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A regularisation approach to causality theory for $C^{1,1}$-Lorentzian metrics

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Abstract We show that many standard results of Lorentzian causality theory remain valid if the regularity of the metric is reduced to $C^{1,1}$. Our approach is based on regularisations of the metric adapted to the causal structure.

Keywords Causality theory, low regularity

1 Introduction

Traditionally, general relativity as a geometric theory has been formulated for smooth space-time metrics. However, over the decades the PDE point of view has become more and more prevailing. After all, general relativity as a physical theory is governed by field equations and questions of regularity are essential in the context of solving the initial value problem. Already the classical local existence theorem for the vacuum Einstein equations ([1]) deals with space-time metrics in H^s_{loc} with $s > 5/2$ (which merely guarantees the metric on the spatial slices to be C^1) and more recent studies have significantly lowered the regularity ([2–4]).

Also from the physical point of view non-smooth solutions are of vital interest. For example, one would like to study systems where different regions of space-time have different matter contents, e.g. inside and outside a star, or in the case of shock waves. On matching these regions the matter variables become discontinuous, which via the field equations forces the differentiability of the metric to be below C^2. E.g. a metric of regularity $C^{1,1}$ (continuously differentiable with locally Lipschitz first order derivatives, often also denoted by C^{2-}) corresponds to finite jumps of the matter variables. In the standard approach ([5]) one deals with metrics which are piecewise C^3 but globally are only C^1. Even more extreme situations are exemplified by impulsive waves (e.g. [6, Ch. 20]) where the metric is still C^3 off the impulse but globally is merely C^0.

On the other hand, in the bulk of the literature in general relativity it seems to be assumed (sometimes implicitly) that the differentiability of the space-time metric is at least C^2, especially so in the standard references on causality theory. More precisely, the presentations in [7–11] generally (seem to) assume smoothness, while [12–15] assume C^2-differentiability. This mismatch in regularity—the quest for low regularity from physics and analysis versus the need for higher regularity to maintain standard results from geometry—has of course been widely noted, see e.g. [12, 11, 16, 13–15, 17] for a review of various approaches to causal structures and discussions of regularity assumptions. The background of this “annoying problem” ([14, §2]) is that for C^2-metrics

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the existence of totally normal (convex) neighbourhoods is guaranteed. Furthermore, as emphasised by Senovilla, C^2-differentiability of the metric is one of the fundamental assumptions of the singularity theorems (see [13, §6.1] for a discussion of regularity issues in this context). Finally, in [15] it has recently been explicitly demonstrated that assuming the metric to be C^2 allows one to retain many of the standard causality properties of smooth metrics.

However, if one attempts to lower the differentiability of the metric below C^2 one encounters serious problems. It *is* possible to develop some of the elements of causality theory in low regularity: E.g., smooth time functions exist on domains of dependence even for continuous metrics ([18, 19]) and the space of causal curves is still compact in this case ([17]). On the other hand it is well-known that some essential building blocks of the theory break down for general C^1-metrics. Explicit counterexamples by Hartman and Wintner, [20, 21] (in the Riemannian case) show that for connections of Hölder regularity $C^{0,\alpha}$ with $0 < \alpha < 1$ convexity properties in small neighbourhoods may fail to hold. For example, radial geodesics may fail to be minimising between any two points they contain. Also recently a study of the causality of continuous metrics in [19] has revealed a dramatic failure of fundamental results of smooth causality: e.g., light cones no longer need to be topological hypersurfaces of codimension one. In fact, for any $0 < \alpha < 1$ there are metrics of regularity $C^{0,\alpha}$, called 'bubbling metrics', whose light-cones have nonempty interior, and for whom the push-up principle ceases to hold (there exist causal curves that are not everywhere null but for which there is no fixed-endpoint deformation into a timelike curve).

For these reasons there has for some time been considerable interest in determining the minimal degree of regularity of the metric for which standard results of Lorentzian causality remain valid. A reasonable candidate is the regularity class $C^{1,1}$ since it marks the threshold where one still has unique solvability of the geodesic equation, and the above remarks show that lower regularity will in general prevent reasonable convexity properties. However, the main ingredient for studying local causality, the exponential map, is now only locally Lipschitz and while it was well-known ([22]) that it is a local homeomorphism, only recently in [23] it was shown to be in fact bi-Lipschitz. More precisely, using approximation techniques and employing new methods of Lorentzian comparison geometry ([24]) it was shown in [23, Th. 2.1] that the exponential map retains maximal regularity in the following sense:

Theorem 1.1. *Let M be a smooth manifold with a $C^{1,1}$-pseudo-Riemannian metric g and let $p \in M$. Then there exist open neighbourhoods $\tilde{U}$ of $0 \in T_pM$ and U of p in M such that*

$$\exp_p : \tilde{U} \to U$$

is a bi-Lipschitz homeomorphism.

It then follows from Rademacher's theorem that both $\exp_p$ and $\exp_p^{-1}$ are differentiable almost everywhere. If $\exp_p : \tilde{U} \to U$ is a bi-Lipschitz homeomorphism and $\tilde{U}$ is star-shaped around 0 we call U a normal neighbourhood of p. If U is a normal neighbourhood of each of its elements then it is called totally normal. In the literature (e.g., [8]), totally normal sets are also called convex sets. Any totally normal set U is geodesically convex in the sense that for any two points in U there is a unique geodesic contained in U that connects them. Totally normal sets play an important role in local causality theory, see Section 2 below. The following result, proved in [23, Th. 4.1] ensures that locally there always exist such neighbourhoods:

Theorem 1.2. *Let M be a smooth manifold with a $C^{1,1}$-pseudo-Riemannian metric g. Then each point $p \in M$ possesses a basis of totally normal neighbourhoods.*

The aim of this paper is to develop the key elements of causality theory for $C^{1,1}$-Lorentzian metrics based on the above results as well as on refined regularisation techniques, extending the approach of [19], thereby demonstrating that indeed $C^{1,1}$ is the minimal degree of regularity where a substantial part of smooth causality theory remains valid.

While we were in the final stages of preparing the present paper we learned that an alternative approach to causality theory for $C^{1,1}$-Lorentzian metrics by E. Minguzzi had recently appeared in

[25]. This paper also establishes the fact $\exp_p$ is a bi-Lipschitz homeomorphism, and in addition shows that exp is a bi-Lipschitz homeomorphism on a neighbourhood of the zero-section in TM and is strongly differentiable over this zero section [25, Th. 1.11]. In this work, the required properties of the exponential map are derived from a careful analysis of the corresponding ODE problem based on Picard-Lindelöf approximations, as well as from an inverse function theorem for Lipschitz maps. In [25] the author also goes on to establish the Gauss Lemma and to develop the essential elements of $C^{1,1}$-causality, thereby obtaining many of the results that are also contained in the present work, some even in greater generality.

