A Geometric Approach to Fault Identification in Linear Repetitive Processes

Sepehr Maleki[1], Zijian Shang[2] and Paolo Rapisarda[3]

***Abstract*— We investigate the fault detection and isolation (FDI) problem for discrete-time linear repetitive processes (DLRP) using a geometric approach. We propose a 2-D model for these systems that incorporates the failure description. Based on this model, we formulate the FDI problem in geometric language and state sufficient conditions for solvability of the problem. We also develop a FDI procedure based on an asymptotic observer of the state.**

I. INTRODUCTION

Repetitive processes represent an extensive class of important industrial applications such as long-wall coal cutting, metal rolling, printing, and modelling of fluid dynamics in distribution pipelines such as gas networks [3]. In such applications, because of the repetitive nature of the process, a failure unless promptly detected and fixed affects not only the current process but also the following ones. Thus fault detection and isolation (FDI) is an important problem that needs to be addressed.

Although the FDI problem for linear systems has been studied intensively due to its significance, and a variety of different methods have been developed to address it, there has been no attempt to specifically investigate the problem in repetitive processes. A comprehensive survey of various FDI techniques can be found in [9] and [10]. One of these techniques is the geometric approach developed by Massoumnia [5] for 1-D systems based on the Beard-Jones detection filter problem (BJDFP) [12], [13]. This approach later was extended to address the problem in 3-D systems in [1]. In this paper, we investigate this problem specifically for repetitive processes. We build an asymptotic observer that by observing the output and the input of the system, asymptotically reconstructs the state. Once a failure occurs, the reconstructed state starts to deviate from the actual state space. Then by using a geometric approach, we propose a fault detection and isolation technique that under suitable assumptions, can detect and uniquely isolate a failure in the system.

The paper is organized as follows: Firstly, the notation used in this paper is presented. In Section II, we briefly review linear repetitive processes. In Section III we recall some preliminary geometric concepts that are used to address the FDI problem. Presence of a failure in the model presented in Section II, is discussed and formulated in IV. Finally, the FDI problem for linear repetitive processes is addressed in Section V and a practical example of various failure types is given in VI.

Notation

Calligraphic letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ are used to denote real vector spaces and subspaces. Capital italic letters $A, B, C, \ldots$ are used to denote matrices and linear maps. $\operatorname{im} A$ denotes the image, or the column space of A. $\ker C$ denotes the kernel, or the null space of C. Restriction of the map A to a linear subspace $\mathcal{W}$ of a vector space $\mathcal{X}$ is denoted by $A\mid_{\mathcal{W}}$, and $A\mid_{\mathcal{X}/\mathcal{W}}$ denotes the map induced by A on the quotient space $\mathcal{X}/\mathcal{W}$. The identitiy matrix is denoted by I_n. Direct sum of subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ is denoted by $\mathcal{V}_1 \oplus \mathcal{V}_2$. We denote by $A^\dagger$ the Moore-Penrose pseudo-inverse of A.

II. LINEAR REPETITIVE PROCESSES

Repetitive processes, also termed as multi-pass processes, are characterised by a series of sweeps (passes), through a set of dynamics defined over a fixed finite duration (pass length) [2]. At each pass, an output is produced which contributes also to the dynamics of next passes. If it is the previous pass only which contributes to the current pass, the process is called *unit memory*, whereas if the previous M pass profiles contribute to the current one, M is the *memory length*.

A state-space model of a discrete linear repetitive process with pass length α has the following form

$$\begin{aligned} x_{k+1}(p+1) &= A_1 x_{k+1}(p) + B u_{k+1}(p) + A_2 x_k(p) \ , \\ y_k(p) &= C x_k(p) \ , \end{aligned} \tag{1}$$

where, $k \geq 0$ is the index of passes, α is the pass length and $0 \leq p \leq \alpha$ is the index along each pass. $x_k(p) \in \mathbb{R}^n$, $u_k(p) \in \mathbb{R}^l$, and $y_k(p) \in \mathbb{R}^m$ are state vector, input vector, and output on the pass k at the time instant p respectively. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, and $C \in \mathbb{R}^{m\times n}$ are state, input and output matrices respectively. Moreover, $A_2 := B_0 C$. In model (1), the current state and pass profile only depend on the previous pass profile and therefore the model is termed as unit-memory.

In a more general case, a state-space model of a discrete linear repetitive process with pass length α and memory length M is:

$$\begin{aligned} x_{k+1}(p+1) &= A x_{k+1}(p) + B u_{k+1}(p) + \sum_{j=0}^{M-1} B_j y_{k-j}(p) \\ y_k(p) &= C x_k(p) \ , \end{aligned} \tag{2}$$

[1]Sepehr Maleki, [2]Zijian Shang, and [3]Paolo Rapisarda are with the Communications, Signal Processing and Control Research Group, University of Southampton, SO17 1Bj, Southampton, Great Britain. email: sm13g10@ecs.soton.ac.uk, zijianshang@gmail.com, pr3@ecs.soton.ac.uk

where M is the memory length and $B_j \in \mathbb{R}^{n\times m}$ are the Memory matrices, $j = 0, 1, \ldots, M-1$.
For simplicity of discussion, in this paper we consider model (1) to be our nominal system model and with small adaptations, the framework presented here can be used for the general model (2).

Following [2], the boundary conditions for this model are:

$$\begin{aligned} x_k(0) &= d_k, k \geq 0 \ , \\ y_{1-j}(p) &= \hat{y}_{1-j}(p), 1 \leq j \leq M, 0 \leq p \leq \alpha \ , \end{aligned} \qquad (3)$$

where d_k is a $n\times 1$ known constant vector, $\hat{y}_{1-j}(p)$ is a $m \times 1$ vector whose entries are known functions of p and $0 \leq p \leq \alpha$.

