AN AUTOMATIC DOMAIN SPLITTING TECHNIQUE TO PROPAGATE UNCERTAINTIES IN HIGHLY NONLINEAR ORBITAL DYNAMICS

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Current approaches to uncertainty propagation in astrodynamics mainly refer to linearized models or Monte Carlo simulations. Naive linear methods fail in nonlinear dynamics, whereas Monte Carlo simulations tend to be computationally intensive. Differential algebra has already proven to be an efficient compromise by replacing thousands of pointwise integrations of Monte Carlo runs with the fast evaluation of the arbitrary order Taylor expansion of the flow of the dynamics. However, the current implementation of the DA-based high-order uncertainty propagator fails in highly nonlinear dynamics or long term propagation. We solve this issue by introducing automatic domain splitting. During propagation, the polynomial of the current state is split in two polynomials when its accuracy reaches a given threshold. The resulting polynomials accurately track uncertainties, even in highly nonlinear dynamics. The method is tested on the propagation of (99942) Apophis post-encounter motion.

INTRODUCTION

Nonlinear propagation of uncertainties plays a key role in astrodynamics. Orbit determination is affected by measurement errors; consequently, the knowledge of the state of any spacecraft or celestial body is characterized by an estimable level of uncertainty. Even the dynamical models used to propagate the motion are synthesized by adopting approximations that affect the accuracy of the orbital predictions. In addition, the size of the uncertainty set tends to quickly increase along the trajectory due to nonlinearities. Nonlinearities are not confined to object dynamics: even simple conversions between different coordinate systems (e.g. the conversion from polar to Cartesian coordinates that forms the foundation for the observation models of many sensors) introduce significant nonlinearities and, thus, affect the accuracy of classical uncertainty propagation techniques.

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Uncertainty propagation and estimation in nonlinear systems is extremely difficult. Present-day approaches mainly refer to linearized propagation models\textsuperscript{1,2,3} or full nonlinear Monte Carlo simulations.\textsuperscript{4} The linear assumption significantly simplifies the problem, but the accuracy of the solution drops off in case of highly nonlinear systems and/or long time propagations. On the other hand, Monte Carlo simulations provide true trajectory statistics, but are computationally intensive and therefore, in many cases, unmanageable. A significant step towards the solution of such issues was taken by Park and Scheeres,\textsuperscript{5} who derived an alternate way to map the statistics by approximating the flow of the dynamics in Taylor series. This approach attains a good agreement with Monte Carlo simulations. However, it suffers of high computational complexity to derive the dynamics of the high order tensor, especially for high fidelity dynamics.

Alternative methods based on the use of differential algebra were already presented by the authors in past works. Differential algebra (DA) supplies the tools to compute the derivatives of functions within a computer environment.\textsuperscript{6} More specifically, by substituting the classical implementation of real algebra with the implementation of a new algebra of Taylor polynomials, any function $f$ of $n$ variables is expanded into its Taylor polynomial up to an arbitrary order $k$ with limited computational effort. As a main consequence, the Taylor expansion of the flow of any ordinary differential equation can be readily obtained by carrying out all the operations of any explicit integration scheme in the DA framework. Nonlinear uncertainty propagation can take advantage of the resulting high order expansions. For example, as the accuracy of the Taylor expansion can be tuned by adjusting the expansion order, the approach of classical Monte Carlo simulations can be enhanced by replacing thousands of integrations with evaluations of the Taylor expansion of the flow. Moreover, since the DA approach can provide the higher order derivatives of any function, also higher order propagation schemes can be enhanced and speeded up.

Differential algebra has already proven its efficiency in the non-linear propagation of uncertainties within different dynamical models, including two-body dynamics,\textsuperscript{7} $(n+1)$-body dynamics,\textsuperscript{8} and geocentric models (including Earth’s gravitational harmonics, solar radiation pressure, shadows, and third body perturbations).\textsuperscript{9} Nonetheless, the accuracy of the method tends to decrease drastically in highly nonlinear dynamics. Examples of this kind can be found in the propagation of asteroids motion after a close encounter with a major body. As reported by Valsecchi et al.,\textsuperscript{10} the asteroid Apophis will have an extremely close approach to the Earth on 13 April 2029. The asteroid orbit will suffer a very large perturbation, opening the door to the possibility of a resonant return in 2036. The nonlinearities of the close encounter and of the post-encounter motion will make any uncertainty in the direction anti-parallel to the Earth heliocentric motion in 2029 diverge by a factor of 40000 in 2036. While performing better than classical linearized methods, the current implementation of the DA uncertainty propagator tends to be inaccurate and impractical in such cases, due to the extremely high order required to describe the resulting uncertainty sets by a single polynomial.

In the field of verified numerics, domain decomposition is a common tool used in interval arithmetic as well as Taylor Model integration.\textsuperscript{11} This paper introduces the concept of automatic domain splitting to the non-verified DA framework and applies it to the problem of uncertainty propagation in order to overcome the previously described issues. In the course of the integration of the initial condition, the polynomial representing the current state is constantly monitored. When the nonlinearities of the system cause the high order terms of the polynomial to grow too large, integration is paused and the domain of the polynomial is split into two halves along one of the variables. This yields two new polynomials, one covering each half of the initial conditions. Since the split of the domain of one variable into half causes the $n$-th order terms of that variable to shrink by a factor of
2, this method efficiently reduces the size of the highest order terms. Integration is then resumed on each one of the two new polynomials until either further splits are required or the final state is reached. The final result of this procedure is a list of final state polynomials, each describing the evolution of some automatically determined subset of the initial condition.

