

# Modelling of switching dynamics in electrical systems

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**Abstract**—In this paper, we use the switched linear differential systems framework [8] to model electrical devices with switching dynamics. Modularity, i.e. independent modelling and incremental combination of complex dynamics, is an important feature of this approach, since we can incorporate new dynamic modes to the bank without altering the existing ones. This makes our approach ideal for describing complex systems (e.g. energy distribution networks). Our modelling approach differs fundamentally from the traditional representation-based theory where the use of a global state space is required.

## I. INTRODUCTION

There exist electrical systems whose physical laws change abruptly due to several reasons that are implicit in their design or application. For example, consider the DC-DC boost converter (see [1], Sec. 2.3, p.22) in Fig. 1, where a physical switch, consisting of solid state devices, changes the electrical configuration of the circuit.

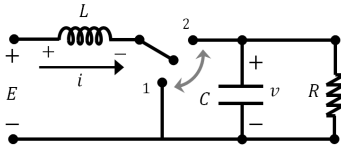


Fig. 1. DC-DC boost converter

When the switch is in position 1, the dynamic equations of the system are

$$\begin{cases} L \frac{d}{dt} i - E = 0, \\ C \frac{d}{dt} v + \frac{v}{R} = 0; \end{cases} \quad (1)$$

on the other hand, when the switch is in position 2, the equations are

$$\begin{cases} L \frac{d}{dt} i + v - E = 0, \\ C \frac{d}{dt} v + \frac{v}{R} - i = 0. \end{cases} \quad (2)$$

Note that the voltage across the capacitor and the current through the inductor in the sets of equations (1) and (2), qualify automatically as state variables; consequently it becomes natural to model a DC-DC boost converter as a switched system using a traditional state space form (see [5], [6]), i.e.  $\frac{d}{dt}x = A_k x + B_k u$ , where the index  $k = 1, 2$  denotes

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the position of the switch in Fig. 1, and determines from the set of matrices  $\{(A_1, A_2), (B_1, B_2)\}$ , the corresponding mathematical description of the system according to the electrical configuration that is active. In the case of the DC-DC boost converter and similar devices, a state space representation can be straightforwardly obtained from first principles. However, electrical systems may exhibit more complex dynamics where further elaboration in the modelling stage is required. For instance, it is well-known that parallel/series connections between capacitors/inductors induce discontinuities in the voltages and currents of switched electrical systems. In those cases, approaches such as the switched differential algebraic equations framework (see [15]) and the complementarity framework (see e.g. [17]) can be used. An even more fundamental issue in the modelling of dynamical systems is that state space representations are hardly ever obtained from physical principles automatically, and we usually need to derive such models from higher-order equations [18]. Moreover, in the case of switched systems, the switching phenomenon may also induce different state space dimensions for the corresponding dynamic regimes. Consider for instance the separately-excited DC motor (see [4], Sec. 2.4, p.78) in Fig. 2. In this case, we select the armature current  $i_a$  and the rotor position  $\theta$  as the variables of interest.

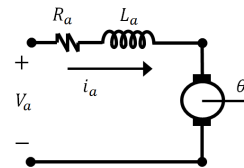


Fig. 2. Separately excited DC motor (armature winding)

The mechanical part of the system can be modelled using Newton's second law for rotatory masses, while the electrical part can be modelled using Kirchhoff voltage law, then we obtain

$$\begin{cases} J \frac{d^2}{dt^2} \theta + B_L \frac{d}{dt} \theta - L_{af} i_f i_a = 0; \\ L_a \frac{d}{dt} i_a - L_{af} i_f \frac{d}{dt} \theta + R_a i_a + V_a = 0. \end{cases} \quad (3)$$

where  $J$  is the rotor inertia,  $B_L$  the rotor viscous friction constant,  $L_{af}$  the mutual inductance,  $L_a$  the armature inductance,  $R_a$  the armature resistance,  $i_f$  the constant field winding current and  $V_a$  the voltage across the terminals of the armature winding that can be manipulated freely. If at an arbitrary time instant we connect a discharged capacitor  $C$

to the terminals of the motor armature winding, we add an additional dynamic equation  $C \frac{d}{dt} v + i_a = 0$ , with  $v = V_a$ .

Differently from the DC-DC boost converter, in the case of the DC motor, we obtain higher-order differential equations in a natural way when modelling the system. Moreover, the switching phenomenon induces a change in the state space dimension of the system. If we would want to construct a minimal state space representation for the DC motor, the state variables can be chosen to be  $x := [\theta \ \frac{d}{dt}\theta \ i_a]^\top$ . On the other hand, following the same criteria, when we attach the capacitor, the new dynamic regime has a set of state variables  $x' := [\theta \ \frac{d}{dt}\theta \ i_a \ v]^\top$ . In the traditional representation-based approaches, we need to satisfy the structural property that enforces the dynamic regimes to share a global state space, and thus we would be forced to construct two representations using the state space with the highest dimension  $x'$ , even for the case when the capacitor is disconnected. Moreover, the complication of constructing state representations with increased sizes becomes more evident when we also increase the number of loads that could be attached to the DC motor at some instant of time. However, there is no compelling reason to construct inflated state space representations if we can study the continuous-time properties of the system in higher-order terms, see for instance [7],[8],[11],[13],[16].

Moreover, in practical situations we are not particularly interested in monitoring the dynamics of the state variable associated to the attached capacitor, but it is of interest to study the dynamics that this new interconnection induces to the variables of the motor that were originally modelled. For the latter case, a straightforward elimination of such variable results in the following set of higher-order dynamic equations

$$\begin{cases} J \frac{d^2}{dt^2} \theta + B_L \frac{d}{dt} \theta - L_{af} i_f i_a = 0 ; \\ L_{af} i_f C \frac{d^2}{dt^2} \theta - L_a C \frac{d^2}{dt^2} i_a - R_a C \frac{d}{dt} i_a + i_a = 0 . \end{cases} \quad (4)$$

Thus the variables of interest in (4) are the same that those in (3). Note also that the set of equations (3) is not required to be modified artificially to make it fit into a representation with a globally shared state space.

