From outside in: external structures and internal properties in linear systems

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joint work with Jan C. Willems, Paul Fuhrmann, Yutaka Yamamoto, Harry L. Trentelman, Arjan van der Schaft, and Shodhan Rao

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Dedicated to the memory of Jan C. Willems

Part I: introduction

A *Leitmotiv* in systems and control

External: arising from system-environment interaction.
External variables (inputs, outputs) involved.

Internal: arising from within the system.
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  $\leftrightsquigarrow$ *Special realizations*

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Today: a *representation-free*, *trajectory-based* approach

Systems, functionals and representations

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"**Image representation**"

"**Kernel representation**"

Systems, functionals and representations

Bilinear- and quadratic differential forms:
functionals of w and a finite number of its derivatives.

Bilinear functional

$$L_\Phi(w_1, w_2) := w_1^\top \Phi_{00} w_2 + \ldots + \frac{d^k w_1}{dt^k}^\top \Phi_{k,m} \frac{d^m w_2}{dt^m} + \ldots$$

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- Powers of ζ: differentiation on the left;
- Powers of η: differentiation on the right;
- Multiplication by $\zeta + \eta \leftrightsquigarrow \frac{d}{dt}$ of functional.

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Quadratic differential form definition straightforward

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Philosophy/methodology:

from trajectory-level concept/property
to representation-level concept/property

Part II: Constructing an intrinsic “state”

The state property

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$$\underset{0}{\wedge} \quad \text{reads } (f_1 \underset{0}{\wedge} f_2)(t) := \begin{cases} f_2(t) & \text{for } t < 0 \\ f_2(t) & \text{for } t \geq 0 \end{cases}$$

Graphically...

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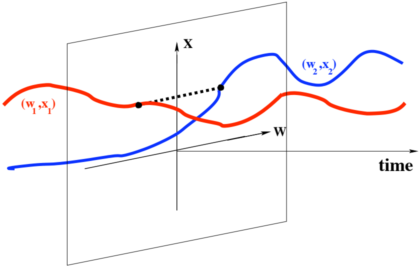

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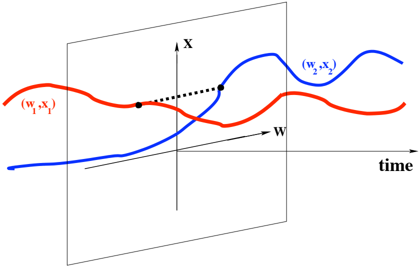

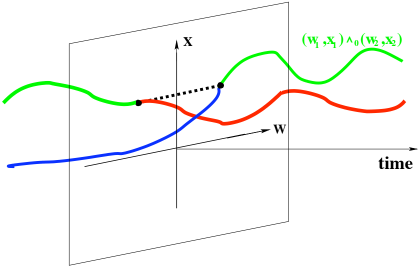

The state property revisited

For ***linear*** case, state property equivalent to:

$$\boxed{\begin{gathered}(w, x) \in \mathfrak{B}_{\text{full}} \text{ and } x(0) = 0 \text{ and } x \text{ continuous at } 0 \\ \Downarrow \\ (0, 0) \underset{0}{\wedge} (w, x) \in \mathfrak{B}_{\text{full}}\end{gathered}}$$

- ***Concatenability with zero*** is key.
- Algebraic characterization?

Kernel representations and the remainder

$w \in \mathcal{L}_1^{loc} \Longrightarrow$ **weak solution**:

$$R\left(\frac{d}{dt}\right) w = 0 \Longleftrightarrow \int_{-\infty}^{+\infty} w^\top R^\top \left(-\frac{d}{dt}\right) f \, dt = 0$$

for all ∞-ly differentiable $f(\cdot)$ of compact support.

