UNIVERSITY OF SOUTHAMPTON

Robust Stability for Nonlinear Control: State-Space and Input-Output Synthesis

by

Jing Liu

A thesis submitted for the degree of Doctor of Philosophy

in the
Faculty of Physical Sciences and Engineering
School of Electronics and Computer Science
University of Southampton

September 2014

ABSTRACT

In this thesis we consider the development of a general nonlinear input-output theory which encompasses systems with initial conditions.

Appropriate signal spaces (i.e., interval spaces, extended spaces and ambient spaces) are introduced with some fundamental assumptions to constitute a framework for the study of input-output systems with abstract initial conditions. Both systems and closed-loop systems are defined in a set theoretic manner from input-output pairs on a doubly infinite time axis, and a general construction of the initial conditions (i.e., a state at time zero) is given in terms of an equivalence class of trajectories on the negative time axis. Fundamental properties (such as existence, uniqueness, well-posedness and causality) of both systems and closed-loop systems are defined and discussed from a very natural point of view. Input-output operators are then defined for given initial conditions, and a suitable notion of input-output stability on the positive time axis with initial conditions is given. This notion of stability is closely related to the ISS/IOS concepts of Sontag.

A fundamental robust stability theorem is derived which represents a generalisation of the input-output operator robust stability theorem of Georgiou and Smith to include the case of initial conditions; and can also be viewed as a generalisation of the ISS approach to enable a realistic treatment of robust stability in the context of perturbations which fundamentally change the structure of the state space. This includes a suitable generalisation of the nonlinear gap metric. Generalisations of this robust stability result are also extended to finite-time reachable systems and to systems with potential for finite-time escape by extending signals on extended spaces to a wider space (ambient space). Some linear and nonlinear applications are given to show the effects of the robust stability results.

We also present a generalised nonlinear ISS-type small-gain result in this input-output structure set up in this thesis, which is established without extra observability conditions and with complete disconnection between the stability property and the existence, uniqueness properties of systems.

Connections between Georgiou and Smith's robust stability type theorems and the nonlinear small-gain theorems are also discussed. An equivalence between a small-gain theorem and a slight variation on the fundamental robust stability result of Georgiou and Smith is shown.

Contents

List of Figures

Declaration of Authorship

I, Jing Liu, declare that the thesis entitled *Robust Stability for Nonlinear Control: State-Space and Input-Output Synthesis* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: [Liu and French, 2014d], [Liu and French, 2014a], [Liu and French, 2014b], Liu and French [2014c], [Liu and French, 2013] and [Buchstaller et al., 2014]

Signed:...

Date:..

Nomenclature

Symbol	**Short description**
$\forall$; $\exists$; $\in$; $\notin$	for all; there exists; belongs to; does not belong to
$<, \leq, >, \geq$	inequality signs
$\bigcup, \cup; \bigcap, \cap$	union; intersection
$\subseteq, \supseteq$	inclusion signs
$A \times B$	Cartesian product
$\emptyset$; $A - B$	empty set; difference set ($x \in A$ and $x \notin B$)
$F : S \subseteq X \to Y$	operator from the set S into the set Y with $S \subseteq X$
$\mathrm{dom}(F)$; $\ker(F)$	domain of F; kernel of F
$\mathrm{range}(F)$; $F(A)$	range of F; image of the set A
$F\vert_{X_0}$	restriction of the operator to the set X_0
$F_2 \circ F_1$	composition of F_2 and F_1, $(F_2 \circ F_1)(x) = F_2(F_1(x))$
$\equiv$; $\triangleq$, $:=$	identically equal, defined as
$\to$; $\Rightarrow$; $\Leftarrow$; $\Leftrightarrow$	converges to; implies, implied by, equivalent to (or if and only if)
$\sim$; $[a]$	equivalence relation; equivalence class of a under $\sim$
$\mathbb{Z}$; $\mathbb{R}$; $\mathbb{C}$	set of integers; set of real numbers; set of complex numbers
$\mathbb{N}$; $\mathbb{N}_{>0}$	set of natural numbers including 0; set of positive intergers
$\mathrm{Re}(z)$; $\mathrm{Im}(z)$	real part of z; imaginary part of z
$\overline{a+bj}$	complex conjugate of $a + bj$, which equals $a - bj$
$\mathbb{R}_+$	set of nonnegative real numbers, $\{x \in \mathbb{R} : x \geq 0\}$
$\mathbb{R}_-$	set of nonpositive real numbers, $\{x \in \mathbb{R} : x \leq 0\}$
$\mathbb{C}_0^+$	set of complex numbers with positive real parts, $\{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$
$\mathbb{R}^n$	the n-dimensional Euclidean space
$\mathbb{R}^{n \times m}$	set of real $n \times m$ matrix
$\sum_{i=1}^n a_i$	summation of a_i for $i = 1, 2, \ldots, n$
$\prod_{i=1}^n a_i$	product of a_i for $i = 1, 2, \ldots, n$
$\{x_n\}_{n=1}^\infty$	sequence of x_n for $n = 1, 2, \ldots$
$\{x \vert P\}$, $\{x : P\}$	set of all elements x with property P
A^T	transpose of a vector (or a matrix) A
A^*	complex conjugate and transpose of A, which equals $\overline{A}^T$
$F^\sim(s)$	para-Hermitian conjugate of $F(s)$, which equals $\left(\overline{F(-\overline{s})}\right)^T$

$\mathrm{col}(x, y)$	the vector $(x^T, y^T)^T \in \mathbb{R}^{m+p}$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$
0_n; $0_{n\times m}$	zero vector in $\mathbb{R}^n$; zero matrix in $\mathbb{R}^{n\times m}$
0	zero scalar, zero vector, zero matrix or zero operator
I	identity operator or identity matrix or time interval
$L^q(T)$	Lebesgue spaces of q-integrable functions: $f : T \to \mathbb{R}$
$L^q(T, \mathbb{R}^n)$	Lebesgue spaces of q-integrable functions: $f : T \to \mathbb{R}^n$
$\wedge$; $\wedge_\tau$	concatenation at time 0; concatenation at time τ
$\mathcal{V}$	normed signal space defined on time interval $(-\infty, \infty)$
$\mathcal{V}^+$; $\mathcal{V}^-$	restriction of $\mathcal{V}$ to $[0, \infty)$; restriction of $\mathcal{V}$ to $(-\infty, 0]$
$\mathcal{V}[0, \tau)$	restriction of $\mathcal{V}$ to $[0, \tau)$ for any $\tau > 0$
$\mathcal{V}_e$; $\mathcal{V}_e^+$	extended space of $\mathcal{V}$; extended space of $\mathcal{V}^+$
$\mathcal{V}_a$; $\mathcal{V}_a^+$	ambient space of $\mathcal{V}$; ambient space of $\mathcal{V}^+$
$\lvert\cdot\rvert$	absolute value or Euclidian norm of a vector in $\mathbb{R}^n$
$\lVert\cdot\rVert$	norm of a vector in a normed signal space $\mathcal{V}$
exp	exponential function
Hurwitz	all eigenvalues have negative real parts
max; min	maximum; minimum
sup; inf	supremum (least upper bound); infimum (greatest lower bound)
ess sup; ess inf	essential supremum; essential infimum
LTI	linear time-invariant
resp.	respectively
w.r.t.	with respect to
s.t.	such that
i.e.	that is to say
e.g.	for example
a.e.	almost everywhere
et al.	and other people or things
$\square$	designation of the end of proofs, examples, etc.

Acknowledgements

During the period of my Ph.D., I have had cause to be grateful for the advice, support and understanding of many people. This thesis would not have been possible without the help of them.

Most importantly, I would like to express my sincere appreciation and gratitude to my thesis supervisor, Prof. Mark French, for his continuous moral and technical support, as well as his enthusiasm. I am indebted to him for providing the initial motivation for my research topic of this thesis, for answers to any of my questions, constructive suggestions and stimulating discussions, and for his painstaking review of this thesis.

Many thanks are due to Dr. Paolo Rapisarda for his thoughtful review of the manuscript and for his valuable suggestions, great insight and encouragement.

I have enjoyed it very much to be a member of the Communications, Signal Processing and Control (CSPC) group in ECS. Many thanks go to all the people in the group who made it such an interesting place to work. In particular, I have benefited greatly from the weekly seminar series in the control group organised by Dr. Bing Chu, Dr Christopher Freeman, Prof. Mark French, Dr. Ivan Markovsky, Dr. Paolo Rapisarda and Prof. Eric Rogers.

I am always deeply grateful to my Master tutor, Prof. Jun-Min Wang, for initially raising my interest in mathematical control theory and for his kind advice, help and encouragement whilst at the Beijing Institute of Technology (BIT), Department of Mathematics.

Lastly I am indebted to my family and friends for their continuous support and encouragement in my life.

The research of this thesis was carried out at the University of Southampton, School of Electronics and Computer Science (ECS) with financial support provided by the China Scholarship Council (CSC), the Department of Business, Innovation & Skills (BIS) of UK, and the University of Southampton for a joint UK-China Scholarship for Excellence (SfE) over the entire period of research.

TO MY PARENTS AND MY WIFE YING

When the answers to a mathematical problem cannot be found, then the reason is frequently the fact that we have not recognized the general idea, from which the given problem appears only as a single link in a chain of related problems.

David Hilbert (1862-1943)

Chapter 1

Introduction

Control theory is an interdisciplinary subject of mathematics and engineering that manage the performance of dynamical systems. Precisely what forms a meaningful notion of good performance is definitely a debatable topic. The problem lies in how to convert the intuitive idea of a good performance into a exact mathematical definition that can be applied to given dynamical systems. We later define more precisely what we mean by 'system' or 'dynamical system', but roughly speaking, it is more like a black box which produce output signals when applied to input signals (Figure 1.1).

1.1 Feedback Control

Feedback is one of the most important concepts from control and systems theory. There are many control tasks such as tracking, disturbance rejection and coping with model uncertainties that require the use of feedback. Today feedback theory has seen a wide range of applications in diverse fields including mechanical engineering, electronic engineering, bioengineering, chemical engineering, economics, social science and so on.

History

Roots of modern feedback control can be traced to J. C. Maxwell's early work on the stability of Watt's flyball governor (see [Maxwell, 1868]), which is the first rigorous mathematical analysis of a feedback control system. Maxwell showed that the system is stable if the roots of certain characteristic equation have negative real parts. E. J.

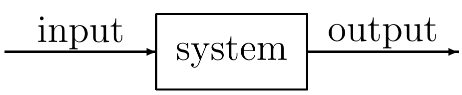

Figure 1.1: Input-output system

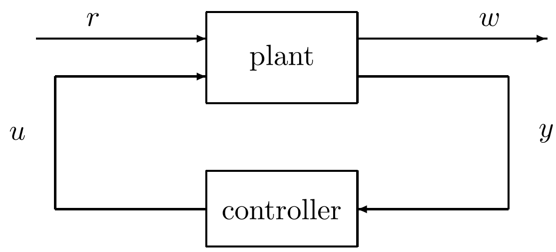

Figure 1.2: A classical closed-loop feedback system

Routh proposed in 1877 a test for determining whether all the roots of the characteristic equation have negative real parts (see [Routh, 1877]). Unaware of the work of Maxwell and Routh, A. Hurwitz solved independently in 1895 the problem of determining the stability of the characteristic equation (see [Hurwitz, 1895]). In electrical engineering and control theory, the idea of feedback was first introduced at the Bell Laboratory by an electrical engineer named H. S. Black (see [Black, 1934, Kline, 1993]). He proposed usage of the negative feedback amplifiers to reduce noise and distortion introduced by the nonlinearities in the problems of telephone transmission, especially of transcontinental communications. Harold S. Black conceived the idea of feedback in a flash of insight while he was aboard the Lackawanna Ferry on his way to work on August 2, 1927. The invention had been submitted to the U. S. Patent Office on August 8, 1928. However, it took more than nine years for the patent to be issued on December 21, 1937. [Black, 1977] later wrote:

> "One reason for the delay was that the concept was so contrary to established beliefs that the Patent Office initially did not believe it would work."

A detailed introduction of the history of feedback control theory can be found in e.g., [Lewis, 1992, Chapter 1].

Why Feedback?

A classical closed-loop feedback system is shown in Figure 1.2 (see e.g., [Doyle et al., 1990]). It consists of two components: a plant to be controlled and a controller to be designed such that some pre-specified properties are satisfied by the whole system. In this configuration, r represents the exogenous inputs such as references, disturbance and so on. w are signals we wish to control. The main idea of feedback control is that the value of the control input u for plant (as output of controller) is based on the observed output y of plant (as input for controller). This is very different from the point of open-loop control where one choose u as an explicit function of time.

Jan W. Polderman and Jan C. Willems give an intuitive example about climbing stairs to indicate that feedback control in general leads to superior performances [Polderman and Willems, 1998, Example 9.1.1]. Disturbances and model uncertainties can be taken into consideration by feedback control, but not by open-loop control. Reducing distortion of amplifier [Black, 1934], rejecting disturbances and handling a variety of model uncertainties are also very important properties of feedback control.

1.2 Two Different Views of Stability

One of the most important topics in control theory is the stability property of a general dynamical system. In the literature, there are two main different competing ways to study nonlinear stability questions: the *state space approach* in the sense of Lyapunov, and the *input-output theory* of which G. Zames and I.W. Sandberg are the most notable contributors.

State Space Approach

The state space approach is usually associated with the name of the Russian mathematician A.M. Lyapunov (1857-1918), who published his famous book *The General Problem of Stability of Motion* in Russian in 1892. His work was largely unknown for many years in the Western world and elsewhere until the Cold War (1953-1962) period, and almost all the work in Lyapunov stability theory until that time are conducted by Russian mathematicians. Translations of Lyapunov's work was first appeared in French and most recently in English [Lyapunov, 1992]. The Lyapunov theory plays a crucial role in control and systems theory, which mainly deals with stability of equilibria for the unforced system (without inputs or controls) described by nonlinear time-varying ordinary differential equations (see e.g., [Michel, 1996]):

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n$ with $t \geq 0$ represent states.

In simple forms, Lyapunov stability theory states that if a system of which the initial state is near an equilibrium remains stay near the equilibrium forever, then the equilibrium is called Lyapunov stable. Today the foundations of the theory are well established and a myriads of publications expanding upon this theory appeared in the control and systems literature (see e.g., [Khalil, 2002, Sastry, 1999, Vidyasagar, 1993]).

The principal advantage of state space approach or Lyapunov stability theory is that it is direct and hence one does not need to solve the differential equation explicitly. It is prepared to the study of systems without inputs. The main limitation for the application of this method is that it requires finding the so-called Lyapunov function which is usually very difficult.

Input-Output Theory

The general nonlinear input-output theory is much more recent in origin than the Lyapunov stability theory, initiated in the 1960s by [Zames, 1963, 1966b,c] and [Sandberg, 1964, 1965a] using the techniques of functional analysis. It deals with systems described by operators mapping from input signals to output signals, similar to "black box" represented graphically as shown in Figure 1.1. The essence of input-output theory is laid in that only relations between inputs and outputs are relevant. It only considers the external structures of a system and ignores the internal system description.

The main advantage of the input-output theory is that it appears possible to make useful assessments of qualitative properties for poorly defined systems, especially meaningful regarding robustness analysis when analysing nonlinear systems which might contain more complicated unstructured uncertainties than those of linear systems, or even not easily represented by state space model. On the other hand, many of the arguments in input-output theory are conceptually clearer than Lyapunov stability theory, at the cost of requiring great background in mathematics. In this theory, the concepts of *causality*, *extended spaces* (recently, *ambient space* for discussions of finite escape times phenomenon) and *truncation operators* play a very important role. These are also frequently encountered notions and will be carefully defined for our work in this thesis.

1.3 Bridge the Gap Across Them

State space model and input-output model are two different types of realisations of looking at the same *system* (physical devices), both of which gives a different kind of insight into how the system works. On one hand, the state space approach deals with equilibrium points for a system governed by unforced ordinary differential equations describing time evolution of state variables without inputs evolving under the influence of a nonzero initial state. On the other hand, current input-output approach deals with forced systems (with inputs) focusing attentions on the influence of inputs upon outputs without mentioning any concept of state at all, thus no initial conditions can be considered in detail. Our objective is to give a framework to study purely input-output systems incorporated with initial conditions in terms of bringing these input-output approach and state space approach together. Of course, how to define the initial conditions in a purely input-output system is the first difficult problem we need to solve.

Input-to-State Stability

States and initial conditions have been introduced into input-output reasoning via the well-known input-to-state stability (ISS) theory introduced by [Sontag, 1989] and its many variants (see e.g., [Sontag, 2008, Sontag and Wang, 1995, 1996]). ISS is fundamental a state space approach in which systems are assumed to have a known state space representation:

$$\dot{x}(t) = f(x(t), u(t)), \quad y = h(x(t)),$$

where $u(t) \in \mathbb{R}^m$ $(t \geq 0)$ representing inputs, $x(t) \in \mathbb{R}^n$ $(t \geq 0)$ representing states and $y(t) \in \mathbb{R}^p$ $(t \geq 0)$ representing outputs. The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is called the *state evolution function* (typical nonlinear) and the map $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is called the *read-out map*.

The notion of stability of these forced systems combines Lyapunov stability from Lyapunov theory and bounded-input-bounded-output stability (BIBO) from input-output theory. Essentially, the method is still a state space approach, and many of investigations and proofs for ISS are using Lyapunov-like methods (i.e., the so-called ISS-Lyapunov function for forced systems which is a natural generalisation of classical Lyapunov function for unforced systems). Generalisations of ISS in this framework to many other stability notions including input-to-output stability (IOS), integral input-to-state stability (iISS), input-output-to-state stability (IOSS), etc. can be found in e.g., [Krichman et al., 2001, Sontag, 2008, Sontag and Wang, 2000].

1.4 Recent Work on Input-Output Theory

In a benchmark paper of [Georgiou and Smith, 1997b], the authors developed an input-output approach to uncertainty in the gap metric for robustness analysis of nonlinear feedback systems. We remark that a priori assumption of systems defined on semi-infinite time axis mapping zero input into zero output implicitly require that the systems have zero initial conditions. For closed-loop systems with nonzero responses to zero disturbances, we cannot directly use Georgiou and Smith's robust stability theory. There are a number of later extensions which permit consideration of nonzero responses to zero disturbances, e.g., [French and Bian, 2009, 2012, Georgiou and Smith, 1997a], however, neither of these approaches are directly aimed at the case of initial conditions, and cannot directly be used to establish fading memory properties.

Explicit robust stability results are given in [French, 2008] and [French et al., 2009] for a specific case of a linear plant and a nonlinear controller with initial conditions. A more general construction for nonlinear plants can be found in [French and Mueller, Section 7], and this forms the basis for this contribution.

1.5 Our Objectives

Our purpose is in a sense of closing the gap in studying input-output theory by incorporating initial conditions into purely input-output systems. We define the initial conditions by equivalent past system input-output trajectories (see Section 3.5.1 on page 53) from an intuitive idea of 'state', i.e., the state at any time together with the future input completely determine the future output. This is the so-called *axiom of state* (or *property of state*) in Willems's behavioural framework (see e.g., [Willems, 1989, Section 2], [Polderman and Willems, 1998, Chapter 4]). In other words, the state is the memory of the system; or say, the state is a classifier of input-output pasts (see e.g., [Zames, 1963]). Note that the notion of state is postulated axiomatically in theories like differential equation theory and difference equation theory. This simplifies the formulation of certain problems and yields very successful and well-established stability theory such as Lyapunov stability theory and ISS theory.

However, the state space approach including ISS doesn't handle certain robustness issues.[1] Essentially, robust stability is concerned with perturbations to nominal systems which induce significant (and potentially unknown) changes to the underlying state space (e.g., changing its dimension, as occurs with a finite dimensional multiplicative perturbation, or shifting from a finite dimensional state space for the nominal model to an infinite dimensional system).

As a concrete example consider the following nominal plant Σ with one dimensional state space:

$$\Sigma : \dot{x}(t) = \phi(x(t)) + u(t), \quad y(t) = x(t), \tag{1.1}$$

and the perturbed plant Σ_τ with infinite dimensional state space:

$$\Sigma_\tau : \dot{x}(t) = \phi(x(t)) + u(t-\tau), \quad y(t) = x(t), \quad 0 < \tau \leq \tau_0, \tag{1.2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a memoryless nonlinear function satisfying the so-called sector condition $\phi \in \textbf{Sector}\ (k_1, k_2)$ with $k_1, k_2 \in \mathbb{R}$ and $k_1 \leq k_2$, i.e., $[\phi(x) - k_1 x][\phi(x) - k_2 x] \leq 0$ for all $x \in \mathbb{R}$. The nominal plant Σ and the perturbed plant Σ_τ are also very close in the sense of gap metric ($\delta(\Sigma, \Sigma_\tau) \to 0$ as $\tau \to 0$), but with different dimensional state spaces, and one would anticipate that a satisfactory feedback controller for Σ will also work for Σ_τ for any $0 < \tau \leq \tau_0$ provided τ_0 is sufficiently small. In terms of the usual state-space method, the initial condition in Σ can be taken to be $x(0)$. However, for Σ_τ the initial condition is necessarily infinite dimensional, e.g., $(x(0), u|_{(-\tau,0]})$. Intuitively, even when initial conditions are taken into consideration, the nominal plant Σ when stabilised by

[1] Robust control is an advanced topic in control theory that explicitly deals with model uncertainty. The model uncertainty is usually characterised as perturbations of a nominal model. The objective for robust control is to design, for a given nominal plant, a controller that stabilises all plants in a neighborhood of the nominal plant in an appropriate sense [Trentelman et al., 2011].

a controller should remain stabilised when replaced by any of the perturbed plants Σ_τ, $0 < \tau \leq \tau_0$. Clearly to quantify such statements, we need appropriate notions of stability together with an appropriate quantification of the notion of 'size' of initial conditions which can be consistently applied across Σ and Σ_τ for any $0 < \tau \leq \tau_0$. Additionally these concepts must also be applicable to all other 'reasonable' perturbations (multiplicative, additive etc.) which often change the state space structure.

Detailed discussion of this example will be considered in Example 4.22 on page 99. Roughly speaking, we identify the nominal (resp., perturbed) plant with the set $\mathfrak{B}_\Sigma$ (resp., $\mathfrak{B}_{\Sigma_\tau}$) which consists of all input-output pairs (u, y) defined in the time domain $(-\infty, \infty)$ satisfying Eq. (1.1) (resp., Eq. (1.2)) for some time function x. For any $w := (u, y) \in \mathfrak{B}_{\Sigma_\tau}$, the restriction of w to the time domain $(-\infty, 0)$ will be a representative of some equivalence class identified with some initial state at time zero. Example 4.22 shows that, under any controller $u(t) = -k \cdot [y(t) + y_0(t)] + u_0(t)$ with any real constant $k > k_2$ (note that u_0 and y_0 represent the input and output disturbances of the plant, respectively), the closed-loop system will always be input to output stable if for any given $\varepsilon \in (0, k - k_2)$ the time delay $\tau < 1/\omega$ with $\omega \triangleq (1 + \max\{|k_1|, |k_2|\})(1 + k + \frac{1+k}{k-k_2-\varepsilon})$. Note that we view the external signals $w_0 := (u_0, y_0)$ (i.e., external disturbances) as the (closed-loop) input and the internal signal $w := (u, y)$ (i.e., input-output trajectories of the plant) as the (closed-loop) output.

1.6 Summary of Contents

- **Chapter 2** (pages 11–36) This chapter contains the mathematical preliminaries which will be used in the rest of this thesis. We present here some basic concepts such as sets, (nonlinear) operators, metric spaces, normed vector spaces, equivalence relations, partitions, and classes $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions. A type of Schauder fixed-point theorem for nonlinear operators is reviewed due to the requirement of establishing properties of existence and boundedness simultaneously for a closed-loop system in Chapters 4 and 6. Nerode equivalence for scalar continuous-time transfer functions is discussed in Section 2.5 on page 21, which gives a key insight to the abstract construction of initial conditions in this thesis (see Section 3.5.1 on page 53). Input-to-state stability and input-to-output stability in state space models are also reviewed. Some of the work in this chapter has been submitted for publication in [Liu and French, 2014c].