Based on the previous discussion, we conclude that although the construction of state space representations arises in a straightforward way when modelling some systems as the boost converter, such approach is not always justified even for very simple cases as the one discussed for the DC motor, where higher-order descriptions suggest a more consistent approach. In this paper we show an alternative modelling approach for electrical systems with switching using the *switched linear differential systems* (SLDS) framework [8]. One of the main features of this framework is its *modularity*; every time a dynamic mode is added to the underlying bank, there is no need to modify the mathematical description of the existing modes. This is in sharp contrast with the traditional approach where we need to write every new description in the same fashion, i.e. considering a global state space, resulting in a more complex dynamical model

(with more variables and more equations), which has an impact in the complexity of stability analysis, simulation, control, etc.

## II. ELEMENTS OF THE BEHAVIOURAL APPROACH

In the following we introduce some concepts of the behavioural approach and state maps as well as the notation that will be repeatedly used along this paper. Detailed elaboration on these topics can be found in [10] and [12].

### A. Notation

The space of real vectors with  $n$  components is denoted by  $\mathbb{R}^n$ , and the space of  $n \times m$  real matrices by  $\mathbb{R}^{n \times m}$ . The ring of polynomials with real coefficients in the indeterminate  $\xi$  is denoted by  $\mathbb{R}[\xi]$ .  $\mathbb{R}^{n \times m}[\xi]$  is the space of  $n \times m$  polynomial matrices in  $\xi$ . If  $A, B$  are matrices with the same number of columns,  $\text{col}(A, B)$  denotes the matrix obtained by stacking  $A$  over  $B$ . The set of infinitely-differentiable functions from  $\mathbb{R}$  to  $\mathbb{R}^w$  is denoted by  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ . If  $f$  is a function defined in a neighbourhood  $[t - \epsilon, t)$  of  $t \in \mathbb{R}$ , we set for  $f : [t - \epsilon, t) \rightarrow \mathbb{R}^\bullet$  the notation  $f(t^-) := \lim_{\tau \nearrow t} f(\tau)$ ; and similarly for  $f : (t, t + \epsilon] \rightarrow \mathbb{R}^\bullet$  we set  $f(t^+) := \lim_{\tau \searrow t} f(\tau)$ , provided that these limits exist.

### B. Linear differential behaviours

A *linear differential behaviour* is a linear subspace  $\mathfrak{B}$  of  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$  consisting of all solutions  $w$  of a given system of linear constant-coefficient differential equations, represented by

$$R \left( \frac{d}{dt} \right) w = 0, \quad (5)$$

where  $R \in \mathbb{R}^{\bullet \times w}[\xi]$ . Equation (5) is called a *kernel representation* of the behaviour  $\mathfrak{B} := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid w \text{ satisfies (5)}\}$ , and  $w$  is called the *external variable* of  $\mathfrak{B}$ . A behaviour defined in such a way is normally denoted by  $\mathfrak{B} =: \ker R \left( \frac{d}{dt} \right)$ . The class of all such behaviours is denoted by  $\mathfrak{L}^w$ .

We recall the notion of  $R$ -canonical representative of a differential operator. Let  $R \in \mathbb{R}^{\bullet \times w}[\xi]$  be nonsingular, and let  $f \in \mathbb{R}^{1 \times w}[\xi]$ ;  $f$  can be uniquely written as  $fR^{-1} = s + n$ , where  $s \in \mathbb{R}^{1 \times w}(\xi)$  is a vector of strictly proper rational functions, and  $n \in \mathbb{R}^{1 \times w}[\xi]$ . We call  $sR \in \mathbb{R}^{1 \times w}[\xi]$  the (uniquely defined) *canonical representative of  $f$  modulo  $R$* , denoted by  $f \bmod R$ . Note that the polynomial differential operators  $f \left( \frac{d}{dt} \right)$  and  $f' \left( \frac{d}{dt} \right)$ , with  $f' = f \bmod R$ , are *equivalent along*  $\ker R \left( \frac{d}{dt} \right)$  in the sense that  $f \left( \frac{d}{dt} \right) w = f' \left( \frac{d}{dt} \right) w$  for all  $w \in \ker R \left( \frac{d}{dt} \right)$ . The definition of  $R$ -canonical representative extends in a natural way to polynomial matrices.

### C. State maps

A latent variable  $\ell$  is a *state variable* for  $\mathfrak{B}$  if and only if there exist  $E, F \in \mathbb{R}^{\bullet \times \bullet}$ ,  $G \in \mathbb{R}^{\bullet \times w}$  such that  $\mathfrak{B} = \{w \mid \exists \ell \text{ s.t. } E \frac{d\ell}{dt} + F\ell + Gw = 0\}$ , i.e. if  $\mathfrak{B}$  has a representation of first order in  $\ell$  and zeroth order in  $w$ . The minimal number of state variables necessary to represent  $\mathfrak{B}$

in this way is an invariant called the *McMillan degree* of  $\mathfrak{B}$ , denoted by  $n(\mathfrak{B})$ .

In [12] it has been shown that a state variable for  $\mathfrak{B}$  can be computed as the image of a polynomial differential operator called a *state map*. We now review the construction of state maps for autonomous behaviours. In this case, a state map acts on the external variable  $w$ . Let  $\mathfrak{B}$  be represented as in (5) with  $R \in \mathbb{R}^{w \times w}[\xi]$  nonsingular, and consider the set defined by  $\mathfrak{X}(R) := \{f \in \mathbb{R}^{1 \times w}[\xi] \mid fR^{-1} \text{ is strictly proper}\}$ .  $\mathfrak{X}(R)$  is a finite-dimensional subspace of the vector space  $\mathbb{R}^{1 \times w}[\xi]$  over  $\mathbb{R}$ . To construct a state map  $X \in \mathbb{R}^{n \times w}[\xi]$  for  $\mathfrak{B}$ , choose a set of generators  $x_i \in \mathbb{R}^{1 \times w}[\xi]$ ,  $i = 1, \dots, N$  of  $\mathfrak{X}(R)$ , and define  $X := \text{col}(x_i)_{i=1, \dots, N}$ ; to obtain a *minimal* state map, choose  $\{x_i\}$  so that they form a basis of  $\mathfrak{X}(R)$ .

It can be shown (see Cor. 6.7 of [12]) that the McMillan degree  $n(\mathfrak{B})$ , i.e. the minimal dimension of a state variable for  $\mathfrak{B}$ , equals  $\deg(\det(R))$ . Note that if  $X \in \mathbb{R}^{n \times w}[\xi]$  is a state map for  $\mathfrak{B} = \ker R \left(\frac{d}{dt}\right)$ , then for every  $F \in \mathbb{R}^{n \times w}[\xi]$ , also  $X'(\xi) := X(\xi) + F(\xi)R(\xi)$  is a state map, since  $X' \left(\frac{d}{dt}\right) w = X \left(\frac{d}{dt}\right) w$  for all  $w \in \mathfrak{B}$ .