The paper is organized as follows. In the next section, a brief introduction is given about DA techniques. The use of DA to compute high order expansion of ODEs flow is then described. A simple application to the propagation of uncertainties in Kepler’s dynamics is presented to show the advantages of high order propagation with respect to linear methods and to illustrate its limits for large uncertainty sets and nonlinear dynamics. Automatic domain splitting is then introduced and the uncertainty propagation in Kepler’s dynamics is resumed to show the advantages of domain splitting over standard high order propagation. Lastly, the performances of the resulting splitting DA-based integrator are assessed on the propagation of asteroid (99942) Apophis.

NOTES ON DIFFERENTIAL ALGEBRA

DA techniques find their origin in the attempt to solve analytical problem by an algebraic approach. Historically, the treatment of functions in numerics has been based on the treatment of numbers, and the classical numerical algorithms are based on the mere evaluation of functions at specific points. DA techniques are based on the observation that it is possible to extract more information on a function rather than its mere values. The basic idea is to bring the treatment of functions and the operations on them to the computer environment in a similar way as the treatment of real numbers. Referring to Figure 1, consider two real numbers $a$ and $b$. Their transformation into the floating point representation, $\overline{a}$ and $\overline{b}$ respectively, is performed to operate on them in a computer environment. Then, given any operation $\times$ in the set of real numbers, an adjoint operation $\otimes$ is defined in the set of FP numbers such that the diagram in figure commutes. Consequently, transforming the real numbers $a$ and $b$ in their FP representation and operating on them in the set of FP numbers returns the same result as carrying out the operation in the set of real numbers and then transforming the achieved result in its FP representation. In a similar way, suppose two sufficiently regular functions $f$ and $g$ are given. In the framework of differential algebra, the computer operates on them using their Taylor series expansions, $F$ and $G$ respectively. Therefore, the transformation of real numbers in their FP representation is now substituted by the extraction of the Taylor expansions of $f$ and $g$. For each operation in the function space, an adjoint operation in the space of Taylor polynomials is defined such that the corresponding diagram commutes; i.e., extracting the Taylor expansions of $f$ and $g$ and operating on them in the function space returns the same result as operating on $f$ and $g$ in the original space and then extracting the Taylor expansion of the resulting function. The straightforward implementation of differential algebra in a computer allows to compute the Taylor coefficients of a function up to a specified order $n$, along with the function evaluation, with a fixed amount of effort. The Taylor coefficients of order $n$ for sums and product of functions, as well as scalar products with reals, can be computed from those of summands and factors; therefore, the set of equivalence classes of functions can be endowed with well-defined operations, leading to the so-called truncated power series algebra.

Similarly to the algorithms for floating point arithmetic, the algorithm for functions followed, including methods to perform composition of functions, to invert them, to solve nonlinear systems explicitly, and to treat common elementary functions. In addition to these algebraic operations,
also the analytic operations of differentiation and integration are introduced, so finalizing the definition of the DA structure. The differential algebra sketched in this section was implemented in the software COSY INFINITY.\footnote{16}

HIGH ORDER EXPANSION OF ODES FLOW

Differential algebra allows the derivatives of any function $f$ of $n$ variables to be computed up to an arbitrary order $k$, along with the function evaluation. This has an important consequence when the numerical integration of an ODE is performed by means of an arbitrary integration scheme. Any integration scheme is based on algebraic operations, involving the evaluation of the ODE right hand side at several integration points. Therefore, carrying out all the evaluations in the DA framework allows differential algebra to compute the arbitrary order expansion of the flow of a general ODE with respect to the initial condition.

Without loss of generality, consider the scalar initial value problem

$$
\begin{align*}
\dot{x} &= f(x, t) \\
x(t_0) &= x_0
\end{align*}
$$

and the associated phase flow $\varphi(t; x_0)$. We now want to show that, starting from the DA representation of the initial condition $x_0$, differential algebra allows us to propagate the Taylor expansion of the flow in $x_0$ forward in time, up to the final time $t_f$.

To this aim, replace the point initial condition $x_0$ by the DA representative of its identity function up to order $k$, which is a $(k+1)$-tuple of Taylor coefficients. (Note that $x_0$ is the flow evaluated at the initial time; i.e, $x_0 = \varphi(t_0; x_0)$.) As for the identity function only the first two coefficients, corresponding to the constant part and the first derivative respectively, are non zeros, we can write $[x_0]$ as $x_0 + \delta x_0$, where $x_0$ is the reference point for the expansion. If all the operations of the numerical integration scheme are carried out in the DA framework, the phase flow $\varphi(t; x_0)$ is approximated, at each fixed time step $t_i$, as a Taylor expansion in $x_0$.