- **Chapter 3** (pages 37–76) In this chapter we develop a general input/output framework which incorporates a general concept of initial conditions characterised by a purely input-output formalism drawn from [Willems, 1989]. These allow us to deal with model perturbations which are often associated with changes in the underling state space structure. In this thesis, both systems and closed-loop systems are defined in a set theoretic manner from input-output pairs on a doubly infinite time axis, and the construction of initial conditions is given in terms of an equivalent class of input-output trajectories on the negative time axis. Comparison with classical initial conditions are also given for both systems (Section 3.5.3 on pages 58–64) and closed-loop systems (Section 3.7.2 on page 69). Fundamental notions of causality, well-posedness (existence and uniqueness) and graph are discussed for both systems and closed-loop systems in the presenting input/output framework. A specific consideration of the uniqueness property of a system is given in Section 3.4.3 on page 51, which will be very useful in the proof of Theorem 4.8 in Chapter 4 on page 81. Relationships between initial conditions, the well-posedness and casuality of open-loop subsystems and closed-loops systems are given in Section 3.7.1 on page 66, Section 3.7.4 on page 72, and Section 3.7.5 on page 73, respectively. A suitable concept of input-output stability on the positive time axis with initial conditions is given for both systems (Section 3.6 on page 64) and closed-loops systems (Section 3.8 on page 74), which is closely related to the ISS/IOS notions initiated by Sontag [1989]. Theorem 3.36 on page 75 summarises several alterative characterisation of this notion of stability for closed-loop systems. Some of the work in this chapter appears in [Liu and French, 2014d], [Liu and French, 2013].

- **Chapter 4** (pages 77–107) This chapter establishes essentially the main results of this thesis (Theorems 4.8 and 4.18) based on the general input/output framework set up in Chapter 3. Theorem 4.8 is a fundamental robust stability result generalising the operator based robust stability theorem of [Georgiou and Smith, 1997b] to include the case of a general initial condition within, in particular, the nonlinear gap formalism of [Georgiou and Smith, 1997b]; this also includes a suitable generalisation of the nonlinear gap metric. Theorem 4.8 can also be viewed as a generalisation of the ISS approach to enable a realistic treatment of robust stability in the context of perturbations which fundamentally change the structure of the state space. Theorem 4.8 is presented in two different versions: one requires the well-posedness of the perturbed closed-loop system, which is a typical assumption in the classical literature; while the other one requires only the uniqueness property of the perturbed closed-loop system, which significantly eases the real-time application of the robust stability result. We remark that in the second case the existence property of the perturbed closed-loop system is established via a type of (Schauder) fixed-point theorem, one of the most important *existence principles* in mathematics. Several technical assumptions are imposed in order to use the Schauder fixed-point theorem, such as a compactness requirement for the plant perturbations and a relative continuity requirement for the nominal closed-loop system. These stronger technical requirements on the plant perturbations and the nominal closed-loop system in turn result in substantially weaker requirements on the perturbed closed-loop system, i.e., the uniqueness property of the perturbed closed-loop system, which is often far easier to be verified than the existence property. This strategy dealing with the existence issue in robust stability analysis first appeared in French and Bian [2012] to establish a bias version of robust stability result. Theorem 4.12 on page 89 in Section 4.3 discusses the relation between Theorem 4.8 and [Georgiou and Smith, 1997b, Theorem 1]. In Section 4.4, a notion of finite-time reachability for a system is defined, and a more applicable robust stability result than Theorem 4.8 in this framework is established (see Theorem 4.18 on page 92). Applications of the main results (Theorems 4.8, 4.18) to linear time-invariant systems for both finite-time reachable situation and general situation are given in Section 4.5 on page 93. Application of Theorem 4.8 to general nonlinear plants with input delay is given in Section 4.6 on page 97. At the end of this chapter, a generalisation of the results from previous sections is given for systems with potential for finite escape times. This is done by using a wider signal space (named ambient space) than the extended space, which is defined in Section 3.2 on page 39 in Chapter 3. Definitions of systems, closed-loop systems, initial conditions, causality, existence and uniqueness properties are all slightly modified in this setting. A suitable notion of locally input to output stability is given by Definition 4.28 on page 103. Similarly, several equivalent characterisation of this notion of stability are summarised in Theorem 4.31 on page 104. The main result of this section is given by Theorem 4.33 on page 105, which are also presented in two different frameworks: one requires the well-posedness of the perturbed closed-loop system; while the other one requires only the uniqueness property of the perturbed closed-loop system. The work in this chapter has been submitted for publication in [Liu and French, 2014d].

- **Chapter 5** (pages 109–118) In this chapter we consider the development of a general nonlinear ISS-type small-gain theorem based on the input/output framework set up in Chapter 3. The main result in this chapter is Theorem 5.2 on page 112, which is established without extra "observability" conditions and with complete disconnection between the stability property and the existence, uniqueness properties. The main idea of the proof of Theorem 5.2 is motivated by [Jiang et al., 1994]. On one hand this small-gain result can be reviewed as a generalisation of the classical input/output operator type small-gain theorems to incorporate abstract initial conditions, and on the other hand a generalisation of the ISS/IOS framework type small-gain theorems to incorporate more general system classes. An illustrative example is given for systems with time delay and nonzero initial conditions to show the utility of Theorem 5.2 at the end of this chapter (Example 5.3 on page 116). The work in this chapter has been submitted for publication as [Liu and French, 2013].

- **Chapter 6** (pages 119–137) In this chapter, we discuss the connections between Georgiou and Smith's robust stability type theorems and the nonlinear small-gain theorems. Three versions of the nonlinear small-gain theorem are discussed in this chapter. The first version is the usual one regarding systems as relations (one-to-many mapping) on signal spaces and using $\mathcal{K}_\infty$ functions, in which the stability property is stated without referring to the existence and uniqueness properties of the corresponding feedback systems. A special case of this result (feedback systems with parts of zero input disturbances) is shown to be equivalent to a fundamental robust stability theorem of Georgiou and Smith [Georgiou and Smith, 1997b, Theorem 6] with a slight modification (Theorem 6.6 (on page 127) and Theorem 6.7 (on page 128) in Section 6.2). The second version of the nonlinear small-gain theorem establishes the existence and boundedness properties simultaneously, which increases greatly its applicability. However, an extra compact condition is imposed due to the use of Schauder's fixed point theorem in the proof. A type of Georgiou and Smith's robust stability theorem establishing boundedness and existence simultaneously is given by applying a special case of the second version of the nonlinear small-gain theorem (see Section 6.3 on page 129). The third one is a local version of the nonlinear small-gain theorem also establishing the existence and boundedness properties simultaneously by still using the Schauder's fixed point theorem, which is used to show a corresponding local version of Georgiou and Smith's robust stability theorem (see Section 6.4 on page 134). The work in this chapter has been submitted for publication in [Liu and French, 2014a].

- **Chapter 7** (pages 139–141) The last chapter contains conclusions and future directions of research.

Riemann has shown us that proofs
are better achieved through ideas
than through long calculations.

David Hilbert (1862-1943)

Chapter 2

Preliminaries

This chapter is to collect some mathematical preliminaries which will be used in the rest of this thesis. We present here some basic concepts such as sets, (nonlinear) operators, metric spaces, normed vector spaces, equivalence relations, partitions, and classes $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions. A type of Schauder fixed-point theorem for nonlinear operators is reviewed due to the requirement of establishing properties of existence and boundedness simultaneously for a closed-loop system in Chapters 4 and 6. Nerode equivalence for scalar continuous-time transfer functions is discussed in Section 2.5 on page 21, which gives a key insight to the abstract construction of initial conditions in this thesis (see Section 3.5.1 on page 53). Input-to-state stability and input-to-output stability in state space models are also reviewed.

2.1 Sets, Operators, Metric Spaces, and Vector Spaces

A *set* is a collection of objects which are called *elements* or *members* or *points* of the set. Two sets are equal if and only if they have the same elements. If x is an element of a set A, we write $x \in A$. If every element of set A is also a member of set B, then A is said to be a *subset* of B, written $A \subseteq B$ (or $B \supseteq A$). We denote by $\emptyset$ the *empty set* which has no elements, and thus the empty set $\emptyset$ is a subset of every set. Every set is a subset of itself. The following are several fundamental operations for constructing new sets from given sets: (1) The *union* of set A and set B: $A \cup B \triangleq \{x \mid x \in A \text{ or } x \in B\}$; (2) The *intersection* of set A and set B: $A \cap B \triangleq \{x \mid x \in A \text{ and } x \in B\}$; (3) The *difference set* $A - B$ of set A and set B: $A - B \triangleq \{x \mid x \in A \text{ and } x \notin B\}$; (4) The *Cartesian product* of set A and set B: $A \times B \triangleq \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Given two sets X and Y, an *operator* from X to Y is a set F of ordered pairs in the Cartesian product $X \times Y$ such that the following property is satisfied:

$$(x, y),\ (x, z) \in F \;\Rightarrow\; y = z. \tag{2.1}$$

The set of all elements of X that can occur as first items of elements in F is called the *domain* of F, denoted by $\mathrm{dom}(F)$. We know from (2.1) that, for any element $x \in \mathrm{dom}(F)$, there exists one and only one element in Y, which we call the image of x under F, written Fx or $F(x)$. The image of any set $A \subseteq \mathrm{dom}(F)$ under F is denoted by $F(A) \triangleq \{F(x) : x \in A\}$. The set of all images of elements in $\mathrm{dom}(F)$ is called the *range* of F. We often write the operator F as

$$F : \mathrm{dom}(F) \subseteq X \to Y;$$

and for ease of notation, if $\mathrm{dom}(F) = X$, we just write $F : X \to Y$.

An operator is sometimes also called a function, a map, a mapping, or a transformation. Basic results and properties on (nonlinear) operators defined on (normed) vector spaces will be reviewed in later sections. We first introduce the concept of metric spaces which is a natural generalisation of the idea of distance between two locations in the real world.

Definition 2.1. A *metric space*, denoted by (X, d), is a set X with a *metric* or *distance function* $d : X \times X \to \mathbb{R}$ such that, for any $x, y, z \in X$ the following axioms hold: (1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$; (3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

We say that a sequence $\{y_n\}$ in a metric space (X, d) *converges* to $y \in X$ if $d(y_n, y) \to 0$ as $n \to \infty$. It can be easily verified that this limit is unique. A sequence $\{x_n\} \subseteq X$ is called a *Cauchy sequence* if the *Cauchy condition* is satisfied, i.e., $d(x_n, x_m) \to 0$ as $m, n \to \infty$. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X has a limit in X, i.e., for any sequence $\{x_n\} \subseteq X$ satisfying $d(x_n, x_m) \to 0$ as $m, n \to \infty$, there exists an $x \in X$ such that $\{x_n\}$ converges to x.

The concept of *vector spaces* is also needed in order to provide the space with certain algebraic structure, e.g., operations of element addition and scalar multiplication.

Definition 2.2. A *vector space* over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) is a set X together with two operations: the addition operation $+ : X \times X \to X$ and the scalar multiplication operation $\cdot : \mathbb{K} \times X \to X$, such that, for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{K}$ we have: (1) $x + y = y + x$ (commutativity of addition); (2) $x + (y + z) = (x + y) + z$ (associativity of addition); (3) $\exists\, 0 \in X$ such that $x + 0 = 0 + x = x$ (existence of additive identity); (4) $\exists - x \in X$ such that $x + (-x) = 0$ (existence of additive inverse); (5) $\exists\, 1 \in \mathbb{K}$ such that $1 \cdot x = x$; (6) $\lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$ (associativity of scalar multiplication); (7) $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$ (first distributive property); (8) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$ (second distributive property).

The vector space X is called a real (or complex) vector space if it is over the field $\mathbb{R}$ (or $\mathbb{C}$). Alternative names for vector spaces are *linear vector spaces* and *linear spaces*.

Let X be a vector space. A finite set $\{x_1, \ldots, x_n\} \subseteq X$ is said to be *linearly dependent* if there is a set of scalars $\{\lambda_1, \ldots, \lambda_n\}$, not all zero, such that $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$. On the other hand, if $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$ implies that $\lambda_k = 0$ for each $k = 1, \ldots, n$, the set $\{x_1, \ldots, x_n\}$ is said to be *linearly independent*. We say that a set (finite or infinite) of vectors $B \subseteq X$ is a *basis* of X if (1) every finite subset $B_0 \subseteq B$ is linearly independent; (2) and every vectors in X is a linear combination of finite elements in B. A vector space that has a finite basis is called *finite-dimensional*.

A subset M of a vector space X over the field $\mathbb{R}$ (or $\mathbb{C}$)is said to be a *(vector) subspace* of X if M satisfies the following two conditions: (1) If $x, y \in M$ then $x + y \in M$; (2) If $x \in M$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) then $\lambda \cdot x \in M$. Loosely speaking, M is a subspace of X if it is a vector space in its own right. Any vector space is a subspace of itself.

Given sets X, Y, and Z, an operator $F : X \to Y$ is called *injective* if for any $x_1, x_2 \in X$ with $Fx_1 = Fx_2$ we have $x_1 = x_2$; and *surjective* if the range of F is the whole of Y. It is called *bijective* if it is both injective and surjective. Let X_0 be a subset of X, then the operator $F_0 : X_0 \to Y$ defined by $F_0 x = Fx$ for every $x \in X_0$ is called the *restriction* of F to X_0 (often denoted by $F|_{X_0}$). On the other hand, an operator $F : X \to Y$ coinciding with F_0 on $X_0 \subseteq X$ is called an *extension* of F_0. The composition of operators $F_2 : Y \to Z$ and $F_1 : X \to Y$ is the operator $F_2 \circ F_1 : X \to Z$ defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for all $x \in X$.

Let $F_1 : X \to Y$ and $F_2 : X \to Y$ be two operators between two real (or complex) vector spaces X and Y, then the addition $F_1 + F_2 : X \to Y$ is an operator defined by $(F_1 + F_2)(x) = F_1(x) + F_2(x)$ for all $x \in X$. Let λ be a real (or complex) constant. Then $\lambda F_1 : X \to Y$ is an operator defined by $(\lambda F_1)(x) = \lambda \cdot F_1(x)$ for all $x \in X$.

Assume that X is a vector space, and G, H and K are three operators from X to X. Then we know that addition and composition always have the right distributive property $(G + H) \circ K = G \circ K + H \circ K$, but not necessarily to have the left distributive property $K \circ (G + H) = K \circ G + K \circ H$ unless K is *linear*.

Note that an operator $F : \mathrm{dom}(F) \subseteq X_1 \to X_2$ between two vector spaces X_1 and X_2 over the field $\mathbb{R}$ (or $\mathbb{C}$) is said to be *linear* if $\mathrm{dom}(F)$ is a vector subspace of X_1 and $F(\alpha x + \beta y) = \alpha(Fx) + \beta(Fy)$ for all $x, y \in X_1$ and scalars $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$).

We next introduce the concepts of *normed vector spaces* and *Banach spaces*. The following notion of norm generalises the absolute value of numbers.

Definition 2.3. A *normed vector space* is a pair $(X, \|\cdot\|)$, where X is a vector space over the field $\mathbb{R}$ (or $\mathbb{C}$) and $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ is a real-valued function defined on X such that: (1) $0 \leq \|x\| < \infty, \forall x \in X$; $\|x\| = 0$ if and only if $x = 0$; (2) $\|\lambda x\| = |\lambda| \, \|x\|$, $\forall x \in X, \forall \lambda \in \mathbb{R}$ (or $\mathbb{C}$); (3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$ (triangle inequality).

Note that every normed vector space $(X, \|\cdot\|)$ can be regarded as as a metric space with the natural distance defined by $d(x, y) = \|x - y\|$ for any x, y in X. A normed vector space $(X, \|\cdot\|)$ is said to be a *Banach space* if the corresponding metric space with the natural distance is complete, i.e., for any Cauchy sequence $\{x_n\} \subseteq X$, there exist an $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

The set $\mathbb{R}^n$ consisting of all n-tuples of real numbers over the field $\mathbb{R}$ is a real vector space if we define addition "+" by component-wise addition, i.e., $x + y = (x_1 + y_1, \cdots, x_n + y_n)^T$, and scalar multiplication "·" by component-wise scalar multiplication, i.e., $\lambda \cdot x = (\lambda x_1, \cdots, \lambda x_2)^T$ for any $x = (x_1, \cdots, x_n)^T$, $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}$. Further, for any q with $1 \leq q \leq \infty$, the real-valued function $|\cdot|_q$ known as the q-norm [1] in $\mathbb{R}^n$ makes this vector space a Banach space, where $|x|_q \triangleq (|x_1|^q + \cdots + |x_n|^q)^{1/q}$ if $1 \leq q < \infty$; and $|x|_\infty \triangleq \max_{1 \leq i \leq n} |x_i|$ if $q = \infty$. The 2-norm in $\mathbb{R}^n$ is often called *Euclidean norm*;

Let $X := C([0, 1], \mathbb{R})$ be the space of all continuous functions $u : [0, 1] \to \mathbb{R}$ with norm $\|u\|_2 := (\int_0^1 |x(t)|^2 \, dt)^{1/2}$. It can be verified that $(X, \|\cdot\|_2)$ is a normed vector space but not a Banach space. A Cauchy sequence $\{u_n\}_{n=1}^\infty$ in X which is not convergent in X can be found in e.g., [Curtain and Zwart, 1995, Example A.2.19, p. 574]. In fact, the *completion* of X with respect to the norm $\|\cdot\|_2$ is the Lebesgue space of 2-integrable functions $L^2([0, 1], \mathbb{R})$.

The following is the concept of *isometric isomorphism* between two normed vector spaces (see e.g., [Zeidler, 1986, p. 771], [Adams and Fournier, 2003, p. 5]).

Definition 2.4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. An operator $F : X \to Y$ is called an *isometric isomorphism* between X and Y if F is a bijective linear operator such that

$$\|Fx\|_Y = \|x\|_X \quad \text{for all } x \in X.$$

We say that X and Y are *isometrically isomorphic* if there exists an isometric isomorphism F between X and Y. Isometrically isomorphic normed vector spaces can be identified with each other, since they have identical structure and only differ in the nature of their elements.

For example, *Fourier transform* $\mathfrak{F}$ is an isometric isomorphism between the time domain signal space $\mathcal{L}_2(\mathbb{R})$ to the frequency domain signal space $\mathcal{L}_2(j\mathbb{R})$ (i.e., Paley-Wiener theorem, see Section 2.5).

If X is a normed vector space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), then there always exists a Banach space Y over $\mathbb{K}$ with $X \subseteq Y$ and $Cl_Y(X) = Y$, where $Cl_Y(X)$ is the closure of X in Y; see Definition 2.6 on page 15. This Y is unique up to isometric isomorphism (see

[1] We often use the symbol $|\cdot|_q$ instead of $\|\cdot\|_q$ to denote the q-norm in $\mathbb{R}^n$.

e.g., [Zeidler, 1986, p. 771]). The standard construction of the completion is through Cauchy sequences (see e.g., [Kato, 1995, p. 129]).

2.2 Continuity, Boundedness, and Compactness for Nonlinear Operators

Now consider a nonlinear operator F between two Banach spaces. The concepts of continuity, boundedness, and compactness of F will be used in later sections. Some of the results are quoted without proof; and usually the detailed proofs of the results can be found in standard textbooks on nonlinear functional analysis (see e.g., [Zeidler, 1986, Appendix], [Adams and Fournier, 2003, Chapter 1], [Deimling, 1985, Chapter 2]). Counterexamples in analysis can be found in e.g., [Gelbaum and Olmsted, 2003].

Definition 2.5. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. An operator (possibly nonlinear) $F : \text{dom}(F) \subseteq X \to Y$ is said to be *continuous at* $x_0 \in \text{dom}(F)$ if any sequence $\{x_n\}_{n=1}^{\infty} \subseteq \text{dom}(F)$ with $\|x_n - x_0\| \to 0$ implies $\|Fx_n - Fx_0\| \to 0$ as $n \to \infty$. The operator F is said to be *continuous* if it is continuous everywhere in its domain. The operator F is said to be *Cauchy continuous* if, given any Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subseteq \text{dom}(F)$ in X, the sequence $\{Fx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. The operator F is said to be *uniformly continuous* if for every pair of sequences $\{x_n\}_{n=1}^{\infty} \subseteq \text{dom}(F)$ and $\{y_n\}_{n=1}^{\infty} \subseteq \text{dom}(F)$ in X such that $\|x_n - y_n\| \to 0$ as $n \to \infty$, then $\|Fx_n - Fy_n\| \to 0$ as $n \to \infty$.

That F is continuous at $x_0 \in \text{dom}(F)$ is equivalent to: for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|Fx - Fx_0\| < \varepsilon$ for all $x \in \text{dom}(F)$ satisfying $\|x - x_0\| < \delta$, see e.g., [Zeidler, 1995, Proposition 1.9.3, p. 27], [Zeidler, 1986, p. 770]. Similarly, that F is uniformly continuous is equivalent to: for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|Fx - Fy\| < \varepsilon$ for all $x, y \in \text{dom}(F)$ satisfying $\|x - y\| < \delta$. Note that uniformly continuous implies Cauchy continuous, and Cauchy continuous implies continuous.[2] Conversely, if X is complete and $\text{dom}(F)$ is closed in X, then continuous implies Cauchy continuous too.

Next we give a brief view on some topological notions, including *bounded, open, closed, (relatively) compact, and convex sets*, associated with a Banach space.

Definition 2.6. Let M be a subset of a Banach space $(X, \|\cdot\|)$. The set M is called *bounded* if and only if there is a number $r \geq 0$ such that $\|u\| \leq r$ for all $u \in M$. The set M is called *open* if and only if, for each $x_0 \in M$, there exists $\delta > 0$ such that all $x \in X$ satisfying $\|x - x_0\| < \delta$ will also belong to M. The set M is called *closed* if and only if, the difference set $X - M$ is open. (This is equivalent to the condition that for every sequence $\{x_n\}_{n=1}^{\infty}$ in M and $x \in X$ with $x_n \to x$ as $n \to \infty$, the limit x also belongs

[2] It suffices to notice that, if $\|x_n - x\| \to 0$ as $n \to \infty$, then the sequence $\{x_1, x, x_2, x, x_3, x, ...\}$ is a Cauchy sequence.

to M.[3]) The *closure* of M in X, denoted by $Cl_X(M)$, is the smallest closed set in X containing M (i.e., the intersection of all closed sets in X containing M). The set M is called *relatively compact* if every sequence in M contains a convergent subsequence. If the limit of this subsequence always belongs to M, then M is said to be *compact*.[4] Further, the set M is called *convex* if and only if, any $u, v \in M$ and any $\alpha \in [0, 1]$ imply $\alpha u + (1 - \alpha)v \in M$.

Suppose that M is a subset of a Banach space $(X, \|\cdot\|)$. The set M is relatively compact if and only if the closure $Cl_X(M)$ of M in X is compact. If M is compact, then it is closed and bounded. If M is relatively compact, then it is bounded. If every closed and bounded subset of the Banach space X is compact (i.e., the Heine–Borel property), then X must be finite dimensional (see e.g., [Deimling, 1985, p. 40]).[5] If M is compact and $N \subset M \subseteq X$ is closed, then N is also compact. (In other words, a closed subset of a compact set of a Banach space is compact.)

The well-known Arzelà-Ascoli theorem (see e.g., [Zeidler, 1986, p. 772], [Adams and Fournier, 2003, p. 11]) states that if M is a *bounded*[6] and *equicontinuous*[7] subset of the space $X := C([a, b], \mathbb{K})$, $(\mathbb{K} := \mathbb{R}, \mathbb{C})$ of all continuous functions $u : [a, b] \to \mathbb{K}$ with norm $\|u\|_\infty := \max_{a \le \tau \le b} |u(\tau)|$, then M is a relatively compact subset of X.

Definition 2.7. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed vector spaces. An operator (possibly nonlinear) $F : \mathrm{dom}(F) \subseteq X \to Y$ is said to be *bounded* if the image of any bounded set in $\mathrm{dom}(F)$ is a bounded set in Y. (That is to say, for any $r_1 > 0$, there exists an $r_2 > 0$ such that $x \in \mathrm{dom}(F)$ and $\|x\|_X \le r_1$ imply $\|Fx\|_Y \le r_2$.)