III. Geometric Background

To formulate the FDI problem in geometric language, we need some preliminary concepts.
Conditioned invariant subspaces for 1-D systems were first introduced in [7]. These subspaces play an important role in solving state estimation problems [8]. For any externally stabilisable conditioned invariant subspace $\mathcal{V}$ there exists an observer that asymptotically reconstructs the state modulo $\mathcal{V}$ [6, Chapter 4].

For the pair $(A_i, C), i = 1, 2$ of the linear repetitive process (1), a conditioned-invariant subspace $\mathcal{V} \subseteq \mathcal{X}$ is defined as follows:

***Definition** 1:* A subspace $\mathcal{V} \subseteq \mathcal{X}$ is an (A_i, C)-invariant subspace if

$$A_i(\mathcal{V} \cap \mathcal{C}) \subseteq \mathcal{V} \ , \quad \text{with } \mathcal{C} := \ker \mathrm{C} \ . \qquad (4)$$

Denote the family of conditioned invariant subspaces containing a given subspace $\mathcal{L}$ by $\underline{\mathcal{W}}(\mathcal{L})$. Clearly, the family $\underline{\mathcal{W}}(\mathcal{L})$ is closed under intersection. Consider $\mathcal{W}_1 \in \underline{\mathcal{W}}(\mathcal{L})$ and $\mathcal{W}_2 \in \underline{\mathcal{W}}(\mathcal{L})$, then $\mathcal{W}_1 \cap \mathcal{W}_2 \in \underline{\mathcal{V}}(\mathcal{L})$. Therefore, there exist a smallest subspace in the family $\underline{\mathcal{W}}(\mathcal{L})$, called the infimal element and denoted by $\mathcal{W}^\star(\mathcal{L})$.

A recursive algorithm to find the subspace $\mathcal{W}^\star(\mathcal{L})$ is given below [5]:

$$\mathcal{V}_i := \begin{cases} \{0\}_n & i = 0 \\ \begin{bmatrix} A_H & B & 0_{n\times l} \end{bmatrix} \mathcal{S} & i > 0 \end{cases} \qquad (5)$$

where $\mathcal{S} := \left(\mathcal{V}_{i-1} \oplus \mathcal{V}_{i-1} \oplus \mathbb{R}^{2l} \cap \ker \begin{bmatrix} C_D & 0_{2(m\times l)} \end{bmatrix}\right)$.

Similar to the 1-D case [7], the following theorem establishes a fundamental result for the decomposition of the system matrices with respect to an invariant subspace.

***Theorem** 1:* The following statements are equivalent:

I. $\mathcal{V} \subseteq \mathbb{R}^n$ is an (A_i, C)-invariant subspace of dimension m, $i = 1, 2$.
II. There exists a similarity transformation (change of basis) $T \in \mathbb{R}^{n\times n}$, such that

$$\hat{A}_i = T^{-1}(A_i + G_iC)T = \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ 0_{(n-m)\times m} & \hat{A}_i^{22} \end{bmatrix} \ ,$$

where, $G_i \in \mathbb{R}^{n\times m}$ is the output-injection matrix.

From Theorem 1, using a similarity transformation $T \in \mathbb{R}^{n\times n}$ for a conditioned invariant subspace $\mathcal{V} \subseteq \mathcal{X}$ of the repetitive process described by (1), it immediately follows:

$$\begin{bmatrix} x'_{k+1}(p+1) \\ x''_{k+1}(p+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_1^{11} & \hat{A}_1^{12} \\ 0 & \hat{A}_1^{22} \end{bmatrix} \begin{bmatrix} x'_{k+1}(p+1) \\ x''_{k+1}(p+1) \end{bmatrix} + \begin{bmatrix} \hat{A}_2^{11} & \hat{A}_2^{12} \\ 0 & \hat{A}_2^{22} \end{bmatrix} \begin{bmatrix} x'_k(p) \\ x''_k(p) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{k+1}(p) \ . \qquad (6)$$

We use representation (6) in order to introduce and study the concept of *internal* and *external stability*.

For $L \in \mathbb{Z}$ we define

$$\mathcal{S}_L := \{(k, p) \in \mathbb{Z}^2 \mid k + p = L\} \ .$$

***Definition** 2:* A conditioned invariant subspace $\mathcal{V}$ is *internally (asymptotically) stable* if

$$[\{x_k(p) \mid (k,p) \in \mathcal{S}_0\} \subset \mathcal{V}] \text{ and } \left[u_k(p) := 0, (k,p) \in \mathbb{Z}^2\right] \implies \lim_{k,p\to\infty} \|x'_k(p)\| = 0 \ .$$

It follows from standard results in nD systems theory (see for example [4, Prop. 3]) that $\mathcal{V}$ is internally stable if and only if the matrices $\hat{A}_i^{11}$, $i = 1, 2$ satisfy

$$\begin{aligned} &\det(I_n - \hat{A}_1^{11}\lambda - \hat{A}_2^{11}\mu) \neq 0 \\ &\text{for all } (\lambda, \mu) \in \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_i| \leq 1, i = 1, 2\}. \end{aligned} \qquad (7)$$

A conditioned invariant subspace $\mathcal{V}$ is *internally stabilisable* if there exist $G_i \in \mathbb{R}^{n\times m}, i = 1, 2$ such that $(A_i + G_iC)\,\mathcal{V} \subseteq \mathcal{V}$ with $(A_i + G_iC)\,|_{\mathcal{V}}$ stable.