For the sake of clarity, consider the forward Euler’s scheme

$$
x_i = x_{i-1} + f(x_{i-1})\Delta t
$$

and substitute the initial value with the DA identity $[x_0] = x_0 + \delta x_0$. At the first time step we have

$$
[x_1] = [x_0] + f([x_0]) \cdot \Delta t.
$$

Figure 1. Analogy between the floating point representation of real numbers in a computer environment (left figure) and the introduction of the algebra of Taylor polynomials in the differential algebraic framework (right figure).
If the function \( f \) is evaluated in the DA framework, the output of the first step, \([x_1]\), is the \(k\)-th order Taylor expansion of the flow \( \varphi(t; x_0) \) in \( x_0 \) for \( t = t_1 \). Note that, as a result of the DA evaluation of \( f([x_0]) \), the \((k + 1)\)-tuple \([x_1]\) may include several non zero coefficients corresponding to high order terms in \( \delta x_0 \). The previous procedure can be inferred through the subsequent steps. The result of the final step, \([x_f]\), is the \(k\)-th order Taylor expansion of \( \varphi(t; x_0) \) in \( x_0 \) at the final time \( t_f \). Thus, the flow of a dynamical system can be approximated, at each time step \( t_i \), as a \(k\)-th order Taylor expansion in \( x_0 \) in a fixed amount of effort.

The conversion of standard integration schemes to their DA counterparts is straightforward both for explicit and implicit solvers. This is essentially based on the substitution of the operations on real numbers with those on DA numbers. In addition, whenever the integration scheme involves iterations (e.g. iterations required in implicit and predictor-corrector methods), step size control, and order selection, a measure of the accuracy of the Taylor expansion of the flow needs to be included. For the numerical integrations presented in this paper, a DA version of the Dormand-Prince (8-th order solution for propagation, 7-th order solution for step size control) implementation of the Runge-Kutta scheme is used. In this case, a weighted norm of the coefficients of the Taylor expansion is used in the step size control procedure.

The main advantage of the DA-based approach is that there is no need to write and integrate variational equations in order to obtain high order expansions of the flow. This result is basically obtained by the substitution of operations between real numbers with those on DA numbers, and therefore the method is ODE independent. Furthermore, the efficient implementation of the differential algebra in COSY INFINITY allows us to obtain high order expansions with limited computational time.

Test case: Uncertainty Propagation in Kepler’s Dynamics

The previous DA-based numerical integrators pave the way to numerous practical applications. An example is given hereafter pertaining the propagation of errors on initial conditions. The Taylor polynomials resulting from the use of DA-based numerical integrators expand the solution of the initial value problem of Eq. (1) with respect to the initial condition. Thus, at each step \( i \), the dependence of the solution on the initial condition is available in terms of an high order polynomial map \([x_i] = \mathcal{M}_{x_i}(\delta x_0)\), where \( \delta x_0 \) is the displacement of the initial condition from its reference value \( x_0 \). Suppose now the reference value \( x_0 \) represents a nominal initial condition for a dynamical system and assume an error \( \delta x_0 \) occurs between the actual initial condition and the nominal one. The mere evaluation of the Taylor polynomial \( \mathcal{M}_{x_i}(\delta x_0) \) supplies the new solution \( x_i \) at time \( t_i \) corresponding to the displaced initial condition. More precisely, the Taylor polynomial \( \mathcal{M}_{x_i}(\delta x_0) \) delivers a Taylor approximation of the new solution \( x_i \), whose accuracy depends on the expansion order \( n \) and the size of the displacement \( \delta x_0 \). The main advantage of the DA-based integrator is that the new solution is obtained by means of the evaluation of a polynomial, so avoiding a new numerical integration corresponding to the displaced initial condition. Moreover, the same Taylor polynomial can be used to compute the solution corresponding to any error \( \delta x_0 \). Consequently, if many values of \( \delta x_0 \) are to be processed, the polynomial evaluations efficiently replace the multiple necessary numerical integrations.

The results of the application of the previous procedure are illustrated in the following example. The dynamics of an object moving in the solar system is integrated in the framework of the two
body problem:

\[
\begin{align*}
\dot{r} &= v \\
\dot{v} &= -\frac{\mu}{r^3} r,
\end{align*}
\]

where \(r\) and \(v\) are the object position and velocity vectors respectively, and \(\mu\) is the Sun gravitational parameter. The nominal initial conditions are set such that the object starts moving from the pericenter of an elliptic orbit, lying on the ecliptic plane (see the dotted line in Figure 2a). The pericenter radius is 1 AU, whereas the magnitude of the initial velocity is selected to have a resulting orbit of eccentricity 0.5. The DA-based integrator is used to compute a 6th order expansion of the ODEs flow along the orbit.

The \(x\) and \(y\) components of the initial position are then supposed to lie in an uncertainty box of size 0.008 AU and 0.08 AU in the \(x\) and \(y\) direction respectively. This is a rather unrealistic uncertainty set, which has been exaggerated for illustrative purposes. The evolution of the resulting initial box is now investigated by propagating its boundary. More specifically, a uniform sampling of the boundary is performed. Then, for each sample, the displacement with respect to the nominal initial conditions is computed and the 6th order polynomial maps obtained with the DA-based integrator are evaluated. In this way, for each integration time, the evolved box can be plotted by means of mere polynomial evaluations. The evolved box is reported in Figure 2a corresponding to 9 integration times uniformly distributed over the orbital period. The time required by COSY-Infinity for the computation of the 6th order map is about 0.15 s on a 2.4 GHz Intel Core i5 MacBook Pro running Mac OS X 10.9.1.