The above condition for bounded operator (possibly nonlinear) is equivalent to the condition that the image of any bounded sequence in $\mathrm{dom}(F)$ is a bounded sequence in Y. (Firstly, a bounded sequence is a bounded set. Secondly, if there exists a bounded set $M \subseteq \mathrm{dom}(F)$ such that $F(M)$ is not bounded in Y, then for any natural number $n > 0$, there exists some $x_n \in M$ such that $\|Fx_n\| \ge n$, and therefore, F transforms a bounded sequence $\{x_n\}$ into an unbounded sequence $\{Fx_n\}$.)

Definition 2.7 of a bounded nonlinear operator generalises the definition of a bounded linear operator: A linear operator $F : \mathrm{dom}(F) \subseteq X \to Y$ between normed vector spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ with $\mathrm{dom}(F)$ a subspace of X is said to be *bounded* if there exists a nonnegative constant r such that $\|Fx\|_Y \le r \cdot \|x\|_X$ for all $x \in \mathrm{dom}(F) \subseteq X$. For linear operators, both definitions of boundedness are equivalent to each other.

[3] i.e., M contains all of its limit points.

[4] This definition of a compact subset is equivalent in Banach spaces to the definition of compactness in a general topological space: M is compact if each of its open covers has a finite subcover (see e.g., [Adams and Fournier, 2003, p. 7], [Deimling, 1985, p. 40]).

[5] Note that the classical Heine–Borel theorem states that a subset of Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

[6] i.e., $\|u\|_\infty \le r$ for all $u \in M$ and fixed $r \ge 0$.

[7] i.e., by definition, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|u(t) - u(\tau)| < \varepsilon$ for all $u \in M$ and $|t - \tau| < \delta$.

Note that a linear operator is continuous if and only if the linear operator is continuous at 0 if and only if the linear operator is bounded (see e.g., [Kato, 1995, p. 145]). A discontinuous and unbounded linear operator can be found in [Gelbaum and Olmsted, 2003, p. 33]. The following nonlinear operator $F_1 : \mathbb{R} \to \mathbb{R}$ defined by (see e.g., [Gelbaum and Olmsted, 2003, p. 22])

$$F_1(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ -x, & \text{if } x \text{ is irrational,} \end{cases}$$

is continuous at the point $x = 0$ only. The nonlinear operator $F_2 : (0, 1) \subseteq \mathbb{R} \to \mathbb{R}$ defined by $F_2(x) = 1/x$ is a continuous but unbounded operator. In fact, for any infinite dimensional Banach space X, there exists a continuous but unbounded nonlinear operator $F_3 : X \to \mathbb{R}$ defined on the whole domain X (see e.g., [Deimling, 1985, Example 2.8.1, p. 55]). The following nonlinear operator $F_4 : \mathbb{R} \to \mathbb{R}$ defined by

$$F_4(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ -1, & \text{if } x \text{ is irrational,} \end{cases} \tag{2.2}$$

is a bounded (even compact) but discontinuous (discontinuous at any point in $\mathbb{R}$) operator.

An important class of nonlinear operators between Banach spaces is the set of *compact operators* which appear in many applications. A compact operator is an operator which transforms bounded sets in the definition of domain into relatively compact sets. The notion of a compact operator plays an essential role in the theory of fixed points of a nonlinear operator (e.g., the Schauder fixed-point theorem in the next section).

Definition 2.8. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. An operator (possibly nonlinear) $F : \text{dom}(F) \subseteq X \to Y$ is said to be *compact* if the image $F(M)$ is relatively compact in Y whenever $M \subseteq \text{dom}(F)$ is bounded.[8]

Let $F : \text{dom}(F) \subseteq X \to Y$ be an operator (possibly nonlinear) between Banach spaces X and Y. If F is compact, then it is also bounded. Recall that a linear operator is bounded if and only if it is continuous. Therefore, every compact linear operator F with $\text{dom}(F)$ a vector subspace of X is also bounded and continuous. However, for a nonlinear operator, compactness does not in general imply continuity (e.g., the nonlinear operator $F_4 : \mathbb{R} \to \mathbb{R}$ given by (2.2)). Let F be a linear operator with $\text{dom}(F)$ a vector subspace of X. If F is *finite-dimensional*,[9] then it is automatically compact (see e.g., [Deimling, 1985, p. 55]). The identity operator $I : X \to X$ on the Banach space X is a compact operator if and only if X is a finite-dimensional vector space (Riesz's Lemma),

[8] This is equivalent to the condition that the image $\{Fx_n\}$ of any bounded sequence $\{x_n\}$ of $\text{dom}(F)$ contains a Cauchy subsequence, see e.g., [Zeidler, 1986, Appendix].

[9] Note that F is said to be *finite-dimensional* if the range $F(\text{dom}(F))$ is contained in a finite-dimensional subspace of Y.

see e.g., [Rynne and Youngson, 2008, p. 47], [Deimling, 1985, p. 40]. Other properties for linear compact operators can be found in e.g., [Curtain and Zwart, 1995, Lemma A.3.22, p. 587].

We conclude this section by reviewing some standard (nonlinear) compact operators in nonlinear functional analysis.

Example 2.9. [Compact and continuous nonlinear integral operators] *Let us consider the integral operators* $F_1 : u \mapsto F_1 u$ *and* $F_2 : u \mapsto F_2 u$ *defined by*

$$(F_1 u)(t) := \int_a^b K\left(t, \tau, u(\tau)\right) \mathrm{d}\tau \quad \text{for all } t \in [a, b],$$
$$(F_2 u)(t) := \int_a^t K\left(t, \tau, u(\tau)\right) \mathrm{d}\tau \quad \text{for all } t \in [a, b],$$

where $-\infty < a < b < \infty$*. If* $K(t, \tau, x)$ *is nonlinear in* x*, then* F_1, F_2 *are usually called nonlinear Urysohn operators. Suppose that we have a continuous function*

$$K : [a, b] \times [a, b] \times [-r, r] \to \mathbb{R},$$

where $0 < r < \infty$*. Set* $X := C([a, b], \mathbb{R})$ *and*

$$M := \left\{u \in X : \|u\|_\infty \leq r\right\},$$

where $\|u\|_\infty := \max_{a \leq \tau \leq b} |u(\tau)|$ *and* $C([a, b], \mathbb{R})$ *is the space of continuous maps* $u : [a, b] \to \mathbb{R}$*. Then the integral operators* $F_1 : M \subseteq X \to X$ *and* $F_2 : M \subseteq X \to X$ *are continuous and compact.*[10] *Nonlinear Urysohn operators acts on some other function spaces (e.g.,* $L^q([a, b], \mathbb{R})$*,* $1 \leq q \leq \infty$*) under suitable restrictions on the function* $K(t, s, x)$ *are also continuous and compact.*

A very important class of linear continuous and compact operator is the set of Fredholm integral operators.

Example 2.10. [Fredholm integral operators] *Consider the linear integral operator*

$$(Fu)(t) := \int_a^b K(t, \tau) u(\tau)\, \mathrm{d}\tau \quad \text{for all } t \in [a, b],$$

where $-\infty < a < b < \infty$*. Set* $X_1 := C([a, b], \mathbb{R})$*,* $X_2 := L^2([a, b], \mathbb{R})$*, and* $X_3 := L^\infty([a, b], \mathbb{R})$*. The linear integral operator* $F : X_1 \to X_1$ *is continuous and compact if* K *is continuous on* $[a, b] \times [a, b]$*. The linear integral operator* $F : X_2 \to X_2$ *is continuous*

[10] That the integral operators F_1 and F_2 are indeed compact follows from the well-known Arzelà-Ascoli theorem (see e.g., [Zeidler, 1986, pp. 54, 772]).

and compact if K *is* Lebesgue measurable[11] *on* $[a, b] \times [a, b]$ *and if*

$$\int_a^b \int_a^b |K(t,\tau)|^2 \, \mathrm{d}t \, \mathrm{d}\tau < \infty.$$

The linear integral operator $F : X_3 \to X_3$ *is continuous and compact if* K *is Lebesgue measurable on* $[a, b] \times [a, b]$ *and if*

$$\operatorname*{ess\,sup}_{a \le t, \tau \le b} |K(t,\tau)| < \infty.$$

Note that linear systems with strictly proper transfer functions define such linear continuous and compact operators (see e.g., [Georgiou and Smith, 1997b, Proposition 4]). The linear integral operator $F : X_i \to X_i$ *with* $i = 1, 2, 3$ *is called a* Fredholm integral operator *in* X_i *(see e.g., [Rynne and Youngson, 2008, Chapter 8]). A special type of Fredholm integral operator is the so-called* Volterra integral operator *having the form:*

$$(Fu)(t) := \int_a^t K(t,\tau)u(\tau) \, \mathrm{d}\tau \quad \textit{for all } t \in [a, b],$$

where $-\infty < a < b < \infty$*, and the upper limit of the integral in the definition of* F *is variable.*

2.3 The Schauder Fixed-Point Theorem

The following Schauder fixed-point theorem which is a well known result in the literature will be used in the proof of the main Theorem 4.8 in Chapter 4 on page 81.

Lemma 2.11. *(Schauder Fixed-Point Theorem (1930)). Let* $\mathcal{M}$ *be a nonempty, closed, bounded, convex subset of a Banach space* $\mathcal{X}$ *(i.e.,* $\mathcal{X}$ *is a complete normed vector space), and suppose* $T : \mathcal{M} \to \mathcal{M}$ *is a continuous and compact operator (possibly nonlinear). Then* T *has a fixed point (i.e., there exists* $x \in \mathcal{M}$ *such that* $Tx = x$*).*

Proof. The proof of this theorem can be found in many nonlinear functional analysis books, see e.g., [Zeidler, 1986, p. 56] or [Zeidler, 1995, p. 61]. Note that the definition of compactness in these two references already requires the operator being continuous. We also remark that the completeness of the normed vector space $\mathcal{X}$ is also important here, although any normed vector space $\tilde{\mathcal{Y}}$ can be *completed* (see e.g., [Zeidler, 1986, p. 771], [Kato, 1995, p. 129]) by a Banach space $\mathcal{Y}$ which contains $\tilde{\mathcal{Y}}$. Because a set which is closed in $\tilde{\mathcal{Y}}$ is not necessarily closed in $\mathcal{Y}$. □

The following well-known facts will be useful for the application of the Schauder fixed-point theorem in proving Theorem 4.8 on page 81.

[11] A brief review of the Lebesgue measure theory can be found in e.g., [Adams and Fournier, 2003, pp. 13–19] or [Tao, 2011, Chapter 1]

Lemma 2.12. *Let $(X, \|\cdot\|)$ be a normed vector space, and let $x_0 \in X$ and $0 \le r < \infty$ be given. Then the set $B_r(x_0) \triangleq \{x \in X : \|x - x_0\| \le r\}$ is nonempty, bounded, closed and convex.*

Proof. Note that $x_0 \in B_r(x_0)$ and $\|x\| \le \|x_0\| + r$ for any $x \in B_r(x_0)$. This implies that the set $B_r(x_0)$ is nonempty and bounded. We next show that $B_r(x_0)$ is closed. To this end, let y_0 be an element of the set $B_r^c(x_0)$, the complement of $B_r(x_0)$, i.e., $y_0 \in X$ and $\|y_0 - x_0\| > r$. Define $\delta \triangleq (\|y_0 - x_0\| - r)/2$, it follows easily from the triangle inequality of norm that for any $y \in X$ with $\|y - y_0\| \le \delta$ we have $\|y - x_0\| \ge \|y_0 - x_0\| - \|y - y_0\| \ge (\|y_0 - x_0\| + r)/2 > r$, i.e., $y \in B_r^c(x_0)$. This implies that $B_r^c(x_0)$ is open, and hence $B_r(x_0)$ is closed. The convexity of $B_r(x_0)$ can also be easily shown by using the triangle inequality of norm and the definition of convexity, [12] see e.g., [Zeidler, 1995, p. 29]. □

The following lemma shows that the composition of a compact operator and a bounded operator is compact (thus also bounded).

Lemma 2.13. *Let X, Y, Z be Banach spaces and $B : \mathrm{dom}(B) \subseteq X \to Y$ be a bounded operator (possibly nonlinear) and $C : \mathrm{dom}(C) \subseteq Y \to Z$ be a compact operator (possibly nonlinear) with $B(\mathrm{dom}(B)) \subseteq \mathrm{dom}(C)$. Then the composition operator $C \circ B : \mathrm{dom}(B) \subseteq X \to Z$ is compact.*

Proof. The proof is similar to the proof for linear operators [Kato, 1995, Theorem III-4.8, p. 158]. Let $\{x_n\}$ be a bounded sequence in $\mathrm{dom}(B)$. Then $\{Bx_n\}$ is bounded in $\mathrm{dom}(C)$ and therefore contains a subsequence $\{Bx'_n\}$ such that $\{C(Bx'_n)\}$ is a Cauchy sequence. This shows that $C \circ B : \mathrm{dom}(B) \subseteq X \to Z$ is compact. □

Recall that a continuous (possibly nonlinear) operator transforms compact (resp., bounded, relatively compact) sets into compact (resp., bounded, relatively compact) sets (see e.g., [Zeidler, 1986, p. 756]). Therefore, if T is a compact operator and S is a continuous operator, then both $T \circ S$ and $S \circ T$ are compact operators. But these results are not needed in this thesis.

2.4 Equivalence Relations and Partitions

An important notion when defining the 'state' of a purely input-output system in this thesis is that of *equivalence relation*.

Definition 2.14. Let X be a nonempty set. A *binary relation* R on X is a subset of the Cartesian product $X \times X$ (see e.g., [Vidyasagar, 1993]).

[12] Note that $\alpha x + (1-\alpha)y - x_0 = \alpha(x - x_0) + (1-\alpha)(y - x_0)$ for any $x, y \in X$ and any $0 \le \alpha \le 1$.

Suppose R is a binary relation on X, then we say that $x \in X$ is related to $y \in X$ if the ordered pair $(x, y) \in R$. Suppose $f : X \to X$ is a map, then f defines a binary relation R_f on X, namely $R_f \triangleq \{(x, f(x)) \mid x \in X\}$.

Definition 2.15. A given binary relation R on a set X is said to be an *equivalence relation* if and only if it is *reflexive*, *symmetric* and *transitive*. Equivalently, for all a, b and c in X: (1) $(a, a) \in R$ (reflexivity); (2) if $(a, b) \in R$ then $(b, a) \in R$ (symmetry); (3) if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ (transitivity). In this case, we denote by $\sim$ the equivalence relation R, and $(x, y) \in R$ by $x \sim y$.

The equivalence class of $a \in X$ under $\sim$, denoted by $[a]$, is defined as $[a] = \{b \in X : a \sim b\}$. The set of all possible equivalence classes of X by $\sim$, denoted by $X/\sim \triangleq \{[a] : a \in X\}$, is the quotient set of X by $\sim$. Note that, if X is a topological space, then there is a natural way of transforming $X/\sim$ into a topological space. The projection of $\sim$ in X is the function $\pi : X \to X/\sim$ defined by $\pi(x) = [a]$ which maps elements of X into their respective equivalence classes by $\sim$.

A notion directly related to *equivalence relation* is that of *partition* of a set.

Definition 2.16. Given any set X, let N be a collection of subsets of X. Then N is called a *partition* of X if, and only if, the empty set $\emptyset \notin N$ and $\bigcup_{A \in N}\{A\} = X$, and $A \cap B = \emptyset$ if $A \in N$, $B \in N$ with $A \neq B$.

Note that from any partition N of X we can define an equivalence relation on X by setting $x \sim y$ when x and y are in the same part of N. Conversely, for any equivalence relation on a set X, the set of its equivalence classes is a partition of X. Thus the notions of equivalence relation and partition are essentially equivalent.

2.5 Nerode Equivalence for Scalar Continuous-Time Transfer Functions

In [Nerode, 1958] the author introduced an abstract approach via *Nerode equivalence* (which originated in automata theory) to state space realisation methods. The essence of Nerode equivalence is that the state space can be identified with a set of equivalent classes of past input signals. This gives a key insight to the abstract construction of initial conditions in this thesis (see Section 3.5.1 on page 53). In this section, we shall show how to formalise this concept. For the construction of state maps in the context of Willems' behavioural theory, see e.g., [Fuhrmann et al., 2007, Rapisarda and Willems, 1997]. The concrete method used here is often cast in a more algebraic system theory, see e.g., [Fuhrmann, 1981, Chapter III], [Kalman et al., 1969, Chapter 10].

Nerode equivalence for the conceptually simpler scalar discrete-time transfer functions has been fully discussed previously in [Kailath, 1980, Section 5.1, p. 315] (see also Nerode construction for two-dimensional (2-D) linear filters defined by formal power series in two variables [Fornasini and Marchesini, 1976]). Here we consider Nerode equivalence for scalar continuous-time transfer functions. Consider the case in which the denominator polynomial of the transfer function associated with multiple repeated roots:

$$\hat{G}(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_0}{(s+p_1)^{r_1}(s+p_2)^{r_2}\ldots(s+p_m)^{r_m}} \tag{2.3}$$

for which $\sum_{i=1}^{m} r_i = n$ with $r_i \in \mathbb{N}_{>0}$, $b_j \in \mathbb{C}$, $p_i \in \mathbb{C}_0^+ \triangleq \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$[13] for any $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n-1$, and $p_l \neq p_k$ for $l \neq k$, and p_l $(l = 1, 2, \ldots, m)$ are not zeros of $\hat{G}$. The above transfer function can be rewritten by *partial fraction expansion* (see e.g., [Polderman and Willems, 1998, Theorem B.2.1, p. 417]) as follows:

$$\hat{G}(s) = \sum_{i=1}^{m}\sum_{j=1}^{r_i} \frac{a_{ij}}{(s+p_i)^j} \tag{2.4}$$

for which $a_{ij} \in \mathbb{C}$ with $a_{ir_i} \neq 0$ for any $j = 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, m$.

We first introduce some notations about the usual time and frequency domain signal spaces (see e.g., [Francis, 1987, Vinnicombe, 2001]). Let $\mathcal{L}_2(j\mathbb{R})$ (resp., $\mathcal{L}_2(\mathbb{R})$) denote the frequency domain (resp., time domain) space of all complex-valued signals (resp., real-valued signals) square integrable on the imaginary axis $j\mathbb{R}$ (resp., on the whole time domain $\mathbb{R}$). The time domain space $\mathcal{L}_2(\mathbb{R})$ is related to the frequency domain space $\mathcal{L}_2(j\mathbb{R})$ via *Fourier transform* denoted by $\mathfrak{F}$. Indeed, the Fourier transform $\mathfrak{F}$ is an *isometric isomorphism*[14] between $\mathcal{L}_2(\mathbb{R})$ and $\mathcal{L}_2(j\mathbb{R})$ (i.e., Paley-Wiener Theorem), and so $\mathcal{L}_2(j\mathbb{R}) \equiv \mathfrak{F}\mathcal{L}_2(\mathbb{R})$. In the time domain, we have the following decomposition $\mathcal{L}_2(\mathbb{R}) = \mathcal{L}_2(\mathbb{R}_+) \oplus \mathcal{L}_2(\mathbb{R}_-)$, where $\mathcal{L}_2(\mathbb{R}_+)$ (resp., $\mathcal{L}_2(\mathbb{R}_-)$) is the space of signals defined for positive (resp., negative) time and zero for negative (resp., positive) time. Similarly, we have in the frequency domain the decomposition $\mathcal{L}_2(j\mathbb{R}) = \mathcal{H}_2 \oplus \mathcal{H}_2^{\perp}$, where $\mathcal{H}_2$ (resp., $\mathcal{H}_2^{\perp}$) is the usual Hardy space of all signals in $\mathcal{L}_2(j\mathbb{R})$ which can be continued analytically into the open right-half (resp., open left-half) of the complex plane. Note that $\mathcal{H}_2$ (resp., $\mathcal{H}_2^{\perp}$) can also be regarded as the space of Fourier transforms of signals in $\mathcal{L}_2(\mathbb{R}_+)$ (resp., $\mathcal{L}_2(\mathbb{R}_-)$), i.e., $\mathcal{H}_2 \equiv \mathfrak{F}\mathcal{L}_2(\mathbb{R}_+)$ and $\mathcal{H}_2^{\perp} \equiv \mathfrak{F}\mathcal{L}_2(\mathbb{R}_-)$. The time domain signal spaces $\mathcal{L}_2(\mathbb{R})$, $\mathcal{L}_2(\mathbb{R}_+)$, $\mathcal{L}_2(\mathbb{R}_-)$ and the frequency domain signal spaces $\mathcal{L}_2(j\mathbb{R})$, $\mathcal{H}_2$, $\mathcal{H}_2^{\perp}$ are all Hilbert spaces endowed with the standard inner products.

Let $\mathcal{L}_\infty(j\mathbb{R})$ denote the standard frequency domain Lebesgue space of all complex-valued functions essentially bounded on the imaginary axis $j\mathbb{R}$ and let $\mathcal{H}_\infty$ (resp., $\mathcal{H}_\infty^-$) denote the standard Hardy space of all functions in $\mathcal{L}_\infty(j\mathbb{R})$ with analytic continuation in the open right-half (resp., open left-half) of the complex plane. Note that the frequency domain function spaces $\mathcal{L}_\infty(j\mathbb{R})$, $\mathcal{H}_\infty$ and $\mathcal{H}_\infty^-$ are all Banach spaces endowed with the standard norms. For the real rational case, $\mathcal{RL}_\infty(j\mathbb{R})$ denotes the subspace of $\mathcal{L}_\infty(j\mathbb{R})$

[13]This assumption can be relaxed, see Remark 2.18 below.
[14]see Definition 2.4 on page 14

whose elements are real rational functions; and similar for $\mathcal{RH}_\infty$ and $\mathcal{RH}_\infty^-$. Note that $\mathcal{RL}_\infty(j\mathbb{R})$ can also be regarded as the space of all continuous-time transfer functions which are real rational, proper, and without poles on the imaginary axis. Similarly, $\mathcal{RH}_\infty$ (resp., $\mathcal{RH}_\infty^-$) is identified with the space of all continuous-time transfer functions which are real rational, proper, and without poles in the closed right-half (resp., open left-half) of the complex plane.

If $\hat{F} \in \mathcal{L}_\infty(j\mathbb{R})$ and $\hat{u} \in \mathcal{L}_2(j\mathbb{R})$ then $\hat{F}\hat{u} \in \mathcal{L}_2(j\mathbb{R})$. Similarly, if $\hat{F} \in \mathcal{H}_\infty$ (resp., $\hat{F} \in \mathcal{H}_\infty^-$) and $\hat{u} \in \mathcal{H}_2$ (resp., $\hat{u} \in \mathcal{H}_2^\perp$) then $\hat{F}\hat{u} \in \mathcal{H}_2$ (resp., $\hat{F}\hat{u} \in \mathcal{H}_2^\perp$). In addition, $\hat{u}_1 \in \mathcal{H}_2$ (resp., $\hat{F}_1 \in \mathcal{H}_\infty$) if and only if $\hat{u}_2 \in \mathcal{H}_2^\perp$ (resp., $\hat{F}_2 \in \mathcal{H}_\infty^-$), where $\hat{u}_1(s) = \hat{u}_2(-s)$ (resp., $\hat{F}_1(s) = \hat{F}_2(-s)$) for $s \in \mathbb{C}$ (see e.g., [Curtain and Zwart, 1995, Theorems A.6.22 and A.6.26, pp. 645 and 647]).

For the construction of Nerode equivalence for scalar continuous-time transfer functions, we need the following notion of *Hankel operator* of the corresponding transfer function.