***Definition** 3:* A conditioned invariant subspace $\mathcal{V}$ is *externally stable* if

$$[\{x_k(p) \mid (k,p) \in \mathcal{S}_0\} \nsubseteq \mathcal{V}] \text{ and } \left[u_k(p) := 0, (k,p) \in \mathbb{Z}^2\right] \implies \lim_{k,p\to\infty} x''_k(p) \in \mathcal{V} \ .$$

It is straightforward to check that $\mathcal{V}$ is externally stable if and only if the pair $(\hat{A}_1^{22}, \hat{A}_2^{22})$ is asymptotically stable in the sense of (7).
A conditioned invariant subspace $\mathcal{V}$ is said to be *externally stabilisable* if there exist $G_i \in \mathbb{R}^{n\times m}, i = 1, 2$ such that $(A_i + G_iC)\,\mathcal{V} \subseteq \mathcal{V}$ with $(A_i + G_iC)\,|_{\mathcal{X}/\mathcal{V}}$ stable.

As for the 1-D case, it can be proved that the family of externally stabilisable conditioned invariant subspaces is closed under intersection (see [6, p. 214]).

The condition (7) is rather difficult to check, and is not easy to use in the synthesis of stabilising controllers. These issues have led to the use of LMIs, see for example [17], [16], for this purpose.

The following result is a restatement of the main result in [16].

***Proposition** 1:* If there exist $P_i = P_i^\top \in \mathbb{R}^n$, $P_i > 0$, $i = 1, 2,$, such that the following LMI holds:

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_1^\top \\ A_2^\top \end{bmatrix} (P_1 + P_2) \begin{bmatrix} A_1 & A_2 \end{bmatrix} > 0 \,, \tag{8}$$

then the system described by (6) is asymptotically stable.

Our framework for fault isolation depends on the concept of *input-containing* conditioned invariant subspaces [6], [15].

***Definition** 4:* $\mathcal{V} \subset \mathbb{R}^n$ is an *input-containing conditioned invariant* subspace for (2), if

$$\begin{bmatrix} A_H & B \end{bmatrix} \left((\mathcal{V} \oplus \mathcal{V} \oplus \mathbb{R}^l) \cap \ker \begin{bmatrix} C_D & 0_{2(m \times l)} \end{bmatrix} \right) \subseteq \mathcal{V} \,, \tag{9}$$

where

$$A_H := \begin{bmatrix} A_1 & A_2 \end{bmatrix} \,, \quad C_D := \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \,.$$

The following characterisations of input-containing subspaces hold.

***Proposition** 2:* Let $\mathcal{V}$ be an r-dimensional subspace of $\mathbb{R}^n$, and let $Q \in \mathbb{R}^{(n-r)\times n}$ be a full row-rank matrix such that $\ker Q = \mathcal{V}$. Then the following statements are equivalent:

I. $\mathcal{V}$ is an input-containing conditioned invariant for (2) ;

II. There exist matrices $\Gamma := \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}$ and $\Lambda := \begin{bmatrix} \Lambda_1 & \Lambda_2 \end{bmatrix}$ with $\Gamma_i \in \mathbb{R}^{(n-r)\times(n-r)}$ and $\Lambda_i \in \mathbb{R}^{(n-r)\times p}, i = 1, 2$, such that

$$\begin{aligned} Q \begin{bmatrix} A_H & B \end{bmatrix} = & \Gamma \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \end{bmatrix} \\ & + \Lambda \begin{bmatrix} C_D & 0_{2(m \times l)} \end{bmatrix} \,, \end{aligned} \tag{10}$$

where $Q_D := \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}$;

III. There exist a matrix $G := \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ with $G_i \in \mathbb{R}^{n\times m}, i = 1, 2$, such that

$$\begin{bmatrix} A_H + GC_D & B \end{bmatrix} \left(\mathcal{V} \oplus \mathcal{V} \oplus \mathbb{R}^l \right) \subseteq \mathcal{V} \,. \tag{11}$$

Proof: similar to [1, Prop. 4]. ■

Following Proposition 2, for an input-containing conditioned invariant subspace $\mathcal{V}$, existence of an output-injection matrix G is guaranteed. Now our problem reduces to construction of a matrix G, if it exists, such that $\mathcal{V}$ is an internally and externally stable $(A_H + GC_D)$-invariant.

A. Construction of a stabilising output-injection G

We aim to construct, if exists, an output-injection $G := \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ such that $\ker Q := \mathcal{V}$ is an internally and externally stable, input-containing $(A_i + G_iC, C)$-invariant subspace. From (10) it follows:

$$Q \begin{bmatrix} A_H & B \end{bmatrix} = \begin{bmatrix} \Gamma & \Lambda \end{bmatrix} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix} \,. \tag{12}$$

The solution for (12) is given by:

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = Q \begin{bmatrix} A_H & B \end{bmatrix} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix}^\dagger + KH \,, \tag{13}$$

where

$$\ker H = \operatorname{im} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix} \,,$$

H has linearly independent rows, and K is an arbitrary matrix of suitable size which represents a first degree of freedom in construction of G that can be exploited for external stabilisation of $\mathcal{V}$.

Using (10), we compute the solutions of $\Lambda = -QG$ as $G = G_\Lambda + \Omega U$, where, $G_\Lambda := -Q^\top (QQ^\top)^{-1}\Lambda$, matrix Ω is a basis for $\ker Q$ and U is an arbitrary matrix of suitable size which represents a second degree of freedom in construction of G that can be exploited for internal stabilisation of $\mathcal{V}$.

