unipotent element if the field has characteristic zero, recall that in the characteristic zero case we set $p = \infty$.). The main result of this section is the following.

Proposition 18.2.1. *Let $x \in H$ be a unipotent element of order p.*

(i) Assume $t < n$. If $p > n/2$ then

$$f_\Omega(x) \geq \frac{t}{n}$$

with equality if, and only if $x \in [J_{n/t}^{t-1}, z]^G$ for any unipotent $z \in \mathrm{GL}_{n/t}$.

(ii) Assume $p \leq n/2$ if $t < n$, and $p \leq n$ if $t = n$. Then

$$f_\Omega(x) \geq \frac{1}{p}$$

Remark 18.2.2. The bound in case (ii) of Proposition 18.2.1 is sharp. In fact, for $p \mid n$, we shall characterise unipotent elements $x \in H$ for which $f_\Omega(x) = 1/p$.

Let $x \in G$ be of prime order p. Then, up to G-conjugacy, we write $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ with $\sum_i ia_i = n$. Notice that in the characteristic zero case the largest Jordan block is J_n. Recall that $[x_1, \ldots, x_l]$ denotes the block diagonal matrix with blocks $x_1, \ldots, x_l$ down the diagonal. Similarly, $[J_i^{a_i}]$ denotes the block diagonal matrix with a_i unipotent Jordan blocks, of size i, down the diagonal.

Notation. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in G$. Assume $x^G \cap H^\circ \neq \emptyset$. When we write $x = [x_1, \ldots, x_t] \in H^\circ$, we set

$$x_i = [J_p^{a_{i,p}}, \ldots, J_1^{a_{i,1}}] \tag{187}$$

so for all i we have $\sum_j ja_{i,j} = n/t$ and $a_j = \sum_{i=1}^t a_{i,j}$ for $1 \leq j \leq p$.

For $x = [x_1, \ldots, x_t] \in H^\circ$ in the above notation, using Theorem 5.2.1, we compute

$$\dim x^{H^\circ} = \frac{n^2}{t} - 2 \sum_{1 \leq i < j \leq p} i(a_{1,i}a_{1,j} + \ldots + a_{t,i}a_{t,j}) - \sum_{i=1}^{p} i(a_{1,i}^2 + \ldots + a_{t,i}^2)$$

The following clarifies the dichotomy in Proposition 18.2.1. Here we assume $t < n$. Notice that in Section 18.4.2.2 we give all the information needed for the case $t = n$.

Lemma 18.2.3. *Assume $p > n/2$ and $t < n$. Let $x \in G$ be of order p. Assume $a_i > 0$ for some $n/2 < i \leq p$. Then $x^G \cap H = \emptyset$.*

PROOF. Recall that $H^\circ = \left(\mathrm{GL}_{n/t}\right)^t$ and $t \geq 2$. Therefore if $x^G \cap H^\circ \neq \emptyset$ then $a_i = 0$ for all $i > n/t$. Since $a_i \neq 0$ for some $i > n/t$ we deduce $x^G \cap H^\circ = \emptyset$. By Proposition 17.4.8, if $x^G \cap (H \setminus H^\circ) \neq \emptyset$ then $a_p = (n/t)h$ for some $h > 0$. In particular, $a_p \geq n/t$. Hence $n \geq a_p p \geq pn/t > n^2/2t$, which is absurd since this would imply $n/t < 2$. *q.e.d.*

Let $x \in G$ be an element of order p such that $x^G \cap H \neq \emptyset$. Then the largest Jordan block that may appear in x is J_p if $p \leq n/2$ (in Example 17.4.10 we have seen that for any p in the case $t = n$ there are unipotent elements in H). If $p > n/2$ then the largest Jordan block that may appear in x is $J_{n/t}$, by Lemma 17.3.1. In view of this fact it is natural to consider these two cases.

Assume $t = n$. Let $x \in G$ be of order p. If $p > n$ then $x^G \cap H = \emptyset$. If $p \leq n$, thanks to Lemma 18.2.3, we see that $x^G \cap H^\circ = \emptyset$.

18.2.1. Case $p > n$. As observed above, in this case any element $x \in H$ of order p is such that $x^G \cap H = x^G \cap H^\circ$, by Lemma 17.3.1 and Lemma B.3.1. Therefore we may assume $x \in H^\circ$ and, up to conjugation, $x = [J_{n/t}^{a_{n/t}}, \ldots, J_1^{a_1}]$.

First we prove the following, in which we compute the f_Ω-values of certain elements.

Lemma 18.2.4. *Let $x \in [J_{n/t}^{t-1}, z]^G$ with $z \in \mathrm{GL}_{n/t}$ unipotent. Then $f_\Omega(x) = t/n$.*

PROOF. First consider the case $z = [J_{n/t}]$. Then $x = [J_{n/t}^t]$ and, using Theorem 5.2.1, we compute

$$\dim x^G = n^2 - nt, \ \dim(x^G \cap H) = \dim x^{H^\circ} = t \dim[J_{n/t}]^{\mathrm{GL}_{n/t}} = \frac{1}{t}(n^2 - nt)$$

Therefore using the formula in Proposition 7.1.8, $f_\Omega(x) = t/n$.

Now assume $z = [J_{n/t-1}^{a_{n/t-1}}, \ldots, J_1^{a_1}]$, with $\sum_{i<n/t} ia_i = n/t$. Then we have $a_{n/t} = t-1$ and we compute

$$\begin{aligned}
\dim x^G &= n^2 - 2\sum_{1\leq i<j\leq n/t} ia_ia_j - \sum_{i\leq n/t} ia_i^2 \\
&= n^2 - 2(t-1)\sum_{i<j<n/t} ia_i - 2\sum_{i<j<n/t} ia_ia_j - \frac{n}{t}(t-1)^2 - \sum_{i<n/t} ia_i^2 \\
\dim(x^G \cap H) &= \dim x^{H^\circ} = (t-1)\dim[J_{n/t}]^{\mathrm{GL}_{n/t}} + \dim z^{\mathrm{GL}_{n/t}} \\
&= \frac{n^2}{t} - \frac{n}{t}(t-1) - 2\sum_{1\leq i<j<n/t} ia_ia_j - \sum_{i<n/t} ia_i^2
\end{aligned}$$

Thus $\dim x^G - \dim(x^G \cap H) = n^2\left(1 - \frac{1}{t}\right) - n(t-1)$. Therefore $f_\Omega(x) = t/n$. *q.e.d.*

The key tool in order to prove Proposition 18.2.1 is the following technical result.

Lemma 18.2.5. *Let $x \in G$ be of order p. Assume $x = [x_1, \ldots, x_t] \in H^\circ$ and $x_1, x_2 \neq J_{n/t}$. Suppose $\dim(x^G \cap H) = \dim x^{H^\circ}$. Let $y = [J_{n/t}, x_2, \ldots, x_t]$. Then $f_\Omega(x) > f_\Omega(y)$.*

PROOF. Say $h \geq 0$ the number of blocks in x equal to $J_{n/t}$. Up to a permutation of the blocks we may assume $x_1, x_2, \ldots, x_{t-h} \neq J_{n/t}$ and $x_{t-h+1} = \ldots = x_t = J_{n/t}$. We use the notation defined in (187) for the x_i's with $i \leq t-h$. Thus $x = [J_{n/t}^h, J_{n/t-1}^{a_{n/t}-1}, \ldots, J_1^{a_1}]$ and, for $j < n/t$, we have $a_j = \sum_{i=1}^{t-h} a_{i,j}$. In particular, $\sum_{i<n/t} ia_i = (t-h)n/t$. Using

Theorem 5.2.1, we compute

$$\begin{aligned}\dim x^G &= n^2 - 2\sum_{i<j\le n/t} i a_i a_j - \sum_i i a_i^2\\ &= n^2 - 2\frac{n}{t}h(t-h) - \frac{n}{t}h^2 - 2\sum_{i<j<n/t} i a_i a_j - \sum_{i<n/t} i a_i^2\end{aligned}$$

By hypothesis we have $\dim(x^G \cap H) = \dim x^{H^\circ}$, and

$$\dim x^{H^\circ} = h\dim[J_{n/t}]^{\mathrm{GL}_{n/t}} + \sum_{i=1}^{t-h}\dim x_i^{\mathrm{GL}_{n/t}}$$

Now, let $y = [y_1, x_2, \ldots, x_t]$ where $y_1 = J_{n/t}$. In particular, in y, the block $J_{n/t}$ has multiplicity $h+1$, while J_i (with $i < n/t$) has multiplicity $a_i - a_{1,i}$. Say $b_i = a_i - a_{1,i}$, for $i < n/t$; then $\sum_i ib_i = (t-h-1)n/t$.

Assume that $\dim(y^G \cap H) > \dim y^{H^\circ}$. Then there exists a block decomposition $\bar{y} = [y_1, A_2, \ldots, A_{t-h}, \ldots, x_t]$ such that $\dim(y^G \cap H) = \dim \bar{y}^{H^\circ} > \dim y^{H^\circ}$. This is absurd since, for $\bar{x} = [x_1, A_2, \ldots, A_{t-h}, x_{t-h+1}, \ldots, x_t] \in x^G$, we would get $\dim \bar{x}^{H^\circ} > \dim x^{H^\circ} = \dim(x^G \cap H)$. Therefore $\dim(y^G \cap H) = \dim y^{H^\circ}$.

Hence, in the same way as above, we compute $\dim y^G$ and $\dim(y^G \cap H)$. We have

$$\begin{aligned}\dim y^G &= n^2 - 2\sum_{i<j\le n/t} i b_i b_j - \sum_{i\le n/t} i b_i^2\\ &= n^2 - 2(h+1)\sum_{i<n/t} i b_i - 2\sum_{i<j<n/t} i b_i b_j - \frac{n}{t}(h+1)^2 - \sum_{i<n/t} i b_i^2\end{aligned}$$

and

$$\dim(y^G \cap H) = (h+1)\dim[J_{n/t}]^{\mathrm{GL}_{n/t}} + \sum_{i=2}^{t-h}\dim x_i^{\mathrm{GL}_{n/t}}$$

Now, we have $f_\Omega(x) > f_\Omega(y)$ if, and only if, $\dim x^G - \dim y^G < \dim(x^G \cap H) - \dim(y^G \cap H)$. By the above computations, using $a_j = \sum_{i=1}^{t-h} a_{i,j}$ and $b_j = \sum_{i=1}^{t-h-1} a_{i,j}$, it is an easy calculation to deduce the following

$$\begin{aligned}\dim x^G - \dim y^G &= 2\frac{n}{t}(t-h) - \frac{n}{t} - \sum_{i<n/t} i a_{1,i}^2 - 2\sum_{i<n/t} i a_{1,i}(a_{2,i} + \ldots + a_{t-h,i})\\ &\quad - 2\sum_{1\le i<j<n/t} i a_{1,i} a_{1,j} - 2\sum_{1\le i<j<n/t} i a_{1,i}(a_{2,j} + \ldots + a_{t-h,j})\\ &\quad - 2\sum_{1\le i<j<n/t} i a_{1,j}(a_{2,i} + \ldots + a_{t-h,i})\\ \dim x^{H^\circ} - \dim y^{H^\circ} &= \dim x_1^{\mathrm{GL}_{n/t}} - \dim[J_{n/t}]^{\mathrm{GL}_{n/t}}\\ &= \frac{n}{t} - 2\sum_{1\le i<j<n/t} i a_{1,i} a_{1,j} - \sum_{i<n/t} i a_{1,i}^2\end{aligned}$$

Simplifying we see that $\dim x^G - \dim y^G < \dim(x^G \cap H^\circ) - \dim(y^G \cap H^\circ)$ is equivalent to

(188)

$$\sum_{i \le j} i a_{1,i}(a_{2,j} + \ldots + a_{t-h,j}) + \sum_{i<j} i a_{1,j}(a_{2,i} + \ldots + a_{t-h,i}) > \frac{n}{t}(t-h-1)$$
$$= \sum_{i<n/t} i(a_{2,i} + \ldots + a_{t-h,i})$$

To prove (188) we shall show that, for all $2 \le l \le t-h$, the coefficient of $a_{l,i}$ in the left hand side is strictly greater than the coefficient of $a_{l,i}$ in the right hand side.

Let $i \in \{1, \ldots, n/t - 1\}$. Then for all $l \in \{2, \ldots, t-h\}$, the coefficient of $a_{l,i}$ in the left hand side is

$$\sum_{j \le i} j a_{1,j} + \sum_{i<j} i a_{1,j}$$

and the coefficient of $a_{l,i}$ in the right hand side is i. We claim

(189)
$$\sum_{j \le i} j a_{1,j} + \sum_{i<j} i a_{1,j} > i$$

which leads to the result. If there is $j > i$ for which $a_{1,j} \ne 0$ then $j a_{1,j} > i$ hence (189) is satisfied. If for all $j > i$ we have $a_{1,j} = 0$ then $\sum_{j \le i} j a_{1,j} = n/t > i$. Hence (189) is proved. The result follow. *q.e.d.*

We now prove part (i) of Proposition 18.2.1.

Lemma 18.2.6. *Let $x \in H$ be of order $p > n$. Then*

$$f_\Omega(x) \ge \frac{t}{n}$$

with equality if, and only, $x \in [J_{n/t}^{t-1}, z]^G$ where $z \in \mathrm{GL}_{n/t}$ is a unipotent element.

PROOF. As remarked above, by Lemma 17.3.1, we have $x^G \cap H = x^G \cap H^\circ$. Hence we may assume $x = [x_1, \ldots, x_t] \in H^\circ$ and $x_1, \ldots, x_t \in \mathrm{GL}_{n/t}$ are unipotent of order p, in the notation defined in (187).

If $t-1$ blocks of x are equal to $J_{n/t}$ then, by Lemma 18.2.4, we have $f_\Omega(x) = t/n$.

Hence, we may assume that at least two blocks, say x_1, x_2, are different from $J_{n/t}$. Then, by Lemma 18.2.5, we construct $y = [y_1, x_2, \ldots, x_t]$ with $y_1 = J_{n/t}$ and we have $f_\Omega(x) > f_\Omega(y)$. We have two cases.

Case 1. If $x_3 = \ldots = x_t = J_{n/t}$ then, by Lemma 18.2.4, we have $f_\Omega(x) > f_\Omega(y) = t/n$ and the result follows.

Case 2. If in y there are at least two blocks, say x_2, x_3, different from $J_{n/t}$ then, using Lemma 18.2.5, we construct a new element $y' = [y_1, y_2, x_3, \ldots, x_t]$, with $y_1 = y_2 = J_{n/t}$, for which $f_\Omega(x) > f_\Omega(y) > f_\Omega(y')$. Again, either $f_\Omega(y') = t/n$ or there are at least two blocks among the x_i's different to $J_{n/t}$. Iterating the construction of Lemma 18.2.5 we have, in finitely many steps,

$$f_\Omega(x) > f_\Omega(y) > f_\Omega(y') > \ldots > f_\Omega([J_{n/t}^{t-1}, x_t]) = t/n$$

And also the characterisation follows. *q.e.d.*

Remark 18.2.7. In the characteristic zero case, we have that an element is unipotent if, and only if, its only eigenvalue is 1. Moreover, by Lemma B.3.1, if $x \in H$ is unipotent then $x^G \cap H = x^G \cap H^\circ$. Therefore the zero characteristic case falls in the hypothesis $p > n$.

Remark 18.2.8. Assume $p > n/2$ and $t < n$. Let $x \in H$ be of order p. Then, by the proof of Lemma 18.2.3, $x^G \cap H = x^G \cap H^\circ$.

18.2.2. Case $p \leq n$. In this case, the largest Jordan block of any element of order p in G has size p. Thus, up to G-conjugacy, $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ with $\sum_i ia_i = n$.

We directly prove that for any $x \in H$ of order p we have $f_\Omega(x) \geq 1/p$.

Lemma 18.2.9. *Let $x \in H$ be of order p. Then $f_\Omega(x) \geq 1/p$.*

Proof. For some $0 \leq h \leq t/p$, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$. Recall that $\pi_h \in S_t$ is any permutation with cycle shape $(p^h, 1^{t-hp})$, see Lemma 17.4.1.

By Lemma 17.4.2, we have that, for suitable $x_{hp+1}, \ldots, x_t \in \mathrm{GL}_{n/t}$ of order p,

$$\dim(x^G \cap H^\circ \pi_h) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{hp+1}, \ldots, x_t]\pi_h)^{H^\circ} \tag{190}$$

Hence, by (184),

$$\dim(x^G \cap H) = h(p-1)\dim \mathrm{GL}_{n/t} + \sum_{i=hp+1}^{t} \dim x_i^{\mathrm{GL}_{n/t}} \tag{191}$$

Say $x_i = [J_p^{a_{i,p}}, \ldots, J_1^{a_{i,1}}]$, for all i. We define $l = hp+1$. Then, by Proposition 17.4.8, we have

$$x = \left[J_p^{nh/t + \sum_{i \geq l} a_{i,p}}, J_{p-1}^{\sum_{i\geq l} a_{i,p-1}}, \ldots, J_1^{\sum_{i \geq l} a_{i,1}}\right]$$

Notice that $\sum_{i \leq p} i(a_{l,i} + \ldots + a_{t,i}) = n - \frac{n}{t}hp$. Using Theorem 5.2.1, we have

$$\begin{aligned}
\dim x^G &= n^2 - 2\Big(\frac{n}{t}h + \sum_{i \geq l} a_{i,p}\Big)\sum_{i<p} i(a_{l,i} + \ldots + a_{t,i}) \\
&\quad - 2\sum_{1 \leq i < j < p} i(a_{l,i} + \ldots + a_{t,i})(a_{l,j} + \ldots + a_{t,j}) \\
&\quad - p\Big(\frac{n}{t}h + \sum_{i \geq l} a_{i,p}\Big)^2 - \sum_{i<p} i(a_{l,i} + \ldots + a_{t,i})^2 \\
&= n^2 - 2\Big(\frac{n}{t}\Big)^2 ht + \Big(\frac{n}{t}h\Big)^2 p \\
&\quad - 2\sum_{1 \leq i < j \leq p} i(a_{l,i} + \ldots + a_{t,i})(a_{l,j} + \ldots + a_{t,j}) - \sum_{i<p} i(a_{l,i} + \ldots + a_{t,i})^2
\end{aligned}$$

Using (191), we compute

$$\dim(x^G \cap H) = \Big(\frac{n}{t}\Big)^2(t-1) - 2\sum_{1 \leq i<j \leq p} i(a_{l,i}a_{l,j} + \ldots + a_{t,i}a_{t,j}) - \sum_{i \leq p} i(a_{l,i}^2 + \ldots + a_{t,i}^2)$$

We have $f_\Omega(x) \geq 1/p$ if, and only if,

$$-\dim x^G + \dim(x^G \cap H) \geq -\dim \Omega\Big(1 - \frac{1}{p}\Big) \tag{192}$$

Substituting the values of $\dim x^G$ and $\dim(x^G\cap H)$ computed above in (192) after a few calculations we get that (192) is equivalent to

$$
\begin{aligned}
(193)\qquad & 2\sum_{1\leq i<j\leq p} i(a_{l,i}+\ldots+a_{t,i})(a_{l,j}+\ldots+a_{t,j})\\
&\qquad -2\sum_{1\leq i<j\leq p} i(a_{l,i}a_{l,j}+\ldots+a_{t,i}a_{t,j})\\
&+\sum_{i=1}^{p} i(a_{l,i}+\ldots+a_{t,i})^2-\sum_{i=1}^{p} i(a_{l,i}^2+\ldots+a_{t,i}^2)\geq \frac{n^2(t-hp)(t-hp-1)}{t^2p}
\end{aligned}
$$

If $hp=t$ then both sides of (192) are 0 (the left hand side is 0 because from (190) we see that x is G-conjugate to π_h and so there are no x_i's).

If $hp+1=t$ then, again, both sides of (192) are 0, observe that in (190) there is only one block x_t (since $l=t$).

Thus, assume $hp<t-1$. It requires only a few steps to see that the following is equivalent to (193):

$$\sum_{l\leq\alpha<\beta\leq t}\Big(\sum_{1\leq i<j\leq p} i(a_{\alpha,i}a_{\beta,j}+a_{\alpha,j}a_{\beta,i})+\sum_{i=1}^{p} ia_{\alpha,i}a_{\beta,i}\Big)\geq \frac{n^2(t-hp)(t-hp-1)}{2t^2p}$$

Say $\mathcal{I}=\{(\alpha,\beta): l\leq\alpha<\beta\leq t\}$ then $|\mathcal{I}|=\frac{(t-l-1)(t-l-2)}{2}=\frac{(t-hp)(t-hp-1)}{2}$. Therefore, fix a pair $(\alpha,\beta)\in\mathcal{I}$, it is enough to show that each summand of the external summation is greater than or equal to $\frac{1}{p}(n/t)^2$. Fix $(\alpha,\beta)\in\mathcal{I}$, and in order of simplifying notation, set $a_{\alpha,i}=a_i$ and $a_{\beta,i}=b_i$. Thus we need to show

$$(194)\qquad \sum_{1\leq i<j\leq t} i(a_ib_j+a_jb_i)+\sum_{i=1}^{p} ia_ib_i\geq \frac{1}{p}\Big(\frac{n}{t}\Big)^2=\frac{n}{tp}\sum_{i=1}^{p} ia_i$$

where we used $n/t=\sum_i ia_i$. In order to do this we shall prove that, for all $i\in\{1,\ldots,p\}$, the coefficient of a_i in the left hand side of (194) is greater than or equal to the coefficient of a_i in the right hand side of (194).

Fix $i\in\{1,\ldots,p\}$. The coefficient of a_i in the right hand side of (194) is $i\frac{n}{tp}$. The coefficient of a_i in the left hand side of (194) is

$$(195)\qquad i\sum_{j\geq i} b_j+\sum_{j<i} jb_j$$

If for every $j\geq i$ we have $b_j=0$, then (195) is $\sum_j jb_j=\frac{n}{t}\geq i\frac{n}{tp}$, since $i\leq p$.

Assume there exists $j\geq i$ such that $b_j\neq 0$. Then we claim

$$i\sum_{j\geq i} b_j+\sum_{j<i} jb_j\geq \frac{i}{p}\Big(\frac{n}{t}\Big)=\frac{i}{p}\sum_{j=1}^{p} jb_j$$

which is satisfied since

$$i\sum_{j\geq i} b_j+\sum_{j<i} jb_j-\frac{i}{p}\sum_{j=1}^{p} jb_j=i\sum_{j\geq i} b_j\Big(1-\frac{j}{p}\Big)+\sum_{j<i} jb_j\Big(1-\frac{i}{p}\Big)\geq 0$$

being $i,j\leq p$. Hence, for all $(\alpha,\beta)\in\mathcal{I}$, (194) is proved. The result follows. *q.e.d.*

Remark 18.2.10. Notice that the proof of Proposition 18.2.9 applies verbatim for any p (possibly $p > n$). In particular the same argument holds if $p \geq n/2$ and $t = n$.

Now the aim is to show that the lower bound $1/p$ is the best possible. First we assume $p \mid n$ and we characterise conjugacy classes of elements $x \in G$ such that $f_\Omega(x) = 1/p$, see Proposition 18.2.16. Then we assume $p \nmid n$ and we show that there exists $x \in H^\circ$ such that $f_\Omega(x) = 1/p + \epsilon$, for a small $\epsilon \geq 0$, see Proposition 18.2.17.

Assume $p \mid n$. First we explicitly compute the f_Ω-value of certain elements.

Lemma 18.2.11. *Assume $p \mid n$.*

(i) If p does not divide n/t, let $x = [J_p^{n/p}]$. Then $f_\Omega(x) = 1/p$.
(ii) If p divides n/t, let $x = \left[J_p^{\frac{n}{pt}(t-1)}, z\right]$, with $z \in \mathrm{GL}_{n/t}$ of order p. Assume $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_h)$ as in (190) where $x_t = z$. Then $f_\Omega(x) = 1/p$.

Proof. Let us compute the f_Ω-values of the given elements separately.

Case (i). First assume $p \nmid n/t$, so that $x = [J_p^{n/p}]$. So, there is no integer a such that $[J_p^a] \in \mathrm{GL}_{n/t}$. Using Theorem 5.2.1 we compute $\dim x^G = n^2(1 - 1/p)$. There exists h such that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_h)$. For suitable $x_{hp+1}, \dots, x_t \in \mathrm{GL}_{n/t}$, we have

$$\dim(x^G \cap H) = \dim([I_{n/t}, \dots, I_{n/t}, x_{hp+1}, \dots, x_t]\pi_h)^{H^\circ}$$

The above observation forces $hp = t$, so $h = t/p$. Hence $\pi_h \in x^G$ and, using (184), we have $\dim(x^G \cap H) = \dim x^G/t$. Therefore $f_\Omega(x) = 1/p$.

Case (ii). Now assume $p \mid n/t$. Let x be as in the hypothesis. Then

$$\dim(x^G \cap H) = \dim([I_{n/t}, \dots, I_{n/t}, x_{hp+1}, \dots, x_t]\pi_h)^{H^\circ}$$

for some $h \leq t/p$ and $x_t = z$. This forces $x_{hp+1} = \dots = x_{t-1} = J_p^{n/(pt)}$. Up to conjugation, say $z = [J_p^{a_p}, \dots, J_1^{a_1}] \in \mathrm{GL}_{n/t}$. Using (184) and Theorem 5.2.1, we compute

$$\dim(x^G \cap H) = \frac{n^2}{t}\Big(1 - \frac{1}{p}\Big) + \frac{1}{p}\Big(\frac{n}{t}\Big)^2 - 2\sum_{i<j} ia_ia_j - \sum_i ia_i^2$$

We have $x = [J_p^{n(t-1)/(pt)+a_p}, J_{p-1}^{a_{p-1}}, \dots, J_1^{a_1}]$. Hence, using Theorem 5.2.1, we have

$$\dim x^G = n^2\Big(1 - \frac{1}{p}\Big) + \frac{1}{p}\Big(\frac{n}{t}\Big)^2 - 2\sum_{i<j\leq p} ia_ia_j - \sum_{i\leq p} ia_i^2$$

Therefore $f_\Omega(x) = 1/p$. *q.e.d.*

Here we characterise elements in H of order p whose f_Ω-value is $1/p$ in the case p divides n.

Lemma 18.2.12. *Assume $p \mid n$. Let $x \in H$ be an element of order p with $f_\Omega(x) = 1/p$. Then one of the following occurs:*

(i) $p \nmid n/t$ and $x \in [J_p^{n/p}]^G$;
(ii) $p \mid n/t$ and $x \in [J_p^{\frac{n}{tp}(t-1)}, z]^G$ for some $z \in \mathrm{GL}_{n/t}$ of order p such that $\dim(x^G \cap H) = \dim([I_{n/t}, \dots, I_{n/t}, x_{hp+1}, \dots, x_{t-1}, z]\pi_h)^{H^\circ}$ for some $h \leq t/p$ and $x_i \in [J_p^{n/tp}]^G$ for all $hp < i < t$.

PROOF. Let $x \in H$ with $f_\Omega(x) = 1/p$. For some $h \le t/p$ we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$, i.e.

$$\dim(x^G \cap H) = h(p-1)\left(\frac{n}{t}\right)^2 + \sum_{i=hp+1}^{t} \dim x_i^{\mathrm{GL}_{n/t}}$$

for suitable x_i's such that

$$[I_{n/t}, \ldots, I_{n/t}, x_{hp+1}, \ldots, x_t]\pi_h \in x^G \cap H^\circ \pi_h \tag{196}$$

We have three cases depending on h.

Case 1. If $hp = t$ then x is H°-conjugate to π_h (by Lemma 17.4.2), hence $x \in [J_p^{n/p}]^G$.

Case 2. If $hp = t-1$ then x is H°-conjugate to $[I_{n/t}, \ldots, I_{n/t}, x_t]\pi_h$. By Lemma 17.4.5, we may write $x = [J_p^{\frac{n}{tp}(t-1)}, z]$ for some $z \in \mathrm{GL}_{n/t}$ of order p such that $\dim(x^G \cap H) = \dim([I_{n/t}, \ldots, I_{n/t}, z]\pi_h)^{H^\circ}$.

Case 3. Assume $hp < t-1$. Let $hp+1 \le \alpha < \beta \le t$, and say a_i, b_i are the multiplicities of J_i in x_α, x_β, respectively.

Then, by the proof of Lemma 18.2.9, for all $i \in \{1, \ldots, p\}$, either $a_i = 0$ or

$$i\sum_{j\ge i} b_j + \sum_{j<i} jb_j = \frac{i}{p}\cdot\frac{n}{t} \tag{197}$$

Let i be such that $a_i \neq 0$. If $i = p$ then (197) is always satisfied, for any b_i's.

If $i < p$, then, using $n/t = \sum_{j=1}^{p} jb_j$, (197) is equivalent to

$$i\sum_{j\ge i} b_j\left(1-\frac{j}{p}\right) + \sum_{j<i} jb_j\left(1-\frac{i}{p}\right) = 0$$

which is satisfied if, and only if, $b_j = 0$ for all $j < p$. Therefore $n/t = pb_p$. This holds for all the pairs (α, β) with $hp+1 \le \alpha < \beta \le t$. Hence, at most one block can be different from $[J_p^{n/pt}]$. Hence, using Lemma 17.4.5, we deduce that $x \in [J_p^{(n/tp)(t-1)}, x_t]^G$ for some $x_t \in \mathrm{GL}_{n/t}$ for which

$$\dim(x^G \cap H) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{hp+1}, \ldots, x_t]\pi_h)^{H^\circ}$$

The result follows. *q.e.d.*

In order to gain a full characterisation of the elements $x \in H$ of order p for which $f_\Omega(x) = 1/p$ – in the case p divides n/t – we need to characterise the elements $[J_p^{\frac{n}{tp}(t-1)}, z]$, with $z \in \mathrm{GL}_{n/t}$ of order p, such that

$$\dim(x^G \cap H^\circ) = (t-1)\dim\left[J_p^{\frac{n}{tp}}\right]^{\mathrm{GL}_{n/t}} + \dim z^{\mathrm{GL}_{n/t}}$$

Clearly, we need only to find these properties for $\dim(x^G \cap H^\circ)$ since for the computation of $\dim(x^G \cap H)$ we will need to understand the block decomposition that maximises the H°-class dimension of $[I_{n/t}, \ldots, I_{n/t}, x_{hp+1}, \ldots, x_t]\pi_h$.

For this purpose let us recall that conjugacy classes of unipotent elements of $G = \mathrm{GL}_n$ are in one-to-one correspondence with partitions of n.

For the purpose of the next result we introduce the following notation on partitions. Let $\lambda \vdash n$ so that $\lambda = (n^{a_n}, \dots, 1^{a_1})$. We say $\lambda = \mu \oplus \eta$ with $\mu = (n^{b_n}, \dots, 1^{b_1})$ and $\eta = (n^{c_n}, \dots, 1^{c_1})$ if $a_i = b_i + c_i$ for all i.

Lemma 18.2.13. *Let $n/t = pm$ and $x = [J_p^m, \dots, J_p^m, z] \in H^\circ$. Say $\lambda = (p^{a_p}, \dots, 1^{a_1})$ the partition of n/t associated to z. Assume*

(1) $a_p < m$;
(2) $\lambda = \mu \oplus \eta$ where $\mu = (p^{b_p}, \dots, 1^{b_1}) \vdash l_1 p$, $\eta = (p^{c_p}, \dots, 1^{c_1}) \vdash l_2 p$ for some $0 < l_1, l_2 < m$, with $b_i, c_j > 0$ for some $i, j < p$.

Then

$$\dim(x^G \cap H^\circ) > \dim x^{H^\circ}$$

Proof. Observe that $m = l_1 + l_2$. Define $\lambda_1 = (p^{m-l_1}) \oplus \mu$ and $\lambda_2 = (p^{m-l_2}) \oplus \eta$. Then $\lambda_1, \lambda_2 \vdash n/t$. Let $y_1, y_2 \in \mathrm{GL}_{n/t}$ be the unipotent elements associated to λ_1, λ_2, respectively. Set $y = [y_1 y_2, y_3, \dots, y_t]$ with $y_i = [J_p^m]$ for $i > 2$. Then $y \in x^G$.

Claim: $\dim y^{H^\circ} > \dim x^{H^\circ}$.

Clearly, if we prove the claim we obtain the result since $\dim(x^G \cap H^\circ) \geq \dim y^{H^\circ}$.

We have $\dim y^{H^\circ} = \dim y_1^{\mathrm{GL}_{n/t}} + \dim y_2^{\mathrm{GL}_{n/t}} + (t-2)\dim[J_p^m]^{\mathrm{GL}_{n/t}}$ and, similarly $\dim x^{H^\circ} = \dim z^{\mathrm{GL}_{n/t}} + (t-1)\dim[J_p^m]^{\mathrm{GL}_{n/t}}$. Therefore the claim is equivalent to

$$\dim y_1^{\mathrm{GL}_{n/t}} + \dim y_2^{\mathrm{GL}_{n/t}} - \dim[J_p^m]^{\mathrm{GL}_{n/t}} - \dim z^{\mathrm{GL}_{n/t}} > 0 \tag{198}$$

In order to compute the dimensions of the $\mathrm{GL}_{n/t}$-conjugacy classes in (198), let us recall the partitions of $n/t = pm$ associated to some of those elements:

$$\begin{aligned} y_1 &\longleftrightarrow \left(p^{m-l_1+b_p}, (p-1)^{b_{p-1}}, \dots, 1^{b_1}\right) \\ y_2 &\longleftrightarrow \left(p^{m-l_2+c_p}, (p-1)^{c_{p-1}}, \dots, 1^{c_1}\right) \\ z &\longleftrightarrow \left(p^{b_p+c_p}, \dots, 1^{b_1+c_1}\right) \end{aligned}$$

Recall that $\sum_i ib_i = l_1 p$ and $\sum_i ic_i = l_2 p$ and $(l_1 + l_2)p = mp = n/t$.

Using Theorem 5.2.1, we compute

$$\begin{aligned} \dim[J_p^m]^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - \left(\frac{n}{t}\right)m \\ \dim z^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - 2\sum_{i<j\leq p} i(b_i+c_i)(b_j+c_j) - \sum_{i\leq p} i(b_i+c_i)^2 \end{aligned}$$

In the same way we compute

$$\begin{aligned} \dim y_1^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - 2(m-l_1+b_p)\sum_{i<p} ib_i - 2\sum_{i<j<p} ib_ib_j - p(m-l_1+b_p)^2 - \sum_{i<p} ib_i^2 \\ \dim y_2^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - 2(m-l_2+c_p)\sum_{i<p} ic_i - 2\sum_{i<j<p} ic_ic_j - p(m-l_2+c_p)^2 - \sum_{i<p} ic_i^2 \end{aligned}$$

Using $\sum_{i<p} ib_i = l_1p - pb_p$ we simplify

$$\begin{aligned}\dim y_1^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - 2(m-l_1)\sum_{i<p} ib_i - 2\sum_{i<j\le p} ib_ib_j - p(m-l_1)^2 - 2pb_p(m-l_1) - \sum_{i\le p} ib_i^2 \\ &= \left(\frac{n}{t}\right)^2 - p(m^2-l_1^2) - 2\sum_{i<j\le p} ib_ib_j - \sum_{i\le p} ib_i^2\end{aligned}$$

Similarly

$$\begin{aligned}\dim y_1^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - p(m^2-l_2^2) - 2\sum_{i<j\le p} ic_ic_j - \sum_{i\le p} ic_i^2 \\ \dim z^{\mathrm{GL}_{n/t}} &= \left(\frac{n}{t}\right)^2 - 2\sum_{i<j\le p} i(b_ib_j + c_ic_j + b_ic_j + b_jc_i) - \sum_{i\le p} i(b_i^2 + 2b_ic_i + c_i^2)\end{aligned}$$

It is a straightforward calculation to see that (198) is equivalent to

$$-p(m^2 - l_1^2 - l_2^2) + 2\sum_{i<j\le p} i(b_ic_j + b_jc_i) + 2\sum_{i\le p} ib_ic_i > 0 \tag{199}$$

Using $m = l_1 + l_2$ and $\sum_i ib_i = l_1p$ we see that (199) is equivalent to

$$\sum_{i<j\le p} i(b_ic_j + b_jc_i) + \sum_{i\le p} ib_ic_i > l_2\sum_{i\le p} ib_i$$

As usual, to prove the previous inequality, we show that the coefficient of b_i in the left hand side is strictly greater than the coefficient of b_i in the right hand side for all $i<p$ (for $i=p$ equality holds). Fix $i \in \{1,\dots,p-1\}$. We need to show

$$i\sum_{j\ge i} c_j + \sum_{j<i} jc_j > l_2 i \tag{200}$$

We have two cases, either $c_j = 0$ for all $j \ge i$ or there exists $j \ge i$ such that $c_j \ne 0$.

Case 1. Assume $c_j = 0$ for all $j \ge i$. Then the left hand side of (200) is $\sum_{j<i} jc_j = l_2p$ which is strictly greater than l_2i, being $i<p$.

Case 2. Assume $c_j \ne 0$ for some $j \ge i$. Then, writing, $l_2i = l_2p\frac{i}{p} = \frac{i}{p}\sum_{j\le p} jc_j$ we deduce that (200) is equivalent to

$$i\sum_{i\le j\le p} c_j\left(1-\frac{j}{p}\right) + \sum_{j<i} jc_j\left(1-\frac{i}{p}\right) > 0$$

since $i,j<p$ and there exists $j<p$ such that $c_j>0$. *q.e.d.*

Remark 18.2.14. Assume $n/t = pm$. Let $x = [J_p^{\frac{n}{pt}(t-1)}, z]$, where $z \in \mathrm{GL}_{n/t}$ has order p. Say $\lambda = (p^{a_p},\dots,1^{a_1}) \vdash n/t$ is the partition associated to z. Assume $a_p < m$ and that the condition (2) in Lemma 18.2.13 does not hold. Then z does not contain two proper sub-partitions of multiples of p not comprising only p parts.

Assume $n/t = pm$. Thanks to Lemma 18.2.13, we have the following.

Proposition 18.2.15. *Assume $n/t = pm$. Let $x = [J_p^{\frac{n}{pt}(t-1)}, z]$, where $z \in \mathrm{GL}_{n/t}$ has order p. Say $\lambda = (p^{a_p},\dots,1^{a_1}) \vdash n/t$ is the partition associated to z. Then*

$$\dim(x^G \cap H) = \dim([I_{n/t},\dots,I_{n/t},x_{hp+1},\dots,x_{t-1},z]\pi_h)^{H^\circ}$$

for some $h \leq t/p$, if, and only if, either $a_p = m$ or condition (2) in Lemma 18.2.13 does not hold.

Proof. If $x = [J_p^{n/p}]$ then $z = [J_p^{n/(tp)}]$ and the result trivially follows. Similarly, let $\lambda \vdash n/t$, the partition associated to z. Assume λ satisfies neither (1) nor (2) of Lemma 18.2.13. Then for any decomposition $x = [A_1, \ldots, A_t] \in H^\circ$ we must have $A_t = z$ (up to a permutation of the blocks), since z does not contain two proper sub-partitions of any multiples of p, both of them with some non-p parts, see Remark 18.2.14.

Conversely, the contrapositive is proved in Lemma 18.2.13. *q.e.d.*

Now we can state a complete classification of elements $x \in H$ of order p for which $f_\Omega(x) = 1/p$, in the case p divides n. Recall $t < n$, the case $t = n$ is dealt with in Section 18.4.2.2.

Proposition 18.2.16. *Assume $p \mid n$. Let $x \in G$ be of order p. Then $f_\Omega(x) = 1/p$ if, and only if, one of the following holds*

(i) $x \in [J_p^{n/p}]^G$;

(ii) $n/t = pm$ *and* $x \in [J_p^{m(t-1)}, z]^G$ *for some* $z \in \mathrm{GL}_{n/t}$ *of order p whose associated partition* $\lambda = (p^{a_p}, \ldots, 1^{a_1}) \vdash n/t$ *satisfies one of the following*

  (a) $a_p = m$; *or,*

  (b) $a_p < m$ *and whenever* $\lambda = \mu \oplus \eta$ *with* $\mu \vdash l_1 p$, $\eta \vdash l_2 p$, *for some* $l_1, l_2 < m$, *either* $\mu = (p^{l_1})$ *or* $\eta = (p^{l_2})$.

Proof. If $x^G \cap H = \emptyset$ then $f_\Omega(x) = 0$. Hence we may assume $x \in H$. If x is one of the elements in the hypothesis then, by Lemma 18.2.11, we have $f_\Omega(x) = 1/p$.

Conversely, if $x \in H$ has order p and $f_\Omega(x) = 1/p$ then, by Lemma 18.2.12 and Proposition 18.2.15, the result follows. *q.e.d.*

If p does not divide n we have the following.

Proposition 18.2.17. *Assume $p \nmid n$. Then there exists $x \in H^\circ$ such that*

$$f_\Omega(x) \leq \frac{1}{p} + \frac{p}{4}\Big(\frac{t}{n}\Big)^2$$

Proof. Let $n/t = ap + b$ with $0 \leq b < p$. Let $x = [J_p^{at}, J_b^t]$. Then by Lemma 17.3.4 we have $\dim(x^G \cap H) = \frac{1}{t} \dim x^G$. Using Theorem 5.2.1, we compute

$$\dim x^G = n^2\Big(1 - \frac{1}{p}\Big) - bt^2\Big(1 - \frac{b}{p}\Big)$$

Therefore

$$f_\Omega(x) = \frac{1}{p} + \frac{b}{p}(p-b)\Big(\frac{t}{n}\Big)^2 \leq \frac{1}{p} + \frac{p}{4}\Big(\frac{t}{n}\Big)^2$$

where the inequality follows from $b(p-b) \leq p^2/4$. *q.e.d.*

18.3. Semisimple elements: lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order $r \neq p$. Up to G-conjugacy, we may write

(201) $$x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$

where $\omega \in k$ is a primitive r-th root of unity.

We shall prove the following.

Proposition 18.3.1. *Let $x \in H$ be an element of prime order r.*

(i) If $r > n$ then $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$.
(ii) If $r \leq n$ then

$$f_\Omega(x) \geq \frac{1}{r} - \epsilon$$

where $\epsilon = 0$ if $n \equiv 0 \pmod{rt}$ and $\epsilon = \frac{rt^2}{4n^2(t-1)}$ otherwise.

Remark 18.3.2. If $r \leq n$ the bound is not the best possible. In Proposition 18.3.19, we shall give the best possible bound on $f_\Omega^\circ(x)$. Notice that in the case $n \equiv 0 \pmod{rt}$ the lower bound is sharp for f_Ω°, thanks to Lemma 18.3.18. In the case where rt does not divide n our bound is often negative (never if $r = 2$). Notice that the lower bound in (ii) is decreasing in r, namely it is maximal when $r = 2$.

Thanks to Lemma 17.3.1 it is natural to consider the two cases $r > n$ and $r \leq n$.

18.3.1. Case $r > n$**.** Let $x \in H$ be an element of prime order $r > n$ as in (201). By Lemma 17.3.1, we have $x^G \cap H = x^G \cap H^\circ$. Hence we may assume $x = [x_1, \ldots, x_t] \in H^\circ$ such that $\dim(x^G \cap H) = \dim x^{H^\circ}$.

Let $x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \ldots, \omega^{r-1} I_{a_{i,r-1}}]$ for all $i \in \{1, \ldots, t\}$. Then

$$\dim x^G = n^2 - \sum_{j=0}^{r-1} \Big(\sum_{i=1}^{t} a_{i,j} \Big)^2, \ \dim(x^G \cap H) = \frac{n^2}{t} - \sum_{i=1}^{t} \sum_{j=0}^{r-1} a_{i,j}^2$$

Therefore we have

$$f_\Omega(x) = \frac{2t}{n^2(t-1)} \sum_{l=0}^{r-1} \Big(\sum_{1 \leq i < j \leq t} a_{i,l} a_{j,l} \Big) \tag{202}$$

Recall that Proposition 17.3.13 gives conditions on the eigenvalues of the x_i's in order to have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. In particular $|a_{i,l} - a_{j,l}| \leq 1$, for all i, j, l.

Lemma 18.3.3. *Let $x \in H$ of prime order r. Assume $\nu(x) = n-1$. Then $f_\Omega(x) = 0$.*

Proof. If $\nu(x) = n-1$ then x is a regular element, i.e. $C_G(x) = (\mathrm{GL}_1)^n$. Without loss of generality we may assume $x = [1, \omega, \ldots, \omega^{n-1}]$. Hence using Theorems 5.3.1 and 17.3.8 we compute $f_\Omega(x) = 0$. *q.e.d.*

In order to complete the characterisation of elements of prime order r with vanishing f_Ω we need the following.

Lemma 18.3.4. *Let $x = [x_1, \ldots, x_t] \in H^\circ$ with $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Assume $\nu(x) < n-1$ and let ω^l be the eigenvalue with largest multiplicity. Then, there exist $i \neq j \in \{1, \ldots, t\}$ such that $a_{i,l}, a_{j,l} \neq 0$.*

Proof. Since $\nu(x) < n-1$ we have $a_l \geq 2$ for some $l \geq 0$. Since $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, we have, by Proposition 17.3.13, $|a_{i,l} - a_{j,l}| \leq 1$ for all $i, j \in \{1, \ldots, t\}$. Let i be such that $a_{i,l} \neq 0$. If for all $j \neq i$ we have $a_{j,l} = 0$ then $a_{i,l} = a_l$ and $a_{i,l} - a_{j,l} = a_l \geq 2$ which is absurd. Hence there exists $j \neq i$ such that $a_{j,l} \neq 0$. *q.e.d.*

Now, we can complete the characterisation.

Lemma 18.3.5. *Let $x \in H$ be of order r. Assume $f_\Omega(x) = 0$. Then $\nu(x) = n - 1$.*

Proof. Assume $x = [x_1, \dots, x_t] \in H^\circ$ is such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. In (202) we have an explicit formula for $f_\Omega(x)$. Assume $\nu(x) < n - 1$. Then by Lemma 18.3.4, given ω^l the eigenvalue with largest multiplicity in x, there exist $i \neq j \in \{1, \dots, t\}$ such that $a_{i,l}a_{j,l} \neq 0$. So $f_\Omega(x) \geq \frac{2t}{n^2(t-1)} a_{i,l}a_{j,l} > 0$, which is absurd. Therefore $\nu(x) = n - 1$. *q.e.d.*

Part (i) of Proposition 18.3.1 follows from Lemmas 18.3.3 and 18.3.5.

18.3.2. Case $r \leq n$. In this section we derive a lower bound on $f_\Omega(x)$ for $x \in H$ an element of prime order $r \leq n$. Recall that

$$f^\circ_\Omega(x) = \frac{\dim \Omega - \dim x^G + \dim(x^G \cap H^\circ)}{\dim \Omega}$$

And, in general, $f_\Omega(x) \geq f^\circ_\Omega(x)$. Moreover, for all $x \in G$ of prime order r we have $x^G \cap H^\circ \neq \emptyset$, since H contains a maximal torus. Therefore our strategy is to find a lower bound on f°_Ω; recall that, by Corollary 17.3.20, $f_\Omega(x) - f^\circ_\Omega(x) \leq \frac{t}{n(t-1)} \leq \frac{2}{n}$.

Remark 18.3.6. Assume $r > t$. Let $x \in H$ be of order r. Then, by Lemma 17.3.1, we have $x^G \cap H = x^G \cap H^\circ$. In particular, $f_\Omega(x) = f^\circ_\Omega(x)$.

We have the following.

Proposition 18.3.7. *Let $x \in H$ of prime order $r \leq n$. Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{4n^2(t-1)}$$

Proof. Let $x \in H$ of prime order r. Then, by Remark 17.3.14, we have $\dim(x^G \cap H^\circ) = (\dim x^G - \sum_i b_i(t - b_i))/t$, where $a_i = c_i t + b_i$ with $0 \leq b_i < t$. Therefore

$$f^\circ_\Omega(x) = \frac{n^2\left(1 - \frac{1}{t}\right) - \dim x^G\left(1 - \frac{1}{t}\right) - \frac{1}{t}\sum_{i=0}^{r-1} b_i(t - b_i)}{n^2\left(1 - \frac{1}{t}\right)} = \frac{\sum_{i=0}^{r-1} a_i^2}{n^2} - \frac{\sum_{i=0}^{r-1} b_i(t - b_1)}{n^2(t-1)}$$

Using Proposition B.2.1, we have $\sum_i a_i^2 \geq \left(\sum_i a_i\right)^2/r = \frac{n^2}{r}$. Moreover, for all i, we have $b_i(t - b_i) \leq t^2/4$. The result follows. *q.e.d.*

Remark 18.3.8. Notice that Proposition 18.3.7 holds in the case $r > n$, as well. However, for $r > n$ we have $\frac{1}{r} - \frac{rt^2}{4n^2(t-1)} < \frac{1}{n} - \frac{t^2}{4n(t-1)} \leq 0$.

In fact, we can derive a better lower bound. We shall construct elements $y \in H$ of prime order r such that for all $x \in H$ of order r we have $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$. Recall that for $x \in G$ of order r we denote $a_i = \dim V_{\omega^i}$.

Definition 18.3.9. Let $x \in G$ be of order r. We say that x is *special* if $|a_i - a_j| \leq 1$ for all $i, j \in \{0, \dots, r-1\}$.

Observe that if x is not special there exist $i, j \in \{0, \dots, r-1\}$ such that $a_i - a_j \geq 2$.

Claim. Let $x \in H$ be of order r. Then $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$ where $y \in H$ is special of order r.

Lemma 18.3.10. *Let $x \in H$ be of order r. Then, either x is special or there exists $y = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}] \in H$ such that $f^\circ_\Omega(x) = f^\circ_\Omega(y)$ and $b_0 - b_1 \geq 2$.*

Proof. Up to G-conjugacy we may write $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H$. Assume x is not special. Then $a_i - a_j \geq 2$ for some $i \neq j$, without loss of generality we may assume $i < j$. Let us define

$$y = [I_{a_i}, \omega I_{a_j}, \ldots, \omega^i I_{a_0}, \ldots, \omega^j I_{a_1}, \omega^{r-1} I_{a_{r-1}}]$$

Since $C_G(x) \cong C_G(y)$ we have $\dim x^G = \dim y^G$. Moreover $\dim(x^G \cap H^\circ) = \dim(y^G \cap H^\circ)$, thanks to Corollary 17.3.15. Therefore $f^\circ_\Omega(x) = f^\circ_\Omega(y)$. *q.e.d.*

The key tool in order to prove the claim is the following technical lemma for non-special elements. Thanks to Lemma 18.3.10 if x is non-special we may assume that $a_0 - a_1 \geq 2$.

Lemma 18.3.11. *Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H$ of prime order r. Assume $a_0 = \max_i\{a_i\}$, $a_1 = \min_i\{a_i\}$ and $a_0 - a_1 \geq 2$. Let $y = [I_{a_0-1}, \omega I_{a_1+1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$$

with equality if, and only if, $a_0 - a_1 = 2$ and $a_0 \equiv 1 \pmod t$.

Proof. Write $a_0 = a_1 + h$ for some $h \geq 2$. The result is equivalent to

$$\dim y^G - \dim x^G > \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \tag{203}$$

We have $\dim y^G - \dim x^G = 2(a_0 - a_1 - 1)$ and using Theorem 17.3.8 we compute:

$$\begin{aligned}\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = &\Big(\Big\lfloor \frac{a_0 - 1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_0}{t} \Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{a_0 - 1}{t} \Big\rfloor - \Big\lfloor \frac{a_0}{t} \Big\rfloor\Big)(t - 2a_0) \\ &+ \Big(\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_1}{t} \Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big)(t - 2a_1) \\ &+ 2\Big\lfloor \frac{a_0 - 1}{t} \Big\rfloor - 2\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor\end{aligned}$$

Thus, we have four cases, depending on the values of the floor functions.

Case 1. For $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$, then (203) is equivalent to

$$a_0 - a_1 - 1 > \Big\lfloor \frac{a_0}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor \tag{204}$$

Case 2. If $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$, (203) is equivalent to

$$(t-1)\Big(\Big\lfloor \frac{a_0}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor - 1\Big) > 0 \tag{205}$$

Case 3. If $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$, the inequality (203) is equivalent to

$$a_1 < t\Big\lfloor \frac{a_0}{t} \Big\rfloor + \Big\lfloor \frac{a_1}{t} \Big\rfloor - \Big\lfloor \frac{a_0}{t} \Big\rfloor \tag{206}$$

Case 4. Finally, for $\lfloor \frac{a_0-1}{t} \rfloor = \lfloor \frac{a_0}{t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$, (203) is equivalent to

$$a_0 > \Big\lfloor \frac{a_0}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor + \Big\lfloor \frac{a_1}{t} \Big\rfloor t + t \tag{207}$$

We now prove the inequalities in each of these four cases.

Case 1. Write $a_0 = b_0t + c_0$ and $a_1 = b_1t + c_1$ where $0 < c_0 < t$ and $0 \leq c_1 < t-1$. Moreover, since $a_0 - a_1 \geq 2$ we also have $b_0 \geq b_1$. Then (204) is equivalent to $a_0 - a_1 - 1 > b_0 - b_1$.

If $b_0 = b_1$, we immediately get the result, since $a_0 - a_1 \geq 2$, and hence $a_0 - a_1 - 1 > 0$.

Thus, assume $b_0 > b_1$. Then there exists an integer $l \geq 1$ such that $b_0 = b_1 + l$. Thus the left hand side of (204) is $a_0 - a_1 - 1 = lt + c_0 - c_1 - 1$, and the right hand side is l. Therefore $lt - l + c_0 - c_1 - 1 > 0$ is equivalent to (204). Since $c_0 \geq 1$ and $-c_1 \geq 2 - t$, we have

$$lt - l + c_0 - c_1 - 1 \geq lt - l + 1 + 2 - t - 1 = l(t-1) - t + 2$$

So, we need to show $l(t-1) - t + 2 > 0$, which is clearly true since it is linear in l and $t - 1 > 0$. Thus $l(t-1) - t + 2 \geq (t-1) - t + 2 > 0$. Hence, (204) has been proved.

Case 2. In the second case we have $a_0 = b_0t$ and $a_1 = b_1t - 1 = (b_1 - 1)t + (t-1)$. Therefore the inequality (205) is equivalent to $b_0 - b_1 > 0$. Assume $b_0 \leq b_1$. Then $a_0 - a_1 = b_0t - b_1t + 1 \leq b_1t - b_1t + 1 = 1$, a contradiction since $a_0 \geq a_1 + 2$.

Case 3. In this case we have $a_0 = b_0t$ and $a_1 = b_1t + c_1$, with $c_1 \in \{0, \dots, t-2\}$. Hence the inequality (206) is equivalent to

$$(b_0 - b_1)(t-1) > c_1$$

Let us assume $b_0 \leq b_1$. Then $b_0t \leq b_1t \leq b_1t + c_1$, i.e. $a_0 \leq a_1$, which contradicts the hypothesis, thus $b_0 - b_1 \geq 1$. Hence $(b_0 - b_1)(t-1) \geq t - 1 > c_1$.

Case 4. In this case we have $a_0 = b_0t + c_0$, with $c_0 \in \{1, \dots, t-1\}$, and $a_1 = b_1t - 1 = (b_1 - 1)t + (t-1)$. And the inequality (207) is equivalent to $(b_0 - b_1)(t-1) + c_0 - 1 > 0$.

Also in this case we have $b_0 \geq b_1$. Suppose $b_0 < b_1$, then $b_0 \leq b_1 - 1$ and $b_0t \leq (b_1 - 1)t$, hence $a_0 = b_0t + c_0 \leq b_0t + t - 1 \leq (b_1 - 1)t + (t-1) = a_1$, which is absurd since $a_0 > a_1$.

If $b_0 > b_1$, since $t \geq 2$, we have $(b_0 - b_1)(t-1) + c_0 - 1 \geq t - 1 + c_0 - 1 \geq c_0 > 0$. Let us assume $b_0 = b_1$, then the inequality (207) is equivalent to $c_0 > 1$. Hence for $c_0 > 1$ we get the result, while if $c_0 = 1$ we have $f^\circ_\Omega(x) = f^\circ_\Omega(y)$. Notice that for $b_0 = b_1$ and $c_0 = 1$ we have $a_0 = b_0t + 1$ and $a_1 = b_0t - 1$, therefore $a_0 - a_1 = 2$.

