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AN EXPERIMENTAL INVESTIGATION OF THE NATURAL FREQUENCY STATISTICS OF A BEAM WITH SPATIALLY CORRELATED RANDOM MASSES

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Abstract. *Experimental investigations into the dynamic response of structures with material or geometrical random fields usually depend upon an initial characterization of this variability, with very little control over the statistics at its early manufacturing stage. This provides the need of a minimal number of samples to generate an ensemble of dynamic responses, making such experimental data scarcely found in the literature. In this work, a cantilever beam with small masses attached along its length according to a given discrete random field has an ensemble of natural frequencies measured for a number of correlation lengths. The results can be used to investigate the effects of the correlation length on the subsequent natural frequency statistics. The experimental results are compared with a wave approximation for flexural waves using a continuous random field for the mass density, in order to approximate the mass distribution. Issues concerning this approximation are discussed. In addition, results are also compared with a simple added mass approximation with assumed modes from a FE solution.*

Keywords. *Structural Dynamics, Random Fields, Waveguides, Experimental Characterization*

1. INTRODUCTION

Manufacturing processes often result in inherent variability of properties compared to the nominal designed product. As the requirements for optimum design increase, it might be important to improve prediction capability by including effects due to this variability. It is usual that mechanical properties of composite structures are modelled by analytical models taking into account mean or nominal properties of the structure, although they have inherent spatial variability.

Irrespective of what analytical approach is used, when predicting the response of structures with variability, some statistical model is required. Random fields theory (Vanmarcke 2010) provides the elements for this representation and it typically involves expressions for the mean and probability density function, together with a model for the spatial variability of the properties. This model is usually given by expressions of spatial variability by a given correlation function and correlation length. The most used methods of representing random field in a mechanical model includes the use of series expansions, like the Karhunen-Loeve decomposition or the Polynomial Chaos expansion (Ghanem & Spanos 1991), and also point discretization or average discretization methods (Sudret & Der Kiureghian 2000).

The characterization of the spatial variability becomes even more relevant when dealing with, for instance, fibre reinforced composite materials, for which different fibre arrangements can affect mechanical properties (Lei et al. 2012). Moreover, the use of purely deterministic approaches to model and, consequently, design structures using these materials can much limit applicability and necessitate the use of higher safety factors (Sriramula & Chryssanthopoulos 2009). Even though the inclusion of spatial variability and uncertainties of material and geometrical properties in mechanical models has received significant attention (e.g. (Der Kiureghian & Ke 1988, Ghanem & Spanos 1991, Schueller 1997, Stefanou 2009, Sudret & Der Kiureghian 2000, Zehn & Saitov 2003)) it is very difficult to measure and quantify them from experimental data. In general, little experimental data is available for practical structures, and the choice of models for spatial variability is made for convenience. A few measurement procedures for characterization of the spatial variability in fibre reinforced composite materials have been recently proposed, based on the volume fraction or fibre distribution (Baxter & Graham 2000, Guilleminot et al. 2008, 2009; Machina & Zehn 2007, Mehrez et al. 2012a,b).

As the requirements for optimum design increase and a broader frequency range is of interest, it is important to improve the prediction capability. Element based techniques, like the Finite Element (FE) method (Petyt 2010, Zienkiewicz & Morgan 1983), are the main prediction tools for structural dynamics in industrial applications. The inclusion of spatially random variability in FE models has also received much attention (Der Kiureghian & Ke 1988, Shang & Yun 2013, Sudret & Der Kiureghian 2000, Zehn & Saitov 2003). However, higher frequency ranges require higher mesh densities in FE analysis, leading to increased computational cost. The pollution effect, i.e. when the accuracy of the FE solution degenerates as the wavenumber increases (Deraemaeker et al. 1999), must also be taken into account, further increasing the computation cost. Wave-based methods, using analytical (Doyle 1997, Lee 2009), semi-analytical (Gavrić 1995) or numerical approximations (Mace et al. 2005), have been used to overcome these issues on waveguides and increase the computational efficiency from the FE like methods.