Nevertheless, we believe that our approach is of interest, and that in fact the approach in [25] and ours nicely complement each other, for the following reasons: Our methods are a direct continuation of the regularisation approach of P. Chrusciel and J. Grant ([19]) and are completely independent from those employed in [25]. The basic idea is to approximate a given metric of low regularity (which may be as low as C^0) by two nets of smooth metrics $\check{g}_\epsilon$ and $\hat{g}_\epsilon$ whose light cones sandwich those of g. We then continue the line of argument of [19, 23] to establish the key results of causality theory for a $C^{1,1}$-metric (thereby answering a corresponding question in [19] which mainly motivated this work, namely whether the results of [15] remain true for $C^{1,1}$-metrics). The advantage of these methods is that they quite easily adapt to regularity below $C^{1,1}$, which as far as we can see is the natural lower bound for the applicability of those employed in [25]. As an example, we note that the push-up lemmas from [19], cf. Prop. 3.6 and 3.7 below, in fact even hold for $C^{0,1}$-metrics (or, more generally, for causally plain C^0-metrics), whereas the corresponding results in [25, Sec. 1.4] require the metric to be $C^{1,1}$.

Furthermore, although considerable work still needs to be done, we believe that the regularisation approach adopted here, together with methods from Lorentzian comparison geometry as used in [24] and [23], will allow us to address some of the other results required (such as curvature estimates, variational properties of curves, and existence of focal points) in order to establish singularity theorems for $C^{1,1}$-metrics, where so far only limited results are available ([13]). Indeed, we note that the relevance of the kind of approximation techniques advocated in [19, 23] for such questions was already pointed out in [12, Sec. 8.4].

The plan of the paper is as follows. In section 2 we introduce the regularisation techniques and show how they may be applied to establish the Gauss Lemma (Theorem 2.7) for a $C^{1,1}$-pseudo-Riemannian metric. Section 3 deals with the key elements of $C^{1,1}$-causality theory and in Theorem 3.9 we again use regularisation methods to show that the local causal structure is given by the image of the null cone under the exponential map. This is then used to show that if a causal curve from p ends at a point in $\partial J^+(p)$ then it is a null geodesic. We then go on to deduce the basic elements of causality theory using standard methods. Finally in section 4 we refer to the results of [19] to show that all the major building blocks are in place to follow the C^2-proofs as given in [15] to establish those elements of causality theory that do not rely on continuity of the curvature.

2 Regularisation techniques

Throughout this paper we assume M to be a C^∞-manifold and only lower the regularity of the metric. This is no loss of generality since any C^k-manifold with $k \geq 1$ possesses a unique C^∞-structure that is C^k-compatible with the given C^k-structure on M (see [26, Th. 2.9]).

As already mentioned in the introduction a fundamental tool in our approach is approximating a given metric of regularity $C^{1,1}$ by a net g_ε of C^∞-metrics, in the following sense:

Remark 2.1. We cover M by a countable and locally finite collection of relatively compact chart neighbourhoods and denote the corresponding charts by (U_i, ψ_i) $(i \in \mathbb{N})$. Let $(\zeta_i)_i$ be a subordinate partition of unity with $\mathrm{supp}(\zeta_i) \Subset U_i$ (i.e., $\mathrm{supp}(\zeta_i)$ is a compact subset of U_i) for all i and choose a family of cut-off functions $(\chi_i)_i \in \mathcal{D}(U_i)$ with $\chi_i \equiv 1$ on a neighbourhood of $\mathrm{supp}(\zeta_i)$. Finally, let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a test function with unit integral and define the standard mollifier $\rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$

($\varepsilon > 0$). Then denoting by f_* (resp. f^*) push-forward (resp. pullback) under a map f, the following formula defines a family $(g_\varepsilon)_\varepsilon$ of smooth sections of $T^0_2(M)$

$$g_\varepsilon := \sum_i \chi_i \, g^i_\varepsilon := \sum_i \chi_i \, \psi_i^* \Big(\big(\psi_{i\,*}(\zeta_i \, g)\big) * \rho_\varepsilon \Big)$$

which satisfies

(i) g_ε converges to g in the C^1-topology as $\varepsilon \to 0$, and
(ii) the second derivatives of g_ε are bounded, uniformly in ε, on compact sets.

On any compact subset of M, therefore, for ε sufficiently small the g_ε form a family of pseudo-Riemannian metrics of the same signature as g whose Riemannian curvature tensors R_ε are bounded uniformly in ε. Indeed, properties (i) and (ii) were the only ones required to derive all results given in [23].

Also observe that the above procedure can be applied even to distributional sections of any vector bundle $E \to M$ (using the corresponding vector bundle charts) and that the usual convergence properties of smoothings via convolution are preserved.

To distinguish exponential maps stemming from metrics g_ε, etc., we will write $\exp_p^{g_\varepsilon}$, etc.. For brevity we will drop this superscript for the $C^{1,1}$-metric g itself, though. We shall need the following properties of the exponential maps corresponding to an approximating net as above:

Lemma 2.2. *Let g be a $C^{1,1}$-pseudo-Riemannian metric on M and let g_ε be a net of smooth pseudo-Riemannian metrics that satisfy conditions (i) and (ii) of Remark 2.1. Then any $p \in M$ has a basis of normal neighbourhoods U such that, with $\exp_p : \tilde U \to U$, all $\exp_p^{g_\varepsilon}$ are diffeomorphisms with domain $\tilde U$ for ε sufficiently small. Moreover, the inverse maps $(\exp_p^{g_\varepsilon})^{-1}$ also are defined on a common neighbourhood of p for ε small, and converge locally uniformly to $\exp_p^{-1}$.*

Proof. The claims about the common domains of $\exp_p^{g_\varepsilon}$, resp. of $(\exp_p^{g_\varepsilon})^{-1}$ follow from [23, Lemma 2.3 and 2.8]. To obtain the convergence result, we first note that without loss, given a common domain V of the $(\exp_p^{g_\varepsilon})^{-1}$ for $\varepsilon < \varepsilon_0$, we may assume that $\bigcup_{\varepsilon<\varepsilon_0}(\exp_p^{g_\varepsilon})^{-1}(V)$ is relatively compact in $\tilde U$: this follows from the fact that the maps $(\exp_p^{g_\varepsilon})^{-1}$ are Lipschitz, uniformly in ε (see [23], the argument following Lemma 2.10).

Now if $(\exp_p^{g_\varepsilon})^{-1}$ did not converge uniformly to $\exp_p^{-1}$ on some compact subset of V then by our compactness assumptions we could find a sequence q_k in V converging to some $q \in V$ and a sequence $\varepsilon_k \searrow 0$ such that $w_k := (\exp_p^{g_{\varepsilon_k}})^{-1}(q_k) \to w \neq \exp_p^{-1}(q)$. But since $(\exp_p^{g_\varepsilon}) \to \exp_p$ locally uniformly (by [23, Lemma 2.3]), we arrive at $q_k = \exp_p^{g_{\varepsilon_k}}(w_k) \to \exp_p(w) \neq q$, a contradiction. □

In the particular case of g being Lorentzian, a more sophisticated approximation procedure, adapted to the causal structure of g, was given in [19, Prop. 1.2].