The result follows. *q.e.d.*

In fact we can give more information about non-special elements $x \in H$ for which $f^\circ_\Omega(x) = f^\circ_\Omega(y)$, where y is as defined in Lemma 18.3.11.

Observe that the case $\max_i\{a_i\} - \min_i\{a_i\} = 0$ may occur only if r divides n and, up to conjugation, $x = [I_{n/r}, \omega I_{n/r}, \dots, \omega^{r-1} I_{n/r}]$.

Lemma 18.3.12. *Assume r divides n. Let $x = [I_{n/r}, \omega I_{n/r}, \dots, \omega^{r-1} I_{n/r}]$ and define $y = [I_{n/r-1}, \omega I_{n/r+1}, \dots, \omega^{r-1} I_{n/r}]$. Then $f^\circ_\Omega(x) \leq f^\circ_\Omega(y)$, with equality if, and only if, rt divides n.*

Proof. This is very similar to Lemma 18.3.11. For details, see Appendix B.4.1 *q.e.d.*

Remark 18.3.13. Thanks to the equality part in the statement of Lemma 18.3.11, the argument of Lemma 18.3.12 extends to any element $x \in H$ of order r whose multiplicities of eigenvalues satisfy $n/r - 1 \leq a_i \leq n/r + 1$, for all $i \in \{0, \dots, r-1\}$.

Now, we prove the claim.

Lemma 18.3.14. *Let $x \in H$ be of order $r < n$. Then*

$$f^{\circ}_{\Omega}(x) \geq f^{\circ}_{\Omega}(z)$$

where $z \in H$ is special of order r.

Proof. Let $x \in H$ of prime order r; if x is special the result follows, hence we may assume x is not special. Thanks to Lemma 18.3.10 we may write $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$ with $a_0 = \max_i\{a_i\}$ and $a_1 = \min_i\{a_i\}$ and $a_0 - a_1 \geq 2$. Therefore, following Lemma 18.3.11, we construct y for which $f^{\circ}_{\Omega}(x) \geq f^{\circ}_{\Omega}(y)$. If y is special the result follows. If y in not special we apply Lemma 18.3.11 again. Eventually, iterating this process, in a finite number of steps we get

$$f^{\circ}_{\Omega}(x) \geq f^{\circ}_{\Omega}(y) \geq \dots \geq f^{\circ}_{\Omega}(z)$$

where z is special. *q.e.d.*

Now we aim to characterise special elements, up to centraliser structure.

Lemma 18.3.15. *Let $x \in G$ be a special element of order r. Assume $a_0 = \max\{a_i\}$.*

(i) If r divides n, then $a_i = n/r$ for all i.
(ii) If r does not divide n, then $a_0 = \lfloor n/r \rfloor + 1$

Proof. Since x is special we have $a_0 - a_i \in \{0, 1\}$ for all i. Therefore, for all i, we have $a_0 - 1 \leq a_i \leq a_0$. Summing over i we deduce $(a_0 - 1)r \leq n \leq ra_0 - 1$. Therefore $\frac{n+1}{r} \leq a_0 \leq \frac{n+r}{r}$. The result follows recalling that $\frac{n-r+1}{r} \leq \left\lfloor \frac{n}{r} \right\rfloor \leq \frac{n}{r}$. *q.e.d.*

Thanks to the previous result the general structure of any special element is clear.

Proposition 18.3.16. *Write $n = qr + c$, where $0 \leq c < r$. Let $z \in G$ be special of order r. Then $C_G(z) \cong C_G(x)$, where*

$$x = \left[I_{\lfloor n/r \rfloor + 1}, \dots, \omega^{c-1} I_{\lfloor n/r \rfloor + 1}, \omega^{c} I_{\lfloor n/r \rfloor}, \dots, \omega^{r-1} I_{\lfloor n/r \rfloor}\right] \tag{208}$$

In view of Lemma 18.3.14 and Proposition 18.3.16 in order to get a lower bound on f°_{Ω} for elements of prime order r in H we need to compute $f^{\circ}_{\Omega}(x)$, for x as in (208). Using Theorems 5.3.1 and 17.3.8, we get

$$\begin{aligned}
\dim x^G &= n^2 - n - 2n\left\lfloor \frac{n}{r} \right\rfloor + r\left\lfloor \frac{n}{r} \right\rfloor^2 + r\left\lfloor \frac{n}{r} \right\rfloor \\
\dim(x^G \cap H^{\circ}) &= \frac{n^2}{t} - n + lt\Big(\left\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \right\rfloor^2 - \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor^2\Big) \\
&\quad + l\Big(t - 2\left\lfloor \frac{n}{r} \right\rfloor\Big)\Big(\left\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \right\rfloor - \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor\Big) \\
&\quad - 2l\left\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \right\rfloor + rt\left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor^2 + rt\left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor - 2r\left\lfloor \frac{n}{r} \right\rfloor\left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor
\end{aligned}$$

It is natural to consider two cases. Either t divides $\lfloor n/r \rfloor + 1$ or it does not. In the latter case, we have

$$\left\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \right\rfloor = \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor$$

Thus

$$\begin{aligned}\dim C^{\circ}_{\Omega}(x) &= \dim \Omega - \dim x^G + \dim(x^G \cap H^{\circ}) \\ &= 2n\Big(\left\lfloor \frac{n}{r} \right\rfloor - \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor\Big) - r\Big(\left\lfloor \frac{n}{r} \right\rfloor^2 - \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor^2 t\Big) - r\Big(\left\lfloor \frac{n}{r} \right\rfloor - \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor t\Big)\end{aligned}$$

We write $n = (at+b)r + c$ with $c \in \{0, 1, \ldots, r-1\}$ and $b \in \{0, 1, \ldots, t-1\}$, notice that

$$\left\lfloor \frac{n}{r} \right\rfloor = at + b, \qquad \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor = a$$

Moreover, observe that if $b = t-1$ we are in the case in which t divides $\lfloor n/r \rfloor + 1$. Using *Mathematica*, we compute

$$f^{\circ}_{\Omega}(x) = \frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} \tag{209}$$

Similarly, in the first case, i.e. t divides $\lfloor n/r \rfloor + 1$, we have

$$\left\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \right\rfloor = \left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor + 1$$

Thus

$$\begin{aligned}\dim C^{\circ}_{\Omega}(x) &= \dim \Omega - \dim x^G + \dim(x^G \cap H^{\circ}) \\ &= 2nt\Big(1 - \frac{1}{t}\Big) - 2rt\left\lfloor \frac{n}{r} \right\rfloor\Big(1 - \frac{1}{t}\Big) + 2nt\left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor\Big(1 - \frac{1}{t}\Big) \\ &\quad + r\left\lfloor \frac{n}{r} \right\rfloor^2 - r\left\lfloor \frac{n}{r} \right\rfloor + rt\left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor^2 + rt\left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor - 2rt\left\lfloor \frac{n}{r} \right\rfloor\left\lfloor \frac{\lfloor n/r \rfloor}{t} \right\rfloor\end{aligned}$$

As before, let us write $n = (at+b)r + c$ with $c \in \{0, 1, \ldots, r-1\}$ and $b = t-1$. Again, using *Mathematica*, we compute

$$f^{\circ}_{\Omega}(x) = \frac{1}{r} - \frac{(r-c)^2}{n^2 r} \tag{210}$$

Observe that if in (209) we substitute the value $b = t-1$ we get (210).

Remark 18.3.17. The right hand side in (210) is clearly strictly less than $1/r$. Similarly, we see that the right hand side of (209) is less than or equal to $1/r$ because $r > c$ and we may assume $b \leq t-2$, hence

$$br(t-b) \geq 2br > 2bc$$

We can indeed characterise special elements x for which $f^{\circ}_{\Omega}(x) = 1/r$.

Lemma 18.3.18. *Write $n = (at+b)r + c$, where $0 \leq c < r$ and $0 \leq b < t$. Let $x \in H$ be special of order r. Then $f^{\circ}_{\Omega}(x) = 1/r$ if, and only if, $b = c = 0$.*

Proof. By Remark 18.3.17, we need to be in the case $b < t-1$. Assume $b = c = 0$, then, using (209), we compute $f^{\circ}_{\Omega}(x) = 1/r$.

Conversely, assume that for the special element x we have $f^\circ_\Omega(x) = 1/r$, as observed above, we may assume $b \le t-2$. Since $c \ge 0$ and $br(t-b) - 2bc > 0$ we must have $c = 0$. Therefore, $br(t-b) = 0$, i.e. $b = 0$ since $t - b > 0$. *q.e.d.*

The above discussion is summarised in the following.

Proposition 18.3.19. *Let $x \in H$ be of order $r < n$. Then*

$$f^\circ_\Omega(x) \ge \frac{1}{r} - \frac{br(t-b)-2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} \tag{211}$$

where we write $n = (at+b)r + c$, with $0 \le c < r$ and $0 \le b < t$.

Proof. Let $x \in H$ be an element of prime order $r < n$. Then, by Lemma 18.3.14, $f_\Omega(x) \ge f_\Omega(z)$ for some special element $z \in H$ of order r. We computed $f_\Omega(z)$ in (209), hence, the bound (211) holds. *q.e.d.*

Remark 18.3.20. With some further comments, using Lemmas 18.3.11, 18.3.12 and Remark 18.3.13, we can characterise (up to conjugacy) elements that realise the bound in Proposition 18.3.19. It turns out that $f^\circ_\Omega(x)$ realises (211) if, and only if, $x \in [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]^G$ and one of the following holds:

(i) for all i, $\lfloor n/r \rfloor \le a_i \le \lfloor n/r \rfloor + 1$;
(ii) $b = c = 0$ and for all i, $n/r - 1 \le a_i \le n/r + 1$;
(iii) $b = t-1$ and there exist j, j' such that $a_j = \lfloor n/r \rfloor + 2, a_{j'} = \lfloor n/r \rfloor$ and for all $i \ne j$, $\lfloor n/r \rfloor \le a_i \le \lfloor n/r \rfloor + 1$.

By the previous discussion the lower bound given in Proposition 18.3.19 for f°_Ω is the best possible. In particular, we deduce that if rt divides n then the best possible lower bound is $1/r$. However, for $r \le n$ we see that the lower bound should have the form $\frac{1}{r^2} - \epsilon$ for some small ϵ, at least when r is close to n.

Example 18.3.21. Let us compute the f_Ω-value (or f°_Ω-value) for special elements in two different cases.

1. Assume $r = n$. Thus, $t = n$ and $\dim \Omega = n(n-1)$. Let $x = [1, \omega, \ldots, \omega^{n-1}]$. We compute $\dim x^G = n^2 - n$ and $\dim(x^G \cap H^\circ) = 0$, since $H^\circ = (\mathrm{GL}_1)^n$. Notice that, by Lemma 17.4.6, x is G-conjugate to π_1. Using (184) we have $\dim(x^G \cap H^\circ \pi_1) = n-1$. Therefore $f_\Omega(x) = 1/r$ whereas $f^\circ_\Omega(x) = 0$.

2. Let $r = n-1$. And $x = [I_2, \omega, \ldots, \omega^{n-2}]$. Then $\dim x^G = n^2 - n - 2$ and $\dim(x^G \cap H^\circ) = n^2/t - n$, since the block decomposition $x = [x_1, \ldots, x_t]$ that maximises $\dim x^{H^\circ}$ consists entirely of regular blocks. Notice that $x^G \cap (H \setminus H^\circ) \ne \emptyset$ only if $t = n$, by Lemma 17.4.6. We compute $f^\circ_\Omega(x) = 2t/n^2(t-1) \ge \frac{2}{n^2} \approx \frac{2}{r^2}$.

The proof of the following allows us to deduce that the lower bound of Proposition 18.3.7 is close to best possible in several cases.

Lemma 18.3.22. *Let $n = (at+b)r + c$, where $0 \le c < r$ and $0 \le b < t$. Let $\epsilon = \frac{br(t-b)-2bc}{n^2(t-1)} + \frac{c^2}{n^2 r}$. Then*

$$0 \le \epsilon \le \frac{rt^2}{4n^2(t-1)}$$

Proof. Let us start by showing that $\epsilon \ge 0$. If $b = 0$ it is clear. Hence we may assume $b \ne 0$, and we claim $r(t-b) - 2c \ge 0$, which leads the result. For $b \le t-2$ we have $r(t-b)-2c \geq 2r-2c>0$ because $c<r$. If $b=t-1$ we have

$$\frac{br(t-b)-2bc}{n^2(t-1)}+\frac{c^2}{n^2r}=\frac{r(t-1)-2c(t-1)}{n^2(t-1)}+\frac{c^2}{n^2r}=\frac{(r-c)^2}{n^2r}>0$$

In order to compute an upper bound on ϵ, let us study the one-variable function $g_1(b)=brt-b^2r-2bc$. It is straightforward to see that $g_1(b) \leq g_1\left(\frac{t}{2}-\frac{c}{r}\right)$. Substituting $b=\frac{t}{2}-\frac{c}{r}$ we compute

$$\epsilon \leq \frac{c^2-crt}{rn^2(t-1)}+\frac{c^2}{rn^2}+\frac{rt^2}{4n^2(t-1)}=\frac{rt^2}{4n^2(t-1)}-\frac{c(r-c)}{n^2r(t-1)}$$

The result follows. *q.e.d.*

Remark 18.3.23. Observe that from the proof of Lemma 18.3.22 we have $\epsilon \leq \frac{rt^2}{4n^2(t-1)}-\frac{c(r-c)}{n^2r(t-1)}$. And $\frac{c(r-c)}{n^2r(t-1)} \leq \frac{r}{4n^2(t-1)}$. Therefore

$$\frac{rt^2}{4n^2(t-1)}-\frac{c(r-c)}{n^2r(t-1)} \geq \frac{r(t+1)}{4n^2}$$

In view of the proof of Lemma 18.3.22 and Remark 18.3.23 we deduce that the lower bound $\frac{1}{r}-\frac{rt^2}{4n^2(t-1)}$ is very accurate when $b \approx \frac{t}{2}-\frac{c}{r}$ and $c \approx r$ (we deduce this also from the proof of Proposition 18.3.7).

18.4. Local upper bounds

For the convenience of the reader we recall that $\mathcal{V}_s=\{x \in G : \nu(x)=s\}$. In addition, we denote by $\mathcal{V}_{s,r}$ the set of elements of $\mathcal{V}_s$ of order r.

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_s$ of odd prime order. We deal with involutions in Section 18.7. The main result of this section is Proposition 18.4.1 below.

Proposition 18.4.1. *Assume r is an odd prime. Let $x \in H \cap \mathcal{V}_{s,r}$. If $s \leq n/2$ then*

$$f_\Omega(x) \leq 1-2\frac{s}{n}+2\left(\frac{s}{n}\right)^2$$

If $s>n/2$ then

$$f_\Omega(x) \leq 1-\frac{s}{n}$$

Proof. Thanks to Proposition 17.2.1, we have

$$f_\Omega(x) \leq \frac{\dim\Omega-\dim x^G\left(1-\frac{1}{t}\right)}{\dim\Omega}=1-\frac{\dim x^G}{n^2}$$

By Proposition 5.4.1, we have $\dim x^G \geq \max\{2s(n-s), ns\}$. The result follows. *q.e.d.*

Remark 18.4.2. In some cases these bounds are sharp. Indeed the bounds are realised if there exists $x \in \mathcal{V}_{s,r} \cap H$ such that $\dim x^G=\max\{ns, 2s(n-s)\}$ and $\dim(x^G \cap H)=\frac{1}{t}\dim x^G$.

Corollary 18.4.3. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \leq 1-\frac{s}{n+1}$$

PROOF. If $s > n/2$ then it is clear that $1 - s/n < 1 - s/(n+1)$. Assume $s \leq n/2$. Then $1 - s/(n+1) - \left(1 - 2s(n-s)/n^2\right) = \frac{s(n^2-2ns+2n-2s)}{n^2(n+1)}$. For $0 \leq s \leq n/2$, we see that $g(s) = n^2 - 2ns + 2n - 2s \geq \min\{g(0), g(n/2)\} = 0$. The result follows. *q.e.d.*

We do not have a complete characterisation of the elements that realise the upper bound, in general.

In Section 18.4.1 we deal with semisimple elements. We show that for every $y \in H \cap \mathcal{V}_{s,r}$ we have $f_\Omega^\circ(y) \leq f_\Omega^\circ(\bar{x})$, for $\bar{x}$ defined in (212). In particular, we see that $f_\Omega(\bar{x})$ is close to the upper bound of Proposition 18.4.1.

In Section 18.4.2 we will give more results on unipotent elements. Let $x \in H \cap \mathcal{V}_{s,p}$. Assume $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ then $\nu(x) = s$ means $n - s = \sum_i a_i$. Unfortunately, in this case we do not deal with this problem in full generality. The condition $\nu(x) = s$ is hard to deal with. However we can deduce some results if we impose stronger hypotheses on n, t, s.

18.4.1. Semisimple elements. Let us assume $r \neq p$. The main result of this section is Proposition 18.4.4, where we show the existence of $x \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega(x)$ is close to the bound of Proposition 18.4.1. In the following two results we write U for the upper bound established in Proposition 18.4.1.

Proposition 18.4.4. *Write $n = (n-s)l + m$, with $0 \leq m < n - s$. Then there exists $x \in H \cap \mathcal{V}_{s,r}$ such that*

$$f_\Omega(x) \geq U - \frac{2}{n} - \iota$$

where $\iota = m/n$ if $s > n/2$, and $\iota = 0$ otherwise.

It is straightforward to deduce the following.

Corollary 18.4.5. *There exists $x \in H \cap \mathcal{V}_{s,r}$ such that*

$$f_\Omega(x) \geq U - \frac{2}{n} - \frac{1}{16}$$

Let s be such that $H \cap \mathcal{V}_{s,r} \neq \emptyset$. Let us write $n = (n-s)l + m$, with $0 \leq m < n - s$. We define

$$\bar{x} = [I_{n-s}, \omega I_{n-s}, \ldots, \omega^{l-1} I_{n-s}, \omega^l I_m] \qquad (212)$$

Claim. Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega^\circ(x) \leq f_\Omega^\circ(\bar{x})$.

The following result allows us to consider only elements with 1-eigenspace of dimension $n - s$.

Lemma 18.4.6. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then there exists $y = [I_{n-s}, \omega I_{b_1}, \ldots, \omega^{r-1} I_{b_{r-1}}] \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega^\circ(x) = f_\Omega^\circ(y)$, $b_1 = \min\{b_i : b_i \neq 0\}$ and $b_2 = \max\{b_i : b_i < n - s\}$.*

PROOF. Let $a_j = \min\{a_i : a_i \neq 0\}$ and $a_l = \max\{a_i : a_i < n - s\}$. Define $y = [I_{n-s}, \omega I_{a_j}, \omega^2 I_{a_l}, \omega^3 I_{a_3}, \ldots, \omega^i I_{a_1}, \omega^j I_{a_2}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. It is clear that $C_G(x) \cong C_G(y)$. Therefore by Theorems 5.3.1 and 17.3.8, we have $f_\Omega^\circ(x) = f_\Omega^\circ(y)$. *q.e.d.*

By Lemma 18.4.6, that for $x \in H \cap \mathcal{V}_{s,r}$, up to G-conjugacy, we may assume

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$

with $a_i \leq n-s$ for all i.

In order to prove the claim we need the following technical result.

Lemma 18.4.7. *Let* $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H \cap \mathcal{V}_{s,r}$. *Assume* $C_G(x) \ncong C_G(\bar{x})$, *for* $\bar{x}$ *defined in (212). Assume* $a_1 = \min\{a_i : a_i \neq 0\}$ *and* $a_2 = \max\{a_i : a_i < n-s\}$. *Define* $y = [I_{n-s}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \omega^3 I_{a_3}, \dots, \omega^{r-1} I_{a_{r-1}}]$. *Then*

$$f_\Omega^\circ(x) \leq f_\Omega^\circ(y)$$

with equality if, and only if, $a_1 = a_2$ *and* $a_1 \equiv 0$ *or* $t-1 \pmod t$.

PROOF. This is very similar to Lemma 18.3.11. For all the details see Appendix B.4.2. *q.e.d.*

We can now prove the claim.

Proposition 18.4.8. *Let* $x \in H \cap \mathcal{V}_{s,r}$. *Then*

$$f_\Omega^\circ(x) \leq f_\Omega^\circ(\bar{x})$$

where $\bar{x}$ *is defined in (212).*

PROOF. Thanks to Lemma 18.4.6, we may assume $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. If $C_G(x) \cong C_G(\bar{x})$ then $f_\Omega^\circ(x) = f_\Omega^\circ(\bar{x})$. Thus we may assume $C_G(x) \ncong C_G(\bar{x})$ and $a_1 = \min\{a_i : a_i \neq 0\}$, $a_2 = \max\{a_i : a_i < n-s\}$. Hence, by Lemma 18.4.7, there exists $y \in H \cap \mathcal{V}_{s,r}$ such that $f_\Omega^\circ(x) \leq f_\Omega^\circ(y)$. If $C_G(y) \ncong C_G(\bar{x})$ we may apply, again, Lemma 18.4.7. Iterating this construction we get

$$f_\Omega^\circ(x) \leq f_\Omega^\circ(y) \leq \ldots \leq f_\Omega^\circ(\bar{x})$$

The result follows. *q.e.d.*

In view of Proposition 18.4.8, the element $\bar{x}$ as defined in (212) is the natural choice in order to prove Proposition 18.4.4. We first study the case $s \leq n/2$ and then $s > n/2$.

Lemma 18.4.9. *Assume* $s \leq n/2$. *Then there exists* $x \in H \cap \mathcal{V}_{s,r}$ *such that*

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

In particular, $f_\Omega(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{2}{n}$.

PROOF. Let $x = [I_{n-s}, \omega I_s]$. Then $x \in H \cap \mathcal{V}_{s,r}$. Using Theorems 5.3.1 and 17.3.8, we compute $\dim x^G = 2s(n-s)$ and

$$\dim(x^G \cap H^\circ) = 2\left\lfloor \frac{s}{t} \right\rfloor \left(\left\lfloor \frac{s}{t} \right\rfloor t - 2s + t \right) + 2s\left(\frac{n}{t} - 1 \right)$$

Write $s = at + b$, with $0 \leq b < t$. Using *Mathematica*, we compute

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Notice that for $r > 2$ we have $x^G \cap H = x^G \cap H^\circ$ hence $f_\Omega(x) = f_\Omega^\circ(x)$.

The difference between the upper bound in Proposition 18.4.1 and $f_\Omega(x)$ is

$$E = \frac{t^2}{2n^2(t-1)}$$

It is straightforward to check that E is maximal when t is maximal, hence $0 \le E \le \frac{1}{2(n-1)} < \frac{2}{n}$. The result follows. *q.e.d.*

In the case $s > n/2$ the computation of $f_\Omega(\bar{x})$ does not follow as easily. We will use Proposition 5.4.1 and Proposition 17.2.1 to simplify the calculations.

Lemma 18.4.10. *Assume $s > n/2$. Then there exists $x \in H \cap \mathcal{V}_{s,r}$ such that*

$$f^\circ_\Omega(x) \ge 1 - \frac{s}{n} - \frac{2}{n} - \frac{m}{n}$$

where $n \equiv m \pmod{n-s}$ and $0 \le m < n - s$.

Proof. Write $n = (n-s)l + m$, $0 \le m < n - s$. Let $x = [I_{n-s}, \ldots, \omega^{l-1} I_{n-s}, \omega^l I_m]$. Using Theorems 5.3.1 and 17.3.8, we compute

$$\begin{aligned}
\dim x^G &= l(n-s)(n+s-l(n-s)) \\
\dim(x^G \cap H^\circ) &= \frac{n^2}{t} - n + l\Big(\Big\lfloor \frac{n-s}{t} \Big\rfloor^2 t + (t - 2n + 2s) \Big\lfloor \frac{n-s}{t} \Big\rfloor\Big) \\
&\quad + \Big\lfloor \frac{n-(n-s)l}{t} \Big\rfloor^2 t + (t - 2n + 2l(n-s)) \Big\lfloor \frac{n-(n-s)l}{t} \Big\rfloor
\end{aligned}$$

Let us observe that we can write

$$\dim x^G = ns + \delta$$

where $\delta = l(n-s)(n+s-l(n-s)) - ns = l(n-s)(m+s) - ns$. Furthermore, thanks to Proposition 5.4.1 we deduce $\delta \ge 0$. Therefore, using $l \le n/(n-s)$, we have

$$\delta \le \frac{n}{n-s}(n-s)(m+s) - ns = nm$$

We divide the analysis into two cases: either t divides s or it does not.

Case 1. Assume $t \mid s$. Then, by Lemma 17.3.16 we get $\dim(x^G \cap H^\circ) = \frac{1}{t} \dim x^G$. Notice that by Proposition 17.2.1, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$ therefore $f_\Omega(x) = f^\circ_\Omega(x)$. And we have

$$f_\Omega(x) = 1 - \frac{s}{n} - \frac{\delta}{n^2} \ge 1 - \frac{s}{n} - \frac{m}{n}$$

Case 2. Assume $t \nmid s$. Then for some $\epsilon \ge 0$ (thanks to Proposition 17.2.1 and $\dim(x^G \cap H) \ge \dim(x^G \cap H^\circ)$)

$$\dim(x^G \cap H^\circ) = \frac{\dim x^G}{t} - \epsilon$$

Then, Proposition 17.3.17 yields $\epsilon \le \dim x^G/n < n$ (it is clear that $\dim x^G < n^2$). Hence

$$f^\circ_\Omega(x) = 1 - \frac{s}{n} - \Big(\frac{\delta}{n^2} + \frac{\epsilon t}{n^2(t-1)}\Big)$$

The difference between the upper bound given in Proposition 18.4.1 and the value of $f^\circ_\Omega(x)$ is

$$0 \le E = \frac{\delta}{n^2} + \frac{\epsilon t}{n^2(t-1)} \le \frac{\delta + 2\epsilon}{n^2} < \frac{nm+n}{n^2}$$

where we used $t/(t-1) \le 2, \delta \le nm, \epsilon < n$. The result follows. *q.e.d.*

Thanks to the computation made in the proof of Lemma 18.4.10 we can prove Corollary 18.4.5.

Proof of Corollary 18.4.5. If $s \leq n/2$ the result follows from Lemma 18.4.10.

Assume $s > n/2$. With a similar argument to that used for ϵ (in the proof of Lemma 18.4.10), we have $\delta(l) = l(n-s)(n+s-l(n-s)) - ns$, so $\delta(l) \leq \delta(\frac{n+s}{2(n-s)}) = (n-s)^2/4$. Therefore $\delta \leq (n-s)^2/4 \leq n^2/16$, since $s \geq n/2$ and $E \leq 2/n + 1/16$. This yields

$$f_\Omega(x) \geq 1 - \frac{s}{n} - \frac{2}{n} - \frac{1}{16}$$

The result follows. *q.e.d.*

Proposition 18.4.4 establishes that the gap between the bounds and the f_Ω-value of some particular element in $H \cap \mathcal{V}_{s,r}$ is small. Furthermore, by the proofs of Lemma 18.4.9 and 18.4.10, we can give infinitely many examples in which the bounds of Proposition 18.4.1 are realised. It is enough to consider elements with the same centralizer structure of $[I_{n-s}, \ldots, \omega^{l-1} I_{n-s}, \omega^l I_m]$ satisfying one of the following conditions:

(i) if $s \leq n/2$, then $s \mid t$;
(ii) if $s > n/2$, then $t \mid n-s$ and $n-s \mid n$ (i.e. $\epsilon = \delta = 0$ in the proof of Lemma 18.4.10).

18.4.2. Unipotent elements. Let $x \in H \cap \mathcal{V}_{s,p}$. Here we assume $t < n$. The case $t = n$ will be studied in Section 18.4.2.2. Up to G-conjugacy we may assume $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$, and $n - s = \sum_i a_i$ (cf. Proposition 5.1.7). By Proposition 18.4.1

$$f_\Omega(x) \leq 1 - \frac{s\gamma}{n}$$

where $\gamma = 1$ is $s > n/2$, or $2(1 - s/n)$ for $s \leq n/2$.

The following result is the analogue of Proposition 18.4.4. We do not cover all the possible cases. In fact the following holds only for $s \leq n/2$.

Proposition 18.4.11. *Assume $H^\circ \cap \mathcal{V}_{s,p} \neq \emptyset$ and $s \leq n/2$.*

(i) Assume that either n/t is even, or n/t is odd and $n - 2s \geq t$. Then there exists $x \in H^\circ \cap \mathcal{V}_{s,p}$ such that

$$f_\Omega^\circ(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{1}{n}$$

(ii) Assume n/t is odd and $n - 2s < t$. Then there exists $x \in H^\circ \cap \mathcal{V}_{s,p}$ such that

$$f_\Omega^\circ(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{t}{2(t-1)}$$

Proof. Let $x = [J_2^s, J_1^{n-2s}]$. If n/t is even we clearly have $x \in H^\circ$. Assume n/t is odd and $n - 2s \geq t$, in particular the number of J_1 blocks is greater than t; hence there exists a block decomposition $[x_1, \ldots, x_t] \in H^\circ$. Write $s = at + b$, where $0 \leq b < t$. In both cases, we define

$$x_1 = \ldots = x_b = [J_2^{a+1}, J_1^{n/t-2a-2}]$$

$$x_{b+1} = \ldots = x_t = [J_2^a, J_1^{n/t-2a}]$$

Hence $\dim(x^G \cap H^\circ) \geq b \dim[J_2^{a+1}, J_1^{n/t-2a-2}]^{\mathrm{GL}_{n/t}} + (t-b)\dim[J_2^a, J_1^{n/t-2a}]^{\mathrm{GL}_{n/t}}$. Using Theorem 5.2.1 we compute

$$\dim x^G = 2s(n-s),\ \dim x_1^{\mathrm{GL}_{n/t}} = 2(a+1)\Big(\frac{n}{t} - a - 1\Big),\ \dim x_{b+1}^{\mathrm{GL}_{n/t}} = 2a\Big(\frac{n}{t} - a\Big)$$

Therefore

$$f_\Omega^\circ(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)} \geq 1 - \frac{2s(n-s)}{n^2} - \frac{1}{n}$$

We have $2b(t-b) \leq t^2/2$ and it is straightforward to show that $t^2/2 \leq n(t-1)$, using $n, t \geq 2$. Hence $\frac{2b(t-b)}{n^2(t-1)} \leq \frac{1}{n}$.

Now assume n/t is odd and $n - 2s < t$. Then no conjugate of $[J_2^s, J_1^{n-2s}]$ lies in H°, since the number of J_1 blocks is not enough to get a block decomposition in H°. Hence, we consider the family of elements $y_i = [J_3^i, J_2^{s-2i}, J_1^{n-2s+i}] \in H \cap \mathcal{V}_{s,p}$. In order to have $y_i \in H^\circ$ we need $n - 2s + 2i \geq t$, i.e. the number of J_3 and J_1 blocks has to be at least t, in order to create a block decomposition in H° (recall that $t < n$ and n/t is odd). The smallest integer l for which (up to conjugacy) $y_{l-1} \notin H^\circ$ and $y_l \in H^\circ$ is $l = \frac{t-n+2s}{2}$. Let $x = y_{\frac{t-n+2s}{2}}$. We consider $x = [x_1, \ldots, x_t]$ with

$$x_1 = \ldots = x_l = [J_3, J_2^{(n/t-3)/2}]$$
$$x_{l+1} = \ldots = x_t = [J_2^{(n/t-1)/2}, J_1]$$

As above $\dim(x^G \cap H^\circ) \geq \dim x^{H^\circ}$. Using *Mathematica* we get

$$f_\Omega^\circ(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{(n-2s-t)(nt-t^2+2t-2s)}{2n^2(t-1)}$$

And it is straightforward to check that

$$f_\Omega^\circ(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{t}{2(t-1)}$$

q.e.d.

18.4.2.1. *Case $t = 2$.* Let us assume $n > 2$ is even and $t = 2$. For $x \in H$ of odd order p we have, by Lemma 17.3.1, $x^G \cap H = x^G \cap H^\circ$.

In the case $s < n/2$ the conditions in (i) of Proposition 18.4.11 hold. Hence we assume $s \geq n/2$. We prove the following.

Proposition 18.4.12. *Assume $s \geq n/2$. Then there exists $x \in H \cap \mathcal{V}_{s,p}$ such that*

$$f_\Omega(x) \geq 1 - \frac{s}{n} - \epsilon$$

where $\epsilon = \frac{(n-s)^2}{4n^2}$ if s is even, and $\epsilon = 1/n$ otherwise.

PROOF. First assume s is even. Write $\frac{n}{2} = \frac{n-s}{2}a + b$ where $0 \leq b < \frac{n-s}{2}$. Let $x = [\bar{x}, \bar{x}] \in H^\circ$ where

$$\bar{x} = [J_{a+1}^b, J_a^{\frac{n-s}{2}-b}]$$

So $\nu(\bar{x}) = s/2$, hence $\nu(x) = s$ and $x \in H \cap \mathcal{V}_{s,p}$. Hence $x = [J_{a+1}^{2b}, J_a^{n-s-2b}]$ and, by Proposition 17.3.4, we have $\dim(x^G \cap H) = \frac{1}{2}\dim x^G$. A straightforward computation, using Theorem 5.2.1, leads to $\dim x^G = ns + \delta$, where $\delta = 2b(n-s-2b)$. Notice that, by Proposition 5.4.1, $\delta \geq 0$. In addition, we see $\delta(b) \leq \delta((n-s)/4) = (n-s)^2/4$. Therefore

$$f_\Omega(x) = 1 - \frac{s}{n} - \frac{\delta}{n^2} \geq 1 - \frac{s}{n} - \frac{(n-s)^2}{4n^2}$$

Now assume s is odd. Write

$$\frac{n}{2} = a_1\frac{n-s-1}{2} + b_1, \quad \frac{n}{2} = a_2\frac{n-s+1}{2} + b_2$$

Let us define $x = [x_1, x_2]$ where

$$x_1 = [J_{a_1+1}^{b_1}, J_{a_1}^{\frac{n-s-1}{2}-b_1}], \quad x_2 = [J_{a_2+1}^{b_2}, J_{a_2}^{\frac{n-s+1}{2}-b_2}]$$

Hence $x \in H \cap \mathcal{V}_{s,p}$. Using $\dim(x^G \cap H^\circ) \geq \dim x^{H^\circ}$, and *Mathematica* we compute

$$f_\Omega(x) \geq 1 - \frac{s}{n} - \frac{1}{n}$$

q.e.d.

18.4.2.2. *Case* $t = n$. In this situation $H = \mathrm{GL}_1 \wr S_n$. Therefore the set of unipotent elements in H is given by $\{\pi_i \mid i = 1, \ldots, \lfloor n/p \rfloor\}$. Recall that π_i is any permutation in S_t with cycle shape $(p^i, 1^{t-pi})$. By Lemma 17.4.5, π_i is G-conjugate to $x = [J_p^i, J_1^{n-ip}]$. Therefore $\nu(\pi_i) = i(p-1)$.

Using Theorem 5.2.1 and (184) we compute

$$\dim \pi_i^G = i(p-1)(2n - ip), \ \dim(\pi_i \cap H) = \dim \pi_i^{H^\circ} = i(p-1)$$

We summarise the previous computation in the following.

Proposition 18.4.13. *Assume $t = n$. Let $x \in H \cap \mathcal{V}_{s,p}$. Then $s = i(p-1)$, for some $0 < i \leq n/p$. Moreover*

$$f_\Omega(\pi_i) = 1 - \frac{i(p-1)(2n - ip - 1)}{n(n-1)}$$

Notice that $f_\Omega(\pi_i)$ as a function decreases in i. The following is a straightforward consequence of Proposition 18.4.13.

Corollary 18.4.14. *Assume $t = n$. Let $x \in H$ be of order p. Then*

$$\frac{1}{p} \leq f_\Omega(x) \leq 1 - \frac{2(p-1)}{n-1} + \frac{p^2 - 1}{n(n-1)}$$

18.5. Local lower bounds

In this section we derive lower bounds on $f_\Omega(x)$, for $x \in H \cap \mathcal{V}_{s,r}$ where $r \neq p$ is an odd prime. The main result of this section is Proposition 18.5.1. Involutions will be studied in Section 18.7.1.

For elements of prime order p we do not have general local bound on f_Ω. In Section 18.4.2.2 we have widely studied the case $t = n$. We shall study the particular case $t = 2$ in Section 18.5.1 below.

Proposition 18.5.1. *Assume $r \neq p$ odd. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n - s)}{n^2} - \frac{2}{n}$$

Let $x \in H \cap \mathcal{V}_{s,r}$, thanks to Lemma 18.4.6 we may assume, without loss of generality

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$

where $a_i \leq n - s$ for all $i > 0$. Notice that we may assume $a_1 = \max_{i>0}\{a_i\}$ and $a_2 = \max_{i>0}\{a_i\}$.

As usual, for any semisimple x we have $x^G \cap H^\circ \neq \emptyset$. Therefore we shall derive lower bounds on f°_Ω; recall that for all $x \in H$ we have $f_\Omega(x) \geq f^\circ_\Omega(x)$. We give the following.

Definition 18.5.2. Let $x \in \mathcal{V}_{s,r}$. Assume $a_0 = n - s$. We say that x is *special* if $|a_i - a_j| \leq 1$ for all $i, j > 0$.

We make the following.

Claim. Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$ where $z \in H \cap \mathcal{V}_{s,r}$ is any special element.

It is clear that any special element of order r has centraliser isomorphic to the following:

(213)

$$z = \begin{cases} [I_{n-s}, \omega, \omega^2, \ldots, \omega^s] & r-1 \geq s \\ [I_{n-s}, \omega I_{\lfloor \frac{s}{r-1} \rfloor +1}, \ldots, \omega^l I_{\lfloor \frac{s}{r-1} \rfloor +1}, \omega^{l+1} I_{\lfloor \frac{s}{r-1} \rfloor}, \ldots, \omega^{r-1} I_{\lfloor \frac{s}{r-1} \rfloor}] & r-1 < s \end{cases}$$

here $s = \lfloor s/(r-1) \rfloor (r-1) + l$ and $0 \leq l < r-1$. In particular, for fixed s, r all the special elements have the same f°_Ω-value.

If $x \in H \cap \mathcal{V}_{s,r}$ is not special then there exist $i, j > 0$ such that $a_i - a_j \geq 2$. The following technical result is the key tool for proving our claim.

Lemma 18.5.3. *Let $x \in H \cap \mathcal{V}_{r,s}$. Assume $a_0 = n - s, a_1 = \max_{i>0}\{a_i\}$ and $a_2 = \min_{i>0}\{a_i\}$, and $a_1 - a_2 \geq 2$. Let $y = [I_{n-s}, \omega I_{a_1 -1}, \omega^2 I_{a_2+1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$$

with equality if, and only if, $a_1 - a_2 = 2$ and $a_1 \equiv 1 \mod t$.

Proof. This is the same as Lemma 18.3.11. *q.e.d.*

An important consequence of Lemma 18.5.3 is the following, which proves the claim introduced before.

Lemma 18.5.4. *Let $x \in H \cap \mathcal{V}_{r,s}$. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$$

where z is special.

Proof. The proof is the same as Proposition 18.3.14, using Lemma 18.5.3. In fact, using the construction of the lemma starting from any $x \in \mathcal{V}_{r,s}$ in a finite number of steps we get a special element z, with $f_\Omega(x) \geq f_\Omega(z)$. *q.e.d.*

Therefore, in view of Lemma 18.5.4, a lower bound on f_Ω is given by $f^\circ_\Omega(z)$ for z defined in (213).

First we study the case $s \leq r-1$; here we compute $f_\Omega(z)$ rather that $f^\circ_\Omega(z)$, unless we are in the case $r - 1 = s$ and $t = n$.

Proposition 18.5.5. *Let $s \leq r-1$ and let $x \in \mathcal{V}_{r,s} \cap H$. Write $s = at + b$, where $0 \leq b < t$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{b(t-b)}{n^2(t-1)}$$

Proof. By Lemma 18.5.4 we have that for all $x \in \mathcal{V}_{r,s}$, $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$ where $z = [I_{n-s}, \omega, \ldots, \omega^s]$. Notice that if $r-1 > s$, by Corollary 17.4.9 $f_\Omega(z) = f^\circ_\Omega(z)$. Using Theorems 5.3.1 and 17.3.8, we compute $\dim z^G = s(2n-s) - s$ and

$$\dim(z^G \cap H^\circ) = \frac{2ns}{t} - 2as - 2s + a^2t + at$$

Notice that $\dim(x^G \cap H^\circ) = 0$ if $t = n$. Thus we get

$$f^\circ_\Omega(z) = 1 - \frac{(2n-s)s}{n^2} - \frac{b(t-b)}{n^2(t-1)}$$

The result follows. *q.e.d.*

Remark 18.5.6. As observed, if $r-1 > s$ then $f_\Omega(z) = f^\circ_\Omega(z)$. If $r-1 = s$, then we need to distinguish two cases. For $t < n$, using Corollary 17.4.9 we deduce that $f_\Omega(z) = f^\circ_\Omega(z)$. Assume $r-1 = s$ and $t = n$. Then for $z = [I_{n-s}, \omega, \ldots, \omega^s]$ we see $z^G \cap H = (z^G \cap H^\circ) \cup (z^G \cap H^\circ \pi_1)$. We have $\dim(z^G \cap H^\circ) = 0$. An easy computation shows $\dim(z^G \cap H) = \dim(z^G \cap H^\circ \pi_1) = r - 1 = s$. Therefore

$$f_\Omega(z) = \frac{n^2 - n - 2ns + s^2 + 2s}{n^2 - n} = 1 - \frac{s(2n-s-2)}{n^2-n}$$

However it is not clear that for all $x \in H \cap \mathcal{V}_{s,r}$ we have $f_\Omega(x) \geq f_\Omega(z)$.

Remark 18.5.7. In order to derive the bound given in Proposition 18.5.1 in the case $s \leq r-1$, we need to show $\frac{b(t-b)}{n^2(t-1)} \leq \frac{2}{n}$. Observe that $b(t-b) \leq \frac{t^2}{4}$, hence we claim $\frac{t^2}{4n^2(t-1)} \leq \frac{2}{n}$. We have $\frac{t^2}{t-1} \leq \frac{n^2}{(n-1)}$, since $t \leq n$. Therefore $\frac{t^2}{4n^2(t-1)} \leq \frac{1}{4(n-1)} \leq \frac{2}{n}$. Thus, for $s \leq r-1$, conclusion of Proposition 18.5.1 holds.

Now assume $s > r-1$. We have the following.

Proposition 18.5.8. *Assume $s > r-1$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{st(2n-s)}{n^3(t-1)}$$

Proof. For all $x \in H \cap \mathcal{V}_{s,r}$ we have $f_\Omega(x) \geq f^\circ_\Omega(x)$. In addition, by Lemma 18.5.4, we have $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$ for any special element $z \in H \cap \mathcal{V}_{s,r}$. Hence it is enough to compute $f^\circ_\Omega(z)$ for z as defined in (213). Using Proposition 17.3.17 we have

$$f_\Omega(z) \geq 1 - \frac{\dim z^G}{n^2} - \frac{t \dim z^G}{n^3(t-1)} = \ell$$

Let $s = a(r-1) + b$ where $0 \leq b < r-1$. Then

$$z = [I_{n-s}, \omega I_{a+1}, \ldots, \omega^b I_{a+1}, \omega^{b+1} I_a, \ldots, \omega^{r-1} I_a]$$

So, using Theorem 5.3.1 we have $\dim z^G = 2ns - s^2 - a^2(r-1) - 2ab - b$. Therefore we compute

$$\begin{aligned}\ell &= 1 - \frac{\dim x^G\big(1-\frac{1}{t}\big)}{\dim\Omega} - \frac{\dim x^G}{n\dim\Omega}\\ &= 1 - \frac{2ns-s^2}{n^2} + \frac{a^2(r-1)+2ab+b}{n^2} - \frac{2ns-s^2-a^2(r-1)-2ab-b}{n^3\big(1-\frac{1}{t}\big)}\\ &= \frac{(n-s)^2}{n^2} - \frac{2ns-s^2}{n^3\big(1-\frac{1}{t}\big)} + \frac{a^2(r-1)+2ab+b}{n^3\big(1-\frac{1}{t}\big)}\Big(1-\frac{1}{t}+\frac{1}{n}\Big)\\ &\geq \frac{(n-s)^2}{n^2} - \frac{2ns-s^2}{n^3\big(1-\frac{1}{t}\big)}\end{aligned}$$

where the inequality follows from the fact that $a^2(r-1)+2ab+b \geq 0$. *q.e.d.*

Remark 18.5.9. Notice that, since $s < n$,

$$\frac{s(2n-s)}{n^3\big(1-\frac{1}{t}\big)} < \frac{n^2 t}{n^3(t-1)} \leq \frac{2}{n}$$

hence $f^\circ_\Omega(z) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{2}{n}$. In particular, Proposition 18.5.1 holds.

Let us also observe that the bound $f^\circ_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{2}{n}$, for any $x \in H \cap \mathcal{V}_{s,r}$ is close to best possible. Now, we bound the term $\frac{a^2(r-1)+2ab+b}{n^3\left(1-\frac{1}{t}\right)}$. Using $a \leq \frac{s}{r-1}$ and $b < r-1 < s$ we have

$$a^2(r-1) + 2ab + b = as + b(a+1) \leq \frac{2s^2}{r-1} + s$$

Therefore, using $s < n, r-1 \geq 2$ and $\frac{t}{t-1} \leq 2$, we have

$$\frac{a^2(r-1)+2ab+b}{n^3\big(1-\frac{1}{t}\big)} < \frac{2t}{n(r-1)(t-1)} + \frac{t}{n^2(t-1)} \leq \frac{2}{n} + \frac{2}{n^2}$$

Thus, for L the lower bound in Proposition 18.5.8 we have $L - \ell \leq 2/n + 2/n^2$.

18.5.1. Unipotent elements, case $t = 2$. Assume $p \neq 2$. Any unipotent element $x \in H \cap \mathcal{V}_{s,p}$ can be written, up to G-conjugacy, $x = [J_p^{a_p}, \dots, J_1^{a_1}]$ with $n - s = \sum_i a_i$. Assume $t = 2$, so that $H = \mathrm{GL}_{n/2} \wr S_2$ is a $\mathscr{C}_2$-subgroup of GL_n. Observe that, by Lemma 17.3.1, we have $x^G \cap H = x^G \cap H^\circ$. We have the following.

Proposition 18.5.10. *Let $x \in H \cap \mathcal{V}_{s,p}$.*

(i) If $s \leq n/2 - 2$ then

$$f_\Omega(x) \geq 1 - \frac{2s}{n}$$

(ii) If $s > n/2 - 2$ then

$$f_\Omega(x) \geq \frac{2}{n}$$

Proof. Assume $s \leq n/2 - 2$. Let $x = [x_1, x_2]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$, say $x_1 = [J_p^{a_p}, \dots, J_1^{a_1}]$ and $x_2 = [J_p^{b_p}, \dots, J_1^{b_1}]$. Then, as in the proof of Lemma 18.2.9, see (193), $f_\Omega(x) \geq \frac{n-2s}{n}$ if, and only if,

$$\sum_{1\leq i<j\leq p} i(a_i b_j + a_j b_i) + \sum_{i=1}^{p} i a_i b_i \geq \frac{(n-2s)n}{4} = \Big(\sum_i b_i\Big)\Big(\sum_i i a_i\Big) - \frac{n}{2}\sum_i (i-1)a_i \tag{214}$$

where – for the right hand side – we used $s = n - \sum_i (a_i + b_i)$.

Fix $i \in \{1, \dots, p\}$. The coefficient of a_i in the left hand side of (214) is $i \sum_{i \le j} b_j + \sum_{j<i} j b_j$, while in the right hand side of (214) it is $i \sum_{j \ge 1} b_j - \frac{n}{2}(i-1)$.

In order to show (214) we need to prove

$$i \sum_{i \le j} b_j + \sum_{j<i} j b_j \ge i \sum_{j \ge 1} b_j - \frac{n}{2}(i-1)$$

which is equivalent to

$$\frac{n}{2}(i-1) \ge \sum_{j<i} (i-j) b_j \tag{215}$$

Writing $\frac{n}{2} = \sum_j j b_j = \sum_{j<i} j b_j + \sum_{j \ge i} j b_j$ and substituting this value in the inequality (215) we find that (215) is equivalent to

$$i \sum_{j<i} (j-1) b_j + (i-1) \sum_{j \ge i} j b_j \ge 0 \tag{216}$$

which is clearly true, since $i, j \ge 1$.

Now assume $s > n/2 - 2$. Here for any $s \in \{n/2 - 1, \dots, n-2\}$ we have that there exists $x = [J_{n/2}, x_2] \in H^\circ \cap \mathcal{V}_{s,p}$ and, by Lemma 18.2.6, we have $f_\Omega(x) = 2/n$ and, for any other $z \in H^\circ \cap \mathcal{V}_{s,p}$, $f_\Omega(z) \ge 2/n$. The result follows. *q.e.d.*

18.6. Further comments on local bounds

Using the bounds in Corollay 18.4.3 and Proposition 18.5.1 we have the following, for elements $x \in H \cap \mathcal{V}_{s,r}$ with $r \ne 2$ an odd prime.

Proposition 18.6.1. *Assume $r \ne p$ is an odd prime. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{s(n-s)}{n^2} + \frac{2}{n}$$

PROOF. Using *Mathematica*, we compute the difference between the upper bound in Proposition 18.4.1 and the lower bound in Proposition 18.5.1. We have

$$1 - \frac{s}{n} - \left(1 - \frac{s(2n-s)}{n^2} - \frac{1}{n}\right) = \frac{s(n-s)}{n^2} + \frac{2}{n}, \quad s > \frac{n}{2}$$
$$1 - \frac{2s}{n} + \frac{2s^2}{n^2} - \left(1 - \frac{s(2n-s)}{n^2} - \frac{1}{n}\right) = \frac{s^2}{n^2} + \frac{2}{n}, \quad s \le \frac{n}{2}$$

The result follows since $s^2 \le s(2n-s)$. *q.e.d.*

Remark 18.6.2. If $s \le \sqrt{n}$ or $s \ge n - \sqrt{n}$ we have $\frac{s(n-s)}{n^2} < \frac{1}{\sqrt{n}} - \frac{1}{n}$. Therefore, for all $x, y \in H \cap \mathcal{V}_{s,r}$, we have $|f_\Omega(x) - f_\Omega(y)| < \frac{1}{\sqrt{n}} + \frac{1}{n}$.

18.6.1. A conjecture on local bounds for unipotent elements.

For $x \in H \cap \mathcal{V}_{s,p}$ it is hard to give lower bound and sharp upper bounds on $f_\Omega(x)$. The main issue is the difficulty of understanding the block decomposition of $x \in H^\circ$ that maximises $\dim x^{H^\circ}$. We give a conjecture which would establish the best possible upper and lower bounds for any s.

Recall that given a partition $\lambda = (p^{a_p}, \dots, 1^{a_1}) \vdash n$ there exists a G-conjugacy class of unipotent elements whose representative, in Jordan form, is given by $x_\lambda = [J_p^{a_p}, \dots, J_1^{a_1}]$, and vice-versa.

Let $s \in \mathbb{Z}$ such that $H \cap \mathcal{V}_{s,p} \neq \emptyset$. Define

$$\mathcal{P}_s(n) = \{\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-s}) \vdash n : 0 < \lambda_i \leq p,\ x_\lambda \in H\}$$

Then $(\mathcal{P}_s(n), \preccurlyeq)$ is a partially ordered set, where $\preccurlyeq$ is the usual dominance ordering on partitions, see Section 5.2.

Conjecture 18.6.3. *Let $x \in \mathcal{V}_{p,s} \cap H$ then*

$$f_\Omega(x_\mu) \leq f_\Omega(x) \leq f_\Omega(x_\lambda)$$

where $\lambda = \min \mathcal{P}_s(n)$ and $\mu = \max \mathcal{P}_s(n)$

One of the difficulties in dealing with this conjecture is that, in general, it is hard to understand the conditions that a certain partition has to satisfy in order to lie in $\mathcal{P}_s(n)$.

Remark 18.6.4. Conjecture 18.6.3 provides also the best possible global bounds. It is enough to consider the maximal and minimal partition in $\bigcup_s \mathcal{P}_s(n)$. Notice that we have strong evidence of Conjecture 18.6.3 for global bounds for unipotent elements and for global and local bounds for semisimple elements. In addition, this conjecture can be generalised to any other classical group just by considering partitions in $\mathcal{P}_s(n) \cap \mathcal{P}_{Cl_n}$, as defined in Section 17.3.1

The above conjecture is implied by the following more general conjecture:

Conjecture 18.6.5. *Let $y \in \overline{x^G}$. Then $\dim C_\Omega(y) \geq \dim C_\Omega(x)$.*

Clearly Conjecture 18.6.5 holds if $y \in x^G$, indeed in this case $\dim C_\Omega(y) = \dim C_\Omega(x)$ since $\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$.

18.7. Involutions

In this section we shall give explicit formulae for $f_\Omega(x)$ when $x \in H$ is an involution. The main result of this section is Theorem 18.7.1, below.

Let $x \in G$ be an involution. Then, for $p \neq 2$, we may write, up to G-conjugacy, $x = [I_{n-s}, -I_s]$ and we may assume $\nu(x) = s \leq n/2$: in fact if $s \geq n/2$ then $\nu(-x) \leq n/2$ and $\dim C_\Omega(x) = \dim C_\Omega(-x)$, by Lemma 7.1.3. If $p = 2$, up to G-conjugacy, we have $x = [J_2^s, J_1^{n-2s}]$ and $\nu(x) = s \leq n/2$.

Theorem 18.7.1. *Let $x \in G$ be an involution with $\nu(x) = s$. Write $s = at + b$, where $0 \leq b < t$.*

(i) If n/t is even or, $n/t > 1$ is odd and $s < \max\{n/t, (n-t)/2\}$, then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

(ii) If n/t is odd and $s \geq \max\{n/t, (n-t)/2\}$, then

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n^2\left(1 - \frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$$

Remark 18.7.2. Notice that if $t = n$ then (ii) applies, as $n/t = 1$ and $s \geq 1$.

The following is a straightforward consequence.

Corollary 18.7.3. *Let $x \in G$ be an involution. Then*

$$\frac{1}{2} \le f_\Omega(x) \le 1 - \frac{2}{n} + \frac{\delta_{n,t}}{n(n-1)}$$

Furthermore $f_\Omega(x) = 1/2$ if, and only if, $\nu(x) = \lfloor n/2 \rfloor$, and $f_\Omega(x) = 1 - 2/n + \delta_{n,t}/n(n-1)$ if, and only if, $\nu(x) = 1$.

Proof. If $t = n$ then differentiating the formula for $f_\Omega(x)$ with respect to s we see that it is decreasing. The result follows by evaluating the formula for $s = 1$ and $s = n/2$ or $(n-1)/2$.

Similarly when $t < n$. Here, to be formal, several cases are needed, due to the presence of b. The strategy is to show that, given $x_s = [I_{n-s}, -I_s]$, then $f_\Omega(x_s) \ge f_\Omega(x_{s+1})$. Notice that this is also implied by Lemma 18.3.14 in the case $p \neq 2$ and $f_\Omega(x) = f_\Omega^\circ(x)$. Again, $f_\Omega(x_s) \ge f_\Omega(x_{\lfloor n/2 \rfloor})$. The result follows. *q.e.d.*

Remark 18.7.4. Notice that if we do not seek a characterisation, for the lower bound, we can observe the following, in case (i)

$$f_\Omega(x) \ge 1 - \frac{2s(n-s)}{n^2} - \frac{t^2}{2n^2(t-1)} \ge 1 - \frac{2s(n-s)}{n^2} - \frac{1}{4(n-2)}$$

from which we deduce $f_\Omega(x) \ge \frac{1}{2} - \frac{1}{4(n-2)}$.