Although homogeneous waveguides have been extensively studied, analytical solutions for non-uniform waveguides are only possible for very particular cases. Much attention has been given to acoustic horns, ducts, rods and beams - e.g. (Eisenberger 1991, Elishakoff & Candan 2001, Guo & Yang 2012, Li 2000, Nagarkar & Finch 1971), particularly when the variation is small over a wavelength, such that there are negligible reflections due to the local impedance changes. Langley (Langley 1999) has shown, using a perturbation approach, that the amplitude of a wave travelling along a non-homogeneous one dimensional waveguide changes and the power is conserved, and he also addressed the issue of wave reflection at a cut off section. Lee et al. (Lee et al. 2007) has shown that the velocity of energy propagation is different to the group velocity for a class of non-homogeneity in one-dimensional waveguides. Scott (Scott 1985) considered the statistics of the wave intensity and phase for a class of one dimensional random waveguides, in specific cases, assuming negligible scattering. Manohar and Keane (Manohar & Keane 1993) have derived expressions for the PDF of the natural frequencies and mode shapes of a class of stochastic rods and also the flow of energy between coupled stochastic rods (Keane & Manohar 1993), in the context of SEA. Moreover, a numerical approach, using the WFE method, has been proposed by Ichchou (Ichchou et al. 2011) to include spatially homogeneous variability in waveguides by using a first order perturbation (Ben Souf et al. 2013, Bouchoucha et al. 2013).

The classical WKB approximation (named after Wentzel, Kramers and Brillouin), a method for finding suitable modifications of plane-wave solutions for propagation in slowly varying media (Pierce 1970), was initially developed for solving the Schrödinger equation in quantum mechanics, and is also called the semi-classical method (Firouz-Abadi et al. 2007) or the geometrical optics approximation (Nayfeh 1973, Whitham 1974). The approximation assumes that the waveguide properties vary slowly enough such that there is no or negligible reflections due to these locals changes, even if the net change is big over large distances. It has been applied in many fields of engineering, including ocean acoustics (Jensen et al. 2011), acoustics (Arenas & Crocker 2001, Biggs 2012, Gaulter & Biggs 2012, Ovenden 2005, Rienstra 2003), structural dynamics (Burr et al. 2000, Chatjigeorgiou 2008, Firouz-Abadi et al. 2007, Gaulter & Biggs 2012, Saito & Oshida 1959) and also for cochlear models (Elliott et al. 2013, Steele & Taber 1979).

The application of the WKB method, or eikonal approximation, is relatively simple, although it can rapidly become arduous for more complicated structures, and has the advantage of preserving the interpretation of travelling waves. This allows the use of the same systematic approach used for homogenous waveguides, as developed by Mace et al. (Harland et al. 2001, Mace 1984, 1992). Bretherton (Bretherton 1968) has extended the approximation for general linear systems, and shown that the variations in amplitude of the waves along rays are governed by conservation of an adiabatic invariant. The method consists of writing the solution in terms of an asymptotic expansion in powers of a small parameter, using local wavenumber and wave amplitude, and then matching the asymptote to a certain order, usually the first order. Pierce (Pierce 1970) has derived WKB solutions for Euler-Bernoulli and Timoshenko beams as well as for thin plates by using a conservation of energy approach, with less mathematic formalism.

However, the WKB approximation breaks down when the traveling wave reaches a local cut-on section. This transition, also known as turning point, leads to an internal reflection, breaking down its main assumption, requiring a different approximation for certain frequency bands - e.g. (Nayfeh 1973). In addition, uniformly valid solutions, i.e. solutions also valid for frequency bands away from the cut off frequency, have been derived using a slight modification of the WKB method for different applications (Biggs 2012, Gauter & Biggs 2013, Ovenden 2005).

In this paper, measurements of the natural frequencies of an ensemble of cantilever beams are performed. The mass per unit length of the beam is randomized by adding small discrete masses to an otherwise uniform beam. In Section 2, the methodology to generate the random masses distribution and its approximation to a continuous random field is presented. In Section 3, a simple model using assumed modes and lumped mass approximation is presented and Section 4 presents the WKB approximation for free vibration of flexural waves. In Section 5, some issues of representing a discrete implementation of a continuous random field are also discussed along with the results obtained, using the WKB formulation. Finally, Section 6 gives the concluding remarks.