The accuracy of the Taylor expansion of the flow is better highlighted in Figure 2b. Focusing on the integration time \(t_i = 929.8\) day, the figure reports the box obtained with a multiple point-wise integration of the samples (solid line). The propagated boxes obtained by the evaluation of the polynomial maps representing the flow of the ODE in Eq. (4) are then plotted for comparison, corresponding to different expansion orders. The figure shows that a 6th order expansion of the flow is necessary to achieve a visually accurate representation of the exact box.

Unfortunately, the accuracy of the 6th order Taylor expansion drastically decreases for longer integration times. Figure 3a focuses on the integration time \(t_i = 2025.9\) day, which corresponds to only about 1.96 revolutions. The figure compares the box obtained by a multiple point-wise
integration of the samples with that resulting from the evaluation of the 6th order polynomial map. The 6th order expansion is not able to accurately describe the exact box. Even increasing the order of the Taylor expansion does not improve the accuracy. This is confirmed in Figure 3b, where the results of a 14th order expansion of the ODEs flow are compared with the exact box.

Figures 3a and 3b demonstrate that a single Taylor expansion of reasonable order is not always able to capture the typical nonlinearities of orbital mechanics problems. Consequently, while performing better than classical linearized methods, the high order integrator described above may fail to accurately track uncertainties depending on the nonlinearity of the dynamics, the size of the uncertainty set to be propagated, and the propagation time.

Automatic domain splitting can play a crucial role to solve the previously described issues. In the course of the integration of the initial conditions, the uncertainty set is split along its variables when the nonlinearities of the system cause the Taylor expansion to lose accuracy. This yields the final set to be described with new polynomials, each one covering a subset of the initial conditions. The technique and its advantages with respect to the current implementation of the DA-based integrator are described in the next section.

AUTOMATIC DOMAIN SPLITTING

The approximation error between an \( n + 1 \) times differentiable function \( f \in C^{n+1} \) and its Taylor expansion \( P_f \) of order \( n \), without loss of generality taken around the origin, is given by Taylor’s theorem:

\[
|f(\delta x) - P_f(\delta x)| \leq C \cdot |\delta x|^{n+1},
\]

for some constant \( C > 0 \). We remark in passing that Taylor’s theorem does not require \( f \) to be analytic, it is sufficient that \( f \in C^{n+1} \).

Consider now the maximum error \( e_r \) of \( P_f \) on a domain \( B_r \) of radius \( r > 0 \) around the expansion point. By equation 5 we have that

\[
|f(\delta x) - P_f(\delta x)| \leq C \cdot r^{n+1} = e_r.
\]

If the domain of \( P_f \) is reduced from \( B_r \) to a ball \( B_{r/2} \) of radius \( r/2 \), the maximum error of \( P_f \)
over $B_{r/2}$ will decrease by a factor of $1/2^{n+1}$, i.e.

$$|f(\delta x) - P_f(\delta x)| \leq C \cdot \delta x^{n+1} \leq C \cdot \left(\frac{r}{2}\right)^{n+1} = C \cdot \frac{e_r}{2^{n+1}}.$$

We observe that for sufficiently large expansion orders, such as e.g. $n = 9$, the effect of reducing the size of the domain by half is thus greatly amplified and the maximum error is reduced by a factor of $\frac{1}{2^{10}} \approx 10^{-4}$. One solution to the previously described problem of non-convergence of the polynomial expansion over its initial domain is therefore to subdivide the initial box into smaller boxes and compute the Taylor expansion around the center point of each of the new boxes. Then the error of the new polynomial expansions in each sub box is greatly reduced, while taken in their entirety, the expansions still cover the entire initial set.

This process is often referred to as a divide and conquer strategy, and is very common in the field of numerical analysis. However, the method described before suffers from an important drawback. By manually subdividing the initial box into smaller subsets of a predefined size, it is necessary to know a priori the required size of the subdivided boxes to obtain the desired error. If the initial boxes are chosen too small, precious computational time is wasted computing expansions over several small boxes where one large box would have sufficed.

Furthermore, for practical reasons such subdivisions are typically performed in a uniform manner, producing a uniform grid of boxes. This adds to the computational cost as often times the dynamical behavior of the function $f$ being expanded differs significantly over the various parts of its domain. In some regions larger boxes will yield the required accuracy, while other regions may be more critical and require a more finely spaced box cover.

Lastly, in the case of the expansion of the flow of an ODE the a priori splitting of the initial box into sub boxes causes computational yet additional unnecessary overhead due to the fact that the flow at the beginning of the integration ($t = t_0$) is just the identity, i.e. $\Phi_{t_0}(x) = x$.

At this initial time, the entire flow over the initial condition box can be accurately represented by the identity polynomial. As the integration of the dynamics progresses, the flow $\Phi$ is distorted away from the identity until such a time $t^*$ at which the polynomial approximation $P_0$ surpasses some pre-specified maximum error. Up until that time $t^*$, however the flow is described well by just one polynomial expansion over the entire initial condition, there is no need to perform the integration between $t_0$ and $t^*$ twice using two separate expansions.