Definition 2.17. For any continuous-time transfer function $\hat{F} \in \mathcal{L}_\infty(j\mathbb{R})$, we define the *Hankel operator with symbol* $\hat{F}$ as the operator $H_{\hat{F}} : \mathcal{H}_2^\perp \to \mathcal{H}_2$ given by

$$H_{\hat{F}}\hat{u} = \Pi_+ M_{\hat{F}} \Pi_- \hat{u} = \Pi_+ M_{\hat{F}} \hat{u}, \quad \forall\, \hat{u} \in \mathcal{H}_2^\perp$$

where Π_+ (resp., Π_-) is the orthogonal projection operator from $\mathcal{L}_2(j\mathbb{R})$ onto $\mathcal{H}_2$ (resp., $\mathcal{H}_2^\perp$); and $M_{\hat{F}} : \mathcal{L}_2(j\mathbb{R}) \to \mathcal{L}_2(j\mathbb{R})$ is the *multiplication operator with symbol* $\hat{F}$ given by $M_{\hat{F}}\hat{g} = \hat{F}\hat{g}$ for any $\hat{g} \in \mathcal{L}_2(j\mathbb{R})$.

Note that the Hankel operator defined here is slightly different from the one given in some other texts (see e.g., [Curtain and Zwart, 1995, Chapter 8, p. 387], [Nikol'skiĭ, 1986, Appendix 4, p. 299], [Partington, 1988]). We summarise some properties of the Hankel operator with symbol $\hat{F} \in \mathcal{L}_\infty(j\mathbb{R})$ relevant to later discussions as follows (see e.g., [Curtain and Zwart, 1995, Chapter 8, pp. 388–389]):

(a). The Hankel operator $H_{\hat{F}}$ is a linear bounded operator from $\mathcal{H}_2^\perp$ to $\mathcal{H}_2$.

(b). If $\hat{F}_1, \hat{F}_2 \in \mathcal{L}_\infty(j\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$ (or $\mathbb{C}$), then $H_{c_1\hat{F}_1 + c_2\hat{F}_2} = c_1 H_{\hat{F}_1} + c_2 H_{\hat{F}_2}$, i.e., the Hankel operator is linear with respect to the symbol over the field $\mathbb{R}$ (or $\mathbb{C}$).

(c). If $\hat{F} \in \mathcal{H}_\infty^-$ then $H_{\hat{F}} = 0$, here 0 denotes the zero operator.

(d). Consider $\hat{F}(s) = 1/(s+p)^r$, where the real part of the complex number p is positive (i.e., $\mathrm{Re}(p) > 0$) and r is a positive integer (i.e., $r \in \mathbb{N}_{>0}$). Clearly, $\hat{F} \in \mathcal{H}_\infty \subseteq \mathcal{L}_\infty(j\mathbb{R})$ and any $\hat{u} \in \mathcal{H}_2^\perp$ has an expansion

$$\hat{u}(s) = \hat{u}(-p) + \sum_{k=1}^{r-1} \hat{u}^{(k)}(-p)\frac{(s+p)^k}{k!} + (s+p)^r \hat{v}(s) \tag{2.5}$$

for some $\hat{v} \in \mathcal{H}_2^\perp$ and s on the open left-half of the complex plane (i.e., $s \in \mathbb{C}$ with $\mathrm{Re}(s) < 0$) and s on the imaginary axis almost everywhere (i.e., $s \in j\mathbb{R}$ a.e.). Thus we have [15]

$$(H_{\hat{F}}\hat{u})(s) = \frac{\hat{u}(-p)}{(s+p)^r} + \sum_{k=1}^{r-1} \frac{\hat{u}^{(k)}(-p)}{k!(s+p)^{r-k}}$$

and the dimension of the range of $H_{\hat{F}}$ is r (i.e., $\dim(\mathrm{range}(H_{\hat{F}})) = r$). It is easily verified that $H_{\hat{F}}\hat{u} = 0$ if and only if $\hat{u}^{(k)}(-p) = 0$ for all $k = 0, 1, 2, \ldots, r-1$.

Here and in what follows the notation $\hat{u}^{(0)}(-p)$ indicates $\hat{u}(-p)$, and $\hat{u}^{(k)}(-p)$, $k > 0$ indicates the k-th derivative of $\hat{u}(s)$ at the point $s = -p$.

We are finally in a position to give the construction of Nerode equivalence for scalar continuous-time transfer function given by (2.4), i.e.,

$$\hat{G}(s) = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \frac{a_{ij}}{(s+p_i)^j} \tag{2.6}$$

for which $\sum_{i=1}^m r_i = n$, and $r_i \in \mathbb{N}_{>0}$, $p_i \in \mathbb{C}_0^+$, $a_{ij} \in \mathbb{C}$ with $a_{ir_i} \neq 0$ for any $j = 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, m$, and $p_l \neq p_k$ for $l \neq k$.

Suppose that the function $G : \mathbb{R}_+ \to \mathbb{R}$ (i.e., impulse response) denotes the inverse (unilateral) Laplace transform of $\hat{G}$. Clearly, the function G is Lebesgue integral (i.e., $\int_0^\infty |G(t)|\,\mathrm{d}t < \infty$) since $\hat{G}$ has all poles on the open left-half of the complex plane; and G *causally* associates with each input signal $u \in \mathcal{L}_2(\mathbb{R})$ an output signal $y \in \mathcal{L}_2(\mathbb{R})$ on the time domain by the following convolution:[16]

$$y(t) = (G * u)(t) = \int_{-\infty}^{t} G(t-\tau)u(\tau)\,\mathrm{d}\tau, \quad \forall\, t \in \mathbb{R}.$$

From [Curtain and Zwart, 1995, Lemma 8.2.3, p. 397] we know that for any $u \in \mathcal{L}_2(\mathbb{R})$ we have $(\mathfrak{F}f)(j\omega) = \hat{G}(j\omega) \cdot (\mathfrak{F}u)(j\omega)$ for all $\omega \in \mathbb{R}$ with $f := G * u$, and that for any $v \in \mathcal{L}_2(\mathbb{R}_-)$ we have $H_{\hat{G}}\hat{v} = \mathfrak{F}(g_{[0,\infty)})$ with $g := G * v$.

Nerode proposed the following approach for introducing the concept of state: Pick some reference time, say $t = 0$, any input signals $u_1 \in \mathcal{L}_2(\mathbb{R}_-)$, $u_2 \in \mathcal{L}_2(\mathbb{R}_-)$, ... can be said to leave the system in the *same state at time* $t = 0$ if the corresponding output signals $y_1 := G * u_1 \in \mathcal{L}_2(\mathbb{R})$, $y_2 := G * u_2 \in \mathcal{L}_2(\mathbb{R})$, ... are all the same for $t \geq 0$ a.e.. (i.e., any inputs $\hat{u}_1 \in \mathcal{H}_2^\perp$, $\hat{u}_2 \in \mathcal{H}_2^\perp$, ... can be said to leave the system in the *same state at time* $t = 0$ if $H_{\hat{G}}\hat{u}_1 \in \mathcal{H}_2$, $H_{\hat{G}}\hat{u}_2 \in \mathcal{H}_2$, ... are all the same.) Thus, the space $\mathcal{L}_2(\mathbb{R}_-)$ can be broken up into *classes* such that for all inputs in any class the corresponding output

[15] see e.g., [Curtain and Zwart, 1995, Example 8.1.5, p. 389]), [Zhu and Stoorvogel, 1989]; and in discrete time case see e.g., [Nikol'skiĭ, 1986, p. 305].

[16] We assume that the system is causal time-invariant and initially at rest at time $t = -\infty$, and that the zero input signal gives a zero output signal.

is the same for $t \geq 0$ a.e.. (i.e., the space $\mathcal{H}_2^\perp$ can be broken up into *classes* such that for all inputs $\hat{u}$ in any class the value $H_{\hat{G}}\hat{u}$ is the same.) We associate with *each class* a *state at time* $t = 0$ for the system. This is all done with reference to the states at $t = 0$, but we can replace $t = 0$ by any other time because of time-invariance.

By the linearity of the Hankel operator $H_{\hat{G}}$ we know that the zero class in $\mathcal{H}_2^\perp$ (i.e., zero state at time $t = 0$) is identified with the kernel of $H_{\hat{G}}$, denoted by $\ker(H_{\hat{G}})$, which is the set of all $\hat{u} \in \mathcal{H}_2^\perp$ such that $H_{\hat{G}}\hat{u} = 0$. From properties (b) and (d) of the Hankel operator, we obtain

$$(H_{\hat{G}}\hat{u})(s) = \sum_{i=1}^{m}\sum_{j=1}^{r_i} a_{ij} \sum_{k=0}^{j-1} \frac{\hat{u}^{(k)}(-p_i)}{k!(s+p_i)^{j-k}} \tag{2.7}$$

and thus [17]

$$\ker(H_{\hat{G}}) = \left\{ \hat{u} \in \mathcal{H}_2^\perp \;\middle|\; \hat{u}^{(k)}(-p_i) = 0,\ \forall\, k = 0, 1, \ldots, r_i - 1 \text{ for } i = 1, 2, \ldots, m \right\}. \tag{2.8}$$

It can be verified that the set on the right hand side of (2.8) is equal to the following one: [18]

$$\left\{ \hat{q}\hat{v} \;\middle|\; \hat{q}(s) = \prod_{i=1}^{m} \frac{(s+p_i)^{r_i}}{(s-\overline{p_i})^{r_i}} \text{ and } \hat{v} \in \mathcal{H}_2^\perp \right\}. \tag{2.9}$$

This is consistent with the fact that $\ker(H_{\hat{G}}) = \Theta\mathcal{H}_2^\perp$ for some *inner* function Θ (i.e., $|\Theta(j\omega)| = 1$ for $\omega \in \mathbb{R}$ a.e.) in $\mathcal{H}_\infty^-$, which is a direct consequence of the *Beurling-Helson theorem* (see e.g., [Nikol'skiĭ, 1986, p. 10], [Partington, 1988, Corollary 6.5, p. 58]).

For any input $\hat{u} \in \mathcal{H}_2^\perp$, we know that $\hat{u}^{(k)}(-p_i)$, $\forall\, k = 0, 1, \ldots, r_i - 1$ for $i = 1, 2, \ldots, m$ are all well-defined. It follows from the interpolation by rational functions theory (see e.g., [Walsh, 1969, Chapter VIII, p. 184]) that there always exists a unique rational function $\hat{f} \in \mathcal{H}_2^\perp$ of the form

$$\hat{f}(s) = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_1 s + c_0}{(s-1)^n} \tag{2.10}$$

with $n = r_1 + \cdots + r_m$ such that

$$\hat{f}^{(k)}(-p_i) = \hat{u}^{(k)}(-p_i),\ \forall\, k = 0, 1, \ldots, r_i - 1 \text{ for } i = 1, 2, \ldots, m. \tag{2.11}$$

[17] Note that by assumption $a_{ir_i} \neq 0$ for any $i = 1, 2, \ldots, m$.

[18] Sketch of proof: ($\supseteq$) Since $\hat{q}(s) = \prod_{i=1}^{m} \frac{(s+p_i)^{r_i}}{(s-\overline{p_i})^{r_i}} \in \mathcal{H}_\infty^-$, we have $\hat{q}\hat{v} \in \ker(H_{\hat{G}})$ for any $\hat{v} \in \mathcal{H}_2^\perp$. ($\subseteq$) For any $\hat{u} \in \ker(H_{\hat{G}})$, from the expansion of $\hat{u}$ similar to (2.5) we obtain $\hat{u}(s) = \hat{v}(s)\prod_{i=1}^{m}(s+p_i)^{r_i}$ for some $\hat{v} \in \mathcal{H}_2^\perp$, and thus $\hat{u}(s) = \hat{q}(s)\hat{w}(s)$ with $\hat{w}(s) := \hat{v}(s)\prod_{i=1}^{m}(s-\overline{p_i})^{r_i} = \hat{q}^\sim(s)\hat{u}(s) \in \mathcal{H}_2^\perp$, since $\hat{q}^\sim(\cdot) \in \mathcal{H}_\infty \subseteq \mathcal{L}_\infty(j\mathbb{R})$, $\hat{u} \in \mathcal{H}_2^\perp \subseteq \mathcal{L}_2(j\mathbb{R})$ (so $\hat{w} = \hat{q}^\sim\hat{u} \in \mathcal{L}_2(j\mathbb{R})$), $\hat{v} \in \mathcal{H}_2^\perp$ and $\mathrm{Re}(p_i) > 0$ for $i = 1, 2, \ldots, m$. Here $F^\sim(s) := \left(\overline{F(-\overline{s})}\right)^T$ denotes the *para-Hermitian conjugation of* $F(s)$.

Note that the complex numbers $c_0, c_1, \dots, c_{n-1}$ in (2.10) are uniquely determined by the complex values $\hat{u}^{(k)}(-p_i)$, $\forall\, k = 0, 1, \dots, r_i - 1$ for $i = 1, 2, \dots, m$. By using the inner-outer factorisation theorem (see e.g., [Partington, 2004, Corollary 1.3.7 (F. Riesz), p. 11], [Partington, 1988, Theorem 2.12 (F. Riesz), p. 21]), there exists some $\hat{g} \in \mathcal{H}_2^\perp$ such that

$$\hat{u} = \hat{g}\hat{B} + \hat{f} \quad \text{with} \quad \hat{B}(s) := \prod_{i:\, p_i = -1} \frac{(1+s)^{r_i}}{(1-s)^{r_i}} \cdot \prod_{i:\, p_i \neq -1} \frac{\left|1 - p_i^2\right|^{r_i}}{(1-p_i^2)^{r_i}} \frac{(s+p_i)^{r_i}}{(s-\overline{p_i})^{r_i}} \tag{2.12}$$

where $\hat{B}(s)$ is the so-called *Blaschke product* for $\mathrm{Re}(s) < 0$ formed using the zeros p_i with multiplicity r_i for $i = 1, 2, \dots, m$ of $\hat{u} - \hat{f} \in \mathcal{H}_2^\perp$ (see (2.11)); and we see from (2.8) and (2.9) that $\hat{g}\hat{B} \in \ker(H_{\hat{G}})$. This means that both $\hat{u}$ and $\hat{f}$ belong to the same class in $\mathcal{H}_2^\perp$ and hence define the same state at time $t = 0$.

The above decomposition of any $\hat{u} \in \mathcal{H}_2^\perp$ given by (2.12) is similar to the one obtained via Euclidean algorithm in discrete-time case (see e.g., [Kailath, 1980, Section 5.1, p. 317]); and we see that the collection of classes (or equivalently the collection of states at time $t = 0$) can be represented by a collection of rational functions of the form (2.10); and each such rational function represents a distinct class. Since $c_0, c_1, \dots, c_{n-1}$ in (2.10) are one-to-one related to $\hat{u}^{(k)}(-p_i)$, $\forall\, k = 0, 1, \dots, r_i - 1$ for $i = 1, 2, \dots, m$, we can use the n values $\hat{u}^{(k)}(-p_i)$, $\forall\, k = 0, 1, \dots, r_i - 1$ for $i = 1, 2, \dots, m$ to specify any of these rational functions and thus any of the states at time $t = 0$. Therefore, the state space is n-dimensional with $n = r_1 + \cdots + r_m$, i.e., the state at time $t = 0$ can be represented by an n-vector, say

$$\begin{aligned} x(0) &= [x_1(0), \dots, x_{r_1}(0), \dots\dots, x_{n-r_m+1}(0), \dots, x_n(0)]^T \\ &\triangleq \left[\frac{\hat{f}^{(r_1-1)}(-p_1)}{(r_1-1)!}, \dots, \frac{\hat{f}^{(0)}(-p_1)}{0!}, \dots\dots, \frac{\hat{f}^{(r_m-1)}(-p_m)}{(r_m-1)!}, \dots, \frac{\hat{f}^{(0)}(-p_m)}{0!}\right]^T \\ &= \left[\frac{\hat{u}^{(r_1-1)}(-p_1)}{(r_1-1)!}, \dots, \frac{\hat{u}^{(0)}(-p_1)}{0!}, \dots\dots, \frac{\hat{u}^{(r_m-1)}(-p_m)}{(r_m-1)!}, \dots, \frac{\hat{u}^{(0)}(-p_m)}{0!}\right]^T. \end{aligned} \tag{2.13}$$

We next show how the state evolves with time as future inputs are applied. Suppose that $\hat{u}(s)$ in $\mathcal{H}_2^\perp$ is a representative of the Nerode equivalence class identified with the state $x(0)$ at $t = 0$ (*origin*), where $x(0)$ is defined by (2.13). After a future input u restricted to the time domain $[0, \tau)$ (resp., $\widehat{u|_{[0,\tau)}}(s) \triangleq \int_{0^-}^{\tau} u(t)e^{-st}\,\mathrm{d}t$ in the frequency domain) is applied, we redefine $t = \tau$ as the new *origin* (note that the system considered here is time-invariant), and thus the state $x(\tau)$ at $t = \tau$ can be associated with the following input signal (in the frequency domain):

$$e^{s\tau}\left[\hat{u}(s) + \widehat{u|_{[0,\tau)}}(s)\right] := e^{s\tau}\left[\hat{u}(s) + \int_{0^-}^{\tau} u(t)e^{-st}\,\mathrm{d}t\right],$$

where the multiplication operator $e^{s\tau}$ in the frequency domain corresponds to the time shifting by $-\tau$ in the time domain. Since the system considered here is linear, we can associate the derivative of the state $\dot{x}(0) \triangleq \frac{\mathrm{d}x(t)}{\mathrm{d}t}|_{t=0}$ at $t=0$ with the following *derivative* input signal (in the frequency domain):

$$\begin{aligned} \hat{u}_d(s) &\triangleq \lim_{\tau \downarrow 0} \frac{1}{\tau} \left\{ e^{s\tau} \left[\hat{u}(s) + \widehat{u|_{[0,\tau)}}(s) \right] - \hat{u}(s) \right\} \\ &= \left\{ \frac{\mathrm{d}}{\mathrm{d}\tau} e^{s\tau} \left[\hat{u}(s) + \widehat{u|_{[0,\tau)}}(s) \right] \right\} \Big|_{\tau=0} \\ &= s\hat{u}(s) + u(0). \end{aligned} \tag{2.14}$$

It is easily verified that

$$\hat{u}_d(-p_i) = -p_i \hat{u}(-p_i) + u(0) \quad \text{and} \quad \hat{u}_d^{(k)}(-p_i) = -p_i \hat{u}^{(k)}(-p_i) + k\hat{u}^{(k-1)}(-p_i)$$

for $k = 1, \ldots, r_i - 1$ with $i = 1, \ldots, m$. In matrix notation we get,

$$\begin{bmatrix} \frac{\hat{u}_d^{(r_i-1)}(-p_i)}{(r_i-1)!} \\ \vdots \\ \frac{\hat{u}_d^{(1)}(-p_i)}{1!} \\ \frac{\hat{u}_d^{(0)}(-p_i)}{0!} \end{bmatrix} = \underbrace{\begin{bmatrix} -p_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & -p_i \end{bmatrix}}_{A_i \triangleq J(-p_i,\, r_i)} \begin{bmatrix} \frac{\hat{u}^{(r_i-1)}(-p_i)}{(r_i-1)!} \\ \vdots \\ \frac{\hat{u}^{(1)}(-p_i)}{1!} \\ \frac{\hat{u}^{(0)}(-p_i)}{0!} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B_i} u(0) \tag{2.15}$$

for $i = 1, \ldots, m$. Note that $J(-p_i, r_i)$ is the $r_i \times r_i$ *Jordan block* with eigenvalues $-p_i$. We define an $n \times n$ matrix A_{jo} (note that $n = r_1 + \cdots + r_m$) and an $n \times 1$ matrix B_{jo} as follows

$$A_{jo} \triangleq \text{block diag}\left\{A_i,\ i = 1, \ldots, m\right\}, \quad B_{jo} \triangleq \left[B_1^T, \cdots, B_m^T\right]^T. \tag{2.16}$$

From (2.13)–(2.15) and above discussions, we know that $\dot{x}(0) = A_{jo}x(0) + B_{jo}u(0)$; and it follows from (2.7) and the initial value theorem[19] that

$$\begin{aligned} y(0) &= \sum_{i=1}^{m} \sum_{j=1}^{r_i} a_{ij} \frac{\hat{u}^{(j-1)}(-p_i)}{(j-1)!} \\ &= \underbrace{\left[a_{1r_1}, \cdots, a_{11}, \cdots\cdots, a_{mr_m}, \cdots, a_{m1}\right]}_{C_{jo}} x(0), \end{aligned} \tag{2.17}$$

where the last equality uses (2.13). Therefore, by time-invariance, we obtain a state space realisation in the *Jordan (modified-diagonal) canonical form* [Ogata, 2002, Chapter 11,

[19] i.e., $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$, where $F(s)$ denotes the Laplace transform of $f(t)$ (see e.g., [Cannon, 2003, Section 17.8, p. 567]).

p. 755], [Kalman, 1965] as follows:

$$\dot{x}(t) = A_{jo}x(t) + B_{jo}u(t), \quad y(t) = C_{jo}x(t), \quad \forall\, t \in \mathbb{R}, \tag{2.18}$$

with A_{jo}, B_{jo} and C_{jo} defined as in (2.15)–(2.17), respectively.

Remark 2.18. It is tacitly assumed that all poles of the transfer function (2.3) lie on the open left-half of the complex plane when we began. However, this assumption can be relaxed, and we could obtain the same state space realisation (2.18) by slightly modifying the definitions of signal spaces, function spaces and Hankel operators. For example, assume that $\mathrm{Re}(p_i) < \lambda$, $\forall i = 1, 2, \dots, m$, for some $\lambda > 0$. We denote $\mathcal{L}_{2,\lambda}(\mathbb{R}) := \{e_\lambda v \mid v \in \mathcal{L}_2(\mathbb{R})\}$ with norm $\|u\|_{\mathcal{L}_{2,\lambda}(\mathbb{R})} := \|e_{-\lambda}u\|_{\mathcal{L}_2(\mathbb{R})}$ for any $u \in \mathcal{L}_{2,\lambda}(\mathbb{R})$, where e_λ is an operator defined by $(e_\lambda v)(t) = e^{\lambda t}v(t)$, $\forall t \in \mathbb{R}$. Similarly, we denote $\mathcal{L}_{2,\lambda}(\mathbb{R}_+) := e_\lambda \mathcal{L}_2(\mathbb{R}_+)$ and $\mathcal{L}_{2,\lambda}(\mathbb{R}_-) := e_\lambda \mathcal{L}_2(\mathbb{R}_-)$ (see e.g., [Weiss, 1994]). The modified frequency domain spaces $\mathcal{H}_{2,\lambda}$, $\mathcal{H}_{2,\lambda}^\perp$, $\mathcal{L}_{2,\lambda}(j\mathbb{R})$, $\mathcal{H}_{\infty,\lambda}$, $\mathcal{H}_{\infty,\lambda}^-$, and $\mathcal{L}_{\infty,\lambda}(j\mathbb{R})$ are obtained by replacing the imaginary axis $j\mathbb{R}$ with the translated imaginary axis $\lambda + j\mathbb{R}$ in the definitions from $\mathcal{H}_2$, $\mathcal{H}_2^\perp$, $\mathcal{L}_2(j\mathbb{R})$, $\mathcal{H}_\infty$, $\mathcal{H}_\infty^-$, and $\mathcal{L}_\infty(j\mathbb{R})$ respectively. (e.g., $\mathcal{H}_{2,\lambda}^\perp$ is the space of all complex-valued functions square integrable on the axis $\lambda + j\mathbb{R}$ with analytic continuation in the left open half-plane in $\mathbb{C}$ delimited by λ (i.e., $\{s \in \mathbb{C} : \mathrm{Re}(s) < \lambda\}$).) Clearly, the modified frequency domain signal spaces are related to the corresponding modified time domain signal spaces via bilateral Laplace transform $\mathfrak{L}$, i.e., $\mathcal{H}_{2,\lambda} \equiv \mathfrak{L}\mathcal{L}_{2,\lambda}(\mathbb{R}_+)$, $\mathcal{H}_{2,\lambda}^\perp \equiv \mathfrak{L}\mathcal{L}_{2,\lambda}(\mathbb{R}_-)$ and $\mathcal{L}_{2,\lambda}(j\mathbb{R}) \equiv \mathfrak{L}\mathcal{L}_{2,\lambda}(\mathbb{R})$. In addition, $\mathcal{L}_{2,\lambda}(\mathbb{R}) = \mathcal{L}_{2,\lambda}(\mathbb{R}_+) \oplus \mathcal{L}_{2,\lambda}(\mathbb{R}_-)$ and $\mathcal{L}_{2,\lambda}(j\mathbb{R}) = \mathcal{H}_{2,\lambda} \oplus \mathcal{H}_{2,\lambda}^\perp$. The *modified Hankel operator* $\tilde{H}_{\hat{G}}$ *with symbol* $\hat{G}$ is defined as

$$\tilde{H}_{\hat{G}} : \mathcal{H}_{2,\lambda}^\perp \to \mathcal{H}_{2,\lambda}, \quad \hat{u} \mapsto \Pi_{\mathcal{H}_{2,\lambda}} \hat{G}\hat{u},$$

where $\Pi_{\mathcal{H}_{2,\lambda}}$ is the orthogonal projection operator from $\mathcal{L}_{2,\lambda}(j\mathbb{R})$ onto $\mathcal{H}_{2,\lambda}$. Thus we could obtain the same Jordan canonical form (2.18) by using similar arguments as above only with the Fourier transform $\mathfrak{F}$ and the imaginary axis $j\mathbb{R}$ replaced by the bilateral Laplace transform $\mathfrak{L}$ and the translated imaginary axis $\lambda + j\mathbb{R}$, respectively.