2. DISTRIBUTION OF RANDOM MASSES

Random fields (Ghanem & Spanos 1991, Stefanou 2009, Sudret & Der Kiureghian 2000, Vanmarcke 2010) are multidimensional random processes and can be used to model spatially distributed variability using a probability measure. A numerical solution to the physical problem might require the variability to be incorporated within a spatial discretization of the physical domain (Zeldin & Spanos 1998). A number of discretization methods are available, among them being the midpoint method. This was first introduced by Der Kiureguian (Der Kiureghian & Ke 1988) and is used in this paper. It consists of approximating the random field in a spatial domain, previously discretized by a given mesh, by using a constant, but random, value within each element, or group of elements. This value is given by a sample of the random field specified at the geometrical centre of the element. This approach is very appealing for big lengths with respect to element size, because it does not require any modifications of the FE code making it suitable to be used with commercial FE in a framework of Monte Carlo sampling (Sudret & Der Kiureghian 2000).

The position of the attached masses are evenly distributed along the beam, i.e. their positions $_i$ always remain the same, as shown in Figure 1, but the values of each individual mass m_i vary according to a Gaussian homogeneous random field $H(x)$ with exponentially decaying correlation function, i.e. $R(x_i, x_j) = exp(-|x_i - x_j|/b\,)$, where x_i and $_j$ are any two points within the beam.

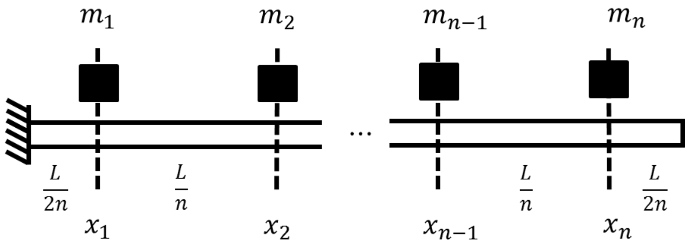

Figure 1: Evenly distributed masses along a cantilever beam, with values varying from 0g to 12g added at each location. Measurements are made with $n = 10$.

A collocation method was used to calculate a correlation matrix. This matrix was then used to solve the associated Karhunen-Loeve eigenproblem (Ghanem & Spanos 1991) into eigenvectors f_i and their respective eigenvalues λ_i, such that the zero mean random field is given by

$$H = \sum_{i=1}^{N_{KL}} \xi_i \sqrt{\lambda_i} f_i, \tag{1}$$

where ξ_i is a zero mean unity standard deviation Gaussian random variable and N_{KL} is the number of terms needed in the KL expansion for an appropriate representation of the random field (Huang et al. 2001).

The masses were added at 10 different locations as multiples of 1 g, such that the nominal or reference beam had $m_0 = 6$ g added at each location, adding up 60 g total. Their values $m_i = m_0 + \mu_i$ are calculated by individual realisations of the zero-mean random field μ_i, with a given correlation length, at the point x_i, i.e. using the mid-point approach for the random field discretisation, and adding it to the baseline value of 6g. This random field is numerically generated using a solution of the KL expansion, discretising the domain and solving an eigenproblem from the correlation matrix (Ghanem & Spanos 1991, Sudret & Der Kiuereghian 2000). The marginal distribution of this random field is given such that the values between -6g to 6g fall inside the 3σ region, i.e. they represent 99.73 % of the samples. Then they are rounded to the nearest integer, and any value sampled outside of this region is rounded to 6g or -6g accordingly. Figure 2 gives the normalized histogram obtained from this procedure.

The uniform beam is $L = 0.4$ m long with rectangular cross section, width $b = 39.85$ mm and thickness $h = 2$ mm. It has Young's modulus $E = 19094 \times 10^7$ Pa, estimated using the measured natural frequencies and a mass density $\rho = 7813.8\ \mathrm{kg/m^3}$ estimated by weighing the beam.

Five different correlation lengths were used to generate the mass distributions: $b = 0$, meaning that the values generated for the masses are statistically independent or uncorrelated; $b = 0.10\ L$, equal to the distance between two adjacent masses; $b = 0.25\ L$; $b = 0.60\ L$; and $b = \infty$, meaning that all of the masses are the same.

The mass distribution is expected to approximate a continuous mass density spatial distribution, for the flexural vibration of a straight beam, and also for a range of correlation lengths. This spatial distribution is represented in the form

$$\rho(x) = \rho_0 \left(1 + \eta_\rho + \sigma H(x)\right), \tag{2}$$

where η_ρ represent the effects due to the mean mass loading, m_0 and σ is a dispersion parameter for the random field. The offset value $\eta_\rho = 0.2409$ is calculated using the reference beam, i.e. the nominal properties, having 6g at each location, and the dispersion parameter $\sigma = 0.0803$ is calculated based on the 3σ requirement.