To formulate this result, we first recall that a space-time is a time-oriented Lorentzian manifold (of signature $(- + \cdots +)$), with time-orientation determined by some continuous timelike vector field. In what follows, all Lorentzian manifolds will be supposed to be time-oriented. Also we recall from [19] that for two Lorentzian metrics g, h, we say that h has strictly larger light cones than g, denoted by $g \prec h$, if for any tangent vector $X \neq 0$, $g(X,X) \leq 0$ implies that $h(X,X) < 0$.

We will also need the following technical tools:

Lemma 2.3. *Let (K_m) be an exhaustive sequence of compact subsets of a manifold M ($K_m \subseteq K^\circ_{m+1}$, $M = \bigcup_m K_m$), and let $\varepsilon_1 \geq \varepsilon_2 \geq \cdots > 0$ be given. Then there exists some $\psi \in C^\infty(M)$ such that $0 < \psi(p) \leq \varepsilon_m$ for $p \in K_m \setminus K^\circ_{m-1}$ (where $K_{-1} := \emptyset$).*

Proof. See, e.g., [27, Lemma 2.7.3]. □

For what follows, recall that $K \Subset M$ denotes that K is a compact subset of M.

Lemma 2.4. *Let M, N be manifolds, and set $I := (0, \infty)$. Let $u : I \times M \to N$ be a smooth map and let (P) be a property attributable to values $u(\varepsilon, p)$, satisfying:*

(i) For any $K \Subset M$ there exists some $\varepsilon_K > 0$ such that (P) holds for all $p \in K$ and $\varepsilon < \varepsilon_K$.
(ii) (P) is stable with respect to decreasing K and ε: if $u(\varepsilon, p)$ satisfies (P) for all $p \in K \Subset M$ and all ε less than some $\varepsilon_K > 0$ then for any compact set $K' \subseteq K$ and any $\varepsilon_{K'} \leq \varepsilon_K$, u satisfies (P) on K' for all $\varepsilon \leq \varepsilon_{K'}$.

Then there exists a smooth map $\tilde{u} : I \times M \to N$ such that (P) holds for all $\tilde{u}(\varepsilon, p)$ ($\varepsilon \in I$, $p \in M$) and for each $K \Subset M$ there exists some $\varepsilon_K \in I$ such that $\tilde{u}(\varepsilon, p) = u(\varepsilon, p)$ for all $(\varepsilon, p) \in (0, \varepsilon_K] \times K$.

Proof. See [28, Lemma 4.3]. □

Based on these auxiliary results, we can prove the following refined version of [19, Prop. 1.2]:

Proposition 2.5. *Let (M, g) be a space-time with a continuous Lorentzian metric, and h some smooth background Riemannian metric on M. Then for any $\varepsilon > 0$, there exist smooth Lorentzian metrics $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ on M such that $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ and $d_h(\check{g}_\varepsilon, g) + d_h(\hat{g}_\varepsilon, g) < \varepsilon$, where*

$$d_h(g_1, g_2) := \sup_{0 \neq X, Y \in TM} \frac{|g_1(X, Y) - g_2(X, Y)|}{\|X\|_h \|Y\|_h}.$$

Moreover, $\hat{g}_\varepsilon$ and $\check{g}_\varepsilon$ depend smoothly on ε, and if $g \in C^{1,1}$ then $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ additionally satisfy (i) and (ii) from Rem. 2.1.

Proof. First we use time-orientation to obtain a continuous timelike one-form $\tilde{\omega}$ (the g-metric equivalent of a continuous timelike vector field). Using the smoothing procedure of Rem. 2.1, on each U_i we can pick $\varepsilon_i > 0$ so small that $\tilde{\omega}_{\varepsilon_i}$ is timelike on U_i. Then $\omega := \sum_i \zeta_i \tilde{\omega}_{\varepsilon_i}$ is a smooth timelike one-form on M. By compactness we obtain on every U_i a constant $c_i > 0$ such that

$$|\omega(X)| \geq c_i \quad \text{for all } g\text{-causal vector fields } X \text{ with } \|X\|_h = 1. \tag{2.1}$$

Next we set on each U_i and for $\eta > 0$ and $\lambda < 0$

$$\hat{g}^i_{\eta,\lambda} = g^i_\eta + \lambda\, \omega \otimes \omega, \tag{2.2}$$

where g^i_η is as in Remark 2.1 (set $\varepsilon := \eta$ there and $g^i_\eta := g_\eta|_{U_i}$). Let Λ_k ($k \in \mathbb{N}$) be a compact exhaustion of $(-\infty, 0)$. For each k, there exists some $\eta_k > 0$ such that $\eta_k < \min_{\lambda \in \Lambda_k} |\lambda|$, $\eta_k > \eta_{k+1}$ for all k, and

$$|g^i_\eta(X, X) - g(X, X)| \leq |\lambda| \frac{c_i^2}{2} \tag{2.3}$$

for all g-causal vector fields X on U_i with $\|X\|_h = 1$, all $\lambda \in \Lambda_k$, and all $0 < \eta \leq \eta_k$. Thus by Lemma 2.3 there exists a smooth function $\lambda \mapsto \eta(\lambda, i)$ on $(-\infty, 0)$ with $0 < \eta(\lambda, i) \leq |\lambda|$ and such that (2.3) holds for all g-causal vector fields X on U_i with $\|X\|_h = 1$, all λ, and all $0 < \eta \leq \eta(\lambda, i)$.

Combining (2.1) with (2.3) we obtain

$$\hat{g}^i_{\eta,\lambda}(X, X) = g(X, X) + (g^i_\eta - g)(X, X) + \lambda\, \omega(X)^2 \leq 0 + \Big(|\lambda| \frac{c_i^2}{2} + \lambda c_i^2\Big) \|X\|_h^2 < 0,$$

for all g-causal X and hence $g \prec \hat{g}^i_{\eta,\lambda}$ for all $\lambda < 0$ and $0 < \eta \leq \eta(\lambda, i)$.

Given a compact exhaustion E_k ($k \in \mathbb{N}$) of $(0, \infty)$, for each k there exists some $\lambda_k < 0$ such that $|\lambda_k| < \min_{\varepsilon \in E_k} \varepsilon$, $\lambda_k < \lambda_{k+1}$ for all k, and

$$d_{U_i}(\hat{g}^i_{\eta(\lambda,i),\lambda}, g) := \sup_{0 \neq X, Y \in TU_i} \frac{|\hat{g}^i_{\eta(\lambda,i),\lambda}(X, Y) - g(X, Y)|}{\|X\|_h \|Y\|_h} < \frac{\varepsilon}{2^{i+1}}.$$

for all $\varepsilon \in E_k$ and all $\lambda_k \le \lambda < 0$. Again by Lemma 2.3 we obtain a smooth map $(0,\infty) \to (-\infty, 0)$, $\varepsilon \to \lambda_i(\varepsilon)$ such that $|\lambda_i(\varepsilon)| < \varepsilon$ for all ε, and $d_{U_i}(\hat{g}^i_{\eta(\lambda_i(\varepsilon),i),\lambda_i(\varepsilon)}, g) < \frac{\varepsilon}{2^{i+1}}$ for all $\varepsilon > 0$. We now consider the smooth symmetric $(0,2)$-tensor field on M,

$$g_\varepsilon := \sum_i \chi_i \hat{g}^i_{\eta(\lambda_i(\varepsilon),i),\lambda_i(\varepsilon)}.$$

By construction, $(\varepsilon, p) \mapsto g_\varepsilon(p)$ is smooth, and g_ε converges to g locally uniformly as $\varepsilon \to 0$. Therefore, for any $K \Subset M$ there exists some ε_K such that for all $0 < \varepsilon < \varepsilon_K$, g_ε is of the same signature as g, hence a Lorentzian metric on K, with strictly larger lightcones than g. We are thus in a position to apply Lemma 2.4 to obtain a smooth map $(\varepsilon, p) \mapsto \hat{g}_\varepsilon(p)$ such that for each fixed ε, $\hat{g}_\varepsilon$ is a globally defined Lorentzian metric which on any given $K \Subset M$ coincides with g_ε for sufficiently small ε.