Automatic Domain Splitting, originally introduced by Berz and Makino in the verified Taylor Model integrator COSY VI, builds on this observation to circumvent these problems by automatically detecting at which time the flow expansion over a given box of initial conditions is becoming too inaccurate. Once this case has been detected, the domain of the original polynomial expansion is divided into two domains of half their original size each, with two separate polynomial expansions on each. Each of the new polynomials represents the exact same function over its domain as the original polynomial, but expanded around the center point of the new domain.

More specifically let $P(x)$ be the polynomial representation of the flow $\Phi_{t^*}(x)$ at some time $t^*$, with the domain $x \in [-1, 1]$. Then the split of $P$ into $P_1$ and $P_2$ is defined as

$$P_1(x) = P \left( \frac{1}{2} \cdot x - \frac{1}{2} \right),$$

$$P_2(x) = P \left( \frac{1}{2} \cdot x + \frac{1}{2} \right).$$
with the domains of \( P_1 \) and \( P_2 \) again being taken as \( x \in [-1, 1] \).

From this definition it is evident that \( P_1 \) is covering the left half \((-1, 0]\) of the domain of \( P \) and \( P_2 \) covering the right half \([0, 1)\). Since both \( P_1 \) and \( P_2 \) are again polynomials of the same degree as \( P \), this splitting operation can be performed exactly in DA arithmetic without adding any truncation errors. The new polynomials \( P_1 \) and \( P_2 \) represent exactly the same graph as that of \( P \), just expanded around a different expansion point. However, in accordance with equation 5, the terms of any order \( n \) in \( P_1 \) and \( P_2 \) will be smaller by a factor of \( 2^n \) than the corresponding terms in \( P \).

After such a split occurs, the integration can continue on each one of \( P_1 \) and \( P_2 \) in the same manner as described in the previous section until further splits are required or the final integration time is reached. The result is a list of polynomial expansions, each covering a specific part of the domain of initial conditions.

The decision when exactly a polynomial needs to be split, and in the case of multivariate polynomials the direction of the split, is in general difficult to answer. We use a heuristic method which estimates the size of the \( n + 1 \) order of the polynomial based on an exponential fit of the size of the known coefficients up to order \( n \). If the size of this truncated order becomes too large, we decide to split the polynomial.

In the case of multivariate polynomials \( P(x_1, x_2, \ldots, x_v) \), the split is only performed in one variable \( x_j \). The splitting direction \( x_j \) is determined by factoring the known coefficients of order up to \( n \) with respect to each \( x_i \), i.e. writing

\[
P(x_1, x_2, \ldots, x_v) = \sum_{k=0}^{n} x_i^k \cdot q_{i,k}(x_1, x_2, \ldots, x_v)
\]

where the polynomials \( q_{i,k} \) do not depend on \( x_i \). Then the size of the coefficients in \( q_{i,n+1} \) are estimated again by an exponential fit, and the direction \( x_i \) with the largest \( q_{i,n+1} \) is chosen as the splitting direction.

In this way, all splits are performed in the direction of the variable that has the largest estimated contribution to the total truncation error of the polynomial \( P \), and thus the splits have the maximal impact on reducing the approximation error.

**Test case: Uncertainty Propagation in Kepler’s Dynamics**

To demonstrate the domain splitting technique described in the previous section, we apply it to the problem of propagating Kepler’s dynamics as in the previous section. The computations are performed at order 14 with the same initial condition box as in the previous example. The splitting precision is set to \( \varepsilon = 3 \cdot 10^{-4} \), meaning that when the estimated truncation error of an expansion exceeds this limit a split is triggered. We remark that for actual applications this limit is typically chosen much lower, it was chosen this high to allow for a better visualization of the splitting process.

Integrating the dynamics from time \( t_0 = 0 \) to time \( t_f = 2906.6 \) day (2.81 revolutions), the entire computation takes about 22 seconds on the same machine used for the example in the previous section, and produces 23 final polynomial expansions covering the initial condition.

Figure 4 shows the resulting boxes at various times during the integration. Up until time \( t_a = 930.1 \) day (0.90 revolutions), the entire set is well described by a single DA expansion. At time \( t_b = 988.3 \) day (0.96 revolutions), just before completing the first revolution 2 splits have occurred,
leading to three polynomial patches. Another split is performed at time $t_c = 1918.4$ day (1.86 revolutions). Figure 4d shows the 15 DA patches that are necessary to accurately track the uncertainty set at time $t_d = 2325.3$ day (2.25 revolutions). Then, the number of patches increases to 23 at the final integration time.

For the sake of completeness, similarly to the previous section, Figure 5 focuses on the integration time $t_i = 2025.9$ day. More specifically, Figure 5a illustrates how automatic domain splitting subdivides the initial domain in 9 boxes during the 14th order integration. On the other hand, Figure 5b reports all 9 propagated boxes at $t_i = 2025.9$ day and the box resulting from a single 14th order Taylor expansion of the flow on the entire uncertainty set. The comparison between Figure 5b and Figure 3b shows that automatic domain splitting allows the exact propagated set to be accurately described with the 9 Taylor polynomials.