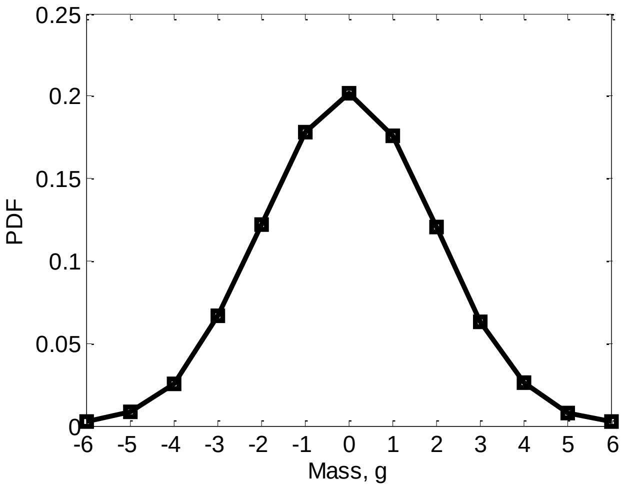

Figure 2: Marginal distribution of the values for the added masses m_i to the baseline value 6g.

3. ASSUMED MODES WITH THE LUMPED MASS APPROXIMATION MODEL

A simple model with assumed modes from an analytical solution of a cantilever Euler-Bernoulli beam with added lumped mass is used to give an approximation of the natural frequencies. It assumes point mass, with no rotation inertia and no added stiffening.

If $\phi_j(x)$ represents the j^{th} mass normalized mode of the cantilever beam, i.e. for the nominal system $\int_0^L \rho A\phi_i(x)\phi_j(x)dx = \delta_{ij}$, then the modal stiffness is $k_{ij} = \int_0^L EI_{yy}\phi_i''(x)\phi_j''(x)dx$. The perturbation of the natural frequencies due to changes in the added masses can be approximated by adding their kinetic energy contributions to each modal mass, using the mode shape of the homogeneous system, so

$$\mu_{ij} = \sum_{k=1}^{N_m} m_k\phi_i(x_k)\phi_j(x_k) \tag{3}$$

where $\phi_i(x)$ is the i^{th} mode and μ_k is the k^{th} mass at x_k. Additionally, the diagonal terms can be used to estimate the natural frequencies of the cantilever beam with added masses

$$\omega_j = \sqrt{\frac{k_{jj}}{1+\mu_{jj}}}. \tag{4}$$

This expression is used to derive statistics of the natural frequencies by generating samples of the discrete random field for the mass distribution, in a Monte Carlo framework (Rubinstein & Kroese 2007).

4. WKB APPROXIMATION FOR FLEXURAL WAVES

The WKB approximation (Pierce 1970) can also be used to estimate the natural frequencies of the cantilever beam using the continuous mass density distribution approximation. Considering a thin beam undergoing flexural vibration, with slowly changing mass density along its length, such that a propagating wave suffers negligible reflections due to the local impedance change, it is possible to derive the governing (Euler-Bernoulli) equation (Cremer et al. 2010)

$$\frac{\partial^2}{\partial x^2}\left(EI_{yy}(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{5}$$

where $EI_{yy}(x)$ and $\rho A(x)$ are the spatially varying flexural stiffness and mass per unit length, respectively, and $w(x,t)$ is the transverse displacement along the beam. Assuming a time harmonic solution $w(x,t) = \hat{w}(x)e^{i\omega t}$, thus, the eikonal $S(x) = ln\,\widetilde{U}(x) + i\theta(x)$ is introduced, in order to find wave solutions of the kind

$$\hat{w}(x) = e^{S(x)} = W(x)e^{\pm i\theta(x)}. \tag{6}$$

Back substituting this into Eq.(5), and neglecting the higher order terms it is possible to find solutions of the kind (Pierce 1970)

$$\widehat{w}(x) = \left(\rho A(x)\right)^{-\frac{3}{8}}\left(EI_{yy}(x)\right)^{-\frac{1}{8}}\Big[C_1 e^{-i\int_{x_0}^{x} k_B(x)dx} + C_2 e^{-\int_{x_0}^{x} k_B(x)dx} + C_3 e^{i\int_{x_0}^{x} k_B(x)dx} + C_4 e^{\int_{x_0}^{x} k_B(x)dx}\Big], \tag{7}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants and the four terms correspond to positive going and negative going propagating and evanescent waves. Moreover the exponential terms $\theta(x) = \pm i\int_{x_0}^{x} k_B(x)dx$ correspond to a phase change of the respective waves, the term $W(x) = \left(\rho A(x)\right)^{-\frac{3}{8}}\left(EI_{yy}(x)\right)^{-\frac{1}{8}}$ corresponds to an amplitude change due to the changes of the beams properties, and $k_B(x) = \left(\frac{\rho A(x)}{EI_{yy}(x)}\right)^{1/4}\sqrt{\omega}$ is the local free wavenumber for bending waves.