Finally, we introduce the notion of *standard state map*. Let  $\mathfrak{B} = \ker R \left(\frac{d}{dt}\right)$  with  $R \in \mathbb{R}^{w \times w}[\xi]$  nonsingular, and assume that  $R$  is *column-reduced*, see [3] section 6.3.2. Denote the *column degrees* of  $R$  by  $\delta_j$ ,  $j = 1, \dots, w$ , and consider the polynomial vectors

$$e_j \xi^k, \quad j = 1, \dots, w, \quad k = 0, \dots, \delta_j - 1, \quad (6)$$

where  $e_j$  is the  $j$ -th vector of the canonical basis of  $\mathbb{R}^{1 \times w}$ . It can be proved that the vectors (6) form a basis for the state space of  $\mathfrak{B}$ . Any matrix whose rows are the vectors (6) (or any permutation thereof) is called a *standard state map* for  $\mathfrak{B}$ .

### III. SWITCHED LINEAR DIFFERENTIAL SYSTEMS

#### A. Main definitions

We start by introducing some basic definitions of switched linear differential systems that can be found in [8].

*Definition 1:* A *switched linear differential system (SLDS)*  $\Sigma$  is a quadruple  $\Sigma = \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$  where:

- $\mathcal{P} = \{1, \dots, N\} \subset \mathbb{N}$ , is the set of *indices*;
- $\mathcal{F} = \{\mathfrak{B}_1, \dots, \mathfrak{B}_N\}$ , with  $\mathfrak{B}_j \in \mathfrak{L}^w$  for  $j \in \mathcal{P}$ , is the *bank of behaviours*;
- $\mathcal{S} = \{s : \mathbb{R} \rightarrow \mathcal{P}\}$  with  $s$  piecewise constant and right-continuous, is the set of admissible *switching signals*;
- $\mathcal{G} = \{(G_{k \rightarrow \ell}^-(\xi), G_{k \rightarrow \ell}^+(\xi)) \in \mathbb{R}^{n \times w}[\xi] \times \mathbb{R}^{n \times w}[\xi]\}$  with  $1 \leq k, \ell \leq N$ ,  $k \neq \ell$ , is the set of *gluing conditions*.

The set of *switching instants* associated with  $s \in \mathcal{S}$  is defined by  $\mathbb{T}_s := \{t \in \mathbb{R} \mid s(t^-) \neq s(t^+)\} = \{t_1, t_2, \dots\}$ , where  $t_i < t_{i+1}$ . ■

In this paper we will consider autonomous behaviours represented in kernel form. We also make the standard assumption that the switching signal is arbitrary and *well-defined*, i.e. every finite interval of  $\mathbb{R}$  contains only a finite number of switching instants (see [14], Sec. 1.3.3).

A SLDS induces a switched behaviour, defined as follows.

*Definition 2:* Let  $\Sigma$  be a SLDS and  $s \in \mathcal{S}$ . The *s-switched behaviour*  $\mathfrak{B}^s$  with respect to  $\Sigma$  is the set of trajectories satisfying the following conditions:

- 1) For all  $t_i, t_{i+1} \in \mathbb{T}_s$ ,  $w|_{[t_i, t_{i+1})} \in \mathfrak{B}_{s(t_i)}|_{[t_i, t_{i+1})}$ ;
- 2)  $w$  satisfies the *gluing conditions*  $\mathcal{G}$  at the switching instants:  $(G_{s(t_{i-1}) \rightarrow s(t_i)}^+ \left(\frac{d}{dt}\right)) w(t_i^+) = (G_{s(t_{i-1}) \rightarrow s(t_i)}^- \left(\frac{d}{dt}\right)) w(t_i^-)$  for each  $t_i \in \mathbb{T}_s$ .

The *switched behaviour*  $\mathfrak{B}^\Sigma$  of  $\Sigma$  is defined by  $\mathfrak{B}^\Sigma := \bigcup_{s \in \mathcal{S}} \mathfrak{B}^s$ . ■

The following example illustrates a typical modelling procedure of the mode behaviours of a SLDS.

*Example 1:* Consider the DC-DC boost converter in Fig. 1 with the dynamic regimes in (1) and (2). If we consider a constant voltage source  $E$ , we can associate a dynamic equation  $\frac{d}{dt} E = 0$  to the model. Then, considering the external variable  $w := [E \quad i \quad v]^\top$ , the mode behaviours in  $\mathcal{F}$  can be expressed as  $\mathfrak{B}_i := \ker R_i \left(\frac{d}{dt}\right)$ ,  $i = 1, 2$  with

$$R_1(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L\xi & 0 \\ 0 & 0 & C\xi + \frac{1}{R} \end{bmatrix},$$

$$R_2(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L\xi & 1 \\ 0 & -1 & C\xi + \frac{1}{R} \end{bmatrix}.$$

□

For the SLDS in Ex. 1, we still need to specify the gluing conditions to define the concatenation of the external variables at switching instants according to Def.s 1 and 2. The gluing conditions reflect the physical laws that dictate the evolution of the system. In the following, we clarify the role of the gluing conditions by studying the concept of *well-posedness*.

#### B. Gluing conditions and well-posedness

The gluing conditions in Def.s 1 and 2 are in general algebraic constraints imposed to the external variables at the switching instants. However, for the case of autonomous modes, they can also be expressed in terms of the state.