We see that for any involution $x \in G$, $x^G \cap H \neq \emptyset$ (we will explicitly construct an involution $y \in H$ given $x \in G$). Therefore in the following we will consider involutions $x \in H$.

We divide the analysis according to whether $p \neq 2$ or $p = 2$. We start with the case $p \neq 2$.

18.7.1. Semisimple involutions. Assume $p \neq 2$. Let $x = [I_{n-s}, -I_s]$ with $\nu(x) = s$. Notice that $x^G \cap H^\circ \neq \emptyset$. Using Theorem 5.3.1 we compute

$$\dim x^G = 2s(n-s) \tag{217}$$

Let $s = at + b$, with $0 \le b < t$. Then $n - s = \left(\frac{n}{t} - a - 1\right)t + (t - b)$. Hence, using Theorem 17.3.8, or the alternative formula for $\dim(x^G \cap H^\circ)$ given in Remark 17.3.14, we have

$$\dim(x^G \cap H^\circ) = \frac{2s(n-s)}{t} - \frac{2b(t-b)}{t}$$

Combining together the above formulae we have the following.

Lemma 18.7.5. *Let $x \in H$ be an involution. Assume $\nu(x) = s$ and $s = at + b$ with $0 \le b < t$. Then*

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

In the case $t = n$, we have $H = \mathrm{GL}_1 \wr S_n$. Given a decomposition $x = [x_1, \ldots, x_t] \in H^\circ$ we have $\dim x_i^{\mathrm{GL}_1} = 0$, thus $\dim(x^G \cap H^\circ) = 0$. In this case it is easy to compute an explicit formula for $f_\Omega(x)$.

Lemma 18.7.6. *Assume $t = n$. Let $x \in H$ be an involution with $\nu(x) = s$. Then*

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n(n-1)}$$

Proof. Let $x = [I_{n-s}, -I_s] \in H$, we have computed in (217), above, $\dim x^G = 2s(n-s)$. Thanks to Proposition 17.4.8, we have $x^G \cap H = \bigcup_{i=0}^{s}(x^G \cap H^\circ \pi_i)$, recall that π_i is any permutation in S_n with cycle shape $(2^i, 1^{n-2i})$ and π_0 is the identity. Using (184), and the fact that $\dim y^{\mathrm{GL}_1} = 0$ for all $y \in \mathrm{GL}_1$, we have

$$\dim(x^G \cap H^\circ \pi_i) = i$$

Therefore $\dim(x^G \cap H) = s$. The result follows. *q.e.d.*

In the following we assume $t < n$ and we shall study $\dim(x^G \cap H^\circ \pi_i)$ for $i > 0$, in order to compute $\dim(x^G \cap H) = \max_{i \geq 0}\{\dim(x^G \cap H^\circ \pi_i)\}$.

Assume n/t is even.

Lemma 18.7.7. *Assume n/t is even. Let $x \in H$ be an involution. Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$.*

Proof. Let $i < h$, where h, defined in (186), is such that $x^G \cap H^\circ \pi_h \neq \emptyset$ and $x^G \cap H^\circ \pi_{h+1} = \emptyset$. Then, by Lemma 17.4.2,

$$\dim(x^G \cap H^\circ \pi_{i+1}) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2i+3}, \ldots, x_t]\pi_{i+1})^{H^\circ}$$

where $x_j \in \mathrm{GL}_{n/t}$ and $|\nu(x_j) - \nu(x_{j'})| \leq 1$ for all $j, j' \geq 2i+3$, by Proposition 17.3.13.

Let $z = [I_{n/2t}, -I_{n/2t}] \in \mathrm{GL}_{n/t}$. Using Lemma 17.4.2 we have

$$\bar{x} = [I_{n/t}, \ldots, I_{n/t}, z, z, z_{2i+3}, \ldots, x_t]\pi_i \in x^G \cap H^\circ \pi_i$$

Hence $\dim(x^G \cap H^\circ \pi_i) \geq \dim \bar{x}^{H^\circ}$. And, using (184) we have

$$\dim(x^G \cap H^\circ \pi_{i+1}) = (i+1)\Big(\frac{n}{t}\Big)^2 + \sum_{j \geq 2i+3} \dim x_j^{\mathrm{GL}_{n/t}}$$

$$\dim \bar{x}^{H^\circ} = i\Big(\frac{n}{t}\Big)^2 + 2\dim z^{\mathrm{GL}_{n/t}} + \sum_{j \geq 2i+3} \dim x_j^{\mathrm{GL}_{n/t}} = \dim(x^G \cap H^\circ \pi_{i+1})$$

where the last equality follows from $\dim z^{\mathrm{GL}_{n/t}} = (n/t)^2/2$. Therefore

$$\dim(x^G \cap H^\circ \pi_i) \geq \dim \bar{x}^{H^\circ} = \dim(x^G \cap H^\circ \pi_{i+1})$$

for all $0 \leq i < h$. Therefore $\dim(x^G \cap H^\circ \pi_0) \geq \dim(x^G \cap H^\circ \pi_i)$ for all $0 \leq i \leq h$. In particular, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. *q.e.d.*

Proposition 18.7.8. *Assume n/t is even. Let $x \in H$ be an involutions with $\nu(x) = s$. Write $s = at + b$, where $0 \leq b < t$. Then*

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Proof. Thanks to Lemma 18.7.7, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Then, by Lemma 18.7.5, the result follows. *q.e.d.*

Assume n/t is odd.

Lemma 18.7.9. *Assume n/t is odd. Let $x \in H$ be an involution with $\nu(x) = s$. Let $l = s - \frac{n-t}{2}$. Then*

$$\dim(x^G \cap H) = \begin{cases} \dim(x^G \cap H^\circ \pi_l) & \text{if } s \geq \max\{\frac{n}{t}, \frac{n-t}{2}\} \\ \dim(x^G \cap H^\circ) & \text{otherwise} \end{cases}$$

Proof. If $s < n/t$ then, by Corollary 17.4.9, we have $x^G \cap H = x^G \cap H^\circ$.

Therefore we may assume $s \geq n/t$. Let $i < h$. Then

$$\dim(x^G \cap H^\circ \pi_{i+1}) = \dim([I_{n/t}, \dots, I_{n/t}, x_{2i+3}, \dots, x_t]\pi_{i+1})^{H^\circ}$$

where $x_j \in \mathrm{GL}_{n/t}$ and $|\nu(x_j) - \nu(x_{j'})| \leq 1$ for all $j, j' \geq 2i+3$, by Proposition 17.3.13. Therefore $\dim(x^G \cap H^\circ \pi_i) = \dim([I_{n/t}, \dots, I_{n/t}, y_{2i+1}, \dots, y_t]\pi_i)^{H^\circ}$ for suitable involutions $y_j \in \mathrm{GL}_{n/t}$.

We define the following two involutions in $\mathrm{GL}_{n/t}$:

$$\bar{x} = \left[I_{\frac{n/t+1}{2}}, -I_{\frac{n/t-1}{2}}\right], \quad \bar{x}' = \left[I_{\frac{n/t-1}{2}}, -I_{\frac{n/t+1}{2}}\right] \tag{218}$$

Using Theorem 5.3.1, we compute $\dim \bar{x}^{\mathrm{GL}_{n/t}} + \dim \bar{x}'^{\mathrm{GL}_{n/t}} = (n/t)^2 - 1$.

Since $s \geq n/t$ we have $h \geq 1$, in particular $x^G \cap H^\circ \pi_1 \neq \emptyset$. Let $x = [x_1, \dots, x_t] \in H^\circ$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Assume, moreover, $x_1 = \bar{x}, x_2 = \bar{x}'$ (the case $x_i \neq \bar{x}'$ for all i will be studied later). Then

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_1) = \dim([I_{n/t}, I_{n/t}, x_3, \dots, x_t]\pi_1)^{H^\circ} &= \left(\frac{n}{t}\right)^2 + \sum_{i=3}^{t} \dim x_i^{\mathrm{GL}_{n/t}} \\ = \dim \bar{x}^{\mathrm{GL}_{n/t}} + \dim \bar{x}'^{\mathrm{GL}_{n/t}} + 1 + \sum_{i=3}^{t} \dim x_i^{\mathrm{GL}_{n/t}} &= \dim(x^G \cap H^\circ) + 1\end{aligned}$$

where the first equality is given by the fact that we cannot increase $\sum_{i=3}^{t} \dim x^{\mathrm{GL}_{n/t}}$ without increasing, at the same time, $\dim x^{H^\circ}$ and, by the hypothesis, $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$; the second equality is given by (184).

Observe that if $\dim(x^G \cap H^\circ) = \sum_i \dim x_i^{\mathrm{GL}_{n/t}}$ and $x_1 = \bar{x}, x_2 = \bar{x}'$ then, by Proposition 17.3.13, $x_i \in \{\bar{x}, \bar{x}'\}$ for all $1 \leq i \leq t$. Therefore it is necessary to understand when the blocks $\bar{x}, \bar{x}'$ occur in the block decomposition of x that maximises its H°-class dimension, i.e. $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.

Let $x = [I_{n-s}, -I_s]$ with $s \leq n/2$. Let $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ with $x_1 = \bar{x}'$ and $x_t = \bar{x}$. Then for some l, $x_1 = \ldots = x_l = \bar{x}'$ and $x_{l+1} = \ldots = x_t = \bar{x}$ (up to permutation of the blocks). Then we have

$$n - s = \frac{n/t+1}{2}(t-l) + \frac{n/t-1}{2}l$$

Notice that $l \leq t - l$ since $n - s \geq s$. In addition, by the same reason, if $\bar{x}'$ arises in the block decomposition of x then there must be at least one $\bar{x}$. Thus we have

$$t - l = \frac{t + (n-s) - s}{2}, \quad l = \frac{t - (n-s) + s}{2}$$

By the previous observations we have that $(\bar{x}, \bar{x}')$ arises in the block decomposition of x if, and only if, $l \geq 0$, that is $s \geq \frac{n-t}{2}$.

Therefore, for $s \geq \max\{\frac{n}{t}, \frac{n-t}{2}\}$, we have $x^G \cap H^\circ\pi_i \neq \emptyset$ for $0 \leq i \leq l$. Moreover, by the same argument given before we have, for $i < l$:

$$\dim(x^G \cap H^\circ\pi_{i+1}) = \dim(x^G \cap H^\circ\pi_i) + 1$$

Therefore $\dim(x^G \cap H^\circ\pi_l) = \max\{\dim(x^G \cap H^\circ\pi_i) : i = 0, \ldots, l\}$. It remains to show that $\dim(x^G \cap H^\circ\pi_l) \geq \dim(x^G \cap H^\circ\pi_i)$ for all $i > l$.

Notice that we can regard $\dim(x^G \cap H^\circ\pi_i) - \dim(x^G \cap H^\circ\pi_{i+1})$, for all $1 \leq i \leq h-1$, as $\dim(\bar{x}^L \cap K^\circ) - \dim(\bar{x}^L \cap K^\circ\pi_1)$ where $L = \mathrm{GL}_m$ and K is a $\mathscr{C}_2$-subgroup of L, for $m = (n/t)(t-2i)$. Therefore, in order to complete the proof it is enough to show that for $n/t \leq s < (n-t)/2$ we have $\dim(x^G \cap H^\circ) \geq \dim(x^G \cap H^\circ\pi_1)$. Here the assumption $s > n/t$ is equivalent to requiring that $x^G \cap H^\circ\pi_1 \neq \emptyset$, similarly, $s < (n-t)/2$ corresponds to the fact that x does not contain blocks $\bar{x}'$ when we write it in block form $x = [x_1, \ldots, x_t]$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.

Let $x = [I_{n-s}, -I_s]$ with $s \leq n/t$ and $n/t \leq s \leq (n-t)/2$. Let $x = [x_1, \ldots, x_t]$ such that $x_i \in \mathrm{GL}_{n/t}$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Then, by the previous discussion, $x_i \neq \bar{x}'$, for all $i \geq 1$. By Proposition 17.3.13, we have $x_1 = \ldots = x_m = \bar{y}'$ and $x_{m+1} = \ldots = x_t = \bar{y}$, where

$$\bar{y} = \left[I_{\frac{n/t+a}{2}}, -I_{\frac{n/t-a}{2}}\right], \quad \bar{y}' = \left[I_{\frac{n/t+a-2}{2}}, -I_{\frac{n/t-a+2}{2}}\right]$$

for some odd integer $a > 1$ (if $a = 1$ then $\bar{y}' = \bar{x}'$). We have

$$n - s = \frac{n/t + a}{2}(t - m) + \frac{n/t + a - 2}{2}m$$

And we get $m = \frac{at-(n-s)+s}{2}$ and $t - m = \frac{t(2-a)+(n-s)-s}{2}$. Therefore

$$\begin{aligned}\dim(x^G \cap H^\circ) &= (t-m)\dim \bar{y}^{\mathrm{GL}_{n/t}} + m \dim \bar{y}'^{\mathrm{GL}_{n/t}} \\ &= \frac{a^2 t}{2} - an + 2as - at + \frac{n^2}{2t} + n - 2s\end{aligned}$$

Moreover, by Proposition 17.3.13 and the proof of Theorem 17.3.8, $a = n/t - 2\lfloor s/t \rfloor$.

Then, by Lemma 17.4.2 we have $[I_{n/t}, I_{n/t}, \bar{z}, \ldots, \bar{z}, \bar{z}', \ldots, \bar{z}'] \in x^G \cap H^\circ\pi_1$ where

$$\bar{z} = \left[I_{\frac{n/t+a'}{2}}, -I_{\frac{n/t-a'}{2}}\right], \quad \bar{z}' = \left[I_{\frac{n/t+a'-2}{2}}, -I_{\frac{n/t-a'+2}{2}}\right]$$

as above, $a' = n/t - 2\lfloor s'/t' \rfloor$, where $s' = s - n/t$ and $t' = t - 2$. And, thanks to Proposition 17.3.13 and (184) we have

$$\dim(x^G \cap H^\circ\pi_1) = \dim([I_{n/t}, I_{n/t}, z_3, \ldots, z_t]\pi_1)^{H^\circ} = \left(\frac{n}{t}\right)^2 + \sum_{i=3}^{t} \dim z_i^{\mathrm{GL}_{n/t}}$$

where $z_i \in \{\bar{z}, \bar{z}'\}$ for all $i \geq 3$.

We have

$$\dim(x^G \cap H^\circ) > \dim([\bar{x}, \bar{x}', z_3, \ldots, z_t])^{H^\circ} = \dim(x^G \cap H^\circ\pi_1) - 1 \tag{219}$$

where $\bar{x}, \bar{x}'$ are defined in (218). Moreover in (219) we have a strict inequality because $\dim(x^G \cap H^\circ) = \sum_{i\geq 1} \dim x_i^{\mathrm{GL}_{n/t}}$ and $x_i \neq \bar{x}'$ for all i, hence

$$\sum \dim x_i^{\mathrm{GL}_{n/t}} > \dim \bar{x}^{\mathrm{GL}_{n/t}} + \dim \bar{x}'^{\mathrm{GL}_{n/t}} + \sum_{i\geq 3} \dim z_i^{\mathrm{GL}_{n/t}}$$

The previous argument proves that, for $n/t \leq s \leq (n-t)/2$, $\dim(x^G \cap H^\circ) \geq \dim(x^G \cap H^\circ \pi_i)$ for all $i \geq 0$.

The result follows. *q.e.d.*

Proposition 18.7.10. *Assume n/t is odd. Let $x \in H$ be an involution with $\nu(x) = s$. Write $s = at + b$, where $0 \leq b < t$.*

(i) If $s \geq \max\{n/t, (n-t)/2\}$, then

$$\dim(x^G \cap H) = \frac{n^2}{2t} - \frac{n}{2} + s$$

and

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n^2\left(1 - \frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$$

(ii) If $s < \max\{n/t, (n-t)/2\}$, then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$ and

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Proof. In the case $s < \max\{n/t, (n-t)/2\}$, by Lemma 18.7.9, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Thus $f_\Omega(x) = f_\Omega^\circ(x)$, the result follows by Lemma 18.7.5.

Now assume $s \geq \max\{n/t, (n-t)/2\}$. Then, by Lemma 18.7.9, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_l)$, where $l = s - \frac{n-t}{2}$. Moreover, by the proof of the aforementioned result, $\dim(x^G \cap H^\circ) = (t-l) \dim \bar{x}^{\mathrm{GL}_{n/t}} + l \dim \bar{x}'^{\mathrm{GL}_{n/t}}$, and

$$\dim(x^G \cap H^\circ \pi_l) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2l+1}, \ldots, x_t]\pi_l)^{H^\circ}$$

where $x_{2i+1} = \ldots = x_t = \bar{x}$. Therefore, using (184), we compute

$$\dim(x^G \cap H^\circ \pi_l) = l\left(\frac{n}{t}\right)^2 + (t-2l) \dim \bar{x}^{\mathrm{GL}_{n/t}} = \frac{n^2}{2t} - \frac{n}{2} + s$$

Using $\dim \Omega = n^2(t-1)/t$ and $\dim x^G = 2s(n-s)$ we compute

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n^2\left(1 - \frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$$

q.e.d.

Remark 18.7.11. Observe that for $t = n$ we have $n/t = 1$ is odd and $n - t = 0$. Therefore for any involution $x \in H$ we are in case (i) of Proposition 18.7.10. Substituting the value $t = n$ in the value of $f_\Omega(x)$ we get the formula computed in Lemma 18.7.6.

18.7.2. Unipotent involutions. Here we assume $p = 2$. Let $x \in G$ be an involution. Then, up G-conjugacy, $x = [J_2^s, J_1^{n-2s}]$ with $\nu(x) = s \leq n/2$. Notice that we may have $x^G \cap H^\circ = \emptyset$. However, as we will see, $x^G \cap H \neq \emptyset$.

In the case $x^G \cap H^\circ \neq \emptyset$ we can compute $\dim(x^G \cap H^\circ)$. In order to do this, we need the following technical result, which is the equivalent of Proposition 17.3.13.

Lemma 18.7.12. *Let $x = [x_1, \dots, x_t] \in H^\circ$ be an involution, with $\nu(x_i) = s_i$. Then $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ if, and only if, $|s_i - s_j| \le 1$ for all $1 \le i, j \le t$.*

PROOF. Let $x = [x_1, \dots, x_t] \in H^\circ$ be such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. We claim $|s_i - s_j| \le 1$ for all i, j. Seeking a contradiction, assume $s_i - s_j \ge 2$ for some i, j. After a permutation of the blocks, if necessary, we may assume $(i, j) = (1, 2)$. Write $s_1 = s_2 + 2 + h$, for some $h \ge 0$. Define involutions $y_1, y_2 \in \mathrm{GL}_{n/t}$ such that $\nu(y_1) = s_1 - 1$ and $\nu(y_2) = s_2 + 1$. Then $y = [y_1, y_2, x_3, \dots, x_t] \in x^G$. By Theorem 5.2.1,

$$\begin{aligned}\dim y^{H^\circ} - \dim x^{H^\circ} &= \dim y_1^{\mathrm{GL}_{n/t}} - \dim x_1^{\mathrm{GL}_{n/t}} + \dim y_2^{\mathrm{GL}_{n/t}} - \dim x_2^{\mathrm{GL}_{n/t}} \\ &= 4(s_1 - s_2 - 1) = 4(s_2 + 2 + h - s_2 - 1) = 4(h+1) > 0\end{aligned}$$

which is absurd since $\dim(x^G \cap H^\circ) \ge \dim y^{H^\circ}$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.

Conversely, let $x \in H^\circ$ be an involution with $\nu(x) = s$. Assume $x = [x_1, \dots, x_t] \in H^\circ$ is such that $|s_i - s_j| \le 1$ for all i, j and $\sum_i s_i = s$. Then $\dim(x^G \cap H^\circ) \ge \dim x^{H^\circ}$. Furthermore, there exists $z = [z_1, \dots, z_t] \in x^G \cap H^\circ$ such that $\dim(x^G \cap H^\circ) = \dim z^{H^\circ}$, say $\nu(z_i) = s'_i$, so that $\sum_i s'_i = s$. Then, by the first part of the proof, $|s'_i - s'_j| \le 1$ for all i, j. Moreover, by Lemma 17.3.12, we have $s_i, s_j, s'_i, s'_j \in \{\lfloor s/t \rfloor, \lfloor s/t \rfloor + 1\}$. Therefore, up to the permutation of the blocks, $s_i = s_{i'}$. Hence

$$\dim x_i^{\mathrm{GL}_{n/t}} = \dim z_i^{\mathrm{GL}_{n/t}} = 2s_i(n/t - s_i)$$

for all i. In particular, $\dim x^{H^\circ} = \dim z^{H^\circ}$. So $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. *q.e.d.*

Proposition 18.7.13. *Let $x \in G$ be an involution. Assume $x^G \cap H^\circ \ne \emptyset$. Write $\nu(x) = s = at + b$, where $0 \le b < t$. Then*

$$\dim(x^G \cap H^\circ) = \frac{2s(n-s)}{t} - \frac{2b(t-b)}{t}$$

PROOF. Let $x = [x_1, \dots, x_t]$ where $x_1 = \dots = x_b = [J_2^{a+1}, J_1^{n/t-2a-2}]$ and $x_{b+1} = \dots = x_t = [J_2^a, J_1^{n/t-2a}]$. Then, by Lemma 18.7.12, we have

$$\begin{aligned}\dim(x^G \cap H^\circ) = \dim x^{H^\circ} &= b \dim x_1^{\mathrm{GL}_{n/t}} + (t-b)\dim x_t^{\mathrm{GL}_{n/t}} \\ &= 2b(a+1)\Big(\frac{n}{t} - a - 1\Big) + 2a(t-b)\Big(\frac{n}{t} - a\Big)\end{aligned}$$

where the computation follows using the formula in Theorem 5.2.1. The result follows, since $a = (s-b)/t$. *q.e.d.*

Recall that for $x = [J_2^s, J_1^{n-2s}]$ we have $\nu(x) = s$ and $\dim x^G = 2s(n-s)$.

Lemma 18.7.14. *Let $x \in G$ be an involution. Assume $x^G \cap H^\circ \ne \emptyset$. Write $\nu(x) = s = at + b$, where $0 \le b < t$. Then*

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Also here we first give an explicit formula for $f_\Omega(x)$ in the case $t = n$.

Lemma 18.7.15. *Assume $t = n$. Let $x \in H$ be an involution with $\nu(x) = s$. Then*

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n(n-1)}$$

Proof. Let $x = [J_2^s, J_1^{n-2s}]$, by Lemma 17.4.5, we have that $\pi_s \in x^G$. Moreover, since $H = \mathrm{GL}_1 \wr S_n$, we have $x^G \cap H = \pi_s^{H^\circ}$. In fact, assume $x^G \cap H^\circ\pi_i \neq \emptyset$ for some $0 \leq i < s$. Then, by Lemma 17.4.2, we have $[I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_i$ with $x_i \in \mathrm{GL}_1$, but this is not possible since $[x_{2i+1}, \ldots, x_t]$ should comprise at least one J_2 block and each x_i has size 1. Therefore, for $x \in H$ involution, with $\nu(x) = s$ we have, using Theorem 5.2.1 and (184)

$$\dim x^G = 2s(n-s), \quad \dim(x^G \cap H) = \dim \pi_s^{H^\circ} = s$$

Using $\dim \Omega = n(n-1)$, the result quickly follows. *q.e.d.*

Assume n/t is even. As for the semisimple case, we first study the case n/t even. In fact, in this case, for any involution $x \in G$, $x^G \cap H^\circ \neq \emptyset$, since we can always find at least one block decomposition $x = [x_1, \ldots, x_t] \in H^\circ$. We have the following.

Lemma 18.7.16. *Assume n/t is even. Let $x \in H$ be an involution. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$$

Proof. Recall from (186) the definition of h. We claim that if $i < h$ then

$$\dim(x^G \cap H^\circ\pi_i) \geq \dim(x^G \cap H^\circ\pi_{i+1})$$

which leads to the result.

Let $x = [J_2^s, J_1^{n-2s}]$. For $i < h$, by Lemma 17.4.2, we have

$$\dim(x^G \cap H^\circ\pi_{i+1}) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2i+3}, \ldots, x_t]\pi_{i+1})^{H^\circ}$$

for suitable involutions $x_j \in \mathrm{GL}_{n/t}$, $j \geq 2i+3$. Let $z = [J_2^{n/2t}] \in \mathrm{GL}_{n/t}$. By Lemma 17.4.5

$$\bar{x} = [I_{n/t}, \ldots, I_{n/t}, z, z, x_{2i+3}, \ldots, x_t]\pi_i \in x^G \cap H^\circ\pi_i$$

Using Theorem 5.2.1, we compute $\dim z^{\mathrm{GL}_{n/t}} = (n/t)^2/2$. Thus

$$\begin{aligned}\dim(x^G \cap H^\circ\pi_i) \geq \dim \bar{x}^{H^\circ} &= i\left(\frac{n}{t}\right)^2 + 2\dim z^{\mathrm{GL}_{n/t}} + \sum_{i \geq 2i+3} \dim x_i^{\mathrm{GL}_{n/t}} \\ &= (i+1)\left(\frac{n}{t}\right)^2 + \sum_{i \geq 2i+3} \dim x_i^{\mathrm{GL}_{n/t}} = \dim(x^G \cap H^\circ\pi_{i+1})\end{aligned}$$

The result follows. *q.e.d.*

The following is a straightforward consequence of Lemma 18.7.16.

Proposition 18.7.17. *Assume n/t is even. Let $x \in H$ be an involutions. Write $\nu(x) = s = at + b$, where $0 \leq b < t$. Then*

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

Proof. Thanks to Lemma 18.7.16, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Then, by Lemma 18.7.14, the result follows. *q.e.d.*

Assume n/t is odd. In Lemma 18.7.20, below, we show that $\dim(x^G \cap H^\circ\pi_i)$ is decreasing in i (whenever it is non-empty). First, we make some observations.

Remark 18.7.18. Assume n/t is odd. Let $x = [J_2^s, J_1^{n-2s}]$ be an involution. Then $x^G \cap H^\circ \neq \emptyset$ if, and only if, $n-2s \geq t$. In fact, if the number of J_1-blocks is strictly less than t we can not find any block decomposition $x = [x_1, \ldots, x_t] \in H^\circ$ with $x_i \in \mathrm{GL}_{n/t}$.

The observation of Remark 18.7.18, thanks to Lemma 17.4.5, extends to any cosets $H^\circ \pi_i$ in the following sense.

Remark 18.7.19. Assume $i < h$ is such that $x^G \cap H^\circ \pi_i \neq \emptyset$. Then for all $i \leq j \leq h$ we have $x^G \cap H^\circ \pi_j \neq \emptyset$. Say $\nu(x) = s$, so that $x = [J_2^s, J_1^{n-2s}]$. By Lemma 17.4.2, we have $[I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_i \in x^G \cap H^\circ \pi_i$, where $[x_{2i+1}, \ldots, x_t] \in \left(\mathrm{GL}_{n/t}\right)^{t-2i}$ is a block decomposition of $[J_2^{s-ni/t}, J_1^{n-2s}]$. So $n-2s \geq t-2i$ by Remark 18.7.18. For all $j \in \{i, i+1, \ldots, h\}$, by Lemma 17.4.2, we have $[I_{n/t}, \ldots, I_{n/t}, y_{2j+1}, \ldots, y_t]\pi_j \in x^G \cap H^\circ \pi_j$ where $[y_{2j+1}, \ldots, y_t] \in \left(\mathrm{GL}_{n/t}\right)^{t-2j}$ is a block decomposition of $[J_2^{s-nj/t}, J_1^{n-2s}]$, and such a block decomposition does exist since $n - 2s \geq t - 2i \geq t - 2j$.

Lemma 18.7.20. *Assume n/t is odd. Let $x \in H$ be an involution. Let $i < h$ such that $x^G \cap H^\circ \pi_i \neq \emptyset$. Then*

$$\dim(x^G \cap H^\circ \pi_i) > \dim(x^G \cap H^\circ \pi_{i+1})$$

Proof. By Remark 18.7.19, we have $x^G \cap H^\circ \pi_{i+1} \neq \emptyset$. Then by Lemma 17.4.2

$$\begin{aligned} \dim(x^G \cap H^\circ \pi_{i+1}) &= \dim([I_{n/t}, \ldots, I_{n/t}, x_{2i+3}, \ldots, x_t]\pi_{i+1})^{H^\circ} \\ &= (i+1)\left(\frac{n}{t}\right)^2 + \sum_{i=2i+3}^{t} \dim x_i^{\mathrm{GL}_{n/t}} \end{aligned} \tag{220}$$

for suitable $x_{2i+3}, \ldots, x_t \in \mathrm{GL}_{n/t}$. Say $\nu(x_j) = s_j$. Hence $x_j = [J_2^{s_j}, J_1^{n/t-2s_j}] \in \mathrm{GL}_{n/t}$. In particular, $n/t - 2s_j$ is odd, since n/t is odd. By Lemma 17.4.5, we must have that at least for one $j \in \{2i+3, \ldots, t\}$, $n/t - 2s_j \geq 3$. In fact, if $n/t - 2s_j = 1$ for all j then $x^G \cap H^\circ \pi_i = \emptyset$, by Remark 18.7.18. Without loss of generality we may assume $n/t - 2s_{2i+3} \geq 3$. For the purpose of a lighter notation set $s_{2i+3} = a$.

Let $\bar{x} = [J_2^{\frac{n/t-1}{2}}, J_1]$ and $\bar{x}_{2i+3} = [J_2^{a+1}, J_1^{n/t-2a-2}]$. Hence, by Lemmas 17.4.2 and 17.4.5, $[I_{n/t}, \ldots, I_{n/t}, \bar{x}, \bar{x}, \bar{x}_{2i+3}, x_{2i+4}, \ldots, x_t]\pi_i \in x^G \cap H^\circ \pi_i$. Thus

$$\begin{aligned} \dim(x^G \cap H^\circ \pi_i) &\geq \dim([I_{n/t}, \ldots, I_{n/t}, \bar{x}, \bar{x}, \bar{x}_{2i+3}, x_{2i+4}, \ldots, x_t]\pi_i)^{H^\circ} \\ &= \left(\frac{n}{t}\right)^2 i + 2\dim \bar{x}^{\mathrm{GL}_{n/t}} + \dim \bar{x}_{2i+3}^{\mathrm{GL}_{n/t}} + \sum_{j \geq 2i+4} \dim x_j^{\mathrm{GL}_{n/t}} \end{aligned} \tag{221}$$

Using (220) and (221) we have

$$\begin{aligned} \dim(x^G \cap H^\circ \pi_i) - \dim(x^G \cap H^\circ \pi_{i+1}) &\geq 2\dim \bar{x}^{\mathrm{GL}_{n/t}} + \dim \bar{x}_{2i+3}^{\mathrm{GL}_{n/t}} \\ &\quad - \left(\frac{n}{t}\right)^2 - \dim x_{2i+3}^{\mathrm{GL}_{n/t}} = 2\frac{n}{t} - 4a - 3 \end{aligned}$$

And, since $n/t - 2a \geq 3$, we have $2(n/t - 2a) - 3 \geq 6 - 3 > 0$. *q.e.d.*

Observe that if n/t is odd then $n \equiv t \pmod 2$, hence $n - t \equiv 0 \pmod 2$.

Corollary 18.7.21. *Let $x \in H$ be an involution with $\nu(x) = s$. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_l)$$

where $l = \max\{0, s - \frac{(n-t)}{2}\}$.

Proof. Thanks to Lemma 18.7.20, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_l)$ where $l = \min\{i : x^G \cap H^\circ \pi_i \neq \emptyset\}$. Let $x = [J_2^s, J_1^{n-2s}]$. Then, as said in Remark 18.7.18 $x^G \cap H^\circ \neq \emptyset$ if, and only if $n - 2s \geq t$, i.e. $s \leq (n-t)/2$.

Assume $s > (n-t)/2$, then $x^G \cap H^\circ = \emptyset$. By Lemma 17.4.2, $x^G \cap H^\circ \pi_i \neq \emptyset$ if, and only if, there exists a block decomposition of $[J_2^{s-ni/t}, J_1^{n-2s}]$ in $t - 2i$ blocks $x_{2i+1}, \dots, x_t \in \mathrm{GL}_{n/t}$, so that $[I_{n/t}, \dots, I_{n/t}, x_{2i+1}, \dots, x_t]\pi_i \in x^G \cap H^\circ \pi_i$. Recall that π_i is G-conjugate to $[J_2^{ni/t}, J_1^{n-2n/ti}]$. Therefore $x^G \cap H^\circ \pi_i \neq \emptyset$ if, and only if, $n - 2s \geq t - 2i$, see Remark 18.7.19. Hence the smallest integer l such that $x^G \cap H^\circ \pi_l \neq \emptyset$ and $x^G \cap H^\circ \pi_{l-1} = \emptyset$ is given by $l = (t - n + 2s)/2$. The result follows. *q.e.d.*

From Lemma 18.7.20 and the subsequent Corollary 18.7.21 we quickly deduce the explicit formula for $f_\Omega(x)$ in the case n/t is odd and $x \in G$ an involution.

Proposition 18.7.22. *Assume n/t is odd. Let $x \in G$ be an involution. Write $\nu(x) = s$.*

(i) Assume $s \leq (n-t)/2$. Then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

(ii) Assume $s > (n-t)/2$. Then $s \geq n/t$ and

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n^2\left(1 - \frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$$

Proof. In the case $s \leq (n-t)/2$ we have, by Corollary 18.7.21, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Hence $f_\Omega(x) = f^\circ_\Omega(x)$ and the result follows by Lemma 18.7.14.

Assume $s > (n-t)/2$. If $s < n/t$ then $s \leq n/t - 1$. Then $n/t - 1 \geq (n-t)/2$; this is not possible since $2 \leq t < n$. Then, by Corollary 18.7.21, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_l)$ where $l = s - (n-t)/2$. So, by Lemma 17.4.2, we have

$$\dim(x^G \cap H^\circ \pi_l) = \dim([I_{n/t}, \dots, I_{n/t}, x_{2l+1}, \dots, x_t]\pi_l)^{H^\circ}$$

where $[x_{2l+1}, \dots, x_t] \in \left(\mathrm{GL}_{n/t}\right)^{t-2l}$ is a block decomposition of $[J_2^{s-nl/t}, J_1^{n-2s}]$. The unique such block decomposition is

$$x_{2l+1} = \ldots = x_t = [J_2^{\frac{n/t-1}{2}}, J_1]$$

since the number of blocks is $t - 2l = n - 2s$, which is the number of the J_1-blocks. Using Theorem 5.2.1, we compute $\dim x_t^{\mathrm{GL}_{n/t}} = \frac{1}{2}(n/t-1)(n/t+1)$. Thus, by (184)

$$\dim(x^G \cap H^\circ \pi_l) = \frac{n^2}{2t} - \frac{n-2s}{2}$$

Finally, using $\dim x^G = 2s(n-s)$ we compute

$$f_\Omega(x) = 1 - \frac{2s(n-s) - s}{n^2\left(1 - \frac{1}{t}\right)} + \frac{n-t}{2n(t-1)}$$

q.e.d.

Remark 18.7.23. In the case $t = n$ we have that $n/t = 1$ is odd, in addition $s \geq n/t$ and $s > (n-t)/2 = 0$. Substituting the value $t = n$ in the formula given in Proposition 18.7.22(ii) we find the formula proved in Lemma 18.7.15.

Remark 18.7.24. Assume n/t is odd. It is clear from the proof of the previous result that if $x \in G$ is an involution then $x^G \cap H \neq \emptyset$. Moreover, let $x \in H$ and assume $\nu(x) = (n-t)/2$. It is a straightforward computation to see that the two formulae given in Proposition 18.7.22 agree and $f_\Omega(x) = 1/2 - (t/n)^2/2$.

Remark 18.7.25. Notice that, thanks to Remark 18.7.24, for n/t odd, the condition in Proposition 18.7.22(i) is equivalent to requiring that $s < \max\{n/t, (n-t)/2\}$. On the other hand, the condition $x \in H$ and $s > (n-t)/2$ (in part (ii) of the aforementioned proposition), is equivalent to requiring that $s \geq \max\{n/t, (n-t)/2\}$. In particular, Theorem 18.7.1 follows in the case $p = 2$, as well.

CHAPTER 19

Symplectic group

Throughout this chapter we set $G = \mathrm{Sp}_n$, with $n > 2$, $H = \mathrm{Sp}_{n/t} \wr S_t$ a $\mathscr{C}_2$-subgroup of G, and $\Omega = G/H$. In this chapter we derive bounds on $f_\Omega(x)$ for $x \in G$ of prime order or any unipotent element in the characteristic zero case, proving the theorems stated in Chapter 16.

19.1. Upper bounds

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H$ of prime order. In particular we shall prove the following.

Proposition 19.1.1. *Let $x \in G$ be of prime order r. Let $\iota = 1$ if $r = p$ and $\iota = 2$ otherwise.*

(i) Assume $p = 2$ if $(t, r) = (n/2, 2)$. Then either

$$f_\Omega(x) \leq 1 - \frac{2\iota}{n}$$

or $f_\Omega(x) \leq U$ as listed in Table 19.1.1; here equality holds if, and only if, $C_G(x)$ is as in the last column.

n	t	r	U	x^G *or* $C_G(x)$
4	2	$= p = 2$	$3/4$	a_2^G
4	2	$\neq p, \neq 2$	$1/2$	GL_2
6	3	$\neq p$	$1/2$	GL_3
8	2	$= 2 \neq p$	$5/8$	$\mathrm{Sp}_4 \times \mathrm{Sp}_4$

Table 19.1.1

Furthermore $f_\Omega(x) = 1 - 2\iota/n$ if, and only if, $\nu(x) = \iota$ or $(n, x^G) = (4, [J_2^2]^G)$ or $(n, C_G(x)) = (8, \mathrm{GL}_4)$.

(ii) Assume $(t, r) = (n/2, 2)$ and $p \neq 2$. Then

$$f_\Omega(x) \leq 1 - \frac{2\iota}{n} + \frac{4\iota - 2}{n(n-2)}$$

with equality if, and only if, $\nu(x) = \iota$.

Remark 19.1.2.

(i) Thanks to Lemma 7.1.1 and Corollary 7.1.11, Proposition 19.1.1 extends to any non-central element in G. In particular, Theorems 16.1.1, 16.1.3 and 16.1.4 follow.

(ii) Notice that in Table 19.1.1 we have, for $n = 8$, the elements with centraliser $\mathrm{Sp}_4 \times \mathrm{Sp}_4$ are conjugate to $x = [I_4, -I_4]$. It is easy to compute $f_\Omega([I_4, -I_4]) = 1 - \frac{2\iota}{n} + \frac{4\iota-2}{n(n-2)} = 5/8$.

As usual, we spread the proof of Proposition 19.1.1 over several lemmas. The same observations made in Remark 10.1.3 hold. So, we will study elements in H of prime order r.

Lemma 19.1.3. *Let $x \in H$ be of order r. Assume $\nu(x) = \iota$, where $\iota = 1$ if $r = p$ and $\iota = 2$ otherwise. Assume $C_G(x) \ncong \mathrm{GL}_2$ if $r \neq p$ and $n = 4$. Then*

$$f_\Omega(x) = 1 - \frac{2\iota}{n}$$

unless $r = 2 \neq p$ and $t = n/2$ in which case $f_\Omega(x) = 1 - \frac{2\iota}{n} + \frac{4\iota-2}{n(n-2)}$.

Proof. This is an easy computation. If $r \neq p$, up to conjugation, $x = [I_{n-2}, \omega, \omega^{-1}]$ if $r \neq 2$, and $x = [I_{n-2}, -I_2]$ if $r = 2$. When $r = p$, up to conjugation, $x = [J_2, J_1^{n-2}]$ (which is of type b_1 when $p = 2$). *q.e.d.*

Remark 19.1.4. Let $n = 4$ and $x = [\omega I_2, \omega^{-1} I_2]$. Then $\nu(x) = 2$ and we easily compute $f_\Omega(x) = 1/2$.

Now we prove that, with prescribed exceptions, given $x \in H$ of prime order with $\nu(x) > \iota$, $f_\Omega(x)$ is strictly less than the values given in Proposition 19.1.1.

Recall that, given $x \in H$ of prime order r, we defined $\iota = 1$ if $r = p$ and $\iota = 2$ otherwise.

For convenience we first study odd prime order elements.

Lemma 19.1.5. *Let $x \in H$ be of odd order r. Assume $\nu(x) > \iota$. Then either $n > 8$ and*

$$f_\Omega(x) < 1 - \frac{2\iota}{n}$$

or $n \leq 8$ and $f_\Omega(x) \leq 1/2$. Moreover, if $n \leq 8$, $f_\Omega(x) = 1/2$ if, and only if, x^G or $C_G(x)$ is as in the last column of Table 19.1.2.

n	t	r	$x^G, C_G(x)$
4	2	$= p \neq 2$	$[J_2^2]^G$
6	3	$\neq p$	GL_3
8	$2, 4$	$\neq p$	GL_4

Table 19.1.2. Exceptional upper bounds for odd prime order elements

Proof. Thanks to Proposition 17.2.1 we have

$$f_\Omega(x) \leq 1 - \frac{\dim x^G}{\dim \Omega}\left(1 - \frac{1}{t} - \zeta\right) \tag{222}$$

where ζ is recorded in Table 17.2.1. We need to show that the right hand side of (222) is strictly smaller than $1 - 2\iota/n$.

Let $\nu(x) = s$. Then, by Proposition 5.4.1, $\dim x^G \geq \max\{s(n-s), ns/2\}$. Let us study separately the two cases $r = p$ and $r \neq p$.

Case 1. Assume $r = p$ and $\nu(x) \geq 2$. If $n \leq 8$ then we can easily check all the possibilities (up to G-conjugacy) for $x \in H$ and compute $f_\Omega(x)$. Thus we assume

$n > 8$. Then, up to G-conjugacy, we may write $x = [J_p^{a_p}, \dots, J_1^{a_1}]$. Therefore (222) is

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2} + \frac{n - \sum_{i \text{ odd}} a_i}{n^2}$$

In particular, since $\nu(x) \geq 2$ we have $\dim x^G \geq 2n - 4$, (recall that $n > 4$). Hence, we need to show

$$1 - \frac{4(n-2)}{n^2} + \frac{n - \sum_{i \text{ odd}} a_i}{n^2} < 1 - \frac{2}{n}$$

which is satisfied if

$$n > 8 - \sum_{i \text{ odd}} a_i$$

The previous inequality is clearly true since $n \geq 10$.

Case 2. Assume $r \neq p$ and $\nu(x) \geq 3$. As above, if $n \leq 10$ it is easy to compute $f_\Omega(x)$ for any prime order $x \in H$ (up to the centraliser structure). Thus we assume $n > 10$. Up to G-conjugacy, we may write $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$ with a_0 even and $a_i = a_{r-i}$ for all $0 < i \leq \frac{r-1}{2}$. Therefore (222) becomes

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2} + \frac{n - a_0}{n^2}$$

Notice that $\nu(x) = 3$ only if $n = 4, 6$. Thus, we may assume $\nu(x) \geq 4$, which implies $\dim x^G \geq \max\{4n - 16, 2n\} = 4n - 16$, for $n \geq 8$. Therefore, we need to show

$$1 - \frac{8(n-4)}{n^2} + \frac{n - a_0}{n^2} < 1 - \frac{4}{n}$$

which is satisfied if, and only if, $3n > 32 - a_0$, which is clearly true since $n \geq 12$. *q.e.d.*

Here we prove a similar result for involutions. Recall that $\iota = 1$ if $p = 2$ and $\iota = 2$ otherwise.

Lemma 19.1.6. *Let $x \in H$ be an involution. Assume $\nu(x) > \iota$. Then either*

$$f_\Omega(x) < 1 - \frac{2\iota}{n}$$

or, (n, t) together with the upper bound on $f_\Omega(x)$ are listed in Table 19.1.3. In the last column of the table we record the G-class or the G-centraliser of all (and the only) elements that realise the bound.

n	t	p	$f_\Omega(x) \leq$	$x^G, C_G(x)$
4	2	$= 2$	$3/4$	a_2^G
8	2	$= 2$	$3/4 = 1 - 2/n$	a_2^G
8	4	$\neq 2$	$7/12$	$\mathrm{Sp}_4 \times \mathrm{Sp}_4$
8	2	$\neq 2$	$5/8$	$\mathrm{Sp}_4 \times \mathrm{Sp}_4$
10	5	$\neq 2$	$11/20$	$\mathrm{Sp}_4 \times \mathrm{Sp}_6$

Table 19.1.3. Exceptional upper bounds for involutions

Proof. We use the same argument as in Lemma 19.1.5. In a similar spirit we study separately the cases $p = 2$ and $p \neq 2$. Recall that given x with $\nu(x) = s$ then $\dim x^G \geq \max\{s(n-s), ns/2\}$.

Case 1. Assume $p = 2$. As above, we assume $n > 8$. Then for all involutions $x \in H$ with $\nu(x) = s$ we have that (222) is equivalent to

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2} + \frac{2s}{n^2}$$

Now, for $x \in H$ such that $\nu(x) \geq 2$ we have $\dim x^G \geq 2(n-2)$, recall that $n \geq 4$. Therefore we need to show

$$1 - \frac{4(n-2)}{n^2} + \frac{2s}{n^2} < 1 - \frac{2}{n}$$

which is true if, and only if, $n > s + 4$. Recall that $s \leq n/2$. Therefore, for $n > 8$, $n > s + 4$ holds.

Case 2. Assume $p \neq 2$. As above, we assume $n > 10$. Let $x \in H$ be an involution. Then, by (222),

$$f_\Omega(x) \leq 1 - \frac{\dim x^G}{\dim \Omega}\Big(1 - \frac{1}{t} - \frac{1}{n}\Big)$$

Assume $\nu(x) \geq 4$, for $n \geq 8$ we have $\dim x^G \geq 4(n-4)$. Hence, we need to show

$$n(n-8)\Big(1 - \frac{1}{t}\Big) - 2n + 8 > 0$$

Since $1 - 1/t \geq 1/2$, we have $n(n-8)\big(1-\frac{1}{t}\big) - 2n + 8 > \frac{n}{2}(n-12)$. Hence, $f_\Omega(x) < 1 - 4/n$, since $n \geq 12$. *q.e.d.*

Remark 19.1.7.

(i) Notice that in Table 19.1.3 the cases $(n,t) = (4,2), (8,2)$ for $p = 2$ are not exceptional upper bounds, in view of Proposition 19.1.1.In fact, for $(n,t) = (4,2)$, the element listed in the table together with elements with ν-value equal to 1 realise this bound.

(ii) Similarly, when $(n,t) = (8,4)$ or $(10,5)$ the bounds $7/12$ and $11/20$, respectively, are not exceptional upper bounds since they are strictly smaller than $1 - \frac{4}{n} + \frac{4\iota - 2}{n(n-2)}$.

Notice that we made no assumption on the order of x when $x \in H$ is unipotent. Hence the bounds given hold for unipotent elements in the characteristic zero case. Combining Lemmas 19.1.3, 19.1.5, 19.1.6 and Remark 19.1.7 we deduce Proposition 19.1.1.

19.2. Unipotent elements: lower bounds

In this section we shall derive lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order p (or unipotent in the characteristic zero case). Recall that in the characteristic zero case we set $p = \infty$.

The main result of this section is the following.

Proposition 19.2.1. *Let $x \in H$ be of order p.*

(i) If $p > n/2$ then

$$f_\Omega \geq \frac{t}{n}$$

with equality if, and only if, $x \in [J_{n/t}^{t-1}, z]^G$ for any unipotent $z \in \mathrm{Sp}_{n/t}$.

(ii) If $p \leq n/2$ then

$$f_\Omega(x) \geq \frac{1}{p}$$

Remark 19.2.2. The lower bound in case (ii) is sharp. In fact, provided $2 \neq p \mid n$, we shall characterise elements $x \in H$ of order p that realise it, see Proposition 19.2.10.

We use the same notation introduced in Section 18.2. So, if $x \in G$ has order p then, up to G-conjugacy $x = [J_p^{a_p}, \dots, J_1^{a_1}]$ with $n = \sum_i ia_i$ and a_i even for i odd (see Theorem 5.2.1). If $x \in H^\circ$ then $x = [x_1, \dots, x_t]$ where

$$x_i = [J_p^{a_{i,p}}, \dots, J_1^{a_{i,1}}] \in \mathrm{Sp}_{n/t} \tag{223}$$

and, for all $1 \leq i \leq t$, we have $\sum_j a_{i,j} = n/t$ and $a_{i,j}$ even for j odd. Moreover for all $1 \leq j \leq p$, $a_j = \sum_i a_{i,j}$. Hence

$$\dim x^G = \frac{n}{2}(n+1) - \sum_{1\leq i<j\leq p} ia_ia_j - \frac{1}{2}\sum_{i\leq p} ia_i^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i$$

$$\dim x^{H^\circ} = \frac{n}{2}\left(\frac{n}{t}+1\right) - \sum_{i<j} i(a_{1,i}a_{1,j} + \ldots + a_{t,i}a_{t,j}) - \frac{1}{2}\sum_{i\leq p} i(a_{1,i}^2 + \ldots + a_{t,i}^2) - \frac{1}{2}\sum_{i \text{ odd}} a_i$$

The following clarifies the dichotomy in Proposition 19.2.1, for a proof see Lemma 18.2.3.

Lemma 19.2.3. *Assume $n/2 < p < \infty$ and let $x = [J_p^{a_p}, \dots, J_1^{a_1}] \in G$ with $a_i > 0$ for some $n/2 < i \leq p$. Then $x^G \cap H = \emptyset$.*

19.2.1. Case $p > n/2$. Let $x \in H$ be an element of prime order p, or any unipotent element if $p = \infty$. Since $p > n/2 \geq t$, Lemma 17.3.1 implies $x^G \cap H = x^G \cap H^\circ$. Hence we may assume $x \in H^\circ$. In particular, up to G-conjugacy, $x = [J_{n/t}^{a_{n/t}}, \dots, J_1^{a_1}]$.

Lemma 19.2.4. *Let $x = [J_{n/t}^{t-1}, z] \in H^\circ$ with $z \in \mathrm{Sp}_{n/t}$ unipotent. Then $f_\Omega(x) = t/n$.*

Proof. This is an easy computation as done in Lemma 18.2.4. *q.e.d.*

The key tool for proving Proposition 19.2.1(i) is the following technical lemma.

Lemma 19.2.5. *Let $x = [x_1, \dots, x_t] \in H^\circ$ be unipotent. Assume $x_1, x_2 \neq J_{n/t}$ and $\dim(x^G \cap H) = \dim x^{H^\circ}$. Let $y = [J_{n/t}, x_2, \dots, x_t]$. Then $f_\Omega(x) > f_\Omega(y)$.*

Proof. The proof is totally similar to that of Lemma 18.2.5. Thanks to the formula in Proposition 7.1.8, the result is equivalent to

$$\dim x^G - \dim y^G < \dim(x^G \cap H) - \dim(y^G \cap H) \tag{224}$$

Notice that $\dim(x^G \cap H) = x^{H^\circ}$ implies $\dim(y^G \cap H) = y^{H^\circ}$, by the same argument given in Lemma 18.2.5. We set up the notation as in Lemma 18.2.5: $x_1, \dots, x_{t-h} \neq J_{n/t}$ and $x_{t-h+1} = \ldots = x_t = J_{n/t}$. Notice $0 \leq h \leq t-2$ since, by hypothesis, $x_1, x_2 \neq J_{n/t}$. We have $x_i = [J_{n/t-1}^{a_{i,n/t-1}}, \dots, J_1^{a_{i,1}}]$ for all $i \leq t-h$. Hence we have $x = [J_{n/t}^h, J_{n/t-1}^{a_{n/t-1}}, \dots, J_1^{a_1}]$ and $y = [J_{n/t}^{h+1}, J_{n/t-1}^{b_{n/t-1}}, \dots, J_1^{b_1}]$, where $a_i = \sum_{j=1}^t a_{j,i}$ and $b_i = a_i - a_{1,i}$. We compute $\dim x^G - \dim y^G$ and $\dim x^{H^\circ} - \dim y^{H^\circ}$. Eventually (224) is equivalent to

$$\sum_{i\leq j} ia_{1,i}(a_{2,j} + \ldots + a_{t-h,j}) + \sum_{i<j} ia_{1,j}(a_{2,i} + \ldots + a_{t-h,i}) > \sum_i i(a_{2,i} + \ldots + a_{t-h,i})$$

The inequality can be proved using the same argument of Lemma 18.2.5. *q.e.d.*

Now, Proposition 19.2.1(i) easily follows.

Lemma 19.2.6. *Let $x \in H$ be of order $p > n$. Then*

$$f_\Omega(x) \geq \frac{t}{n}$$

with equality if, and only if, $x \in [J_{n/t}^{t-1}, z]^G$, for any unipotent $z \in \mathrm{Sp}_{n/t}$.

Proof. The same as Lemma 18.2.6. *q.e.d.*

19.2.2. Case $p \leq n/2$. In this case the largest Jordan block that may appear in an element of prime order p in H is J_p. Let $x \in H$ of order p; up to G-conjugacy, we have $x = [J_p^{a_p}, \dots, J_1^{a_1}]$.

As in Section 18.2.2 we directly prove that $f_\Omega(x) \geq 1/p$. First we make some observations on the dimensions of conjugacy classes of involutions.

Remark 19.2.7. Assume $p = 2$. Let $x \in G$ be an involution. Then x has Jordan form $[J_2^{m_2}, J_1^{m_1}]$ with $s = n - m_2 - m_1$. Recall the formula for $\dim x^G$ in Proposition 5.2.5.

(i) Let $x \in G$ be an involution of type a_s. It is straightforward to check

$$\dim a_s^G = \frac{n(n+1)}{2} - \sum_{i<j} i m_i m_j - \frac{1}{2}\sum_i i m_i^2 - \frac{1}{2}\sum_{i \text{ odd}} m_i - m_2$$

(ii) Let $x \in H$ be an involution of type b_s or c_s. Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$ for some $h \leq \lfloor t/2 \rfloor$. Thus, thanks to Lemma 17.4.2, for suitable involutions $x_{2h+1}, \dots, x_t \in \mathrm{Sp}_{n/t}$

$$\dim(x^G \cap H) = \dim([I_{n/t}, \dots, I_{n/t}, x_{2h+1}, \dots, x_t]\pi_h)^{H^\circ}$$

Since $\dim b_s^G, \dim c_s^G > \dim a_s^G$ we must have that each x_i is either b_{s_i} or c_{s_i}-type. In addition, we point out that the formula in Theorem 5.2.1 for unipotent elements of odd order is the same as the formulae for b_s^G and c_s^G given in Proposition 5.2.5.

Lemma 19.2.8. *Let $x \in H$ be of order p. Then $f_\Omega(x) \geq 1/p$.*

Proof. Let $x \in H$ be of prime order p. Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$ for some $0 \leq h \leq \lfloor t/p \rfloor$. Recall that $\pi_h \in S_t$ is any permutation with cycle shape $(p^h, 1^{t-rh})$.