Then, the phase and amplitude changes of the positive going and negative going propagating and evanescent waves traveling through a distance L, from $x = 0$ to $x = L$, are given by

$$b^+ = \frac{\widetilde{W}(L)}{\widetilde{W}(0)} e^{-i\int_0^L k_B(x)dx} a^+, \tag{8}$$

$$b^- = \frac{\widetilde{W}(L)}{\widetilde{W}(0)} e^{i\int_0^L k_B(x)dx} a^-, \tag{9}$$

$$b_N^+ = \frac{\widetilde{W}(L)}{\widetilde{W}(0)} e^{-\int_0^L k_B(x)dx} a_N^+, \tag{10}$$

$$b_N^- = \frac{\widetilde{W}(L)}{\widetilde{W}(0)} e^{\int_0^L k_B(x)dx} a_N^-, \tag{11}$$

where $W(0)$ and $W(L)$ are the amplitudes given by $W(x) = \left(\rho A(x)\right)^{-\frac{3}{8}}\left(EI_{yy}(x)\right)^{-\frac{1}{8}}$ evaluated at $\;= 0$ and $\;= L$, respectively.

Considering a finite waveguide of length L, undergoing free flexural wave behaviour, then the positive-going and the negative going wave amplitudes on the left hand boundary, $\mathbf{a}^+$ and $\mathbf{a}^-$, and on the right hand boundary, $\mathbf{b}^+$ and $\mathbf{b}^-$, can be related from Eq. (8)-(11), by

$$\mathbf{b} = \mathbf{\Lambda}\mathbf{a}, \tag{12}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_{11} & 0 \\ 0 & \mathbf{\Lambda}_{22} \end{bmatrix}, \tag{13}$$

is a propagation matrix and

$$\mathbf{\Lambda}_{11} = \begin{bmatrix} e^{-i\theta_T+\gamma_T} & 0 \\ 0 & e^{-\theta_T+\gamma_T} \end{bmatrix}, \tag{14}$$

$$\mathbf{\Lambda}_{22} = \begin{bmatrix} e^{i\theta_T+\gamma_T} & 0 \\ 0 & e^{\theta_T+\gamma_T} \end{bmatrix}. \tag{15}$$

The wave amplitudes are given by the vectors

$$\mathbf{a} = \begin{Bmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \end{Bmatrix}, \mathbf{b} = \begin{Bmatrix} \mathbf{b}^+ \\ \mathbf{b}^- \end{Bmatrix}. \tag{16}$$

where the propagating and evanescent waves are, respectively, given as $\mathbf{a}^+ = [a^+ \quad a_N^+]^{\mathrm{T}}$, $\mathbf{a}^- = [a^- \quad a_N^-]^{\mathrm{T}}$, $\mathbf{b}^+ = [b^+ \quad b_N^+]^{\mathrm{T}}$ and $\mathbf{b}^- = [b^- \quad b_N^-]^{\mathrm{T}}$. The total phase and amplitude change for a travelling wave propagating from one boundary to the other can be given by

$$\theta_T = \int_0^L k_B(x)dx, \tag{17}$$

and

$$\gamma_T = ln\left[\frac{\widetilde{W}(L)}{\widetilde{W}(0)}\right], \tag{18}$$

respectively.