Define kernel representations  $\mathfrak{B}_j := \ker R_j \left(\frac{d}{dt}\right)$ , with  $R_j \in \mathbb{R}^{n_j \times w}[\xi]$  nonsingular,  $j = 1, \dots, N$  for the mode behaviours in  $\mathcal{F}$ . We also define  $n_j := \deg(\det(R_j))$ ,  $j = 1, \dots, N$ , and we fix minimal state maps  $X_j \in \mathbb{R}^{n_j \times w}[\xi]$ ,  $j = 1, \dots, N$ . We compute the uniquely defined  $R_j$ -*canonical representative*  $G' \left(\frac{d}{dt}\right)$  of  $G \left(\frac{d}{dt}\right)$ , denoted by  $G' = G \bmod R$ , such that  $G' \left(\frac{d}{dt}\right) w = G \left(\frac{d}{dt}\right) w$  for all  $w \in \mathfrak{B}_j$ . Now let  $(G_{k \rightarrow \ell}^-, G_{k \rightarrow \ell}^+) \in \mathcal{G}$ ; then  $(G_{k \rightarrow \ell}^- \bmod R_k, G_{k \rightarrow \ell}^+ \bmod R_\ell)$  are equivalent to  $(G_{k \rightarrow \ell}^- \bmod R_k, G_{k \rightarrow \ell}^+ \bmod R_\ell)$ , in the sense that the algebraic conditions imposed by the one pair are satisfied if and only if they are satisfied by the other. Moreover, since  $G_{k \rightarrow \ell}^- \bmod R_k$  and  $G_{k \rightarrow \ell}^+ \bmod R_\ell$  are  $R_k$ -, respectively  $R_\ell$ -canonical, there exist constant matrices  $F_{k \rightarrow \ell}^-$  and  $F_{k \rightarrow \ell}^+$  of suitable

dimensions such that  $G_{k \rightarrow \ell}^-(\xi) \bmod R_k = F_{k \rightarrow \ell}^- X_k(\xi)$  and  $G_{k \rightarrow \ell}^+(\xi) \bmod R_\ell = F_{k \rightarrow \ell}^+ X_\ell(\xi)$ . We call

$$\mathcal{G}' := \{(F_{k \rightarrow \ell}^- X_k(\xi), F_{k \rightarrow \ell}^+ X_\ell(\xi)) \mid 1 \leq k, \ell \leq N, k \neq \ell\},$$

the *normal form* of  $\mathcal{G}$ .

Note that when switching occurs, the gluing conditions must specify a full set of ‘‘initial conditions’’ for the behaviour after the switching instant. Such property is called *well-posedness* according to the following definition that can be found in [8].

*Definition 3:* Let  $\Sigma$  be a SLDS with  $\mathfrak{B}_i = \ker R_i \left( \frac{d}{dt} \right)$  autonomous,  $i = 1, \dots, N$ . The normal form gluing conditions  $\mathcal{G}' := \{(F_{k \rightarrow \ell}^- X_k(\xi), F_{k \rightarrow \ell}^+ X_\ell(\xi))\}_{k, \ell=1, \dots, N, k \neq \ell}$  are *well-posed* if for all  $k, \ell = 1, \dots, N, k \neq \ell$ , and for all  $v_k \in \mathbb{R}^{n_k}$  there exists at most one  $v_\ell \in \mathbb{R}^{n_\ell}$  such that  $F_{k \rightarrow \ell}^- v_k = F_{k \rightarrow \ell}^+ v_\ell$ . ■

Thus when we switch from  $\mathfrak{B}_k$  to  $\mathfrak{B}_\ell$  at  $t_j$ , the admissible trajectory according to Def. 2 has a ‘‘final state’’  $v_k := X_k \left( \frac{d}{dt} \right) w(t_j^-)$ , then there exists at most one ‘‘initial state’’ for  $\mathfrak{B}_\ell$ , defined by  $v_\ell := X_\ell \left( \frac{d}{dt} \right) w(t_j^+)$ , compatible with the gluing conditions. Well-posedness implies that for all  $k, \ell = 1, \dots, N, k \neq \ell$ ,  $F_{k \rightarrow \ell}^\pm$  is full column rank, and consequently there exists a *re-initialisation map*  $L_{k \rightarrow \ell} : \mathbb{R}^{n_k} \rightarrow \mathbb{R}^{n_\ell}$  defined by  $L_{k \rightarrow \ell} := F_{k \rightarrow \ell}^{+*} F_{k \rightarrow \ell}^-$ , where  $F_{k \rightarrow \ell}^{+*}$  is a left inverse of  $F_{k \rightarrow \ell}^+$ . Given a switching signal such that  $s(t_{j-1}) = k$  and  $s(t_j) = \ell$ ; for all  $t_j \in \mathbb{T}_s$  and all admissible  $w \in \mathfrak{B}^\Sigma$  it holds that

$$\begin{aligned} & \left[ G_{k \rightarrow \ell}^+ \left( \frac{d}{dt} \right) w(t_j^+) = G_{k \rightarrow \ell}^- \left( \frac{d}{dt} \right) w(t_j^-) \right] \\ \implies & \left[ X_\ell \left( \frac{d}{dt} \right) w(t_j^+) = L_{k \rightarrow \ell} \left( X_k \left( \frac{d}{dt} \right) w(t_j^-) \right) \right]. \end{aligned} \quad (7)$$

*Example 2:* We continue with Ex. 1. The physics of the system imposes that the external variables are continuous, i.e. the instantaneous value of the external variables does not change at switching instants. Since  $R_i(\xi)$ ,  $i = 1, 2$  is column reduced, then  $X \left( \frac{d}{dt} \right) = I_3$  is a standard state map for  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$ , consequently well-posed gluing conditions can be defined as  $(G_{1 \rightarrow 2}^-, G_{1 \rightarrow 2}^+) := (X(\xi), X(\xi))$ ,  $(G_{2 \rightarrow 1}^-, G_{2 \rightarrow 1}^+) := (X(\xi), X(\xi))$ . Note that such algebraic constraints imply that

$$\begin{bmatrix} E(t_j^+) \\ i(t_j^+) \\ v(t_j^+) \end{bmatrix} = \begin{bmatrix} E(t_j^-) \\ i(t_j^-) \\ v(t_j^-) \end{bmatrix},$$

for every switching instant  $t_j \in \mathbb{T}_s$ . □

In Ex. 2, since the mode behaviours share the same state space and moreover, their trajectories are always continuous, then the re-initialisation maps as in (7) are equal to the identity. However in general, when modelling switched electrical systems, there may exist discontinuities in the voltages and currents and/or the state space of the system can change at switching instants.

Switched (or hybrid) systems with discontinuities are often called *systems with impulse effects* (see e.g. [19]). In

such systems, state trajectories are smooth in any interval between two consecutive switching times but *state jumps* can appear at switching instants. In the SLDS framework, the discontinuities of the state trajectories at switching instants are modelled implicitly via the gluing conditions. Consider the following example.