By Lemma 17.4.2, for suitable $x_{hr+1}, \dots, x_t \in \mathrm{Sp}_{n/t}$ of order p we have

$$\dim(x^G \cap H^\circ \pi_h) = \dim([I_{n/t}, \dots, I_{n/t}, x_{hr+1}, \dots, x_t]\pi_h)^{H^\circ} \tag{225}$$

Thus, using (184), we have

$$\dim(x^G \cap H) = h(p-1)\dim \mathrm{Sp}_{n/t} + \sum_{i=hp+1}^{t} \dim x_i^{H^\circ}$$

Say $a_{i,m}$ the multiplicity of the block J_m in x_i and write $l = hp+1$. Then, by Proposition 17.4.8 we have, up to conjugation,

$$x = \left[J_p^{nh/t + \sum_{i\geq l} a_{i,p}}, J_{p-1}^{\sum_{i\geq l} a_{i,p-1}}, \dots, J_1^{\sum_{i \geq l} a_{i,1}}\right]$$

Notice that $\sum_{i\leq p} i(a_{l,i}+\ldots+a_{t,i}) = n-\frac{n}{t}hp$. Using Theorem 5.2.1 we compute $\dim x^G$ and $\dim(x^G\cap H)$. Eventually, as in Lemma 18.2.9, the result is equivalent to

$$(226)\quad \sum_{l\leq\alpha<\beta\leq t}\Big(\sum_{1\leq i<j\leq p} i(a_{\alpha,i}a_{\beta,j}+a_{\alpha,j}a_{\beta,i})+\sum_{i\leq p} ia_{\alpha,i}a_{\beta,i}\Big)\geq \frac{n^2}{2t^2p}(t-hp)(t-hp-1)$$

Let $\mathcal{I}=\{(\alpha,\beta): hp<\alpha<\beta\leq t\}$, then $|\mathcal{I}|=\frac{(t-hp)(t-hp-1)}{2}$. Fix $(\alpha,\beta)\in\mathcal{I}$, and write $a_i=a_{\alpha,i}$ and $b_i=a_{\beta,i}$. Then we claim

$$(227)\quad \sum_{1\leq i<j\leq p} i(a_ib_j+a_jb_i)+\sum_{i\leq p} ia_ib_i\geq\frac{n^2}{t^2p}$$

Notice that, assuming the claim is true, $f_\Omega(x)=1/p$ if, and only if, for every $(\alpha,\beta)\in\mathcal{I}$, in (227) equality holds. Arguing as in Lemma 18.2.9, we prove the claim.

Notice that, by Remark 19.2.7(ii), the previous argument holds in the case $p=2$ for involutions of type b_s and c_s. Therefore we only need to check for a_s-type involutions.

Let x be of a_s-type. Then, by Proposition 5.2.5, $\dim x^G=s(n-s)$ and for some $h\leq\lfloor t/2\rfloor$ we have $\dim(x^G\cap H)=\dim(x^G\cap H^\circ\pi_h)$. Hence, using (184),

$$\dim(x^G\cap H)=h\dim \mathrm{Sp}_{n/t}+\sum_{i=2h+1}^{t}\dim x_i^{\mathrm{Sp}_{n/t}}$$

where each x_i is of type a_{s_i}, otherwise x would not be of a_s-type. Moreover, by Lemma 17.4.5, we have $s=\sum_i s_i+nh/t$. In view of Remark 19.2.7(i) we write $x=[J_2^{m_2},J_1^{m_1}]$ and $x_i=[J_2^{m_{2,i}},J_1^{m_{1,2}}]$. Thus $m_2=\frac{n}{t}h+\sum_i m_{i,2}$. Then, using the same argument as above and the formula for the G-conjugacy class of x given in Remark 19.2.7(i). We see that $f_\Omega(x)\geq 1/2$ if, and only if,

$$\sum_{l\leq\alpha<\beta\leq t}\Big(\sum_{1\leq i<j\leq p} i(m_{\alpha,i}m_{\beta,j}+m_{\alpha,j}m_{\beta,i})+\sum_{i\leq p} im_{\alpha,i}m_{\beta,i}\Big)+\frac{n}{t}h\geq\frac{n^2}{2t^2p}(t-hp)(t-hp-1)$$

which is clearly true thanks to the same argument given above. *q.e.d.*

Remark 19.2.9. Assume $p=2$ and x is an a_s type involution. Then, by the previous proof $f_\Omega(x)>1/2$ if $\dim(x^G\cap H)=\dim(x^G\cap H^\circ\pi_h)$ for some $h>0$.

Now, we focus on the characterization of elements $x\in H$ of odd order p for which $f_\Omega(x)=1/p$. If $p\mid n$ we shall succeed in this aim, see Proposition 19.2.10. For $p=2$ more information will be provided in Section 19.7.2.

As done for the case $G=\mathrm{GL}_n$ we can show the analogue of Lemmas 18.2.11, 18.2.12 and 18.2.13 and Proposition 18.2.15.

Eventually, also here we have the following. Recall the notation introduced before Lemma 18.2.13: if $\mu=(p^{a_p},\ldots,1^{a_1})\vdash n$ and $\eta=(p^{b_p},\ldots,1^{b_1})\vdash m$ then we write $\mu\oplus\eta=(p^{a_p+b_p},\ldots,1^{a_1+b_1})\vdash n+m$. Recall the definition of $\mathcal{P}_G$ given in Section 17.3.1.

Proposition 19.2.10. *Assume $p\mid n$ is odd. Let $x\in H$ be of order p. Then $f_\Omega(x)=1/p$ if, and only if, one of the following holds*

(i) $x\in[J_p^{n/p}]^G$;

(ii) $n/t = pm$ *and* $x \in [J_p^{m(t-1)}, z]^G$ *for any* $z \in \mathrm{Sp}_{n/t}$ *of order* p *whose associated partition* $\lambda = (p^{a_p}, \dots, 1^{a_1}) \vdash n/t$ *satisfies one of the following*

(a) $a_p = m$*; or,*

(b) $a_p < m$ *and whenever* $\lambda = \mu \oplus \eta$ *with* $\mu \vdash l_1 p$, $\eta \vdash l_2 p$ *for some* $l_1, l_2 < m$ *even such that* $\mu \oplus (p^{l_2}), \mu \oplus (p^{l_1}) \in \mathcal{P}_{\mathrm{Sp}_{n/t}}$ *then* $\eta = (p^{l_1})$ *or* $\eta = (p^{l_2})$*.*

Recall that if $p > t$ and $x \in H$ has order p then $x^G \cap H = x^G \cap H^\circ$. So if $p \geq n/t$ then $f_\Omega(x) \geq t/n$. The following result deals with the case $p < t$ and $p \nmid n$.

Proposition 19.2.11. *Assume* $p \nmid n$ *is odd. Then there exists* $x \in H$ *of order* p *such that*

$$f_\Omega(x) \leq \begin{cases} \frac{1}{p} + \frac{p}{n} & \frac{n}{t} < p < t \\ \frac{9}{4p} & \text{otherwise} \end{cases}$$

Proof. As observed above, we may assume $p < t$. Let us study separately the two cases $p < n/t$ and $n/t < p < t$.

Case 1. Assume $p < n/t$. Then $n/t = a(2p) + b$, where $0 \leq b < 2p$ is even. In the case $b < p$ we define $\bar{x} = [J_p^{2a}, J_b] \in \mathrm{Sp}_{n/t}$. So $x = [\bar{x}, \dots, \bar{x}] \in H^\circ$. And we have

$$\begin{aligned} \dim x^G &= \frac{n}{2}(n+1) - 2abt^2 - \frac{bt^2}{2} - 2pa^2t^2 - at \\ \dim x^{H^\circ} &= \frac{n}{2}\Big(\frac{n}{t} + 1\Big) - 2abt - \frac{bt}{2} - 2pa^2 t - at \end{aligned}$$

It is easy to check that $\dim x^{H^\circ} = \big(\frac{1}{t} + \frac{n-2at}{2\dim x^G}\big(1 - \frac{1}{t}\big)\big) \dim x^G$. Therefore, by Proposition 17.2.1 we have $\dim(x^G \cap H) = \dim x^{H^\circ}$. Using Proposition 7.1.8 and *Mathematica* we compute

$$f_\Omega(x) = \frac{t^2}{n^2}(4a^2p + 4ab + b) = \frac{1}{p} + \frac{bt^2}{pn^2}(p-b) \leq \frac{1}{p} + \frac{p}{4}\cdot\frac{t^2}{n^2} < \frac{5}{4p} \tag{228}$$

where the second equality follows substituting $a = (n/t - b)/2p$; the first inequality follows from $b(p-b) \leq p^2/4$, the last inequality follows from $t/n < 1/p$.

If $b > p$, we define $\bar{x} = [J_p^{2a}, J_{p-1}, J_{b-p+1}] \in \mathrm{Sp}_{n/t}$. Notice that $b \leq 2p - 2$ hence $\bar{x}$ has order p. Thus $x = [\bar{x}, \dots, \bar{x}] \in H^\circ$. And

$$\begin{aligned} \dim x^G &= \frac{n}{2}(n+1) - t^2(b-p+1)(2a+1) - (p-1)2at^2 \\ &\quad - \frac{b-p+1}{2}t^2 - \frac{p-1}{2}t^2 - 2a^2t^2p - at \end{aligned}$$

As above, we see $\dim x^{H^\circ} = \big(\frac{1}{t} + \frac{n-2at}{2\dim x^G}\big(1 - \frac{1}{t}\big)\big) \dim x^G$. With the aid of *Mathematica* we compute

$$f_\Omega(x) = \frac{1}{p} - \frac{t^2}{n^2p}(b^2 - 3bp + 2p^2 - 2p) \leq \frac{1}{p} + \frac{t^2}{n^2}\Big(2 + \frac{p}{4}\Big) < \frac{9}{4p} \tag{229}$$

where the first inequality follows from the observation that $g(b) := b^2 - 3bp + 2p^2 - 2p$ is maximal when $b = 3p/2$, then we used $t/n < 1/p$.

Case 2. Now assume $n/t < p < n$ and define $h = \lfloor t/p \rfloor$. Let us observe that $\frac{t-p+1}{p} \leq h \leq \frac{t}{p}$. We consider $x = [J_p^{nh/t}, J_{n/t}^{t-hp}] \in H$. By the hypothesis $p \neq t$. Assume $p > t$, then $x^G \cap (H \setminus H^\circ) = \emptyset$, by Lemma 17.3.1; therefore $x^G \cap H = x^G \cap H^\circ$ and

$f_\Omega(x) \geq t/n$ by Proposition 19.2.1(i). Thus we may assume $p < t$, hence $h \geq 1$. In addition, $p > n/t$ implies that in $\mathrm{Sp}_{n/t}$ there are no unipotent elements with J_p blocks. Hence $x^G \cap H = x^G \cap H^\circ \pi_h$. Using Theorem 5.2.1 and (184), we compute

$$\dim x^G = \frac{n}{2}(n+1) - \Big(\frac{n}{t}\Big)^2 h(t-hp) - \frac{n}{2t}(t-hp)^2 - \frac{p}{2}\Big(\frac{n}{t}h\Big)^2 - \frac{n}{2t}h$$
$$\dim(x^G \cap H) = \frac{n^2}{2t}\Big(1 - \frac{h}{t}\Big) + \frac{n}{2t}h(p-1)$$

Using *Mathematica* we compute

$$f_\Omega(x) = \frac{t}{n} + \frac{h(n-pt)(2t-1-hp)}{nt(t-1)} \tag{230}$$

Define $g(h) := \frac{h(n-pt)(2t-1-hp)}{nt(t-1)}$. Then we have

$$g'(h) = \frac{(2hp-2t+1)(pt-n)}{nt(t-1)}$$

Since $\frac{t-p+1}{p} \leq h \leq \frac{t}{p}$, we deduce that $g(h) \leq \max\{g(t/p), g(\frac{t-p+1}{p})\}$. We compute

$$g\Big(\frac{t}{p}\Big) = \frac{1}{p} - \frac{t}{n}, \quad g\Big(\frac{t-p+1}{p}\Big) = \frac{(t-p+1)(2-p-t)(pt-n)}{npt(t-1)}$$

And

$$g\Big(\frac{t-p+1}{p}\Big) - g\Big(\frac{t}{p}\Big) = \frac{(pt-n)(2+p(p-3))}{npt(t-1)} > 0$$

Therefore

$$f_\Omega(x) \leq \frac{t}{n} + g\Big(\frac{t-p+1}{p}\Big) = \frac{1}{p} + \frac{(p-n/t)(p-2)(p-1)}{np(t-1)} < \frac{1}{p} + \frac{p}{n}$$

where we used $(p-2)(p-1) < p^2$ and $p - n/t \leq p-2 < p-1 < t-1$. *q.e.d.*

19.3. Semisimple elements: lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order $r \neq p$. The main result we shall prove is the following.

Proposition 19.3.1. *Let $x \in H$ be of prime order $r \neq p$.*

(i) Assume $r \geq n-1$. Then $f_\Omega(x) = 0$ if, and only if, one of the following holds
  (a) $\nu(x) = n-1$; or,
  (b) $C_G(x) \cong \mathrm{Sp}_2 \times \mathrm{GL}_1^{n/2-1}$.

(ii) If $r < n-1$ then

$$f_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{4n^2(t-1)} - \frac{1}{n}$$

Remark 19.3.2. The lower bound in (ii) is not the best possible. In fact it is often negative if $r \neq 2$. However we shall derive the best possible lower bound on $f^\circ_\Omega(x)$ for elements of prime order $r < n-1$, see Proposition 19.3.16. Notice that the lower bound in (ii) is decreasing in r, namely it is maximal when $r = 2$.

Let $x \in G$ be an element of prime order $r \neq p$. Up to G-conjugacy, we may write

$$x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \tag{231}$$

where a_0 is even and $a_i = a_{r-i}$ for all $0 < i \leq \frac{r-1}{2}$. We shall also use the notation

$$x = [I_{a_0}, (\omega, \omega^{-1}) I_{a_1}, \dots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{232}$$

Observe that for $r > n/2$ there are no elements $x \in H$ of order r such that $x^G \cap (H \setminus H^\circ) \neq \emptyset$. For future reference we state the following.

Lemma 19.3.3. *Assume $r \geq n/2$. Let $x \in H$ be of order r. Then*

(i) for $r \neq n/2$, $x^G \cap H = x^G \cap H^\circ$;
(ii) if $r = n/2$ then $x^G \cap (H \setminus H^\circ) \neq \emptyset$ if, and only if, $x \in [I_2, \omega I_2, \dots, \omega^{n/2-1} I_2]^G$ and $t = n/2$.

Proof. First notice that $t \leq n/2$. If $r > n/2$ then Lemma 17.3.1 implies $x^G \cap H = x^G \cap H^\circ$. If $r = n/2$ and $x^G \cap (H \setminus H^\circ) \neq \emptyset$ then $t = n/2$ and, by Proposition 17.4.8, $x = [I_2, \omega I_2, \dots, \omega^{n/2-1} I_2]$; the converse is clear by Lemma 17.4.6. *q.e.d.*

The dichotomy in Proposition 19.3.1 is essentially linked to the existence in G of regular elements, i.e. elements with centraliser of minimal dimension.

Recall that H° contains a maximal torus. Thus given $x \in G$ of prime order $r \neq p$ we have $x^G \cap H^\circ \neq \emptyset$. In the case $r < n-1$ we shall study the related ratio

$$f^\circ_\Omega(x) = \frac{\dim \Omega - \dim x^G + \dim(x^G \cap H^\circ)}{\dim \Omega}$$

and we derive lower bounds on it. Then thanks the general inequality $f_\Omega(x) \geq f^\circ_\Omega(x)$ we deduce lower bounds on $f_\Omega(x)$. Notice that $f_\Omega(x) = f^\circ_\Omega(x)$ for any $x \in G$ of prime order $r > t$, by Lemma 19.3.3.

19.3.1. Case $r \geq n-1$. We use the same argument given for GL_n in Section 18.3.1. Let $x \in H$ of prime order r, then by Lemma 19.3.3 we have $x^G \cap H = x^G \cap H^\circ$. In particular, $\dim(x^G \cap H) = \dim x^{H^\circ}$, for a suitable block decomposition $x = [x_1, \dots, x_t]$. Let us denote

$$x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \dots, \omega^{r-1} I_{a_{i,r-1}}]$$

with $a_{i,0}$ even and $a_{i,j} = a_{i,r-j}$ for all $1 \leq j \leq \frac{r-1}{2}$. Moreover $\sum_i a_{i,j} = a_j$ and $\sum_j a_{i,j} = n/t$. Using Theorem 5.3.1 we compute

$$\dim x^G = \frac{n(n+1)}{2} - \frac{a_0(a_0+1)}{2} - \frac{1}{2}\sum_{i=1}^{r-1} a_i^2$$

$$\dim(x^G \cap H) = \frac{n}{2}\Big(\frac{n}{t}+1\Big) - \sum_{i=1}^{t}\Big(\frac{a_{i,0}(a_{i,0}+1)}{2} + \frac{1}{2}\sum_{j=1}^{r-1} a_{i,j}^2\Big)$$

Using Proposition 7.1.8, we compute

$$\dim C_\Omega(x) = \sum_{l=0}^{r-1} \sum_{1 \leq i < j \leq t} a_{i,l} a_{j,l} \tag{233}$$

Hence we can prove the following.

Proposition 19.3.4. *Let $x \in H$ of prime order $r \geq n-1$. Then $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$ or $C_G(x) \cong \mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n/2-1}$.*

PROOF. Assume $\nu(x) = n-1$. Up to the centraliser structure, we may write $x = [\omega, \omega^{-1}, \dots, \omega^{n/2}, \omega^{-n/2}]$. Let $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H) = \dim x^{H^\circ}$. For all $l \in \{1, \dots, r-1\}$ there exists only one block x_i such that $a_{i,l} \neq 0$, in fact $a_{i,l} = 1$. Therefore, using (233), $f_\Omega(x) = 0$. Similarly if $x = [I_2, \omega, \omega^{-1}, \dots, \omega^{n/2-1}, \omega^{-n/2+1}]$.

Now assume $x \in H$ has order r and $f_\Omega(x) = 0$. Write $x = [x_1, \dots, x_t]$ such that $\dim(x^G \cap H) = \dim x^{H^\circ}$. Then Proposition 17.3.10 yields $|a_{i,l} - a_{j,l}| \leq 1$ and $|a_{i,0} - a_{j,0}| \leq 2$ for all $i, j \in \{1, \dots, t\}$ and $l \in \{1, \dots, r-1\}$. Assume $a_l \geq 2$ for some $l \in \{1, \dots, r-1\}$, then there exists $i, j \in \{1, \dots, t\}$ such that $a_{i,l}, a_{j,l} \neq 0$, which implies $f_\Omega(x) > 0$, by (233). Therefore for all l we have $a_l \in \{0, 1\}$. Similarly if $a_0 > 2$, hence $a_0 \geq 4$ we have $a_{i,0}, a_{j,0} \neq 0$ for some $i, j \in \{1, \dots, t\}$ and again $f_\Omega(x) > 0$. Therefore $a_0 \in \{0, 2\}$. The result follows, as $n = \sum_i a_i$. *q.e.d.*

19.3.2. Case $r < n-1$. Let $x \in H$ be of prime order r. We shall derive lower bound on the related ratio $f_\Omega^\circ(x)$.

As in Proposition 18.3.7, using Theorem 17.3.8, we can easily show the following two results. Note that Lemmas 19.3.5 and 19.3.6 with Remark 19.3.6, below, complete the proof of Proposition 19.3.1.

Lemma 19.3.5. *Let $x \in H$ be of odd order $r < n-1$. Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{4n^2(t-1)} - \frac{1}{n}$$

PROOF. Up to conjugation, write $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H$. For $i \in \{1, \dots, r-1\}$ write $a_i = c_i t + b_i$ where $0 \leq b_i < t$. Similarly $a_0/2 = c_0 t + b_0$, where $0 \leq b_0 < t$. Using the formula of $\dim(x^G \cap H^\circ)$ given in Remark 17.3.14 and Theorem 5.3.1 we compute

$$\begin{aligned} f_\Omega^\circ(x) &= \frac{\frac{n^2}{2}\left(1-\frac{1}{t}\right) - \dim x^G\left(1-\frac{1}{t}\right) + \frac{n-a_0}{2}\left(1-\frac{1}{t}\right)}{\frac{n^2}{2}\left(1-\frac{1}{t}\right)} - \frac{4b_0(t-b_0) + \sum_{i>0} b_i(t-b_i)}{n^2(t-1)} \\ &= \frac{\sum_{i\geq 0} a_i^2}{n^2} - \frac{4b_0(t-b_0) + \sum_{i>0} b_i(t-b_i)}{n^2(t-1)} \geq \frac{1}{r} - \frac{rt^2}{4n^2(t-1)} - \frac{3t^2}{4n^2(t-1)} \end{aligned}$$

where we used $\sum_{i=0}^{r-1} a_i^2 \geq \frac{1}{r}(\sum_i a_i)^2 = \frac{n^2}{r}$, by Proposition B.2.1, and $b(t-b) \leq t^2/4$. It is easy to check that, for $2 \leq t \leq n/2$, $\frac{t^2}{t-1} \leq \frac{n^2}{2n-4}$. Thus $\frac{3t^2}{4n^2(t-1)} \leq \frac{3}{8n-16} \leq \frac{1}{n}$ for $n \geq 4$. The result follows. *q.e.d.*

Lemma 19.3.6. *Let $x \in H$ be an involution. Then*

$$f_\Omega(x) \geq \frac{1}{2} - \frac{2t^2}{n^2(t-1)}$$

PROOF. Up to conjugation, write $x = [I_s, -I_{n-s}] \in H$, where $s \leq n/2$. Then $\dim x^G = s(n-s)$ by Theorem 5.3.1. Write $s/2 = at + b$ with $0 \leq b < t$. By Theorem 17.3.8, we have $\dim(x^G \cap H^\circ) = \frac{s(n-s)}{t} - \frac{4b(t-b)}{t}$. Therefore

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)} \geq \frac{1}{2} - \frac{2t^2}{n^2(t-1)}$$

where, as in Lemma 19.3.5, we used $b(t-b) \leq t^2/4$. *q.e.d.*

Remark 19.3.7. It is easy to check that $\frac{1}{2} - \frac{2t^2}{n^2(t-1)} \geq \frac{1}{2} - \frac{t^2}{2n^2(t-1)} - \frac{1}{n}$ for $n \geq 8$. For $n = 4, 6$ the only involutions, up to conjugacy, are $[I_2, -I_2]$ and $[I_2, -I_4]$ and a straightforward computation shows that the previous inequality holds. Therefore the proof of Proposition 19.3.1 is now complete.

Remark 19.3.8. If $n > 4$ and $x \in H$ is an involution, by Lemma 19.3.5, $f_\Omega(x) \geq \frac{1}{2} - \frac{1}{n-2}$.

Although the bound for involutions is acceptable (it is always non-negative); the lower bound given in Lemma 19.3.5 is often negative.

For the rest of the section we shall assume r > 2.

As in Section 18.3.1, we can derive much better bounds. Namely, for any odd prime r we construct a class of elements $z \in H$ of order r, with the property that for any $x \in H$ of order r we have $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$.

The following definition has already been used for the analogous analysis when $G = \mathrm{GL}_n$, see Definition 18.3.9. Recall that for $x \in G$ of order r we write $a_i = \dim V_{\omega^i}$.

Definition 19.3.9. Let $x \in G$ be of order r. We say that x is *special* if $|a_i - a_j| \leq 1$ for all $i, j \in \{0, \dots, r-1\}$.

We give the following.

Claim. Let $x \in H$ be of order r. Then $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$ where $z \in H$ is special of order r.

If x is not special there exist $i \neq j$ such that $a_i - a_j \geq 2$. Notice that because of the centraliser structure of semisimple elements, it is important to distinguish the cases i or $j = 0$ and $i, j \neq 0$.

The following is the analogue of Lemma 18.3.10.

Lemma 19.3.10. *Let $x \in G$ be of order r. Then, either x is special or there exists $y = [I_{b_0}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}] \in H$ such that $f^\circ_\Omega(x) = f^\circ_\Omega(y)$ and one of the following holds*

(i) $|b_0 - b_1| \geq 2$*; or,*
(ii) $b_1 - b_2 \geq 2$.

Proof. It is enough to relabel the eigenvalue in a suitable way. *q.e.d.*

Also for $G = \mathrm{Sp}_n$ a similar version of Lemma 18.3.11 holds. However, here, due to the centraliser structure, more cases arise.

Let $x \in H$ be non-special of order r. Thanks to Lemma 19.3.10 we may assume $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}]$ and $|a_0 - a_1| \geq 2$ or $a_1 - a_2 \geq 2$ where in all these cases we may assume that a_i is either $\max_j\{a_j\}$ or $\min_j\{a_j\}$ for $i = 0, 1, 2$. For each of these three cases we construct a particular element.

Case 1. Assume $a_1 - a_2 \geq 2$. Then we define

$$y = [I_{a_0}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \omega^3 I_{a_3}, \dots, \omega^{r-2} I_{a_{r-2}+1}, \omega^{r-1} I_{a_{r-1}-1}] \tag{234}$$

Case 2. Assume $a_0 - a_1 \geq 2$. Then we define

$$y = [I_{a_0-2}, \omega I_{a_1+1}, \omega^2 I_{a_2}, \dots, \omega^{r-1} I_{a_{r-1}+1}] \tag{235}$$

Case 3. Assume $a_1 - a_0 \geq 2$. Then we define

$$y = [I_{a_0+2}, \omega I_{a_1-1}, \omega^2 I_{a_2}, \ldots, \omega^{r-1} I_{a_{r-1}-1}] \tag{236}$$

With the notation introduced above we prove the following technical lemma which is the key tool for the proof of the claim.

Lemma 19.3.11. *Let $x \in H$ be of order r. Assume x is not special. Then*

$$f_\Omega^\circ(x) \geq f_\Omega^\circ(y)$$

where y is defined in (234), (235) or (236).

Proof. The proof is very similar to that of Lemma 18.3.11. We give all the details in Appendix B.4.3. *q.e.d.*

Now we can prove the claim.

Lemma 19.3.12. *Let $x \in H$ be of order r. Then*

$$f_\Omega^\circ(x) \geq f_\Omega^\circ(z)$$

for any special element $z \in H$ of order r.

Proof. The argument is totally similar to that of Lemma 18.3.14. *q.e.d.*

Now we characterise special elements, up to centraliser structure.

Lemma 19.3.13. *Let $x \in H$ be a special element of order r. Say $a_l = \max\{a_i\}$.*

(i) If $r \mid n$, then $a_i = n/r$ for all i.
(ii) If $r \nmid n$, then $a_l = \lfloor n/r \rfloor + 1$. In particular $a_0 = \lfloor n/r \rfloor + \iota$ where $\iota = 0, 1$ depending on whether $\lfloor n/r \rfloor$ is even or odd.

Proof. The argument of Lemma 18.3.15 can be repeated verbatim here. *q.e.d.*

The following is a straightforward consequence of Lemma 19.3.13. Notice that if we write $n = \lfloor n/r \rfloor r + c$ with $0 \leq c < r$ then $c \equiv \lfloor n/r \rfloor \pmod 2$.

Proposition 19.3.14. *Let $n = \lfloor n/r \rfloor r + c$. Let $x \in G$ be special of order r. Then $C_G(x) \cong C_G(z)$, where*

$$z = \left[I_{\lfloor \frac{n}{r} \rfloor + \delta}, \omega I_{\lfloor \frac{n}{r} \rfloor + 1}, \omega^{-1} I_{\lfloor \frac{n}{r} \rfloor + 1}, \ldots, \omega^{-\lfloor c/2 \rfloor} I_{\lfloor \frac{n}{r} \rfloor + 1}, \omega^{\lfloor c/2 \rfloor + 1} I_{\lfloor \frac{n}{r} \rfloor}, \ldots, \omega^{-\frac{r-1}{2}} I_{\lfloor \frac{n}{r} \rfloor}\right] \tag{237}$$

and $\delta = 0$ if $\lfloor n/r \rfloor$ is even and 1 otherwise.

Thanks to Lemma 19.3.12, in order to have the best possible lower bound on f_Ω° we need to compute $f_\Omega^\circ(z)$, where z is defined in (237). We study two cases depending on the parity of $\lfloor n/r \rfloor$.

Case A. Assume $\lfloor n/r \rfloor$ is even and write $n = \lfloor n/r \rfloor r + c$. Then

$$z = [I_{\lfloor n/r \rfloor}, \omega I_{\lfloor n/r \rfloor + 1}, \omega^{-1} I_{\lfloor n/r \rfloor + 1}, \ldots, \omega^{-c/2} I_{\lfloor n/r \rfloor + 1}, \omega^{c/2+1} I_{\lfloor n/r \rfloor}, \ldots, \omega^{-(r-1)/2} I_{\lfloor n/r \rfloor}]$$

Using Theorem 5.3.1, we compute

$$\dim z^G = \frac{n}{2}(n+1) - \frac{1}{2}\left\lfloor \frac{n}{r} \right\rfloor \left(\left\lfloor \frac{n}{r} \right\rfloor + 1 \right) - \frac{c}{2}\left(\left\lfloor \frac{n}{r} \right\rfloor + 1 \right)^2 - \frac{r-1-c}{2}\left\lfloor \frac{n}{r} \right\rfloor^2$$

And, thanks to Theorem 17.3.8, we compute

$$\begin{aligned}\dim(z^G\cap H^\circ) = {} & \frac{n^2}{2t} - \left\lfloor\frac{n}{r}\right\rfloor + 2\Big(\left\lfloor\frac{\lfloor n/r\rfloor}{2t}\right\rfloor^2 t + (t-\lfloor n/r\rfloor)\left\lfloor\frac{\lfloor n/r\rfloor}{2t}\right\rfloor\Big)\\ & + \frac{c}{2}\Big(\left\lfloor\frac{\lfloor n/r\rfloor+1}{t}\right\rfloor^2 t + (t-2\lfloor n/r\rfloor-2)\left\lfloor\frac{\lfloor n/r\rfloor+1}{t}\right\rfloor\Big)\\ & + \frac{r-1-c}{2}\Big(\left\lfloor\frac{\lfloor n/r\rfloor}{t}\right\rfloor^2 t + (t-2\lfloor n/r\rfloor)\left\lfloor\frac{\lfloor n/r\rfloor}{t}\right\rfloor\Big)\end{aligned}$$

Let us write $\lfloor n/r\rfloor = at+b$ with $0\le b<t$. Notice that

$$\left\lfloor\frac{n}{r}\right\rfloor = at+b, \quad \left\lfloor\frac{\lfloor n/r\rfloor}{t}\right\rfloor = a$$

Moreover if we write $\lfloor n/r\rfloor/2 = a't+b'$, then either $b'<t/2$ or $t/2\le b'<t$. Hence, if $b'<t/2$ we have

$$a=2a', \quad b=2b'$$

in particular $\lfloor\frac{\lfloor n/r\rfloor}{2t}\rfloor = a/2 = \frac{1}{2}\lfloor\frac{\lfloor n/r\rfloor}{t}\rfloor$. For $b'\ge t/2$ we have

$$a=2a'+1, \quad b=2b'-t$$

and $\lfloor\frac{\lfloor n/r\rfloor}{2t}\rfloor = (a-1)/2 = \frac{1}{2}\lfloor\frac{\lfloor n/r\rfloor}{t}\rfloor - \frac{1}{2}$.

Then, as in Section 18.3, we have that either $b<t-1$ or $b=t-1$, i.e. either

$$\left\lfloor\frac{\lfloor n/r\rfloor+1}{t}\right\rfloor = \left\lfloor\frac{\lfloor n/r\rfloor}{t}\right\rfloor, \quad \text{or} \quad \left\lfloor\frac{\lfloor n/r\rfloor+1}{t}\right\rfloor = \left\lfloor\frac{\lfloor n/r\rfloor}{t}\right\rfloor+1$$

Together with the two cases on b' analysed above we get four different cases. In all the cases we use

$$\dim z^G = \frac{n}{2}(n+1) - \frac{1}{2}(at+b)(at+b+1) - \frac{c}{2}(at+b+1)^2 - \frac{r-1-c}{2}(at+b)^2$$

Case A1. Assume $b'<t/2$ and $\lfloor\frac{\lfloor n/r\rfloor+1}{t}\rfloor = \lfloor\frac{\lfloor n/r\rfloor}{t}\rfloor$. Therefore

$$\begin{aligned}\dim(z^G\cap H^\circ) = {} & \frac{n^2}{2t} - (at+b) + 2\Big(\frac{a^2}{4}t + (t-at-b)\frac{a}{2}\Big)\\ & - ac + \frac{r-1}{2}\Big(a^2t + (t-2at-2b)a\Big)\end{aligned}$$

Case A2. Assume $b'\ge t/2$ and $\lfloor\frac{\lfloor n/r\rfloor+1}{t}\rfloor = \lfloor\frac{\lfloor n/r\rfloor}{t}\rfloor$. Therefore

$$\begin{aligned}\dim(z^G\cap H^\circ) = {} & \frac{n^2}{2t} - (at+b) + 2\Big(\frac{(a-1)^2}{4}t + (t-at-b)\frac{a-1}{2}\Big)\\ & - ac + \frac{r-1}{2}\Big(a^2t + (t-2at-2b)a\Big)\end{aligned}$$

With the aid of *Mathematica* we compute

$$f^\circ_\Omega(z) = \frac{1}{r} - \frac{c^2+br(r+1)}{n^2r\left(1-\frac{1}{t}\right)} + \frac{(br+c)^2}{n^2r(t-1)} - \iota\frac{t-2b}{n^2\left(1-\frac{1}{t}\right)} \tag{238}$$

where $\iota=0$ in **Case A1** and $\iota=1$ in **Case A2**.

Case A3. Assume $b' < t/2$ and $\lfloor \frac{\lfloor n/r \rfloor+1}{t} \rfloor = \lfloor \frac{\lfloor n/r \rfloor}{t} \rfloor + 1$. Therefore

$$\begin{aligned} \dim(z^G \cap H^\circ) = &\frac{n^2}{2t} - (at+b) + 2\Big(\frac{a^2}{4}t + (t-at-b)\frac{a}{2}\Big) \\ &+ \frac{c}{2}\Big(2at + t - 2a + t - 2\lfloor n/r \rfloor - 2\Big) \\ &+ \frac{r-1}{2}\Big(a^2t + (t - 2at - 2b)a\Big) \end{aligned}$$

(in this case $b = t-1$).

Case A4. Assume $b' \geq t/2$ and $\lfloor \frac{\lfloor n/r \rfloor+1}{t} \rfloor = \lfloor \frac{\lfloor n/r \rfloor}{t} \rfloor + 1$. Then $b = t-1$ and $b = 2b' - t$, hence $2(b'-t) = -1$, which is absurd. Therefore this case never occurs.

Again, using *Mathematica*, we compute $f^\circ_\Omega(z)$ for **Case A3**. Recall that $n = (at+b)r + c$ and we are assuming $b = t-1$. Thus

$$f^\circ_\Omega(z) = \frac{1}{r} - \frac{(r-c)^2}{n^2 r} - \frac{t}{n^2} \tag{239}$$

Notice that if we substitute the value $b = t-1$ in (238) we get (239).

Case B. Assume $\lfloor n/r \rfloor$ is odd and write $n = \lfloor n/r \rfloor r + c$. Then

$$z = [I_{\lfloor n/r \rfloor+1}, \omega I_{\lfloor n/r \rfloor+1}, \omega^{-1} I_{\lfloor n/r \rfloor+1}, \ldots, \omega^{-(c-1)/2} I_{\lfloor n/r \rfloor+1}, \omega^{(c+1)/2} I_{\lfloor n/r \rfloor}, \ldots, \omega^{-(r-1)/2} I_{\lfloor n/r \rfloor}]$$

And, by Theorem 17.3.8 we have

$$\begin{aligned} \dim(z^G \cap H^\circ) = &\frac{n^2}{2t} - \Big\lfloor \frac{n}{r} \Big\rfloor - 1 + 2\Big(\Big\lfloor \frac{\lfloor n/r \rfloor + 1}{2t} \Big\rfloor^2 t + (t - \lfloor n/r \rfloor - 1)\Big\lfloor \frac{\lfloor n/r \rfloor + 1}{2t} \Big\rfloor\Big) \\ &+ \frac{c-1}{2}\Big(\Big\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \Big\rfloor^2 t + (t - 2\lfloor n/r \rfloor - 2)\Big\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \Big\rfloor\Big) \\ &+ \frac{r-c}{2}\Big(\Big\lfloor \frac{\lfloor n/r \rfloor}{t} \Big\rfloor^2 t + (t - 2\lfloor n/r \rfloor)\Big\lfloor \frac{\lfloor n/r \rfloor}{t} \Big\rfloor\Big) \end{aligned}$$

Write $\lfloor n/r \rfloor = at + b$ with $0 \leq b < t$. Again, we have that $\lfloor n/r \rfloor/2 = a't + b'$ with $0 \leq b' < t$ and

$$\begin{aligned} a = 2a',\ b = 2b', &\quad \text{if } 1 \leq b' < t/2 \\ a = 2a' + 1,\ 2b' - t = b, &\quad \text{if } t/2 \leq b' < t \end{aligned}$$

Therefore

$$\frac{1}{2}\Big\lfloor \frac{n}{r} \Big\rfloor + \frac{1}{2} = \begin{cases} \frac{a}{2}t + \frac{b+1}{2} & b' < t/2,\ b \neq t-1 \\ \frac{a+1}{2}t & b' < t/2,\ b = t-1 \\ \frac{a-1}{2}t + \frac{t+b}{2} & b' \geq t/2,\ b \neq t-1 \end{cases}$$

Notice that the case $b' \geq t/2$ and $b = t-1$, as above, does not occur. Let us compute $f^\circ_\Omega(z)$ in each of these three cases. We record the dimension of z^G:

$$\begin{aligned} \dim z^G &= \frac{n}{2}(n+1) - \frac{1}{2}\Big(\Big\lfloor \frac{n}{r} \Big\rfloor + 1\Big)\Big(\Big\lfloor \frac{n}{r} \Big\rfloor + 2\Big) - \frac{c-1}{2}\Big(\Big\lfloor \frac{n}{r} \Big\rfloor + 1\Big)^2 - \frac{r-c}{2}\Big\lfloor \frac{n}{r} \Big\rfloor^2 \\ &= \frac{n}{2}(n+1) - \frac{1}{2}(at+b+1)(at+b+2) - \frac{c-1}{2}(at+b+1)^2 - \frac{r-c}{2}(at+b)^2 \end{aligned}$$

Case B1. Assume $b' < t/2$ and $\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \rfloor = \lfloor \frac{\lfloor n/r \rfloor}{t} \rfloor$. Therefore

$$\begin{aligned}\dim(z^G \cap H^\circ) = \frac{n^2}{2t} - (at+b) - 1 + 2\Big(\frac{a^2}{4}t + (t - at - b - 1)\frac{a}{2}\Big) \\ - a(c-1) + \frac{r-1}{2}\Big(a^2 t + (t - 2at - 2b)a\Big)\end{aligned}$$

Case B2. Assume $b' \geq t/2$ and $\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \rfloor = \lfloor \frac{\lfloor n/r \rfloor}{t} \rfloor$. Therefore

$$\begin{aligned}\dim(z^G \cap H^\circ) = \frac{n^2}{2t} - (at+b) - 1 + 2\Big(\frac{(a-1)^2}{4}t + (t - at - b - 1)\frac{a-1}{2}\Big) \\ - a(c-1) + \frac{r-1}{2}\Big(a^2 t + (t - 2at - 2b)a\Big)\end{aligned}$$

With the aid of *Mathematica*, using $a = ((n-c)/r - b)/t$, we compute

$$f^\circ_\Omega(z) = \frac{1}{r} - \frac{c^2 + br(r+1)}{n^2 r\big(1 - \frac{1}{t}\big)} + \frac{(br+c)^2}{n^2 r(t-1)} - \frac{1}{n^2\big(1 - \frac{1}{t}\big)} - \iota\frac{t - 2b - 4}{n^2\big(1 - \frac{1}{t}\big)} \tag{240}$$

where $\iota = 0$ if $b' < t/2$ and $\iota = 1$ otherwise.

Case B3. Assume $b' < t/2$ and $\lfloor \frac{\lfloor n/r \rfloor + 1}{t} \rfloor = \lfloor \frac{\lfloor n/r \rfloor}{t} \rfloor + 1$, so that $b = t-1$. We compute

$$f^\circ_\Omega(z) = \frac{1}{r} - \frac{(r-c)^2}{n^2 r} + \frac{2}{n^2\big(1 - \frac{1}{t}\big)} \tag{241}$$

A direct check shows that (241) has the same value as (240) with $\iota = 1$ when we substitute the value $b = t-1$ in it.

Remark 19.3.15. If rt divides n then $n = art$ for some even a and $f^\circ_\Omega(z) = 1/r$.

We summarise the previous discussion in the following.

Proposition 19.3.16. *Let $x \in H$ be of order r. Write $n = (at+b)r + c$, where $0 \leq b < t$ and $0 \leq c < r$. Then*

$$f^\circ_\Omega(x) \geq \frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} - \frac{2t}{n^2}$$

Proof. Thanks to Proposition 19.3.14 we have $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$ where z is a special element. Notice that

$$\frac{1}{r} - \frac{c^2 + br(r+1)}{n^2 r\big(1 - \frac{1}{t}\big)} + \frac{(br+c)^2}{n^2 r(t-1)} = \frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} - \frac{bt}{n^2(t-1)} \tag{242}$$

We immediately see that

$$\frac{t(t-b)}{n^2(t-1)}, \frac{t(t-b-3)}{n^2(t-1)} \leq \frac{t^2}{n(t-1)} \leq \frac{2t}{n^2}$$

Using $f^\circ_\Omega(z)$ computed in (238) and (240) with the previous estimations we deduce the desired inequality. *q.e.d.*

Remark 19.3.17. Notice the similarity of the lower bounds on f_Ω for odd order semisimple elements given in Proposition 19.3.16 and Proposition 18.3.19. In addition, the best possible lower bounds on f_Ω are given in (238) and (240), when $\lfloor n/r \rfloor$ is even and odd, respectively.

19.4. Local upper bounds

Recall, we defined $\mathcal{V}_s = \{x \in G : \nu(x) = s\}$ and $\mathcal{V}_{s,r}$ the set of elements of $\mathcal{V}_s$ of order r. In this section we shall derive upper bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_s$ of odd prime order. We deal with involutions in Section 19.7.

The main result of this section is the following.

Proposition 19.4.1. *Assume r is an odd prime. Let $x \in H \cap \mathcal{V}_{s,r}$. If $s \leq n/2$ then*

$$f_\Omega(x) \leq 1 - 2\frac{s}{n} + 2\Big(\frac{s}{n}\Big)^2 + \frac{1}{n}$$

If $s > n/2$ then

$$f_\Omega(x) \leq 1 - \frac{s}{n} + \frac{1}{n}$$

Proposition 19.4.1 is proved in Lemma 19.4.2 for unipotent elements and in Lemma 19.4.4 for semisimple elements. We use the same argument given for Proposition 18.4.1. One of the main tools is Proposition 17.2.1.

19.4.1. Unipotent elements. Assume $p \neq 2$. Let $x \in H \cap \mathcal{V}_{s,p}$. Up to G-conjugacy, $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ and $s = n - \sum_i a_i$. Notice that all the results we prove hold in the characteristic zero case as well.

Lemma 19.4.2. *Let $x \in H \cap \mathcal{V}_{s,p}$. Then the conclusion of Proposition 19.4.1 holds.*

Proof. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$. By Proposition 5.4.1, $\dim x^G \geq \max\{s(n-s), ns/2\}$. Using Proposition 17.2.1 we have

$$f_\Omega(x) \leq 1 - \frac{2\dim x^G}{n^2} + \frac{n - \sum_{i \text{ odd}} a_i}{n^2} \leq 1 - \frac{2\dim x^G}{n^2} + \frac{1}{n}$$

If $s < n/2$, then $\dim x^G \geq s(n-s)$. Therefore

$$f_\Omega(x) \leq 1 - \frac{2\dim x^G}{n^2} + \frac{1}{n} \leq 1 - \frac{2s(n-s)}{n^2} + \frac{1}{n}$$

If $s \geq n/2$ then $\dim x^G \geq ns/2$. Thus

$$f_\Omega(x) \leq 1 - \frac{2\dim x^G}{n^2} + \frac{1}{n} \leq 1 - \frac{s}{n} + \frac{1}{n}$$

q.e.d.

For $s \leq n/2$ it is easy to construct elements $x \in H \cap \mathcal{V}_{s,p}$, for instance $x = [J_2^s, J_1^{n-2s}]$. In the following we show that the upper bound of Lemma 19.4.2 is close to best possible.

Lemma 19.4.3. *Let $s \leq n/2$. Write $s = at + b$ with $0 \leq b < t$. Then there exists $x \in H \cap \mathcal{V}_{s,p}$ such that*

$$f_\Omega(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{1}{n}$$

Proof. Let $x = [J_2^s, J_1^{n-2s}]$. Using Theorem 5.2.1 we compute $\dim x^G = s(n-s+1)$. Consider the following block decomposition $x = [y, \ldots, y, z, \ldots, z] \in H^\circ$ where $y = [J_2^{a+1}, J_1^{n/t-2-2a}]$ and $z = [J_2^a, J_1^{n/t-2a}]$. Then $\dim(x^G \cap H^\circ) \geq \dim x^{H^\circ}$. We compute

$$\dim x^{H^\circ} = b \dim y^{\mathrm{Sp}_{n/t}} + (t-b)\dim z^{\mathrm{Sp}_{n/t}} = \frac{s(n-s)}{t} - s - \frac{b(t-b)}{t}$$

We see that $b(t-b) \leq t^2/4$ and $t^2/(t-1) \leq n^2/(2(n-2))$. Therefore

$$f_\Omega(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)} \geq 1 - \frac{2s(n-s)}{n^2} - \frac{1}{4(n-2)}$$

The result follows from observing that for $n \geq 4$ we have $1/4(n-2) \leq 1/n$. *q.e.d.*

19.4.2. Semisimple elements. Assume $r \neq 2$. Let $x \in \mathcal{V}_{s,r}$. If $s < n/2$ the largest eigenspace has to be the 1-eigenspace, since $a_i = a_{r-i}$ for all $1 \leq i \leq \frac{r-1}{2}$. Up to G-conjugacy,

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$

with $a_i \leq n-s$ for all i. For $s \geq n/2$ we have the following two possibilities

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$
$$x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \ldots, \omega^{r-2} I_{a_{r-2}}, \omega I_{n-s}]$$

with $a_0 \leq n-s$ even and $a_i \leq n-s$ for all i. In this section, for $x \in H$ of order r, we shall also use the notation

$$x = [I_{a_0}, (\omega, \omega^{-1}) I_{a_1}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{(r-1)/2}}]$$

We have the following.

Lemma 19.4.4. *Let $x \in H \cap \mathcal{V}_{s,r}$. If $s < n/2$ then*

$$f_\Omega(x) \leq 1 - \frac{2s(n-s)}{n^2} + \frac{s}{n^2}$$

If $s \geq n/2$ then

$$f_\Omega(x) \leq 1 - \frac{s-1}{n}$$

Proof. Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$. By Proposition 5.4.1, $\dim x^G \geq \max\{s(n-s), ns/2\}$. Using Proposition 17.2.1 we have

$$f_\Omega(x) \leq 1 - \frac{2\dim x^G}{n^2} + \frac{n-a_0}{n^2}$$

If $s < n/2$, by the previous discussion we have $a_0 = n-s$ hence

$$f_\Omega(x) \leq 1 - \frac{2\dim x^G}{n^2} + \frac{s}{n^2} \leq 1 - \frac{2s(n-s)}{n^2} + \frac{s}{n^2}$$

If $s \geq n/2$ then $a_0 \geq 0$ hence

$$f_\Omega(x) \leq 1 - \frac{2\dim x^G}{n^2} + \frac{1}{n} \leq 1 - \frac{s}{n} + \frac{1}{n}$$

q.e.d.

Now, the aim is to show that the bounds given in Lemma 19.4.4 are close to the best possible. In fact, we shall give more information. Write $n = (n-s)l + m$ with $0 \leq m < n-s$. Notice that if s is odd then $l \equiv m \pmod 2$; if, instead, s is even then m is even.

Claim. Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f^\circ_\Omega(x) \le f^\circ_\Omega(\bar{x})$ where $\bar{x}$ is one of the following:
(243)

$$[I_{n-s}, (\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\frac{l-1}{2}} I_{n-s}, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{m/2}] \quad l \text{ odd}$$
$$[I_{n-s}, (\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\frac{l}{2}-1} I_{n-s}, (\omega,\omega^{-1})^{\frac{l}{2}} I_{(n-s)/2}, (\omega,\omega^{-1})^{\frac{l}{2}+1} I_{m/2}] \quad l \text{ even}$$
$$[I_m, (\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\frac{l}{2}} I_{n-s}] \quad l \text{ odd}$$
$$[I_{m+1}, (\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\frac{l-1}{2}} I_{n-s}] \quad l \text{ even}$$
$$[I_{n-s-1}, (\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\frac{l-1}{2}} I_{n-s}, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{(m+1)/2}] \quad l \text{ odd}$$
$$[I_{n-s-1}, (\omega,\omega^{-1})I_{n-s}, \dots, (\omega,\omega^{-1})^{\frac{l}{2}-1} I_{n-s}, (\omega,\omega^{-1})^{\frac{l}{2}} I_{(n-s+m+1)/2}] \quad l \text{ even}$$

Assume $x \in H \cap \mathcal{V}_{s,r}$ has the 1-eigenspace of dimension $n-s$. Then, there exists $y = [I_{n-s}, \omega I_{b_1}, \dots, \omega^{r-1} I_{b_{r-1}}] \in H \cap \mathcal{V}_{s,r}$ with 1-eigenspace of dimension $n-s$ such that $a_1 = \min\{a_i : a_i \ne 0\}$ and $a_2 = \max\{a_i : a_i < n-s\}$ and $f^\circ_\Omega(x) = f^\circ_\Omega(y)$, to construct y it is enough to relabel in a suitable way the eigenvalues of x.

Lemma 19.4.5. *Let $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H \cap \mathcal{V}_{s,r}$. Assume $a_1 = \min\{a_i : a_i \ne 0\}$ and $a_2 = \max\{a_i : a_i < n-s\}$. Let $y = [I_{n-s}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \dots]$. Then $f^\circ_\Omega(x) \le f^\circ_\Omega(y)$.*

Proof. The result is equivalent to $\dim y^G - \dim x^G \le \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ)$. Using Theorem 5.3.1 we have $\dim y^G - \dim x^G = 2(a_1 - a_2 - 1)$. Using Theorem 17.3.8 we compute $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ)$. We see that we get the same inequality as in the proof of Lemma 18.4.7. Therefore the result follows using the same argument. *q.e.d.*

In case the 1-eigenspace has not the largest dimension we may assume $x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \dots, \omega^{r-1} I$
The following result is trivial (it is enough to relabel the eigenvalues of x in a suitable way).

Lemma 19.4.6. *Let $x \in \mathcal{V}_{s,r}$, with $a_i = n-s$ for some $i > 0$. Assume $C_G(x) \not\cong C_G(\bar{x})$. Then there exists $y = [I_{b_0}, \omega I_{n-s}, \dots, \omega^{r-1} I_{n-s}] \in H \cap \mathcal{V}_{s,r}$ such that $f^\circ_\Omega(x) = f^\circ_\Omega(y)$ and one of the following holds*

(i) $b_0 = \max\{b_i : b_i < n-s-1\}$ *and* $b_2 = \min\{b_i : b_i > 0\}$*;*
(ii) $b_0 = \min\{b_i : b_i > 0\}$ *and* $b_2 = \max\{b_i : b_i < n-s\}$*;*
(iii) $b_2 = \max\{b_i : b_i < n-s\}$ *and* $b_3 = \min\{b_i : b_i > 0\}$*;*

Let $x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \dots, \omega^{r-1} I_{n-s}] \in H \cap \mathcal{V}_{s,r}$. Let us consider each of the three cases occurring in Lemma 19.4.6 for x.

Case (i). Assume $a_0 = \max\{a_i : a_i < n-s-1\}$ and $a_2 = \min\{a_i : a_i > 0\}$, then we define

(244) $$y = [I_{a_0+2}, \omega I_{n-s}, \omega^2 I_{a_2-1}, \dots, \omega^{r-2} I_{a_2-1} \omega^{r-1} I_{n-s}]$$

Case (ii). Assume $a_0 = \min\{a_i : a_i > 0\}$ and $a_2 = \max\{a_i : a_i < n-s\}$, then we define

(245) $$y = [I_{a_0-2}, \omega I_{n-s}, \omega^2 I_{a_2+1}, \dots, \omega^{r-2} I_{a_2+1} \omega^{r-1} I_{n-s}]$$

Case (iii). Assume $a_2 = \max\{a_i : a_i < n-s\}$ and $a_3 = \max\{a_i : a_i > 0\}$, then we define

$$y = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2+1}, \omega^3 I_{a_3-1}, \dots, \omega^{r-2} I_{a_2+1} \omega^{r-1} I_{n-s}] \tag{246}$$

Lemma 19.4.7. *Let $x \in H \cap \mathcal{V}_{s,r}$. Assume $a_1 = n-s$ and $C_G(x) \not\cong C_G(\bar{x})$. Let y as in (244)–(246). Then*

$$f_\Omega^\circ(x) \le f_\Omega^\circ(y)$$

Proof. This is very similar to the usual argument given for semisimple elements. For all the details see Appendix B.4.4. *q.e.d.*

The claim quickly follows.

Proposition 19.4.8. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega^\circ(x) \le f_\Omega^\circ(\bar{x})$, for one of the $\bar{x}$ defined in (243).*

Proof. The proof is totally similar to the one of Lemma 18.4.8. *q.e.d.*

Remark 19.4.9. The main difference with Section 18.4.1 is that here we have more different elements in the definition of $\bar{x}$. This is due to the centraliser structure of semisimple elements in G, e.g. a_0 must be even.

Thanks to Proposition 19.4.8 we reduce the computation of upper bounds on $f_\Omega^\circ(x)$ for $x \in H \cap \mathcal{V}_{s,r}$ to the computation of $f_\Omega^\circ(\bar{x})$. We do not compute $f_\Omega^\circ(\bar{x})$ in all the possible cases. However, in the following particular case we show that the upper bound given in Lemma 19.4.4 is very accurate.

Lemma 19.4.10. *Assume $s \le 2n/3$ is even. Then there exists $x \in H \cap \mathcal{V}_{s,r}$ such that*

$$f_\Omega(x) \ge 1 - \frac{2s(n-s)}{n^2} - \frac{s^2}{2n^2} - \frac{3t^2}{2n^2(t-1)}$$

Proof. Let $x = [I_{n-s}, \omega I_{s/2}, \omega^{-1} I_{s/2}] \in H \cap \mathcal{V}_{s,r}$. Then we compute

$$\dim x^G = ns - \frac{3}{4}s^2 + \frac{s}{2}$$

$$\dim(x^G \cap H^\circ) = \frac{ns}{t} - s - 3s\left\lfloor\frac{s}{2t}\right\rfloor + 3t\left\lfloor\frac{s}{2t}\right\rfloor^2 + 3t\left\lfloor\frac{s}{2t}\right\rfloor$$

Differentiating in $\lfloor s/2t \rfloor$ we see that $\dim(x^G \cap H^\circ)$ is minimal when $\lfloor s/2t \rfloor = s/2t - 1/2$. Hence

$$\dim(x^G \cap H^\circ) \ge \frac{ns}{t} - \frac{3s^2}{4t} + \frac{s}{2} - \frac{3t}{4}$$

Therefore

$$f_\Omega^\circ(x) \ge 1 - \frac{2s(n-s)}{n^2} - \frac{s^2}{2n^2} - \frac{3t^2}{2n^2(t-1)}$$

The result follows since $f_\Omega(x) \ge f_\Omega^\circ(x)$. *q.e.d.*

19.5. Local lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$ where r is an odd prime other than p. We deal with involutions in Section 19.7. Recall, by Proposition 19.3.4, if $\nu(x) = n-1$ or $n-2$ then $f_\Omega(x) = 0$, unless $\nu(x) = 2$ and $n = 4$ in which case for $f_\Omega([I_2,\omega,\omega^{-1}]) = 0$ and $f_\Omega([\omega I_2,\omega^{-1}I_2]) = 1/2$. Therefore we may assume $\nu(x) < n-2$ and $n > 4$.

We have the following.