The waveguide natural frequencies can be determined by applying the wave train closure principle, tracing the round-trip of a traveling wave (Cremer et al. 2010, Mace 1984). Thus, from Eq. (12)

$$\mathbf{b}^+ = \mathbf{\Lambda}_{11}\mathbf{a}^+ \tag{19}$$
$$\mathbf{b}^- = \mathbf{\Lambda}_{22}\mathbf{a}^- \tag{20}$$

where $(\cdot)_{ij}$ is the i^{th} row and j^{th} column element of the corresponding matrix, applying the reflections at the boundaries and noting that $\mathbf{\Lambda}_{12} = \mathbf{\Lambda}_{21} = \mathbf{0}$, where $\mathbf{0}$ is a 2×2 zero matrix, gives

$$\mathbf{b}^+ = \mathbf{\Lambda}_{11}\mathbf{\Gamma_L}\mathbf{a}^-, \tag{21}$$
$$\mathbf{\Gamma_R}\mathbf{b}^+ = \mathbf{\Lambda}_{22}\mathbf{a}^-. \tag{22}$$

Rearranging and substituting, the natural frequencies correspond to the zeros of the characteristic equation,

$$\det|\mathbf{\Lambda}_{22}^{-1}\mathbf{\Gamma_R}\mathbf{\Lambda}_{11}\mathbf{\Gamma_L} - \mathbf{I}| = 0. \tag{23}$$

where $\mathbf{I}$ is a 2×2 identity matrix and the reflection matrices $\mathbf{\Gamma_R}$ and $\mathbf{\Gamma_L}$ follow from the relations between the waves in the boundaries $\mathbf{a}^+ = \mathbf{\Gamma_L}\mathbf{a}^-$ and $\mathbf{b}^- = \mathbf{\Gamma_R}\mathbf{b}^+$. They can be found by writing the equations for equilibrium and continuity in a systematic framework (Mace 1984). Solving Equation (23) will usually lead to the problem of finding the n^{th} root of a transcendental equation θ_{Tn}, such that the n^{th} natural frequency ω_n is given by

$$\omega_n = \frac{\theta_{Tn}^2}{\left(\int_0^L \sqrt[4]{\frac{\rho A(x)}{EI_{yy}(x)}}\, dx\right)^2}. \tag{24}$$

This transcendental equation depends on the boundary conditions, or the matrices $\mathbf{\Gamma_L}$ and $\mathbf{\Gamma_R}$, and they are analogous to the homogenous case, whose roots for typical boundary conditions can be found in tables (Graff 1991), for instance, where the results can be directly applied in Eq. (24). This is shown for a particular boundary condition by Pierce (Pierce 1970).

In this paper, only the mass density is considered to be spatially varying, reducing the expression on the denominator to $\rho A(x) = \rho(x)A$ and $EI_{yy}(x) = EI_{yy}$. Moreover, this expression is also be used to derive statistics of the natural frequencies in a Monte Carlo framework (Rubinstein & Kroese 2007), using the continuous random field approximation for the mass density, from Section 2, and constant flexural stiffness.

5. EXPERIMENTAL RESULTS AND DISCUSSION

The natural frequencies of the 2nd to the 7th mode of the cantilever beam with added masses were measured using 20 statistically independent samples for the mass configuration, for each correlation length except for the uniform sample corresponding to $b = \infty$. A slightly different procedure was used for the latter case. Because there are only 13 possible integer values for the masses between -6g and 6g, the mean value and standard deviation for this case were calculated by using the normalized discrete PDF, i.e.

$$\overline{\omega} = \sum_{i=1}^{13} \omega_i p_i, \tag{25}$$

$$\sigma_\omega = \sqrt{\sum_{i=1}^{13} (\omega_i - \overline{\omega})^2 p_i}, \tag{26}$$

where p_i is the normalized PDF for the discrete set of masses, as shown in Figure 2. Table 1 and Table 2 give the mean value and standard deviation of the measured natural frequencies for 2nd, 3rd and 4th modes and 5th , 6th and 7th modes, respectively, for each correlation length. The 95% confidence interval for the mean value is calculated using a t-student distribution, for the cases with 20 samples. Note that for all the correlation lengths, the mean values of the natural frequencies are almost the same for each mode. That is expected for small values of σ. from Eq. (2). Moreover, the standard deviation increases with increasing the correlation length for each individual mode. It also increases with increasing mode number.

Table 1: Mean value with 95% confidence interval and the estimated standard deviation of the measured natural frequencies, in Hz, for the 2nd, 3rd and 4th modes for each correlation length.