*Example 3:* Consider the DC-DC converter in Fig. 3 obtained from [20]. The electrical configuration of the converter is similar to the one in the previous examples where the operation of the circuit allows only two dynamic regimes, i.e. when the switches are in position (1, 1) or (2, 2). Moreover, the electrical configuration in Fig. 3 contains *switched capacitors*, i.e. the physical switches can be manipulated in such a way that a parallel connection between the capacitors  $C_1$  and  $C_2$  is induced, this happens in particular when the switches are in the position (1, 1).

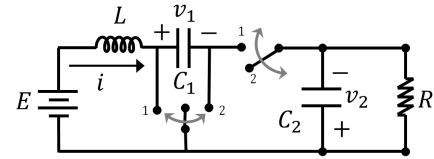


Fig. 3. DC-DC converter with switched capacitors

Considering the external variable  $w := [E \ i \ v_1 \ v_2]^\top$ , the mode behaviours can be modelled as  $\mathfrak{B}_i := \ker R_i \left( \frac{d}{dt} \right)$ ,  $i = 1, 2$ , with

$$R_1(\xi) := \begin{bmatrix} \xi & 0 & 0 & 0 \\ -1 & L\xi & 0 & 0 \\ 0 & 0 & (C_1 + C_2)\xi + \frac{1}{R} & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix};$$

$$R_2(\xi) := \begin{bmatrix} \xi & 0 & 0 & 0 \\ -1 & L\xi & 1 & 0 \\ 0 & -1 & C_1\xi & 0 \\ 0 & 0 & 0 & C_2\xi + \frac{1}{R} \end{bmatrix}.$$

Since the voltages across the capacitors  $C_1$  and  $C_2$  are discontinuous due to their sudden parallel connection, when we switch from  $\mathfrak{B}_2$  to  $\mathfrak{B}_1$ , the gluing conditions can be modelled taking into account the principle of conservation of charge (cf. [2])

$$v_1(t^+) = v_2(t^+) = \frac{C_1 v_1(t^-) + C_2 v_2(t^-)}{C_1 + C_2}.$$

Then we define  $(G_{1 \rightarrow 2}^-, G_{1 \rightarrow 2}^+) := (I_4, I_4)$ ,  $(G_{2 \rightarrow 1}^-, G_{2 \rightarrow 1}^+) := (L_{2 \rightarrow 1} I_4, I_4)$ , with

$$L_{2 \rightarrow 1} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{C_1}{C_1 + C_2} & \frac{C_2}{C_1 + C_2} \\ 0 & 0 & \frac{C_1}{C_1 + C_2} & \frac{C_2}{C_1 + C_2} \end{bmatrix}.$$

When we switch from  $\mathfrak{B}_1$  to  $\mathfrak{B}_2$ , the trajectories of the external variables are continuous and the gluing conditions

can be straightforwardly modelled to be the identity. On the other hand, when we switch from  $\mathfrak{B}_2$  to  $\mathfrak{B}_1$ , since the voltages across  $C_1$  and  $C_2$  change instantaneously after the switch, the principle of conservation of charge is modelled via the re-initialisation map  $L_{2 \rightarrow 1}$ .  $\square$

*Remark 1:* Note that in Ex. 3 the number of external variables is increased compared to that of Ex. 1. This occurs due to the fact that the variables  $v_1$  and  $v_2$  are fundamental attributes of the converter where we focus attention to apply the “principle of conservation of charge”. A similar modelling approach for switched systems in the presence of impulses and discontinuities in the state space setting is presented in [15] where algorithms to compute “consistency projectors” whose image is the subspace of consistent initial values at switching instants are established.  $\square$

*Remark 2:* Note that the mode behaviours in Ex. 3 do not share the same state space. A standard state map for  $\mathfrak{B}_2$  can be computed from  $X_2(\xi) := I_4$ , while a standard state map for  $\mathfrak{B}_1$  can be computed from  $X_1(\xi) := [I_3 \ 0_{3 \times 1}]$ .  $\square$

In the following section we will study a special class of SLDS whose mode behaviours do not share the same state space.

#### IV. MODULAR SLDS

Physical systems with modular structure are systems consisting of a main device that has been designed independently from other module devices. Furthermore, when such modules are interconnected with the main device at some instant of time, they induce a change in the state space dimension of the original system. We now discuss some mathematical properties of systems with modular structure using the SLDS framework. Consider the following lemma.

*Lemma 1:* Let  $\mathfrak{B}_i = \ker R_i \left( \frac{d}{dt} \right)$ ,  $i = 1, 2$ . Assume that  $R_1, R_2 \in \mathbb{R}^{w \times w}[\xi]$  are nonsingular, and that  $R_2 R_1^{-1}$  is strictly proper. Let  $n_i := \deg(\det(R_i))$ ; then  $n_2 < n_1$ . There exist  $X'_1 \in \mathbb{R}^{(n_1 - n_2) \times w}[\xi]$ ,  $X_2 \in \mathbb{R}^{n_2 \times w}[\xi]$  such that  $X_2 \left( \frac{d}{dt} \right)$  is a minimal state map for  $\mathfrak{B}_2$ , and  $X_1 \left( \frac{d}{dt} \right) := [X_2 \left( \frac{d}{dt} \right) \ X'_1 \left( \frac{d}{dt} \right)]^\top$ , is a minimal state map for  $\mathfrak{B}_1$ . Moreover, there exists  $\Pi \in \mathbb{R}^{(n_1 - n_2) \times n_2}$  such that  $X'_1(\xi) \bmod R_2 = \Pi X_2(\xi)$ .

*Proof:* See [8], Lemma 1.  $\blacksquare$

In [8], a SLDS consisting of two mode behaviours as in Lemma 1 in the bank is called *standard SLDS*. Such definition can be extended for more complex systems that contain more than one combination of pairs of mode behaviours in the bank, with the properties specified in Lemma 1.