Proposition 19.5.1. *Assume $\nu(x) < n-2$ and $n > 4$. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{s(2n-s+1)}{n^3\left(1-\frac{1}{t}\right)}$$

PROOF. Up to G-conjugacy we have $x = [I_{a_0},\ldots,\omega^{r-1}I_{a_{r-1}}]$ where $a_i \leq n-s$ for all i. In particular $n - a_0 \geq s$. By Proposition 5.4.1 $\dim x^G \leq \frac{s}{2}(2n-s+1/s)$. In addition, by Proposition 17.3.17, we have

$$\dim(x^G \cap H^\circ) \geq \left(\frac{1}{t} - \frac{1}{n}\right)\dim x^G + \frac{n-a_0}{2}\left(1 - \frac{1}{t} + \frac{1}{n}\right)$$

Therefore we have

$$\begin{aligned} f_\Omega^\circ(x) &\geq \frac{\dim\Omega - \dim x^G\left(1-\frac{1}{t}+\frac{1}{n}\right)}{\dim\Omega} + \frac{n-a_0}{2\dim\Omega}\left(1-\frac{1}{t}+\frac{1}{n}\right) \\ &\geq \frac{\dim\Omega - \frac{s}{2}(2n-s+1/s)\left(1-\frac{1}{t}+\frac{1}{n}\right)}{\dim\Omega} + \frac{s}{n^2} + \frac{st}{n^3(t-1)} \\ &\geq 1 - \frac{s(2n-s)}{n^2} - \frac{s(2n-s+1)}{n^3\left(1-\frac{1}{t}\right)} \end{aligned}$$

q.e.d.

From the lower bound given in Proposition 19.5.1, we quickly deduce the following.

Corollary 19.5.2. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{4}{n}$$

PROOF. This is an immediate calculation using the lower bound in Proposition 19.5.1. *q.e.d.*

As done in Section 19.3 we can, in fact, give more informations. We construct a family of elements in $H \cap \mathcal{V}_{s,r}$ with the property that some of them realises the best possible lower bound on f_Ω°. The following is very similar to Definition 19.3.9.

Definition 19.5.3. Let $x \in \mathcal{V}_{s,r}$. Then $a_l = a_{r-l} = n-s$, for some $0 \leq l \leq (r-1)/2$. We say that x is *special* if $|a_i - a_j| \leq 1$ for all $i,j \in \{0,\ldots,r-1\} \setminus \{l\}$.

It is quite easy to see that the discussion made in Section 19.3 may be applied here. In particular, we can easily modify Lemma 19.3.11 and the definitions of the elements y before it. Let $x \in H \cap \mathcal{V}_{s,r}$ and assume $a_l = a_{r-l} = n-s$ for some $l \leq (r-1)/2$, possibly $l = 0$. If x is not special then $a_i - a_j \geq 2$ for some $i, j \neq l$. Hence we imitate the construction before Lemma 19.3.11 using a_i, a_j. Thus a version of Lemma 19.3.11 holds. Therefore we deduce the following, which is totally similar to Lemma 19.3.12.

Lemma 19.5.4. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then there exists $z \in \mathcal{V}_{s,r}$ special such that $f_\Omega^\circ(x) \geq f_\Omega^\circ(z)$.*

With an argument very similar to the one given in Section 19.3 we can classify, up to G-centraliser, special elements in $H \cap \mathcal{V}_{s,r}$.

Lemma 19.5.5. *Let $x \in H \cap \mathcal{V}_{s,r}$ and assume x is special. Write $s = a(r-1)+b$, $0 \le b < r-1$, and $2s - n = l(r-2)+m$, $0 \le m < r-2$. Then $C_G(x) \cong C_G(z)$ where z is one of the following:*

$$z = [I_{n-s}, (\omega,\omega^{-1})I_{a+1}, \dots, (\omega,\omega^{-1})^{b/2}I_{a+1}, (\omega,\omega^{-1})^{b/2+1}I_a, \dots, (\omega,\omega^{-1})^{(r-1)/2}I_a]$$
$$z = [I_a, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_a, \dots, (\omega,\omega^{-1})^{(r-b-1)/2}I_a,$$
$$(\omega,\omega^{-1})^{(r-b+1)/2}I_{a+1}, \dots, (\omega,\omega^{-1})^{(r-1)/2}I_{a+1}]$$
$$z = [I_{a+1}, (\omega,\omega^{-1})I_{n-s}, (\omega,\omega^{-1})^2 I_a, \dots, (\omega,\omega^{-1})^{(r-b)/2}I_a,$$
$$(\omega,\omega^{-1})^{(r-b)/2+1}I_{a+1}, \dots, (\omega,\omega^{-1})^{(r-1)/2}I_{a+1}]$$

In order to have the best possible lower bound, by Lemma 19.5.4, it is enough to compute $f_\Omega^\circ(z)$ for z as defined in Lemma 19.5.5. In general it is hard to get an easy and readable expression for such a lower bound; however giving some restrictions on the value of s we can easily get an explicit expression for it.

Proposition 19.5.6. *Assume $s \le r-1$ if s is even and $2s-n \le r-1$ for s odd. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*
$$f_\Omega(x) \ge \iota\Big(1 - \frac{s(2n-s)}{n^2} - \frac{1}{n}\Big)$$
where $\iota = 1$ if $s \le n/2$ and $\iota = 2$ if $s > n/2$.

Proof. Thanks to Lemma 19.5.4 we need to compute the f_Ω°-value of a special element $z \in H \cap \mathcal{V}_{s,r}$. By Lemma 19.5.5 we may assume $z = [I_{n-s}, \omega, \omega^{-1}, \dots, \omega^{s/2}, \omega^{-s/2}]$ for s even and $z = [\omega I_{n-s}, \omega^{-1}I_{n-s}, \omega^2, \omega^{-2}, \dots, \omega^{s-n/2+1}, \omega^{-s+n/2-1}]$ for s odd.

Assume s is even. Then, for $z = [I_{n-s}, \omega, \omega^{-1}, \dots, \omega^{s/2}, \omega^{-s/2}]$, using Theorem 5.3.1 and Theorem 17.3.8, we compute
$$\dim z^G = \frac{s}{2}(2n-s)$$
$$\dim(z^G \cap H^\circ) = \frac{ns}{t} - s - 2s\Big\lfloor\frac{s}{2t}\Big\rfloor + 2t\Big\lfloor\frac{s}{2t}\Big\rfloor + 2t\Big\lfloor\frac{s}{2t}\Big\rfloor^2$$
Differentiating with respect to $\lfloor s/2t \rfloor$ we see that $\dim(z^G \cap H^\circ)$ is minimal for $\lfloor s/2t\rfloor = s/2t - 1/2$. Hence
$$\dim(z^G \cap H^\circ) \ge \frac{s}{2t}(2n-s) - \frac{t}{2}$$
Therefore
$$f_\Omega^\circ(z) \ge 1 - \frac{s(2n-s)}{n^2} - \frac{t^2}{n^2(t-1)} \ge 1 - \frac{s(2n-s)}{n^2} - \frac{1}{n}$$
where the second inequality follows from the fact that $t^2/(t-1) \le n^2/(2(n-2))$ since $t \le n/2$.

Assume s is odd. In particular $s > n/2$. And, by Lemma 19.5.5, we may assume $z = [\omega I_{n-s}, \omega^{-1}I_{n-s}, \omega^2, \omega^{-2}, \dots, \omega^{s-n/2+1}, \omega^{-s+n/2-1}]$. We compute
$$\dim z^G = 2ns - s^2 - s + n - \frac{n^2}{2}$$
$$\dim(z^G \cap H^\circ) = -\frac{n^2}{2t} + \frac{2ns}{t} + n + \Big(1 + \Big\lfloor\frac{s}{t}\Big\rfloor\Big)\Big(\Big\lfloor\frac{s}{t}\Big\rfloor t - 2s\Big)$$

Again we see that $\dim(z^G \cap H^\circ)$ is minimal when $\lfloor s/t \rfloor = s/t - 1/2$. Hence

$$\dim(x^G \cap H^\circ) \geq -\frac{n^2}{2t} + \frac{2ns}{t} + n - s - \frac{s^2}{t} - \frac{t}{4}$$

Therefore

$$f_\Omega^\circ(z) \geq 2 - \frac{2s(2n-s)}{n^2} - \frac{t^2}{2n^2(t-1)} \geq 2 - \frac{2s(2n-s)}{n^2} - \frac{1}{2n}$$

The result follows. *q.e.d.*

In the general case, $s > r$ for s even and $2s - n > r - 1$ for s odd we can easily compute $f_\Omega^\circ(z)$, where z is a special element, using the formulae in Theorems 5.3.1 and 17.3.8.

For example, assume $s < n/2$ is even and $s > r - 1$. Hence, by Lemma 19.5.5, any special element $x \in H \cap \mathcal{V}_{s,r}$ has centraliser isomorphic to that of

$$z = [I_{n-s}, (\omega, \omega^{-1})I_{a+1}, \ldots, (\omega, \omega^{-1})^{b/2}I_{a+1}, (\omega, \omega^{-1})^{b/2+1}I_a, \ldots, (\omega, \omega^{-1})^{(r-1)/2}I_a]$$

where we write $s = a(r-1) + b$, with $0 \leq b < r - 1$ is even. Then we compute

$$\dim z^G = ns - \frac{s^2 r}{2(r-1)} + \frac{s}{2} - \frac{b(r-1-b)}{2(r-1)} \leq \frac{s}{2}(2n - s + 1)$$

And $\dim z^G \geq ns - s^2 + \frac{s}{2} - \frac{r-1}{8}$. In particular $\dim z^G$ is an integer which lies in an interval of size

$$\frac{s}{2}(2n - s + 1) - \left(ns - s^2 + \frac{s}{2} - \frac{r-1}{8}\right) = \frac{s^2}{2} + \frac{r-1}{8} < \frac{s}{2}\left(s + \frac{1}{4}\right)$$

In particular, using the upper bound on $\dim(z^G \cap H)$ in Proposition 17.2.1, we have

$$\begin{aligned} f_\Omega(z) &\leq 1 - \frac{\dim z^G(1 - \frac{1}{t})}{\dim \Omega} + \frac{s}{2\dim\Omega}\left(1 - \frac{1}{t}\right) = 1 - \frac{2s(n-s)}{n^2} + \frac{2(r-1)}{n^2} \\ &< 1 - \frac{2s(n-s-1)}{n^2} \end{aligned}$$

So, if ℓ is the lower bound of $f_\Omega(x)$ given in Proposition 19.5.1 we have

$$f_\Omega(z) - \ell \leq \frac{s(s+1)}{n^2}$$

19.6. Further comments on local bounds

Let $r \neq p$ be an odd prime. Let $x \in H \cap \mathcal{V}_{s,r}$. Then, by Propositions 19.4.1 and 19.5.1 we have upper and lower bounds on $f_\Omega(x)$ depending on $\nu(x) = s$. In particular, given $x, y \in H \cap \mathcal{V}_{s,r}$ we see that $f_\Omega(x) - f_\Omega(y)$ is bounded.

Proposition 19.6.1. *Assume $r \neq p$ is odd. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| \leq \frac{s(n-s)}{n^2} + \frac{5}{n}$$

Proof. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega(x), f_\Omega(y) \leq U$ where U is given in Proposition 19.4.1, and $f_\Omega(x), f_\Omega(y) \geq \ell$ where we take ℓ as in Corollary 19.5.2.

Assume $s > n/2$ then

$$|f_\Omega(x) - f_\Omega(y)| \leq 1 - \frac{s-1}{n} - \left(1 - \frac{s(2n-s)}{n^2} - \frac{4}{n}\right) = \frac{s(n-s)}{n^2} + \frac{5}{n}$$

If $s \leq n/2$ then

$$|f_\Omega(x) - f_\Omega(y)| \leq 1 - \frac{2s(n-s)}{n^2} + \frac{1}{n} - \Big(1 - \frac{s(2n-s)}{n^2} - \frac{4}{n}\Big) = \frac{s^2}{n^2} + \frac{5}{n}$$

It is straightforward to check that for $s \leq n/2$ we have $s^2 \leq s(n-s)$. Therefore

$$\frac{s^2}{n^2} \leq \frac{s(n-s)}{n^2}$$

The result follows. *q.e.d.*

19.7. Involutions

Let $x \in G$ be an involution, write $\nu(x) = s$. The aim of this section is to find the best possible function $g(n,t,s)$ and $\epsilon \geq 0$ such that

$$g(n,t,s) \leq f_\Omega(x) \leq g(n,t,s) + \epsilon$$

Recall that if $p \neq 2$ then for any involutions $x, y \in G$ with $\nu(x) = \nu(y)$ we have that either $y \in x^G$ or $-y \in x^G$; in particular $f_\Omega(x) = f_\Omega(y)$. If $p = 2$ and $x \in G$ is an involution then $\nu(x)$ identifies x^G only if it is odd; if $\nu(x) = s$ is even there are precisely two G-conjugacy classes of involutions whose representatives are denoted a_s, c_s, see Section 5.2.1.

Before stating the main result of this section we need to define some more notation. Let $\ell \in \{0, \dots, \lfloor t/2 \rfloor\}$. Let $x \in G$ be an involution. We define

$$f_\Omega^\ell(x) = \frac{\dim \Omega - \dim x^G + \dim(x^G \cap H^\circ \pi_\ell)}{\dim \Omega}$$

where, as usual, $\pi_\ell \in S_t$ is any involution with cycle shape $(2^\ell, 1^{t-2\ell})$. Notice that for $\ell = 0$ we have $f_\Omega^\circ(x)$.

Let $x \in G$ be an involution with $\nu(x) = s$. In the case where $p = 2$ and x is an a_s-type involution we define

$$h' = \max\Big\{0, \frac{s+t}{2} - \frac{n}{4}\Big\}$$

otherwise we set $h' = 0$. The motivation for this notation and this definition will be clarified by Proposition 19.7.5 below.

Theorem 19.7.1. *Let $x \in G$ be an involution, write $\nu(x) = s$.*

(A) Let $\epsilon = 1$ if $s - h'\frac{n}{t} \geq \frac{n}{t}$ and 0 otherwise.

(i) Assume $p \neq 2$. Then

$$f_\Omega^\circ(x) \leq f_\Omega(x) \leq f_\Omega^\circ(x) + \epsilon \frac{t}{n(t-1)}\Big(\frac{1}{2} + 2\frac{t}{n}\Big)$$

(ii) Assume $p = 2$ and x is a_s-type. Then

$$f_\Omega^{h'}(x) \leq f_\Omega(x) \leq f_\Omega^{h'}(x) + \epsilon \frac{t\iota}{2n(t-1)}$$

where $\iota = 2$ if $n/2t$ is odd and $n - 2s \geq 2t$, $\iota = 1$ otherwise.

(iii) Assume $p = 2$ and x is not a_s-type. Then

$$f_\Omega(x) = f_\Omega^\circ(x)$$

(B) Write $s/\iota = \lfloor s/\iota t \rfloor t + b$ where $\iota = 1$ if x is b_s or c_s-type, and $\iota = 2$ otherwise. Then the following hold.

(i) Assume $p \neq 2$. Then

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

(ii) Assume $p = 2$ and x is of type b_s or c_s. Then

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

(iii) Assume $p = 2$ and x is a_s-type. If $n/2t$ is even, or $n/2t$ is odd and $n-2s \geq 2t$ then

$$f_\Omega^\circ(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

Otherwise $x^G \cap H^\circ \pi_i = \emptyset$ for all $i < h'$ and

$$f_\Omega^{h'}(x) = 1 - \frac{2st(n-s-1)}{n^2(t-1)} + \frac{n+2(s-t)}{4n(t-1)}$$

Remark 19.7.2. Assume $p \neq 2$. For some special value of t, namely $t \in \{2, n/4, n/2\}$, we can compute $f_\Omega(x)$ for any involution $x \in G$, see Proposition 19.7.10.

Notice that $n/t \geq 2, t/(t-1) \leq 2$ and $t^2/(t-1) \leq n^2/2n(n-2)$. Using these inequalities the following result quickly follows from Theorem 19.7.1(A).

Corollary 19.7.3. *Let $x \in G$ be an involution. Then*

$$f_\Omega^{h'}(x) \leq f_\Omega(x) \leq f_\Omega^{h'}(x) + \frac{3}{n}$$

Notice that, thanks to the formulae given in Theorem 19.7.1 we quickly deduce lower bounds on $f_\Omega(x)$. Denote by x_s an involution with $\nu(x_s) = s$. Then it is easy to show $f_\Omega^{h'}(x_s) \geq f_\Omega^{h'}(x_{s+\iota})$ where $\iota = 2$ when $p \neq 2$ or x_s is a_s-type and $\iota = 1$ otherwise. In particular we deduce that the best possible lower bound on f_Ω is realised by an involution with largest ν-value.

Remark 19.7.4. Recall the best possible lower bounds on $f_\Omega(x)$, for $x \in H$ of prime order, given in Lemma 19.2.8 (for $p = 2$) and in Lemma 19.3.6 (when $p \neq 2$). Here we show that the lower bounds given are close to the best possible. Let $x \in G$ be an involution with $\nu(x) = n/2$. Using Theorem 19.7.1(B), we compute $f_\Omega^{h'}(x)$. Notice that $f_\Omega(x) \geq f_\Omega^{h'}(x)$. In the three cases of the theorem we have:

(i) $f_\Omega^\circ(x) = 1/2$ if $n/2t$ is even and $f_\Omega^\circ(x) = 1/2 - \frac{2t^2}{n^2(t-1)}$ otherwise;
(ii) $f_\Omega^\circ(x) = 1/2$;
(iii) $f_\Omega^\circ(x) = 1/2$ when $h' = 0$. If $h' > 0$ then $h' = t/4$ and $f_\Omega^{h'}(x) = 1/2 + \frac{t}{2n(t-1)}$

If $p \neq 2$, or $p = 2$ and x of $a_{n/2}$-type we need $\nu(x) = n/2$ to be even. Similarly, in the case $n/2$ is odd, one can compute $f_\Omega(x)$ when $\nu(x) = n/2 - 1$.

In the same spirit as the definition of the integer h, see (186), we define, for $x \in G$,

$$h' = \min\{i : x^G \cap H^\circ \pi_i \neq \emptyset\} \tag{247}$$

thus $x^G \cap H^\circ \pi_{h'} \neq \emptyset$ and $x^G \cap H^\circ \pi_i = \emptyset$ for all $i < h'$.

Proposition 19.7.5. *Let $x \in G$ be an involution. Then $x^G \cap H \neq \emptyset$. Furthermore:*

(i) Assume $p = 2$, $n/2t$ is odd and x is a_s-type. Then $h' = \max\{0, \frac{s+t}{2} - \frac{n}{4}\}$.
(ii) In all the other cases $h' = 0$.

Remark 19.7.6. Although for any semisimple involution $x \in H$ we have $x^G \cap H^\circ \neq \emptyset$; in the case $p = 2$, namely for a_s-type involutions we may have $x^G \cap H^\circ = \emptyset$. Lemma 19.7.12 and the discussion before it show that for an a_s-type involution $x \in G$ we have $x^G \cap H^\circ = \emptyset$ if, and only if, $n - 2s < 2t$ and $n/2t$ odd.

19.7.1. Semisimple involutions. First we study the case $p \neq 2$. Let $x \in G$ be an involution. Then, up to G-conjugacy, $x = [I_s, -I_{n-s}]$. Notice that, by Lemma 7.1.3, we may assume $s \leq n/2$ so that $\nu(x) = s$.

In this case we do not give, in general, an explicit formula for $f_\Omega(x)$, as we did in Section 18.7 in the case $G = \mathrm{GL}_n$. However, for special values of t we shall compute it, see Section 19.7.1.1.

First we give the following, which proves Theorem 19.7.1(B)(i).

Lemma 19.7.7. *Let $x \in G$ be an involution with $\nu(x) = s$. Write $s/2 = at + b$ where $0 \leq b < t$. Then*

$$f^\circ_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

Proof. Using Proposition 5.3.1, we have $\dim x^G = s(n-s)$. In addition, Theorem 17.3.8 provides a formula for $\dim(x^G \cap H^\circ)$. The result follows with an easy computation. *q.e.d.*

In order to show Theorem 19.7.1 we need the following technical lemma.

Lemma 19.7.8. *Let $x \in G$ be an involution. Then*

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{4} + t$$

Proof. If $x^G \cap H = x^G \cap H^\circ$ the result trivially follows.

Assume, for $i > 0$, $x^G \cap H^\circ \pi_i \neq \emptyset$. Then, by Lemma 17.4.2 we have, for suitable involutions $x_{2i+1}, \ldots, x_t \in \mathrm{Sp}_{n/t}$:

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_i) &= \dim([I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_i)^{H^\circ} \\ &= i \dim \mathrm{Sp}_{n/t} + \sum_{j>2i} \dim x_j^{\mathrm{Sp}_{n/t}}\end{aligned}$$

Define $z = z' = [I_{n/2t}, -I_{n/2t}]$ if $n/2t$ is even, and $z = [I_{n/2t-1}, -I_{n/2t+1}] = -z'$ otherwise. Recall definition (23), $\delta_{a;b} = 1$ if $b \mid a$ and 0 otherwise. We compute

$$\dim z^{\mathrm{Sp}_{n/t}} + \dim(z')^{\mathrm{Sp}_{n/t}} = \frac{1}{2}\left(\frac{n}{t}\right)^2 - 2\delta_{n/2t;2}$$

Notice that $\pi_i = \pi_{i-1}(2i-1, 2i)$ and $(2i-1, 2i)$ is S_t-conjugate to $\pi_1 \in [I_{n-n/t}, -I_{n/t}]^G$. Then

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_{i-1}) &\geq \dim([I_{n/t}, \ldots, I_{n/t}, z, z', x_{2i+1}, \ldots, x_t]\pi_{i-1})^{H^\circ} \\ &= (i-1)\dim \mathrm{Sp}_{n/t} + \dim z^{\mathrm{Sp}_{n/t}} + \dim(z')^{\mathrm{Sp}_{n/t}} + \sum_{j>2i} \dim x_j^{\mathrm{Sp}_{n/t}} \\ &= \dim(x^G \cap H^\circ \pi_i) - \frac{n}{2t} - 2\delta_{n/2t;2}\end{aligned}$$

Therefore

$$\dim(x^G \cap H^\circ \pi_i) \leq \dim(x^G \cap H^\circ \pi_{i-1}) + \frac{n}{2t} + 2$$

For some $0 \leq \ell \leq t/2$, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_\ell)$. So

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_\ell) &\leq \dim(x^G \cap H^\circ \pi_{\ell-1}) + \frac{n}{2t} + 2 \leq \dim(x^G \cap H^\circ \pi_{\ell-1}) + \frac{n}{t} + 4 \\ &\leq \ldots \leq \dim(x^G \cap H^\circ \pi_0) + \ell\frac{n}{2t} + 2\ell\end{aligned}$$

where we denote $\pi_0 = 1 \in S_t$. Since $\ell \leq t/2$ we deduce

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{4} + t$$

q.e.d.

Thanks to Lemma 19.7.8 we deduce Theorem 19.7.1(A)(i).

Proposition 19.7.9. *Let $x \in G$ be an involution. Then*

$$f_\Omega^\circ(x) \leq f_\Omega(x) \leq f_\Omega^\circ(x) + \epsilon \frac{t}{n(t-1)}\Big(\frac{1}{2} + 2\frac{t}{n}\Big)$$

where $\epsilon = 1$ if $\nu(x) \geq n/t$ and 0 otherwise.

Proof. Using Lemma 19.7.8 we have

$$f_\Omega(x) \leq f_\Omega^\circ(x) + \frac{n/4 + t}{\dim \Omega} = f_\Omega^\circ(x) + \frac{t}{n(t-1)}\Big(\frac{1}{2} + 2\frac{t}{n}\Big)$$

In the case $\nu(x) < n/t$, Corollary 17.4.9 implies $x^G \cap H = x^G \cap H^\circ$. In particular, $f_\Omega(x) = f_\Omega^\circ(x)$. *q.e.d.*

19.7.1.1. *Some special value of t.* Assume $t \in \{2, n/4, n/2\}$. Then it is easy to compute an explicit formula of $f_\Omega(x)$, for an involution $x \in G$. Recall, from (186), $h = \max\{i : x^G \cap H^\circ \pi_i \neq \emptyset\}$.

Proposition 19.7.10. *Assume $t \in \{2, n/4, n/2\}$. Let $x \in G$ be an involution with $\nu(x) = s$. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$$

Moreover

(i) Assume $t = 2$. Then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8(1-\delta_{s;4})}{n^2}$$

(ii) Assume $t = n/4$. Then

$$f_\Omega(x) = 1 - \frac{2s(n-s) + 5s}{n(n-4)} - \frac{2\delta_{s;4}}{n(n-4)}$$

(iii) Assume $t = n/2$. Then

$$f_\Omega(x) = 1 - \frac{2s(n-s) - 3s}{n(n-2)}$$

PROOF. We only prove the result for $t = n/2$. The argument is very similar in the other cases. Up to G-conjugacy, we have $x = [I_s, -I_{n-s}]$ and $s \le n/2$ is even.

Using Theorem 5.3.1, we have $\dim x^G = s(n-s)$. Using Lemmas 17.4.2 and 17.4.6 we see that x is G-conjugate to $\pi_{s/2}$. In particular, $x^G \cap H^\circ \pi_i \neq \emptyset$ for all $0 \le i \le s/2$ and it is empty for $i > s/2$; here we denote $\pi_0 = 1$. By Lemma 17.4.2, $x \in ([I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_i)^{H^\circ}$ where $x_j = \pm I_2$ for $j > 2i$. Thus, using (184), we compute

$$\dim(x^G \cap H^\circ \pi_i) = i\frac{n}{2t}\Big(\frac{n}{t} + 1\Big) = 3i$$

Therefore $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_{s/2}) = 3s/2$. The result follows using Proposition 7.1.8 and $\dim \Omega = \frac{n}{2}(n-2)$. *q.e.d.*

19.7.2. Unipotent involutions. Now we assume $p = 2$. Let $x \in G$ be an involution with $\nu(x) = s$. Then, in Jordan form, $x = [J_2^s, J_1^{n-2s}]$. As explained in Section 5.2.1, the Jordan form does not always identify conjugacy classes. If s is odd then there exists only one conjugacy class of involutions with $\nu(x) = s$ whose representative is denoted by b_s. If s is even there are precisely two conjugacy classes of involutions with $\nu(x) = s$, whose representatives are denoted a_s and c_s. Recall that a_s-type involution are defined to be those involutions x such that for any $v \in V$,

$$(x.v, v) = 0$$

Let $x \in G$ be an involution. If x is b_s or c_s-type or x is of a_s-type and $n/2t$ is even then $x^G \cap H^\circ \neq \emptyset$. In fact we will explicitly construct a block decomposition $[x_1, \ldots, x_t] \in x^G$ in Sections 19.7.2.1 and 19.7.2.2. In Lemma 19.7.12 we characterise the a_s-type involutions for which $a_s^G \cap H^\circ \neq \emptyset$ when $n/2t$ is odd.

The following is the key result in order to compute an explicit formula of $\dim(x^G \cap H^\circ)$; notice that it is the analogue of Proposition 17.3.13

Proposition 19.7.11. *Let $x \in G$ be an involution. Assume $x^G \cap H^\circ \neq \emptyset$. Suppose $x = [x_1, \ldots, x_t] \in H^\circ$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.*

- *(i) If x is a_s-type then x_i is a_{s_i}-type and $|s_i - s_j| \le 2$ for all $i, j \in \{1, \ldots, t\}$;*
- *(ii) If x is b_s or c_s-type, then x_i is of type b_{s_i} or c_{s_i} and $|s_i - s_j| \le 1$ for all $i, j \in \{1, \ldots, t\}$.*

In addition, if $[z_1, \ldots, z_t] \in x^G$ is a block decomposition which satisfies conditions in (i) or (ii) then $\dim(x^G \cap H^\circ) = \sum_i \dim z_i^{\mathrm{Sp}_{n/t}}$.

PROOF. First assume $x \in H^\circ$ is a_s-type. Let $x = [x_1, \ldots, x_t] \in H^\circ$ be any block decomposition. If x_i is not a-type then we immediately deduce that x is not a_s-type. So each $x_i \in \mathrm{Sp}_{n/t}$ is of type a_{s_i} with $\sum_i s_i = s$. Assume $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Seeking a contradiction, we assume $s_1 - s_2 > 2$, so that $s_1 - s_2 = 4 + 2h$ for some integer $h \ge 0$. Define $y = [y_1, y_2, x_3, \ldots, x_t]$, where y_1 is a_{s_1-2} and y_2 is a_{s_2+2}. Then $y \in x^G$. Using Proposition 5.2.5 we compute $\dim y^{H^\circ} - \dim x^{H^\circ} = 4(s_1 - s_2 - 2) = 4(2+2h) \ge 8$, which is absurd as it would imply $\dim(x^G \cap H) < \dim y^{H^\circ}$. Therefore $|s_i - s_j| \leq 2$ for all i, j. The result follows.

Now, assume x is of type b_s or c_s. Let $x = [x_1, \ldots, x_t] \in H^\circ$ with $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ then each x_i is of type b_{s_i} or c_{s_i} since

$$\dim a_{s_i}^{\mathrm{Sp}_{n/t}} < \dim c_{s_i}^{\mathrm{Sp}_{n/t}}$$

by Proposition 5.2.5. Seeking a contradiction, assume $s_1 - s_2 > 1$, so that $s_1 - s_2 = 2 + h$ for some $h \geq 0$. Define $y = [y_1, y_2, x_t, \ldots, x_t]$, where $\nu(y_1) = s_1 - 1$ and $\nu(y_2) = s_2 + 1$. Then $\dim y^{H^\circ} - \dim x^{H^\circ} = 2(s_1 - s_2 - 1) = 2(h+1) \geq 2$. This implies $\dim(x^G \cap H^\circ) < \dim y^{H^\circ}$ which is absurd. Therefore $|s_i - s_j| \leq 1$ for all i, j.

Now assume that we have a block decomposition $x = [z_1, \ldots, z_t]$ where z_i is a_{s_i} if x is a-type and z_i is b_{s_i} or c_{s_i} if x is not a-type. Assume moreover that $|s_i - s_j| \leq \iota$ for all i, j, where $\iota = 2$ if x is a-type and 1 otherwise. We have $\dim(x^G \cap H^\circ) \geq \dim x^{H^\circ}$ and $\dim(x^G \cap H^\circ) = \dim y^{H^\circ}$ for some $y = [y_1, \ldots, y_t] \in x^G$ (in particular $\nu(x) = \nu(y) = s$). Then, by the first part $|\nu(y_i) - \nu(y_j)| \leq \iota$. Notice that given $s \leq n/2$ there exists only one t-tuple, up to permutation of its elements, $(s_1, \ldots, s_t)$ that satisfies the following conditions

- $|s_i - s_j| \leq \iota$ for all i, j;
- $s_1, \ldots, s_t$ are even (if $\iota = 2$);
- $\sum_i s_i = s$.

To see this it is enough to apply the same argument given for Lemma 17.3.12, for example. Therefore we conclude that for all i there exists j such that $\nu(y_i) = \nu(z_j)$; in other words, the block decomposition $[y_1, \ldots, y_t]$ is obtained by permuting (and possibly by conjugating) the blocks z_i's in $x = [z_1, \ldots, z_t]$. Hence $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. *q.e.d.*

In order to complete the proof of Theorem 19.7.1 we divide the analysis into two cases. First we study a_s-type involutions then b_s or c_s-type.

19.7.2.1. *a_s-type involutions.* Let $x \in G$ be a_s-type. If $n/2t$ is odd we may have $x^G \cap H^\circ = \emptyset$.

Lemma 19.7.12. *Assume $n/2t$ is odd. Let $x \in G$ be a_s-type. Then $x^G \cap H^\circ \neq \emptyset$ if, and only if,*

$$n - 2s \geq 2t$$

Proof. In a suitable basis, $x = [J_2^s, J_1^{n-2s}]$. Assume $x^G \cap H^\circ \neq \emptyset$. Write $x = [x_1, \ldots, x_t] \in H^\circ$ where $x_i \in \mathrm{Sp}_{n/t}$ is a_{s_i}-type for all $i \leq t$ and $\sum_i s_i = s$. Then each x_i has Jordan form $x_i = [J_2^{s_i}, J_1^{n/t - 2s_i}]$. It is clear that $n/t - 2s_i > 0$ is even, since $n/t \equiv 2 \pmod 4$ and $2s_i \equiv 0 \pmod 4$. Thus $n - 2s = \sum_{i=1}^t \left(\frac{n}{t} - 2s_i\right) \geq 2t$

Conversely, assume $n - 2s \geq 2t$. In particular $s \leq \left(\frac{n}{2t} - 1\right)t$ thus $l = \left\lfloor \frac{s}{n/2t - 1} \right\rfloor \leq t$. Write $s = \left(\frac{n}{2t} - 1\right)l + m$ where $0 \leq m < \frac{n}{2t} - 1$. Since s and $\frac{n}{2t} - 1$ are even we deduce that m is even. We define $[x_1, \ldots, x_t] \in H^\circ$ such that $x_1, \ldots, x_l$ are $a_{n/2t-1}$-type, x_{l+1} is a_m-type (or $I_{n/t}$ if $m = 0$) and $x_i = I_{n/t}$ for $i > l + 1$. Then $[x_1, \ldots, x_t] \in x^G \cap H^\circ$. *q.e.d.*

Recall the definition of the integer h' associated to x given in (247). If x is an involution of type b_s or c_s then $h' = 0$. Let $x \in G$ be a_s-type. We can characterise h' via numerical data.

Corollary 19.7.13. *Assume $n/2t$ is odd. Let $x \in G$ be a_s-type. Then*

$$h' = \max\Big\{0, \frac{s+t}{2} - \frac{n}{4}\Big\}$$

In particular, $x^G \cap H \neq \emptyset$.

PROOF. In a suitable basis we have $x = [J_2^s, J_1^{n-2s}]$. Let $i \leq t/2$ such that $s \geq \frac{n}{t}i$. Then by Lemmas 17.4.2 and 17.4.5, we have

$$y = [I_{n/t}, \ldots, I_{n/t} \mid J_2^{s-in/t}, J_1^{n-2s}]\pi_i \in x^G$$

So $y \in H^\circ\pi_i$ if, and only if, $[J_2^{s-in/t}, J_1^{n-2s}] \in \big(\mathrm{Sp}_{n/t}\big)^{t-2i}$. Hence, Lemma 19.7.12 implies $y \in H^\circ\pi_i$ if, and only if, $n - 2s \geq 2(t-2i)$. Therefore $h' = \min\{i : n - 2s \geq 2(t-2i)\}$. Notice that $h' = s/2 + t/2 - n/4 \in \mathbb{Z}$ since $n/2t$ is odd; also, $0 \leq h' \leq t/2$. Thus $x^G \cap H \neq \emptyset$. *q.e.d.*

Remark 19.7.14. Lemma 19.7.12 and Corollary 19.7.13 hold also if $G = \mathrm{O}_n$.

Thanks to Proposition 19.7.11 and Corollary 19.7.13 we can compute $f_\Omega^{h'}(x)$ for any involution $x \in G$. Then, studying separately the two cases depending on the parity of $n/2t$ we can give an upper bound on $f_\Omega(x)$, as done in Proposition 19.7.9 for semisimple involutions.

We shall prove the following, which immediately implies Theorem 19.7.1(A)(ii).

Proposition 19.7.15. *Let $x \in G$ be a_s-type. Then*

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ\pi_{h'}) + \frac{n}{4}\iota$$

where $\iota = 2$ if $n/2t$ is odd and $n - 2s \geq 2t$, and $\iota = 1$ otherwise.

In the following we assume $x \in G$ is a_s-type. The aim is to compute $f_\Omega^{h'}(x)$ and prove Proposition 19.7.15.

Case 1. Assume $x^G \cap H^\circ = \emptyset$. Corollary 19.7.13 implies $x^G \cap H^\circ\pi_{h'} \neq \emptyset$ and $x^G \cap H^\circ\pi_i = \emptyset$ for all $i < h' = \frac{2(s+t)-n}{4}$. Moreover, by the proof of Corollary 19.7.13, h' is the least integer such that

$$[I_{n/t}, \ldots, I_{n/t}, x_{2h'+1}, \ldots, x_t]\pi_{h'} \in x^G \cap H^\circ\pi_{h'}$$

where $x_{2h'+1} = \ldots = x_t$ are $a_{n/2t-1}$-involutions in $\mathrm{Sp}_{n/t}$.

By Proposition 5.2.5 $\dim(a_{n/2t-1})^{\mathrm{Sp}_{n/t}} = (n/2t-1)(n/2t+1)$. So, using (184), we have

$$\dim(x^G \cap H^\circ\pi_{h'}) = \frac{n(n+2(s-t))}{8t} + s \tag{248}$$

Again, by Proposition 5.2.5, $\dim x^G = s(n-s)$. Thus

$$f_\Omega^{h'}(x) = 1 - \frac{2st(n-s-1)}{n^2(t-1)} + \frac{n(n+2(s-t))}{4n^2(t-1)}$$

This proves Theorem 19.7.1(B)(iii).

Before showing Proposition 19.7.15 for a_s-type involutions with $h' > 0$ we try to understand how $\dim(a_s^G \cap H^\circ \pi_i)$ behaves depending on i.

Lemma 19.7.16. *Let $x \in G$ be a_s-type. Assume $h' > 0$. Let $i \geq 0$ such that $x^G \cap H^\circ \pi_{h'+i} \neq \emptyset$ and assume $i \leq n/6 - s/3$. Then*

$$\dim(x^G \cap H^\circ \pi_{h'+i}) = \dim(x^G \cap H^\circ \pi_{h'}) + i\Big(\frac{n}{2t} - 6\Big)$$

PROOF. As seen above

$$\dim(x^G \cap H^\circ \pi_{h'}) = \dim([I_{n/t}, \ldots, I_{n/t}, a_{n/2t-1}, \ldots, a_{n/2t-1}]\pi_{h'})^{H^\circ}$$

where there are exactly $2h'$ identity blocks.

Using Lemma 17.4.5, we have

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_{h'+1}) &= \dim([I_{n/t}, \ldots, I_{n/t}, a_{n/2t-3}, a_{n/2t-1}, \ldots, a_{n/2t-1}]\pi_{h'+1})^{H^\circ} \\ &= \dim(x^G \cap H^\circ \pi_{h'}) + \frac{n}{2t} - 6\end{aligned}$$

where in the first line there are $2(h'+1)$ identity blocks. Moreover the first equality is given by Proposition 19.7.11, since $|n/2t - 3 - (n/2t - 1)| = 2$; the second equality is a straightforward calculation.

In general, for $i \leq (t - 2h')/3 = n/6 - s/3$,

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_{h'+i}) &= \dim([I_{n/t}, \ldots, I_{n/t}, a_{n/2t-3}, \ldots, a_{n/2t-3}, a_{n/2t-1}, \ldots, a_{n/2t-1}]\pi_{h'+i})^{H^\circ} \\ &= \dim(x^G \cap H^\circ \pi_{h'}) + i\Big(\frac{n}{2t} - 6\Big)\end{aligned}$$

where in the first line there are $2(h'+i)$ identity blocks, i blocks of type $a_{n/2t-3}$, and $(t-2h'-3i)$ blocks of type $a_{n/2t-1}$; thus we have the upper bound $i \leq (t-2h')/3$. *q.e.d.*

Remark 19.7.17. The proof of Lemma 19.7.16 relies on Proposition 19.7.11, thanks to which we know the block decomposition which maximises the dimension of $\dim(x^G \cap H^\circ \pi_i)$. Moreover, another fundamental tool is the fact that the block decomposition which maximises $\dim(x^G \cap H^\circ \pi_{h'})$ is given by $(t - 2h')$-blocks of $a_{n/2t-1}$-type. After the limit value $i = n/6 - s/3$ it is not clear – in the case $x^G \cap H^\circ \pi_{h'+i+1} \neq \emptyset$ – how to construct a decomposition which maximises $\dim(x^G \cap H^\circ \pi_{h'+i+1})$. In order to gain an explicit formula for $\dim(x^G \cap H)$ in the case where $n/2t$ is odd and $h' > 0$ we need to understand the behaviour of $\dim(x^G \cap H^\circ \pi_i)$ when $i > n/6 - s/3$.

Now we show Proposition 19.7.15 for a_s-type involutions with $h' > 0$.

Lemma 19.7.18. *Assume $n/2t$ is odd. Let $x \in G$ be a_s-type. Assume $x^G \cap H^\circ = \emptyset$. Then*

$$\dim(x^G \cap H) < \dim(x^G \cap H^\circ \pi_{h'}) + \frac{n}{4} - t$$

PROOF. Assume $n/t = 2$. Thus $H = \mathrm{Sp}_2 \wr S_{n/2}$. Then, Lemma 17.4.5 implies that x is G-conjugate to $\pi_{s/2}$. (Notice $h' = s/2$). If $x^G \cap H^\circ \pi_i \neq \emptyset$, for some $i \neq s/2$, then Lemma 17.4.2 would imply $[I_2, \ldots, I_2, x_{2i+1}, \ldots, x_t]\pi_i \in x^G$, for suitable $x_{2i+1}, \ldots, x_t \in \mathrm{Sp}_2$; since x is a_s-type then each x_i should be a-type: this is absurd because Sp_2 does not contain any a-type involution. Therefore $x^G \cap H = x^G \cap H^\circ \pi_{h'}$ and the result trivially holds.

For the reminder of the proof we may assume $n/t \geq 6$. The hypothesis $x^G \cap H^\circ = \emptyset$ implies $h' > 0$. For some $h' \leq \ell \leq t/2$, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_\ell)$.

Notice that if $s = n/2$ then $h' = t/2$ hence $x^G \cap H = x^G \cap H^\circ \pi_{h'}$. Similarly if $s = n/2 - 1$ then $h' = (t-1)/2$ hence $x^G \cap H = x^G \cap H^\circ \pi_{h'}$. In both cases the result trivially follows.

Assume $s < n/2 - 1$. In particular $\ell \neq t/2$. By Lemmas 17.4.2 and 17.4.5, for suitable blocks of type a_{s_i}, with $\sum_i s_i = s - \frac{n}{t}\ell$, we have

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_\ell) &= \dim([I_{n/t}, \ldots, I_{n/t}, x_{2\ell+1}, \ldots, x_t]\pi_\ell)^{H^\circ} \\ &= \ell \frac{n}{2t}\Big(\frac{n}{t} + 1\Big) + \sum_{i > 2\ell} s_i \Big(\frac{n}{t} - s_i\Big) \\ &\leq \ell \frac{n}{2t}\Big(\frac{n}{t} + 1\Big) + \frac{n}{t}\Big(s - \frac{n}{t}\ell\Big) - \frac{1}{t - 2\ell}\Big(s - \frac{n}{t}\ell\Big)^2 = g(\ell)\end{aligned}$$

where we used $\sum_{i>2\ell} s_i^2 \geq \frac{1}{t-2\ell}\big(\sum s_i\big)^2$, by Proposition B.2.1. With some elementary analysis we deduce that $g(\ell) \leq g\big(\frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}}\big)$. Notice that $0 < h' \leq \frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}} < t/2$. We use *Mathematica* and the value of $\dim(x^G \cap H^\circ \pi_{h'})$ computed in (248) to get the following simplified formula

$$g\Big(\frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}}\Big) = \dim(x^G \cap H^\circ \pi_{h'}) + \frac{n-2s}{8t}\Big(n - 4t\big(\sqrt{n/t} - 1\big)\Big)$$

By Lemma 19.7.12, $n - 2s < 2t$, since $h' > 0$. Moreover since $n/t \geq 6$ we have $\sqrt{n/t} > 2$. The result follows. *q.e.d.*

If $x^G \cap H^\circ \neq \emptyset$ then $h' = 0$. Now we deal with this case. We shall prove a result similar to Lemma 19.7.18 using the same technique for $n/2t$ odd; for $n/2t$ even the proof will be very similar to that of Lemma 19.7.8.

Case 2. Assume $x^G \cap H^\circ \neq \emptyset$. First we compute $\dim(x^G \cap H^\circ)$.

Write $s/2 = \lfloor s/2t \rfloor t + b$. Notice that $\mathrm{Sp}_{n/t}$ contains $a_{2\lfloor s/2t \rfloor}$-type involutions and, if $b \neq 0$, it contains $a_{2\lfloor s/2t \rfloor + 2}$-type involutions. Let z be an $a_{2\lfloor s/2t \rfloor}$-involution, and z' be an $a_{2\lfloor s/2t \rfloor + 2}$-involution. Define $x_1 = \ldots = x_b = z' \in \mathrm{Sp}_{n/t}$ and $x_{b+1} = \ldots = x_t = z \in \mathrm{Sp}_{n/t}$. Then $[x_1, \ldots, x_t] \in x^G \cap H^\circ$. Proposition 19.7.11 yields $\dim(x^G \cap H^\circ) = \sum_i \dim x_i^{\mathrm{Sp}_{n/t}}$. Using Proposition 5.2.5 We compute

$$\dim x^G = s(n-s),\ \dim(x^G \cap H^\circ) = \frac{s(n-s)}{t} - \frac{4b(t-b)}{t} \tag{249}$$

Therefore

$$f^\circ_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

This proves Theorem 19.7.1(B)(iii).

The following is the analogue of Lemma 19.7.8 proved for semisimple involutions.

Lemma 19.7.19. *Assume $n/2t$ is even. Let $x \in G$ be a_s-type. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{4}$$

Proof. The proof is very similar to that of Lemma 19.7.8.

Assume $x^G \cap H^\circ\pi_i \neq \emptyset$, for some $i > 0$. Then for suitable a-type involutions $x_{2i+1}, \dots, x_t \in \mathrm{Sp}_{n/t}$ we have, using (184),

$$\dim(x^G \cap H^\circ\pi_i) = \dim([I_{n/t}, \dots, I_{n/t}, x_{2i+1}, \dots, x_t]\pi_i)^{H^\circ} = i \dim \mathrm{Sp}_{n/t} + \sum_{j>2i} \dim x_j^{\mathrm{Sp}_{n/t}}$$

Let $z \in \mathrm{Sp}_{n/t}$ be an $a_{n/2t}$-type involution (recall that $n/2t$ is even). We compute $2\dim z^{\mathrm{Sp}_{n/t}} = (n/t)^2/2$. Then

$$\begin{aligned}\dim(x^G \cap H^\circ\pi_{i-1}) &\geq \dim([I_{n/t}, \dots, I_{n/t}, z, z, x_{2i+1}, \dots, x_t]\pi_{i-1})^{H^\circ} \\ &= i \dim \mathrm{Sp}_{n/t} - \frac{n}{2t} + \sum_{j>2i} \dim x_j^{\mathrm{Sp}_{n/t}} = \dim(x^G \cap H^\circ\pi_i) - \frac{n}{2t}\end{aligned}$$

For some $0 \leq \ell \leq t/2$, $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_\ell)$. By the previous computation

$$\dim(x^G \cap H^\circ\pi_\ell) \leq \ldots \leq \dim(x^G \cap H^\circ\pi_0) + \ell\frac{n}{2t} \leq \dim(x^G \cap H^\circ) + \frac{n}{4}$$

q.e.d.

Thus we have just proved Proposition 19.7.15 for a-type involution when $n/2t$ is even. Now we deal with the case $n/2t$ odd.

Lemma 19.7.20. *Assume $n/2t$ is odd. Let $x \in G$ be a_s-type. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{2}$$

Proof. If $n/t = 2$ then the same argument of Lemma 19.7.18 holds. Therefore we assume $n/t \geq 6$. In addition Lemma 19.7.12 implies $n - 2s \geq 2t$. Recall the formula of $\dim(x^G \cap H^\circ)$ given in (249).

We mirror the proof of Lemma 19.7.18 and we use some of the notation defined in the aforementioned result. For some $0 \leq \ell \leq t/2$, we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_\ell)$. Assume $\ell < t/2$, the case $\ell = t/2$ will be studied later. So

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi_\ell) \leq g(\ell) \leq g\Big(\max\Big\{0, \frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}}\Big\}\Big)$$

First assume $\max\{0, \frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}}\} = 0$. Then

$$g(\ell) \leq g(0) = \frac{s(n-s)}{t} = \dim(x^G \cap H^\circ) + \frac{4b(t-b)}{t} \leq \dim(x^G \cap H^\circ) + t$$

The result follows.

Now assume $\max\{0, \frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}}\} = \frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}} > 0$. Combining this inequality with $n - 2s \geq 2t$, we deduce

$$\frac{n - \sqrt{nt}}{2} < s < \frac{n}{2} - t \tag{250}$$

And, using *Mathematica*, we have

$$\begin{aligned} g(\ell) \leq g\Big(\frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}}\Big) &= \frac{s(n-s)}{t} + \frac{(n-2s)^2 + nt - 2t\sqrt{n/t}(n-2s)}{4t} \\ &< \frac{s(n-s)}{t} + \frac{(n-2s)^2 + nt}{4t} - 2t < \frac{s(n-s)}{t} + \frac{n}{2} - 2t \end{aligned}$$

where we use $n-2s \geq 2t$ and $\sqrt{n/t} > 2$ for the first inequality; the last inequality follows using the lower bound on s given in (250). Therefore

$$\dim(x^G \cap H) < \dim(x^G \cap H^\circ) + \frac{n}{2} - 2t + \frac{4b(t-b)}{t} \leq \dim(x^G \cap H^\circ) + \frac{n}{2} - t$$

In the case $\ell = t/2$ we have $s = n/2$, because Lemma 17.4.5 implies that $\pi_{t/2}$ and $[J_2^{n/2}]$ are G-conjugate. Notice that $n/4 = \lfloor n/4t \rfloor t + t/2$ since $n/2t$ is odd. Using (184) we compute $\dim(x^G \cap H^\circ \pi_\ell) = \dim \pi_{t/2}^{H^\circ} = \frac{s(n-s)}{t} + \frac{n}{4}$. Therefore

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ) + \frac{n}{4} + \frac{4b(t-b)}{t} \leq \dim(x^G \cap H^\circ) + \frac{n}{4} + t$$

The proof is complete. *q.e.d.*

The results in Lemmas 19.7.18, 19.7.19 and 19.7.20 immediately imply Theorem 19.7.1(A)(ii).

19.7.2.2. *b_s, c_s-type involutions.* For involutions of type b_s or c_s we can provide more informations. In fact we shall compute the value of $\dim(x^G \cap H)$.

Lemma 19.7.21. *Let $x \in G$ be b_s or c_s-type. Assume $x^G \cap H^\circ \pi_i \neq \emptyset$ for some $i > 0$. Then $x^G \cap H^\circ \pi_{i-1} \neq \emptyset$ and*

$$\dim(x^G \cap H^\circ \pi_i) < \dim(x^G \cap H^\circ \pi_{i-1})$$

Proof. Using Lemma 17.4.2 we may assume

$$\dim(x^G \cap H^\circ \pi_i) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_i)^{H^\circ}$$

for some involution $x_{2i+1}, \ldots, x_t \in \mathrm{Sp}_{n/t}$ each of them is never a-type, by Proposition 19.7.11. Define $z = [J_2^{n/2t}]$, which is an involution of type $b_{n/2t}$ or $c_{n/2t}$. Using Proposition 5.2.5 we compute $2\dim z^{\mathrm{Sp}_{n/t}} = \frac{n}{2t}\left(\frac{n}{t}+1\right) + \frac{n}{2t}$. As usual

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_{i-1}) &\geq \dim([I_{n/t}, \ldots, I_{n/t}, z, z, x_{2i+1}, \ldots, x_t]\pi_{i-1})^{H^\circ} \\ &= \dim(x^G \cap H^\circ \pi_i) + \frac{n}{2t}\end{aligned}$$

The result follows. *q.e.d.*

Corollary 19.7.22. *Let $x \in G$ be an involution of type b_s or c_s. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$$

In particular, $f_\Omega(x) = f_\Omega^\circ(x)$.

Let us compute $f_\Omega^\circ(x)$ when $x \in G$ is of type b_s or c_s. Write $s = \lfloor s/t \rfloor t + b$ where $0 \leq b < t$. Let $x = [x_1, \ldots, x_t] \in H^\circ$ where each x_i is either b or c-type with $\nu(x_1) = \ldots = \nu(x_b) = \lfloor s/t \rfloor + 1$ and $\nu(x_{b+1}) = \ldots = \nu(x_t) = \lfloor s/t \rfloor$. Thanks to Proposition 19.7.11 we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. So we compute

$$\dim(x^G \cap H^\circ) = \frac{s(n-s)}{t} + s - \frac{b(t-b)}{t}$$

By Proposition 5.2.5 we have $\dim x^G = s(n-s+1)$. So

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

This concludes the proof of Theorem 19.7.1.

CHAPTER 20

Orthogonal group

Throughout this chapter we set $G = \mathrm{O}_n$, $H = \mathrm{O}_{n/t} \wr S_t \leq G$ a $\mathscr{C}_2$-subgroup, where $1 < t < n$, and $\Omega = G/H$. Notice, we avoid the case $t = n$ for otherwise H would be finite; in particular, n is never prime. In this chapter we derive bounds on $f_\Omega(x)$ for $x \in G$ of prime order or any unipotent element in the characteristic zero case, proving the theorems stated in Chapter 16.

Recall that, in the case $p = 2$, n/t is even.

20.1. Upper bounds

In this section we derive upper bounds on $f_\Omega(x)$ for $x \in G$ of prime order or any unipotent element in the characteristic zero case. In particular we shall prove the following.

Proposition 20.1.1. *Assume $n > 4$. Let $x \in G$ be of prime order r.*

(i) Assume $r \neq 2$. Then

$$f_\Omega(x) \leq 1 - \frac{4}{n}$$

with equality if, and only if, $\nu(x) = 2$ or one of $x^G, C_G(x)$ is as in Table 20.1.1.

n	t	r	$x^G, C_G(c)$
6	2	p	$[J_3^2]^G$
6	3	$p = 3$	$[J_3^2]^G$
6	2	$\neq p$	$\mathrm{O}_2 \times \mathrm{GL}_2$
6	3	$3 \neq p$	$\mathrm{O}_2 \times \mathrm{GL}_2$
8	$2, 4$	$\neq p$	GL_4

Table 20.1.1

(ii) Assume $r = 2$. Then

$$f_\Omega(x) \leq 1 - \frac{2}{n}$$

with equality if, and only if, $\nu(x) = 1$

Remark 20.1.2. Let us make some comments on Proposition 20.1.1.

(i) In all the cases we have $f_\Omega(x) \leq 1 - \frac{2}{n}$, and equality holds if, and only if, $\nu(x) = 1$. Notice that $\nu(x) = 1$ necessarily implies that x is an involution.

(ii) Thanks to Lemma 7.1.1 and Corollary 7.1.11, Proposition 20.1.1 extends to any non-central element in G. In particular, Theorems 16.1.1, 16.1.3 and 16.1.4 follow.

In the case $n = 4$ we give the following.

Lemma 20.1.3. *Assume $n = 4$. Let $x \in G$ be of prime order r.*

(i) If $p \neq 2$, then $f_\Omega(x) \leq 1/2$. Moreover equality holds if, and only if, $C_G(x) \cong \mathrm{O}_2 \times \mathrm{O}_2$ or $\mathrm{O}_1 \times \mathrm{O}_3$ or GL_2.

(ii) Assume $p = 2$. Then $f_\Omega(x) \leq 3/4$ with equality if, and only if, x is an a_2-type involution.

PROOF. In this case $G = \mathrm{O}_4$ hence $H = \mathrm{O}_2 \wr S_2$. Notice, when $p \neq 2$ no unipotent element lies in H. The result follows with easy computations. *q.e.d.*

For the remainder of the section we assume $n > 4$.

As usual, we spread the proof of Proposition 20.1.1 in the following several lemmas. The same observations made in Remark 10.1.3 hold. Hence, we will only consider elements in H. In the following r is always a prime.

Lemma 20.1.4. *Let $x \in H$ be of odd order r. Assume $\nu(x) = 2$. Then*

$$f_\Omega(x) = 1 - \frac{4}{n}$$

PROOF. Up to G-conjugacy, the only two unipotent elements with ν-value equal 2 are $x = [J_2^2, J_1^{n-4}]$ and $y = [J_3, J_1^{n-3}]$. Up to the centraliser structure, the only semisimple element $z \in H$ with $\nu(z) = 2$ is $z = [I_{n-2}, \omega, \omega^{-1}]$. With an easy computation we get the result. *q.e.d.*

Lemma 20.1.5. *Let $x \in H$ be an involution. Assume $\nu(x) = 1$. Then*

$$f_\Omega(x) = 1 - \frac{2}{n}$$

PROOF. This is a straightforward computation using Proposition 5.2.5 (if $p = 2$) and Theorem 5.3.1 (if $p \neq 2$). *q.e.d.*

Now we complete the proof of Proposition 20.1.1. As above, also here we first study odd order unipotent and semisimple elements and then involutions.