	Mode 2		**Mode 3**		**Mode 4**	
Correlation Length b	ϖ	σ_ω	ϖ	σ_ω	ϖ	σ_ω
0	57.4±0.3	0.6	159.6±0.9	2.1	310.5±1.3	2.8
0.10 L	57.5±0.4	0.8	160.4±1.0	2.2	310.5±2.1	4.3
0.25 L	57.7±0.6	1.3	160.1±1.5	3.2	310.7±3.0	6.2
0.60 L	57.9±0.7	1.5	161.0±1.9	3.9	311.1±3.5	7.4
∞	57.6	1.6	160.1	4.3	309.9	8.1

Table 2: Mean value with 95% confidence interval and the estimated standard deviation of the measured natural frequencies, in Hz, for the 5th, 6th and 7th modes for each correlation length.

	Mode 5		**Mode 6**		**Mode 7**	
Correlation Length b	ϖ	σ_ω	ϖ	σ_ω	ϖ	σ_ω
0	506.7±1.9098	4.2	760.0±2.5	5.6	1072.6±4.4	9.8
0.10 L	506.3±2.5714	5.4	759.8±4.3	8.9	1075.0±7.0	14.7
0.25 L	508.4±4.3596	9.1	761.1±6.3	13.0	1074.7±9.8	20.5
0.60 L	509.3±5.6755	11.8	764.0±8.5	17.8	1077.0±12.0	25.1
∞	505.6	12.7	757.9	18.6	1069.1	27.3

The sample mean ϖ and standard deviation σ_ω for each mode and correlation length were used to calculate the Coefficient of Variation (COV) as

$$COV = \frac{\sigma_\omega}{\varpi}. \tag{27}$$

Figure 3 and Figure 4 show the COV for the 2nd, 3rd and 4th modes as well as for the 5th, 6th and 7th modes as functions of the correlation length, along with the COV obtained with 5000 Monte Carlo sampling using the simple added mass theory, Eq. (4), and the WKB approximation, Eq. (24).

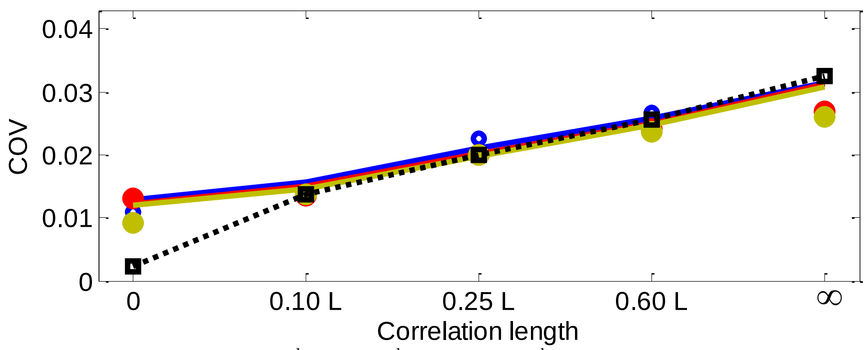

Figure 3: Coefficient of Variation for the 2nd (red), 3rd (blue) and 4th (yellow) mode for each correlation length from the measurements (circle), simple added mass theory (full line) and WKB (dotted black for all of the modes).

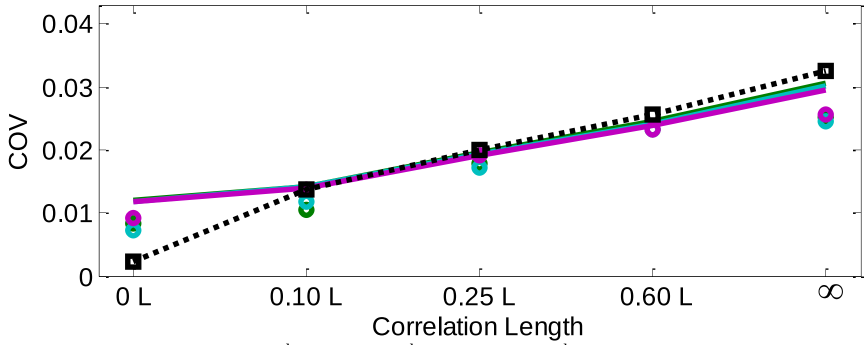

Figure 4: Coefficient of Variation for the 5th (green), 6th (cyan) and 7th (magenta) mode for each correlation length from the measurements (circle), simple added mass theory (full line) and WKB (dotted black for all of the modes).

Note that, in general, the results agree well for all of the modes and correlation lengths except for the uncorrelated case, $b = 0$, when the slowly varying properties assumption is no longer valid, therefore breaking down the WKB approximation. Moreover, it can be shown that representing a random field using the midpoint method along with a coarse mesh, i.e. the distance between two consecutive masses is longer than the correlation length, may introduce higher variability into the stochastic response (Zeldin & Spanos 1998). This result agrees with the much higher COV found for the experimental and added mass theory prediction.