Following from Theorem 1, for $i = 1, 2$, we have

$$T \begin{bmatrix} A_i + G_iC \end{bmatrix} T^{-1} = \begin{bmatrix} \Delta_i^{11}(K,U) & \Delta_i^{12}(K,U) \\ 0 & \Delta_i^{22}(K,U) \end{bmatrix} \,, \tag{14}$$

where $T := \begin{bmatrix} T_c \\ Q \end{bmatrix}$, and the rows of T_c are linearly independent from those of Q. It follows from [15, Lemma 3.2] that the choice of K affects $\Delta_i^{22}(K,U)$ but not $\Delta_i^{11}(K,U)$ and the choice of U affects $\Delta_i^{11}(K,U)$ but not t $\Delta_i^{22}(K,U)$.

***Proposition** 3:* Let $\Gamma_i,\ \Lambda_i, i = 1, 2$, satisfy (10), Then $\Gamma_i = \Delta_i^{22}(K,U)$, the $(2,2)$-block of (14).

Proof: From (10), it follows

$$Q \begin{bmatrix} A_i & B \end{bmatrix} - \Lambda_i \begin{bmatrix} C & 0_{m\times l} \end{bmatrix} = \Gamma_i \begin{bmatrix} Q & 0_{(n-r)\times l} \end{bmatrix} \,,$$

from which we can write:

$$\begin{aligned} & Q(\begin{bmatrix} A_i & B \end{bmatrix} \underbrace{-(Q^\dagger + VK)\Lambda}_{:=G_i} \begin{bmatrix} C & 0_{m\times l} \end{bmatrix}) = \\ & \Gamma_i \begin{bmatrix} Q & 0_{(n-r)\times l} \end{bmatrix} \,, \end{aligned}$$

where, $\operatorname{im} V := \mathcal{V}$, and K is an arbitrary matrix of suitable size.

Now consider (14), and partition T as $T := \begin{bmatrix} T_c \\ Q \end{bmatrix}$. The second block of (14) yields

$$Q \begin{bmatrix} A_i + G_iC \end{bmatrix} T^{-1} = \begin{bmatrix} 0 & \Delta_i^{22}(K,U) \end{bmatrix} = \Gamma\ Q\ T^{-1} \,.$$

Conclude that $\Gamma_i Q = \begin{bmatrix} 0 & \Delta_i^{22}(K,U) \end{bmatrix} T = \Delta_i^{22}(K,U)Q$. Since Q has full row-rank, this implies $\Gamma_i = \Delta_i^{22}(K,U)$. ■

***Proposition** 4:* Let $\Gamma_i,\ \Lambda_i, i = 1, 2$, satisfy (10), Then $T_c(A_i + \Omega UC)T_c^\top = \Delta_i^{11}(K,U)$.

Proof: Similar to that of Proposition 3. ■

To build a stabilising output-injection matrix G, write (13) as:

$$\begin{bmatrix}\Gamma & \Lambda\end{bmatrix} = \begin{bmatrix}V_1 & V_2 & \bar{V}\end{bmatrix} + K\begin{bmatrix}H_1 & H_2 & \bar{H}\end{bmatrix} , \qquad (15)$$

where,

$$\begin{bmatrix}V_1 & V_2 & \bar{V}\end{bmatrix} := Q\begin{bmatrix}A_H & B\end{bmatrix}\begin{bmatrix}Q_D & 0_{2(n-r)\times 2l}\\ C_D & 0_{2n\times 2l}\end{bmatrix}^{\dagger} ,$$

and

$$\ker\ \begin{bmatrix}H_1 & H_2 & \bar{H}\end{bmatrix} = \operatorname{im}\ \begin{bmatrix}Q_D & 0_{2(n-r)\times 2l}\\ C_D & 0_{2n\times 2l}\end{bmatrix} ,$$

are partitioned with respect to $\begin{bmatrix}\Gamma & \Lambda\end{bmatrix}$. Thus, $\Gamma_i = V_i + KH_i, i = 1, 2$, and $\Lambda = \bar{V} + K\bar{H}$. If $\begin{bmatrix}Q_D & 0_{2(n-r)\times 2l}\\ C & 0_{2(m\times l)}\end{bmatrix}$ has full rank, there are no degrees of freedom to exploit for stabilisation.

We intend to compute, if exists, an externally stabilising G for the conditioned invariant subspace $\mathcal{V}$. From the previous discussion, the problem reduces to find matrices K such that $\Gamma = V_i + KH_i$ is asymptotically stable. With respect to Proposition 1, one solution to determine K is to solve the following LMI for K [15]:

$$\begin{bmatrix}\Phi & 0\\ 0 & \Psi-\Phi\end{bmatrix} - \begin{bmatrix}\Gamma_1^{\top}\\ \Gamma_2^{\top}\end{bmatrix}\Psi\begin{bmatrix}\Gamma_1 & \Gamma_2\end{bmatrix} > 0 , \qquad (16)$$

for some $\Phi := P_1 > 0$ and $\Psi := P_1 + P_2 > 0$. By the well-known Schur complement and using $\Gamma_i = V_i + KH_i$, for $i = 1, 2$, (16) is equivalent to

$$\begin{bmatrix}\Phi & 0 & (\Psi V_1 + \Theta H_1)^{\top}\\ 0 & \Psi-\Phi & (\Psi V_2 + \Theta H_2)^{\top}\\ \Psi V_1 + \Theta H_1 & \Psi V_2 + \Theta H_2 & \Psi\end{bmatrix} > 0 , \qquad (17)$$

for some $\Phi > 0, \Psi > 0$ and Θ of suitable dimensions, where $\Theta := \Psi K$.

In the same way, we can set up another LMI and by exploiting U, the second degree of freedom, internally stabilise the conditioned invariant subspace $\mathcal{V}$.