*Definition 4:* A SLDS  $\Sigma$  with pairs of behaviours in  $\mathcal{F}$  defined as  $\mathfrak{B}_i := \ker R_i \left( \frac{d}{dt} \right)$ ,  $i = k, \ell$  with  $k, \ell \in \mathcal{P}$ ,  $k \neq \ell$ ; and such that  $R_\ell R_k^{-1}$  is strictly proper, is said to have a *modular structure*. Moreover, the gluing conditions associated to such pairs can be written as

$$\begin{cases} (G_{k \rightarrow \ell}^-, G_{k \rightarrow \ell}^+) := (X_\ell(\xi), X_\ell(\xi)), \\ (G_{\ell \rightarrow k}^-, G_{\ell \rightarrow k}^+) := \left( \begin{bmatrix} X_\ell(\xi) \\ \Pi_{\ell \rightarrow k} X_\ell(\xi) \end{bmatrix}, \begin{bmatrix} X_\ell(\xi) \\ X'_k(\xi) \end{bmatrix} \right); \end{cases} \quad (8)$$

where  $\Pi_{\ell \rightarrow k} \in \mathbb{R}^{(n_k - n_\ell) \times n_\ell}$  is such that  $X'_k(\xi) \bmod R_\ell = \Pi_{\ell \rightarrow k} X_\ell(\xi)$ .  $\blacksquare$

According to Def. 4, when we switch from a mode behaviour  $\mathfrak{B}_\ell$  with a lower state space dimension to one with a higher dimension  $\mathfrak{B}_k$ , the gluing conditions demand the continuity of the  $n_k - 1$  components of the state of  $\mathfrak{B}_k$  at the switching instants, by setting “initial conditions” that respect the laws imposed by the mode behaviours.

An example of a physical system with a modular structure is illustrated in Fig. 4. In that case, a power source is designed to feed loads that have been independently designed and that are connected/disconnected at certain times. In the following, we will focus on the modelling of electrical systems with a modular structure.

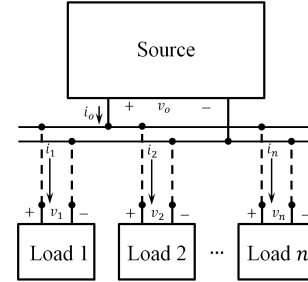


Fig. 4. DC-DC source converter

To model the switched electrical system of Fig. 4, the loads can be described independently with higher-order representations using *one-port impedance/admittances*. On the other hand, to compute the gluing conditions, we need to consider some additional concepts.

Let  $V, I \in \mathbb{R}[\xi]$  represent the voltages and currents across and through the driving-point of a one-port electrical network respectively. Such polynomials can be obtained in a standard way by computing the impedance  $Z(\xi) := \frac{V(\xi)}{I(\xi)}$  or admittance  $Y(\xi) := \frac{I(\xi)}{V(\xi)}$  functions of the electrical network. We now formulate the following definitions.

*Definition 5:* A one-port network is said to be *current-driven* if its impedance function  $Z(\xi)$  is proper.  $\blacksquare$

*Definition 6:* A one-port network is said to be *voltage-driven* if its admittance function  $Y(\xi)$  is proper.  $\blacksquare$

If we interconnect a load to a main power device and the output of the such device is designed to provide a voltage source, then such loads should be *voltage-driven* networks. If such condition does not hold, depending on the electrical configuration, we may produce discontinuities in voltages/currents at the switching instants. In the latter case, a special modelling of the gluing conditions as the ones illustrated in Ex. 3 could be required, taking into account additional physical laws such as the principle of conservation of charge. To illustrate this issue, consider Fig. 4; if we connect the non zero output voltage  $v_o$  of the main power device directly to a certain  $k$ -th current-driven load, we impose the algebraic constraint  $v_o = v_k$ , which in general is not satisfied at the switching instant. Consequently, the gluing conditions must specify the value of such voltages according to the physical situation that is being modelled.

In the following we will consider modular SLDS without discontinuities, and in particular switched voltage/current sources connected to voltage/current driven loads respectively.

### V. MAIN EXAMPLE: INCREMENTAL MODELLING

Consider the following example which summarises the concepts studied in the previous sections.

*Example 4:* Consider Fig. 5 which depicts a DC-DC boost converter acting as a voltage source for a set of loads that can be connected/disconnected at arbitrary time instants. We proceed to model a SLDS with the mode behaviours  $\mathfrak{B}_k := R_k \left( \frac{d}{dt} \right)$ ,  $i = 1, \dots, 8$ , in Table I.

TABLE I  
DYNAMICAL MODES

Case	Switch Position	Behaviours	Attached load
1	1,2	$\mathfrak{B}_1, \mathfrak{B}_2$	None
2	1,2	$\mathfrak{B}_3, \mathfrak{B}_4$	Low-pass filter
3	1,2	$\mathfrak{B}_5, \mathfrak{B}_6$	RL load
4	1,2	$\mathfrak{B}_7, \mathfrak{B}_8$	DC motor

**Case 1: No attached load.** As we know from Ex. 1, taking into account the external variable  $w := [E \ i_L \ v_o]$ , we obtain

$$R_1(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o \xi & 0 \\ 0 & 0 & C_o \xi + \frac{1}{R_o} \end{bmatrix},$$

$$R_2(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o \xi & 1 \\ 0 & -1 & C_o \xi + \frac{1}{R_o} \end{bmatrix};$$

and gluing conditions  $(G_{1 \rightarrow 2}^-, G_{1 \rightarrow 2}^+) := (X_1(\xi), X_2(\xi))$ ,  $(G_{2 \rightarrow 1}^-, G_{2 \rightarrow 1}^+) := (X_2(\xi), X_1(\xi))$  where  $X_1(\xi) = X_2(\xi) = I_3$  induces a state map  $\mathfrak{B}_i$ ,  $i = 1, 2$ .

Now we are interested in modelling the behaviour of the external variables that have been chosen when we attach additional loads, then we proceed to model the interconnection of each load independently.