Then $d_h(\hat{g}_\varepsilon, g) < \varepsilon/2$, and $\varepsilon \to 0$ implies $\lambda_i(\varepsilon) \to 0$ and a fortiori $\eta(\lambda_i(\varepsilon), i) \to 0$ for each $i \in \mathbb{N}$.

From this, by virtue of (2.2), (i) and (ii) of Remark 2.1 hold for $\hat{g}_\varepsilon$ if $g \in C^{1,1}$.

The approximation $\check{g}_\varepsilon$ is constructed analogously choosing $\lambda > 0$. □

Remark 2.6. (i) From Rem. 2.1 and the above proof it follows that, given a Lorentzian metric of some prescribed regularity (e.g., Sobolev, Hölder, etc.), the inner and outer regularisations $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ converge to g as good as regularisations by convolution do locally.

(ii) If g is a metric of general pseudo-Riemannian signature, then since g_ε in Rem. 2.1 depends smoothly on ε, also in this case an application of Lemma 2.4 allows to produce regularisations $\tilde{g}_\varepsilon$ that are pseudo-Riemannian metrics on all of M of the same signature as g and satisfy (i) and (ii) from that remark.

To conclude this section we derive the Gauss lemma for $C^{1,1}$-metrics. This result has first appeared (in a more general form) in [25]. In the spirit of our approach, we include an independent proof using regularisation methods.

Theorem 2.7. *(The Gauss Lemma) Let g be a $C^{1,1}$-pseudo-Riemannian metric on M, and let $p \in M$. Then p possesses a basis of normal neighbourhoods U with the following properties:* $\exp_p : \tilde{U} \to U$ *is a bi-Lipschitz homeomorphism, where $\tilde{U}$ is an open star-shaped neighbourhood of* 0 *in T_pM. Moreover, for almost all $x \in \tilde{U}$, if v_x, $w_x \in T_x(T_pM)$ and v_x is radial, then*

$$\langle T_x \exp_p(v_x), T_x \exp_p(w_x)\rangle = \langle v_x, w_x\rangle.$$

Proof. Let g_ε be approximating smooth metrics as in Rem. 2.1. Take U, $\tilde{U}$ as in Lemma 2.2 and let $x \in \tilde{U}$ be such that $T_x \exp_p$ exists. By bilinearity, we may assume that $x = v_x = v$ and $w_x = w$. Let $f(t,s) := \exp_p(t(v + sw))$. Then $(t,s) \mapsto f(t,s)$ is $C^{0,1}$ hence $(t,s) \mapsto f_t(t,s) \in L^\infty_{\mathrm{loc}}$ and $(t,s) \mapsto f_s(t,s) \in L^\infty_{\mathrm{loc}}$. For any fixed s, however, $t \mapsto f(t,s)$ is C^2, as is $s \mapsto f(t,s)$ for any t fixed (both curves being geodesics).

Let $f^\varepsilon(t,s) := \exp_p^{g_\varepsilon}(t(v + sw))$. Then by the smooth Gauss lemma, for all ε we have:

$$\langle T_v \exp_p^{g_\varepsilon}(v), T_v \exp_p^{g_\varepsilon}(w)\rangle = \langle f^\varepsilon_s(1,0), f^\varepsilon_t(1,0)\rangle = \langle v, w\rangle.$$

By standard ODE estimates (see [23, Lemma 2.3] and the discussion following it) it follows that $\forall v$: $\exp_p^{g_\varepsilon}(tv) \to \exp_p(tv)$ in $C^1(\mathbb{R})$ for $\varepsilon \to 0$. Thus, we have:

$$f^\varepsilon_t(1,0) = \partial_t|_1 \exp_p^{g_\varepsilon}(tv) \to \partial_t|_1 \exp_p(tv) \ (\varepsilon \to 0)$$
$$f^\varepsilon_s(1,0) = \partial_s|_0 \exp_p^{g_\varepsilon}(v + sw) \to \partial_s|_0 \exp_p(v + sw) \ (\varepsilon \to 0)$$

and therefore, whenever $T_v \exp_p$ exists,

$$\langle T_v \exp_p(v), T_v \exp_p(w)\rangle = \langle v, w\rangle.$$

□

3 Causality theory

As in [15] we will base our approach to causality theory on locally Lipschitz curves. We note that this definition differs from that in [25], where the corresponding curves are required to be C^1 (see, however, Cor. 3.10 below). Any locally Lipschitz curve c is differentiable almost everywhere (Rademacher's theorem) and we call c timelike, causal, spacelike or null, if $c'(t)$ has the corresponding property whenever it exists. If the time-orientation of M is determined by a continuous timelike vector field X then a causal curve c is called future- resp. past-directed if $\langle X(c(t)), c'(t)\rangle < 0$ resp. > 0 wherever $c'(t)$ exists. With these notions we have:

Definition 3.1. Let g be a C^0-Lorentzian metric on M. For $p \in A \subseteq M$ we define the relative chronological, respectively causal future of p in A by (cf. [15, 2.4]):

$$I^+(p, A) := \{q \in A |\text{ there exists a future directed timelike curve in } A \text{ from } p \text{ to } q\ \}$$
$$J^+(p, A) := \{q \in A |\text{ there exists a future directed causal curve in } A \text{ from } p \text{ to } q\ \} \cup A.$$

For $B \subseteq A$ we set $I^+(B, A) := \bigcup_{p\in B} I^+(p, A)$ and analogously for $J^+(B, A)$. We set $I^+(p) := I^+(p, M)$. Replacing 'future directed' by 'past-directed' we obtain the corresponding definitions of the chronological respectively causal pasts I^-, J^-.

Below we will formulate all results for I^+, J^+. By symmetry, the corresponding claims for chronological or causal pasts follow in the same way.

As usual, for p, $q \in M$ we write $p < q$, respectively $p \ll q$, if there is a future directed causal, respectively timelike, curve from p to q. By $p \leq q$ we mean $p = q$ or $p < q$.

We now recall some definitions that were introduced in [19] and results there obtained which will be of use in this paper.