Kernel representations and the remainder

Integrating $\int_{t_1}^{t_2} w^\top R\left(-\frac{d}{dt}\right)^\top f\ dt$ **by parts on** f:

$$\int_{t_1}^{t_2} w^\top \left(R_0^\top f - R_1^\top \frac{d}{dt} f + \ldots + (-1)^L R_L^\top \frac{d^L}{dt^L} f\right) dt$$

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$$= \int_{t_1}^{t_2} w^\top R_0^\top f\; dt - \left(w^\top R_1^\top f \,|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt} w^\top R_1^\top \right) f\; dt \right)$$

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Results in

- **bilinear terms in derivatives** of w and f in t_2, t_1
- integrals of bilinear terms in derivatives of w and f

Kernel representations and the remainder

Repeating until f no more differentiated in integral:

$$\int_{t_1}^{t_2} w^\top R\left(-\frac{d}{dt}\right)^\top f dt \;=\; \int_{t_1}^{t_2} f^\top R\left(\frac{d}{dt}\right) w\, dt + B_\Pi(f,w)\big|_{t_1}^{t_2}$$

where ***remainder*** $B_\Pi(f,w)$ defined by

$$\begin{bmatrix} f^\top & \dots & f^{(N-1)\top} \end{bmatrix} \tilde{\Pi} \begin{bmatrix} w \\ \vdots \\ w^{(N-1)} \end{bmatrix}$$

for some constant matrix $\tilde{\Pi}$ of dimension $Np \times Nq$.

A bilinear differential form!

Algebraic construction: kernel representations

Theorem. w concatenable with zero iff $B_\Pi(f, w)|_0^{+\infty} = 0$
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  $\rightsquigarrow$ properties of $\mathfrak{B}$ (external) directly reflected in internal (state) behavior.

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Theorem. Given $w = M\left(\frac{d}{dt}\right)\ell$, with $M \in \mathbb{R}^{q\times m}[s]$, define

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Part III: External structures and state realisations

Dual system

$w_1, w_2 \in \mathfrak{C}^\infty$ of compact support; define **dual** system by:

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\begin{aligned}
< w_1, w_2 > \;&:=\; \int_{-\infty}^{+\infty} w_1^\top w_2 \; dt \\
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$x' := Y\left(\frac{d}{dt}\right) w$ is state for $\mathfrak{B}^\perp$, acting on ℓ' s.t. $w' = R\left(-\frac{d}{dt}\right) \ell'$.

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Also $\frac{d}{dt} x'^\top x = w'^\top w$

Conservative port-Hamiltonian systems

Partition $w =: \begin{bmatrix} u \\ y \end{bmatrix}$, $u(t), y(t) \in \mathbb{R}^m$.

For compact-support $w_1, w_2 \in \mathfrak{C}^\infty$:

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$\mathfrak{B}$ is ***conservative port-Hamiltonian*** if

$$\mathfrak{B} = \mathfrak{B}^{\perp_Q}$$

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Proposition. $\mathfrak{B} = \mathrm{im}\, M\left(\frac{d}{dt}\right)$ is conservative port-Hamiltonian if and only if

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Moreover $\Psi(\zeta,\eta) = \Psi(\eta,\zeta)^\top$, and $\widetilde{\Psi}$ is symmetric.

Conservative port-Hamiltonian systems

Theorem. From $(\zeta+\eta)\Psi(\zeta,\eta) = M(\zeta)^\top Q M(\eta)$ factor

$$\widetilde{\Psi} = \widetilde{Z}^\top Q' \widetilde{Z} \ ,$$

with $\#$ rows $\widetilde{Z} = \mathrm{rank}(\widetilde{\Psi})$ and $Q'^\top = Q'$.

Define $Z(s) := \widetilde{Z} \begin{bmatrix} I_m \\ sI_m \\ \vdots \end{bmatrix}$. Then $x := Z\left(\frac{d}{dt}\right)\ell$ is state for $\mathfrak{B}$.