Lemma 20.1.6. *Assume $p \neq 2$. Let $x \in H$ be unipotent with $\nu(x) > 2$. Then*

$$f_\Omega(x) \leq 1 - \frac{4}{n}$$

with equality if, and only if, $(n, t, p) = (6, 2, any)$ or $(6, 3, 3)$ and $x \in [J_3^2]^G$.

PROOF. In the case $n \leq 10$ the result follows by direct computations. So we assume $n > 10$. For $x \in H$, using Proposition 17.2.1 we have

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2}$$

The only partitions of n in $n-3$ parts are $(2^3, 1^{n-6}), (3, 2, 1^{n-5})$ and $(4, 1^{n-4})$. Thus Theorem 5.2.1 implies that there is no unipotent element $x \in G$ with $\nu(x) = 3$. Hence we may assume $\nu(x) \geq 4$. Thanks to Proposition 5.4.1 we have $\dim x^G \geq \max\{4(n-5), 3n/2\} = 4(n-5)$, as $n > 10$. Therefore

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2} \leq 1 - \frac{8(n-5)}{n^2} < 1 - \frac{4}{n}$$

where the last inequality is satisfied since $n > 10$. *q.e.d.*

Lemma 20.1.7. *Let $x \in H$ be of odd order $r \neq p$. Assume $\nu(x) > 2$. Then*

$$f_\Omega(x) \leq 1 - \frac{4}{n}$$

with equality if, and only if, $(n, t, r, C_G(x)) = (6, 2, -, \mathrm{O}_2 \times \mathrm{GL}_2), (6, 3, 3, \mathrm{O}_2 \times \mathrm{GL}_2)$ *or* $(8, -, -, \mathrm{GL}_4)$.

Proof. We proceed as in Lemma 20.1.6. For the cases $n \leq 10$ it is straightforward to get the result. Thus we assume $n > 10$.

Let $x \in H$ be semisimple of odd prime order. Notice that if $\nu(x) = 3$ then $n = 4$ or 6. So we may assume $\nu(x) \geq 4$. Proposition 17.2.1 implies

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2}$$

By Proposition 5.4.1, $\dim x^G \geq \max\{4(n-5), 3n/2\} = 4(n-5)$, as $n > 10$. Then as in Lemma 20.1.6 we get the result. *q.e.d.*

In order to complete the proof of Proposition 20.1.1 we need to deal with involutions.

Lemma 20.1.8. *Let $x \in H$ be an involution. Assume $\nu(x) \geq 2$. Then*

$$f_\Omega(x) < 1 - \frac{2}{n}$$

Proof. The case $n = 6$ quickly follows. So we assume $n > 6$.

By Proposition 17.2.1, for $x \in H$, we have

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2}$$

Assume $\nu(x) \geq 2$. Then Proposition 5.4.1 implies $\dim x^G \geq \max\{2(n-3), n/2\} = 2(n-3)$. Hence, for $n > 6$ we have

$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2} \leq 1 - \frac{4(n-3)}{n^2} < 1 - \frac{2}{n}$$

q.e.d.

Remark 20.1.9. Notice that all the previous results apply for unipotent elements in characteristic zero.

20.2. Unipotent elements: lower bounds

In this section we derive lower bound on $f_\Omega(x)$ for $x \in H$ of prime order p (or any unipotent element in the characteristic zero case). Recall that in the characteristic zero case we set $p = \infty$.

The main result of this section is the following.

Proposition 20.2.1. *Let $x \in H$ be of order p.*

(i) If $p > n/2$ then

$$f_\Omega(x) \geq \frac{t}{n} + 2\left(\frac{t}{n}\right)^2 \delta_{n/t;2}$$

with equality if, and only if, n/t is even and $x \in [J_{n/t-1}^t, J_1^t]^G$, or n/t is odd and $x \in [J_{n/t}^{t-1}, z]^G$ for any $z \in \mathrm{O}_{n/t}$ unipotent.

(ii) If $p \leq n/2$ then
$$f_\Omega(x) \geq \frac{1}{p} - \epsilon$$
where $\epsilon = \frac{t}{n(t-1)}$ if $p = 2$ and x is of b_s or c_s-type, and $\epsilon = 0$ otherwise.

Remark 20.2.2.

(i) Proposition 20.2.1 extends to any element $x \in G$ such that $x^G \cap H \neq \emptyset$.

(ii) In case (ii) we have a partial characterisation of elements that realise equality, see Proposition 20.2.13. In the case $p = 2$ we shall give more informations in Section 20.7.2. In fact we shall show that $f_\Omega(x) \geq 1/p$ if $p = 2$ and x is an involutions, see Remark 20.7.4.

As always we shall use the following notation. For $x \in G$ of order p we write, up to G-conjugacy $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ where $n = \sum_i a_i$, and a_i is even whenever i is even, see Theorem 5.2.1. If $x^G \cap H^\circ \neq \emptyset$ then we may write $x = [x_1, \ldots, x_t]$ for $x_i \in \mathrm{O}_{n/t}$ unipotent of order p. Moreover, we may assume $\dim(x^G \cap H^\circ) = \sum_i \dim x_i^{\mathrm{O}_{n/t}}$ and, as usual, we write
$$x_i = [J_p^{a_{i,p}}, \ldots, J_1^{a_{i,1}}]$$
where, for $1 \leq j \leq p$, we have $a_j = \sum_i a_{i,j}$ and $n/t = \sum_l a_{i,l}$. Hence
$$\dim x^G = \frac{n}{2}(n-1) - \sum_{i<j} i a_i a_j - \frac{1}{2}\sum_i i a_i^2 + \frac{1}{2}\sum_{i \text{ odd}} a_i$$
$$\dim(x^G \cap H^\circ) = \frac{n}{2}\Big(\frac{n}{t} - 1\Big) - \sum_{i<j} i(a_{1,i}a_{1,j} + \ldots + a_{t,i}a_{t,j})$$
$$- \frac{1}{2}\sum_i i(a_{1,i}^2 + \ldots + a_{t,i}^2) + \frac{1}{2}\sum_{i \text{ odd}} a_i$$

The following is an analogue of Lemma 19.2.3, which motivates the two cases $p > n/2$ and $p \leq n/2$.

Lemma 20.2.3. *Assume $n/2 < p < \infty$ and let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in G$. Suppose $a_i > 0$ for some $i > n/2$. Then $x^G \cap H = \emptyset$.*

We start with the case $p > n/2$.

20.2.1. Case $p > n/2$. Let $x \in H$ be of order p, or any unipotent element if $p = \infty$. Then, by Lemmas 17.3.1, 20.2.3 and Proposition 17.4.11 we have $x^G \cap H = x^G \cap H^\circ$, see Lemma B.3.1 for $p = \infty$. In particular, the largest block allowed has size n/t. Thus we may assume $x \in H^\circ$ and $\dim(x^G \cap H) = \dim x^{H^\circ}$.

Here, we need to distinguish two cases depending on the parity of n/t. For example, if n/t is odd then $[J_{n/t}] \in \mathrm{O}_{n/t}$ and so $[J_{n/t}^t]^G \cap H^\circ \neq \emptyset$; instead, for n/t even we have $[J_{n/t}] \notin \mathrm{O}_{n/t}$, however if t is even we have $[J_{n/t}^t] \in G$ and $[J_{n/t}^t]^G \cap H = \emptyset$.

Lemma 20.2.4. *Assume n/t is odd. Let $x \in [J_{n/t}^{t-1}, z]^G$ where $z \in \mathrm{O}_{n/t}$ is any unipotent element. Then $f_\Omega(x) = t/n$.*

Proof. This is an easy computation, very similar to that in Lemma 18.2.4. *q.e.d.*

When n/t is even, $[J_{n/t-1}, J_1] \in \mathrm{O}_{n/t}$, so $x = [J_{n/t-1}^t, J_1^t]^G \cap H^\circ \neq \emptyset$. In general $f_\Omega(x) \neq f_\Omega([J_{n/t-1}^{t-1}, J_1^{t-1}, z])$ where $z \in \mathrm{O}_{n/t}$ is any unipotent element. (An example is given by $[J_7^2, J_1^2]$ and $[J_7, J_1^9]$ in $H^\circ = \mathrm{O}_8 \times \mathrm{O}_8$.)

Lemma 20.2.5. *Assume n/t is even. Let $x \in [J_{n/t-1}^t, J_1^t]^G$. Then $f_\Omega(x) = t/n + 2(t/n)^2$.*

Proof. This is an easy computation. *q.e.d.*

The following two technical lemmas are the analogues of Lemma 19.2.5, proved in the symplectic case. First we study the case n/t odd. Recall we are assuming $p > n/2$.

Lemma 20.2.6. *Assume n/t is odd. Let $x = [x_1, \ldots, x_t] \in H^\circ$ and assume $x_1, x_2 \neq [J_{n/t}]$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Define $y = [J_{n/t}, x_2, \ldots, x_t] \in H^\circ$. Then $f_\Omega(x) > f_\Omega(y)$.*

Proof. The proof of Lemma 19.2.5 can be repeated verbatim here. *q.e.d.*

Remark 20.2.7. In the case where n/t is even the argument of Lemma 20.2.6 still applies although in this case $y \notin \mathrm{O}_{n/t}$. However, we can still consider the formulae of $f_\Omega(x)$ and $f_\Omega(y)$ from a purely combinatorial point of view. Hence, by Lemma 20.2.9 below, we will deduce for any element $x \in H$ that $f_\Omega(x) \geq t/n$ where t/n is given by $f_\Omega([J_{n/t}^t])$.

A similar result holds when n/t is even. Notice that in the assumptions of the next result we only require one block in $x = [x_1, \ldots, x_t]$ to be different from $[J_{n/t-1}, J_1]$, whereas in Lemma 20.2.6 we needed two blocks different from $[J_{n/t}]$ (or, more formally, not conjugate to it).

Lemma 20.2.8. *Assume $p > n/2$ and n/t is even. Let $x = [x_1, \ldots, x_t] \in H^\circ$ and assume $x_1 \neq [J_{n/t-1}, J_1]$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Define $y = [y_1, x_2, \ldots, x_t] \in H^\circ$ with $y_1 = [J_{n/t-1}, J_1]$. Then $f_\Omega(x) > f_\Omega(y)$.*

Proof. The proof is similar to that of Lemma 19.2.5. Notice that since $\dim(x^G \cap H) = \dim x^{H^\circ}$ we deduce $\dim(y^G \cap H) = \dim y^{H^\circ}$ since otherwise we would increase the dimension of $[x_2, \ldots, x_t]$ which is absurd.

We need to show

$$\dim x^G - \dim y^G < \dim(x^G \cap H) - \dim(y^G \cap H) \tag{251}$$

Notice that $\dim(x^G \cap H) - \dim(y^G \cap H) = \dim x_1^{\mathrm{O}_{n/t}} - \dim[J_{n/t-1}, J_1]^{\mathrm{O}_{n/t}}$.

Permuting the blocks of x we see that there exists $h \geq 0$ such that $x_1, \ldots, x_{t-h} \neq [J_{n/t-1}, J_1]$ and $x_{t-h+1} = \ldots = x_t = [J_{n/t-1}, J_1]$. Notice that since $x_1 \neq [J_{n/t-1}, J_1]$ we have $0 \leq h \leq t-1$. For all $i \leq t-h$ we write

$$x_i = [J_{n/t-2}^{a_{i,n/t-2}}, \ldots, J_1^{a_{i,1}}] \in \mathrm{O}_{n/t}$$

So $x = [J_{n/t-1}^{h}, J_{n/t-2}^{a_{n/t-2}}, \dots, J_2^{a_2}, J_1^{a_1+h}]$ and $y = [J_{n/t-1}^{h+1}, J_{n/t-2}^{b_{n/t-2}}, \dots, J_2^{a_2}, J_1^{b_1+h+1}]$ where $a_j = \sum_i a_{i,j}$ and $b_j = a_j - a_{i,1}$, for all $1 \leq j \leq p$. We compute

$$\begin{aligned}\dim x^G - \dim y^G &= \frac{n}{t}(t-h-1) + 2h - h\sum_i a_{1,i} + \sum_i (a_{2,i} + \ldots + a_{t-h,i}) \\ &\quad + (\dim x^{H^\circ} - \dim y^{H^\circ}) \\ &\quad - \sum_{i\leq j} i a_{1,i}(a_{2,j} + \ldots + a_{t-h,j}) - \sum_{i<j} i a_{1,j}(a_{2,i} + \ldots + a_{t-h,i}) \\ \dim x^{H^\circ} - \dim y^{H^\circ} &= \frac{n}{2t} - \sum_{i<j} i a_{1,i} a_{1,j} - \frac{1}{2}\sum_i i a_{1,i}^2 + \frac{1}{2}\sum_{i \text{ odd}} a_{1,i}\end{aligned}$$

Substituting $\frac{n}{t}(t-h-1) = \sum_i i(a_{2,i} + \ldots + a_{t-h,i})$ and using the above computations we deduce that (251) is equivalent to

$$\sum_{i\leq j} i a_{1,i}(a_{2,j} + \ldots + a_{t-h,j}) + \sum_{i<j} i a_{1,j}(a_{2,i} + \ldots + a_{t-h,i}) \\ + h\Big(\sum_{i\geq 1} a_{1,i} - 2\Big) > \sum_{i\geq 1}(i+1)(a_{2,i} + \ldots + a_{t-h,i})$$

It is clear that $h\big(\sum_{i\geq 1} a_{1,i} - 2\big) \geq 0$ since $x_1 \neq [J_{n/t}]$ and so x_1 comprises at least two blocks.

Therefore we need to show

(252)
$$\sum_{i\leq j} i a_{1,i}(a_{2,j} + \ldots + a_{t-h,j}) + \sum_{i<j} i a_{1,j}(a_{2,i} + \ldots + a_{t-h,i}) > \sum_i (i+1)(a_{2,i} + \ldots + a_{t-h,i})$$

This can be proved using the same argument as in Lemma 18.2.5. Fix $j \in \{1, \ldots, n/t-2\}$ and $\ell \in \{2, \ldots, t-h\}$. The coefficient of $a_{\ell,j}$ in the right hand side of (252) is $(j+1)$ whereas that in the left hand side is

$$\sum_{i\leq j} i a_{1,i} + j\sum_{j<i} a_{1,i}$$

We claim that

$$\sum_{i\leq j} i a_{1,i} + j\sum_{j<i} a_{1,i} > j+1 \tag{253}$$

We split the proof of (253) into two cases. First we assume that for all $i > j$ we have $a_{1,i} = 0$; therefore (253) is equivalent to

$$\sum_{i\leq j} i a_{1,i} + j\sum_{j<i} a_{1,i} = \sum_{i\leq j} i a_{1,i} = \frac{n}{t} > j+1$$

which is true since $j < n/t - 1$. Now assume $a_{1,i} \neq 0$ for some $i > j$. Say $a_{1,i_1}, \ldots, a_{1,i_m}$ are the only non-zero multiplicities with $i_1, \ldots, i_m > j$. Then (253) is equivalent to

$$\sum_{i\leq j} i a_{1,i} + j(a_{1,i_1} + \ldots + a_{1,i_m}) > j+1$$

If $\sum_{i\leq j} i a_{1,i} = 0$ then $a_{1,i_1} + \ldots + a_{1,i_m} \geq 2$ (as $x_1 \neq [J_{n/t-1}, J_1]$), thus $j(a_{1,i_1} + \ldots + a_{1,i_m}) \geq 2j > j+1$. If $\sum_{i\leq j} i a_{1,i} \geq 1$ then one of the following two conditions holds

- $\sum_i i a_{1,i} = 1$ and $a_{1,i_1} + \ldots + a_{1,i_m} \geq 2$ since $x_1 \neq [J_{n/t-1}, J_1]$

- $\sum_i ia_{1,i} \geq 2$ and $a_{1,i_1} + \ldots + a_{1,i_m} \geq 1$.

In both cases inequality (253) is satisfied. *q.e.d.*

Using Lemmas 20.2.6 and 20.2.8, we deduce the following, which is the analogue of Lemma 18.2.6 and 19.2.6.

Lemma 20.2.9. *Let $x \in H$ be of order p. Then the conclusion of Proposition 20.2.1(i) holds.*

PROOF. This is the same as Lemma 18.2.6. *q.e.d.*

20.2.2. Case $p \leq n/2$. Now we assume $p \leq n/2$. In this case the largest Jordan block of any element of order p in H is J_p. As in the case $G = \mathrm{GL}_n$ (Lemma 18.2.9) and in the case $G = \mathrm{Sp}_n$ (Lemma 19.2.8), we directly show that for every element $x \in H$ of order p we have $f_\Omega(x) \geq 1/p$.

Lemma 20.2.10. *Assume $p \neq 2$. Let $x \in H$ be of order p. Then*

$$f_\Omega(x) \geq \frac{1}{p}$$

PROOF. The same argument as in Lemmas 18.2.9 and 19.2.8 applies here. *q.e.d.*

In order to prove the analogue of Lemma 20.2.10 in the case $p = 2$ we need the following.

Remark 20.2.11. Assume $p = 2$.

(i) Let $x \in G$ be an involution of type a_s. Then x has Jordan form $[J_2^{m_2}, J_1^{m_1}]$ with $s = n - m_2 - m_1$. It is straightforward to check that the formula in Theorem 5.2.1 can be written as follows

$$\dim x^G = \frac{n}{2}(n-1) - \sum_{i<j} im_i m_j - \frac{1}{2}\sum_i im_i^2 + \frac{1}{2}\sum_{i\,\mathrm{odd}} m_i$$

(ii) Let $x \in G$ be an involution of type b_s or c_s. Then, by Proposition 5.2.5, $\dim b_s^G = \dim c_s^G = s(n-s)$. As before say $[J_2^{m_2}, J_1^{m_1}]$ is the Jordan form of x. We see that

$$\dim x^G = \frac{n}{2}(n-1) - \sum_{i<j} im_i m_j - \frac{1}{2}\sum_i im_i^2 + \frac{1}{2}\sum_{i\,\mathrm{odd}} m_i + (n - m_1 - m_2)$$

Let $h \leq t/2$ such that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$. Then, for suitable blocks $x_{2h+1}, \ldots, x_t \in \mathrm{O}_{n/t}$, we have, by Lemma 17.4.2,

$$\dim(x^G \cap H) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2h+1}, \ldots, x_t])^{H^\circ}$$

Since $\dim c_s^G > \dim a_s^G$ we must have that x_i is either a b_{s_i} or c_{s_i}-type for $i \geq 2h+1$.

Lemma 20.2.12. *Assume $p = 2$. Let $x \in H$ be an involution. Then*

$$f_\Omega(x) \geq \frac{1}{p} - \epsilon$$

where $\epsilon = 0$ if x is of type a_s and $\epsilon = \frac{t}{n(t-1)}$ otherwise.

PROOF. For an involutions $x \in H$ of type a_s the argument of Lemma 20.2.10 applies, thanks to Remark 20.2.11. Therefore $f_\Omega(x) \geq 1/p$.

Let $x \in H$ be an involution of type b_s or c_s. We follow the proof of Lemma 19.2.8. Thus, for some $h \leq t/p$,

$$\dim(x^G \cap H) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2h+1}, \ldots x_t]\pi_h)^{H^\circ}$$

Say $a_{i,1}$ and $a_{i,2}$ are the multiplicities, in x_i, of the Jordan blocks J_1 and J_2, respectively. Using Remark 20.2.11 we compute $\dim x^G$ and $\dim(x^G \cap H)$. We have $f_\Omega(x) \geq 1/p - \epsilon$ if, and only if,

$$-\dim x^G + \dim(x^G \cap H) \geq -\dim \Omega\Big(1 - \frac{1}{p}\Big) - \epsilon \dim \Omega \tag{254}$$

Following the computation in Lemma 19.2.8 (that could be repeated verbatim for Lemma 20.2.10) we see that (254) is equivalent to (226), with the right hand side given by

$$\frac{n^2}{2t^2p}(t - hp)(t - hp - 1) + \frac{n}{t}h - \epsilon \dim \Omega$$

And we see that

$$\frac{n}{t}h - \epsilon \dim \Omega = \frac{n}{t}h - \frac{n}{2} \leq 0$$

since $h \leq t/2$. Therefore using the same argument as in Lemma 19.2.8, we see that (226) is verified. This leads to the result. *q.e.d.*

As for the case $G = \mathrm{GL}_n$ we can prove the analogue of Lemmas 18.2.11, 18.2.12 and 18.2.13 and Proposition 18.2.15.

Recall the notation introduced before Lemma 18.2.13: if $\mu = (p^{a_p}, \ldots, 1^{a_1}) \vdash n$ and $\eta = (p^{b_p}, \ldots, 1^{b_1}) \vdash m$ then we write $\mu \oplus \eta = (p^{a_p+b_p}, \ldots, 1^{a_1+b_1}) \vdash n + m$.

Assume $p \mid n$. The following characterises elements $x \in H$ of order $p \neq 2$ such that $f_\Omega(x) = 1/p$. Recall the definition of $\mathcal{P}_G$ given in Section 17.3.1.

Proposition 20.2.13. *Assume $p \neq 2$ and $p \mid n$. Let $x \in H$ be of order p. Then $f_\Omega(x) = 1/p$ if, and only if, one of the following holds*

- *(i) $x \in [J_p^{n/p}]^G$;*
- *(ii) $n/t = pm$ and $x = [J_p^{m(t-1)}, z]$ for any $z \in \mathrm{O}_{n/t}$ of order p whose associated partition $\lambda = (p^{a_p}, \ldots, 1^{a_1}) \vdash n/t$ satisfied one of the following*
  - *(a) $a_p = m$; or,*
  - *(b) $a_p < m$ and whenever $\lambda = \mu \oplus \eta$ with $\mu \vdash l_1 p$, $\eta \vdash l_2 p$ for some $l_1, l_2 < m$ such that $\mu \oplus (p^{l_2}), \eta \oplus (p^{l_1}) \in \mathcal{P}_{\mathrm{O}_{n/t}}$ then $\mu = (p^{l_1})$ or $\eta = (p^{l_2})$.*

Recall that if $p > t$ and $x \in G$ has order p then $x^G \cap H = x^G \cap H^\circ$; so if $p \geq n/t$ then $f_\Omega(x) \geq t/n$. In the case where p does not divide n we have the following, in which we assume $p < t$ (thanks to the previous observation).

Proposition 20.2.14. *Assume $p \neq 2$ and $p \nmid n$. Then there exists $x \in H$ of order p such that*

$$f^\circ_\Omega(x) \leq \begin{cases} \frac{1}{p} + \frac{p}{n} & \frac{n}{t} < p < t \\ \frac{5}{4p} + \frac{2}{p^2} & \text{otherwise} \end{cases}$$

Proof. We divide the analysis into two cases, $p < n/t$ and $n/t < p < n$. The proof is very similar to that of Proposition 19.2.11, so we omit the details of the computations.

Case 1. Assume $p < n/t$. Let $n/t = ap + b$.

If b is odd then $\bar{x} = [J_p^a, J_b] \in \mathrm{O}_{n/t}$. We define $x = [J_p^{at}, J_b^t] = [\bar{x}, \ldots, \bar{x}] \in H^\circ$. Notice that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. Using $a = n/tp - b/p$, we compute

$$f_\Omega^\circ(x) = \frac{t^2}{n^2}(pa^2 + 2ab + b) = \frac{1}{p} + \frac{t^2}{n^2 p} b(p-b) \le \frac{1}{p} + \frac{t^2}{4n^2} p < \frac{1}{p} + \frac{1}{4p}$$

If b is even then $\bar{x} = [J_p^a, J_{b-1}, J_1] \in \mathrm{O}_{n/t}$. As above, we define $x = [\bar{x}, \ldots, \bar{x}] \in H^\circ$. We compute

$$f_\Omega^\circ(x) = \frac{t^2}{n^2}(pa^2 + 2ab + b + 2) = \frac{1}{p} + \frac{t^2}{n^2 p} b(p-b) + 2\frac{t^2}{n^2} \le \frac{5}{4p} + \frac{2}{p^2}$$

Case 2. Assume $n/t < p < t$. We define $h = \lfloor t/p \rfloor$. Observe that $\frac{t-p+1}{p} \le h \le \frac{t}{p}$. Then the G-class of one of the following $[J_p^{nh/t}, J_{n/t}^{t-hp}], [J_p^{nh/t}, J_{n/t-1}^{t-hp}, J_1^{t-hp}]$ meets H. Indeed, if n/t is odd then $[J_p^{nh/t}, J_{n/t}^{t-hp}]^G \cap H \neq \emptyset$, otherwise $[J_p^{nh/t}, J_{n/t-1}^{t-hp}, J_1^{t-hp}]^G \cap H \neq \emptyset$.

First assume n/t is odd. Let $x = [J_p^{nh/t}, J_{n/t}^{t-hp}]$. Since $p > n/t$ we have $x^G \cap H = x^G \cap H^\circ \pi_h$. We compute

$$f_\Omega(x) = \frac{t}{n} + \frac{h(n-pt)(2t-1-hp)}{nt(t-1)} < \frac{1}{p} + \frac{p}{n}$$

where the inequality follows by the same argument as in Proposition 19.2.11

Now assume n/t is even. Let $x = [J_p^{nh/t}, J_{n/t-1}^{t-hp}, J_1^{t-hp}]$. Again, $x^G \cap H = x^G \cap H^\circ \pi_h$. So

$$f_\Omega(x) = \frac{t}{n} + 2\frac{t^2}{n^2} + \frac{h(n-pt)(2t-1-hp)}{nt(t-1)} - \frac{2hpt(2t-1-hp)}{n^2(t-1)}$$

We see that $\frac{2hpt(2t-1-hp)}{n^2(t-1)}$ is minimal for h maximal, since $h \le t/p$ we have $\frac{2hpt(2t-1-hp)}{n^2(t-1)} \le 2\frac{t^2}{n^2}$. Therefore

$$f_\Omega(x) \le \frac{t}{n} + \frac{h(n-pt)(2t-1-hp)}{nt(t-1)} < \frac{1}{p} + \frac{p}{n}$$

The result follows. *q.e.d.*

Remark 20.2.15. Notice that Corollary 17.3.20 yields $f_\Omega(x) \le f_\Omega^\circ(x) + 4/n$. Therefore Proposition 20.2.14 shows that the lower bound $1/p$ is close to best possible.

20.3. Semisimple elements: lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H$ of prime order $r \neq p$. If x is an involution then, up to G conjugation, we have $x = [I_s, -I_{n-s}]$. For $x \in G$ of odd prime order, up to G-conjugacy, we write

$$x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \tag{255}$$

where $a_0 \equiv n \pmod 2$ and $a_i = a_{r-i}$ for all $0 < i \le \frac{r-1}{2}$. We shall also use the notation

$$x = [I_{a_0}, (\omega, \omega^{-1}) I_{a_1}, \ldots, (\omega, \omega^{-1})^{\frac{r-1}{2}} I_{a_{\frac{r-1}{2}}}] \tag{256}$$

The main result of this section is the following.

Proposition 20.3.1. *Let $x \in H$ be of order $r \neq p$.*

(i) Assume $r \geq n-1$.

(a) If n/t *is even then* $f_\Omega(x) = 0$ *if, and only if,* $\nu(x) = n-1$ *or* $C_G(x) \cong \mathrm{O}_2 \times \mathrm{GL}_1^{n/2-1}$.

(b) If n/t *is odd then*

$$f_\Omega(x) \geq \left(\frac{t}{n}\right)^2$$

with equality if, and only if $C_G(x) \cong \mathrm{O}_t \times \left(\mathrm{GL}_1\right)^{(n-t)/2}$.

(ii) If $r < n-1$ *then*

$$f_\Omega(x) \geq \frac{1}{r} - \epsilon$$

where $\epsilon = \frac{rt^2}{2n^2(t-1)}$ *if* $r > 2$ *and* $\epsilon = \frac{t^2}{2n^2(t-1)}$ *otherwise.*

Remark 20.3.2.

(i) In the case $r = 2$, more details will be provided in Section 20.7.

(ii) If $r < n-1$ and n/t is even the lower bound derived for the symplectic group in Proposition 19.3.16 for elements of odd prime order holds in this case as well, see Section 20.3.2.1.

First let us observe that for $r > n/2$ there are no elements $x \in H$ of order r such that $x^G \cap (H \setminus H^\circ) \neq \emptyset$. The proof of the following is the same as for Lemma 19.3.3.

Lemma 20.3.3. *Assume* $r \geq n/2$. *Let* $x \in G$ *of order* r. *Then*

(i) for $r \neq n/2$, $x^G \cap H = x^G \cap H^\circ$;

(ii) if $r = n/2$ *then* $x^G \cap (H \setminus H^\circ) \neq \emptyset$ *if, and only if,* $x = [I_2, \omega I_2, \ldots, \omega^{n/2-1} I_2]$ *and* $t = n/2$.

The main reason for which in Proposition 20.3.1 we have the two cases $r \geq n-1$ and $r < n-1$ is linked to the existence in H of regular elements, i.e. elements with centraliser isomorphic to $(\mathrm{GL}_1)^{n/2}$, or elements with centraliser $\mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}$.

Let $x \in G$ have order $r \neq p$. If $r < n-1$, in the case $x^G \cap H^\circ \neq \emptyset$, we shall study the related ratio $f_\Omega^\circ(x)$ deriving lower bounds on it (notice that this lower bound is the best possible). So thanks to the general inequality $f_\Omega(x) \geq f_\Omega^\circ(x)$, we immediately deduce lower bounds on $f_\Omega(x)$.

In the case $r < n-1$ we may have $x^G \cap H^\circ = \emptyset$; however this can only happen if n/t is odd. In this case we deduce a general lower bound which is not the best possible, see Proposition 20.3.9 and Remark 20.3.10.

Notice that for $x \in H$ of order r with $t < r < n-1$ we have $f_\Omega(x) = f_\Omega^\circ(x)$, by Lemmas 17.3.1 and 20.3.3.

20.3.1. Case $r \geq n-1$**.** We use the same argument given for GL_n in Section 18.3.1 and for Sp_n in Section 19.3.1. Let $x \in H$ of prime order r. Then Lemma 17.3.1 yields $x^G \cap H = x^G \cap H^\circ$. In particular, $\dim(x^G \cap H) = \dim x^{H^\circ}$, for a suitable block decomposition $x = [x_1, \ldots, x_t] \in H^\circ$. For all $1 \leq i \leq t$, we denote

$$x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \ldots, \omega^{r-1} I_{a_{i,r-1}}] \tag{257}$$

with $a_{i,0} \equiv n/t \pmod 2$ and $a_{i,j} = a_{i,r-j}$ for all $1 \leq j \leq \frac{r-1}{2}$. Moreover $\sum_i a_{i,j} = a_j$ and $\sum_j a_{i,j} = n/t$. As for the symplectic case, cf. Section 19.3.1, we deduce

$$\dim C_\Omega(x) = \sum_{l=0}^{r-1} \sum_{1\leq i<j\leq t} a_{i,l}a_{j,l} \tag{258}$$

The proof of the following is totally similar to that of Proposition 19.3.4. Notice that in Proposition 19.3.4 the main tool is the fact that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ if, and only if, $|a_{i,j} - a_{i',j}| \leq 1$ (or 2 in the case $j = 0$) for all i, i', j, see Proposition 17.3.13.

Proposition 20.3.4. *Let $x \in H$ of order $r \geq n-1$. Then $f_\Omega(x) = 0$ if, and only if, $\nu(x) = n-1$ or $C_G(x) \cong \mathrm{O}_2 \times (\mathrm{GL}_1)^{n/2-1}$.*

Now assume n/t is odd. Let $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H^\circ$. As already observed, there exists a block decomposition $x = [x_1, \dots, x_t] \in H^\circ$ such that $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. We write each x_i as (257). Since n/t is odd, $a_{i,0} \geq 1$ is odd for all $1 \leq i \leq t$. Therefore

$$\sum_{1\leq i<j\leq t} a_{i,0}a_{j,0} \geq \sum_{1\leq i<j\leq t} 1 = \frac{t(t-1)}{2} \tag{259}$$

This observation leads to the following.

Proposition 20.3.5. *Assume n/t is odd and $r \geq n-1$. Let $x \in H$ be of order r. Then*

$$f_\Omega(x) \geq \left(\frac{t}{n}\right)^2$$

with equality if, and only if, $C_G(x) \cong \mathrm{O}_t \times (\mathrm{GL}_1)^{(n-t)/2}$.

Proof. As observed above $x^G \cap H = x^G \cap H^\circ$ hence $a_0 \geq t$. Let $x = [x_1, \dots, x_t]$ where $x_i \in \mathrm{O}_{n/t}$. Assume $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. In particular, by Proposition 17.3.13, $|a_{i,j} - a_{i',j}| \leq 1$ for all $1 \leq i, i' \leq t$ and $1 \leq j \leq r-1$. With this notation, we have an explicit formula for $\dim C_\Omega(x)$, given in (258). Hence, using (259),

$$\dim C_\Omega(x) = \sum_{1\leq i<j\leq t} a_{i,0}a_{j,0} + \sum_{l=1}^{r-1} \sum_{1\leq i<j\leq t} a_{i,l}a_{j,l} \geq \frac{t(t-1)}{2} \tag{260}$$

Notice that in (260) equality holds if, and only if, $a_{i,0} = 1$ for all i, i.e. $a_0 = t$, and $a_{i,l}a_{j,l} = 0$ for all $l \geq 1$ and $i < j$. The result follows. *q.e.d.*

20.3.2. Case $r < n-1$. Here we shall derive lower bound on $f_\Omega(x)$ for $x \in H$ of prime order $r < n-1$.

In the case where n/t is even, H contains a maximal torus. Hence, for any semisimple elements $x \in G$ of prime order r, $x^G \cap H^\circ \neq \emptyset$. In the case where n/t is odd there are elements x of odd prime order r such that $x^G \cap H = x^G \cap (H \setminus H^\circ)$. For example, let $G = \mathrm{O}_{18}, H = \mathrm{O}_3 \wr S_6, r = 5 \neq p$, and consider

$$x = [I_4, \omega I_4, \omega^2 I_3, \omega^3 I_3, \omega^4 I_4] \in \mathrm{O}_{18} \tag{261}$$

By Lemmas 17.4.2 and 17.4.6 we have, up to G-conjugacy, $x = [I_3, I_3, I_3, I_3, I_3, z]\pi_1 \in H$, where $z = [1, \omega, \omega^4]$. We see that $x^G \cap H^\circ = \emptyset$: in fact any decomposition $x = [x_1, \dots, x_6]$ should comprise blocks $x_i \in \mathrm{O}_3$ containing at least one eigenvalue 1. We shall give a complete characterization of elements $x \in H$ of odd prime order r for which $x^G \cap H^\circ = \emptyset$ in Corollary 20.3.19.

However, in most cases elements of prime order r lie in H°.

Remark 20.3.6. If $r > t$ and $x \in H$ has order r then, by Lemma 17.3.1 we have $x^G \cap H = x^G \cap H^\circ$; similarly, if $n/2 \leq r < n-1$. Therefore, in these cases, $f_\Omega(x) = f^\circ_\Omega(x)$.

We can immediately prove the following result, which is the analogue of Lemma 19.3.5. In view of the above observation, i.e. the existence of elements of prime order r whose G-class does not meet H°, we need the assumption $x \in H^\circ$.

Lemma 20.3.7. *Let $x \in H^\circ$ be of order $r < n-1$. Then*

$$f^\circ_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{4n^2(t-1)} - \frac{2}{n}$$

Proof. Up to conjugation, write $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H$ of order r. Write $a_i = c_i t + b_i$ for $i > 0$ and a_0 as in Remark 17.3.14. We use the formulae of $\dim(x^G \cap H^\circ)$ as in (173) and (174), see Remark 17.3.14. We follow the same argument as for Lemma 19.3.5.

Case 1. First assume n/t is even. Let $a_0/2 = c_0 t + b_0$ where $0 \leq b_0 < t$. We have

$$\begin{aligned} f^\circ_\Omega(x) &= \frac{\sum_{i\geq 0} a_i^2}{n^2} - \frac{4b_0(t-b_0) + \sum_{i>0} b_i(t-b_i)}{n^2(t-1)} \\ &\geq \frac{1}{r} - \frac{rt^2}{4n^2(t-1)} - \frac{3t^2}{4n^2(t-1)} \end{aligned}$$

Then since $g(t) = \frac{t^2}{t-1} \leq g(n/2)$ we deduce the result.

Case 2. Now assume n/t is odd. Using the formula for $\dim(x^G \cap H^\circ)$ given in (174),

$$f^\circ_\Omega(x) = \frac{\sum_{i\geq 0} a_i^2}{n^2} - \frac{t^2 + 4b_0(t-b_0)}{n^2(t-1)} - \frac{\sum_{i>0} b_i(t-b_i)}{n^2(t-1)} \geq \frac{1}{r} - \frac{(r+7)t^2}{4n^2(t-1)}$$

Notice that $\frac{t^2}{t-1}$ is maximal for t maximal, i.e. $t = n/2$. The result follows. *q.e.d.*

Notice that if $x \in G$ is an involution (in the case $p \neq 2$) then $x^G \cap H^\circ \neq \emptyset$.

Lemma 20.3.8. *Let $x \in G$ be an involution. Then*

$$f^\circ_\Omega(x) \geq \frac{1}{2} - \frac{t^2}{2n^2(t-1)}$$

Proof. Let $x \in G$ be an involution. Up to G-conjugacy, we may write, $x = [I_s, -I_{n-s}]$ where $s \leq n/2$. We use the same argument as for Lemma 19.3.6.

We have $\dim x^G = s(n-s)$ and, thanks to Theorem 17.3.8, we have an explicit formula for $\dim(x^G \cap H^\circ)$. Write $s = ct + b$ where $0 \leq b < t$. Then

$$f^\circ_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)} \geq \frac{1}{2} - \frac{t^2}{2n^2(t-1)}$$

where we used $b(t-b) \leq t^2/4$ and $s(n-s) \leq n^2/4$. *q.e.d.*

Also here, as for the symplectic case, we have that the lower bound for involutions in Lemma 20.3.8 is always positive. Instead in the case $r > 2$ we directly check (using GAP, for example) that in several cases the lower bound given in Lemma 20.3.7 is negative. However, an easy calculation shows that for $r \leq n/t$ the previous bounds are positive.

The next result gives close to the best possible lower bound for any element $x \in H$ of odd prime order r (even in the case $x^G \cap H^\circ = \emptyset$). In order to state and prove it we need the following notation. Let $x \in H$ be a semisimple element, in the notation of (255). Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$ for some $h \leq t/r$. Therefore, by Proposition 17.4.8, the eigenvalue ω^i of x has multiplicity $\frac{n}{t}h + \alpha_i$ for all $0 \leq i \leq r-1$, where $\sum_i \alpha_i = n - \frac{n}{t}hr$. Write $\alpha_i = c_i(t-hr) + b_i$ where $0 \leq b_i < t - hr$.

Proposition 20.3.9. *Let $x \in H$ be of odd order r. Assume* $\dim(x^G \cap H) = \dim(x^G \cap H^\circ \pi_h)$ *for some* $0 \leq h \leq t/r$. *Then*

$$f_\Omega(x) \geq \frac{1}{r} - \frac{2t}{n^2(t-1)} \sum_{i=0}^{r-1} b_i \Big(1 - \frac{b_i}{t-hr}\Big)$$

PROOF. Thanks to Proposition 17.4.8, we have, up to G-conjugacy,

$$x = \Big[I_{\frac{n}{t}h+\alpha_0}, \omega I_{\frac{n}{t}h+\alpha_1}, \ldots, \omega^{r-1} I_{\frac{n}{t}h+\alpha_{r-1}}\Big]$$

with $(\frac{n}{t}h + \alpha_0) \equiv n \pmod 2$ and $\alpha_i = \alpha_{r-i}$ for all $1 \leq i \leq \frac{r-1}{2}$. Lemma 17.4.2 implies

(262) $$\dim(x^G \cap H) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{hr+1}, \ldots, x_t]\pi_h)^{H^\circ}$$

for suitable blocks $x_i \in \mathrm{O}_{n/t}$. We write

$$x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \ldots, \omega^{r-1} I_{a_{i,r-1}}]$$

Thanks to Proposition 17.3.13 we have $|a_{i,j} - a_{i',j}| \leq 1$ and $|a_{i,0} - a_{i',0}| \leq 2$ for all $hr + 1 \leq i, i' \leq t$ and $j > 0$.

We compute

$$\dim x^G = \frac{n}{2}(n-1) - \frac{r}{2}\Big(\frac{n}{t}h\Big)^2 - \frac{n}{t}h\Big(n - \frac{n}{t}hr\Big) + \frac{n}{2t}h + \frac{\alpha_0}{2} - \frac{1}{2}\sum_{j=0}^{r-1} \alpha_j^2$$

$$\dim(x^G \cap H) = \frac{n}{2t}\Big(\frac{n}{t} - 1\Big)(t-h) + \frac{\alpha_0}{2} - \frac{1}{2}\sum_{i=hr+1}^{t}\sum_{j=0}^{r-1} a_{i,j}^2$$

Therefore, we deduce

(263) $$\dim C_\Omega(x) = \frac{n^2}{2t^2}h(t-hr-1) + \frac{n^2}{2t}h + \sum_{hr+1 \leq l < m \leq t}\sum_{i=0}^{r-1} a_{l,i}a_{m,i}$$

Let $\mathcal{I}_x = \{(l,m) : hr+1 \leq l < m \leq t\}$. If $h = t/r$ or $(t-1)/r$ then $\mathcal{I}_x = \emptyset$ and a straightforward computation leads to

$$f_\Omega(x) = \frac{1}{r}$$

Assume $h < (t-1)/r$ then $\mathcal{I}_x \neq \emptyset$. Notice that $|\mathcal{I}_x| = \frac{(t-hr)(t-hr-1)}{2}$. Now, Lemma 17.3.12 implies that for all $hr+1 \leq l \leq t$ and $0 \leq i \leq r-1$ we have $a_{l,i} \in \{\lfloor \frac{\alpha_i}{t-hr} \rfloor, \lfloor \frac{\alpha_i}{t-hr} \rfloor + 1\}$. Write $\alpha_i = c_i(t-hr) + b_i$ where $c_i = \lfloor \frac{\alpha_i}{t-hr} \rfloor$ and $0 \leq b_j < t - hr$.

Then $|\{l : a_{l,i} = c_i + 1\}| = b_i$ and $|\{l : a_{l,i} = c_i\}| = t - hr - b_i$. We compute

$$\begin{aligned}
\sum_{(l,m)\in\mathcal{I}_x} \sum_{i=0}^{r-1} a_{l,i}a_{m,i} &= \sum_{i=0}^{r-1} c_i^2 \frac{(t-hr-b_i)(t-hr-b_i-1)}{2} \\
&+ \sum_{i=0}^{r-1}\Big(c_i(c_i+1)b_i(t-hr-b_i) + (c_i+1)^2\frac{b_i(b_i-1)}{2}\Big) \\
&= \frac{1}{2}\Big(1-\frac{1}{t-hr}\Big)\sum_{i=0}^{r-1} a_i^2 - \frac{1}{2}\sum_{i=0}^{r-1} b_i\Big(1-\frac{b_i}{t-hr}\Big) \\
&\geq \Big(\frac{n}{t}\Big)^2\frac{(t-hr)(t-hr-1)}{2r} - \frac{1}{2}\sum_{i=0}^{r-1} b_i\Big(1-\frac{b_i}{t-hr}\Big)
\end{aligned}$$

where we used $\sum_i \alpha_i = (n/t)(t-hr)$ and $\sum_i \alpha_i^2 \geq 1/r\big(\sum_i \alpha_i\big)^2$, by Proposition B.2.1. Therefore, using the previous estimation and (263) we deduce

$$\dim C_\Omega(x) \geq \frac{n^2(t-1)}{2rt} - \sum_{i=0}^{r-1} b_i\Big(1-\frac{b_i}{t-hr}\Big) \tag{264}$$

The result follows. *q.e.d.*

Remark 20.3.10. Notice that the right hand side of (264) is minimal when $h = 0$ and $b_i = t/2$, for all i. Therefore

$$f_\Omega(x) \geq \frac{1}{r} - \frac{rt^2}{2n^2(t-1)}$$

which is slightly better (if $r \leq 7$) than the bound given in Lemma 20.3.7 for elements in H°. Again, a GAP computation shows that this is often negative.

In the following we give an infinite class of examples of elements with the property that the G-conjugacy class does not meet H° and have f_Ω-value equal to $1/r$.

Example 20.3.11. Assume $p \neq 2$. Let n, t such that n/t is odd, t is prime and $n/t < t$. Let $x = [I_{n/t}, \omega I_{n/t}, \ldots, \omega^{r-1}I_{n/t}]$ of prime order $r = t$. Then $x^G \cap H^\circ = \emptyset$ since we need the multiplicity of 1 to be at least t (see also Corollary 20.3.19) and $x^G \cap H = x^G \cap H^\circ \pi_1$ since, by Proposition 17.4.8, $\pi_1 \in x^G$ and $\pi_i \notin x^G$ for $i > 1$. Here $h = 1$ so $t - hr = 0$ and the proof of Proposition 20.3.9 implies $f_\Omega(x) = 1/r$.

In the case where n/t is even, with the same argument as already used for GL_n and Sp_n we can construct an explicit family of elements in H° with the property that each of them realises the best possible lower bound on f_Ω°, and hence on f_Ω.

If n/t is odd, with a similar argument we can reach a similar conclusion, for elements in H°. However, in some cases there exist elements x of prime order r such that $x^G \cap H^\circ = \emptyset$ and $x^G \cap H \neq \emptyset$.

20.3.2.1. *n/t even.* The strategy is to show that the lower bounds derived in the symplectic case are still valid. Let $x \in G$ be of prime order r. Then, up to G-conjugacy, $x = [I_{a_0}, \omega I_{a_{r-1}}, \ldots, \omega^{r-1}I_{a_{r-1}}]$. In the case where n/t is even, $x^G \cap H^\circ \neq \emptyset$, since H contains a maximal torus of G. Notice that (as a diagonal matrix) $x \in \mathrm{Sp}_n$ and $x^{\mathrm{Sp}_n} \cap K^\circ \neq \emptyset$, for $K = \mathrm{Sp}_{n/t} \wr S_t$ a $\mathscr{C}_2$-subgroup of Sp_n. Using Theorem 5.3.1 and

Theorem 17.3.8 it is straightforward to compute

$$\dim x^{\mathrm{Sp}_n} - \dim x^G = n - a_0$$
$$\dim(x^{\mathrm{Sp}_n} \cap K^\circ) - \dim(x^G \cap H^\circ) = n - a_0$$

In particular, if $\Omega' = \mathrm{Sp}_n/K$, then $f^\circ_\Omega(x) = f^\circ_{\Omega'}(x)$. Hence the argument given in Section 19.3.2 holds for $G = \mathrm{O}_n$ and $H = \mathrm{O}_{n/t} \wr S_t$ with n/t even. For future reference we record this fact in the following.

Proposition 20.3.12. *Let n be an even integer and t a non-trivial divisor of n such that n/t is even. Denote $L = \mathrm{Sp}_n, K = \mathrm{Sp}_{n/t} \wr S_t$ a $\mathscr{C}_2$-subgroup of L and set $\Omega' = L/K$. Let x be a semisimple element of prime order r in O_n (or Sp_n). Then*

(i) there exists a G-conjugate y of x in $H \cap K$;
(ii) $f^\circ_\Omega(y) = f^\circ_{\Omega'}(y)$.

Remark 20.3.13. It is clear that Proposition 20.3.12 implies that the lower bounds proved in the symplectic case in Section 19.3.2 holds in this case as well.

As for the symplectic group also here we define special elements. Recall that if $x \in G$ has prime order r then we denote by $a_0, a_1, \ldots, a_{r-1}$ the dimensions of the eigenspaces relative to the eigenvalues $1, \omega, \ldots, \omega^{r-1}$.

Definition 20.3.14. Let $x \in G$ be of order r. We say that x is *special* if $|a_i - a_j| \leq 1$ for all $0 \leq i, j \leq r-1$.

Thanks to Proposition 20.3.12, the following may be deduced from the arguments of Section 19.3.2.

Proposition 20.3.15. *Let $x \in G$ be of order r. Then*

$$f_\Omega(x) \geq f_\Omega(z)$$

where $z \in G$ is special of order r.

The computation in Section 19.3.2 after Proposition 19.3.14 holds. Thus we have the following.

Proposition 20.3.16. *Let $x \in G$ be of order r. Write $n = (at+b)r + c$, where $0 \leq b < t$ and $0 \leq c < r$. Then*

$$f^\circ_\Omega(x) \geq \frac{1}{r} - \frac{br(t-b) - 2bc}{n^2(t-1)} - \frac{c^2}{n^2 r} - \frac{2t}{n^2}$$

20.3.2.2. *n/t odd.* In the case where n/t is odd H does not contain a maximal torus. Thus we may have semisimple elements $x \in H$ of odd prime order r such that $x^G \cap H^\circ \neq \emptyset$.

First we characterise elements $x \in G$ of prime order r such that $x^G \cap H \neq \emptyset$ or $x^G \cap H^\circ \neq \emptyset$. Recall that r is an odd prime.

Lemma 20.3.17. *Let $x \in H$ be of order r. Then $x^G \cap H^\circ = \emptyset$ if, and only if, either $a_0 < t$ or, $a_0 \geq t$ and $a_0 - t$ odd.*

Proof. Up to G-conjugation, we may assume x is as in (255). Therefore we need to show that $x \notin H^\circ$ if, and only if, the condition in the statement holds. We prove the contrapositive in both cases.

Assume $x \notin H^\circ$ and $a_0 \geq t$. Seeking a contradiction we assume $a_0 - t$ even. Write $(a_0 - t)/2 = c_0 t + b_0$ where $0 \leq b_0 < t$ and $c_0 = \lfloor \frac{a_0 - t}{2t} \rfloor$. In addition, for all $1 \leq j \leq r-1$, write $a_j = c_j t + b_j$ where $0 \leq b_j < t$ and $c_j = \lfloor a_j / t \rfloor$. Then, for $1 \leq i \leq t$, define

$$x_i = [I_{a_{i,0}}, \omega I_{a_{i,1}}, \ldots, \omega^{r-1} I_{a_{i,r-1}}]$$

where $a_{i,0} \in \{c_0, c_0 + 2\}$ and $a_{i,j} \in \{c_j, c_j + 1\}$ and the $a_{i,j}$ are such that $\sum_i a_{i,j} = a_j$ and $\sum_j a_{i,j} = n/t$. Then $x_i \in \mathrm{O}_{n/t}$ for all i. Hence $x = [x_1, \ldots, x_t] \in H^\circ$ which is a contradiction. Therefore $a_0 - t$ is odd.

Conversely, up to G-conjugation, assume $x = [x_1, \ldots, x_t] \in H^\circ$. In particular, since n/t is odd and $x_i \in \mathrm{O}_{n/t}$ we deduce, in the above notations, $a_{i,0} \geq 1$ is odd.

We have $a_0 = \sum_i a_{i,0} \geq \sum_i 1 = t$. For all $1 \leq i \leq t$ we have that $a_{i,0} - 1$ is even. Therefore $\sum_i (a_{i,0} - 1) = a_0 - t$ is even, which is absurd. Therefore $x \notin H^\circ$. *q.e.d.*

Now we characterise elements in G of order r whose G-classes meet H. We adopt the standard notation established in (255), i.e. a_i denotes the dimension of the eigenspace of x relative to the eigenvalue ω^i. We define the following set

$$\mathcal{J}_x = \left\{ i \in \{0, \ldots, \lfloor t/r \rfloor\} : a_0 = \frac{n}{t} i + b_0 \text{ and } b_0 \geq t - ir \right\}$$

Proposition 20.3.18. *Let $x \in G$ be of order r. Assume n/t is odd Then $x^G \cap H \neq \emptyset$ if, and only if, $\mathcal{J}_x \neq \emptyset$ and $a_i \geq \frac{n}{t} \min \mathcal{J}_x$ for all $1 \leq i \leq r-1$.*

Proof. Assume $x^G \cap H \neq \emptyset$. Then there exists $h \leq t/r$ such that $x^G \cap H^\circ \pi_h \neq \emptyset$. Hence, thanks to Lemma 17.4.2, up to H°-conjugation, we may write

$$x = [I_{n/t}, \ldots, I_{n/t}, x_{hr+1}, \ldots, x_t]\pi_h$$

We denote by $b_{i,j}$ the multiplicity of ω^j in x_i. Therefore, since $x_i \in \mathrm{O}_{n/t}$ and n/t is odd we have $b_{i,0} \geq 1$ for all $hr + 1 \leq i \leq t$. Thus $b_0 = \sum_{i > hr} b_{i,0} \geq t - hr$. Then, by Proposition 17.4.8 we have that x is G-conjugate to

$$x = \left[I_{\frac{n}{t}h + b_0}, \omega I_{\frac{n}{t}h + b_1}, \ldots, \omega^{r-1} I_{\frac{n}{t}h + b_{r-1}} \right] \tag{265}$$

where $b_j = \sum_{i > hr} b_{i,j}$. In particular, $h \in \mathcal{J}_x$ and the multiplicity of ω^j is at least $\frac{n}{t}h$ for all $1 \leq j \leq r-1$.

Conversely, say $h = \min \mathcal{J}_x$. Then, for some $b_0, \ldots, b_{r-1}$ we have, up to G-conjugacy,

$$x = \left[I_{\frac{n}{t}h + b_0}, \omega I_{\frac{n}{t}h + b_1}, \ldots, \omega^{r-1} I_{\frac{n}{t}h + b_{r-1}} \right]$$

Since $b_0 \geq t - hr$, by Lemma 20.3.17, there exists a block decomposition $[x_{hr+1}, \ldots, x_t]$ of $[I_{b_0}, \omega I_{b_1}, \ldots, \omega^{r-1} I_{b_{r-1}}]$. Thus, by Lemmas 17.4.2 and 17.4.6, up to G-conjugation, $x = [I_{n/t}, \ldots, I_{n/t}, x_{hr+1}, \ldots, x_t]\pi_h$. Hence $x^G \cap H \neq \emptyset$. *q.e.d.*

The following is a particular case of Proposition 20.3.18.

Corollary 20.3.19. *Let $x \in G$ be of order r. Then $x^G \cap H^\circ \neq \emptyset$ if, and only if, $a_0 \geq t$.*

In fact we can give infinitely many examples of triples (n, t, r) for which there exists $x \in \mathrm{O}_{n/t} \wr S_t$ such that x has order r and $x \notin (\mathrm{O}_{n/t})^t$. See Example 20.3.11.

Notice that Lemma 19.3.13 holds in this case as well. So if $x \in G$ is special then $a_i \in \{\lfloor n/r \rfloor, \lfloor n/r \rfloor + 1\}$ for all $0 \leq i \leq r-1$. In particular, thanks to Corollary 20.3.19, we have the following.

Lemma 20.3.20. *Let $x \in G$ be a special element. Then $x^G \cap H^\circ \neq \emptyset$ if, and only if, one of the following holds*

(i) n is even $\lfloor n/r \rfloor + \delta_{\lfloor n/r \rfloor - 1;2} \geq t$;
(ii) n is odd $\lfloor n/r \rfloor + \delta_{\lfloor n/r \rfloor;2} \geq t$.

Assume special elements lie in H°. Then with the usual argument we can show that given $x \in H^\circ$ of order r then $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$ for $z \in H^\circ$ a special element (of order r).

Let $x = [I_{a_0}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}] \in H^\circ$ be not special. In particular we may assume that one of the following holds: $a_1 - a_0 \geq 2$, $a_0 - a_1 \geq 2$ or $a_1 - a_2 \geq 2$. In each of these three cases we define an element y as follows.