Definition 3.2. A locally Lipschitz curve $\alpha : [0, 1] \to M$ is said to be locally uniformly timelike (l.u.-timelike) with respect to the C^0-metric g if there exists a smooth Lorentzian metric $\check{g} \prec g$ such that $\check{g}(\alpha', \alpha') < 0$ almost everywhere. Then for $p \in A \subseteq M$

$$\check{I}^+_g(p, A) := \{q \in A |\text{ there exists a future directed l.u.-timelike curve in } A \text{ from } p \text{ to } q\}.$$

Thus $\check{I}^+_g(A) = \bigcup_{\check{g}\prec g} I^+_{\check{g}}(A)$, hence it is open ([19, Prop. 1.4]). The following definition ([19, Def. 1.8]) introduces a highly useful substitute for normal coordinates in the context of metrics of low regularity

Definition 3.3. Let (M, g) be a smooth Lorentzian manifold with continuous metric g and let $p \in M$. A relatively compact open subset U of M is called a cylindrical neighbourhood of $p \in U$ if there exists a smooth chart (φ, U), $\varphi = (x^0, ..., x^{n-1})$ with $\varphi(U) = I \times V$, I an interval around 0 in $\mathbb{R}$ and V open in $\mathbb{R}^{n-1}$, such that:

1. $\frac{\partial}{\partial x^0}$ is timelike and $\frac{\partial}{\partial x^i}$, $i = 1, ..., n-1$, are spacelike,
2. For $q \in U$, $v \in T_qM$, if $g_q(v, v) = 0$ then $\frac{|v^0|}{\|\mathbf{v}\|} \in (\frac{1}{2}, 2)$ (where $T_q\varphi(v) = (v^0, \mathbf{v})$, and $\|\ \|$ is the Euclidean norm on $\mathbb{R}^{n-1}$),
3. $(\varphi_* g)_{\varphi(p)} = \eta$ (the Minkowski metric).

By [19, Prop. 1.10], every point in a spacetime with continuous metric possesses a basis of cylindrical neighbourhoods. According to [19, Def. 1.16], a Lorentzian manifold M with C^0-metric g is called causally plain if for every $p \in M$ there exists a cylindrical neighbourhood U of p such that $\partial\check{I}^\pm(p, U) = \partial J^\pm(p, U)$. This condition excludes causally 'pathological' behaviour (bubbling metrics). By [19, Cor. 1.17], we have:

Proposition 3.4. *Let g be a $C^{0,1}$-Lorentzian metric on M. Then (M, g) is causally plain.*

The most important property of causally plain Lorentzian manifolds for our purposes is given in the following result ([19, Prop. 1.21]).

Proposition 3.5. *Let g be a continuous, causally plain Lorentzian metric and let $A \subseteq M$. Then*

$$I^{\pm}(A) = \check{I}^{\pm}(A). \tag{3.1}$$

Furthermore, we will make use of the following 'push-up' results ([19, Lemma 1.22], [19, Prop. 1.23]):

Proposition 3.6. *Let g be a causally plain C^0-Lorentzian metric on M and let p, q, $r \in M$ with $p \le q$ and $q \ll r$ or $p \ll q$ and $q \le r$. Then $p \ll r$.*

Proposition 3.7. *Let M be a spacetime with a continuous causally plain metric g. Consider a causal future-directed curve $\alpha : [0,1] \to M$ from p to q. If there exist s_1, $s_2 \in [0,1]$, $s_1 < s_2$, such that $\alpha|_{[s_1,s_2]}$ is timelike, then in any neighbourhood of $\alpha([0,1])$ there exists a timelike future-directed curve from p to q.*

Returning now to our main object of study, for the remainder of the paper g will denote a $C^{1,1}$-Lorentzian metric. Then in particular, g is causally plain by Prop. 3.4. To analyse the local causality for g in terms of the exponential map we first introduce some terminology. Let $\tilde{U}$ be a star-shaped neighbourhood of $0 \in T_pM$ such that $\exp_p : \tilde{U} \to U$ is a bi-Lipschitz homeomorphism (Th. 1.1). On T_pM we define the position vector field $\tilde{P} : v \mapsto v_v$ and the quadratic form $\tilde{Q} : T_pM \to \mathbb{R}$, $v \mapsto g_p(v,v)$. By P, Q we denote the push-forwards of these maps via $\exp_p$, i.e.,

$$\begin{aligned} P(q) &:= T_{\exp_p^{-1}(q)} \exp_p(\tilde{P}(\exp_p^{-1}(q))) \\ Q(q) &:= \tilde{Q}(\exp_p^{-1}(q)). \end{aligned}$$

As $\exp_p$ is locally Lipschitz, P is an L^∞_{loc}-vector field on U, while Q is locally Lipschitz (see, however, Rem. 3.8 below).

Let X be some smooth vector field on U and denote by $\tilde{X}$ its pullback $\exp_p^* X$ (note that $T_v \exp_p$ is invertible for almost every $v \in \tilde{U}$). Then by Th. 2.7, for almost every $q \in U$ we have, setting $\tilde{q} := \exp_p^{-1}(q)$:

$$\langle \mathrm{grad} Q(q), X(q)\rangle = X(Q)(q) = \tilde{X}(\tilde{Q})(\tilde{q}) = \langle \mathrm{grad}\tilde{Q}, \tilde{X}\rangle|_{\tilde{q}} = 2\langle \tilde{P}, \tilde{X}\rangle|_{\tilde{q}} = 2\langle P, X\rangle|_q.$$

It follows that $\mathrm{grad} Q = 2P$.

Remark 3.8. It is proved in [25] that the regularity of both P and Q is better than would be expected from the above definitions. Indeed, [25, Prop. 2.3] even shows that P, as a function of (p,q) is strongly differentiable on a neighbourhood of the diagonal in $M \times M$, and by [25, Th. 1.18], Q is in fact $C^{1,1}$ as a function of (p,q). We will however not make use of these results in what follows and only remark that slightly weaker regularity properties of P and Q (as functions of q only) can also be obtained directly from standard ODE-theory. In fact, setting $\alpha_v(t) := \exp_p(tv)$ for $v \in T_pM$, it follows that $P(q) = \alpha'_{v_q}(1)$, where $v_q := \exp_p^{-1}(q)$. Since $t \mapsto (\alpha_v(t), \alpha'_v(t))$ is the solution of the first-order system corresponding to the geodesic equation with initial value (p,v), and since the right-hand side of this system is Lipschitz-continuous, [29, Th. 8.4] shows that $v \mapsto \alpha'_v(1)$ is Lipschitz-continuous. Since also $q \mapsto v_q$ is Lipschitz, we conclude that P is Lipschitz-continuous. From this, by the above calculation, it follows that Q is $C^{1,1}$.