Conservative port-Hamiltonian systems

Theorem. From $(\zeta+\eta)\Psi(\zeta,\eta) = M(\zeta)^\top Q M(\eta)$ factor

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Q': ***internal energy***; Q (external) ***supply rate***.

Part IV: State equations and minimality

State equations

$$(\zeta+\eta)\Pi(\zeta,\eta) = R(-\zeta) - R(\eta) \text{ and } \Pi(\zeta,\eta) = Y(\zeta)^\top X(\eta)$$

$$\implies (\zeta Y(\zeta))^\top X(\eta) + Y(\zeta)^\top \eta X(\eta) - R(-\zeta) = -R(\eta)$$

with $Y(s) = \begin{bmatrix} Y_0 & \dots & Y_N \end{bmatrix} \begin{bmatrix} I_n \\ I_n s \\ \vdots \end{bmatrix}$

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For all testing functions f

$$\begin{bmatrix} f^\top & \dots \frac{d^N f}{dt^N} \end{bmatrix} \left(\begin{bmatrix} Y_0^\top \\ \vdots \\ Y_{N-1} \\ 0 \end{bmatrix} \frac{d}{dt} x + \begin{bmatrix} 0 \\ Y_0^\top \\ \vdots \\ Y_{N-1} \end{bmatrix} x - \begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix} w \right) = 0$$

State equations

Differential-algebraic equations follow:

$$\underbrace{\begin{bmatrix} Y_0^\top \\ \vdots \\ Y_{N-1} \\ 0 \end{bmatrix}}_{=:E} \frac{d}{dt} x + \underbrace{\begin{bmatrix} 0 \\ Y_0^\top \\ \vdots \\ Y_{N-1} \end{bmatrix}}_{=:F} x - \underbrace{\begin{bmatrix} R_0 \\ -R_1 \\ \vdots \\ (-1)^N R_N \end{bmatrix}}_{=:G} w = 0 \, .$$

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- Realization by inspection.
- State-input-output equations, too.
- Canonical realizations: factorise $\widetilde{\Pi}$ appropriately.

Minimality

Minimal state map: minimal number of components

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Proposition: Let $\Pi(\zeta,\eta) = Y^T(\zeta)X(\eta) = \frac{R(-\zeta)-R(\eta)}{\zeta+\eta}$.
State variable $x = X\left(\frac{d}{dt}\right) w$ is minimal if and only if

$$[fX(s) = h(s)R(s) \text{ with } f \in \mathbb{R}^{\bullet},\ h \in \mathbb{R}^{1\times p}[s]] \Longrightarrow [f = 0]$$

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Minimal state $\Longrightarrow$ Minimal factorisation

Converse **not** true in general: $\texttt{ker}\ R\left(\frac{d}{dt}\right) = \{0\}$ for

$R(s) := \begin{bmatrix} s & 1 \\ s-1 & 1 \end{bmatrix}$, but $\Pi(\zeta,\eta) = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$

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Minimal state $\Longrightarrow$ Minimal factorisation

Converse true if R is **row-proper**

Part V: State from data

State representations from data

Problem: compute state equations of conservative port-Hamiltonian $\mathfrak{B} = \mathrm{im}\, M\left(\frac{d}{dt}\right)$ from

$$w_i(t) = v_i e^{\lambda_i t}\ ,\ i = 1, \ldots, N\ ,\ v_i \in \mathbb{C}^q$$

with $\lambda_i + \lambda_j^* \neq 0$, $i, j = 1, \ldots, N$.

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Approach: Compute **state trajectories** x_i **for** w_i.
State equations straightforward.