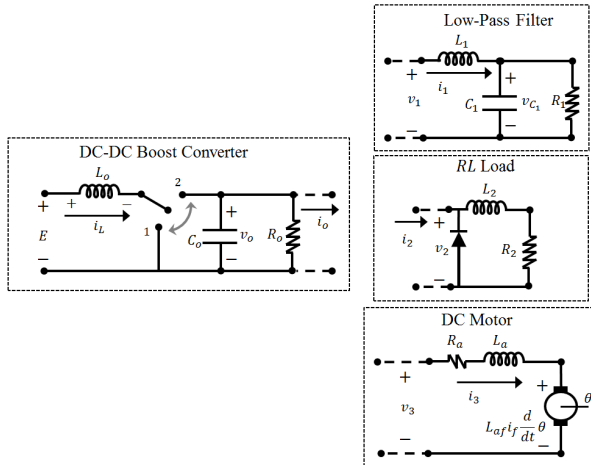


Fig. 5. DC-DC source converter with multiple loads

**Case 2: Low-pass filter.** When we attach the low-pass filter to the boost converter<sup>1</sup>, it is clear that  $v_o = v_1$  and  $i_o = i_1$ . In order to model the mode behaviours associated to such interconnection note that when the switch is in position 1, the output current  $i_o$  can be associated to the dynamic equation

$$i_o = -C_o \frac{d}{dt} v_o - \frac{1}{R_o} v_o. \quad (9)$$

Moreover, the corresponding transfer function of the load can be obtained by straightforward *series/parallel reductions* of one-port admittances, then it follows that

$$Y_1(\xi) := \frac{I_o(\xi)}{V_o(\xi)} := \frac{L_1 C_1 \xi + 1}{R_1 L_1 C_1 \xi^2 + L_1 \xi + R_1},$$

which implies that  $L_1 C_1 \frac{d}{dt} v_o + v_o = R_1 L_1 C_1 \frac{d^2}{dt^2} i_o + L_1 \frac{d}{dt} i_o + R_1 i_o$ . We can eliminate the variable  $i_o$  by substituting equation (9) in the latter equation, then we obtain

$$R_3(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o \xi & 0 \\ 0 & 0 & R_{3,33}(\xi) \end{bmatrix}.$$

where

$$R_{3,33}(\xi) := R_1 L_1 C_o C_1 \xi^3 + \left( L_1 C_o + \frac{R_1}{R_o} L_1 C_1 \right) \xi^2 + \left( L_1 C_1 + R_1 C_o + \frac{L_1}{R_o} \right) \xi + \frac{R_1}{R_o} + 1.$$

Similarly, when the switch is in position 2, we obtain the polynomial matrix

$$R_4(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o \xi & 1 \\ 0 & -R_1 L_1 C_1 \xi^2 - L_1 \xi - R_1 & R_{4,33}(\xi) \end{bmatrix}.$$

where  $R_{4,33}(\xi) = R_{3,33}(\xi)$ . Taking into account the laws of the system, we can define the corresponding gluing conditions using standard state maps. We compute

$$X_3(\xi) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \xi \\ 0 & 0 & \xi^2 \end{bmatrix} =: X_4(\xi).$$

then the gluing conditions associated to the interconnection under study are defined on the basis of physical considerations as

$$(G_{3 \rightarrow 4}^-, G_{3 \rightarrow 4}^+) := (X_3(\xi), X_4(\xi)),$$

$$(G_{4 \rightarrow 3}^-, G_{4 \rightarrow 3}^+) := (X_4(\xi), X_3(\xi)),$$

for the switching between behaviours that share the same state space. In physical terms, the state space of the boost converter is augmented when we attach an additional module that contains reactive elements. As previously discussed, for

<sup>1</sup>In Case 2 of Fig. 5, we include an ‘‘anti-parallel’’ diode only to follow standard designs of inductive loads with continuous currents. However, the diode is always assumed to remain open as the load remains connected to the converter.

modular SLDS such action implies strict properness on the polynomial matrices  $R_1R_3^{-1}$ ,  $R_1R_4^{-1}$ ,  $R_2R_3^{-1}$  and  $R_2R_4^{-1}$  and also that the state space of  $\mathfrak{B}_1, \mathfrak{B}_2$  is included in the state space of  $\mathfrak{B}_3, \mathfrak{B}_4$ . Consequently, for the switching between pairs of behaviours with modular structure we define the gluing conditions as follows

$$\begin{aligned} (G_{3 \rightarrow 1}^-, G_{3 \rightarrow 1}^+) &:= (X_1(\xi), X_1(\xi)) , \\ (G_{1 \rightarrow 3}^-, G_{1 \rightarrow 3}^+) &:= (L_{1 \rightarrow 3}X_1(\xi), X_3(\xi)) , \\ (G_{3 \rightarrow 2}^-, G_{3 \rightarrow 2}^+) &:= (X_2(\xi), X_2(\xi)) , \\ (G_{2 \rightarrow 3}^-, G_{2 \rightarrow 3}^+) &:= (L_{2 \rightarrow 3}X_2(\xi), X_3(\xi)) , \\ (G_{4 \rightarrow 1}^-, G_{4 \rightarrow 1}^+) &:= (X_1(\xi), X_1(\xi)) , \\ (G_{1 \rightarrow 4}^-, G_{1 \rightarrow 4}^+) &:= (L_{1 \rightarrow 4}X_1(\xi), X_4(\xi)) , \\ (G_{4 \rightarrow 2}^-, G_{4 \rightarrow 2}^+) &:= (X_2(\xi), X_2(\xi)) , \\ (G_{2 \rightarrow 4}^-, G_{2 \rightarrow 4}^+) &:= (L_{2 \rightarrow 4}X_2(\xi), X_4(\xi)) . \end{aligned}$$

where the re-initialisation maps are defined according to Def. (4) as  $L_{\ell \rightarrow k} := [L_3 \quad \Pi_{\ell \rightarrow k}]^T$  with  $\Pi_{\ell \rightarrow k} \in \mathbb{R}^{2 \times 3}$ , such that  $X_k(\xi) \bmod R_\ell = L_{\ell \rightarrow k}X_\ell(\xi)$ , i.e.

$$\begin{aligned} L_{1 \rightarrow 3} &:= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{R_o C_o} \\ 0 & 0 & \frac{1}{R_o^2 C_o^2} \end{bmatrix} =: L_{1 \rightarrow 4} , \\ L_{2 \rightarrow 3} &:= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{C_o} & -\frac{1}{R_o C_o} \\ \frac{1}{L_o C_o} & -\frac{1}{R_o C_o^2} & \frac{L_o - R_o^2 C_o}{R_o^2 L_o C_o^2} \end{bmatrix} =: L_{2 \rightarrow 4} . \end{aligned}$$

**Case 3: RL load.** We now take into account the interconnection with the  $RL$  load. The admittance transfer function of the load is

$$Y_2(\xi) := \frac{I_o(\xi)}{V_o(\xi)} := \frac{1}{L_2\xi + R_2} .$$

Using the latter equation, as shown for the previous case, we can obtain the following polynomial matrices.