IV. Failure Modelling in Discrete Linear Repetitive Processes

Consider the unit-memory repetitive system (1). In what follows, we assume the system is detectable in the sense of [11, Th. 5.15]. To model the dynamics of the system after a failure has occurred, we augment the model with additional terms that represent the failure modes:

$$\begin{aligned}x_{k+1}(p+1) =& Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p)\\ &+ \begin{bmatrix}L^1 & L^2 & \cdots & L^l\end{bmatrix}\begin{bmatrix}m^1_{k+1}(p)\\ m^2_{k+1}(p)\\ \vdots\\ m^l_{k+1}(p)\end{bmatrix} . \qquad (18)\end{aligned}$$

where $m^j_{k+1}(p)$ and the matrices L^j, $j = 1, 2, ..., l$, are termed the *failure modes* and *signatures* respectively. Failure modes are unknown arbitrary functions corresponding to the type of the failure in the system. In the absence of failure, these modes are identical to zero while have some non-zero value once a failure has occurred.

Failure signatures together with the failure modes enable modelling a variety number of failures in the system, such as actuator failures, changes in system dynamics and sensor failures [5]. We consider four specific types of failure. The first type is a dead actuator. Suppose the j-th actuator is dead, then the failure signature L^j is the i-th column of the input matrix B, and the failure mode is $m^j_k(p) = -u^j_k(p)$ where $u^j_k(p)$ is the j-th component of the input $u_k(p)$. Secondly, if there is a bias in the j-th actuator, the failure signature L^j is the j-th column of the input matrix B, and $m^j_k(p) = b$ where $b \in \mathbb{R}$ is a non-zero constant. Thirdly, an actuator could be saturated if the input is too large. This case can be easily modelled by a combination of the first two cases. The last case, which is the most complicated one, is that the j-th actuator responds to the input in a wrong way, namely the j-th column of the input matrix B, denote by B^j, is changed to some different column vector $B^{j'}$. In this case, the failure signature is described by $L^j = \begin{bmatrix} B^j & B^{j'} \end{bmatrix}$ and is no longer a column vector, but a matrix. The corresponding failure mode is represented by $m^j_k(p) = \begin{bmatrix} -u^j_k(p) & u^j_k(p) \end{bmatrix}^T$.

We make the following assumptions:

- **Detectability:** The pairs (A, C) and (B_0C, C) are detectable. This guarantees that an asymptotic observer can be designed;
- **Unambiguous failure modes:** The failure signature matrix L^j has full column rank, $j = 1, 2, ..., l$;
- **No simultaneous failures:** If there exist $1 \le \bar{j} \le l$ such that $m^{\bar{j}}_k(p) \neq 0$, then $m^j_k(p) = 0$ for $j \neq \bar{j}$.

The system in the failure situation is modelled as:

$$\begin{aligned}x_{k+1}(p+1) =& Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p)\\ &+ L^j m^j_{k+1} ,\\ y_k(p) =& Cx_k(p) . \qquad (19)\end{aligned}$$

V. Fault Detection and Isolation

Consider designing a full-order observer of the following form for our nominal system model:

$$\begin{aligned}\hat{x}_{k+1}(p+1) =& A\hat{x}_{k+1}(p) + Bu_{k+1}(p) + B_0\hat{y}_k(p)\\ &- G_1\big(y_{k+1}(p) - \hat{y}_{k+1}(p)\big)\\ &- G_2\big(y_k(p) - \hat{y}_k(p)\big) , \qquad (20)\end{aligned}$$

where $G_i, i = 1, 2$, is the output-injection matrix. Moreover, define the error vector as $e_{k+1}(p+1) = x_{k+1}(p+1) - \hat{x}_{k+1}(p+1)$. If no failure is present in the system, the error dynamics can be computed by subtracting (20) from (1), obtaining

$$e_{k+1}(p+1) = (A+G_1C)e_{k+1}(p) + (B_0C+G_2C)e_k(p) \ , \tag{21}$$

which converges asymptotically to zero if $A+G_1C$ and B_0C+G_2C are stable matrices.

In the presence of a failure, the error dynamics is obtained by subtracting (20) from (19):

$$\begin{aligned} e_{k+1}(p+1) =& (A+G_1C)e_{k+1}(p) + (B_0C+G_2C)e_k(p) \\ &+ L^j m^j_{k+1}(p) \ . \end{aligned} \tag{22}$$

In case of a failure, the estimate error does not converge asymptotically to zero even if $A+G_1C$ and B_0C+G_2C are stable, but converges asymptotically to the reachable subspace [4] of the system (22). Note that $e_{k+1}(p)$ represents the error corresponding to the current pass whereas $e_k(p)$ represents the error corresponding to the previous pass. Denote by $\mathcal{L}^i := \mathrm{im}\ L^j$ and by $\mathcal{V}^\star(\mathcal{L}^j)$ the smallest conditioned invariant subspace containing $\mathcal{L}^j$ (i.e., the reachability subspace of $(A+G_1C, B_0C+G_2C, L^j)$). G_2 can be selected as $G_2 = -B_0$ so that the error from the previous pass is cancelled; of course, this is just one possible choice. The desired choice of G_1 should make $\mathcal{V}^\star(\mathcal{L}^j)$ into an externally stabilisable $(A+G_1C)$-invariant subspace. This stabilisability requirement in a fault-free situation described by (21), guarantees the convergence of the error to zero even if the initial error is not congruent. In the case when one fault has occurred, for example corresponding to the error signature L^j, the dynamics of the error is described by (22) with m^j_{k+1} non-zero and the error signature asymptotically lies in $\mathcal{V}^\star(\mathcal{L}^j)$. The internal stabilisability of the conditioned invariant is implied by the assumption that the system is detectable. One possible G_1 can be determined by solving:

$$-A\begin{bmatrix}L^1 & L^2 & \cdots & L^l\end{bmatrix} = G_1C\begin{bmatrix}L^1 & L^2 & \cdots & L^l\end{bmatrix} \ .$$

However, other stabilising gains, upon existence, can be computed as discussed in Section III.