Thus far we have shown that in the scalar case the notion of Nerode equivalence can be used to obtain a state space realisation in the Jordan canonical form in a natural way. The multivariable analog of this results for continuous time transfer function matrices can also be developed by using the concept of Smith-McMillan form (see [Liu and French, 2014c]).

2.6 Comparison Classes of $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ Functions

In this section, we introduce the concept of comparison functions (i.e., class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ functions) (see e.g., [Isidori, 1999] or [Vidyasagar, 1993]), which are

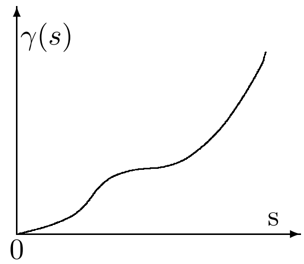

Figure 2.1: Class $\mathcal{K}_\infty$ function γ

used widely in the rest of the thesis.

Definition 2.19. A function $\gamma : [0, a) \to \mathbb{R}_+$ (in most cases we have $a = \infty$) is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and satisfies $\gamma(0) = 0$; moreover, if $a = \infty$ and $\lim_{s\to\infty} \gamma(s) = \infty$, then it is said to be of class $\mathcal{K}_\infty$ (Figure 2.1).

Definition 2.20. A function $\beta : [0, a) \times \mathbb{R}_+ \to \mathbb{R}_+$ (in most cases we have $a = \infty$) is said to be of class $\mathcal{KL}$ if it is such that $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$, and the function $\beta(s, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$ for each fixed $s \in [0, a)$.

For example, $\gamma(s) = 1 - e^{-s}$ for any $s \in \mathbb{R}_+$ is a class $\mathcal{K}$ function but not a class $\mathcal{K}_\infty$ function, and $\gamma(s) = 2s^2$ for any $s \in \mathbb{R}_+$ is a class $\mathcal{K}_\infty$ function, and $\beta(s, t) = 4s^3 \cdot e^{-2t^2}$ for any $s \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ is a class $\mathcal{KL}$ function.

In the following, we will summarise some interesting features about class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ functions.

1. The composition of two class $\mathcal{K}_\infty$ (resp., class $\mathcal{K}$) functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, denoted $\gamma_1 \circ \gamma_2(\cdot)$ or $\gamma_1(\gamma_2(\cdot))$, is still a class $\mathcal{K}_\infty$ (resp., class $\mathcal{K}$) function.
2. For any class $\mathcal{K}$ function $\gamma : [0, a) \to \mathbb{R}_+$ and $\lim_{s\to a} \gamma(s) = b$, there exists a unique function $\gamma^{-1} : [0, b) \to [0, a)$ such that $\gamma^{-1} \circ \gamma(s) = s$ for all $s \in [0, a)$ and $\gamma \circ \gamma^{-1}(s) = s$ for all $s \in [0, b)$. In addition, $\gamma^{-1} \in \mathcal{K}$. If γ belongs to class $\mathcal{K}_\infty$, so does also γ^{-1}.
3. For any functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ and function $\beta \in \mathcal{KL}$, the function $\tilde{\beta} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $(s, t) \mapsto \gamma_1(\beta(\gamma_2(s), t))$ is a class $\mathcal{KL}$ function.

We know that $\beta(s, t) = \gamma(s)e^{-\lambda t}$ with $\lambda > 0$ and $\gamma \in \mathcal{K}_\infty$ is a particular form of class $\mathcal{KL}$ function. To understand class $\mathcal{KL}$ function more clearly, we give the following Lemma 2.21 which says that any class $\mathcal{KL}$ function can be estimated in the sense of the exponential function and of two other class $\mathcal{K}_\infty$ functions, and Lemma 2.22 which states that when a function can be dominated by some class $\mathcal{KL}$ function.

Lemma 2.21. *Suppose that β is a class $\mathcal{KL}$ function. Then, there exist two functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that $\beta(s, t) \le \gamma_1(\gamma_2(s)e^{-t})$ for all $(s, t) \in [0, a) \times \mathbb{R}_+$.*

Proof. See e.g., [Sontag, 1998a, Proposition 7], [Isidori, 1999, Lemma 10.1.1, p. 2], [Karafyllis and Jiang, 2011, Theorem 3.1, p. 124]. ☐

Lemma 2.22. *Let $\phi(s,t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying*

- *for any $r > 0$ and any $\varepsilon > 0$ there exists some $T = T_{r,\varepsilon} > 0$ such that $\phi(s,t) < \varepsilon$ for all $0 \le s \le r$, $t \ge T$;*
- *for any $\varepsilon > 0$, there exists $r > 0$ such that $\phi(s,t) < \varepsilon$ for all $0 \le s \le r$, $t \ge 0$.*

Then a $\mathcal{KL}$ function β exists such that $\phi(s,t) \le \beta(s,t)$ for all $s \ge 0$, $t \ge 0$.

Proof. This lemma is stated in [Albertini and Sontag, 1999, Lemma 15] (see also [Sontag and Ingalls, 2002, Proposition A.1]). It is proved in [Lin et al., 1996, Section 3] but not explicitly presented in the above form (see also proofs of [Lin et al., 1993, Proposition 2.5 and Lemma 3.1]). ☐

The following Lemmas 2.23, 2.24 and 2.25 will be frequently used in this thesis.

Lemma 2.23. *For any function $\lambda : [0, r) \to \mathbb{R}_+$ of class $\mathcal{K}$, any function ρ of class $\mathcal{K}_\infty$ and any two nonnegative real numbers a and b with $a + b < r$, we have the following inequalities:*

$$\lambda(a+b) \le \max\left\{\lambda \circ (I+\rho)(a),\ \lambda \circ (I+\rho^{-1})(b)\right\} \tag{2.19a}$$
$$\lambda(a+b) \le \lambda \circ (I+\rho)(a) + \lambda \circ (I+\rho^{-1})(b) \tag{2.19b}$$

Proof. It follows from considering the two cases, $b \le \rho(a)$ and $b \ge \rho(a)$, and using the fact that the function $\lambda(s)$ is nondecreasing with respect to s, that

$$\begin{aligned} \lambda(a+b) &\le \lambda \circ (I+\rho)(a), \quad \text{if } b \le \rho(a); \\ \lambda(a+b) &\le \lambda \circ (I+\rho^{-1})(b), \quad \text{if } a \le \rho^{-1}(b). \end{aligned}$$

Thus, we have the inequality (2.19). ☐

Lemma 2.24. *Let ρ be a function of class $\mathcal{K}_\infty$, then we have*

$$\left(I - (I+\rho)^{-1}\right)^{-1}(s) = (I+\rho^{-1})(s), \quad \forall s \ge 0; \tag{2.20}$$
$$\left(I - (I+\rho)^{-1}\right)(s) = \rho \circ (I+\rho)^{-1}(s), \quad \forall s \ge 0. \tag{2.21}$$

Proof. We define another function $\Delta(s), s \ge 0$ of class $\mathcal{K}_\infty$ as follows:

$$\Delta(s) = (I+\rho)^{-1}(s), \quad \forall s \ge 0. \tag{2.22}$$

In order to prove that (2.20) holds, it suffices to show that the following two equalities hold

$$(I - \Delta) \circ (I + \rho^{-1})(s) = I(s), \quad \forall s \geq 0; \tag{2.23}$$
$$(I + \rho^{-1}) \circ (I - \Delta)(s) = I(s), \quad \forall s \geq 0. \tag{2.24}$$

By pointwise addition of functions, (2.23) and (2.24) are equivalent to the following equalities (2.25) and (2.26), respectively

$$\rho^{-1}(s) = \Delta \circ (I + \rho^{-1})(s), \quad \forall s \geq 0; \tag{2.25}$$
$$\rho^{-1} \circ (I - \Delta)(s) = \Delta(s), \quad \forall s \geq 0. \tag{2.26}$$

It follows from (2.22), the definition of function Δ, and the equality $(I + \rho) \circ \rho^{-1}(s) = (I + \rho^{-1})(s), \forall s \geq 0$ that (2.25) holds, i.e., (2.23) holds. Note that

$$\rho \circ \Delta(s) = [(\rho + I) - I] \circ \Delta(s) = (I - \Delta)(s), \quad \forall s \geq 0. \tag{2.27}$$

and applying function ρ^{-1} on both side of (2.27), we get (2.26), hence (2.24) holds. Therefore (2.20) follows, and (2.21) can be directly obtained from (2.27). This completes the proof. □

The following technical result is taken from [Jiang et al., 1994, Lemma A.1], which will be used in the proof of Theorem 4.8 on page 81.

Lemma 2.25. *Let $\beta \in \mathcal{KL}$ and $\lambda \in \mathcal{K}_\infty$ with $I - \lambda \in \mathcal{K}_\infty$ be given, and let μ be any real number with $0 < \mu \leq 1$. Then, for any function δ with $\delta - I \in \mathcal{K}_\infty$, there exists a function $\hat{\beta} \in \mathcal{KL}$ such that, for any nonnegative real numbers $s \geq 0$, $d \geq 0$, and for any nonnegative real function $z(t)$ essentially bounded on $[0, \infty)$ and satisfying*

$$z(t) \leq \beta(s,t) + \lambda(\|z\|_{[\mu t, \infty)}) + d, \quad \forall\, t \in [0, +\infty), \tag{2.28}$$

we have

$$z(t) \leq \hat{\beta}(s,t) + (I - \lambda)^{-1} \circ \delta(d), \quad \forall\, t \in [0, +\infty).$$

Proof. See [Jiang et al., 1994, Lemma A.1]. Sketch of proof: Define a new function $\bar{z}(t) := z(t)\chi\big(\|z\|_{[\mu t,\infty)} - (I - \lambda)^{-1} \circ \delta(d)\big)$, where $\chi(x) = 1$ if $x > 0$ and $\chi(x) = 0$ if $x \leq 0$. It can be verified that $z(t) \leq \bar{z}(t) + (I - \lambda)^{-1} \circ \delta(d)$, and that (using (2.28))

$$\bar{z}(t) \leq \beta(s,t) + \big(\lambda + \delta^{-1} \circ (I - \lambda)\big)(\|\bar{z}\|_{[\mu t, \infty)}). \tag{2.29}$$

Thus the conclusion follows if we can show that there exists a function $\hat{\beta}(s,t)$ of class $\mathcal{KL}$ satisfying $\bar{z}(t) \leq \hat{\beta}(s,t)$. This is true by combining Lemma 2.22 and the following two claims (which are the difficult parts of the proof): (1) For any $r > 0$ and $\varepsilon > 0$, there exists some $T = T(r,\varepsilon) > 0$ such that $\bar{z}(t) \leq \varepsilon$, $\forall t \geq T$ if $\bar{z}(t)$ satisfies (2.29) with $s \leq r$; (2) For any $\varepsilon > 0$, there exists $r > 0$ such that $\bar{z}(t) \leq \varepsilon$, $\forall t \geq 0$ if $\bar{z}(t)$ satisfies (2.29) with $s \leq r$. □

The following Lemma 2.26 involves convolution integral and comparison functions. For two functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$, the convolution integral function $f * g : \mathbb{R}_+ \to \mathbb{R}$ is defined as follows:

$$(f * g)(t) \triangleq \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$$

It is easy to see that $f*g = g*f$. Also notice that for any measurable locally essentially bounded functions $f_i : \mathbb{R}_+ \to \mathbb{R}$ with $i = 1,2,3$ we have [20]

$$\begin{aligned}
\big((f_1 * f_2) * f_3\big)(t) &= \int_0^t (f_2 * f_1)(t-\tau) f_3(\tau)\, d\tau = \int_0^t \int_0^{t-\tau} f_2(t-\tau-\theta) f_1(\theta) f_3(\tau)\,\mathrm{d}\theta d\tau \\
&= \int_0^t \int_0^{t-\theta} f_2(t-\tau-\theta) f_1(\theta) f_3(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta = \int_0^t \int_0^x f_2(x-\tau) f_1(t-x) f_3(\tau)\,\mathrm{d}\tau\,\mathrm{d}x \\
&= \int_0^t f_1(t-x) \int_0^x f_2(x-\tau) f_3(\tau)\,\mathrm{d}\tau\,\mathrm{d}x = \int_0^t f_1(t-x)(f_2 * f_3)(x)\,\mathrm{d}x \\
&= \big(f_1 * (f_2 * f_3)\big)(t), \quad \forall t \geq 0.
\end{aligned} \tag{2.30}$$

Note that the third equality in (2.30) uses the Fubini-Tonelli theorem (see e.g., [Krantz, 2011, p. 54]).

Lemma 2.26. *Given any two functions $f_i : \mathbb{R}_+ \to \mathbb{R}$ with $i = 1,2$. If there are four comparison functions $\beta_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}_\infty$ with $i = 1,2$ such that*

$$|(f_i * u)(t)| \leq \beta_i(\|u\|_{[0,h]}, t-h) + \gamma_i(\|u\|_{[h,t]}), \quad \forall t \geq h \geq 0,\ \forall u \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R}),\ \forall i = 1,2$$

for any $0 \leq a \leq b$, where $L^\infty_{loc}(\mathbb{R}_+,\mathbb{R})$ is the space of all measurable locally essentially bounded functions $x : \mathbb{R} \to \mathbb{R}^n$ with $\|u\|_{[a,b]} \triangleq \operatorname{ess\,sup}_{t\in[a,b]} |u(t)|$. Then we have

$$\big|\big((f_1 * f_2) * u\big)(t)\big| \leq \beta(\|u\|_{[0,h]}, t-h) + (\gamma_1 \circ \gamma_2)(\|u\|_{[h,t]}), \quad \forall t \geq h \geq 0,\ \forall u \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R})$$

with $\beta \in \mathcal{KL}$ defined by

$$\beta(r,t) \triangleq \beta_1\big(\gamma_2(r), t\big) + \beta_1\big(\beta_2(r,0), t/2\big) + \gamma_1\big(\beta_2(r,t/2)\big), \quad \forall r \geq 0, \forall t \geq 0. \tag{2.31}$$

Proof. For any $u \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R})$ and any $t \geq h \geq 0$ we have

$$\begin{aligned}
\big((f_1 * f_2) * u\big)(t) =& \big(f_1 * (f_2 * u)\big)(t) \\
=& \int_0^t f_1(t-\theta) \int_0^\theta f_2(\theta-\tau) u(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta = A + B + C + D
\end{aligned}$$

[20] see e.g., [Desoer and Vidyasagar, 2009, p. 239]

with $A \triangleq \int_0^h f_1(t-\theta)\int_0^\theta f_2(\theta-\tau)u(\tau)\,d\tau\,\mathrm{d}\theta$, $B \triangleq \int_h^t f_1(t-\theta)\int_h^\theta f_2(\theta-\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta$, $C \triangleq \int_h^{\frac{t+h}{2}} f_1(t-\theta)\int_0^h f_2(\theta-\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta$, and $D \triangleq \int_{\frac{t+h}{2}}^t f_1(t-\theta)\int_0^h f_2(\theta-\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}X\theta$.

It is easy to see that

$$\begin{aligned}
|A| \leq& \beta_1\left(\left\|\int_0^{\cdot} f_2(\cdot-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{[0,h]}, t-h\right) \leq \beta_1\left(\gamma_2(\|u\|_{[0,h]}), t-h\right)\\
|B| \leq& \gamma_1\left(\left\|\int_h^{\cdot} f_2(\cdot-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{[h,t]}\right) \leq (\gamma_1\circ\gamma_2)(\|u\|_{[h,t]})\\
|C| \leq& \beta_1\left(\left\|\int_0^h f_2(\cdot-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{[h,\frac{t+h}{2}]}, t-\frac{t+h}{2}\right) \leq \beta_1\left(\beta_2(\|u\|_{[0,h]}, 0), \frac{t-h}{2}\right)\\
|D| \leq& \gamma_1\left(\left\|\int_0^h f_2(\cdot-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{[\frac{t+h}{2},t]}\right) \leq \gamma_1\left(\beta_2\left(\|u\|_{[0,h]}, t-\frac{t+h}{2}\right)\right)
\end{aligned}$$

We define a function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by (2.31), thus we have $\beta \in \mathcal{KL}$ and

$$\left|\big((f_1 * f_2) * u\big)(t)\right| \leq \beta(\|u\|_{[0,h]}, t-h) + (\gamma_1\circ\gamma_2)(\|u\|_{[h,t]})$$

for any $t \geq h \geq 0$ and any $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R})$. This completes the proof. □

2.7 Input-to-State Stability in State Space Model

The following notion of input-to-state stability in state space model was introduced by Sontag (see e.g., [Isidori, 1999, Khalil, 2002, Sontag, 1989]).

Consider a nonlinear system

$$\dot{x} = f(x, u) \tag{2.32}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, where $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ and $f(0,0) = 0$. The input function $u : [0,\infty) \to \mathbb{R}^m$ of (2.32) can be any measurable locally essentially bounded functions. The set of all such functions, endowed with the essential supremum norm $\|u\|_\infty = \operatorname{ess\,sup}\{|u(t)|, t \geq 0\}$, is denoted by $L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ (where $|\cdot|$ denotes the usual Euclidean norm).

Definition 2.27. The system (2.32) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function γ, called a gain function, such that, for all input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and all $x_0 \in \mathbb{R}^n$, the response $x(t)$ of (2.32) for the initial state $x(0) = x_0$ and the input u satisfies

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_{[0,t)}) \tag{2.33}$$

for all $t \geq 0$.

The following Lyapunov-like theorem gives a sufficient condition for input-to-state stability in a state space model (see e.g., [Isidori, 1999, Khalil, 2002, Sontag and Wang, 1995]).

Theorem 2.28. *Let $V : \mathbb{R}^m \to \mathbb{R}$ be a continuous differentiable function such that*

$$\underline{\alpha}(|x|) \le V(x) \le \overline{\alpha}(|x|) \quad \text{for all } x \in \mathbb{R}^n \tag{2.34}$$

$$|x| \ge \rho(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x,u) \le -\alpha(|x|), \quad \text{for all } x \in \mathbb{R}^n \tag{2.35}$$

where $\underline{\alpha}$, $\overline{\alpha}$, α are class $\mathcal{K}_\infty$ functions and ρ is a class $\mathcal{K}$ function. Then, the system (2.32) is input-to-state stable, an estimate of the form (2.33) holds with a gain function given by $\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \rho(r)$.

If (2.35) is replaced by the following condition:

$$\frac{\partial V}{\partial x} f(x,u) \le -\alpha(|x|) + \sigma(\|u\|), \quad \text{for all } x \in \mathbb{R}^n \text{ and all } u \in \mathbb{R}^m \tag{2.36}$$

where σ is a class $\mathcal{K}$ function. Then, in view of [Isidori, 1999, Lemma 10.4.2], the system (2.32) is still input-to-state stable and the gain function can be chosen as $\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha^{-1}\big(k\sigma(r)\big)$, where k is any real number satisfying $k > 1$.

Example 2.29. *Consider a linear time-invariant (LTI) system*

$$\dot{x} = Ax + Bu$$

and suppose that A is Hurwitz (i.e., all eigenvalues of the matrix A have negative real parts, see e.g., [Sontag, 1998b, Def. C.5.2]). For any constant symmetric matrix Q with $Q > 0$ (i.e., Q is positive definite, e.g., $Q = I$), since A is Hurwitz, there exists a unique[21] *$P > 0$ satisfying the Lyapunov equation $PA + A^T P = -Q$. Observe that the function $V(x) = x^T P x$ satisfies*

$$\underline{\lambda}_P \cdot |x|^2 \le V(x) \le \overline{\lambda}_P \cdot |x|^2 \tag{2.37}$$

where $\underline{\lambda}_P > 0$ and $\overline{\lambda}_P > 0$ are the smallest and largest eigenvalues of P, respectively. Note that

$$\begin{aligned} \frac{\partial V}{\partial x}(Ax + Bu) &= -x^T Q x + u^T B^T P x + x^T P B u \\ &\le -\underline{\lambda}_Q \cdot |x|^2 + 2\,|x| \cdot \|P\| \cdot \|B\| \cdot \|u\| \end{aligned}$$

where $\underline{\lambda}_Q > 0$ is the smallest eigenvalue of Q. Pick any $\varepsilon \in (0,1)$, then

$$|x| \ge \frac{2}{\varepsilon \cdot \underline{\lambda}_Q} \|P\| \cdot \|B\| \cdot \|u\| \quad \Rightarrow \quad \frac{\partial V}{\partial x}(Ax + Bu) \le -(1-\varepsilon)\underline{\lambda}_Q \cdot |x|^2 \tag{2.38}$$

[21] In fact, $P = \int_0^\infty e^{tA^T} Q e^{tA}\, \mathrm{d}t$.

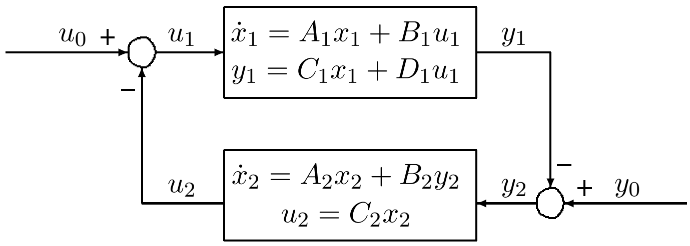

Figure 2.2: Linear time-invariant closed-loop system in state space model

Thus the linear system is input-to-state stable with a gain function given by

$$\gamma(r) = \frac{2 \cdot \underline{\lambda}_P}{\varepsilon \cdot \overline{\lambda}_P \cdot \underline{\lambda}_Q} \|P\| \cdot \|B\| \cdot r$$

which is a linear function.

2.8 Input-to-Output Stability in State Space Model

Consider a nonlinear system with outputs of the general form:

$$\dot{x} = f(x, u), \quad y = h(x) \tag{2.39}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are both locally Lipschitz continuous with $f(0, 0) = 0$ and $h(0) = 0$. The input function $u : [0, \infty) \to \mathbb{R}^m$ of (2.39) can be any measurable locally essentially bounded functions. [Sontag and Wang, 1999] introduced the following notion of input-to-output stability in a state space model.

Definition 2.30. The system (2.39) is said to be input-to-output stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function γ, called a gain function, such that, for all input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and all $x_0 \in \mathbb{R}^n$, the corresponding output $y(t)$ of (2.39) for the initial state $x(0) = x_0$ and the input u satisfies

$$|y(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_{[0,t)}) \tag{2.40}$$

for all $t \geq 0$.

A Lyapunov-like theorem in [Sontag and Wang, 2000, Theorem 1.2] gives a sufficient and necessary condition for input-to-output stability in a state space model. The following is a simple example concerning input-to-output stability of a linear time-invariant closed-loop system with a state-space representation.