For the case when $b = \infty$, the beam mass can be treated as a random variable, instead of a random field, i.e. Eq. (4) and Eq. (24) reduces, respectively, to

$$\tilde{\omega}_j = \sqrt{\frac{k_j}{1 + m\sum_{i=1}^{N_m}\phi_{ij}^2}}, \tag{28}$$

$$\omega_n = \sqrt{\frac{EI_{yy}}{\rho A}\frac{\theta_{Tn}^2}{L^2}}. \tag{29}$$

where the random mass is given by $m = \sigma_m\xi_1$ and σ_m is the standard deviation of the beam mass. The mass density is a random variable given by $\rho = \rho_0(1 + \eta_\rho + \sigma\xi_2)$, from Eq. (2), ξ_1 and ξ_2 being Gaussian independent, zero-mean, unity standard deviation random variables. Using a first order approximation, it is possible to find analytically a COV value for the WKB approach, given by

$$COV = \frac{\sigma}{2(1 + \eta_\rho)}, \tag{30}$$

and also for the added mass approximation,

$$COV = \frac{\sigma_m}{2}\sum_{i=1}^{N_m}\phi_{ij}^2. \tag{31}$$

Moreover, from mode normalization, the summation in Eq. (31) is reduced to N_m/m_{beam}, where m_{beam} is the mass of the beam, therefore the expression can be also reduced to $COV = (N_m\sigma_m)/m_{beam}$. Results are shown in Table 3. Note that they have a good agreement with the results obtained from Monte Carlo sampling of Eq. (4) and (24), also shown in Figure 3 and Figure 4, but they are different from the experimental results. From the modal contribution in Eq. (31), it is possible to note that the positions of the masses along the beam affect the COV values differently for each mode. This bias due to the mode shape is also seen in the other correlation lengths as well as in the experimental results. However, it does not happen when using the WKB approximation, as can be seen from Eq. (30), for $b = \infty$, because the mass is distributed over the beam. This effect is also present for the others correlation lengths, as it can be seen in Figure 3 and Figure 4.

Table 3: Coefficient of Variation for the 2nd, 3rd, 4th, 5th, 6th and 7th modes natural frequencies for the infinite correlation length $b = \infty$, using the first order approximation on the assumed modes and WKB approach.

COV	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7
Assumed modes	0.0311	0.0308	0.0305	0.0303	0.300	0.0297
WKB	0.0324	0.0324	0.0324	0.0324	0.0324	0.0324
Experimental	0.0269	0.0267	0.0258	0.0250	0.0244	0.0254

6. CONCLUDING REMARKS

In this paper, the issue of implementing a random field into a structure according to an analytical model is approached. In the experiment, a set of masses were distributed over the length of an otherwise uniform cantilever beam, according to a random field with an exponentially decaying correlation function, using different correlation lengths, and rounded to integer values in grams with a clipped Gaussian distribution. The Karhunen-Loeve expansion was used to generate the homogeneous Gaussian random field samples. Natural frequencies were measured for five different correlation lengths; one where the masses were homogeneously distributed, i.e. all the masses along the beam had the same value, meaning an infinite correlation length; another such that they were uncorrelated, i.e. statistically independent from each other, meaning zero correlation length. These two are limiting cases, in the sense that they represent, respectively, the greatest and the smallest variability in the natural frequencies of the cantilever beam. Also, the natural frequencies were measured for three other correlation lengths, all equal to or greater than the distance between two consecutive masses. It was shown that the greater the correlation length the higher the standard deviation of the natural frequencies.

A simple model with assumed modes from an analytical solution of a cantilever beam with added lumped mass is used to give an approximation of the natural frequencies, assuming point mass, no rotation inertia and no added stiffening. Moreover, a wave formulation, using the WKB approximation for flexural waves, in which the discrete mass distribution is approximated to a spatially continuous mass density, was also used to calculate the natural frequencies. The approximation also does not take into account the upper and lower limits of the discrete mass distribution. This leads to some discrepancies with the experimental results, but succeed to represent the overall trend in the COV as a function of the correlation length. A Monte Carlo sampling is used to calculate the mean value and standard deviation from both numerical models and the statistics are compared with the experimental results.

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