**LONG TERM PROPAGATION OF (99942) APOPHIS**

The improvements that automated domain splitting brings into the DA-based integration are now investigated in the long-term propagation of uncertainties during Apophis post-encounter motion. The motion of Apophis in the Solar System is modeled including relativistic corrections to the well-known Newtonian forces. Specifically, the full equation of motion is given by
Figure 5. Propagation of the initial uncertainty set in the two-body dynamics to $t = 2025.9$ day with automatic domain splitting: (a) final subdivision of the initial domain in 9 subdomains; (b) comparison between a single 14th order Taylor expansion of the flow and a 14th order Taylor expansion including automatic domain splitting.

\[ \ddot{r} = G \sum_i \frac{m_i (r_i - r)}{r_i^3} \left\{ 1 - \frac{2(\beta + \gamma)}{c^2} G \sum_j \frac{m_j}{r_j} - \frac{2\beta - 1}{c^2} G \sum_{j \neq i} \frac{m_j}{r_{ij}} + \frac{\gamma |\dot{r}|^2}{c^2} \right\} 
+ \frac{(1 + \gamma)|\dot{r_i}|^2}{c^2} - \frac{2(1 + \gamma)}{c^2} \dot{r} \cdot \dot{r}_i - \frac{3}{2c^2} \left[ \frac{(r - r_i) \cdot \dot{r}_i}{r_i} \right]^2 + \frac{1}{2c^2} (r_i - r) \cdot \dot{r}_i 
+ G \sum_i \frac{m_i}{c^2 r_i} \left\{ \frac{3 + 4\gamma}{2} \ddot{r} + \frac{1}{r_i^2} \left[ \frac{(r - r_i) \cdot [(2 + 2\gamma)\dot{r} - (1 + 2\gamma)\dot{r}_i]}{r_i} \right] \left( \dot{r} - \dot{r}_i \right) \right\}, \]

where $r$ is the point of interest, $G$ is the gravitational constant; $m_i$ and $r_i$ are the mass and the Solar System barycentric position of body or planetary system $i$; $r_i = |r_i - r|$; $c$ is the speed of light in vacuum; and $\beta$ and $\gamma$ are the parametrized post-Newtonian parameters measuring the nonlinearity in superposition of gravity and space curvature produced by unit rest mass.\(^{20}\)

In Eq. 6 it is assumed that the object we are integrating is affected by the gravitational attraction of $n$ bodies, but has no gravitational effect on them; i.e., we are adopting the restricted $(n+1)$-body problem approximation. The positions, velocities, and accelerations of the $n$ bodies are considered as given values, computed from the JPL DE405 ephemeris model. In our integrations $n$ includes the Sun, planets, the Moon, Ceres, Pallas, and Vesta. For planets with moons, with the exception of the Earth, the center of mass of the system is considered. The dynamical model is written in the J2000.0 Ecliptic reference frame and is commonly referred to as Standard Dynamical Model.\(^{21}\)

As illustrated in Figure 6a, the asteroid Apophis will have an extremely close approach to the Earth on 13 April 2029 with a nominal closest distance of about $3.8 \cdot 10^4$ km. The asteroid orbit will then suffer a very large perturbation on its orbital parameters, which will cause its orbital period to increase from 323.60 days to 422.33 days (see Figure 6b for a plot of Apophis trajectory before and after 2029 close encounter). This opens the door to the possibility of a resonant return to Earth in 2036.

The nominal initial state and the associated $\sigma$ of Apophis at 3456 MJD2000 (June 18, 2009), expressed in equinoctial variables $p = (a, P_1, P_2, Q_1, Q_2, \ell)$, are used as test case in the followings. More specifically, Table 1 reports the Apophis’ ephemerides derived from the observa-
Figure 6. (a) Apophis’ nominal distance profile from Earth; (b) Apophis’ nominal trajectory before and after the close encounter on April 13th, 2029.

Table 1. Apophis equinoctial variables at 3456 MJD2000 (June 18, 2009) and associated σ values obtained from the Near Earth Object Dynamic Site in September 2009

<table>
<thead>
<tr>
<th>Nom Value</th>
<th>σ</th>
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<tbody>
<tr>
<td>a</td>
<td>0.922438242375914</td>
</tr>
<tr>
<td>P1</td>
<td>-0.093144699837425</td>
</tr>
<tr>
<td>P2</td>
<td>0.166982492089134</td>
</tr>
<tr>
<td>Q1</td>
<td>-0.012032857685451</td>
</tr>
<tr>
<td>Q2</td>
<td>-0.026474053361345</td>
</tr>
<tr>
<td>l</td>
<td>88.3150906433494</td>
</tr>
</tbody>
</table>

The nonlinearities of the close encounter and of the post-encounter motion will make the uncertainty in the direction anti-parallel to the Earth heliocentric motion in 2029 drastically diverge in subsequent epochs. This is clearly illustrated in Figure 7: the uncertain conditions of Table 1 are first propagated to epoch 10550 MJD2000 (before the close encounter) by sampling the edges of the 3σ uncertainty box with 2000 points and carrying out the associated pointwise integrations (see Figure 7a). The integration of Apophis’s motion is then continued to include the post-encounter motion. The resulting sets of final positions are illustrated in Figures 7b to 7d. As can be seen, the uncertainty set tends to quickly spread along the orbit due to the perturbations induced by the close encounter.