Example 2.31. *Consider the closed-loop system shown in Figure 2.2, in which both the plant and controller are (LTI) subsystems. The plant is described by*

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t), \quad y_1(t) = C_1 x_1(t) + D_1 u_1(t) \tag{2.41}$$

while the controller is described by

$$\dot{x}_2(t) = A_2 x_2(t) + B_2 y_2(t), \quad u_2(t) = C_2 x_2(t) \tag{2.42}$$

and the feedback interconnection is described by

$$u_0(t) = u_1(t) + u_2(t), \quad y_0(t) = y_1(t) + y_2(t) \tag{2.43}$$

In the above, it is assumed that $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$ *and* $u_i(t) \in \mathbb{R}^m$, $y_i(t) \in \mathbb{R}^p$ *with* $i = 0, 1, 2$. *The matrices* A_i, B_i, C_i *with* $i = 1, 2$ *and* D_1 *are of appropriate dimensions. A simple calculation shows that the expression of closed-loop system with product state* $x = (x_1^T, x_2^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, *input* $w_0 = (u_0^T, y_0^T)^T \in \mathbb{R}^m \times \mathbb{R}^p$ *and output* $w_1 = (u_1^T, y_1^T)^T \in \mathbb{R}^m \times \mathbb{R}^p$ *is the following*

$$\dot{x} = Ax + Bw_0, \quad w_1 = Cx + Dw_0 \tag{2.44}$$

where the matrices A, B, C, D *are defined by*

$$A = \begin{pmatrix} A_1 & -B_1 C_2 \\ -B_2 C_1 & A_2 + B_2 D_1 C_2 \end{pmatrix}; \quad B = \begin{pmatrix} B_1 & 0 \\ -B_2 D_1 & B_2 \end{pmatrix};$$
$$C = \begin{pmatrix} 0 & -C_2 \\ C_1 & -D_1 C_2 \end{pmatrix}; \quad D = \begin{pmatrix} I & 0 \\ D_1 & 0 \end{pmatrix}.$$

In view of Example 2.29, if the matrix A *is Hurwitz. Then the closed-loop system (2.44) is input-to-state stable with* (x_1, x_2) *as states and* (u_0, y_0) *as inputs. Since* $\|C_1\|$, $\|C_2\|$ *and* $\|D_1\|$ *are bounded, we also obtain that the closed-loop system (2.44) is input-to-output stable with* (u_0, y_0) *as inputs and* (u_1, y_1) *as outputs.*

> Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.
>
> Bertrand Russell (1872-1970)

Chapter 3

Framework for General Input-Output Theory with Initial Conditions

The general nonlinear input-output theory initiated in the 1960s by [Zames, 1963, 1966b,c] and [Sandberg, 1964, 1965a] using the techniques of functional analysis. In this approach, systems were represented by operators mapping from inputs to outputs. A central issue in the input-output theory is robustness. For linear systems, robustness in the gap and graph metrics is initiated by Zames and El-Sakkary in [Zames and El-Sakkary, 1980] (see also [Georgiou and Smith, 1990], [Foias et al., 1993], [Vidyasagar, 2011], etc.). In a seminal work by [Georgiou and Smith, 1997b], the authors developed an input-output approach to uncertainty in the gap metric for robustness analysis of nonlinear feedback systems. A notable limitation of this work is that it implicitly require that the systems have zero initial conditions. The main part of this thesis is to undertake the substantial generalisation of Georgiou and Smith's input-output theory to the case of systems with initial conditions.

This chapter serves to provide a unified framework for general input-output theory with initial conditions which will underlie the future work. Both systems and closed-loop systems are defined in a set theoretic manner from input-output pairs on a doubly infinite time axis, and the construction of initial conditions is given in terms of an equivalent class of input-output trajectories on the negative time axis. Comparison with classical initial conditions are given for both systems and closed-loop systems. Fundamental notions of causality, well-posedness, i.e., existence and uniqueness, and graph are discussed in the presenting input-output framework. After that, a specific consideration of the uniqueness property of a system is given, which will be necessary for the proof of Theorem 4.8 in Chapter 4 on page 81. Relationships between initial conditions, the well-posedness and casuality of open-loop subsystems and closed-loops systems are discussed in subsequent

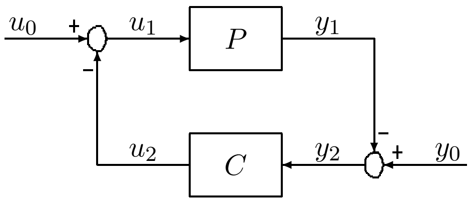

Figure 3.1: Closed-loop system $[P, C]$

sections. A suitable concept of input-output stability on the positive time axis with initial conditions is defined, which is closely related to the ISS/IOS notions initiated by Sontag [1989]. The chapter ends by summarising several alterative characterisation of this notion of input-output stability for closed-loop systems.

3.1 Standard Feedback Configuration and General Time Function Spaces

The standard feedback configuration considered throughout this thesis is shown in Figure 3.1 with the following equations

$$[P, C] : \begin{array}{l} w_i = (u_i, y_i) \text{ for } i = 0, 1, 2, \\ w_1 \in \mathfrak{B}_P, w_2 \in \mathfrak{B}_C, w_0 = w_1 + w_2, \end{array} \tag{3.1}$$

where we choose $\mathcal{U}, \mathcal{Y}$, $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$ to be appropriate signal spaces, and $w_1 \in \mathfrak{B}_P \subseteq \mathcal{W}_e$ (or $\mathcal{W}_a$), $w_2 \in \mathfrak{B}_C \subseteq \mathcal{W}_e$ (or $\mathcal{W}_a$), $w_0 \in \mathcal{W}_e$ (or $\mathcal{W}_a$) (These symbols undefined for the moment will be explained more carefully below). (u_0, y_0) denote external "disturbance" signals; (u_1, y_1) are the input-output signals pair of the plant P to be controlled; and (u_2, y_2) are the output-input signals pair of the controller C. Both plant P and controller C are systems of the closed-loop system $[P, C]$, the precise definition of system and closed-loop system will be defined in the future sections.

We next introduce the concept of general time function spaces $\mathcal{F}_J(X)$. The signal spaces given on the next section including interval, extended, and ambient signal spaces are defined as some suitable time function subspaces of $\bigcup_{J \subseteq \mathbb{R}} \mathcal{F}_J(X)$.

Let J be any time interval which is subinterval of $\mathbb{R}$ (possibly finite, semi-infinite, and doubly-infinite interval), and X be any normed linear space with norm $\|\cdot\|_X$ (typically, $X = \mathbb{R}^n$, $\mathbb{C}^n$ or $L^2([0, 1], \mathbb{R}^m)$), and $f : J \to X$ be any time function which is a X-valued function defined on time interval J, and $\mathcal{F}_J(X)$ be the set of all time functions from J into X. It is easy to see that $\mathcal{F}_J(X)$ is a natural vector space (see Section 2.1) over $\mathbb{R}$ (or $\mathbb{C}$) under pointwise addition and scalar multiplication defined by

$$(f + g)(t) = f(t) + g(t), \quad (\lambda f)(t) = \lambda f(t), \quad \forall f, g \in \mathcal{F}_J(X), \forall t \in J, \forall \lambda \in \mathbb{R} \text{ (or } \mathbb{C}).$$

Given any subintervals I and J of $\mathbb{R}$ with $J \subseteq I$, we define a *truncation operator* T_J from $\mathcal{F}_I(X)$ into $\mathcal{F}_{\mathbb{R}}(X)$ as follows:

$$T_J : \mathcal{F}_I(X) \to \mathcal{F}_{\mathbb{R}}(X), \quad u \mapsto T_J u \triangleq \left(t \mapsto \begin{cases} u(t), & t \in J \\ 0, & \text{otherwise} \end{cases} \right)$$

and a *restriction operator* R_J from $\mathcal{F}_I(X)$ into $\mathcal{F}_J(X)$ as follows:

$$R_J : \mathcal{F}_I(X) \to \mathcal{F}_J(X), \quad u \mapsto u_J \triangleq \Big(t \mapsto u(t), \quad t \in J \Big).$$

3.2 Interval, Extended, and Ambient Signal Spaces

Within the classical approach to input-output analysis, all signals are considered to lie within the extended spaces ($\mathcal{V}_e^+$ below). This forces signals to be defined only on the semi-infinite time domain and hence precludes finite time escape analysis of systems and detailed analysis of systems with nonzero initial conditions. By thinking of a system as defined on a doubly infinite time domain, the past input-output signals corresponding to the system before initial time, say $t = 0$, can be used to characterise its initial state at $t = 0$. This gives a unified framework for the study of initial conditions in a purely input-output theory. Generalisation of the input-output operator based robust stability theorem of [Georgiou and Smith, 1997b] to include the case of initial conditions is the core part of this thesis (see Theorem 4.8 in Chapter 4). Thus the extended space $\mathcal{V}_e$ defined on doubly infinite time axis is necessary in this framework.

At the end of Chapter 4, we also wish to generalise Theorem 4.8 to provide a robust stability theory for nonlinear systems including finite escape times phenomenon. In this case, the ambient space $\mathcal{V}_a$ defined below is more appropriate than the extended space $\mathcal{V}_e$ to capture the behaviour of systems with signals only defined on finite intervals. For example, if $\mathcal{V} = L^p(\mathbb{R}, \mathbb{R})$ with $p = 2, \infty$, the time function

$$x(t) = \begin{cases} \tan(t), & \text{if } t \in (-\pi/2, \pi/2), \\ 0, & \text{otherwise,} \end{cases}$$

does not belong to $L^p(J_1, \mathbb{R}) = \mathcal{V}(J_1) \subseteq \mathcal{V}_e$ with $J_1 = (-\tau_1, \tau_1)$ for any $\tau_1 \geq \pi/2$, but $x \in L^p(J_2, \mathbb{R}) = \mathcal{V}(J_2)$ with $J_2 = (-\tau_2, \tau_2)$ for all $\tau_2 \in (0, \pi/2)$, hence $x \in \mathcal{V}_{(-\pi/2,\pi/2)} \subseteq \mathcal{V}_a$ corresponding to the definition given below.[1] Note that the ambient space $\mathcal{V}_a$ consists of all signals defined on time intervals of both finite and infinite lengths, which is more wider than the extended space $\mathcal{V}_e$.

Formally, let $\mathcal{V} \subseteq \mathcal{F}_{\mathbb{R}}(X)$ be a normed vector space with norm $\|\cdot\|$. For any open subinterval $J = (t_1, t_2)$ of $\mathbb{R}$ with $-\infty \leq t_1 < t_2 \leq +\infty$, we associate with the normed

[1] $\int \tan^2(t)\,\mathrm{d}t = -t + \tan(t) + c$

vector space $\mathcal{V}$ the following *interval spaces*, *extended spaces*, and *ambient space* (for these signal spaces defined only on positive time domain see [French and Bian, 2012]):

- $\mathcal{V}(J) \triangleq \{v \in \mathcal{F}_J(X) \mid \exists\, w \in \mathcal{V} \text{ such that } v = R_J w\} = R_J(\mathcal{V})$: *the interval space with respect to* J;
- $\mathcal{V}_J \triangleq \Big\{v \in \mathcal{F}_J(X) \mid \forall J' \triangleq (\mu_1, \mu_2), t_1 < \mu_1 < \mu_2 < t_2 : R_{J'}v \in \mathcal{V}(J')\Big\}$: *the extended space with respect to* J;
- $\mathcal{V}_e \triangleq \Big\{v \in \mathcal{F}_{\mathbb{R}}(X) \mid \forall J' \triangleq (\mu_1, \mu_2), -\infty < \mu_1 < \mu_2 < +\infty : R_{J'}v \in \mathcal{V}(J')\Big\}$: *the extended space with respect to the doubly infinite time domain*;
- $\mathcal{V}_a \triangleq \bigcup_{\{J'\}} \mathcal{V}_{J'}$: *the ambient space* $\big($where the set $\{J'\}$ consists of all open subintervals $J' \subseteq \mathbb{R}$ (possibly semi-infinite or infinite)$\big)$.

Let $R_+ \triangleq R_{[0,\infty)}$ and $R_- \triangleq R_{(-\infty,0]}$, with the normed vector space $\mathcal{V}$ we also associate the following *interval spaces*, *extended spaces*, and *ambient spaces* on the right semi-time domain from time $t = 0$ and on the left semi-time domain up to time $t = 0$:

- $\mathcal{V}^+ \triangleq R_+\mathcal{V}$ (resp., $\mathcal{V}^- \triangleq R_-\mathcal{V}$): *the restriction of the normed vector space* $\mathcal{V}$ *to the positive (resp., negative) time domain*;
- $\mathcal{V}_e^+ \triangleq R_+\mathcal{V}_e$ (resp., $\mathcal{V}_e^- \triangleq R_-\mathcal{V}_e$): *the extended space with respect to the positive (resp., negative) time domain*;
- $\mathcal{V}[0,\omega) \triangleq R_+\mathcal{V}(-\infty,\omega)$ $\big($resp., $\mathcal{V}(-\omega,0] \triangleq R_-\mathcal{V}(-\omega,\infty)\big)$ for any $\omega \in (0,\infty]$: *the interval space with respect to* $[0,\omega)$ *on the positive time domain (resp.,* $(-\omega,0]$ *on the negative time domain)*;
- $\mathcal{V}_{[0,\omega)} \triangleq R_+\mathcal{V}_{(-\infty,\omega)}$(resp., $\mathcal{V}_{(-\omega,0]} \triangleq R_-\mathcal{V}_{(-\omega,\infty)}$) for any $\omega \in (0,\infty]$: *the extended space with respect to* $[0,\omega)$ *on the positive time domain (resp.,* $(-\omega,0]$ *on the negative time domain)*;
- $\mathcal{V}_a^+ \triangleq \bigcup_{0<t\leq\infty} \mathcal{V}_{[0,t)}$ (resp., $\mathcal{V}_a^- \triangleq \bigcup_{-\infty\leq t<0} \mathcal{V}_{(t,0]}$): *the ambient space on the positive (resp., negative) time domain.*

According to above definitions of signal spaces it is easily verified that

$$\begin{cases} \mathcal{V}(\mathbb{R}) \equiv \mathcal{V} \subseteq \mathcal{V}_{\mathbb{R}} \equiv \mathcal{V}_e \subseteq \mathcal{V}_a \subseteq \mathcal{F}_{\mathbb{R}}(X), \\ \mathcal{V}(\mathbb{R}_+) \equiv \mathcal{V}^+ \subseteq \mathcal{V}_{\mathbb{R}_+} \equiv \mathcal{V}_e^+ \subseteq \mathcal{V}_a^+ \subseteq \mathcal{F}_{\mathbb{R}_+}(X), \\ \mathcal{V}(\mathbb{R}_-) \equiv \mathcal{V}^+ \subseteq \mathcal{V}_{\mathbb{R}_-} \equiv \mathcal{V}_e^+ \subseteq \mathcal{V}_a^+ \subseteq \mathcal{F}_{\mathbb{R}_-}(X). \end{cases}$$

Suppose that for any time interval $J \subseteq \mathbb{R}$, a corresponding (extended) norm,[2] denoted by $\|\cdot\|_J$, is also defined on the extended space $\mathcal{V}_J$, with the following basic assumptions made concerning $\mathcal{V}_{\mathbb{R}} \equiv \mathcal{V}_e$ and all other $\mathcal{V}_J$ with $J \subseteq \mathbb{R}$:

[2] It is possible that the norm of some element in $\mathcal{V}_J$ equals $+\infty$.

Assumption 3.1. *It is assumed throughout that for any time intervals J_1 and J_2 with $J_1 \subseteq J_2 \subseteq \mathbb{R}$ (possibly $J_2 = \mathbb{R}$):*

(1) For any $x \in \mathcal{V} \subseteq \mathcal{V}_{\mathbb{R}} \equiv \mathcal{V}_e$, we have $\|x\| \equiv \|x\|_{\mathbb{R}}$. In order to simplify notation, we will abbreviate $\|\cdot\|_{\mathbb{R}}$ by $\|\cdot\|$ for the extended norm defined on $\mathcal{V}_{\mathbb{R}} \equiv \mathcal{V}_e$.

(2) If $x \in \mathcal{V}_{J_1}$ then $\|x\|_{J_1} = \|T_{J_1}x\| \leq \infty$.

*(3) If $x \in \mathcal{V}_{J_2}$ then $\|R_{J_1}x\|_{J_1} = \|T_{J_1}x\| \leq \|x\|_{J_2} \leq \infty$. (*monotonicity condition*)*

By using Assumption 3.1, we can obtain the following results:

For any $x_i \in \mathcal{V}_{I_i}$, $i = 1, 2$ with $I_1 \subseteq I_2 \subseteq \mathbb{R}$, we have $y_i \triangleq T_{I_1}x_i = T_{I_2}(T_{I_1})x_i \in \mathcal{V}_{\mathbb{R}}$, $i = 1, 2$, and thus

$$\|R_{I_2}T_{I_1}x_1\|_{I_2} = \|R_{I_2}y_1\|_{I_2} \stackrel{(3)}{=} \|T_{I_2}y_1\| \tag{3.2}$$

$$= \|T_{I_2}(T_{I_1}x_1)\| = \|T_{I_1}x_1\| \stackrel{(2)}{=} \|x_1\|_{I_1} \tag{3.3}$$

and

$$\|R_{I_2}T_{I_1}x_2\|_{I_2} = \|R_{I_2}y_2\|_{I_2} \stackrel{(3)}{=} \|T_{I_2}y_2\| = \|T_{I_2}(T_{I_1}x_2)\| \tag{3.4}$$

$$= \|T_{I_1}x_2\| \stackrel{(2)}{=} \|R_{I_1}x_2\|_{I_1} \stackrel{(2)}{\leq} \|x_2\|_{I_2}, \tag{3.5}$$

where the second equalities in (3.2) and (3.4) use assumption (3) with $J_1 \triangleq I_2 \subseteq J_2 \triangleq \mathbb{R}$, the last equality in (3.3) follows directly from assumption (2) with $J_1 \triangleq I_1 \subseteq J_2 \triangleq \mathbb{R}$, and the last two equalities in (3.5) also follow from assumption (3) with $J_1 \triangleq I_1 \subseteq J_2 \triangleq I_2 \subseteq \mathbb{R}$.

We denote by $\mathrm{dom}(x)$ the domain of $x \in \mathcal{V}_a$. For $(x, y) \in \mathcal{V}_a \times \mathcal{V}_a$, the domains of x and y may be different. In this case, we adopt the convention $\mathrm{dom}(x, y) \triangleq \mathrm{dom}(x) \cap \mathrm{dom}(y)$. It can be easily seen that $\mathcal{V}_a^- \oplus \mathcal{V}_a^+ \triangleq \{v \in \mathcal{V}_a \mid 0 \in \mathrm{dom}(v)\} \subseteq \mathcal{V}_a$.

For general interval $J \subset \mathbb{R}$, the relation between $\mathcal{V}(J)$ and $\mathcal{V}_J$ is closely analogous to that between $\mathcal{V}$ and $\mathcal{V}_e$; and the space $\mathcal{V}_J$ has the feature that allows consideration of finite escape times and of initial conditions.

In addition, for any $u \in \mathcal{V}_J$ (where $J \subseteq \mathbb{R}$ is an open subinterval) and any finite subinterval $J' \subseteq J$, if $\overline{J'} \subseteq J$, then $T_{J'}u$ is bounded ($\overline{J'}$ is the closure of J' in $\mathbb{R}$). However, for general J', $T_{J'}u$ does not necessarily have the property. For example, choose an unbounded $u \in \mathcal{V}_J$ and choose $J' = J$, then the restriction of $T_{J'}u$ to J equals to u (i.e., $R_J T_{J'}u = u$) and thus $T_{J'}u$ is also unbounded. The restriction operator $R_{J'}$ has a similar property.

From the definition of $\mathcal{V}_J$ we know that there always exists a map $E_J : \mathcal{V}_J \to \mathcal{V}$ (not necessarily continuous) satisfying $R_J x = R_J(E_J x)$ for any $x \in \mathcal{V}_J$. From the definition of $\mathcal{V}_e$ we know that it is possible that for some $x \in \mathcal{V}_e$ with $\|x\| < \infty$ we still have $x \notin \mathcal{V}$ (see e.g., Section 3.3.4 on page 45 when $\mathcal{V} \equiv BUC(\mathbb{R}, \mathbb{R}^n)$).

Extended spaces appeared first in the context of input-output theory in the works of [Zames, 1966b,c] and [Sandberg, 1965b], in which only those functions whose truncations lie in the normed vector space belong to its corresponding extended space; and this implicitly imposes a *truncation closedness* condition (for a definition see Proposition 4.2 below) on the normed vector space (e.g., not if $\mathcal{V} = BC(\mathbb{R}_+, \mathbb{R})$ of all bounded continuous functions on the positive time domain). Here, the extended space is defined via restriction operators rather than usual truncation operators. It bears a certain similarity to the locally normed vector space used in the theory of differential equations (see e.g., [Delfour and Mitter, 1972]) but with a fundamental difference. In terms of notations in this thesis, the interval space $\mathcal{V}(t_1, t_2)$ used to define the extended space $\mathcal{V}_{(t_1,t_2)}$ is directly induced from the basic normed vector space $\mathcal{V}$; while the interval space $\mathcal{V}(t_1, t_2)$ used to define the locally normed vector space $\mathcal{V}_{loc}(t_1, t_2)$ is usually assigned according to experiences at the same time when assigning the normed vector space $\mathcal{V}$. (See Section 3.3.3 with $\mathcal{V} \equiv BC(\mathbb{R}, \mathbb{R}^n)$ and Section 3.3.4 with $\mathcal{V} \equiv BUC(\mathbb{R}, \mathbb{R}^n)$.)

3.3 Some Special Signal Spaces

It is very useful at this stage to present some special signal spaces and their corresponding properties. Lebesgue integral functions spaces, continuous functions spaces and Sobolev spaces are all discussed in this section.

3.3.1 Spaces of Lebesgue Integrable Functions: $L^q(\mathbb{R}, \mathbb{R}^n)$, $1 \le q < \infty$

For any positive real number q with $1 \le q < \infty$, we let $\mathcal{V} \equiv L^q(\mathbb{R}, \mathbb{R}^n)$ denote the space of all measurable functions $x : \mathbb{R} \to \mathbb{R}^n$ for which $\int_{\mathbb{R}} |x(t)|^q \,\mathrm{d}t < \infty$ and with norm $x \mapsto \|x\| \triangleq \big(\int_{\mathbb{R}} |x(t)|^q \,\mathrm{d}t\big)^{1/q}$.

For any $-\infty \le t_1 < t_2 \le +\infty$ the interval space $\mathcal{V}(t_1, t_2) \equiv L^q\big((t_1, t_2), \mathbb{R}^n\big)$ consists of all those measurable functions $x : (t_1, t_2) \to \mathbb{R}^n$ for which $\int_{t_1}^{t_2} |x(t)|^q \,\mathrm{d}t < \infty$ with norm given by $\|x\|_{(t_1,t_2)} = \left(\int_{t_1}^{t_2} |x(t)|^q \,\mathrm{d}t\right)^{1/q}$; and the extended space $\mathcal{V}_{(t_1,t_2)} \equiv L^q_e\big((t_1, t_2), \mathbb{R}^n\big)$ consists of all those measurable functions $x : (t_1, t_2) \to \mathbb{R}^n$ with the property that $R_{(\tau_1,\tau_2)} x \in L^q\big((\tau_1, \tau_2), \mathbb{R}^n\big)$ for all $-\infty \le t_1 < \tau_1 < \tau_2 < t_2 \le \infty$. The extended space $\mathcal{V}_e \equiv L^q_e(\mathbb{R}, \mathbb{R}^n)$. The ambient space $\mathcal{V}_a \equiv \bigcup_{\{(\tau_1,\tau_2) | -\infty \le \tau_1 < \tau_2 \le \infty\}} L^q_e\big((\tau_1, \tau_2), \mathbb{R}^n\big)$.

For the positive time domain $\mathbb{R}_+ \equiv [0, \infty)$, we have $\mathcal{V}^+ \equiv L^q(\mathbb{R}_+, \mathbb{R}^n)$, $\mathcal{V}^+_e \equiv L^q_e(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{V}^+_a \equiv \bigcup_{\{[0,\tau) | 0 < \tau \le \infty\}} L^q_e\big([0, \tau), \mathbb{R}^n\big)$.