As in the smooth case, we may use $\exp_p$ to introduce normal coordinates. To this end, let e_0, $\dots$, e_n be an orthonormal basis of T_pM and for $q \in U$ set $x^i(q)e_i := \exp_p^{-1}(q)$. The coordinates x^i then are of the same regularity as $\exp_p^{-1}$, i.e., locally Lipschitz. The coordinate vector fields $\frac{\partial}{\partial x^i}\big|_q = T_{\exp_p^{-1}(q)} \exp_p(e^i)$ themselves are in L^∞_{loc}. Note, however, that in the $C^{1,1}$-setting we can no longer use the relation $g_p = \eta$ (the Minkowski-metric in the x^i-coordinates), since it is not clear a priori that $\exp_p$ is differentiable at 0 with $T_0 \exp_p = \mathrm{id}_{T_pM}$[1]. Due to the additional loss in regularity it is also usually not advisable to write the metric in terms of the exponential chart (the metric coefficients in these coordinates would only be L^∞_{loc}).

The following is the main result on the local causality in normal neighbourhoods.

[1] See, however, [25] where it is shown that indeed $\exp_p$ is even strongly differentiable at 0 with derivative id_{T_pM}.

Theorem 3.9. *Let g be a $C^{1,1}$-Lorentzian metric, and let $p \in M$. Then p has a basis of normal neighbourhoods U, $\exp_p : \tilde{U} \to U$ a bi-Lipschitz homeomorphism, such that:*

$$I^+(p,U) = \exp_p(I^+(0) \cap \tilde{U})$$
$$J^+(p,U) = \exp_p(J^+(0) \cap \tilde{U})$$
$$\partial I^+(p,U) = \partial J^+(p,U) = \exp_p(\partial I^+(0) \cap \tilde{U})$$

Here, $I^+(0) = \{v \in T_pM \mid \tilde{Q}(v) < 0\}$, and $J^+(0) = \{v \in T_pM \mid \tilde{Q}(v) \leq 0\}$. In particular, $I^+(p,U)$ (respectively $J^+(p,U)$) is open (respectively closed) in U.

Proof. We first note that the third claim follows from the first two and the fact that $\exp_p$ is a homeomorphism on U. For the proof of the first two claims we take a normal neighbourhood U that is contained in a cylindrical neighbourhood of p. In addition, we pick a regularising net $\hat{g}_\varepsilon$ as in Prop. 2.5 and let U, $\tilde{U}$ as in Lemma 2.2 (fixing a suitable $\varepsilon_0 > 0$).
($\supseteq$) Let $v \in \tilde{U}$ and let $\alpha := t \mapsto \exp_p(tv)$, $t \in [0,1]$. Set $\alpha_\varepsilon(t) := \exp_p^{\hat{g}_\varepsilon}(tv)$. Then by continuous dependence on initial data we have that $\alpha_\varepsilon \to \alpha$ in C^1 (cf. [23, Lemma 2.3]). Hence applying the smooth Gauss lemma for each ε it follows that for each $t \in [0,1]$ we have

$$g(\alpha'(t), \alpha'(t)) = \lim_{\varepsilon \to 0} \hat{g}_\varepsilon(\alpha'_\varepsilon(t), \alpha'_\varepsilon(t)) = \lim_{\varepsilon \to 0} (\hat{g}_\varepsilon)_p(v,v) = g_p(v,v).$$

Also, time-orientation is respected by $\exp_p$ since both $I(0) \cap \tilde{U}$ and $I(p,U)$ (by [19, Prop. 1.10]) have two connected components, and the positive x^0-axis in $\tilde{U}$ is mapped to $I^+(p,U)$.

($\subseteq$): We denote the position vector fields and quadratic forms corresponding to $\hat{g}_\varepsilon$ by $\tilde{P}_\varepsilon$, P_ε and $\tilde{Q}_\varepsilon$, Q_ε, respectively.

If $\alpha : [0,1] \to U$ is a future-directed causal curve in U emanating from p then α is timelike with respect to each $\hat{g}_\varepsilon$. Set $\beta := (\exp_p)^{-1} \circ \alpha$ and $\beta_\varepsilon := (\exp_p^{\hat{g}_\varepsilon})^{-1} \circ \alpha$. By [15, Prop. 2.4.5], $\beta_\varepsilon([0,1]) \subseteq I^+_{\hat{g}_\varepsilon(p)}(0)$ for all $\varepsilon < \varepsilon_0$. Then by Lemma 2.2 we have that $\beta_\varepsilon \to \beta$ uniformly, and that $\tilde{Q}_\varepsilon \to \tilde{Q}$ locally uniformly, so $\tilde{Q}(\beta(t)) = \lim \tilde{Q}_\varepsilon(\beta_\varepsilon(t)) \leq 0$ for all $t \in [0,1]$, and therefore $\beta((0,1]) \subseteq J^+(0) \cap \tilde{U}$. Together with the first part of the proof it follows that $\exp_p(J^+(0) \cap \tilde{U}) = J^+(p,U)$. Now assume that α is timelike. Then by Prop. 3.5, $\alpha((0,1]) \subseteq \check{I}^+(p,U)$. This means that there exists a smooth metric $\check{g} \prec g$ such that α is $\check{g}$-timelike. Let $f_{\check{g}}, f_g$ denote the graphing functions of $\partial I^+_{\check{g}}(p,U)$ and $\partial J^+(p,U)$, respectively (in a cylindrical chart, see [19, Prop. 1.10]). Then by [19, Prop. 1.10], since α lies in $I^+_{\check{g}}(p,U)$, it has to lie strictly above $f_{\check{g}}$, hence also strictly above f_g, and so $\alpha((0,1]) \cap \partial J^+(p,U) = \emptyset$. But then, since $\exp_p$ is a homeomorphism on U, we have that

$$\beta((0,1]) \cap (\partial J^+(0) \cap \tilde{U}) = \beta((0,1]) \cap \exp_p^{-1}(\partial J^+(p,U)) = \exp_p^{-1}(\alpha((0,1]) \cap \partial J^+(p,U)) = \emptyset$$

Hence β lies entirely in $I^+(0) \cap \tilde{U}$, as claimed. □

Corollary 3.10. *Let $U \subseteq M$ be open, $p \in U$. Then the sets $I^+(p,U)$, $J^+(p,U)$ remain unchanged if, in Def. 3.1, Lipschitz curves are replaced by piecewise C^1 curves, or in fact by broken geodesics.*

Proof. Let $\alpha : [0,1] \to U$ be a, say, future directed timelike Lipschitz curve in U. By Th. 1.2 and Th. 3.9 we may cover $\alpha([0,1])$ by finitely many totally normal open sets $U_i \subseteq U$, such that there exist $0 = t_0 < \cdots < t_N = 1$ with $\alpha([t_i, t_{i+1}]) \subseteq U_{i+1}$ and $I^+(\alpha(t_i), U_i) = \exp_{\alpha(t_i)}(I^+(0) \cap \tilde{U}_i)$ for $0 \leq i < N$. Then the concatenation of the radial geodesics in U_i connecting $\alpha(t_i)$ with $\alpha(t_{i+1})$ gives a timelike broken geodesic from $\alpha(0)$ to $\alpha(1)$ in U. □

The following analogue of [15, Cor. 2.4.10] provides more information about causal curves intersecting the boundary of $J^+(p,U)$:

Corollary 3.11. *Under the assumptions of Th. 3.9, suppose that $\alpha : [0,1] \to U$ is causal and $\alpha(1) \in \partial J^+(p,U)$. Then α lies entirely in $\partial J^+(p,U)$ and there exists a reparametrisation of α as a null-geodesic segment.*