Having derived the error dynamics in two situations of a fault-free and faulty system, we can spell out the FDI problem in a geometric language:

Fault Detection and Isolation Problem in Linear Repetitive Processes

Find subspaces $\mathcal{V}^j$, $j=1,2,\dots,l$, such that:

I. There exists a stabilising gain $G_i \in \mathbb{R}^{m\times n}$, such that $(A+G_1C)\,\mathcal{V}^j \subset \mathcal{V}^j$ and $(B_0C+G_2C)\,\mathcal{V}^j \subset \mathcal{V}^j, j=1,2,\dots,l$;

II. $\mathcal{L}^j \subseteq \mathcal{V}^j$;

III. $\mathcal{V}^j \cap \left(\sum_{h\neq j}\mathcal{V}^h\right) = \{0\}$, $j=1,2,\dots,l$.

The first condition guarantees that the subspaces $\mathcal{V}^j$ are internally and externally stable and invariant under the error dynamics. So the error due to a non-zero $m^j_k(p)$ remains inside $\mathcal{V}^j$. The second condition states that the subspaces $\mathcal{V}^j$ should contain the image of the failure signature. The last condition establishes that the subspaces have trivial intersection, which enables unique isolation of the failure. Having conditions I, II, and III satisfied, the procedure to construct an asymptotic observer for the purpose of fault detection as follows:

Construction of an Asymptotic FDI Observer

1. Check the detectability of (A, C) and (B_0C, C). If detectable, proceed to the next step. If not, stop ;

2. Compute the family of smallest conditioned invariant subspaces $\underline{\mathcal{W}}^\star\left(\mathcal{L}^j\right)$, $j=1,2,\dots,l$ containing $\mathcal{L}^j$, $j=1,2,\dots,l$ by using algorithm (5) ;

3. Verify condition (III) for the family $\underline{\mathcal{W}}^\star\left(\mathcal{L}^j\right)$. If not satisfied, stop ;

4. Find stabilising gains $G_i, i=1,2,$ if they exist, such that condition (I) holds (see Section III-A). If not, stop ;

Once the matrices $G_i,\ i=1,2$ have been obtained, we first define a *threshold value* $\varepsilon > 0$. If the norm of the error $e_k(p)$ is greater than ε, it is assumed that a fault has occurred. The determination of an appropriate ε on the basis of the fault description (and in a realistic situation, also on the basis of the size of disturbances and of the noise level) is an important issue which we do not consider here.

Our FDI procedure is as follows:

Algorithm V.1: FDI PROCEDURE()

1 : **for** $k=0$ **to** K^*
2 : **for** $p=1$ **to** α
3 : **if** $||e_k(p)|| > \varepsilon$ **then**
4 : Compute $e'_k(p)$, the projection of $e_k(p)$ down onto $\mathcal{V}^\star(\mathcal{L}^j)$;
5 : Compute $f := arg\ \max\{||e'_j(p)||\},\ j=1,...,l$
6 : **return** (f) ;
7 : **end if**
8 : **end for**
9 : **end for**

$^*K$ is the number of passes.

Note that due to the assumptions, the computation of f in **Step 5** is well-defined.

VI. Example

In this section, we apply the proposed fault detection and isolation technique developed in previous sections to the metal rolling process presented in [14, p. 703]:

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} \ ,$$

$$A' = \begin{bmatrix} \frac{35}{36} & \frac{7}{72} & 0 \\ -\frac{5}{18} & \frac{35}{36} & 0 \\ \frac{35}{5216} & \frac{7}{10432} & \frac{160}{163} \\ \frac{175}{2608} & \frac{35}{5216} & -\frac{30}{163} \\ -\frac{8}{2007} & -\frac{4}{10035} & \frac{67}{10520} \\ -\frac{80}{2007} & -\frac{8}{2007} & \frac{67}{1052} \end{bmatrix} ,$$

$$A'' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{16}{163} & 0 & 0 \\ \frac{160}{163} & 0 & 0 \\ -\frac{27}{42394} & -\frac{135}{137} & \frac{27}{274} \\ -\frac{67}{1052} & -\frac{20}{137} & \frac{135}{137} \end{bmatrix} , \quad B_0 = \begin{bmatrix} \frac{1}{26} \\ \frac{5}{63} \\ -\frac{13}{2608} \\ -\frac{65}{1304} \\ \frac{60}{20263} \\ \frac{127}{4289} \end{bmatrix} ,$$

$$B = 10^{-3} \times \begin{bmatrix} -\frac{5}{18} & 0 & 0 \\ -\frac{25}{9} & 0 & 0 \\ -\frac{71}{37004} & -\frac{30}{163} & 0 \\ -\frac{93}{4847} & -\frac{300}{163} & 0 \\ \frac{47}{41236} & -\frac{11}{9214} & -\frac{20}{137} \\ \frac{72}{6317} & -\frac{55}{4607} & -\frac{200}{137} \end{bmatrix} ,$$

$$C = \begin{bmatrix} \frac{77}{223} & \frac{77}{2230} & -\frac{559}{1023} & -\frac{183}{3349} & \frac{135}{137} & \frac{27}{274} \end{bmatrix} .$$

which are computed using the following parameters:

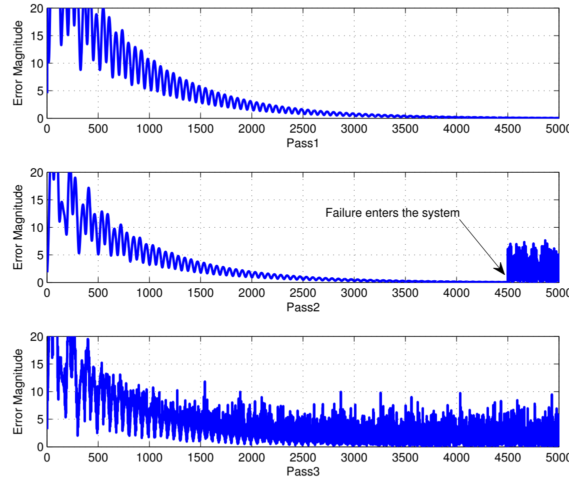

Fig. 1. Error norm for dead actuator failure in passes 1 to 3

$$\hat{A}_i = \frac{1}{1 + a_{0i}T^2} \begin{bmatrix} 1 & T \\ -a_{0i}T & 1 \end{bmatrix} , \quad \hat{B}_i = \frac{c_{0i}T}{1 + a_{0i}T^2} \begin{bmatrix} T \\ 1 \end{bmatrix} ,$$

$$\hat{B}_{0i} = \frac{(-b_{0i} + a_{0i}b_{2i})T}{1 + a_{0i}T^2} \begin{bmatrix} T \\ 1 \end{bmatrix} , \quad C_i = \frac{1}{1 + a_{0i}T^2} \begin{bmatrix} 1 & T \end{bmatrix} ,$$

$$a_{0i} = \frac{\lambda_{1i}\lambda_2}{M_i(\lambda_{1i} + \lambda_2)} , \quad b_{2i} = \frac{-\lambda_2}{\lambda_{1i} + \lambda_2} ,$$

$$b_{0i} = \frac{-\lambda_{1i}\lambda_2}{M_i(\lambda_{1i} + \lambda_2)} , \quad c_{0i} = \frac{-\lambda_{1i}}{M_i(\lambda_{1i} + \lambda_2)} ,$$

$$\lambda_{11} = 40N/m \ , \lambda_{12} = 60N/m \ , \lambda_{13} = 80N/m \ ,$$
$$\lambda_2 = 100N/m \ , M_1 = 10Kg \ , M_2 = 20Kg \ , M_3 = 30Kg$$

There are three actuators in the system. The input is considered to be a decreasing force along each pass. The first 3 passes of the system each having a length of $\alpha = 5000$ are simulated using model (19). We design an asymptotic observer of the form (20) for the system. For the gains G_1 and G_2 of the observer, G_1 is computed as discussed in Section III-A, and G_2 is considered as $G_2 = -B_0$.

Since, the simulations are carried out regardless of disturbances and noise, it is reasonable to set our threshold to 0. We consider two types of faults happening in the system:

A. Dead actuator

The first case we consider is where one of the actuators, say the first one, is dead at pass $k_0 = 4500$ at $p_0 = 2$. Figure 1 shows that the error vector goes to zero from some non zero boundary conditions at the beginning. Thereafter, the error constantly stays at zero until $k_0 = 4500$, $p_0 = 2$ is reached where the first actuator dies.

Now that it has become obvious that a failure has occurred, we isolate the fault. This is done as discussed in Section V by projecting the the error vector to subspaces $\mathcal{W}^{\star}(\mathcal{L}^1)$,

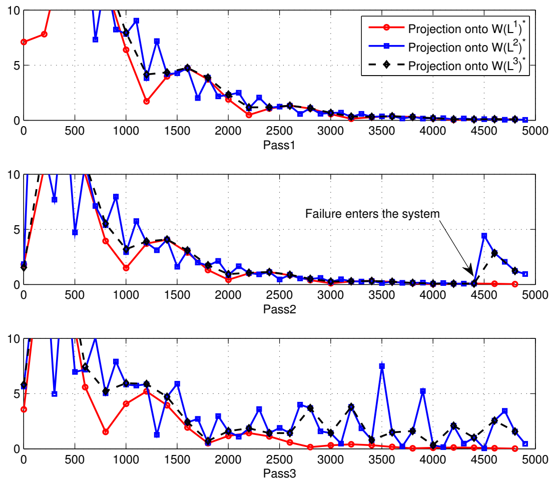

Fig. 2. Norm of projection of the error to the subspaces containing $\mathcal{W}^\star(\mathcal{L}^i)$, $i = 1, 2, 3$ in passes 1 to 3.

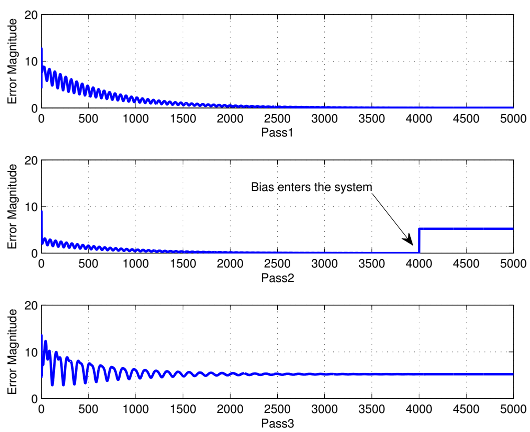

Fig. 3. Error norm for biased actuator failure in passes 1 to 3

$\mathcal{W}^\star(\mathcal{L}^2)$, and $\mathcal{W}^\star(\mathcal{L}^3)$. This is depicted in Figure 2. It can be seen that at the beginning of the process the error converges to zero and lies in $\{0\}$. After the fault happens, the error deviates from zero and lies in $\mathcal{W}^\star(\mathcal{L}^1)$ implying that the fault has occurred in the first actuator.