Note that $\mathcal{V} \equiv L^q(\mathbb{R}, \mathbb{R}^n)$, $\mathcal{V}^+ \equiv L^q(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{V}(t_1, t_2) \equiv L^q\big((t_1, t_2), \mathbb{R}^n\big)$ are all complete normed vector spaces, i.e., Banach spaces. Define $x(t) \triangleq e^t$, $\forall t \in \mathbb{R}$, it can be easily verified that $x \in L^q_e(\mathbb{R}, \mathbb{R})$ with $\|x\| = \infty$.

For any $-\infty < t_1 < t_2 < \infty$ and any $x \in \mathcal{V} \equiv L^q(\mathbb{R}, \mathbb{R}^n)$, we have $T_{(t_1,t_2)}x \in \mathcal{V} \equiv L^q(\mathbb{R}, \mathbb{R}^n)$. The normed vector spaces $\mathcal{V} \equiv L^q(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{V}^+ \equiv L^q(\mathbb{R}_+, \mathbb{R}^n)$ are truncation complete. For any $x \in \mathcal{V}_e \equiv L^q_e(\mathbb{R}, \mathbb{R}^n)$, if $\|x\| < \infty$, then $x \in \mathcal{V} \equiv L^q(\mathbb{R}, \mathbb{R}^n)$; and in this case, we have $\big\|R_{(t_1,t_2)}x\big\|_{(t_1,t_2)} \to \|x\|$ as $t_1 \to -\infty$ and $t_2 \to \infty$. For any $-\infty < t_1 < t_2 < \infty$, there exists a linear uniformly continuous map (zero extensions) $E_{(t_1,t_2)} \triangleq T_{(t_1,t_2)} : L^q\big((t_1, t_2), \mathbb{R}^n\big) \to L^q(\mathbb{R}, \mathbb{R}^n)$ such that $R_{(t_1,t_2)}x = R_{(t_1,t_2)}(E_{(t_1,t_2)}x)$ for any $x \in L^q\big((t_1, t_2), \mathbb{R}^n\big)$.

3.3.2 Spaces of Essentially Bounded Functions: $L^\infty(\mathbb{R}, \mathbb{R}^n)$

We denote by $\mathcal{V} \equiv L^\infty(\mathbb{R}, \mathbb{R}^n)$ the space of all essentially bounded[3] measurable functions $x : \mathbb{R} \to \mathbb{R}^n$ with norm $\|x\| \triangleq \operatorname{ess\,sup}_{t\in\mathbb{R}} |x(t)|$. Then for any $-\infty \le t_1 < t_2 \le +\infty$ the interval space $\mathcal{V}(t_1, t_2) \equiv L^\infty\big((t_1, t_2), \mathbb{R}^n\big)$ consists of all those measurable functions $x : (t_1, t_2) \to \mathbb{R}^n$ for which $\operatorname{ess\,sup}_{t\in(t_1,t_2)} |x(t)| < \infty$ with norm given by $\|x\|_{(t_1,t_2)} = \operatorname{ess\,sup}_{t\in(t_1,t_2)} |x(t)|$; and the extended space $\mathcal{V}_{(t_1,t_2)} \equiv L^\infty_e\big((t_1, t_2), \mathbb{R}^n\big)$ consists of all those measurable functions $x : (t_1, t_2) \to \mathbb{R}^n$ with the property that $R_{(\tau_1,\tau_2)}x \in L^\infty\big((\tau_1, \tau_2), \mathbb{R}^n\big)$ for all $-\infty \le t_1 < \tau_1 < \tau_2 < t_2 \le \infty$. The extended space $\mathcal{V}_e \equiv L^\infty_e(\mathbb{R}, \mathbb{R}^n)$. The ambient space $\mathcal{V}_a \equiv \bigcup_{\{(\tau_1,\tau_2)|-\infty\le\tau_1<\tau_2\le\infty\}} L^\infty_e\big((\tau_1, \tau_2), \mathbb{R}^n\big)$.

For the positive time domain $\mathbb{R}_+ \equiv [0, \infty)$, we have $\mathcal{V}^+ \equiv L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, $\mathcal{V}^+_e \equiv L^\infty_e(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{V}^+_a \equiv \bigcup_{\{[0,\tau)|0<\tau\le\infty\}} L^\infty_e\big([0, \tau), \mathbb{R}^n\big)$.

Note that $\mathcal{V} \equiv L^\infty(\mathbb{R}, \mathbb{R}^n)$, $\mathcal{V}^+ \equiv L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{V}(t_1, t_2) \equiv L^\infty\big((t_1, t_2), \mathbb{R}^n\big)$ are all Banach spaces. Define $x(t) \triangleq t$, $\forall t \in \mathbb{R}$, it can be easily seen that $x \in L^\infty_e(\mathbb{R}, \mathbb{R})$ with $\|x\| = \infty$.

For any $-\infty < t_1 < t_2 < \infty$ and any $x \in \mathcal{V} \equiv L^\infty(\mathbb{R}, \mathbb{R}^n)$, we have $T_{(t_1,t_2)}x \in \mathcal{V} \equiv L^\infty(\mathbb{R}, \mathbb{R}^n)$. The normed vector spaces $\mathcal{V} \equiv L^\infty(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{V}^+ \equiv L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ are truncation complete. For any $x \in \mathcal{V}_e \equiv L^\infty_e(\mathbb{R}, \mathbb{R}^n)$, if $\|x\| < \infty$, then $x \in \mathcal{V} \equiv L^\infty(\mathbb{R}, \mathbb{R}^n)$; and in this case, we have $\big\|R_{(t_1,t_2)}x\big\|_{(t_1,t_2)} \to \|x\|$ as $t_1 \to -\infty$ and $t_2 \to \infty$. For any $-\infty < t_1 < t_2 < \infty$, there exists a linear uniformly continuous map (zero extensions) $E_{(t_1,t_2)} \triangleq T_{(t_1,t_2)} : L^\infty\big((t_1, t_2), \mathbb{R}^n\big) \to L^\infty(\mathbb{R}, \mathbb{R}^n)$ such that $R_{(t_1,t_2)}x = R_{(t_1,t_2)}(E_{(t_1,t_2)}x)$ for any $x \in L^\infty\big((t_1, t_2), \mathbb{R}^n\big)$.

[3] A measurable function x defined on Ω is said to be essentially bounded on Ω if there exists a constant K such that $|x(t)| \le K$ a.e. on Ω.

3.3.3 Spaces of Bounded, Continuous Functions: $BC(\mathbb{R},\mathbb{R}^n)$

Let $\mathcal{V} \equiv BC(\mathbb{R},\mathbb{R}^n)$ denote the space of all continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ for which $\sup_{t\in\mathbb{R}} |x(t)| < \infty$ and with norm $\|x\| \triangleq \sup_{t\in\mathbb{R}} |x(t)|$.

For any $-\infty < t_1 < t_2 < \infty$, the interval space $\mathcal{V}(t_1,t_2) \equiv BC(\mathbb{R},\mathbb{R}^n)|_{(t_1,t_2)} \equiv BUC\big((t_1,t_2),\mathbb{R}^n\big)$ consists of all those uniformly continuous functions $x : (t_1,t_2) \to \mathbb{R}^n$ for which $\sup_{t\in(t_1,t_2)} |x(t)| < \infty$ with norm given by $\|x\|_{(t_1,t_2)} = \sup_{t\in(t_1,t_2)} |x(t)|$. For any $-\infty < t < \infty$, the interval space $\mathcal{V}(-\infty,t) \equiv BC(\mathbb{R},\mathbb{R}^n)|_{(-\infty,t)}$; and the interval space $\mathcal{V}(t,\infty) \equiv BC(\mathbb{R},\mathbb{R}^n)|_{(t,\infty)}$.[4]

For any $-\infty \le t_1 < t_2 \le \infty$, the extended space $\mathcal{V}_{(t_1,t_2)} \equiv BC_e\big((t_1,t_2),\mathbb{R}^n\big)$ consists of all those continuous functions $x : (t_1,t_2) \to \mathbb{R}^n$ with the property that $R_{(\tau_1,\tau_2)}x \in BC\big((\tau_1,\tau_2),\mathbb{R}^n\big)$ for all $-\infty \le t_1 < \tau_1 < \tau_2 < t_2 \le \infty$. It is easily verified that $\mathcal{V}_{(t_1,t_2)} \equiv BC_e\big((t_1,t_2),\mathbb{R}^n\big)$ is the same as the space $C\big((t_1,t_2),\mathbb{R}^n\big)$ of all continuous function (not necessarily bounded) defined on (t_1,t_2). The extended space $\mathcal{V}_e \equiv BC_e(\mathbb{R},\mathbb{R}^n) \equiv C(\mathbb{R},\mathbb{R}^n)$. The ambient space $\mathcal{V}_a \equiv \bigcup_{\{(\tau_1,\tau_2)|-\infty\le\tau_1<\tau_2\le\infty\}} C\big((\tau_1,\tau_2),\mathbb{R}^n\big)$.

For the positive time domain $\mathbb{R}_+$, we have $\mathcal{V}^+ \equiv BC(\mathbb{R}_+,\mathbb{R}^n)$, $\mathcal{V}_e^+ \equiv BC_e(\mathbb{R}_+,\mathbb{R}^n) \equiv C(\mathbb{R}_+,\mathbb{R}^n)$ and $\mathcal{V}_a^+ \equiv \bigcup_{\{[0,\tau)|0<\tau\le\infty\}} C\big([0,\tau),\mathbb{R}^n\big)$.

Note that $\mathcal{V} \equiv BC(\mathbb{R},\mathbb{R}^n)$, $\mathcal{V}^+ \equiv BC(\mathbb{R}_+,\mathbb{R}^n)$ and $\mathcal{V}(t_1,t_2) \equiv BC(\mathbb{R},\mathbb{R}^n)|_{(t_1,t_2)}$ are all Banach spaces (see e.g., [Adams and Fournier, 2003, p. 10]). Define $x(t) \triangleq \tan\frac{\pi(2t-t_2-t_1)}{2(t_2-t_1)}$, $-\infty < t_1 < t < t_2 < \infty$, it can be shown that x belongs to $BC_e\big((t_1,t_2),\mathbb{R}\big)$ but with $\sup_{t\in(t_1,t_2)} |x(t)| = \infty$; and $y(t) \triangleq t$, $(t \in \mathbb{R})$ belongs to $BC_e(\mathbb{R},\mathbb{R})$ but with $\sup_{t\in\mathbb{R}} |x(t)| = \infty$.

Define $x(t) \triangleq e^{-|t|}$ for any $t \in \mathbb{R}$, it can be easily verified that $x \in \mathcal{V} \equiv BC(\mathbb{R},\mathbb{R}^n)$ but $T_{(-1,1)}x \notin \mathcal{V} \equiv BC(\mathbb{R},\mathbb{R}^n)$ since $T_{(-1,1)}x$ is not continuous on $\mathbb{R}$. The normed vector spaces $\mathcal{V} \equiv BC(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{V}^+ \equiv BC(\mathbb{R}_+,\mathbb{R}^n)$ are truncation complete. For any $x \in \mathcal{V}_e \equiv BC_e(\mathbb{R},\mathbb{R}^n) \equiv C(\mathbb{R},\mathbb{R}^n)$, if $\|x\| < \infty$, then $x \in \mathcal{V} \equiv BC(\mathbb{R},\mathbb{R}^n)$; and in this case, we have $\left\|R_{(t_1,t_2)}x\right\|_{(t_1,t_2)} \to \|x\|$ as $t_1 \to -\infty$ and $t_2 \to \infty$.

For any $-\infty < t_1 < t_2 < \infty$ and any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, define a linear uniformly continuous map $E_{(t_1,t_2)}^{\varepsilon_1,\varepsilon_2}$ from $\mathcal{V}(t_1,t_2) \equiv BUC\big((t_1,t_2),\mathbb{R}^n\big)$ into $BUC(\mathbb{R},\mathbb{R}^n) \subset \mathcal{V} \equiv BC(\mathbb{R},\mathbb{R}^n)$ as

[4] Note that $\mathcal{V}(t_1,t_2)$ is not the same as the space $BC\big((t_1,t_2),\mathbb{R}^n\big)$ of all bounded continuous functions on (t_1,t_2) when $(t_1,t_2) \ne \mathbb{R}$. For example, the function $x(t) \triangleq \sin(1/t)$, $(0 < t < 1)$ belongs to $BC\big((0,1),\mathbb{R}\big)$, but x is not continuous extendable to the domain $(-\infty,\infty)$.

follows: [5]

$$(E^{\varepsilon_1,\varepsilon_2}_{(t_1,t_2)}x)(t) = \begin{cases} x(t), & \text{if } t \in (t_1,\, t_2) \\ (t-t_1)\cdot x(t_1^+)/\varepsilon_1 + x(t_1^+), & \text{if } t \in [t_1-\varepsilon_1,\, t_1] \\ (t-t_2)\cdot x(t_2^-)/\varepsilon_2 + x(t_2^-), & \text{if } t \in [t_2,\, t_2+\varepsilon_2] \\ 0, & \text{if } t \in (-\infty,\, t_1-\varepsilon_1)\cup(t_2+\varepsilon_2,\, \infty) \end{cases} \tag{3.6}$$

for all $x \in BUC\big((t_1,t_2),\mathbb{R}^n\big)$; it is easily verified that $R_{(t_1,t_2)}x = R_{(t_1,t_2)}(E^{\varepsilon_1,\varepsilon_2}_{(t_1,t_2)}x)$ for any $x \in BUC\big((t_1,t_2),\mathbb{R}^n\big)$.

3.3.4 Spaces of Bounded, Uniformly Continuous Functions: $BUC(\mathbb{R},\mathbb{R}^n)$

We define the space $\mathcal{V} \equiv BUC(\mathbb{R},\mathbb{R}^n)$ to consist of all those uniformly continuous functions $x:\mathbb{R}\to\mathbb{R}^n$ for which $\sup_{t\in\mathbb{R}}|x(t)| < \infty$ and with norm $\|x\| \triangleq \sup_{t\in\mathbb{R}}|x(t)|$.

For any $-\infty \le t_1 < t_2 \le +\infty$ the interval space $\mathcal{V}(t_1,t_2) \equiv BUC\big((t_1,t_2),\mathbb{R}^n\big)$ consists of all those uniformly continuous functions $x:(t_1,t_2)\to\mathbb{R}^n$ for which $\sup_{t\in(t_1,t_2)}|x(t)| < \infty$ with norm given by $\|x\|_{(t_1,t_2)} = \sup_{t\in(t_1,t_2)}|x(t)|$; and the extended space $\mathcal{V}_{(t_1,t_2)} \equiv BUC_e\big((t_1,t_2),\mathbb{R}^n\big)$ consists of all those uniformly continuous functions $x:(t_1,t_2)\to\mathbb{R}^n$ with the property that $R_{(\tau_1,\tau_2)}x \in BUC\big((\tau_1,\tau_2),\mathbb{R}^n\big)$ for all $-\infty \le t_1 < \tau_1 < \tau_2 < t_2 \le \infty$. It is easily verified that $\mathcal{V}_{(t_1,t_2)} \equiv BUC_e\big((t_1,t_2),\mathbb{R}^n\big) \equiv C\big((t_1,t_2),\mathbb{R}^n\big)$. The extended space $\mathcal{V}_e \equiv C(\mathbb{R},\mathbb{R}^n)$. The ambient space $\mathcal{V}_a \equiv \bigcup_{\{(\tau_1,\tau_2)|-\infty\le\tau_1<\tau_2\le\infty\}} C\big((\tau_1,\tau_2),\mathbb{R}^n\big)$.

For the positive time domain $\mathbb{R}_+$, we have $\mathcal{V}^+ \equiv BUC(\mathbb{R}_+,\mathbb{R}^n)$, $\mathcal{V}_e^+ \equiv BUC_e(\mathbb{R}_+,\mathbb{R}^n) \equiv C(\mathbb{R}_+,\mathbb{R}^n)$ and $\mathcal{V}_a^+ \equiv \bigcup_{\{[0,\tau)|0<\tau\le\infty\}} C\big([0,\tau),\mathbb{R}^n\big)$.

Note that $\mathcal{V} \equiv BUC(\mathbb{R},\mathbb{R}^n)$, $\mathcal{V}^+ \equiv BUC(\mathbb{R}_+,\mathbb{R}^n)$ and $\mathcal{V}(t_1,t_2) \equiv BUC\big((t_1,t_2),\mathbb{R}^n\big)$ are closed subspaces of $BC(\mathbb{R},\mathbb{R}^n)$, $BC(\mathbb{R}_+,\mathbb{R}^n)$ and $BC\big((t_1,t_2),\mathbb{R}^n\big)$, respectively; and thus also Banach spaces (see e.g., [Adams and Fournier, 2003, p. 10]). Note that $x(t) \triangleq 1/t$, $(0 < t < 1)$ belongs to $BUC_e\big((0,1),\mathbb{R}\big)$ but with $\sup_{t\in(t_1,t_2)}|x(t)| = \infty$; and $y(t) \triangleq t^2$, $(t \in \mathbb{R})$ belongs to $BUC_e(\mathbb{R},\mathbb{R})$ but with $\sup_{t\in\mathbb{R}}|x(t)| = \infty$.)

Define $x(t) \triangleq \sin(t)$ for any $t \in \mathbb{R}$, it can be easily verified that $x \in \mathcal{V} \equiv BUC(\mathbb{R},\mathbb{R}^n)$ but $T_{(-1,1)}x \notin \mathcal{V} \equiv BUC(\mathbb{R},\mathbb{R}^n)$ since $T_{(-1,1)}x$ is not even continuous on $\mathbb{R}$. The normed vector spaces $\mathcal{V} \equiv BUC(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{V}^+ \equiv BUC(\mathbb{R}_+,\mathbb{R}^n)$ are truncation complete. Define $y(t) \triangleq \sin(t^2)$ for any $t \in \mathbb{R}$, it is easily verified that $y \in \mathcal{V}_e \equiv BUC_e(\mathbb{R},\mathbb{R}^n) \equiv C(\mathbb{R},\mathbb{R}^n)$ with $\|y\| = \sup_{t\in\mathbb{R}}\big|\sin(t^2)\big| < \infty$ but $y \notin \mathcal{V} \equiv BUC(\mathbb{R},\mathbb{R}^n)$, since y is not uniformly continuous on $\mathbb{R}$. For any $-\infty < t_1 < t_2 < \infty$ and any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists a linear uniformly continuous map $E^{\varepsilon_1,\varepsilon_2}_{(t_1,t_2)}$ defined by (3.6) from $\mathcal{V}(t_1,t_2) \equiv$

[5] We define $x(t_1^+) \triangleq \lim_{t\to t_1^+} x(t)$ for any $x \in BUC\big((t_1,t_2),\mathbb{R}^n\big)$. Note that this is possible, since every bounded and uniformly continuous function on a open interval Ω posses a unique, bounded continuous extension to the closure $\bar{\Omega}$ of Ω (see e.g., [Adams and Fournier, 2003, p. 10]).

$BUC\big((t_1,t_2),\mathbb{R}^n\big)$ into $\mathcal{V} \equiv BUC(\mathbb{R},\mathbb{R}^n)$ such that $R_{(t_1,t_2)}x = R_{(t_1,t_2)}(E^{\varepsilon_1,\varepsilon_2}_{(t_1,t_2)}x)$ for any $x \in \mathcal{V}(t_1,t_2) \equiv BUC\big((t_1,t_2),\mathbb{R}^n\big)$.

3.3.5 Sobolev Spaces: $W^{r,q}(\mathbb{R},\mathbb{R}^n)$, $1 \le q < \infty$

For any positive integer r and any positive real number q with $1 \le q < \infty$, we let $\mathcal{V} \equiv W^{r,q}(\mathbb{R},\mathbb{R}^n)$ denote the Sobolev space of all those r-times weakly (or distributionally) differentiable functions $x : \mathbb{R} \to \mathbb{R}^n$ for which $D^i x \in L^q(\mathbb{R},\mathbb{R}^n)$, $\forall\, 0 \le i \le r$ with the Sobolev norm $\|x\|_{r,q} \triangleq \left(\sum_{0\le i\le r} \left\|D^i x\right\|_q^q\right)^{1/q}$, where $D^i x$ is the i-th weak (or distributional) derivative[6] of x and $\|\cdot\|_q$ is the norm in $L^q(\mathbb{R},\mathbb{R}^n)$ (see e.g., [Adams and Fournier, 2003, p. 59]). Note that the Sobolev space $\mathcal{V} \equiv W^{r,q}(\mathbb{R},\mathbb{R}^n)$ coincides with the space $H^{r,q}(\mathbb{R},\mathbb{R}^n)$ which denotes the completion of the space $CW^{r,q}(\mathbb{R},\mathbb{R}^n) \triangleq \{x \in C^r(\mathbb{R},\mathbb{R}^n) : \|x\|_{r,q} < \infty\}$ with respect to the norm $\|\cdot\|_{r,q}$ (see e.g., [Adams and Fournier, 2003, Theorem 3.17, p. 59]), where $C^r(\mathbb{R},\mathbb{R}^n)$ denotes the space of all those r-times differentiable (in the classical sense) functions $x : \mathbb{R} \to \mathbb{R}^n$ for which the (classical) derivative $x^{(r)}$ is continuous.

For any $-\infty \le t_1 < t_2 \le +\infty$ the interval space $\mathcal{V}(t_1,t_2) \equiv W^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$ consists of all those r-times weakly differentiable functions $x : (t_1,t_2) \to \mathbb{R}^n$ for which $D^i x \in L^q\big((t_1,t_2),\mathbb{R}^n\big)$ with norm given by $\|x\|_{r,q,(t_1,t_2)} = \left(\sum_{0\le i\le r} \left\|D^i x\right\|_{q,(t_1,t_2)}^q\right)^{1/q}$; and the extended space $\mathcal{V}_{(t_1,t_2)} \equiv W_e^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$ consists of all those r-times weakly differentiable functions $x : (t_1,t_2) \to \mathbb{R}^n$ with the property that $R_{(\tau_1,\tau_2)}x \in W^{r,q}\big((\tau_1,\tau_2),\mathbb{R}^n\big)$ for all $-\infty \le t_1 < \tau_1 < \tau_2 < t_2 \le \infty$. The extended space $\mathcal{V}_e \equiv W_e^{r,q}(\mathbb{R},\mathbb{R}^n)$. The ambient space $\mathcal{V}_a \equiv \bigcup_{\{(\tau_1,\tau_2)|-\infty\le\tau_1<\tau_2\le\infty\}} W_e^{r,q}\big((\tau_1,\tau_2),\mathbb{R}^n\big)$.

Note that $\mathcal{V} \equiv W^{r,q}(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{V}(t_1,t_2) \equiv W^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$ are all Banach spaces (see e.g., [Adams and Fournier, 2003, p. 60]).[7] Note that $x(t) \triangleq e^t$, $\forall t \in \mathbb{R}$ belongs to $W_e^{r,q}(\mathbb{R},\mathbb{R})$ but with $\|x\|_{r,q} = \infty$).

For any $-\infty < t_1 < t_2 < \infty$ and any $x \in \mathcal{V} \equiv W^{r,q}(\mathbb{R},\mathbb{R}^n)$, we have $T_{(t_1,t_2)}x \in \mathcal{V} \equiv W^{r,q}(\mathbb{R},\mathbb{R}^n)$. The normed vector spaces $\mathcal{V} \equiv W^{r,q}(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{V}^+ \equiv W^{r,q}(\mathbb{R}_+,\mathbb{R}^n)$ are truncation complete. For any $x \in \mathcal{V}_e \equiv W_e^{r,q}(\mathbb{R},\mathbb{R}^n)$, if $\|x\| < \infty$, then $x \in \mathcal{V} \equiv W^{r,q}(\mathbb{R},\mathbb{R}^n)$; and in this case, we have $\left\|R_{(t_1,t_2)}x\right\|_{(t_1,t_2)} \to \|x\|$ as $t_1 \to -\infty$ and $t_2 \to \infty$.