Case 1. Assume $a_1 - a_0 \geq 2$. Define

$$y = [I_{a_0+2}, \omega I_{a_1-1}, \omega I_{a_2}, \ldots] \tag{266}$$

Case 2. Assume $a_0 - a_1 \geq 2$. Define

$$y = [I_{a_0-2}, \omega I_{a_1+1}, \omega I_{a_2}, \ldots] \tag{267}$$

Case 3. Assume $a_1 - a_2 \geq 2$. Define

$$y = [I_{a_0}, \omega I_{a_1-1}, \omega I_{a_2+1}, \ldots] \tag{268}$$

Notice that if $x \in H^\circ$ then y as in (266) and (268) lies in H°, thanks to Corollary 20.3.19. It is not true, in general, that y as in (267) lies in H° (for example in the case $a_0 = t$).

With the above notation we prove the following technical lemma which is the key tool in the proof of our claim that special elements realise the best possible lower bound on f°_Ω.

Lemma 20.3.21. *Let $x \in H^\circ$ ne not special. Define y as in (266), (267) or (268) and assume $y \in H^\circ$. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$$

Proof. The usual argument applies. For all the details see Appendix B.4.5 *q.e.d.*

With the usual argument, see for example Lemma 18.3.14, we deduce.

Proposition 20.3.22. *Let $x \in H^\circ$ be an element of prime order r. Then*

$$f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$$

where $z \in H^\circ$ is special.

The following is the same of Proposition 19.3.14.

Proposition 20.3.23. *Write $n = \lfloor n/r \rfloor r + c$. Any special element of prime order r in G has centraliser isomorphic to*

(269)
$$z = \left[I_{\lfloor \frac{n}{r} \rfloor + \delta}, \omega I_{\lfloor \frac{n}{r} \rfloor + 1}, \omega^{-1} I_{\lfloor \frac{n}{r} \rfloor + 1}, \ldots, \omega^{-\lfloor c/2 \rfloor} I_{\lfloor \frac{n}{r} \rfloor + 1}, \omega^{\lfloor c/2 \rfloor + 1} I_{\lfloor \frac{n}{r} \rfloor}, \ldots, \omega^{-\frac{r-1}{2}} I_{\lfloor \frac{n}{r} \rfloor} \right]$$

where $\delta = 0$ for c is even, otherwise $\delta = 1$.

In the case $z \in H^\circ$ we can compute $f^\circ_\Omega(z)$ and, thanks to Proposition 20.3.22, this is the best possible lower bound on $f_\Omega(x)$ for $x \in G$ such that $x^G \cap H^\circ \neq \emptyset$.

20.4. Local upper bounds

Recall that we denote $\mathcal{V}_s = \{x \in G : \nu(x) = s\}$ and we write $\mathcal{V}_{s,r}$ for the set of elements of $\mathcal{V}_s$ of order r. Assume r is an odd prime. In this section we derive upper bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$. We shall study involutions in Section 20.7.

The main result of this section is the following.

Proposition 20.4.1. *Assume r is an odd prime. Let $x \in H \cap \mathcal{V}_{s,r}$. If $s < n/2$ then*
$$f_\Omega(x) \leq 1 - \frac{2s(n-s)}{n^2} + \frac{2s}{n^2}$$
If $s \geq n/2$ then
$$f_\Omega(x) \leq 1 - \frac{s-1}{n}$$

Proof. Let $x \in \mathcal{V}_{s,r}$. Then, by Proposition 5.4.1, we have $\dim x^G \geq \max\{s(n-s-1), n(s-1)/2\}$. Using Proposition 17.2.1 we have
$$f_\Omega(x) \leq 1 - \frac{2 \dim x^G}{n^2}$$

The result quickly follows from observing that $\max\{s(n-s-1), n(s-1)/2\} = s(n-s-1)$ if $s < n/2$ and $\max\{s(n-s-1), n(s-1)/2\} = n(s-1)/2$ if $s \geq n/2$. *q.e.d.*

Corollary 20.4.2. *Let $x \in H \cap \mathcal{V}_{s,r}$. Then*
$$f_\Omega(x) \leq 1 - \frac{s}{n+1} + \frac{1}{n}$$

Proof. The result quickly follows with an easy computation using the bounds given in Proposition 20.4.1. *q.e.d.*

20.4.1. Unipotent elements.

Let $x \in H \cap \mathcal{V}_{s,p}$. Then, up to G-conjugacy, $x = [J_p^{a_p}, \ldots, J_1^{a_1}]$ and $s = n - \sum_i a_i$.

With some restriction on s we can construct an element $x \in H \cap \mathcal{V}_{s,p}$ whose f_Ω-value is close to the upper bound of Proposition 20.4.1.

Proposition 20.4.3. *Assume $s \leq n/2$ is even. Then there exists $x \in H \cap \mathcal{V}_{s,p}$ such that*
$$f_\Omega(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{2t^2}{n^2(t-1)}$$

Proof. Let $x = [J_2^s, J_1^{n-2s}]$. Then $x \in G$ (since s is even). Using Theorem 5.2.1 we compute $\dim x^G = s(n-s-1)$. Write $s/2 = at + b$ where $0 \leq b < t$. Let $x = [x_1, \ldots, x_t]$ where $x_1 = \ldots = x_b = [J_2^{2a+2}, J_1^{n/t-4a-4}] \in \mathrm{O}_{n/t}$ and $x_{b+1} = \ldots = x_t = [J_2^{2a}, J_1^{n/t-4a}] \in$

$O_{n/t}$. Then $x \in H^\circ$ and $\dim(x^G \cap H) \geq \dim(x^G \cap H^\circ) \geq \dim x^{H^\circ}$. We compute

$$\dim x^{H^\circ} = b \dim x_1^{O_{n/t}} + (t-b) \dim x_t^{O_{n/t}} = \frac{s}{t}(n-s) - s - 4\frac{b}{t}(t-b)$$

Therefore

$$f_\Omega(x) \geq 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

The result quickly follows from noticing that $b(t-b) \leq t^2/4$. *q.e.d.*

20.4.2. Semisimple elements. Let $x \in \mathcal{V}_{s,r}$. If $s < n/2$ then the largest eigenspace must be the 1-eigenspace, since $a_i = a_{r-i}$ for $1 \leq i \leq \frac{r-1}{2}$. Therefore, up to G-conjugacy, we write

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$

with $a_i \leq n-s$. In the case $s \geq n/2$ then, up to G-conjugacy (and up to centraliser structure), one of the following two possibilities occurs

$$x = [I_{n-s}, \omega I_{a_1}, \ldots, \omega^{r-1} I_{a_{r-1}}]$$
$$x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \ldots, \omega^{r-2} I_{a_{r-2}}, \omega^{r-1} I_{n-s}]$$

with $a_i \leq n-s$ for all i and $a_0 \equiv n \pmod 2$.

In the case where n/t is even, as already observed, if $x \in G$ has prime order r then $x^G \cap H^\circ \neq \emptyset$. Hence we can make the same construction as for the symplectic case. In particular we can construct a family of elements of $H \cap \mathcal{V}_{s,r}$ with the property that some of them realise the best possible upper bound on f_Ω°.

20.4.2.1. *n/t even.* In the case where n/t is even thanks to Proposition 20.3.12 all the arguments given in Section 19.4.2 apply. In particular we have the following.

Proposition 20.4.4. *Let $x \in H \cap \mathcal{V}_{s,r}$. Write $n = (n-s)l + m$ where $0 \leq m < n-s$. Then $f_\Omega^\circ(x) \leq f_\Omega^\circ(\bar{x})$ where $\bar{x}$ is one of the following*

(270)

$$[I_{n-s}, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{\frac{l-1}{2}} I_{n-s}, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{m/2}] \quad l \text{ odd}$$
$$[I_{n-s}, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{\frac{l}{2}-1} I_{n-s}, (\omega,\omega^{-1})^{\frac{l}{2}} I_{(n-s)/2}, (\omega,\omega^{-1})^{\frac{l}{2}+1} I_{m/2}] \quad l \text{ even}$$
$$[I_m, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{\frac{l}{2}} I_{n-s}] \quad l \text{ odd}$$
$$[I_{m+1}, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{\frac{l-1}{2}} I_{n-s}] \quad l \text{ even}$$
$$[I_{n-s-1}, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{\frac{l-1}{2}} I_{n-s}, (\omega,\omega^{-1})^{\frac{l+1}{2}} I_{(m+1)/2}] \quad l \text{ odd}$$
$$[I_{n-s-1}, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{\frac{l}{2}-1} I_{n-s}, (\omega,\omega^{-1})^{\frac{l}{2}} I_{(n-s+m+1)/2}] \quad l \text{ even}$$

20.5. Local lower bounds

In this section we derive lower bounds on $f_\Omega(x)$ for $x \in H \cap \mathcal{V}_{s,r}$ where $r \neq p$ is an odd prime. Recall, by Proposition 20.3.4, if $\nu(x) = n-1$ then $f_\Omega(x) = 0$ and there exists x with $\nu(x) = 2$ such that $f_\Omega(x) = 0$. Moreover if $n = 4$ then $f_\Omega([I_2, \omega, \omega^{-1}]) = 0$ and $f_\Omega([\omega I_2, \omega^{-1} I_2]) = 1/2$. Therefore we may assume $\nu(x) < n-2$ and $n > 4$. Notice, moreover, that $\nu(x) > 1$ for all semisimple elements $x \in G$.

We have the following.

Proposition 20.5.1. *Let $x \in H^\circ \cap \mathcal{V}_{s,r}$. Then*

$$f^\circ_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{s(2n-s-1)}{n^3\big(1-\frac{1}{t}\big)} - \frac{1}{n} + \frac{s-2}{n^2}$$

Proof. Let $x \in H \cap \mathcal{V}_{s,r}$. Up to conjugation, write $x = [I_{a_0}, \dots, \omega^{r-1} I_{a_{r-1}}]$ where $a_i \leq n - s$ for all i. Moreover $n - a_0 \leq n$.

Proposition 5.4.1 implies $\dim x^G \leq \frac{1}{2}(2ns - s^2 - 2s + 1) < \frac{s}{2}(2n - s - 1)$. In addition, by Proposition 17.3.17, we have

$$\dim(x^G \cap H^\circ) \geq \Big(\frac{1}{t} - \frac{1}{n}\Big) \dim x^G - \frac{n-a_0}{2}\Big(1 - \frac{1}{t} + \frac{1}{n}\Big)$$

Therefore

$$\begin{aligned} f^\circ_\Omega(x) &\geq \frac{\dim\Omega - \dim x^G\big(1 - \frac{1}{t} + \frac{1}{n}\big)}{\dim\Omega} - \frac{n-a_0}{2\dim\Omega}\Big(1 - \frac{1}{t} + \frac{1}{n}\Big) \\ &\geq \frac{\dim\Omega - \frac{s}{2}(2n-s-1)\big(1 - \frac{1}{t} + \frac{1}{n}\big)}{\dim\Omega} - \frac{1}{n} - \frac{t}{n^2(t-1)} \\ &= 1 - \frac{s(2n-s)}{n^2} - \frac{s(2n-s-1)}{n^3\big(1-\frac{1}{t}\big)} - \frac{1}{n} + \frac{s}{n^2} - \frac{t}{n^2(t-1)} \end{aligned}$$

The result follows using $t/(t-1) \leq 2$. *q.e.d.*

By Proposition 20.3.12, in the case where n/t is even, the lower bound given for the symplectic case holds, see also Remark 20.3.13. Moreover we have a description of the semisimple elements $z \in G$ of prime order r and $z^G \cap H^\circ \cap \mathcal{V}_s \neq \emptyset$ such that for all $x \in H^\circ$ with $\nu(x) = s$ and order r we have $f^\circ_\Omega(x) \geq f^\circ_\Omega(z)$, see Definition 19.5.3.

Proposition 20.5.2. *Assume n/t is even. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{s(2n-s+1)}{n^3\big(1-\frac{1}{t}\big)}$$

It is convenient, for the purpose of the next section, to state the following simplified lower bound. For a proof we refer the reader to Corollary 19.5.2.

Corollary 20.5.3. *Assume n/t is even. Let $x \in H \cap \mathcal{V}_{s,r}$. Then*

$$f_\Omega(x) \geq 1 - \frac{s(2n-s)}{n^2} - \frac{4}{n}$$

20.6. Further comments on local bounds

The following is a consequence of Propositions 20.4.1 and 20.5.1.

Proposition 20.6.1. *Assume n/t is even and $r \neq p$ is odd. Let $x, y \in H^\circ \cap \mathcal{V}_{s,r}$. Then*

$$|f_\Omega(x) - f_\Omega(y)| < \frac{s(n-s)}{n^2} + \frac{4}{n}$$

Proof. Let $x, y \in H \cap \mathcal{V}_{s,r}$. Then $f_\Omega(x), f_\Omega(y) \leq U$ where U is given in Proposition 19.4.1, and $f_\Omega(x), f_\Omega(y) \geq \ell$ where ℓ is as in Corollary 20.5.3.

Assume $s > n/2$ then

$$\begin{aligned}|f_\Omega(x) - f_\Omega(y)| &\le 1 - \frac{s-1}{n} - \ell = \frac{s+2}{n} + \frac{ns(t+1) - st(s+1)}{n^3(t-1)} - \frac{s^2-2}{n^2}\\ &\le \frac{s(n-s)}{n^2} + \frac{2n(n+1) + 3ns - 2s(s+1)}{n^3}\\ &\le \frac{s(n-s)}{n^2} + \frac{25}{8n} + \frac{1}{2n^2} + \frac{1}{2n^2} < \frac{s(n-s)}{n^2} + \frac{4}{n}\end{aligned}$$

where the second inequality follows from the observation that the first line is maximal when t is minimal, i.e. $t = 2$. The second inequality follows from the fact that

$$g(s) = \frac{2n(n+1) + 3ns - 2s(s+1)}{n^3} \le g\Big(\frac{3n-2}{4}\Big)$$

If $s \le n/2$ then, as in the previous case, we get

$$|f_\Omega(x) - f_\Omega(y)| \le 1 - \frac{2s(n-s)}{n^2} + \frac{2s}{n^2} - \ell \le \frac{s(n-s)}{n^2} + \frac{9n-4}{n^3}$$

q.e.d.

Remark 20.6.2. In the case $s \le \sqrt{n}$ or $s \ge n - \sqrt{n}$ we have $|f_\Omega(x) - f_\Omega(y)| < \frac{1}{\sqrt{n}} + \frac{3}{n}$.

20.7. Involutions

In this section we give an explicit formula for $f_\Omega(x)$, for any involution $x \in H$ (not a_s-type). We mimic Section 5.2.1. Recall that if $p \neq 2$ then for any involutions $x, y \in H$ with $\nu(x) = \nu(y)$ we have that either $y \in x^G$ or $-y \in x^G$, in particular $f_\Omega(x) = f_\Omega(y)$. If $p = 2$ and $x \in H$ is an involution then $\nu(x)$ identifies x^G only if it is odd; if $\nu(x) = s$ is even there are precisely two G-conjugacy classes of involutions whose representatives are denoted by a_s, c_s, see Section 5.2.1. Recall that, in the case $p = 2$, n/t must be even.

Recall the notation established in Section 19.7. For $\ell \in \{0, \dots, \lfloor t/2 \rfloor\}$ and an involution $x \in H$,

$$f_\Omega^{h'}(x) = \frac{\dim\Omega - \dim x^G + \dim(x^G \cap H^\circ \pi_{h'})}{\dim\Omega}$$

In the case where $n/2t$ is odd we set $h' = \max\{0, \frac{s+t}{2} - \frac{n}{4}\}$. If $n/2t$ is even then $h' = 0$.

The main result of this section is the following.

Theorem 20.7.1. *Let $x \in G$ be an involution with $\nu(x) = s$.*

(A) Assume x is not a_s-type and write $s = \lfloor s/t \rfloor t + b$. Then

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

(B) Assume $p = 2$ and x is a_s-type. Write $s/2 = \lfloor s/2t \rfloor + b$. Set $\epsilon = 1$ if $s - h'\frac{n}{t} > \frac{n}{t}$ and $\epsilon = 0$ otherwise. Then

$$f_\Omega^{h'}(x) \le f_\Omega(x) \le f_\Omega^{h'}(x) + \epsilon\frac{t\iota}{2n(t-1)}$$

where $\iota = 2$ if $n/2t$ is odd and $n - 2s \ge 2t$, and $\iota = 1$ otherwise. Furthermore one of the following two conditions holds:

(i) $h' = 0$ *and*

$$f^{\circ}_{\Omega}(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

(ii) $h' > 0$ *and*

$$f^{h'}_{\Omega}(x) = 1 - \frac{2st(n-s-1)}{n^2(t-1)} + \frac{n+2(s-t)}{4n(t-1)}$$

The following is an immediate consequence of Theorem 20.7.1.

Corollary 20.7.2. *Assume* $p = 2$. *Let* $x \in G$ *be an* a_s*-type involution. Then*

$$f^{h'}_{\Omega}(x) \le f_{\Omega}(x) \le f^{h'}_{\Omega}(x) + \frac{2}{n}$$

Using the formulae in Theorem 20.7.1, we immediately deduce lower bounds on $f_{\Omega}(x)$ for $x \in G$ an involution.

Corollary 20.7.3. *Let* $x \in G$ *be an involution. Assume* x *is not* a_s*-type if* $p = 2$. *Then*

$$f_{\Omega}(x) \ge \frac{1}{2} - \frac{t^2}{n^2(t-1)}\delta_{n/t-1;2}$$

Moreover equality holds if $\nu(x) = \lfloor n/2 \rfloor$.

Proof. The result follows using the same argument of Corollary 18.7.3. *q.e.d.*

Remark 20.7.4. If $p = 2$ then n/t is even. Hence $f_{\Omega}(x) \ge 1/2$. If $p \ne 2$, it is straightforward to check, using the lower bound in Corollary 20.7.3, that $f_{\Omega}(x) \ge 1/2 - 1/n$.

For a_s-type involutions we could compute a lower bound in the same way. Also, notice that the lower bound given in Lemma 20.3.8 is close to best possible. For example, when $n/2t$ is even, one computes $f^{\circ}_{\Omega}(a_{n/2}) = 1/2 + t/(2n(t-1))$; and, in the case $n/2t$ is odd then $h' = n/2$ and $f^{h'}_{\Omega}(a_{n/2}) \ge 1/2 - 2t^2/(n^2(t-1))$ with the equality possible.

The following is the analogue, for the orthogonal group, of Proposition 19.7.5.

Proposition 20.7.5. *Let* $x \in G$ *be an involution. Then* $x^G \cap H \ne \emptyset$. *Furthermore:*

(i) Assume $p = 2$, $n/2t$ *is odd and* x *is* a_s*-type. Define* $h' = \max\{0, \frac{s+t}{2} - \frac{n}{4}\}$. *Then* $x^G \cap H^{\circ}\pi_{h'} \ne \emptyset$ *and* $x^G \cap H^{\circ}\pi_i = \emptyset$ *for all* $i < h'$.

(ii) In all the other cases $x^G \cap H^{\circ} \ne \emptyset$.

Remark 20.7.6. In the case $p = 2$ we have that n/t is even. A consequence of Proposition 20.7.5 is that $a_s^G \cap H^{\circ} \ne \emptyset$ if, and only if, $n - 2s \ge 2t$. Notice that by the proof of this result if $h' > 0$ then

$$\dim(a_s^G \cap H^{\circ}\pi_{h'}) = \dim([I_{n/t}, \dots, I_{n/t}, z, \dots, z]\pi_{h'})^{H^{\circ}}$$

where $z \in \mathrm{O}_{n/t}$ is an $a_{n/2t-1}$-type involution.

20.7.1. Semisimple involutions. Assume $p \ne 2$. Let $x \in G$ be an involution. So, up to conjugation, $x = [I_s, -I_{n-s}]$, with $\nu(x) = s \le n/2$. Then $x^G \cap H^{\circ} \ne \emptyset$. The main aim of this section is to give an explicit formula for $f_{\Omega}(x)$; in fact we shall show that $f_{\Omega}(x) = f^{\circ}_{\Omega}(x)$.

Recall the definition of $\delta_{a;b}$ given in (23): $\delta_{a;b} = 1$ if $b \mid a$ and 0 otherwise.

We define

$$z = [I_{\lfloor n/2t \rfloor}, -I_{n/t - \lfloor n/2t \rfloor}],\ z' = -z$$

Notice that $z = z'$ if n/t is even. In addition, $z, z' \in \mathrm{O}_{n/t}$, by Theorem 5.3.1. We compute

$$\dim z^{\mathrm{O}_{n/t}} + \dim(z')^{\mathrm{O}_{n/t}} = \frac{1}{2}\Big(\frac{n}{t}\Big)^2 - \frac{\delta_{n/t-1;2}}{2} = \dim \mathrm{O}_{n/t} + \frac{1}{2}\Big(\frac{n}{t} - \delta_{n/t;2}\Big)$$

Lemma 20.7.7. *Let $x \in G$ be an involution. Assume $x^G \cap H^\circ \pi_i \neq \emptyset$ for some $i > 0$. Then $x^G \cap H^\circ \pi_{i-1} \neq \emptyset$ and*

$$\dim(x^G \cap H^\circ \pi_{i-1}) > \dim(x^G \cap H^\circ \pi_i)$$

PROOF. Up to G conjugation, we write $x = [I_s, -I_{n-s}]$ with $s \leq n/2$. By Lemma 17.4.2, x is H°-conjugate to

$$[I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_i \tag{271}$$

for suitable involutions $x_i \in \mathrm{O}_{n/t}$. In particular, we may choose the x_i's such that

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_i) &= \dim([I_{n/t}, \ldots, I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_i)^{H^\circ} \\ &= i \dim \mathrm{O}_{n/t} + \sum_{j=2i+1}^{t} \dim x_j^{\mathrm{O}_{n/t}}\end{aligned}$$

Thanks to Lemma 17.4.6 we have

$$[I_{n/t}, \ldots, I_{n/t}, I_{n/t}, -I_{n/t}, x_{2i+1}, \ldots, x_t]\pi_{i-1} \in x^G \cap H^\circ \pi_{i-1}$$

where we have replaced, in (271), the last block $I_{n/t}$ with $-I_{n/t}$. Then

$$\begin{aligned}\dim(x^G \cap H^\circ \pi_{i-1}) &\geq \dim([I_{n/t}, \ldots, I_{n/t}, z, z', x_{2i+1}, \ldots, x_t]\pi_{i-1})^{H^\circ} \\ &= (i-1)\dim \mathrm{O}_{n/t} + \dim z^{\mathrm{O}_{n/t}} + \dim(z')^{\mathrm{O}_{n/t}} + \sum_{j=2i+1}^{t} \dim x_j^{\mathrm{O}_{n/t}} \\ &= \dim(x^G \cap H^\circ \pi_i) + \frac{1}{2}\Big(\frac{n}{t} - \delta_{n/t;2}\Big) > \dim(x^G \cap H^\circ \pi_i)\end{aligned}$$

The result follows. *q.e.d.*

Therefore we have the following.

Corollary 20.7.8. *Let $x \in G$ be an involution. Then $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$.*

Corollary 20.7.8 together with Theorems 5.3.1 and 17.3.8 lead to an explicit formula of $f_\Omega(x)$ for any involution $x \in G$.

Proposition 20.7.9. *Let $x \in G$ be an involution with $\nu(x) = s \leq n/2$. Write $s = ct+b$ where $0 \leq b < t$. Then*

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

PROOF. This is a straightforward computation using the formulae for $\dim x^G$ and $\dim(x^G \cap H^\circ)$ given in Theorems 5.3.1 and 17.3.8, respectively. *q.e.d.*

We can easily compute global lower bound on $f_\Omega(x)$, for $x \in G$ a semisimple involution. The following result is rather technical, however it is elementary and it is a quick consequence of Proposition 20.7.9.

Corollary 20.7.10. *Let $x \in G$ be an involution. Then*

$$f_\Omega(x) \geq \frac{1}{2} + \frac{\delta_{n-1;2}}{2n^2} - \frac{t^2}{2n^2(t-1)}\delta_{\frac{n-1}{2};2}$$

with equality if, and only if, $\nu(x) \in \{n/2, (n-1)/2\}$, or $t \mid s$ and $\nu(x) = n/2, n/2-1$.

Proof. Write $x_s = [I_s, -I_{n-s}]$ and $s = ct + b$ where $0 \leq b < t$. Then $s - 1 = ct + (b-1)$ if $b > 0$ and $s - 1 = (c-1)t + (t-1)$ if $b = 0$. We claim $f_\Omega(x_s) \leq f_\Omega(x_{s-1})$. In the case $b > 0$, we compute

$$\begin{aligned} f_\Omega(x_s) - f_\Omega(x_{s-1}) &= -\frac{2}{n^2}(n-2s+1) - \frac{2}{n^2(t-1)}(t-2b+1) \\ &\leq -\frac{2}{n^2}(n-2s+1) + \frac{2(t-3)}{n^2(t-1)} < -\frac{2}{n^2}(n-2s) < 0 \end{aligned}$$

In the case $b = 0$ we have

$$f_\Omega(x_s) - f_\Omega(x_{s-1}) = -\frac{2}{n^2}(n-2s) \leq 0$$

Notice, moreover, that $f_\Omega(x_s) = f_\Omega(x_{s-1})$ if, and only if, $b = 0$ and $s = n/2$.

Therefore, for any involution $x \in G$ we have $f_\Omega(x) \geq f_\Omega(x_{\lfloor n/2 \rfloor})$, with equality if, and only if, $\nu(x) = \lfloor n/2 \rfloor$ or $\nu(x) = n/2 - 1$ and $t \mid n/2$. The result follows from applying the formula in Proposition 20.7.9. Notice that for $s = n/2$ we have that either $t \mid n/2$ or $n/2 \equiv t/2 \pmod t$. So

$$\left\lfloor \frac{n}{2t} \right\rfloor = \frac{n}{2t}, \text{ or } \frac{n/t-1}{2}$$

Therefore

$$f_\Omega(x_{n/2}) = \frac{1}{2} - \frac{t^2}{2n^2(t-1)}\delta_{\frac{n}{t}-1;2}$$

Assume n is odd. Then $\lfloor n/2 \rfloor = (n-1)/2$ and we have

$$\left\lfloor \frac{n-1}{2t} \right\rfloor = \frac{n/t-1}{2}$$

And, again, $n/2 \equiv t/2 \pmod t$. Therefore

$$f_\Omega(x_{(n-1)/2}) = \frac{1}{2} + \frac{1}{2n^2} - \frac{t^2}{2n^2(t-1)}\delta_{\frac{n}{t}-1;2}$$

q.e.d.

A simplified lower bound quickly follows.

Corollary 20.7.11. *Let $x \in G$ be an involution. Then*

$$f_\Omega(x) \geq \frac{1}{2} - \frac{1}{n}$$

Proof. We have

$$\frac{2b(t-b)}{n^2(t-1)} \leq \frac{t^2}{2n^2(t-1)} \leq \frac{1}{4(n-2)} \leq \frac{1}{n}$$

and $2s(n-s) \leq n^2/2$. *q.e.d.*

20.7.2. Unipotent involutions. Now we assume $p = 2$. Then, Proposition 4.1.11 implies that both n and n/t are even. The same discussion given in Section 19.7.2 holds.

The following is the key result needed to compute an explicit formula for $\dim(x^G \cap H^\circ)$. Notice that it is the dual of Proposition 17.3.13 and the proof is totally similar to that for Proposition 19.7.11.

Proposition 20.7.12. *Let $x \in G$ be an involution such that $x^G \cap H^\circ \neq \emptyset$. Assume $x = [x_1, \ldots, x_t] \in H^\circ$ and $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$.*

(i) If x is a_s-type then x_i is a_{s_i}-type and $|s_i - s_j| \leq 2$ for all $i, j \in \{1, \ldots, t\}$;
(ii) If x is b_s or c_s-type, then x_i is of type b_{s_i} or c_{s_i}, with $\sum_i s_i = s$ and $|s_i - s_j| \leq 1$ for all $i, j \in \{1, \ldots, t\}$.

In addition, if $x = [z_1, \ldots, z_t]$ is a block decomposition which satisfies the previous conditions then $\dim(x^G \cap H^\circ) = \sum_i \dim z_i^{\mathrm{O}_{n/t}}$.

20.7.2.1. *a_s-type involutions.* Let $x \in G$ be an a_s-type involution. In the case where $n/2t$ is even we have $x^G \cap H^\circ \neq \emptyset$; in fact we shall construct a block decomposition $[x_1, \ldots, x_t] \in x^G$ at the end of this section, in order to compute $f_\Omega^\circ(x)$. In the case where $n/2t$ is odd most of the results proved in Section 19.7.2.1 hold. We summarise them in the following, which characterises the a_s-type involutions for which $a_s^G \cap H^\circ \neq \emptyset$.

Lemma 20.7.13. *Assume $n/2t$ is odd. Let $x \in G$ be a_s-type. Then $x^G \cap H^\circ \neq \emptyset$ if, and only if $n - 2s \geq 2t$.*

We can also characterise, in terms of combinatorial data, the smallest i such that $x^G \cap H^\circ \pi_i \neq \emptyset$. Let $h' = \max\{0, \frac{s+t}{2} - \frac{n}{4}\}$. Then Corollary 19.7.13 yields

$$x^G \cap H^\circ \pi_{h'} \neq \emptyset, \text{ and } x^G \cap H^\circ \pi_i = \emptyset$$

for all $i < h'$. In particular, if $h' > 0$ then

$$\dim(x^G \cap H^\circ \pi_{h'}) = \dim([I_{n/t}, \ldots, I_{n/t}, x_{2h'+1}, \ldots, x_t]\pi_{h'})^{H^\circ} \tag{272}$$

and x_i is of type $a_{n/2t-1}$ for all $i \geq 2h' + 1$.

Case 1. Assume $x^G \cap H^\circ = \emptyset$. In this case $n/2t$ is odd and $h' = \frac{s+t}{2} - \frac{n}{4} > 0$. Then, thanks to (272), we have

$$\dim(x^G \cap H^\circ \pi_{h'}) = h' \dim \mathrm{O}_{n/t} + (t - 2h') \dim(a_{n/2t-1})^{\mathrm{O}_{n/t}} = \frac{n(n + 2(s - t))}{8t}$$

So, since $\dim x^G = s(n - s - 1)$, we have

$$f_\Omega^{h'}(x) = 1 - \frac{2st(n - s - 1)}{n^2(t - 1)} + \frac{n + 2(s - t)}{4n(t - 1)}$$

As in Section 19.7.2.1, if $x^G \cap H^\circ \pi_{h'+1} \neq \emptyset$ and $i < n/6 - s/3$ we can compute $\dim(x^G \cap H^\circ \pi_{h'+1})$. In fact, Lemma 19.7.16 holds in this case, as well.

The following is the analogue of Lemma 19.7.18 and it is the main tool to prove Theorem 20.7.1(B) in the case where $h' > 0$.

Lemma 20.7.14. *Assume $n/2t$ is odd. Let $x \in G$ be a_s-type. Assume $x^G \cap H^\circ = \emptyset$. Then*

$$\dim(x^G \cap H) < \dim(x^G \cap H^\circ \pi_{h'}) + \frac{n}{4} - t$$

PROOF. The proof is entirely similar to the proof of Lemma 19.7.18. *q.e.d.*

If $h' = 0$ then $x^G \cap H^\circ \neq \emptyset$. Now we deal with this case.

Case 2. Assume $x^G \cap H^\circ \neq \emptyset$. Let $x \in G$ be an a_s-type involution. Write $s/2 = \lfloor s/2t \rfloor t + b$, $0 \leq b < t$. Notice that $\mathrm{O}_{n/t}$ contains $a_{2\lfloor s/2t \rfloor}$-type involutions and, if $b \neq 0$, it contains $a_{2\lfloor s/2t \rfloor + 2}$-type involutions. Let z be an $a_{2\lfloor s/2t \rfloor}$-involution, and z' be an $a_{2\lfloor s/2t \rfloor + 2}$-involution. Define $x_1 = \ldots = x_b = z' \in \mathrm{O}_{n/t}$ and $x_{b+1} = \ldots = x_t = z \in \mathrm{O}_{n/t}$. Then $[x_1, \ldots, x_b, x_{b+1}, \ldots, x_t] \in x^G \cap H^\circ$. Moreover, thanks to Proposition 20.7.12, we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$. We compute

$$\dim(x^G \cap H^\circ) = \frac{s(n-s)}{t} - s - \frac{4b(t-b)}{t}$$

And, since $\dim x^G = s(n-s-1)$, we get

$$f^\circ_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{8b(t-b)}{n^2(t-1)}$$

The main tool in order to prove Theorem 20.7.1(B) for a_s-type involutions whose G-classes meet H°, in the case $n/2t$ is even, is the following.

Lemma 20.7.15. *Assume $n/2t$ is even. Let $x \in G$ be a_s-type. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{4}$$

PROOF. This is totally similar to the proof of Lemma 19.7.19. Also here we deduce $\dim(x^G \cap H^\circ \pi_{i-1}) \geq \dim(x^G \cap H^\circ \pi_i) - \frac{n}{2t}$ for i such that $x^G \cap H^\circ \pi_i \neq \emptyset$. The result follows with the same argument as for Lemma 19.7.19. *q.e.d.*

And, in the case where $n/2t$ is odd we have the following.

Lemma 20.7.16. *Assume $n/2t$ is odd. Let $x \in G$ be a_s-type. Assume $x^G \cap H^\circ \neq \emptyset$. Then*

$$\dim(x^G \cap H) \leq \dim(x^G \cap H^\circ) + \frac{n}{2}$$

PROOF. The proof is very similar to that of Lemma 19.7.20. The only difference is given by the extra term '$-s$' appearing in the formula for $\dim(x^G \cap H^\circ)$. Exploiting the proof of Lemma 19.7.20 and using *Mathematica* we compute (for $\ell < n/2$)

$$g(\ell) \leq g\Big(\frac{t}{2} - \frac{n-2s}{2\sqrt{n/t}}\Big) = \frac{s(n-s)}{t} - s + \frac{(n-2s)^2 + nt - 2t\sqrt{n/t}(n-2s)}{4t}$$

Therefore, arguing as in Lemma 19.7.20, we get

$$\dim(x^G \cap H) < \dim(x^G \cap H^\circ) + \frac{n}{2} - 2t + \frac{4b(t-b)}{t}$$

In the case $\ell = t/2$ we have $s = n/2$ and

$$\dim(x^G \cap H^\circ \pi_\ell) = \dim(x^G \cap H^\circ) + \frac{n}{4} + t$$

The result follows. *q.e.d.*

Now, Theorem 20.7.1(B) follows from applying the upper bounds derived in Lemmas 20.7.14, 20.7.15 and 20.7.16.

20.7.2.2. *b_s, c_s-type involutions.* Mirroring the arguments in Section 19.7.2.2, also here, we compute $\dim(x^G \cap H)$.

Lemma 20.7.17. *Let $x \in G$ be of type b_s or c_s. Assume $x^G \cap H^\circ\pi_i \neq \emptyset$ for some $i > 0$. Then $x^G \cap H^\circ\pi_{i-1} \neq \emptyset$ and*

$$\dim(x^G \cap H^\circ\pi_i) < \dim(x^G \cap H^\circ\pi_{i-1})$$

Proof. The proof is totally similar to that of Lemma 19.7.21. The only difference is that for $z \in \mathrm{O}_{n/t}$ of type $b_{n/2t}$ or $c_{n/2t}$ we have $2\dim z^{\mathrm{O}_{n/t}} = \frac{1}{2}\left(\frac{n}{t}\right)^2$. Then following the same argument we get the result. *q.e.d.*

In particular we have the following.

Corollary 20.7.18. *Let $x \in G$ be of type b_s or c_s. Then*

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$$

As in Section 19.7.2.2, we write $s = \lfloor s/t \rfloor t + b$. We define $x = [x_1, \ldots, x_t] \in H^\circ$ where the blocks x_i are all b- or c-type with $\nu(x_1) = \ldots = \nu(x_b) = \lfloor s/t \rfloor + 1$ and $\nu(x_{b+1}) = \ldots = \nu(x_t) = \lfloor s/t \rfloor$. Then, by Proposition 20.7.12 we have $\dim(x^G \cap H^\circ) = \dim x^{H^\circ}$ and

$$\dim(x^G \cap H^\circ) = \frac{s(n-s)}{t} - \frac{b(t-b)}{t}$$

By Corollary 20.7.18 we have $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Therefore, using $\dim x^G = s(n-s)$, we get an explicit formula for $f_\Omega(x)$.

Proposition 20.7.19. *Let $x \in G$ be of type b_s or c_s. Write $s = \lfloor s/t \rfloor t + b$. Then*

$$f_\Omega(x) = 1 - \frac{2s(n-s)}{n^2} - \frac{2b(t-b)}{n^2(t-1)}$$

This concludes the proof of Theorem 20.7.1.

APPENDIX A

List of results

The purpose of this appendix is to give an overall guide to the results proved in the thesis, for the reader's convenience.

A.1. Background

Let G be a classical group, i.e. $G = \mathrm{GL}_n, \mathrm{Sp}_n$ or O_n defined over an algebraically closed field of characteristic p. Let $H \leq G$ be a closed subgroup in one of the families $\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_6$. In the following we recall the reader the structure of H in the various cases. For more details we refer to Chapters 9, 13, 17, for $\mathscr{C}_6$-, $\mathscr{C}_3$- and $\mathscr{C}_2$-subgroups, respectively.

Assume $H \in \mathscr{C}_2$. Then the possibilities for G and H are as follows

$$\mathrm{GL}_{n/t} \wr S_t < \mathrm{GL}_n,\ \mathrm{Sp}_{n/t} \wr S_t < \mathrm{Sp}_n,\ \mathrm{O}_{n/t} \wr S_t < \mathrm{O}_n$$

Here, S_t acts by permuting the linear factors. Notice that, when $G = \mathrm{O}_n$, we assume $n/t > 1$ and, in the case $p = 2$, we also assume n/t is even.

In the case $H \in \mathscr{C}_3$ then $G = \mathrm{Sp}_n$ or O_n with n even and

$$H = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\}.\langle \tau \rangle \cong \mathrm{GL}_{n/2}.2$$

where $\tau = \begin{pmatrix} & I_{n/2} \\ \epsilon I_{n/2} & \end{pmatrix}$, and $\epsilon = -1$ if $G = \mathrm{Sp}_n$ whereas $\epsilon = 1$ for $G = \mathrm{O}_n$.

If $H \in \mathscr{C}_6$ then the possibilities for (G, H) are:

$$(\mathrm{GL}_n, \mathrm{Sp}_n),\ (\mathrm{GL}_n, \mathrm{O}_n)$$
$$(\mathrm{Sp}_n, \mathrm{O}_n) \text{ and } p = 2$$

In the following we give various tables where we record the results proved for each action.

A.2. How to read the tables

In the first set of tables (Tables A.3.1 – A.3.6) we fix an action – i.e. we fix H to be a subgroup of a particular family – and we give two tables for each action: one in which we give references for results regarding global bounds and the other for local bounds.

In the second set of tables (Tables A.3.7 – A.3.11) we fix a certain type of bound (e.g. global lower bounds for unipotent elements, or local upper bounds) and we give references for results dealing with the corresponding bound for any classical group in any action.

References for involutions are given in Table A.3.12 (see (6), below).

The tables are self-explanatory, here we explain the main notations. Recall that for G a classical group and $H \leq G$ a closed subgroup in $\mathscr{C}_2, \mathscr{C}_3$ or $\mathscr{C}_6$ we provide upper and lower bounds on $f_\Omega(x)$ for $x \in G$ such that $x^G \cap H \neq \emptyset$. Since $f_\Omega(x) = f_\Omega(x^g)$ for all $g \in G$, we usually assume $x \in H$.

(1) r is the (prime) order of the element; p is the characteristic of the field (recall we set $p = \infty$ for the characteristic zero case);

(2) given $x \in H$ we set $\nu(x) = s$;

(3) the notation regarding the results stated for the *accuracy* part is one of three types:

   (i) we write "–" if the result on the bound comprises a characterisation or gives comments on the sharpness of the bound;

   (ii) in black we state either results in which the reader can find the *characterisation* of the elements that realise the bound or results where elements that realise the bound are given;

   (iii) in red we refer to *existence* results: these results will show the existence of an element whose f_Ω-value is close to the bound;

(4) we write $\rightarrow\leftarrow$ if we have not dealt with the corresponding case;

(5) we write "N/A" if such case does not occur, e.g. $\mathscr{C}_3$-actions when $G = \mathrm{GL}_n$;

(6) Table A.3.12 records results for involutions. It comprises two lines: *formula* and *approximation*. The aim here is to show that for any involution $x \in G$ we have

$$g(n, t, s) \leq f_\Omega(x) \leq g(n, t, s) + \epsilon$$

for $\epsilon \geq 0$. We refer to results where we give these data and we write ✓ in the *formula* line if $\epsilon = 0$, otherwise we write ✓ in the *approximation* line.

A.3. The tables

G	Upper bound		Lower bound		
	$r \neq 2$	$r = 2$	$r = p \neq 2$	$p \neq r \neq 2$	$r = 2$
Sp_n					
Bound	14.1.1	14.7.2	14.2.1	14.3.1($r > n$) 14.3.15 ($r < n$)	14.7.2
Accuracy	–	–	– ($p \mid n, p > n/2$) 14.2.8 (o.w.)	–	–
O_n					
Bound	15.1.1	15.7.2	15.2.1	15.3.1 ($r > n$) 15.3.14 ($r < n$)	15.7.2
Accuracy	–	–	– ($p \mid n$) 15.2.5 ($p < n/2, p \nmid n$) 15.2.8 ($p > n/2$)	–	–

Table A.3.1. Global bounds for $\mathscr{C}_3$-actions

G	Upper bound		Lower bound	
	$r = p$	$r \neq p$	$r = p$	$r \neq p$
Sp_n				
Bound	14.4.4	14.4.14	14.5.2	14.5.14($s \leq r-1$) 14.5.15 (o.w.)
Accuracy	14.4.6	14.4.15	14.5.3	14.5.11
O_n				
Bound	15.4.3	15.4.14	15.5.2	15.5.8 ($s \leq r-1$) 15.5.9 (o.w.)
Accuracy	15.4.6	15.4.15	15.5.3	15.5.7

Table A.3.2. Local bounds for $\mathscr{C}_3$-actions

(G, H)	Upper bound	Lower bound		
		$r = p$	$r \neq p$	$r = 2 \neq p$
$(\mathrm{GL}_n, \mathrm{Sp}_n)$				
Bound	10.1.1	10.2.1	10.3.7	10.3.1
Accuracy	–	10.2.5 ($2 < p < n, p \nmid n$) – (o.w.)	10.3.6	–
$(\mathrm{GL}_n, \mathrm{O}_n)$				
Bound	11.1.1	11.2.1	11.3.5	§ 11.3.2
Accuracy	–	11.2.5 ($2 < p < n$ & $p \nmid n$) – (o.w.)	11.3.4	–
$(\mathrm{O}_n, \mathrm{Sp}_n)$				
Bound, Acc.	9.4.1	9.4.3	9.4.3	9.4.3

Table A.3.3. Global bounds for $\mathscr{C}_6$-actions

(G, H)	Upper bound		Lower bound	
	$r = p$	$r \neq p$	$r = p$	$r \neq p$
$(\mathrm{GL}_n, \mathrm{Sp}_n)$				
Bound	10.4.2	10.4.10	10.5.2	10.5.4
Accuracy	10.4.3 ($\frac{n}{n-s} \leq p$) $\to\leftarrow$ (o.w.)	10.4.12	10.5.3 ($s \leq n/2$) $\to\leftarrow$ (o.w.)	10.5.6 ($s \leq n/2$) $\to\leftarrow$ (o.w.)
$(\mathrm{GL}_n, \mathrm{O}_n)$				
Bound	11.4.2	11.4.5	11.5.2	11.5.4
Accuracy	11.4.3 ($s \leq n/2$) $\to\leftarrow$ (o.w.)	11.4.6	11.5.3 ($s \leq p$) $\to\leftarrow$ (o.w.)	11.5.5 ($s \leq n/2$) $\to\leftarrow$ (o.w.)
$(\mathrm{O}_n, \mathrm{Sp}_n)$				
Bound, Acc.	–	9.4.4	–	9.4.4 ($s < n/2$) 9.4.5 (o.w.)

Table A.3.4. Local bounds for $\mathscr{C}_6$-actions

G	Upper bound	Lower bound	
		$r = p$	$r \neq p$
GL_n			
Bound	18.1.1	18.2.1	18.3.1($r > n$) 18.3.19 (o.w.)
Accuracy	–	– ($p > n$) 18.2.16 ($p \nmid n$) 18.2.17 ($p < n, p \nmid n$)	– ($r > n$) 18.3.14 ($2 \neq r \leq n$) 18.7.3 ($r = 2$)
Sp_n			
Bound	19.1.1	19.2.1	19.3.1($r \geq n-1$) 19.3.16 ($2 \neq r < n-2$) 19.3.6 ($r = 2$)
Accuracy	–	– ($p > n/2$) 19.2.10 ($2 \neq p \mid n$) 19.2.11 ($2 \neq p < n/2, p \nmid n$) 19.7.4 ($p = 2$)	– ($r \geq n-1$) 19.3.12 ($2 \neq r < n-1$) 19.7.4 ($r = 2$)
O_n			
Bound	20.1.1	20.2.1	20.3.1($r \geq n-1$) 20.3.16 ($r < n-1, n/t$ even) 20.3.9 ($r < n-1, n/t$ odd) 20.3.8 ($r = 2$)
Accuracy	–	– ($p > n/2$) 20.2.13 ($2 \neq p \mid n$) 20.2.14 ($2 \neq p < n/2, p \nmid n$) 20.7.3 ($p = 2$)	– ($r \geq n-1$) 20.3.15 ($r < n-1, n/t$ even) 20.3.21 ($x \in H^\circ, r < n-1, n/t$ odd) $\rightarrow\leftarrow$ (n/t odd) 20.7.3 ($r = 2$)

Table A.3.5. Global bounds for $\mathscr{C}_2$-actions

G	Upper bound		Lower bound	
	$r = p$	$r \neq p$	$r = p$	$r \neq p$
GL_n				
Bound	18.4.1	18.4.1	18.4.13 $(t = n)$ 18.5.10 $(t = 2)$ $\rightarrow\leftarrow$ $(t \neq 2, n)$	18.5.5$(s \leq r - 1)$ 18.5.8 (o.w.)
Accuracy	18.4.11 $(s \leq n/2)$ 18.4.12 (o.w.) 18.4.13 $(t = n)$	18.4.4	18.4.13 $(t = n)$ $\rightarrow\leftarrow$ (o.w.)	18.5.4
Sp_n				
Bound	19.4.2	19.4.4	$\rightarrow\leftarrow$	19.5.1 19.5.6 $(s \leq r - 1, s$ even; $2s - n \leq r - 1, s$ odd$)$
Accuracy	19.4.3 $(s \leq n/2)$ $\rightarrow\leftarrow$ (o.w.)	19.4.8 19.4.10$(s \leq 2n/3)$	$\rightarrow\leftarrow$	19.5.4
O_n				
Bound	20.4.1	20.4.1	$\rightarrow\leftarrow$	20.5.1$(x \in H^\circ)$ 20.5.2 $(n/t$ even$)$ $\rightarrow\leftarrow$ $(n/t$ odd, $x \notin H^\circ)$
Accuracy	20.4.3 $(s \leq n/2)$ $\rightarrow\leftarrow$ (o.w.)	20.4.4 $(n/t$ even$)$ $\rightarrow\leftarrow$ $(n/t$ odd$)$	$\rightarrow\leftarrow$	$\rightarrow\leftarrow$

Table A.3.6. Local bounds for $\mathscr{C}_2$-actions

G	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_6$
GL_n			
	18.1.1	N/A	10.1.1 (Sp_n) 11.1.1 (O_n)
Sp_n			
	19.1.1	14.1.1 ($r \neq 2$) 14.7.2 (o.w.)	9.4.1
O_n			
	20.1.1	15.1.1 ($r \neq 2$) 15.7.2 (o.w.)	N/A

Table A.3.7. Upper bounds

G	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_6$
GL_n			
Bound	18.2.1	N/A	10.2.1 (Sp_n) 11.2.1 (O_n)
Accuracy	– ($p > n$) 18.2.16 ($p \mid n$) 18.2.17 ($p < n$ & $p \nmid n$) 18.7.3 ($p = 2$)		10.2.5 ($2 \neq p < n$ & $p \nmid n, \mathrm{Sp}_n$) 11.2.5 ($2 \neq p < n$ & $p \nmid n, \mathrm{O}_n$) – (o.w.)
Sp_n			
Bound	19.2.1	14.2.1	9.4.3
Accuracy	– ($p > n/2$) 19.2.10 ($p \mid n$) 19.2.11 ($2 \neq p < n$ & $p \nmid n$) 19.7.4 ($p = 2$)	– ($p > n/2$, or $p \leq n/2$ & $n/2 \equiv 0, p-1 \pmod p$) 14.2.8 (o.w., $p \neq 2$) 14.7.2 ($p = 2$)	–
O_n			
Bound	20.2.1	15.2.1	N/A
Accuracy	– ($p > n/2$), 20.2.13 ($2 \neq p \mid n$) 20.2.14 ($2 \neq p < n$ & $p \nmid n$) 20.7.3 ($p = 2$)	– ($2 \neq p \mid n$) 15.2.5 ($p < n/2$ & $p \nmid n$) 15.2.8 ($p > n/2$) 15.7.2 ($p = 2$)	

Table A.3.8. Global lower bounds for prime order unipotent elements

G	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_6$
GL_n			
Bound	18.3.1 $(r > n)$ 18.3.19 (o.w.)	N/A	10.3.1 (Sp_n) 10.3.7 $(2 \neq r \leq n, \mathrm{Sp}_n)$ 10.3.9 $(r = 2, \mathrm{Sp}_n)$ 11.3.5 $(r \neq 2, \mathrm{O}_n)$ § 11.3.2 $(r = 2, \mathrm{O}_n)$
Accuracy	– $(r > n)$ 18.3.14 $(2 \neq r \leq n)$ 18.7.3 $(r = 2)$		– (Sp_n) – (O_n)
Sp_n			
Bound	19.3.1 $(r \geq n-1)$ 19.3.16 (o.w.)	14.3.1 $(r > n)$ 14.3.15 $(r < n)$ 14.7.2 $(r = 2)$	9.4.3
Accuracy	– $(r \geq n-1)$ 19.3.12 $(r < n-1)$ 19.7.4 $(r = 2)$	–	–
O_n			
Bound	20.3.1 $(r \geq n-1)$ 20.3.16 $(n/t$ even$)$ 20.3.9 $(n/t$ odd$)$	15.3.1 $(r > n)$ 15.3.14 $(2 \neq r < n)$ 15.7.2 $(r = 2)$	N/A
Accuracy	– $(r \geq n-1)$ 20.3.15 $(n/t$ even$)$ 20.3.21 $(x \in H^\circ$ n/t odd$)$ $\rightarrow\leftarrow$ $(n/t$ odd$)$ 20.7.10 $(r = 2)$	–	

Table A.3.9. Global lower bounds for prime order semisimple elements

G	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_6$
GL_n			
Bound	18.4.1	N/A	10.4.2 $(r = p, \mathrm{Sp}_n)$ 10.4.10 $(r \neq p, \mathrm{Sp}_n)$ 11.4.2 $(r = p, \mathrm{O}_n)$ 11.4.5 $(r \neq p, \mathrm{O}_n)$
Accuracy	18.4.11 $(r = p, s \leq n/2)$ 18.4.12 (r=p, o.w.) 18.4.13 $(r = p, t = n)$ 18.4.4 $(r \neq p)$		10.4.3 $(r = p, \frac{n}{n-s} \leq p, \mathrm{Sp}_n)$ $\rightarrow\leftarrow$ $(r = p$, o.w., $\mathrm{Sp}_n)$ 10.4.12 $(r \neq p, \mathrm{Sp}_n)$ 11.4.3 $(r = p, s \leq n/2, \mathrm{O}_n)$ $\rightarrow\leftarrow$ $(r = p$, o.w., $\mathrm{O}_n)$ 11.4.6 $(r \neq p, \mathrm{O}_n)$
Sp_n			
Bound	19.4.2 $(r = p)$ 19.4.4 $(r \neq p)$	14.4.4 $(r = p)$ 14.4.14 $(r \neq p)$	9.4.4
Accuracy	19.4.3 $(r = p, s \leq n/2)$ $\rightarrow\leftarrow$ $(r = p$, o.w.) 19.4.8 $(r \neq p)$ 19.4.10$(r \neq p, s \leq 2n/3)$	14.4.6 $(r = p)$ 14.4.15 $(r \neq p)$	–
O_n			
Bound	20.4.1	15.4.3 $(r = p)$ 15.4.14 $(r \neq p)$	N/A
Accuracy	20.4.3 $(r = p, s \leq n/2)$ $\rightarrow\leftarrow$ $(r = p$, o.w.) 20.4.4 $(r \neq p$, n/t even) $\rightarrow\leftarrow$ $(r \neq p$, o.w.)	15.4.6$(r = p)$ 15.4.15$(r \neq p)$	

Table A.3.10. Local upper bounds

G	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_6$
GL_n			
Bound	18.4.13 $(r = p, t = n)$ 18.5.10 $(r = p, t = 2)$ $\rightarrow\leftarrow$ $(r = p, t \neq 2, n)$ 18.5.5$(r \neq p, s \leq r - 1)$ 18.5.8 $(r \neq p$, o.w.$)$	N/A	10.5.2 $(r = p, \mathrm{Sp}_n)$ 10.5.4 $(r \neq p, \mathrm{Sp}_n)$ 11.5.2 $(r = p, \mathrm{O}_n)$ 11.5.4 $(r \neq p, \mathrm{O}_n)$
Accuracy	– $(r = p, t = n)$ $\rightarrow\leftarrow$ $(r = p$, o.w.$)$ 18.5.4 $(r \neq p)$		10.5.3 $(r = p, s \leq p, \mathrm{Sp}_n)$ $\rightarrow\leftarrow$ $(r = p$, o.w., $\mathrm{Sp}_n)$ 10.5.6 $(r \neq p, s \leq n/2, \mathrm{Sp}_n)$ $\rightarrow\leftarrow$ $(r \neq p$, o.w., $\mathrm{Sp}_n)$ 11.5.3 $(r = p, s \leq p, \mathrm{O}_n)$ $\rightarrow\leftarrow$ $(r = p$ o.w., $\mathrm{O}_n)$ 11.5.5 $(r \neq p, s \leq n/2, \mathrm{O}_n)$ $\rightarrow\leftarrow$ $(r \neq p$, o.w., $\mathrm{O}_n)$
Sp_n			
Bound	$\rightarrow\leftarrow$ $(r = p)$ 19.5.1 19.5.6 $(r \neq p, s \leq r - 1, s$ even; $2s - n \leq r - 1, s$ odd$)$	14.5.2 $(r = p)$ 14.5.14$(r \neq p, s \leq r - 1)$ 14.5.15 $(r \neq p$, o.w.$)$	9.4.4 $(s < n/2)$ 9.4.5 (o.w.)
Accuracy	$\rightarrow\leftarrow$ $(r = p)$ 19.5.4 $(r \neq p)$	14.5.3 $(r = p)$ 14.5.11 $(r \neq p)$	–
O_n			
Bound	$\rightarrow\leftarrow$ $(r = p)$ 20.5.1$(r \neq p, x \in H^\circ)$ 20.5.2 $(r \neq p, n/t$ even$)$ $\rightarrow\leftarrow$ $(r \neq p, n/t$ odd, $x \notin H^\circ)$	15.5.2 $(r = p)$ 15.5.8 $(r \neq p, s \leq r - 1)$ 15.5.9 $(r \neq p$, o.w.$)$	N/A
Accuracy	$\rightarrow\leftarrow$	15.5.3 $(r = p)$ 15.5.7 $(r \neq p)$	

Table A.3.11. Local lower bounds

G	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_6$
GL_n			
	18.7.1	N/A	9.3.1 (Sp_n) 9.3.2 (O_n)
Formula	✓		✓
Approximation			
Sp_n			
	19.7.1	14.7.1	9.3.3
Formula	✓ $(p = 2, x$ not a_s-type$)$	✓	✓
Approximation	✓ (o.w.)		
O_n			
	20.7.1	15.7.1	N/A
Formula	✓ $(x$ not a_s-type$)$	✓	N/A
Approximation	✓ (o.w.)		