The performances of the standard DA-based integrator are first assessed. Apophis’ initial conditions of Table 1 are initialized as DA variables, converted into cartesian coordinates using the relations given in Battin, and then numerically propagated. Table 2 reports the maximum position error of a 9th order Taylor representation of the flow at the corners of the initial set, with respect to the pointwise integration of the same points. The errors are computed for increasing epochs. The table shows that a single Taylor polynomial of relatively high order cannot track uncertainties with sufficient accuracy for practical applications such as impact probability computation.

Automatic domain splitting is then enabled to improve the accuracy of the standard DA-based integration. The initial conditions of Table 1 are propagated until May 1st, 2038 (14000 MJD2000),
Figure 7. (a) Propagation of Apophis’ uncertain initial conditions: (a) November 19th, 2028 (10550 MJD2000); (b) September 15th, 2029 (10850 MJD2000); (c) April 3rd, 2030 (11050 MJD2000); (d) November 8th, 2032 (12000 MJD2000)

Table 2. Maximum position error of a single 9th order Taylor representation of the flow of Apophis’ dynamics over the initial uncertainty set of Table 1 for different epochs.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10550</td>
<td>19 Nov 2028</td>
<td>0.5 \cdot 10^{-11}</td>
</tr>
<tr>
<td>12800</td>
<td>17 Jan 2035</td>
<td>0.5 \cdot 10^{-6}</td>
</tr>
<tr>
<td>12825</td>
<td>11 Feb 2035</td>
<td>0.1 \cdot 10^{-5}</td>
</tr>
<tr>
<td>12850</td>
<td>8 Mar 2035</td>
<td>0.8 \cdot 10^{-3}</td>
</tr>
<tr>
<td>12875</td>
<td>2 Apr 2035</td>
<td>0.5 \cdot 10^{-2}</td>
</tr>
<tr>
<td>12900</td>
<td>27 Apr 2035</td>
<td>0.2 \cdot 10^{-1}</td>
</tr>
</tbody>
</table>
which includes the resonant return of 2036. A 9th order integration is performed and the integrator settings have been tuned to split the initial domain so to meet the requirement of tracking the uncertainties with an accuracy of the order of $10^{-9}$ AU and $10^{-9}$ AU/day for the asteroid position and velocity, respectively. In order to limit the number of generated subdomains and polynomials, domain splitting is disabled on any box whose volume is less than $2^{-12}$ times that of the initial domain. This limit was selected to reduce the computational time for this exploratory work, so to enable a relatively quick parametric analysis of the integrator performances. The limit can be removed in future work, especially considering that automatic domain splitting can take significant advantage of parallelization. Thus, the computational time can be drastically reduced.

Figure 8 plots the results obtained until April 27th, 2035 (12900 MJD2000) in terms of Apophis’ distance from Earth. The solid lines represent the trajectories followed by the center points of each box, whereas the lower and upper bounds of Earth’s distance over the entire uncertainty set are given by the grey band. As can be seen, the integrator is able to propagate uncertainties using only one 9th order Taylor polynomial until March 3rd, 2034 (12490 MJD2000), which already follows the first close encounter of 2029. Then, the nonlinearities over the now relatively large uncertainty set prevent the integrator from meeting the accuracy requirements. Thus, a first split occurs in the semimajor-axis direction, which causes the initial domain to split in two boxes. The resulting two polynomials are propagated forward in time and only one additional split occurs before epoch 12700 MJD2000. The perturbation induced by the close encounter intensifies the nonlinearities and causes a significant number of splits to issue from the three boxes.

The number of splits drastically increases in subsequent epochs. This is clearly illustrated in Figure 9, which reports the results obtained until the final epoch 14000 MJD2000. The figure shows that most splits tend to occur when the trajectories get close to Earth. This is expected as Earth’s gravitational perturbation tends to be maximized. The final number of generated boxes is 2497, whereas the associated computational time is 28.13 hr on an Intel(R) Core(TM) i7-4820K CPU @ 3.70GHz. It is worth highlighting that not all boxes reach the final epoch: this is due to the minimum
allowed box size. In particular, the number of splits tends to drastically increase close to the minima of the distance profile due to the higher nonlinearities. Once the minimum size is achieved, the resulting box can no longer be propagated without failing to meet the accuracy requirements. Thus, its integration is stopped.