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Analogous to *subspace identification* methods:

External trajectories

$\rightsquigarrow$ state trajectories

$\rightsquigarrow$ state equations

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- $w_i = v_i e^{\lambda_i t} = M\left(\frac{d}{dt}\right)\ell_i$;
- $M(\zeta)QM(\eta) = (\zeta + \eta)X(\zeta)^\top Q' X(\eta)$

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Theorem: If $N > \mathrm{n}(\mathfrak{B})$=minimal state dimension, then

$$\mathrm{rank}\left[\frac{v_i^* Q v_j}{\lambda_i^* + \lambda_j}\right]_{i,j=1,\ldots,N} = \mathrm{n}(\mathfrak{B}) \; .$$

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Rank-revealing factorize $\left[\frac{v_i^* Q v_j}{\lambda_i^*+\lambda_j}\right]_{i,j=1,\ldots,N} =: X^* Q' X$, with $X = \begin{bmatrix} x_1 & \ldots & x_N \end{bmatrix}$. Then $x_i e^{\lambda_i t}$ state trajectory for $w_i e^{\lambda_i t}$.

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Solve for E, F, G:

$$E \, X \operatorname{diag}(\lambda_1, \ldots, \lambda_N) + F \, X + G \begin{bmatrix} v_1 & \ldots & v_N \end{bmatrix} = 0$$

State representations from data

Approximate factorization $\left[\frac{v_i^* Q v_j}{\lambda_i^* + \lambda_j}\right]_{i,j=1,\ldots,N} \simeq \widehat{X}^* \widehat{Q}' \widehat{X}$

with $\widehat{X} \in \mathbb{R}^{\widehat{n} \times N}$, $\widehat{n} < \mathrm{n}(\mathfrak{B})$

$\Longrightarrow$ lower order **approximate** state model

also **conservative**, **port-Hamiltonian**

$\Longrightarrow$ **model reduction**

Part VI: Open problems and conclusions

Work in progress and open problems

- Model order reduction:
  - choice of exponential trajectories;
  - error bounds;
  - generalisation to dissipative systems.

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- Model order reduction:
  - choice of exponential trajectories;
  - error bounds;
  - generalisation to dissipative systems.
- Multidimensional systems:
  - **global** state variable available for **2-D** case;
  - **local** state variable?
  - n-D systems, with $n > 2$?

Conclusions

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Conclusions

- A representation-free approach to state construction;
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- State equations by inspection;
- Also applied in identification, model order reduction;

References

- Rapisarda, P. and Willems, J.C. “State Maps for Linear Systems”. *SIAM Journal on Control and Optimization*, vol. 35, no. 3, pp. 1053-1091, 1997.
- Fuhrmann, P. A., Rapisarda, P. and Yamamoto, Y. “On the state of behaviors”, *Linear Algebra and Its Applications* 424(2-3) pp. 570-614, 2007.
- Rapisarda, P. and Trentelman, H.L. “Identification and data-driven model reduction of state-space representations of lossless and dissipative systems from noise-free data”, *Automatica*, vol. 47, pp. 1721-1728, 2011.
- van der Schaft, A.J. and P. Rapisarda, “State maps from integration by parts”, *SIAM J. Contr. Opt.*, 49, (6), 2415–2439, 2011.
- Rapisarda, P. and van der Schaft, A.J. “Canonical realizations by factorization of constant matrices”, *System and Control Letters*, 61, (8), pp. 827-833, 2012.
- Rapisarda, P. and van der Schaft, A. “From integration by parts to state and boundary variables of linear differential and partial differential systems”, in *Mathematical System Theory - Festschrift in Honor of Uwe Helmke on the Occasion of his Sixtieth Birthday*, CreateSpace, pp. 437-448, 2013.
- Rapisarda, P. and Rao, S. “Realization of lossless systems via constant matrix factorizations”, *IEEE Trans. Aut. Contr.*, vol. 58, no. 10, pp. 2632–2636, 2013.
- P. Rapisarda, A.J. van der Schaft, “Identification and Data-Driven Reduced-Order Modeling for Linear Conservative Port and Self-Adjoint Hamiltonian Systems", *Proc. 52nd IEEE CDC*, Florence, Italy, 2013.

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