Proof. Suppose to the contrary that there exists $t_0 \in (0,1)$ such that $\alpha(t_0) \in I^+(p,U)$. Then there exists a future directed timelike curve γ from p to $\alpha(t_0)$. Applying Prop. 3.7 to the concatenation $\gamma \cup \alpha|_{[t_0,1]}$ it follows that there exists a future directed timelike curve from p to $\alpha(1)$. But then $\alpha(1) \in I^+(p,U)$, a contradiction. Thus $\alpha(t) \in \partial J^+(p,U)$, $\forall t \in [0,1]$, implying that $\beta(t) = \exp_p^{-1} \circ\alpha(t) \in \partial J^+(0)$, $\forall t \in [0,1]$, so $\tilde{Q}(\beta(t)) = 0$, $\forall t \in [0,1]$ and for almost all t we have

$$0 = \frac{d}{dt}\tilde{Q}(\beta(t)) = g_p(\mathrm{grad}\tilde{Q}(\beta(t)), \beta'(t)) = g_p(2\tilde{P}(\beta(t)), \beta'(t)).$$

Hence $\beta(t)$ is collinear with $\beta'(t)$ almost everywhere, and it is easily seen that this implies the existence of some $v \neq 0, v \in \partial J^+(0)$, and of some $h : \mathbb{R} \to \mathbb{R}$ such that $\beta(t) = h(t)v$. The function h is locally Lipschitz since β is, and injective since α is (on every cylindrical neighbourhood there is a natural time function). Thus h is strictly monotonous, and in fact strictly increasing since otherwise β would enter $J^-(0)$. Thus $\beta'(t) = f(t)\beta(t)$ where $f(t) := \frac{h'(t)}{h(t)} \in L^\infty_{\mathrm{loc}}$. From here we may argue exactly as in [15, Cor. 2.4.10]: the function $r(s) := \int_0^s f(\tau)\,d\tau$ is locally Lipschitz and strictly increasing, hence a bijection from $[0,1]$ to some interval $[0,r_0]$. Thus so is its inverse $r \to s(r)$, and we obtain $\beta(s(r))' = \beta'(s(r))/f(s(r)) = \beta(s(r))$ a.e., where the right hand side is even continuous. It follows that in this parametrisation, β is C^1 and in fact is a straight line in the null cone, hence α can be parametrised as a null-geodesic segment, as claimed. □

Corollary 3.12. *The relation $\ll$ is open: if $p \ll q$ then there exist neighbourhoods V of p and W of q such that $p' \ll q'$ for all $p' \in V$ and $q' \in W$. In particular, for any $p \in M$, $I^+(p)$ is open in M.*

Proof. Let α be a future-directed timelike curve from p to q and pick totally normal neighbourhoods N_p, N_q of p, q as in Th. 3.9. Now let $p' \in N_p$ and $q' \in N_q$ be points on α. Then $V := I^-(p', N_p)$ and $W := I^+(q', N_q)$ have the required property. □

From this we immediately conclude:

Corollary 3.13. *Let $A \subseteq U \subseteq M$, where U is open. Then*

$$I^+(A,U) = I^+(I^+(A,U)) = I^+(J^+(A,U)) = J^+(I^+(A,U)) \subseteq J^+(J^+(A,U)) = J^+(A,U)$$

A consequence of Prop. 3.7 is that the causal future of any $A \subseteq M$ consists (at most) of A, $I^+(A)$ and of null-geodesics emanating from A:

Corollary 3.14. *Let $A \subseteq M$ and let α be a causal curve from some $p \in A$ to some $q \in J^+(A) \setminus I^+(A)$. Then α is a null-geodesic that does not meet $I^+(A)$.*

Proof. By Prop. 3.7, α has to be a null curve. Moreover, if $\alpha(t) \in I^+(A)$ for some t then for some $a \in A$ we would have $a \ll \alpha(t) \leq q$, so $q \in I^+(A)$ by Prop. 3.6, a contradiction. Covering α by totally normal neighbourhoods as in Cor. 3.10 and applying Cor. 3.11 gives the claim. □

Following [8, Lemma 14.2] we next give a more refined description of causality for totally normal sets. For this, recall from the proof of [23, Th. 4.1] that the map $E : v \mapsto (\pi(v), \exp_{\pi(v)}(v))$ is a homeomorphism from some open neighbourhood S of the zero section in TM onto an open neighbourhood W of the diagonal in $M \times M$. If U is totally normal as in Th. 3.9 and such that $U \times U \subseteq W$ then the map $U \times U \to TM$, $(p,q) \mapsto \overrightarrow{pq} := \exp_p^{-1}(q) = E^{-1}(p,q)$ is continuous.

Proposition 3.15. *Let $U \subseteq M$ be totally normal as in Th. 3.9.*

(i) Let p, $q \in U$. Then $q \in I^+(p,U)$ (resp. $\in J^+(p,U)$) if and only if $\overrightarrow{pq}$ is future-directed timelike (resp. causal).

(ii) $J^+(p,U)$ *is the closure of* $I^+(p,U)$ *relative to* U*.*
(iii) *The relation* $\leq$ *is closed in* $U \times U$*.*
(iv) *If* K *is a compact subset of* U *and* $\alpha : [0,b) \to K$ *is causal, then* α *can be continuously extended to* $[0,b]$*.*

Proof. (i) and (ii) are immediate from Th. 3.9.

(iii) Let $p_n \leq q_n$, $p_n \to p$, $q_n \to q$. By (i), $\overrightarrow{p_n q_n}$ is future-directed causal for all n. By continuity ([23, Th. 4.1]), therefore, $\langle \overrightarrow{pq}, \overrightarrow{pq} \rangle \leq 0$, so $\overrightarrow{pq}$ is future-directed causal as well.

(iv) Let $0 < t_1 < t_2 < \cdots \to b$. Since K is compact, $\alpha(t_i)$ has an accumulation point p and it remains to show that p is the only accumulation point. Suppose that $q \neq p$ is also an accumulation point. Choose a subsequence t_{i_k} such that $\alpha(t_{i_{2k}}) \to p$ and $\alpha(t_{i_{2k+1}}) \to q$. Then since $\alpha(t_{i_{2k}}) \leq \alpha(t_{i_{2k+1}}) \leq \alpha(t_{i_{2k+2}})$, (iii) implies that $p \leq q \leq p$. By (i), then, $\overrightarrow{pq}$ would be both future- and past-directed, which is impossible. □

From this, with the same proof as in [8, Lemma 14.6] we obtain:

Corollary 3.16. *Let* $A \subseteq M$*. Then*

(i) $J^+(A)^\circ = I^+(A)$*.*
(ii) $J^+(A) \subseteq \overline{I^+(A)}$*.*
(iii) $J^+(A) = \overline{I^+(A)}$ *if and only if* $J^+(A)$ *is closed.*

Finally, as in the smooth case, one may introduce a notion of causality also for general continuous curves (cf. [12, p. 184], [10, Def. 8.2.1]):

Definition 3.17. A continuous curve $\alpha : I \to M$ is called future-directed causal (resp. timelike) if for every $t \in I$ there exists a totally normal neighbourhood U of $\alpha(t)$ such that for any $s \in I$ with $\alpha(s) \in U$ and $s > t$, $\alpha(s) \in J^+(\alpha(t)) \setminus \{\alpha(t)\}$ (resp. $\alpha(s) \in I^+(\alpha(t)) \setminus \{\alpha(t)\}$), and analogously for $s < t$ with J^- resp. I^-.