$$R_5(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o\xi & 0 \\ 0 & 0 & R_{5,33}(\xi) \end{bmatrix} ,$$

with  $R_{5,33}(\xi) := L_2C_o\xi^2 + \left(R_2C_o + \frac{L_2}{R_o}\right)\xi + 1$ ; and

$$R_6(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o\xi & 1 \\ 0 & -L_2\xi - R_2 & R_{6,33}(\xi) \end{bmatrix} ,$$

with  $R_{6,33}(\xi) = R_{5,33}(\xi)$

Moreover, the gluing conditions corresponding to the current interconnection can be defined analogously to the latter case, i.e. we compute

$$X_5(\xi) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \xi \end{bmatrix} =: X_6(\xi) ;$$

and the re-initialisation maps

$$\begin{aligned} L_{1 \rightarrow 5} &:= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{R_o C_o} \end{bmatrix} =: L_{1 \rightarrow 6} , \\ L_{2 \rightarrow 5} &:= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{C_o} & -\frac{1}{R_o C_o} \end{bmatrix} =: L_{2 \rightarrow 6} . \end{aligned}$$

**Case 4: DC motor.** Finally, we consider the interconnection of the main power device with a DC motor as in Fig. 5. The *voltage-to-current transfer function* of the DC motor is

$$\frac{I_o(\xi)}{V_o(\xi)} := \frac{J\xi + B_L}{L_a J\xi^2 + (B_L L_a \xi + R_a J)\xi + L_{af}^2 i_f^2 + B_L R_a} .$$

Then, as in the previous cases, we can compute the following polynomial matrices

$$R_7(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o\xi & 0 \\ 0 & 0 & R_{7,33}(\xi) \end{bmatrix} ,$$

with

$$\begin{aligned} R_{7,33}(\xi) &:= \\ &L_a C_o J \xi^3 + \left(B_L L_a C_o s^2 + R_a C_o J s^2 + \frac{L_a J}{R_o}\right) \xi^2 \\ &+ \left(L_{af}^2 i_f^2 C_o + B_L R_a C_o + \frac{B_L L_a + R_a J}{R_o} + J\right) \xi \\ &+ \frac{L_{af}^2 i_f^2}{R_o} + \frac{B_L R_a}{R_o} + B_L ; \end{aligned}$$

and

$$R_8(\xi) := \begin{bmatrix} \xi & 0 & 0 \\ -1 & L_o\xi & 1 \\ 0 & -J\xi - B_L & R_{8,33}(\xi) \end{bmatrix} ,$$

with  $R_{8,33}(\xi) = R_{7,33}(\xi)$ . To define the gluing conditions, we compute

$$X_7(\xi) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \xi \\ 0 & 0 & \xi^2 \end{bmatrix} =: X_8(\xi) .$$

and the re-initialisation maps

$$L_{1 \rightarrow 7} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{R_o C_o} \\ 0 & 0 & \frac{1}{R_o^2 C_o^2} \end{bmatrix} =: L_{1 \rightarrow 8} ,$$

$$L_{2 \rightarrow 7} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{C_o} & -\frac{1}{R_o C_o} \\ \frac{1}{L_o C_o} & -\frac{1}{R_o C_o^2} & \frac{L_o - R_o^2 C_o}{R_o^2 L_o C_o^2} \end{bmatrix} =: L_{2 \rightarrow 8} .$$

Clearly, additional behaviours resulting from the combination of more than one load connected at the same time can be considered by using the same approach. Moreover, note that further incorporation of loads can be considered without altering the previously modelled modes and gluing conditions.  $\square$

*Remark 3:* In the presented examples, the set of external variables is the same for each dynamic mode. However, a “highly” modular SLDS may include dynamic modes that do not share the same set of external variables. Examples of such situations in electrical systems are converters with multiple connectable/disconnectable sources.  $\square$

*Remark 4:* Note that systems as in Ex. 4 can be described using state space representations, however such approach scores low on modularity, i.e. an incremental modelling is not permitted. Consider the boost converter in the Case 1, and consider that we know that the loads in Cases 2 and 3 will be connected/disconnected at some point. The dynamic modes can be represented by  $\frac{d}{dt}x = A_i x$ , with  $A_i \in \mathbb{R}^{6 \times 6}$ ,  $i = 1, \dots, 6$ . Since a global augmented state space  $x := [E \ i_L \ v_o \ i_2 \ v_{C_1} \ i_2]^\top$  is used, then every mode contains the highest possible complexity, even for the Case 1 when the loads are not connected. Furthermore, in practical applications, it is natural to consider interconnections of switched converters as in Ex. 4 with a large number of loads whose dynamical properties are unknown during the design process. In other words, only *nominal* loads are considered when the main power devices are designed, consequently it becomes of interest to study the dynamics induced when new loads are connected during implementations, see for instance [9]. Following this common situation, if we now consider the possibility to connect/disconnect the DC Motor of Case 3 in Ex. 4, the previous state space  $x$  is augmented to include new state variables, thus we define e.g.  $x' := [E \ i_L \ v_o \ i_2 \ v_{C_1} \ i_2 \ i_a \ \frac{d}{dt}i_a]^\top$ . In order to construct the same type of representation for every mode, we write new modes  $\frac{d}{dt}x' = A'_k x'$ , with  $A'_k \in \mathbb{R}^{8 \times 8}$ ,  $k = 1, \dots, 8$ . Consequently, the previously modelled matrices  $A_i$ ,  $i = 1, \dots, 6$  are useless, which is in sharp contrast with the SLDS framework where an incremental modelling is permitted and the existing modes are not altered. In the state space case, every mode is now represented by an 8-th order state space system, whereas in the SLDS approach such level of complexity is required only for the particular case when all the loads are connected at the same time to the converter.  $\square$

## VI. CONCLUSIONS

We have illustrated a new approach for the modelling of switched electrical systems. We have shown that although the use of representations with a global state space is sometimes justified, for the case of systems with modular structure, the traditional representation-oriented theory is neither necessary nor advisable. In the case of modular switched electrical systems, the use of a global state space requires the modification of all the systems in the bank everytime we consider a new module to be interconnected to the main power

device. On the other hand, in the SLDS framework, we are able to model the traditional switched systems with a common state space as a special case, while the modelling of modular SLDS arises in a natural way since behaviours with different state space dimension can be accommodated in a natural incremental way. Future research directions in this area include stability analysis and stabilization of switched electrical systems with a modular structure. Preliminary work on these topics and for the particular type of applications studied in this paper can be found in [8], Ex. 6.

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