B. Biased actuator

The next case we consider is a biased actuator. Suppose one of the actuators, say the second one, is biased. The bias enters the system at $k_0 = 4000$ at $p_0 = 2$. Figure 3 illustrates this bias where it can be observed that the error due to non-zero initial conditions goes to zero and then rises and constantly stays at $b \in \mathbb{R}$ after the bias enters the system.

Then as shown in Figure 4, projecting the error vector onto subspaces $\mathcal{W}^\star(\mathcal{L}^1)$, $\mathcal{W}^\star(\mathcal{L}^2)$, and $\mathcal{W}^\star(\mathcal{L}^3)$, reveals that the bias has occurred in the second actuator.

Additionally, by looking at the error signal, one can also

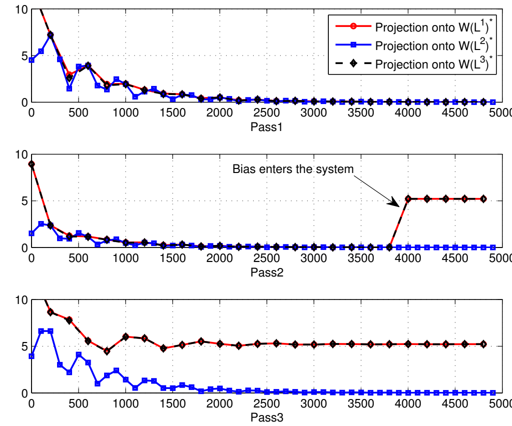

Fig. 4. Norm of projection of the error to the subspaces containing $\mathcal{W}^\star(\mathcal{L}^i)$, $i = 1, 2, 3$ in passes 1 to 3.

recognise the type of the failure (i.e., dead or biased). In case of a dead actuator as can be seen in Figure 2, the behaviour of the error signal depends on the input signal, which is a decreasing signal on each pass here, whereas in the case of biased actuator, the error signal does not depend on the input and stays constantly at $b \in \mathbb{R}$ after the bias enters the system.

VII. CONCLUSIONS

We presented a geometric approach to fault detection and isolation in discrete linear repetitive processes. In our method, the whole system state is reconstructed instead of exploiting just the system output for residual generation. Thus, a wider range of failures can be detected and isolated compared to other methods used in the past (for example [5]). Effectiveness of the proposed approach was illustrated by providing an industrial example in which we detect and isolate a dead and a biased actuator.

REFERENCES

[1] S. Maleki, P. Rapisarda, L. Ntogramatzidis, and E. Rogers, "Fault Detection and Isolation in 3D Linear Systems", *Multidim. Syst. Sign. P.*, Submitted for publication.

[2] E. Rogers, K. Galkowski, D. H. Owens, *Control Systems Theory and Applications for Linear Repetitive Processes*, Berlin Heidelberg: Springer, 2007.

[3] T-P Azevedo-Perdicoulis, G. Jank and P. Lopes dos Santos, "The Good Behaviour of the Gas network: Boundary Control, Observability and Stability", Proc. of the 8th Int. Workshop on Multidim. Syst. , pp.1-6, Sept. 2013.

[4] E. Fornasini, G. Marchesini, "Doubly-indexed Dynamical Systems: State-space Models and Structural Properties", *Math. Syst. Theory*, pp. 59-72, vol. 12, no. 1, 1978.

[5] M. A. Massoumnia, "A Geometric Approach to the Synthesis of Failure Detection Filters", *IEEE Trans. Autom. Control*, pp. 839-849, vol. AC-31, 1986.

[6] G. Basile, G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*, Prentice-Hall, Inc., 1992.

[7] G. Basile, G. Marro, "Controlled and Conditioned Invariant Subspaces in Linear System Theory",*J. Optimiz. Theory App., 3(5), pp.306-315*

[8] G. Basile, G. Marro, "On The Observability of Linear Time-Invariant Systems with Unknown Inputs.", *J. Optimiz. Theory App., 3(6), pp.410-415*

[9] I. Hwang, S. Kim, S. Chze Eng, "A Survey of Fault Detection, Isolation, and Reconfiguration Methods", *IEEE Trans. Control Syst. Technol.*, vol.18, no.3, pp.636-653,2010

[10] A. S. Willsky, "A Survey of Design Methods for Failure Detection in Dynamic Systems", *Automatica*, pp. 601-611, vol.12, 1976.

[11] Galkowski, K. and Wood, J.D., "Multidimensional Signals, Circuits and Systems', *Taylor & Francis*, 2004.

[12] R. V. Beard, "Failure Accommodation in Linear Systems Through Self-Reorganisation", Ph. D dissertation, Dep. Aeronautics and Astronautics, Mass. Inst. Technol., Feb. 1971.

[13] H. L. Jones, "Failure Detection in Linear System", Ph. D dissertation, Dep. Aeronautics and Astronautics, Mass. Inst. Technol., Aug. 1973.

[14] J. Bochniak, K. Galkowski, E. Rogers, and A. Kummert, "Control Law Design for Switched Repetitive Processes with a Metal Rolling Example", *IEEE Intl. Conf. Contr.*, pp.700-705, 1-3 Oct. 2007.

[15] L. Ntogramatzidis, M. Cantoni, "Detectability Subspaces and Observer Synthesis for Two-Dimensional Systems", *Multidim. Syst. Sign. P.*, vol.23, no.1-2, pp. 79-96, 2012

[16] Kar, H. and V. Singh, "Stability of 2-D Systems Described by the Fornasini-Marchesini First Model", *IEEE T. Signal Proces.*, vol. 51, no.6, pp. 1675–1676, 2003.

[17] Gałkowski, K., Lam, J., Xu, S. and Z. Lin, "LMI Approach to State-Feedback Stabilization of Multidimensional Systems", *Intl. J. Contr.*, vol. 76, no. 14, pp. 1428–1436, 2003.