Then the proof of [10, Lemma 8.2.1]) carries over to the $C^{1,1}$-setting, showing that any continuous causal (resp. timelike) curve is locally Lipschitz.

Remark 3.18. While a continuous causal curve α need not be a causal Lipschitz curve in the sense of our definition (cf. [25, Rem. 1.28]), it still follows that $\langle \alpha'(t), \alpha'(t) \rangle \leq 0$ wherever $\alpha'(t)$ exists (however, $\alpha'(t)$ might be 0).

To see this, consider first the case where g is smooth. Set $p := \alpha(t)$, pick a normal neighbourhood U around p and set $\beta := \exp_p^{-1} \circ \alpha$. Then $\beta'(t) = \alpha'(t)$ and by Def. 3.17 and Th. 3.9, $\beta(s) \in J^+(0)$ for $s > t$ small. Therefore, $\beta'(t) \in J^+(0)$, so $\langle \alpha'(t), \alpha'(t) \rangle \leq 0$. In the general case, where g is only supposed to be $C^{1,1}$, pick a regularisation $\hat{g}_\varepsilon$ as in Prop. 2.5. Then $\hat{g}_\varepsilon(\alpha'(t), \alpha'(t)) \leq 0$ for all ε by the above and letting $\varepsilon \to 0$ gives the claim.

4 Further aspects of causality theory

In the previous section we have shown that the fundamental constructions of causality theory remain valid for $C^{1,1}$-metrics. It was demonstrated by P. Chruściel in [15] that to obtain a consistent causality theory for C^2-metrics one needs two main ingredients: on the one hand, a push-up Lemma, as given by Prop. 3.6, 3.7. The second pillar in the development of the theory is the fact that accumulation curves of causal curves are causal again. Here, if $\alpha_n : I \to M$ is a sequence of paths (parametrised curves) then a path $\alpha : I \to M$ is called an accumulation curve of the sequence (α_n) if there exists a subsequence (α_{n_k}) that converges to α uniformly on compact subsets of I. It was shown in [19, Th. 1.6] that limit stability of causal curves holds in fact even for continuous metrics:

Theorem 4.1. *Let g be a C^0-Lorentzian metric on M, and let $\alpha_n : I \to M$ be a sequence of causal curves that accumulate at some $p \in M$ ($\alpha_n(0) \to p$). Then there exists a causal curve α that is an accumulation curve of α_n.*

With these key tools at hand, and the results obtained so far, causality theory for $C^{1,1}$-metrics can be further developed by following the proofs given in [15] for C^2-metrics. In the remainder of this section we list some main results that can be derived in this way.

Extendability of geodesics is characterised as follows (cf. [15, Prop. 2.5.6]):

Proposition 4.2. *Let (M, g) be a spacetime with a $C^{1,1}$-Lorentzian metric g. A geodesic $\alpha : I \to M$ is maximally extended as a geodesic if and only if it is inextendible as a causal curve.*

Furthermore, it is already shown in [15, Th. 2.5.7]) that even if the metric is merely supposed to be continuous, every future directed causal (resp. timelike) curve possesses an inextendible causal (resp. timelike) extension of α. As a direct consequence of Cor. 3.11 we obtain (cf. [15, Prop. 2.6.9]):

Proposition 4.3. *Let g be a $C^{1,1}$-Lorentzian metric on M. If α is an achronal causal curve, then α is a null geodesic.*

For sequences of curves, [15, Prop. 2.6.8, Th. 2.6.10]) give:

Proposition 4.4. *Let g be a $C^{1,1}$-Lorentzian metric on M. If $\alpha_n : I \to M$ is a sequence of maximally extended geodesics accumulating at α, then α is a maximally extended geodesic.*

Theorem 4.5. *Let (M, g) be a spacetime with a $C^{1,1}$-Lorentzian metric g and let $\alpha_n : I \to M$ be a sequence of achronal causal curves accumulating at α. Then α is achronal.*

Causality conditions and notions such as domains of dependence and Cauchy horizons can be defined independently of the regularity of the metric. As an example of the interrelation of causality conditions for metrics of low regularity, we mention [15, Prop. 2.7.4], which shows that if a spacetime with continuous metric is stably causal then it is strongly causal. Turning now to globally hyperbolic spacetimes, [15, Prop. 2.8.1, Cor. 2.8.4, Th. 2.8.5] give:

Proposition 4.6. *Let (M, g) be a globally hyperbolic spacetime with g a $C^{1,1}$-Lorentzian metric and let α_n be a sequence of causal curves accumulating at both p and q. Then there exists a causal curve α which is an accumulation curve of the α_n's and passes through p and q.*

Corollary 4.7. *If M is a spacetime with a $C^{1,1}$-Lorentzian metric g that is globally hyperbolic, then*

$$\overline{I^{\pm}(p)} = J^{\pm}(p).$$

Theorem 4.8. *For a globally hyperbolic spacetime M with a $C^{1,1}$-metric g, if $q \in I^+(p)$, resp. $q \in J^+(p)$, there exists a timelike, resp. causal, future directed geodesic from p to q.*

Moreover, the proof of [15, Th. 2.9.9] can be adapted to show:

Theorem 4.9. *Let M be a spacetime with a $C^{1,1}$-Lorentzian metric g and let S be an achronal hypersurface in (M, g). Suppose that the interior $D_I^\circ(S)$ of the domain of dependence $D_I(S)$ is nonempty. Then $D_I^\circ(S)$ equipped with the metric obtained by restricting g is globally hyperbolic.*

Note that here the definition of domains of dependence is based on timelike curves, as is the definition of Cauchy horizons. Finally, the analogue of [15, Prop. 2.10.6] establishes the existence of generators of Cauchy horizons :

Proposition 4.10. *Let g be a $C^{1,1}$-Lorentzian metric on M and let S be a spacelike C^1-hypersurface in M. For any point p in the Cauchy horizon $H_I^+(S)$ there exists a past directed null geodesic $\alpha_p \subset H_I^+(S)$ starting at p which either does not have an endpoint in M, or has an endpoint in $\bar{S} \setminus S$.*

Based on these foundations, a deeper study of causality theory, in particular in the direction of singularity theorems for metrics of low regularity can be undertaken. As detailed in [13, Sec. 6.1], this will require to solve a whole range of analytical problems that go beyond the results of this paper, in particular concerning variational properties of curves, control of curvature quantities, and a study of focal points. As already mentioned in the introduction, we hope that the techniques developed in [24, 19, 23, 25] as well as in this paper can contribute to this task.

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