Most splits along the integration occur in the semimajor axis and the true longitude directions. Consequently, Figure 10 reports the initial uncertainty set in the $a, l$-plane and all the resulting subdomains. Regions of larger final boxes can be easily distinguished from areas where most splits occur. This figure represents a precious source of information for astrodynamicists. As explained in the previous sections, the splits occur when the nonlinearities are too strong to be managed with a single Taylor polynomial. Consequently, the areas in Figure 10 where most splits concentrate coincide with regions of strong nonlinearity. These regions are intuitively recognized to include boxes that attain the closest distances from Earth. This is confirmed by Figure 11, which superimpose a color map to Figure 10. More specifically, the color map of Figure 11 illustrates the final epoch reached by the integration of each box. As can be seen, the regions of larger boxes of Figure 10 match the black areas in Figure 11. This means that larger boxes smoothly propagate until the final time. Consequently, all the initial conditions lying within have no risk to impact Earth until the final epoch. On the other hand, the propagations of smaller boxes tend to stop before 14000 MJD2000. This means that smaller boxes and the associated colored areas might contain risky initial conditions and deserve additional analysis. Thus, Figure 11 allows astrodynamicists to rule out the possibility of impacts from the black areas and supplies an upper bound for impact probability.

The different prevalent colors of the two colored areas of Figure 11 supply an additional valuable information on the resonances. More specifically, moving from the left to the right side of the figure, the prevalent color of the first colored area points out that the integration of most boxes stops around 2036. Similarly, the colors of the second area indicates that the integrations of the
Figure 10. Final boxes on the initial domain in the $a, \lambda$-plane.

Figure 11. Color map superimposed to the final boxes on the initial domain in the $a, \lambda$-plane: final integration time.
Figure 12. Sample initial conditions from the four areas of Figure 11 (a) and associated Earth distance profiles (b).

Table 3. Apophis equinoctial variables at 3473.5 MJD2000 (July 5, 2009) and associated \( \sigma \) values obtained from the Near Earth Object Dynamic Site in September 2013.

<table>
<thead>
<tr>
<th>Nom Value</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0.922443731280282</td>
</tr>
<tr>
<td>( P_1 )</td>
<td>-0.093137787707699</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>0.166984258496493</td>
</tr>
<tr>
<td>( Q_1 )</td>
<td>-0.012032702063741</td>
</tr>
<tr>
<td>( Q_2 )</td>
<td>-0.026474187976460</td>
</tr>
<tr>
<td>( l )</td>
<td>107.7856397515106</td>
</tr>
</tbody>
</table>

associated boxes are cut around 2037. Consequently, the colored areas clearly mark the regions of possible resonances between Earth and Apophis: the initial conditions lying in the first and second colored areas can bring to impact risk in 2036 and 2037 respectively. This is confirmed in Figures 12a and 12b. Sample initial conditions are taken from the four main areas of Figure 10b as illustrated in Figure 12a. The resulting distance profiles are plotted in Figure 12b: the initial conditions lying in the black areas show safe distances from Earth throughout the integration, whereas the remaining ones get closer distances near the associated resonance epoch.

A last result deserves to be mentioned. The initial conditions of Table 1 are derived from the observations available until late 2009. They allowed to illustrate the advantages and potentialities of the synergy between automatic domain splitting and DA-based integration. Additional optical and radar observations have been made thereafter (see http://neo.jpl.nasa.gov/apophis/). Thanks to the new observations, more accurate initial conditions are available to astrodynamists. Table 3 reports Apophis’ ephemerides on July 5th, 2009 (3473.5 MJD2000) including all recent optical and radar observations. These data were obtained by accessing the Near Earth Object Dynamic Site in October 2013.

As illustrated in the table, the recent observations allowed the standard deviations to be considerably reduced. A 9th order DA-based integration of the initial conditions of Table 3 shows that the new initial conditions can be smoothly propagated until the final epoch 14000 MJD2000 without requiring any split of the initial domain (see Figure 13b for the distance profile associated to the new
Figure 13. Uncertainty set of Table 3 reported on the color map of Figure 11 (a) and associated Earth distance profile (b).

initial conditions). This confirms the results and distances of closest approach reported in the Near Earth Object Dynamic Site (http://newton.dm.unipi.it/neodys) and NASA’s Jet Propulsion Laboratory website (http://neo.jpl.nasa.gov/apophis/). The initial conditions of Table 3 are now propagated backward to epoch 3456 MJD2000 and superimposed to Figure 11. The result is presented in 13a. As expected, the new initial conditions lie in the black areas of the figure, which further confirms that Apophis’ impact in 2036 can be ruled out.

CONCLUSION

This paper introduced automatic domain splitting into the high order DA-based integration to accurately propagate large sets of uncertainties in highly nonlinear dynamics and long term integrations. The resulting integrator splits the initial uncertainty domain in subdomains along the integration when the polynomials representing the current state do not meet the accuracy requirements. The final result is a list of final state polynomials, each describing the evolution of some automatically determined subset of the initial condition. Thus, altogether, the Taylor polynomials accurately map the entire initial domain in the final set. Consequently, given any generated sample in the initial domain, its integration to the final time can be replaced with the fast evaluation of the polynomial map that propagates the subdomain it belongs to. Monte Carlo simulations can then be enhanced by replacing thousands of integrations with evaluations of the Taylor expansions of the flow.

The performances of the integrator have been assessed on the propagation of asteroid (99942) Apophis post-encounter motion. In order to limit the number of generated subdomains and polynomials, a minimum box size was fixed. Consequently, not all boxes achieved the final integration time. These boxes correspond to regions of strong nonlinearity. For the case of Apophis, they turn out to match the regions where risky close encounters can occur. Consequently, future work needs to be dedicated to break this limit and allow impact probability computation for resonant returns.

REFERENCES