Table A.3.12. Involutions

APPENDIX B

Miscellaneous results

B.1. On the dominance ordering of partitions

Let n be a positive integer and $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition of n. We write $\lambda \vdash n$. Then $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\sum_i \lambda_i = n$. Recall, from Section 5.2, given partitions $\lambda, \mu \vdash n$ the *dominance ordering* is defined as follows: $\mu \preccurlyeq \lambda$ if, and only if, for all $1 \leq l \leq n$

$$\sum_{i=1}^{l} \mu_i \leq \sum_{i=1}^{l} \lambda_i$$

The following result gives a minimal element in the set of partitions of an integer n with a fixed number of non-zero parts.

Lemma B.1.1. *Let $n, t \in \mathbb{Z}_{>0}$ such that $t < n$. Write $n = at + b$, $0 \leq b < t$. Let $\mu_1 = \dots = \mu_b = a + 1$ and $\mu_{b+1} = \dots = \mu_t = a$. Set $\mu = (\mu_1, \dots, \mu_t) \vdash n$. Let $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ such that $\lambda_i \neq 0$ for all i. Then $\mu \preccurlyeq \lambda$.*

PROOF. First we show that $\lambda_1 \geq a + 1$ if $\lambda \neq \mu$. Assume $\lambda_i < a$. Then $n = \sum_i \lambda_1 \leq a - 1 + \sum_{i \geq 2} \lambda_i \leq (a-1)t < n$, since $\lambda_i \leq \lambda_1 \leq a - 1$. Hence, we deduce $\lambda_1 \geq a$. Now assume $\lambda_1 = a$. Then, as before, $n = a + \sum_{i \geq 2} \lambda_i \leq at$, which is absurd if, and only if, $b > 0$. For $b = 0$, instead, we deduce $\lambda = \mu$. Therefore either $\lambda_1 = a$ and $\lambda = \mu$ or, $\lambda_1 \geq a + 1$. In the case $\lambda = \mu$ the result trivially follows, hence we may assume $\lambda \neq \mu$ so $\lambda_1 \geq a + 1$.

We define the following notation. Let $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ and $1 \leq l \leq t$, we denote

$$A_l(\lambda) = \sum_{i \geq l} \lambda_i$$

Seeking a contradiction, assume there exists $h \in \{1, \dots, t\}$ such that

$$\sum_{i=1}^{h} \mu_i > \sum_{i=1}^{h} \lambda_i \tag{273}$$

In particular, we may assume h is the least integer with this property, and by the previous discussion $h > 1$. Thus $\sum_{i=1}^{h-1} \mu_i \leq \sum_{i=1}^{h-1} \lambda_i$. Therefore

$$\sum_{i=1}^{h-1} \lambda_i + \mu_h \geq \sum_{i=1}^{h-1} \mu_i + \mu_h > \sum_{i=1}^{h-1} \lambda_i + \lambda_h$$

Thus $\mu_h > \lambda_h$.

In the case $h \leq b$ we have $\mu_h = a + 1 > \lambda_h$. In particular $a \geq \lambda_h$. So, $a > \lambda_i$ for all $i \geq h$ since $\lambda_h \geq \lambda_{h+1} \geq \dots \geq \lambda_t$. If, instead, $h > b$ we have $\mu_h = a > \lambda_h$. Hence $a > \lambda_i$ for all $i \geq h$.

Therefore, in both cases $A_{h+1}(\mu) \geq A_{h+1}(\lambda)$. This with (273) yields

$$n = A_{h+1}(\mu) + \sum_{i=1}^{h} \mu_i > A_{h+1}(\lambda) + \sum_{i=1}^{h} \lambda = n$$

which is a contradiction. The result follows. *q.e.d.*

Example B.1.2. Let $x = [J_p^{a_p}, \ldots, J_1^{a_1}] \in \mathrm{GL}_n$ and assume $\nu(x) = s$, so that $\sum_i a_i = n - s$. Write $n = (n-s)a + b$ where $0 \leq b < n - s$. Let $y = [J_{a+1}^b, J_a^{n-s-b}]$, and notice that $\nu(y) = s$. Then, by Lemma B.1.1 and Proposition 5.2.3, we have $\dim x^G \geq \dim y^G$.

B.2. A useful inequality

Often in our arguments we have been using the following result.

Proposition B.2.1. *Let $a_1, \ldots, a_n$ be positive integers. Then*

$$\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\Big(\sum_{i=1}^{n} a_i\Big)^2$$

B.3. Conjugacy classes of unipotent elements in characteristic zero

Let $G = Cl_n$, $H \leq G$ be a $\mathscr{C}_i$-subgroup for $i = 2, 3, 6$. Several times we observe that if $n < p < \infty$ and $x \in G$ is unipotent then $x^G \cap H = x^G \cap H^\circ$. Here we show that the same holds in the characteristic zero case (recall that we set $p = \infty$).

Lemma B.3.1. *Assume $p = \infty$. Let $H \leq G$ be closed and $x \in G$ be unipotent. Then $x^G \cap H = x^G \cap H^\circ$.*

Proof. Notice that all unipotent elements of G have infinite order. Proposition 2.1.2 implies that H/H° is a finite subgroup. Hence H/H° is an algebraic group over k. Consider $\pi\colon H \to H/H^\circ$ the natural projection. Let $x \in H$ be unipotent. Then $\pi(x)$ is unipotent, by Theorem 2.3.3(iii). Thus, $\pi(x) = 1$ otherwise it would have infinite order. So $x \in H^\circ$. Hence $H \setminus H^\circ$ does not contain unipotent elements. The result follows. *q.e.d.*

B.4. $\mathscr{C}_2$-actions

Here we record some of the technical proofs we have omitted. In the following we assume r is an odd prime and $\omega \in k$ is a primitive r-th root of unity.

B.4.1. Lower bounds on semisimple elements, $G = \mathrm{GL}_n$.

Proof. Assume $r \mid n$. Let $x = [I_{n/r}, \ldots, \omega^{r-1} I_{n/r}]$. Define $y = [I_{n/r-1}, \omega I_{n/r+1}, \ldots, \omega^{r-1} I_{n/r}]$. Then we want to show $f_\Omega^\circ(x) \leq f_\Omega^\circ(y)$ with equality if, and only if, $rt \mid n$. It is enough to show

$$\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \geq \dim y^G - \dim x^G$$

with equality if, and only if, $rt \mid n$. We have $\dim y^G - \dim x^G = -2$. Using Theorem 17.3.8 we compute

$$\begin{aligned}\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = &\Big(\Big\lfloor \frac{n/r-1}{t}\Big\rfloor^2 - \Big\lfloor \frac{n/r}{t}\Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{n/r-1}{t}\Big\rfloor - \Big\lfloor \frac{n/r}{t}\Big\rfloor\Big)\Big(t - 2\frac{n}{r}\Big)\\ &\Big(\Big\lfloor \frac{n/r+1}{t}\Big\rfloor^2 - \Big\lfloor \frac{n/r}{t}\Big\rfloor^2\Big)t + \Big(\Big\lfloor \frac{n/r+1}{t}\Big\rfloor - \Big\lfloor \frac{n/r}{t}\Big\rfloor\Big)\Big(t - 2\frac{n}{r}\Big)\\ &+ 2\Big\lfloor \frac{n/r-1}{t}\Big\rfloor - 2\Big\lfloor \frac{n/r+1}{t}\Big\rfloor\end{aligned}$$

Let us write $n/r = at + b$, with $b \in \{0, \dots, t-1\}$, we consider three cases:

Case 1. If $b = 0$ we have

$$\left\lfloor \frac{n/r-1}{t} \right\rfloor - \left\lfloor \frac{n/r}{t} \right\rfloor = -1 \text{ and } \left\lfloor \frac{n/r+1}{t} \right\rfloor - \left\lfloor \frac{n/r}{t} \right\rfloor = 0$$

and a straightforward computation shows $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = -2$.

Case 2. If $b \in \{1, \dots, t-2\}$ then

$$\left\lfloor \frac{n/r-1}{t} \right\rfloor - \left\lfloor \frac{n/r}{t} \right\rfloor = 0 \text{ and } \left\lfloor \frac{n/r+1}{t} \right\rfloor - \left\lfloor \frac{n/r}{t} \right\rfloor = 0$$

and we deduce that $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = 0$.

Case 3. If $b = t-1$ we have

$$\left\lfloor \frac{n/r-1}{t} \right\rfloor - \left\lfloor \frac{n/r}{t} \right\rfloor = 0 \text{ and } \left\lfloor \frac{n/r+1}{t} \right\rfloor - \left\lfloor \frac{n/r}{t} \right\rfloor = 1$$

and a straightforward computation shows $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = 0$.

The result follows. *q.e.d.*

B.4.2. Local upper bounds, $G = \mathrm{GL}_n$.

Proof of Lemma 18.4.7. Write $n = (n-s)l + m$ with $0 \leq m < n-s$. Here $x = [I_{n-s}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H \cap \mathcal{V}_{s,r}$ with centraliser not isomorphic to the centraliser of $\bar{x} = [I_{n-s}, \omega I_{n-s}, \dots, \omega^{l-1} I_{n-s}, \omega^l I_m]$. We assume $a_1 = \min\{a_i : a_i \neq 0\}$ and $a_2 = \max\{a_i : a_i < n-s\}$ and we define $y = [I_{n-s}, \omega I_{a_1-1}, \omega^2 I_{a_2+1}, \dots, \omega^{r-1} I_{a_{r-1}}]$. Then $y \in H \cap \mathcal{V}_{s,r}$. We claim $f^\circ_\Omega(x) \leq f^\circ_\Omega(y)$ with equality if, and only if, $a_1 = a_2$ or $a_1 \equiv 0, t-1 \pmod t$.

We have $f^\circ_\Omega(x) \leq f^\circ_\Omega(y)$ if, and only if,

$$\dim y^G - \dim x^G \leq \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \tag{274}$$

And using Theorem 17.3.8 we compute:

$$\begin{aligned}
\dim y^G - \dim x^G &= 2(a_1 - a_2 - 1) \\
\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) &= \left(\left\lfloor \frac{a_1-1}{t} \right\rfloor^2 - \left\lfloor \frac{a_1}{t} \right\rfloor^2\right)t + \left(\left\lfloor \frac{a_1-1}{t} \right\rfloor - \left\lfloor \frac{a_1}{t} \right\rfloor\right)t \\
&\quad - 2a_1\left(\left\lfloor \frac{a_1-1}{t} \right\rfloor - \left\lfloor \frac{a_1}{t} \right\rfloor\right) + 2\left\lfloor \frac{a_1-1}{t} \right\rfloor \\
&\quad + \left(\left\lfloor \frac{a_2+1}{t} \right\rfloor^2 - \left\lfloor \frac{a_2}{t} \right\rfloor^2\right)t + \left(\left\lfloor \frac{a_2+1}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor\right)t \\
&\quad - 2a_2\left(\left\lfloor \frac{a_2+1}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor\right) - 2\left\lfloor \frac{a_2+1}{t} \right\rfloor
\end{aligned}$$

As in Lemma 18.3.11 we get four cases depending on the values of the floor functions.

Case 1. For $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor$, the inequality (274) is equivalent to (here we shall prove the strict inequality):

$$a_1 - a_2 - 1 < \left\lfloor \frac{a_1}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor \tag{275}$$

Case 2. If $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor - 1$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor + 1$, the inequality (274) is equivalent to (also here we shall prove the strict inequality):

$$\left\lfloor \frac{a_1}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor - 1 < 0 \tag{276}$$

Case 3. If $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor - 1$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor$, the inequality (274) is equivalent to

$$a_2 \geq \left\lfloor \frac{a_1}{t} \right\rfloor t + \left\lfloor \frac{a_2}{t} \right\rfloor - \left\lfloor \frac{a_1}{t} \right\rfloor \tag{277}$$

Case 4. Finally, for $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor + 1$, the inequality (274) is equivalent to

$$a_1 \leq \left\lfloor \frac{a_1}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor + \left\lfloor \frac{a_2}{t} \right\rfloor t + t - 1$$

Let us observe that, in this case, $a_2 = \lfloor a_2/t \rfloor t + (t-1)$, hence (274) is equivalent to

$$a_1 \leq \left\lfloor \frac{a_1}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor + a_2 \tag{278}$$

Let us consider each of these four cases.

Case 1. In this case we can write $a_1 = b_1 t + c_1$ and $a_2 = b_2 t + c_2$ where $c_1 \in \{1, \dots, t-1\}$ and $c_2 \in \{0, \dots, t-2\}$. With this notation (275) is equivalent to $a_1 - a_2 - 1 < b_1 - b_2$. If $a_1 = a_2$ we have $a_1 - a_2 - 1 = -1 < 0 = \lfloor a_1/t \rfloor - \lfloor a_2/t \rfloor$ hence (275) holds. If $a_1 < a_2$ we consider two cases. Either $b_1 = b_2$ and $c_1 < c_2$, in which case the inequality (275) is equivalent to $c_1 - c_2 - 1 < 0$, which is clearly true. Or $b_1 < b_2$, in which case (275) is equivalent to $(b_1 - b_2)(t-1) + (c_1 - c_2) - 1 < 0$, which is true because $(b_1 - b_2)(t-1) + (c_1 - c_2) - 1 \leq -(t-1) + (t-1) - 1 < 0$, since $b_1 - b_2 \leq -1$ and $c_1 - c_2 \leq t-1$. Hence (275) is proved.

Case 2. In the second case we have $a_1 = b_1 t$ and $a_2 = b_2 t + (t-1)$. If $b_1 > b_2$ we would have $a_1 = b_1 t \geq (b_2 + 1)t > b_2 t + t - 1 = a_2$ which contradicts the hypothesis $a_1 < a_2$. Therefore $b_1 \leq b_2$. The inequality (276) is equivalent to $b_1 - b_2 - 1 < 0$, which is satisfied since $b_1 - b_2 - 1 \leq b_2 - b_2 - 1 = -1 < 0$.

Case 3. In this case we may write $a_1 = b_1 t, a_2 = b_2 t + c_2$, where $c_2 \in \{0, \dots, t-2\}$. And inequality (277) is equivalent to $b_2 t + c_2 \geq b_1 t - b_1 + b_2$. Let us consider separately the case $c_2 = 0$ from the one in which $c_2 \neq 0$.

First assume $c_2 = 0$. If $b_1 = b_2$ then in (277) equality holds, therefore $f_\Omega(x) = f_\Omega(y)$. If $b_1 < b_2$ then (277) is equivalent to $(b_2 - b_1)(t-1) \geq 0$, clearly true (notice that here the strict inequality holds). Notice that the case $b_1 > b_2$ does not arise, since this would imply $a_1 > a_2$.

If $c_2 \neq 0$, again we need to show $b_1 \leq b_2$. Thus (277) is equivalent to $b_2 t + c_2 \geq b_1 t - b_1 + b_2$, which can be written as $(b_2 - b_1)(t-1) + c_2 \geq 0$. Assume $b_1 > b_2$, hence $b_1 \geq b_2 + 1$; then $a_1 = b_1 t \geq (b_2 + 1)t > b_2 t + c_2 = a_2$ which is absurd. Hence $b_1 \leq b_2$ and we have $(b_2 - b_1)(t-1) + c_2 \geq c_2 > 0$, also here the strict inequality holds.

Case 4. In the fourth case we have $a_1 = b_1 t + c_1, a_2 = b_2 t + (t-1)$ where $c_1 \in \{1, \dots, t-1\}$. And inequality (278) is equivalent to $(b_1 - b_2)(t-1) + (c_1 - (t-1)) \leq 0$. This last inequality is always true since $b_1 \leq b_2$ (for the same reasons as in the previous cases) and $c_1 \leq t-1$. Moreover equality holds if, and only if, $b_1 = b_2$ and $c_1 = t-1$, i.e. $a_1 = a_2$ and $a_1 \equiv t-1 \pmod t$. *q.e.d.*

B.4.3. Semisimple elements: lower bounds, $G = \mathrm{Sp}_n$.

Proof of Lemma 19.3.11. Let $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H$ be a non-special element of prime order r. Since x is not special, thanks to Lemma 19.3.10, one of the following holds: $a_1 - a_2 \geq 2, a_0 - a_1 \geq 2$ or $a_1 - a_0 \geq 2$. For each of these three cases we have defined an element y, see (234), (235) and (236). We claim that $f^{\circ}_{\Omega}(x) \geq f^{\circ}_{\Omega}(y)$.

The claim is equivalent to the following

$$\dim y^G - \dim x^G \geq \dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ}) \tag{279}$$

Let us study the three different cases.

Assume $\mathbf{a_1 - a_2 \geq 2}$. Then, for y as in (234) we have $\dim y^G - \dim x^G = 2(a_1 - a_2 - 1)$. Using the formula in Theorem 17.3.8, we have that $\dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ})$ is the same as in the proof of Lemma 18.3.11, with a_1 and a_2 in the place of a_0 and a_1, respectively. The same argument verbatim applies here.

Assume $\mathbf{a_0 - a_1 \geq 2}$. Let y as in (235). We compute $\dim y^G - \dim x^G = 2(a_0 - a_1 - 1)$. Let $a_0 = c_0 t + b_0 1$ and $a_1 = c_1 t + b_1$, with $0 \leq b_i < t$, $i = 1, 2$. Assume $c_0 - c_1 < 0$ then $a_0 - a_1 = (c_0 - c_1)t + b_0 - b_1 \leq -t + b_0 - b_1 < 0$ which is absurd, so $c_0 - c_1 \geq 0$.

Using Theorem 17.3.8 we have

$$\begin{aligned}
\dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ}) = 2 &+ 2t\Big(\Big\lfloor \frac{a_0 - 2}{2t} \Big\rfloor^2 - \Big\lfloor \frac{a_0}{2t} \Big\rfloor^2\Big) + 4\Big\lfloor \frac{a_0 - 2}{2t} \Big\rfloor \\
&+ 2(t - a_0)\Big(\Big\lfloor \frac{a_0 - 2}{2t} \Big\rfloor - \Big\lfloor \frac{a_0}{2t} \Big\rfloor\Big) - 2\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor \\
&+ \Big(\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_1}{t} \Big\rfloor^2\Big)t \\
&+ (t - 2a_1)\Big(\Big\lfloor \frac{a_1 + 1}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big)
\end{aligned}$$

We follow the same argument as that given in Lemma 18.3.11. Hence we have four cases depending on the values of the floor functions appearing in $\dim(y^G \cap H^{\circ}) - \dim(x^G \cap H^{\circ})$. Notice that $a_0/2 = \lfloor a_0/2t \rfloor t + b_0$ and $a_1 = \lfloor a_1/t \rfloor t + b_1$, where $0 \leq b_i < t$ for $i = 0, 1$. In the notation introduced above $c_0 = 2\lfloor a_0/2t \rfloor$ and $c_1 = \lfloor a_1/t \rfloor$.

Case 1. Assume $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$. Hence $b_0 > 0$ and $b_1 < t - 1$. Then (279) is equivalent to

$$a_0 - a_1 - 1 \geq 1 + 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor \tag{280}$$

Recall that $2\lfloor \frac{a_0}{2t} \rfloor - \lfloor \frac{a_1}{t} \rfloor \geq 0$, if equality holds then $a_0 - a_1 = 2b_0 - b_1 \geq 2$, hence (280) is satisfied. If $2\lfloor \frac{a_0}{2t} \rfloor - \lfloor \frac{a_1}{t} \rfloor \geq 1$ then

$$\Big(2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big)(t - 1) + 2b_0 - b_1 - 2 \geq (t - 1) + 2 - (t - 2) - 2 > 0$$

where we used $2b_0 \geq 2$ and $b_1 \leq t - 2$. In particular, (280) is satisfied.

Case 2. If $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$ then $b_0 = 0$ and $b_1 = t - 1$. Thus $a_0 - a_1 = (c_0 - c_1)t - (t - 1) \geq 2$, in particular $c_0 > c_1$. And (279) is equivalent to

$$2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor - 1 \geq 0 \tag{281}$$

which is satisfied by the observation that $2\lfloor \frac{a_0}{2t} \rfloor - \lfloor \frac{a_1}{t} \rfloor \geq 0$.

Case 3. Assume $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$. Hence $b_0 = 0$ and $b_1 < t-1$. Hence (279) is equivalent to

$$a_1 \leq 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor (t-1) + \Big\lfloor \frac{a_1}{t} \Big\rfloor \tag{282}$$

Substituting the value $a_1 = \lfloor a_1/t \rfloor t + b_1$ we have

$$2\Big\lfloor \frac{a_0}{2t} \Big\rfloor (t-1) + \Big\lfloor \frac{a_1}{t} \Big\rfloor - a_1 = 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor (t-1) - \Big\lfloor \frac{a_1}{t} \Big\rfloor (t-1) - b_1 > \Big(2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor - 1\Big)(t-1)$$

In the case $2\lfloor \frac{a_0}{2t} \rfloor - \lfloor \frac{a_1}{t} \rfloor = 0$ we have $a_0 - a_1 = 2b_0 - b_1 = -b_1 \leq 0$ which is absurd since $a_0 - a_1 \geq 2$. Therefore (282) is satisfied.

Case 4. Assume $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$. Hence $b_0 > 0$ and $b_1 = t-1$. Hence (279) is equivalent to

$$a_0 - 1 \geq 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor + \Big\lfloor \frac{a_1}{t} \Big\rfloor (t-1) + t \tag{283}$$

Again, substituting the value $a_0 = 2\lfloor a_0/2t \rfloor t + 2b_0$, we have

$$\begin{aligned} a_0 - 1 - 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor (t-1) - t &= 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor (t-1) - \Big\lfloor \frac{a_1}{t} \Big\rfloor (t-1) - t + 2b_0 - 1 \\ &= \Big(2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor - 1\Big)(t-1) + 2(b_0 - 1) \end{aligned}$$

If $2\lfloor \frac{a_0}{2t} \rfloor - \lfloor \frac{a_1}{t} \rfloor \geq 1$ we see that (283) is satisfied, since $b_0 \geq 1$. Hence, assume $2\lfloor \frac{a_0}{2t} \rfloor - \lfloor \frac{a_1}{t} \rfloor = 0$, in particular $a_0 - a_1 = 2b_0 - b_1$ and since $a_0 - a_1 \geq 2$ we deduce $2(b_0 - 1) \geq b_1 = t - 1$. We have

$$a_0 - 1 - 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor (t-1) - t = -(t-1) + 2(b_0 - 1) \geq 0$$

where the last equality follows from $b_1 = t - 1$. Therefore (283) holds.

Assume $\mathbf{a_1 - a_0 \geq 2}$. Let y as in (236). We compute $\dim y^G - \dim x^G = 2(a_1 - a_0 - 2)$. Write $a_0 = c_0 t + b_0$ and $a_1 = c_1 t + b_1$, with $0 \leq b_i < t$, as before. Again $c_0 \leq c_1$. We have

$$\begin{aligned} \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) &= -2 + 2t\Big(\Big\lfloor \frac{a_0+2}{2t} \Big\rfloor^2 - \Big\lfloor \frac{a_0}{2t} \Big\rfloor^2\Big) - 4\Big\lfloor \frac{a_0+2}{2t} \Big\rfloor \\ &\quad + 2(t - a_0)\Big(\Big\lfloor \frac{a_0+2}{2t} \Big\rfloor - \Big\lfloor \frac{a_0}{2t} \Big\rfloor\Big) + 2\Big\lfloor \frac{a_1-1}{t} \Big\rfloor \\ &\quad + \Big(\Big\lfloor \frac{a_1-1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_1}{t} \Big\rfloor^2\Big)t \\ &\quad + (t - 2a_1)\Big(\Big\lfloor \frac{a_1-1}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big) \end{aligned}$$

We use the same procedure as in the previous case. Write $a_0/2 = \lfloor a_0/2t \rfloor t + b_0$ and $a_1 = \lfloor a_1/t \rfloor t + b_1$, where $0 \leq b_i < t$ for $i = 0, 1$.

Case 1. Assume $\lfloor \frac{a_0+2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor$ and $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$. Hence $b_0 < t-1$ and $b_1 > 0$. So (279) is equivalent to

$$a_1 - a_0 - 2 \geq \Big\lfloor \frac{a_1}{t} \Big\rfloor - 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor - 1 \tag{284}$$

In the case $\lfloor a_1/t \rfloor = 2\lfloor a_0/2t \rfloor$ we have $a_1 - a_0 = b_1 - 2b_0 \geq 2$, and so (284) is satisfied. Assume $\lfloor a_1/t \rfloor > 2\lfloor a_0/2t \rfloor$. Then we have

$$(285) \qquad \begin{aligned} a_1 - a_0 - 1 - \left\lfloor \frac{a_1}{t} \right\rfloor + 2\left\lfloor \frac{a_0}{2t} \right\rfloor &= a_1\Big(1 - \frac{1}{t}\Big) - a_0\Big(1 - \frac{1}{t}\Big) + \frac{b_1 - 2b_0 - t}{t} \\ &= (a_1 - a_0)\Big(1 - \frac{1}{t}\Big) + \frac{b_1 - 2b_0 - t}{t} \\ &\geq 2\Big(1 - \frac{1}{t}\Big) + \frac{1 - 2t + 4 - t}{t} = -1 + \frac{3}{t} \end{aligned}$$

where we used $a_1 - a_0 \geq 2$, $b_1 \geq 1$ and $b_0 \leq t - 2$. Notice that the left hand side of (285) is an integer. Therefore we deduce that (284) is satisfied.

Case 2. Assume $\lfloor \frac{a_0+2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor + 1$ and $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor - 1$. Then $b_0 = t - 1$ and $b_1 = 0$. So that $a_1 - a_0 = (\lfloor a_1/t \rfloor - 2\lfloor a_0/2t \rfloor)t - 2t + 2 \geq 2$, thus $\lfloor a_1/t \rfloor - 2\lfloor a_0/2t \rfloor - 2 \geq 0$. We see that (279) is equivalent to

$$(286) \qquad \left\lfloor \frac{a_1}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor - 2 \geq 0$$

which is satisfied by the previous observation.

Case 3. Assume $\lfloor \frac{a_0+2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor + 1$ and $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$ we have $b_0 = t - 1$ and $b_1 > 0$. Hence (279) is equivalent to

$$(287) \qquad a_1 \geq -1 + \left\lfloor \frac{a_1}{t} \right\rfloor + 2\left\lfloor \frac{a_0}{2t} \right\rfloor (t - 1) + 2t$$

And we have

$$\begin{aligned} a_1 + 1 - \left\lfloor \frac{a_1}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor (t-1) - 2t &= a_1 + 1 - \frac{a_1 - b_1}{t} - \frac{a_0 - 2(t-1)}{t}(t-1) - 2t \\ &= a_1\Big(1 - \frac{1}{t}\Big) + \frac{b_1}{t} - \frac{a_0}{t} - 3 + \frac{2}{t} \\ &\geq (a_0 + 2)\Big(1 - \frac{1}{t}\Big) + \frac{b_1}{t} - \frac{a_0}{t} - 3 + \frac{2}{t} \\ &= a_0 - 2\frac{a_0}{t} + \frac{b_1}{t} - 1 \geq a_0\Big(1 - \frac{2}{t}\Big) + \frac{1}{t} - 1 \\ &\geq \frac{1}{t} - 1 \end{aligned}$$

where the first inequality is due to $a_1 - a_0 \geq 2$, the second to $b_1 \geq 1$ and the last to $t \geq 2$. In particular since $a_1 + 1 - \lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0}{2t} \rfloor (t-1) - 2t$ is an integer we deduce that it is always non-negative. Hence (287) is satisfied.

Case 4. Assume $\lfloor \frac{a_0+2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor$ and $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor - 1$ we have $b_0 < t - 1$ and $b_1 = 0$. Hence (279) is equivalent to

$$(288) \qquad a_0 \leq \left\lfloor \frac{a_1}{t} \right\rfloor (t - 1) + 2\left\lfloor \frac{a_0}{2t} \right\rfloor$$

Notice that $\lfloor \frac{a_1}{t} \rfloor = a_1/t$. Hence we compute

$$\begin{aligned} \left\lfloor \frac{a_1}{t} \right\rfloor (t-1) + 2\left\lfloor \frac{a_0}{2t} \right\rfloor - a_0 &= a_1\Big(1 - \frac{1}{t}\Big) - a_0 + \frac{a_0 - 2b_0}{t} \\ &= (a_1 - a_0)\Big(1 - \frac{1}{t}\Big) - 2\frac{b_0}{t} \geq 2\Big(1 - \frac{1}{t}\Big) - 2\frac{t-2}{t} = \frac{2}{t} > 0 \end{aligned}$$

where we used $a_1 - a_0 \geq 2$ and $b_0 \leq t-2$. Therefore (288) is satisfied and the proof is complete. *q.e.d.*

B.4.4. Local upper bounds, $G = \mathrm{Sp}_n$.

Proof of Lemma 19.4.7. Write $n = (n-s)l + m$, where $0 \leq m < n-s$. Here we assume $x = [I_{a_0}, \omega I_{n-s}, \omega^2 I_{a_2}, \ldots, \omega^{r-1} I_{n-s}] \in H \cap \mathcal{V}_{s,r}$. In addition, we assume that the centraliser of x is not isomorphic to the centraliser of one of the following (depending on the parity of l):

$$[I_m, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{l/2} I_{n-s}],\ [I_{m+1}, (\omega,\omega^{-1})I_{n-s}, \ldots, (\omega,\omega^{-1})^{(l-1)/2} I_{n-s}]$$

So, thanks to Lemma 19.4.6, we may assume that one of the following holds:

(i) $a_0 = \max\{b_i : b_i < n-s-1\}$ and $a_2 = \min\{a_i : b_i > 0\}$;

(ii) $a_0 = \min\{b_i : b_i > 0\}$ and $a_2 = \max\{a_i : b_i < n-s\}$;

(iii) $a_2 = \max\{b_i : b_i < n-s\}$ and $a_3 = \min\{a_i : b_i > 0\}$;

In each of these three cases we define $y \in H \cap \mathcal{V}_{s,r}$ as in (244), (245) or (246). Then we claim $f^\circ_\Omega(x) \leq f^\circ_\Omega(y)$.

We only need to check cases (i) and (ii), since in case (iii) we get the same inequality as that shown in the proof of Lemma 18.4.7, (as in Lemma 19.4.5). We need to show

$$\dim y^G - \dim x^G \leq \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \tag{289}$$

Assume case (i) holds, in particular $a_0 \geq a_2$. Write $a_0 = b_0' t + c_0'$ and $a_2 = b_2 t + c_2$, and assume $b_0' < b_2$, then $a_0 - a_2 = (b_0' - b_2)t + (c_0' - c_2) \leq -t + t - 1 = -1$, therefore $b_0' \geq b_2$. Let y as in (244). Then we compute $\dim y^G - \dim x^G = -2(a_0 - a_2 + 2)$. And, using Theorem 17.3.8, we get

$$\begin{aligned}\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = &-2 + 2t\Big(\Big\lfloor \frac{a_0+2}{2t}\Big\rfloor^2 - \Big\lfloor \frac{a_0}{2t}\Big\rfloor^2\Big) - 4\Big\lfloor \frac{a_0+2}{2t}\Big\rfloor \\ &+ 2(t-a_0)\Big(\Big\lfloor \frac{a_0+2}{2t}\Big\rfloor - \Big\lfloor \frac{a_0}{2t}\Big\rfloor\Big) \\ &+ t\Big(\Big\lfloor \frac{a_2-1}{t}\Big\rfloor^2 - \Big\lfloor \frac{a_2}{t}\Big\rfloor^2\Big) + 2\Big\lfloor \frac{a_2-1}{t}\Big\rfloor \\ &+ (t-2a_2)\Big(\Big\lfloor \frac{a_2-1}{t}\Big\rfloor - \Big\lfloor \frac{a_2}{t}\Big\rfloor\Big)\end{aligned}$$

As usual, we need to study the four cases that arise from the different possible values of the floor functions.

Case 1. If $\lfloor \frac{a_0+2}{2t}\rfloor = \lfloor \frac{a_0}{2t}\rfloor$ and $\lfloor \frac{a_2-1}{t}\rfloor = \lfloor \frac{a_2}{t}\rfloor$. We have that (289) is equivalent to

$$a_0 - a_1 + 2 \geq 1 + 2\Big\lfloor \frac{a_0}{2t}\Big\rfloor - \Big\lfloor \frac{a_2}{t}\Big\rfloor \tag{290}$$

Let us write $a_0/2 = b_0 t + c_0$ and $a_2 = b_2 t + c_2$ where $0 \leq c_0 < t-1$ and $0 < c_2 < t$. In the case $c_0 < t/2$ we have $b_0' = 2b_0'$ and $c_0' = 2c_0$, so $2b_0 \geq b_2$, by the previous discussion. If $t/2 \geq c_0 < t$ then $b_0' = 2b_0 + 1$ and $c_0' = 2c_0 - t$, thus $2b_0 + 1 \geq b_2$.

Substituting these values in (290) we see that it is equivalent to

$$(t-1)(2b_0 - b_2) + (2c_0 - c_2) + 1 \geq 0$$

First assume $c_0 < t/2$. If $2b_0 = b_2$ then $a_0 - a_2 = 2c_0 - c_2$ and the previous inequality is satisfied. If $2b_0 > b_2$ then, using $2c_0 \geq 0,\ c_2 < t$, we have $(t-1)(2b_0 - b_2) + (2c_0 - c_2) + 1 > t - 1 + (0 - t) + 1 = 0$.

Now assume $c_0 \geq t/2$, so that $2b_0+1 \geq b_2$. If $2b_0+1 = b_2$ then $a_0-a_2 = 2c_0-t+c_2 \geq 0$, hence $(t-1)(2b_0-b_2)+(2c_0-c_2)+1 = 2c_0-t-c_2+2 \geq 2$. If, instead, $2b_0+1 > b_2$ then $2b_0-b_2 \geq 0$. In the case $2b_0 = b_2$ we have $a_0 - a_2 = 2c_0 - c_2 \geq 0$, hence $(t-1)(2b_0 - b_2) + (2c_0 - c_2) + 1 = 2c_0 - c_2 + 1 \geq 1$. If $2b_0 > b_2$ then $(t-1)(2b_0 - b_2) + (2c_0 - c_2) + 1 \geq t - 1 + 2c_0 - c_2 + 1 \geq t - 1 + t - (t-1) + 1 = t + 1 > 0$, where we used $2c_0 \geq t$ and $c_2 \leq t - 1$.

Case 2. If $\lfloor \frac{a_0+2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor + 1$ and $\lfloor \frac{a_2-1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor - 1$. Thus, $a_0/2 = b_0t + t - 1$ and $a_2 = b_2t$. We see that (289) is equivalent to

$$2\left\lfloor \frac{a_0}{2t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor + 2 \geq 0 \tag{291}$$

which is clearly true since $2\lfloor a_0/2t \rfloor - \lfloor a_2/t \rfloor \geq -1$ as shown in the previous case.

Case 3. If $\lfloor \frac{a_0+2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor + 1$ and $\lfloor \frac{a_2-1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor$. Thus, $a_0/2 = b_0t + t - 1$ and $a_2 = b_2t + c_2$ where $0 < c_2 < t$, in particular $a_0 = (2b_0 + 1)t + t - 2$. We see that (289) is equivalent to

$$a_2 \leq 2\left\lfloor \frac{a_0}{2t} \right\rfloor (t-1) + \left\lfloor \frac{a_2}{t} \right\rfloor + 2t - 1 \tag{292}$$

which is equivalent to

$$\left(2\left\lfloor \frac{a_0}{2t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor\right)(t-1) - c_2 + 2t - 1 \geq 0$$

Recall that $2\lfloor a_0/2t \rfloor - \lfloor a_2/t \rfloor \geq -1$ and $c_2 \leq t - 1$. Hence

$$\left(2\left\lfloor \frac{a_0}{2t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor\right)(t-1) - c_2 + 2t - 1 \geq -(t-1) - (t-1) + 2t - 1 = 1$$

Case 4. If $\lfloor \frac{a_0+2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor$ and $\lfloor \frac{a_2-1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor - 1$. Thus, $a_0/2 = b_0t + c_0$ and $a_2 = b_2t$ where $0 < c_0 < t$. We see that (289) is equivalent to

$$a_0 \geq 2\left\lfloor \frac{a_0}{2t} \right\rfloor + \left\lfloor \frac{a_2}{t} \right\rfloor (t-1)$$

which is equivalent to

$$\left(2\left\lfloor \frac{a_0}{2t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor\right)(t-1) + 2c_0 \geq 0 \tag{293}$$

Recall that $2\lfloor a_0/2t \rfloor - \lfloor a_2/t \rfloor \geq -1$. If $2\lfloor a_0/2t \rfloor - \lfloor a_2/t \rfloor = -1$ then $a_0 - a_2 = -t + 2c_0$ hence $2c_0 \geq t$. In particular (293) becomes $-(t-1) + 2c_0 \geq 0$ and it is satisfied. In the case $2\lfloor a_0/2t \rfloor - \lfloor a_2/t \rfloor \geq 0$ we immediately deduce that (293) is satisfied since $c_0 \geq 0$.

Assume case (ii) holds, hence $a_0 \leq a_2$. If we write $a_0 = b_0't + c_0$ and $a_2 = b_2t + c_2$ with $0 \leq c_0', c_2 < t$, as above we can show $b_0' \leq b_2$. Let y be as in (245). Then we compute $\dim y^G - \dim x^G = 2(a_0 - a_2 - 1)$. And, using Theorem 17.3.8 we get

$$\begin{aligned}
\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) &= 2 + 2t\left(\left\lfloor \frac{a_0-2}{2t} \right\rfloor^2 - \left\lfloor \frac{a_0}{2t} \right\rfloor^2\right) + 4\left\lfloor \frac{a_0-2}{2t} \right\rfloor \\
&+ 2(t-a_0)\left(\left\lfloor \frac{a_0-2}{2t} \right\rfloor - \left\lfloor \frac{a_0}{2t} \right\rfloor\right) \\
&+ t\left(\left\lfloor \frac{a_2+1}{t} \right\rfloor^2 - \left\lfloor \frac{a_2}{t} \right\rfloor^2\right) - 2\left\lfloor \frac{a_2-1}{t} \right\rfloor \\
&+ (t-2a_2)\left(\left\lfloor \frac{a_2+1}{t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor\right)
\end{aligned}$$

We argue as before.

Case 1. If $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor$. We have that (289) is equivalent to

$$a_0 - a_2 - 1 \geq 1 + 2\left\lfloor \frac{a_0}{2t} \right\rfloor - \left\lfloor \frac{a_2}{t} \right\rfloor \tag{294}$$

Writing $a_0 = 2\lfloor a_0/2t \rfloor + 2c_0$ and $a_2 = \lfloor a_2/2t \rfloor + c_2$ we see that (294) is equivalent to

$$\left(\left\lfloor \frac{a_2}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor\right)(t-1) + c_2 - 2c_0 + 2 \geq 0 \tag{295}$$

In the case $c_0 < t/2$ then $b_0' = 2\lfloor a_0/2t \rfloor$ and $c_0' = c_0$ hence $2\lfloor a_0/2t \rfloor \leq \lfloor a_2/t \rfloor$. If $2\lfloor a_0/2t \rfloor = \lfloor a_2/t \rfloor$ then $a_2 - a_0 = c_2 - 2c_0 \geq 0$, in particular (295) is satisfied. If $2\lfloor a_0/2t \rfloor < \lfloor a_2/t \rfloor$ then

$$\left(\left\lfloor \frac{a_2}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor\right)(t-1) + c_2 - 2c_0 + 2 > (t-1) - t + 2 > 0$$

where we used $c_2 \geq 0$ and $2c_0 < t$.

If $c_0 \geq t/2$ then $b_0' = 2\lfloor a_0/2t \rfloor + 1$ and $c_0' = 2c_0 - t$. Hence $2\lfloor a_0/2t \rfloor + 1 \leq \lfloor a_2/t \rfloor$. First suppose $2\lfloor a_0/2t \rfloor + 1 = \lfloor a_2/t \rfloor$ then $a_2 - a_0 = c_2 - (2c_0 - t) = t + c_2 - 2c_0 \geq 0$. Thus

$$\left(\left\lfloor \frac{a_2}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor\right)(t-1) + c_2 - 2c_0 + 2 = (t-1) + c_2 - 2c_0 + 2 \geq 1$$

Now assume $2\lfloor a_0/2t \rfloor + 1 < \lfloor a_2/t \rfloor$ then

$$\left(\left\lfloor \frac{a_2}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor\right)(t-1) + c_2 - 2c_0 + 2 \geq 2(t-1) + c_2 - 2c_0 + 2 > 0$$

where we used $c_2 \geq 0$ and $c_0 < t$.

Case 2. Assume $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor - 1$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor + 1$. In particular, (289) is equivalent to

$$\left\lfloor \frac{a_2}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor + 1 \geq 0 \tag{296}$$

which is clearly true since $\lfloor a_2/t \rfloor - 2\lfloor a_0/2t \rfloor \geq 0$ as shown in the previous case.

Case 3. If $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor - 1$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor$. Thus (289) is equivalent to

$$a_2 \geq 2\left\lfloor \frac{a_0}{2t} \right\rfloor (t-1) + \left\lfloor \frac{a_2}{t} \right\rfloor$$

which is equivalent to

$$\left(\left\lfloor \frac{a_2}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor\right)(t-1) + c_2 \geq 0 \tag{297}$$

which is satisfied since $c_2 \geq 0$ and $\lfloor a_2/t \rfloor - 2\lfloor a_0/2t \rfloor \geq 0$.

Case 4. Assume $\lfloor \frac{a_0-2}{2t} \rfloor = \lfloor \frac{a_0}{2t} \rfloor$ and $\lfloor \frac{a_2+1}{t} \rfloor = \lfloor \frac{a_2}{t} \rfloor + 1$. We see that (289) is equivalent to

$$a_0 - 1 \leq 2\left\lfloor \frac{a_0}{2t} \right\rfloor + \left\lfloor \frac{a_2}{t} \right\rfloor (t-1) + t$$

using $a_0 = \lfloor a_0/2t \rfloor t + 2c_0$ we see that it is equivalent to

$$\left(\left\lfloor \frac{a_2}{t} \right\rfloor - 2\left\lfloor \frac{a_0}{2t} \right\rfloor\right)(t-1) + t + 1 - 2c_0 \geq 0 \tag{298}$$

In the case $\lfloor a_2/t \rfloor = 2\lfloor a_0/2t \rfloor$ we have $a_2 - a_0 = c_2 - 2c_0 = t - 1 - 2c_0 \geq 0$. Hence (298) is satisfied. If, instead, $\lfloor a_2/t \rfloor > 2\lfloor a_0/2t \rfloor$, then, using $c_0 < t$,

$$\Big(\Big\lfloor \frac{a_2}{t} \Big\rfloor - 2\Big\lfloor \frac{a_0}{2t} \Big\rfloor\Big)(t-1) + t + 1 - 2c_0 > t - 1 + t + 1 - 2t = 0$$

The proof is complete. *q.e.d.*

B.4.5. Semisimple elements: global lower bounds, $G = \mathrm{O}_n$**.** Here $H = \mathrm{O}_{n/t} \wr S_t$, where $1 < t < n$ and n/t is odd.

Proof of Lemma 20.3.21. Let $x = [I_{a_0}, \omega I_{a_1}, \dots, \omega^{r-1} I_{a_{r-1}}] \in H^\circ$ be non-special. Then one of the following three cases occurs: $a_1 - a_0 \geq 2$, $a_0 - a_1 \geq 2$ or $a_1 - a_2 \geq 2$. Then we define $y \in H^\circ$ as in (266), (267) or (268). We claim $f^\circ_\Omega(x) \geq f^\circ_\Omega(y)$.

As done for Lemma 19.3.11 we need to show

$$\dim y^G - \dim x^G \geq \dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) \tag{299}$$

We write $a_1 = \lfloor \frac{a_1}{t} \rfloor t + b_1$ and $\frac{a_0 - t}{2} = \lfloor \frac{a_0 - t}{2t} \rfloor t + b_0$.

Assume $\mathbf{a_1 - a_0 \geq 2}$. Using Theorem 5.3.1 we compute $\dim y^G - \dim x^G = 2(a_1 - a_0 - 1)$. First let us observe that $\lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0 - t}{2t} \rfloor \geq 0$. In fact if the difference is negative then, using $b_1 < t$ and $b_0 \geq 0$,

$$2 \leq a_1 - a_0 = \Big(\Big\lfloor \frac{a_1}{t} \Big\rfloor - 2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor\Big)t + b_1 - 2b_0 - t < b_1 - 2b_0 - t < 0$$

we have a contradiction.

Using Theorem 17.3.8 we compute

$$\begin{aligned}
\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = {} & -2 + t\Big(\Big\lfloor \frac{a_1 - 1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_1}{t} \Big\rfloor^2\Big) + 2\Big\lfloor \frac{a_1 - 1}{t} \Big\rfloor \\
& + (t - 2a_1)\Big(\Big\lfloor \frac{a_1 - 1}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big) - 4\Big\lfloor \frac{a_0 + 2 - t}{2t} \Big\rfloor \\
& + 2t\Big(\Big\lfloor \frac{a_0 + 2 - t}{2t} \Big\rfloor^2 - \Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor^2\Big) \\
& + 2(2t - a_0)\Big(\Big\lfloor \frac{a_0 + 2 - t}{2t} \Big\rfloor - \Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor\Big)
\end{aligned}$$

We have four cases depending on the values of the floor functions.

Case 1. Assume $\lfloor \frac{a_0 + 2 - t}{2t} \rfloor = \lfloor \frac{a_0 - t}{2t} \rfloor$ and $\lfloor \frac{a_1 - 1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$. In this case (299) is equivalent to

$$a_1 - a_0 - \Big\lfloor \frac{a_1}{t} \Big\rfloor + 2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor \geq 0$$

Using $a_1 - a_0 \geq 2, b_0 \leq t - 2$ and $b_1 \geq 1$, we compute

$$\begin{aligned}
a_1 - a_0 - \Big\lfloor \frac{a_1}{t} \Big\rfloor + 2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor &= (a_1 - a_0)\Big(1 - \frac{1}{t}\Big) + \frac{b_1 - 2b_0 - t}{t} \\
&\geq 2\Big(1 - \frac{1}{t}\Big) + \frac{1 - 2t + 4 - t}{t} = 1 + \frac{2}{t}
\end{aligned}$$

since the left hand side of the previous equation is an integer we deduce that is non-negative.

Case 2. Assume $\lfloor \frac{a_0+2-t}{2t} \rfloor = \lfloor \frac{a_0-t}{2t} \rfloor + 1$ and $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor - 1$. Notice that $b_1 = 0$ and $b_0 = t-2$. In this case (299) is equivalent to

$$\Big\lfloor \frac{a_1}{t} \Big\rfloor - 2\Big\lfloor \frac{a_0-t}{2t} \Big\rfloor - 3 \geq 0$$

We have $2 \leq a_1 - a_0 = \big(\lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0-t}{2t} \rfloor - 3\big)t + 4$, hence $\big(\lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0-t}{2t} \rfloor - 3\big)t + 2 \geq 0$.

Seeking a contradiction, assume $\lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0-t}{2t} \rfloor - 3 \leq -1$. Then $-t+2 \geq \big(\lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0-t}{2t} \rfloor - 3\big)t + 2 \geq 0$, which is true if, and only if, $\lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0-t}{2t} \rfloor - 3 = -1$ and $t = 2$. In particular we have $0 = \lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0-t}{2t} \rfloor - 2 = \frac{a_1}{2} - 2\lfloor \frac{a_0-t}{2t} \rfloor - 2$, i.e. $a_1 - 4\lfloor \frac{a_0-t}{2t} \rfloor - 4 = 0$. But $a_0 = 4\lfloor \frac{a_0-t}{2t} \rfloor - 4$ and $2 \leq a_1 - a_0$ hence we get a contradiction.

Case 3. Assume $\lfloor \frac{a_0+2-t}{2t} \rfloor = \lfloor \frac{a_0-t}{2t} \rfloor + 1$ and $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$. Here $b_0 = t-1$ and $b_1 > 0$. In this case (299) is equivalent to

$$a_1 \geq \Big\lfloor \frac{a_1}{t} \Big\rfloor - 2\Big\lfloor \frac{a_0-t}{2t} \Big\rfloor (t-1) + 3t - 2$$

which is equivalent to $a_1 - a_0 \geq \lfloor \frac{a_1}{t} \rfloor - 2\lfloor \frac{a_0-t}{2t} \rfloor = \frac{a_1-b_1}{t} - \frac{a_0-2b_0-t}{t}$. Therefore 299 is equivalent to

$$a_1\Big(1-\frac{1}{t}\Big) - a_0\Big(1-\frac{1}{t}\Big) + \frac{b_1-2b_0-t}{t} \geq 0$$

notice that the left hand side of the previous expression is an integer. Using $a_1 - a_0 \geq 2$ and the bounds on b_i's, we have

$$a_1\Big(1-\frac{1}{t}\Big) - a_0\Big(1-\frac{1}{t}\Big) + \frac{b_1-2b_0-t}{t} \geq 2\Big(1-\frac{1}{t}\Big) - 3\Big(1-\frac{1}{t}\Big) = -\Big(1-\frac{1}{t}\Big)$$

In particular (299) is verified.

Case 4. Assume $\lfloor \frac{a_0+2-t}{2t} \rfloor = \lfloor \frac{a_0-t}{2t} \rfloor$ and $\lfloor \frac{a_1-1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor - 1$. Here $b_0 < t-1$ and $b_1 = 0$. In this case (299) is equivalent to

$$a_1\Big(1-\frac{1}{t}\Big) - a_0 + 2\Big\lfloor \frac{a_0-t}{2t} \Big\rfloor + 1 \geq 0$$

which is true if and only if $(a_1 - a_0)\big(1-\frac{1}{t}\big) - 2\frac{b_0}{t} \geq 0$. We have

$$(a_1 - a_0)\Big(1-\frac{1}{t}\Big) - 2\frac{b_0}{t} \geq 2\Big(1-\frac{1}{t}\Big) - 2\frac{t-2}{t} = \frac{2}{t} > 0$$

Assume $\mathbf{a_0 - a_1 \geq 2}$. Using Theorem 5.3.1 we compute $\dim y^G - \dim x^G = 2(a_0 - a_1 - 2)$. As above, we have $2\lfloor \frac{a_0-t}{2t} \rfloor - \lfloor \frac{a_1}{t} \rfloor \geq 0$. Moreover, by Theorem 17.3.8,

$$\begin{aligned}\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ) = 2 &+ t\Big(\Big\lfloor \frac{a_1+1}{t} \Big\rfloor^2 - \Big\lfloor \frac{a_1}{t} \Big\rfloor^2\Big) - 2\Big\lfloor \frac{a_1+1}{t} \Big\rfloor \\ &+ (t-2a_1)\Big(\Big\lfloor \frac{a_1+1}{t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big) + 4\Big\lfloor \frac{a_0-2-t}{2t} \Big\rfloor \\ &+ 2t\Big(\Big\lfloor \frac{a_0-2-t}{2t} \Big\rfloor^2 - \Big\lfloor \frac{a_0-t}{2t} \Big\rfloor^2\Big) \\ &+ 2(2t-a_0)\Big(\Big\lfloor \frac{a_0-2-t}{2t} \Big\rfloor - \Big\lfloor \frac{a_0-t}{2t} \Big\rfloor\Big)\end{aligned}$$

Again, we have four cases depending on the values of the floor functions. And we can prove each of them as in the previous case.

Case 1. Assume $\lfloor \frac{a_0-2-t}{2t} \rfloor = \lfloor \frac{a_0-t}{2t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$. In this case (299) is equivalent to

$$a_0 - a_1 - 3 \geq 2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor$$

which is equivalent to

$$(a_0 - a_1)\Big(1 - \frac{1}{t}\Big) + \frac{2b_0 - b_1 - 2t}{t} \geq 0$$

Since $a_0 - a_1 \geq 2, b_0 \geq 1$ and $b_1 \leq t - 2$ we have

$$(a_0 - a_1)\Big(1 - \frac{1}{t}\Big) + \frac{2b_0 - b_1 - 2t}{t} \geq 2\Big(1 - \frac{1}{t}\Big) + \frac{2 - t + 2 - 2t}{t} = -1 + \frac{2}{t}$$

Since the left hand side of the previous expression is an integer we deduce that (299) is satisfied.

Case 2. Assume $\lfloor \frac{a_0-2-t}{2t} \rfloor = \lfloor \frac{a_0-t}{2t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$. Hence $b_0 = 0$ and $b_1 = t - 1$. In this case (299) is equivalent to

$$2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor \geq 0$$

which is trivially satisfied.

Case 3. Assume $\lfloor \frac{a_0-2-t}{2t} \rfloor = \lfloor \frac{a_0-t}{2t} \rfloor - 1$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor$. Here $b_0 = 0$ and $b_1 \leq t - 2$. In this case (299) is equivalent to

$$a_1 + 1 \leq 2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor (t-1) + \Big\lfloor \frac{a_1}{t} \Big\rfloor + t$$

which is equivalent to

$$\Big(2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big)(t-1) + t - 1 - b_1 \geq 0$$

And since $b_1 \leq t - 2$ we deduce that the previous inequality is satisfied.

Case 4. Assume $\lfloor \frac{a_0-2-t}{2t} \rfloor = \lfloor \frac{a_0-t}{2t} \rfloor$ and $\lfloor \frac{a_1+1}{t} \rfloor = \lfloor \frac{a_1}{t} \rfloor + 1$. Here $b_0 > 0$ and $b_1 = t - 1$. In this case (299) is equivalent to

$$\Big(2\Big\lfloor \frac{a_0 - t}{2t} \Big\rfloor - \Big\lfloor \frac{a_1}{t} \Big\rfloor\Big)(t-1) + 2(b_0 - 1) \geq 0$$

which is clearly satisfied since $b_0 \geq 1$.

Assume $\mathbf{a_0 - a_1 \geq 2}$. Using Theorem 5.3.1 and Theorem 17.3.8 we have that $\dim y^G - \dim x^G$ and $\dim(y^G \cap H^\circ) - \dim(x^G \cap H^\circ)$ are the same as in the proof of Lemma 18.3.11, with a_1 and a_2 in the place of a_0 and a_1, respectively. The same argument verbatim applies here. *q.e.d.*

List of notations

k	algebraically closed field
p	characteristic of k (we set $p = \infty$ for characteristic zero)
G°	the connected component containing 1 (with respect to the Zariski topology) of the algebraic group G
V	finite dimensional k-vector space
$\mathrm{GL}(V),\ \mathrm{GL}_n$	general linear group on V
$\mathrm{Sp}(V),\ \mathrm{Sp}_n$	symplectic group on V
$\mathrm{O}(V),\ \mathrm{O}_n$	orthogonal group on V
$Cl(V),\ Cl_n$	any classical group on V
$o(x)$	order of x
$\mathscr{R}$	subset of G of elements of prime order, including all nontrivial unipotent element if $p = \infty$
$\nu(x)$	codimension of largest eigenspace of x
$\mathcal{V}_s$	subset of G of the elements x with $\nu(x) = s$
$\mathcal{V}_{s,r}$	subset of $\mathcal{V}_s$ of the elements of order r
S_t	symmetric group on t letters
r	prime integer
ω	a primitive r-th root of unity in k
π_h	permutation in S_t with cycle shape $(r^h, 1^{t-hr})$
Ω	coset variety G/H
$C_\Omega(x)$	fixed point space of $x \in G$, i.e. $\{\omega \in \Omega : x.\omega = \omega\}$
$f_\Omega(x)$	ratio $\dim C_\Omega(x)/\dim\Omega$
$f^\circ_\Omega(x)$	ratio $\dim C^\circ_\Omega(x)/\dim\Omega$, where $\dim C^\circ_\Omega(x) = \dim\Omega - \dim x^G + \dim(x^G \cap H^\circ)$
$\lfloor x \rfloor$	for $x \in \mathbb{R}$, the largest integer smaller than or equal to x
$\delta_{a,b}$	Kronecker delta, $\delta_{a,b} = 1$ if $a = b$ and 0 otherwise
$\delta_{a;b}$	Modular delta, $\delta_{a;b} = 1$ if $b \mid a$ and 0 otherwise

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