From [Adams and Fournier, 2003, p. 146], we know that for any $-\infty < t_1 < t_2 < \infty$, there always exists a linear continuous map $E_{(t_1,t_2)} : W^{r,q}\big((t_1,t_2),\mathbb{R}^n\big) \to W^{r,q}(\mathbb{R},\mathbb{R}^n)$ such that $R_{(t_1,t_2)}x = R_{(t_1,t_2)}(E_{(t_1,t_2)}x)$ for any $x \in W^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$. It is useful to remark that the truncation (or zero extensions) operator $T_{(t_1,t_2)}$ only defines a continuous map from $W_0^{r,q}\big((t_1,t_2),\mathbb{R}^n\big) \subsetneq W^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$ to $W^{r,q}(\mathbb{R},\mathbb{R}^n) \equiv W_0^{r,q}(\mathbb{R},\mathbb{R}^n)$; in fact,

[6] Note that the weak derivative $D^i x$ coincides with the classical derivative $x^{(i)}$ when $x \in C^i(\mathbb{R},\mathbb{R}^n)$.

[7] Note that both $CW^{r,q}(\mathbb{R},\mathbb{R}^n)$ and $CW^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$, $(1 \le q < \infty)$ are normed vector spaces (but not complete) with respect to the norms $\|\cdot\|_{r,q}$ and $\|\cdot\|_{r,q,(t_1,t_2)}$, respectively.

a function x defined on (t_1, t_2) belongs to $W_0^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$ if and only of $T_{(t_1,t_2)}x$ belongs to $W^{r,q}(\mathbb{R},\mathbb{R}^n)$ (see e.g., [Adams and Fournier, 2003, pp. 70, 71, 159]).[8]

3.3.6 Additional Consideration of Special Signal Spaces

For any open subinterval $J \triangleq (t_1, t_2)$ with $-\infty \le t_1 < t_2 \le \infty$ and any positive integer r, we define the space $W^{r,\infty}(J,\mathbb{R}^n)$ to consist of all those r-times weakly differentiable functions $x : J \to \mathbb{R}^n$ for which $D^i x \in L^\infty(J,\mathbb{R}^n)$, $\forall\, 0 \le i \le r$ with norm $\|x\|_{r,\infty,J} \triangleq \max_{0\le i\le r}\big(\operatorname{ess\,sup}_{t\in J}\big|(D^i x)(t)\big|\big)$. Define a space $CW^{r,\infty}(J,\mathbb{R}^n) \triangleq \{x \in C^r(J,\mathbb{R}^n) : \|x\|_{r,\infty,J} < \infty\}$; and let $H^{r,\infty}(J,\mathbb{R}^n)$ denote the completion of $CW^{r,\infty}(J,\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{r,\infty,J}$. Then $W^{r,\infty}(J,\mathbb{R}^n)$ and $H^{r,\infty}(J,\mathbb{R}^n) = CW^{r,\infty}(J,\mathbb{R}^n)$ are Banach spaces; and $H^{r,\infty}(J,\mathbb{R}^n) \subsetneq W^{r,\infty}(J,\mathbb{R}^n)$ (see e.g., [Adams and Fournier, 2003, pp. 10, 61, 67]).[9] We can similarly define the interval spaces, extended spaces and ambient spaces for above discussed normed vector spaces. Other useful normed vector spaces such as absolutely continuous functions spaces, Lipschitz continuous functions spaces and Hölder continuous functions spaces can be found in the same book [Adams and Fournier, 2003].

3.4 Systems

A system in the control sense is an input-output relation, which is viewed as a "black box" mapping inputs to outputs [Zames, 1963]. The essence is that only the relationship between inputs and outputs is a-priori relevant. In this sense, notions of a system and of stability should be made without the axiomatical postulation of state.

We are now in a position to introduce a precise definition of a system, which is defined in a set theoretic manner from input-output pairs on a doubly infinite time axis,[10] i.e., a set of all possible input-output trajectories on the time domain $(-\infty, \infty)$ compatible with the description of the system.

Definition 3.2. Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$, a system Q is defined via the specification of a subset $\mathfrak{B}_Q \subseteq \mathcal{W}_e$.

Note that here we did not exactly give a mathematical definition for the input and the output; $(u, y) \in \mathcal{W}_e$ is called an input-output pair. At this stage, we do not impose any further requirements on the input/output partition. If we consider a system as a black

[8]Note that $W_0^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$ denotes the closure of $C_c^\infty\big((t_1,t_2),\mathbb{R}^n\big)$ in the Banach space $W^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$, where $C_c^\infty\big((t_1,t_2),\mathbb{R}^n\big)$ consists of all those smooth (or infinitely differentiable) functions with compact support in (t_1, t_2). The space $W_0^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$ is a Banach space itself, since it is closed in $W^{r,q}\big((t_1,t_2),\mathbb{R}^n\big)$.

[9]For instance, $x(t) \triangleq |t|$, $(-1 < t < 1)$ belongs to $W^{1,\infty}\big((-1,1),\mathbb{R}\big)$; but $x \notin H^{1,\infty}\big((-1,1),\mathbb{R}\big)$.

[10]This will be slightly modified for systems with potential for finite-time escape (see Section 4.7 on page 100).

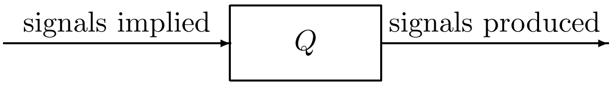

Figure 3.2: A black box

box shown in Figure 3.2, which produces some signal when implying each signal, then it is intuitive to label the signal implied as an input and the one produced as an output.

We call u and $\mathcal{U}_e$ an input variable and the input signal space of Q, respectively; similarly, y and $\mathcal{Y}_e$ an output variable and the output signal space of Q, respectively.

Example 3.3. *Let Q be an input-output operator from $L_e^2(\mathbb{R}, \mathbb{R}^m)$ to $L_e^2(\mathbb{R}, \mathbb{R}^p)$, and define $\mathcal{U}_e \triangleq L_e^2(\mathbb{R}, \mathbb{R}^m)$ and $\mathcal{Y}_e \triangleq L_e^2(\mathbb{R}, \mathbb{R}^p)$. Then the system Q is represented by the set $\mathfrak{B}_Q = \{(u, y) \in \mathcal{U}_e \times \mathcal{Y}_e \mid y = Qu\}$.*

Example 3.4. *Let Q be an input-output operator from $L_e^2(\mathbb{R}_+, \mathbb{R}^m)$ to $L_e^2(\mathbb{R}_+, \mathbb{R}^p)$, and define $\mathcal{U}_e \triangleq L_e^2(\mathbb{R}, \mathbb{R}^m)$ and $\mathcal{Y}_e \triangleq L_e^2(\mathbb{R}, \mathbb{R}^m)$. Then the system Q is represented by the set*

$$\mathfrak{B}_Q = \{(u, y) \in \mathcal{U}_e \times \mathcal{Y}_e \mid R_- y = R_- u = 0, R_+ y = Q(R_+ u)\}$$

Note that the above definition of a system is slightly different from both Zames's representation of input-output systems by operators [Zames, 1960] and Willems's structure of input-output systems by behaviours with input/output partition [Polderman and Willems, 1998, Definition 3.3.1, p. 84], [Willems, 1991].[11] Here, we allow both (u, y_1) and (u, y_2) with $y_1 \neq y_2$ belong to the same set $\mathfrak{B}_Q$. And it does not require that for any $u \in \mathcal{U}_e$ there exists a $y \in \mathcal{Y}_e$ such that $(u, y) \in \mathfrak{B}_Q$.

Example 3.5. *Let $\mathcal{U} = \mathcal{Y} \triangleq L^2(\mathbb{R}; \mathbb{R})$ and consider the system Q represented by the set $\mathfrak{B}_Q = \left\{(u, y) \in \mathcal{U}_e \times \mathcal{Y}_e \mid y^2 = u\right\}$. It is easy to verify that for $u(t) = e^{-2|t|}, t \in \mathbb{R}$ and $y(t) = e^{-|t|}, t \in \mathbb{R}$ we have both (u, y) and $(u, -y)$ belong to $\mathfrak{B}_Q$, and that for $u(t) = -e^{-2|t|}, t \in \mathbb{R}$ (so $u \in \mathcal{U}_e$), there is no $y \in \mathcal{Y}_e$ such that $(u, y) \in \mathfrak{B}_Q$.*

We will see in the subsequent sections that this definition of systems allows us to define initial conditions for systems appropriately and to treat in a unified manner systems with initial conditions of a structurally different type (e.g., both time delay distributed parameter and ODE systems) and to make it compatible with the definition of closed-loop systems.

[11] In Willems's behavioural framework, a system Σ is defined as a triple $\Sigma \triangleq (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} \subseteq \mathbb{R}$ the time axis, $\mathbb{W}$ the values-space of time signals, and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the behaviour ($\mathbb{W}^{\mathbb{T}}$ represents the set of all time functions from $\mathbb{T}$ to $\mathbb{W}$). The behaviour $\mathfrak{B}$ is simply a set of time trajectories compatible with the laws that govern the system. Willems's input-output system $\Sigma_{I/O}$ is defined as a quadruple $\Sigma_{I/O} \triangleq (\mathbb{T}, \mathbb{U}, \mathbb{Y}, \mathfrak{B})$ with $\mathbb{T} \subseteq \mathbb{R}$ the time axis, $\mathbb{U}$ the values-space of input time signals, $\mathbb{Y}$ the values-space of output time signals, and $\mathfrak{B} \subseteq (\mathbb{U}, \mathbb{Y})^{\mathbb{T}}$ the behaviour, such that the following axioms are satisfied: (1) $u \in \mathbb{U}^{\mathbb{T}}$ is free; i.e., for all $u \in \mathbb{U}^{\mathbb{T}}$, there exists a $y \in \mathbb{Y}^{\mathbb{T}}$ such that $(u, y) \in \mathfrak{B}$; (2) Output ($y \in \mathbb{Y}^{\mathbb{T}}$) processes input ($u \in \mathbb{U}^{\mathbb{T}}$), i.e., for any $t_0 \in \mathbb{T}$, $\{(u, y), (u, y') \in \mathfrak{B},\ y(t) = y'(t)$ for $t < t_0$ $(t \in \mathbb{T})\} \Rightarrow \{y = y'\}$.

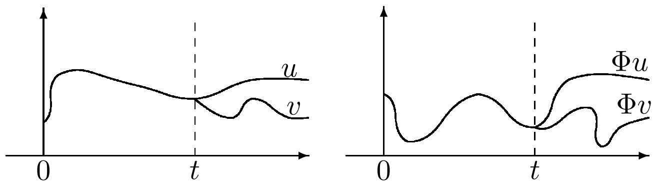

Figure 3.3: Causal operator Φ

The following is the definition of a linear time-invariant system.

Definition 3.6. A system Q (see Definition 3.2) is said to be *linear* if the set $\mathfrak{B}_Q$ is a vector space, i.e., $\lambda_1 w_1 + \lambda_2 w_2 \in \mathfrak{B}_Q$ for any $w_1, w_2 \in \mathfrak{B}_Q$ and any $\lambda_1, \lambda_2 \in \mathbb{R}$. It is said to be *time-invariant* if $w \in \mathfrak{B}_Q$ implies $w(\cdot + \tau) \in \mathfrak{B}_Q$ for all $\tau \in \mathbb{R}$.

3.4.1 Causality of Systems

The notion of causality is a fundamental property of dynamical systems. We start from the definition of a causal operator and then generalise it to the concept of a causal system in the presenting framework.

Given normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$, an operator $\Phi : \mathcal{U}_e^+ \to \mathcal{Y}_e^+$ is said to be *causal* if,

$$\forall u, v \in \mathcal{U}_e^+, \forall t > 0 : \left[u|_{[0,t]} = v|_{[0,t]} \Rightarrow (\Phi u)|_{[0,t]} = (\Phi v)|_{[0,t]} \right].$$

The definition of causality for an operator captures the essence of the idea that the current outputs depend only the past and current inputs and not on future ones. For a clear understanding of the causality condition of an operator, see Figure 3.3 or [Marquez, 2003, Chapter 6].

The following definition of a causal system [Bian et al., 2008] is a generalisation of the concept of a casual operator.

Definition 3.7. A system Q (see Definition 3.2) is said to be *causal* if

$$\forall (u, y_u), (v, y_v) \in \mathfrak{B}_Q, \forall t \in \mathbb{R} : \left[u|_{(-\infty,t]} = v|_{(-\infty,t]} \Rightarrow \mathfrak{B}_Q^u|_{(-\infty,t]} = \mathfrak{B}_Q^v|_{(-\infty,t]} \right],$$

where $\mathfrak{B}_Q^u \triangleq \{(u, y) \in \mathcal{W}_e \mid \exists y \text{ such that } (u, y) \in \mathfrak{B}_Q\}$.

Here the definition of causality is equivalent to the definition of non-anticipation in [Willems, 1991, Definition VIII.4]. Note that any operator $\Phi : \mathcal{U}_e^+ \to \mathcal{Y}_e^+$ with $\Phi(0) = 0$ can be represented by a system $\mathfrak{B}_\Phi = \{w = (u, y) \in \mathcal{U}_e \times \mathcal{Y}_e \mid R_- y = R_- u = 0, R_+ y = \Phi(R_+ u)\}$. According to both above definitions, the operator Φ is causal if and only if the system $\mathfrak{B}_\Phi$ is causal.

3.4.2 Existence, Uniqueness and Well-Posedness of Systems

We will be interested to define system properties using trajectories defined on the positive direction time line $[t, \infty)$. In order to define the well-posedness of a system, we first introduce the two properties of existence and uniqueness of a system.

In the following, we fix the initial time $t = 0$ if not otherwise specified and use the notation $\mathfrak{B}_Q^-$ defined as follows to denote the system Q's past trajectories:

$$\mathfrak{B}_Q^- \triangleq R_- \mathfrak{B}_Q = \left\{ w_- \in \mathcal{W}_e^- \,|\, \exists\, w_+ \in \mathcal{W}_e^+, \text{ s.t. } w_- \wedge w_+ \in \mathfrak{B}_Q \right\}, \tag{3.7}$$

where the *concatenation* $\wedge$ is defined as follows (see e.g., [Chen et al., 2007, Willems, 1991]):

$$(u \wedge_\tau v)(t) \triangleq \begin{cases} u(t), & \text{for } t < \tau, \\ v(t), & \text{for } t \geq \tau, \end{cases} \tag{3.8}$$

for any $\tau \in \mathbb{R}$, and we abbreviate $u \wedge v \triangleq u \wedge_0 v$.

Definition 3.8. A system Q (see Definition 3.2) is said to have the *existence property* if for any $w_- \in \mathfrak{B}_Q^-$ and any $u_+ \in \mathcal{U}_e^+$ there exists a $y_+ \in \mathcal{Y}_e^+$ such that $w_- \wedge (u_+, y_+) \in \mathfrak{B}_Q$; and the *uniqueness property* if for any $w_- \in \mathfrak{B}_Q^-$ and any $w_+ \triangleq (u_+, y_+) \in \mathcal{W}_e^+$, $\tilde{w}_+ \triangleq (\tilde{u}_+, \tilde{y}_+) \in \mathcal{W}_e^+$, we have

$$w_- \wedge w_+, w_- \wedge \tilde{w}_+ \in \mathfrak{B}_Q \text{ with } u_+ = \tilde{u}_+ \;\Rightarrow\; y_+ = \tilde{y}_+,$$

and is *well-posed* if it has both the existence and uniqueness properties.

Well-posedness means that the future output y_+ can be deduced from the set $\mathfrak{B}_Q$ (representing system properties) and the past input-output pair (u_-, y_-) and the future input u_+. Uniqueness property is equivalent to the concept of *output processes input* (see e.g., [Willems, 1991]) defined as

$$(u, y), (u, y') \in \mathfrak{B}_Q, \; y(t) = y'(t) \text{ for } t < 0 \;\Rightarrow\; y = y'.$$

In [Willems, 1991], the property of output processes input together with some other properties are postulated as axioms that need to be satisfied when defining input-output dynamical systems. We remark that this is not appropriate in the context of feedback theory, since properties such as existence and uniqueness are not automatically satisfied by the closed-loop system (see e.g., Section 3.7.4 on page 72).

Note that if we replace the initial time 0 with any time $t_0 \in \mathbb{R}$, then the definitions of a system's existence, uniqueness and well-posedness property also need to be slightly changed by letting separating time 0 to be time t_0.

3.4.3 Additional Consideration of Causality and Uniqueness

Thus far, we have defined the properties of causality and uniqueness separately for a system. These two properties are closely related to each other as can be seen from the following proposition.

Proposition 3.9. *For any system Q (see Definition 3.2), suppose that Q is causal (see Definition 3.7), and that Q has the uniqueness property (see Definition 3.8). Then for any $w_- \triangleq (u_-, y_-) \in \mathfrak{B}_Q^-$, any $w_+ \triangleq (u_+, y_+) \in \mathcal{W}_e^+$, $\tilde{w}_+ \triangleq (\tilde{u}_+, \tilde{y}_+) \in \mathcal{W}_e^+$, and any $\tau \in (0, \infty)$, we have*

$$w_- \wedge w_+, w_- \wedge \tilde{w}_+ \in \mathfrak{B}_Q \text{ with } u_+|_{[0,\tau)} = \tilde{u}_+|_{[0,\tau)} \;\Rightarrow\; y_+|_{[0,\tau)} = \tilde{y}_+|_{[0,\tau)}.$$

Proof. Define $w \triangleq (u, y) \triangleq (u_- \wedge u_+, y_- \wedge y_+)$ and $\tilde{w} \triangleq (\tilde{u}, \tilde{y}) \triangleq (u_- \wedge \tilde{u}_+, y_- \wedge \tilde{y}_+)$; thus $w = w_- \wedge w_+$ and $\tilde{w} = w_- \wedge \tilde{w}_+$. Since the system Q is causal and $u|_{(-\infty,\tau)} = \tilde{u}|_{(-\infty,\tau)}$ (note that $u_+|_{[0,\tau)} = \tilde{u}_+|_{[0,\tau)}$), we obtain that $\mathfrak{B}_Q^u|_{(-\infty,\tau)} = \mathfrak{B}_Q^{\tilde{u}}|_{(-\infty,\tau)}$ with $\mathfrak{B}_Q^u$ defined as in Definition 3.7. It follows from the fact $(u, y)|_{(-\infty,\tau)} \in \mathfrak{B}_Q^u|_{(-\infty,\tau)} = \mathfrak{B}_Q^{\tilde{u}}|_{(-\infty,\tau)}$ that there exists a $\hat{y} \triangleq \hat{y}_- \wedge \hat{y}_+ \in \mathcal{Y}_e$ satisfying $(\tilde{u}, \hat{y}) \in \mathfrak{B}_Q^{\tilde{u}} \subseteq \mathcal{W}_e$ and $(\tilde{u}, \hat{y})|_{(-\infty,\tau)} = (u, y)|_{(-\infty,\tau)} = w|_{(-\infty,\tau)}$. Hence, we have $\hat{y}_- = y_-$ and $\hat{y}_+|_{[0,\tau)} = y_+|_{[0,\tau)}$. To conclude the proof, we only have to show that $\hat{y}_+|_{[0,\tau)} = \tilde{y}_+|_{[0,\tau)}$. This follows directly from the uniqueness property of the system Q and the fact that $w_- \wedge (\tilde{u}_+, \tilde{y}_+) = \tilde{w} \in \mathfrak{B}_Q$ and $w_- \wedge (\tilde{u}_+, \hat{y}_+) = (\tilde{u}, \hat{y}) \in \mathfrak{B}_Q^{\tilde{u}} \subseteq \mathfrak{B}_Q$ (in fact, we have $\hat{y}_+ = \tilde{y}_+$). $\square$

The following result concerning properties of causality and uniqueness of a system will be used in the proof of Theorem 4.8 in Chapter 4 on page 81.

Corollary 3.10. *For any system Q (see Definition 3.2), suppose that Q is causal (see Definition 3.7), and that Q has the uniqueness property (see Definition 3.8). If for any $w_- \in \mathfrak{B}_Q^-$, any $u_+ \in \mathcal{U}_e^+$, and any $\tau \in (0, \infty)$, there exists a $y_+^\tau \in \mathcal{Y}_e^+$ such that $[w_- \wedge (u_+, y_+^\tau)]\big|_{(-\infty,\tau)} \in \mathfrak{B}_Q|_{(-\infty,\tau)}$. Then the system Q is well-posed.*

Proof. We only need to show that the system Q has the existence property. To this end, fix any $w_- \in \mathfrak{B}_Q^-$ and any $u_+ \in \mathcal{U}_e^+$, define a time function $y_+(t)$ on the positive infinite interval, $0 \le t < \infty$ as follows: for any $t \ge 0$, choose some $\tau \in (0, \infty)$ with $\tau > t$, let $y_+(t) \triangleq y_+^\tau(t)$. This function y_+ is well-defined.[12] It follows from the definition of $\mathcal{Y}_e^+$ that $y_+ \in \mathcal{Y}_e^+$, since $y_+|_{[0,\tau)} = y_+^\tau|_{[0,\tau)}$ with $y_+^\tau \in \mathcal{Y}_e^+$ for all $0 < \tau < \infty$. To conclude the proof, we need to show $w_- \wedge (u_+, y_+) \in \mathfrak{B}_Q$. This is obvious since $[w_- \wedge (u_+, y_+)]\big|_{(-\infty,\tau)} = [w_- \wedge (u_+, y_+^\tau)]\big|_{(-\infty,\tau)} \in \mathfrak{B}_Q|_{(-\infty,\tau)}$ for all $0 < \tau < \infty$. $\square$

[12] To see this, it suffices to show that $y_+^{\tau_1}|_{[0,\tau_1)} = y_+^{\tau_2}|_{[0,\tau_1)}$ for any $0 < \tau_1 < \tau_2 < \infty$. This follows directly from Proposition 3.9, since the system Q is causal and has the uniqueness property.

3.4.4 Graph of Systems

The notion of graph of a system plays an important role in the nonlinear input-output robust control theory, which is very useful in the characterisation of model uncertainties via gap metric [Georgiou and Smith, 1997b]. The graph of a system is just the set of all bounded input-output pairs which are compatible with the system. The following is a generalisation of the notion of graph for systems with initial conditions.

The *graph* $\mathcal{G}_Q^{w_-}$ of a system Q (see Definition 3.2) for a given past trajectory $w_- \in \mathfrak{B}_Q^-$ is defined by

$$\mathcal{G}_Q^{w_-} \triangleq \left\{ w_+ \in \mathcal{W}^+ \mid w_- \wedge w_+ \in \mathfrak{B}_Q \right\} \subseteq \mathcal{W}^+. \tag{3.9}$$

This generalises the definition of graph for a system represented by an input-output operator defined on positive time domain, i.e., the graph of an input-output operator P from $\mathcal{U}_e^+$ to $\mathcal{Y}_e^+$ (e.g., from $L_e^2(\mathbb{R}_+, \mathbb{R}^m)$ to $L_e^2(\mathbb{R}_+, \mathbb{R}^p)$) with $P0 = 0$ is defined as

$$\mathcal{G}_P^0 \triangleq \left\{ (u_+, y_+) \in \mathcal{W}^+ \mid y_+ = Pu_+ \text{ with } u_+ \in \mathcal{U}^+,\, y_+ \in \mathcal{Y}^+ \right\} \subseteq \mathcal{W}^+ \triangleq \mathcal{U}^+ \times \mathcal{Y}^+.$$

In [Doyle et al., 1993, Proposition 4], we know that the feedback interconnection $[P, C]$ depicted in Figure 3.1 with two input-output operators $P : \mathcal{U}_e^+ \to \mathcal{Y}_e^+$ and $C : \mathcal{Y}_e^+ \to \mathcal{U}_e^+$ is well-posed and stable[13] if and only if $\mathcal{W}^+$ can be written as a direct sum $\mathcal{W}^+ = \mathcal{G}_P^0 \oplus \mathcal{G}_C^0$, i.e., for any $w_0 \in \mathcal{W}^+$, there exist unique $w_1 \in \mathcal{G}_P^0$ and $w_2 \in \mathcal{G}_C^0$ such that $w_0 = w_1 + w_2$. In this case, the *inverse* graph of C is understood as $\mathcal{G}_C^0 \triangleq \{(u_+, y_+) \in \mathcal{W}^+ \mid Cy_+ = u_+ \text{ with } u_+ \in \mathcal{U}^+,\, y_+ \in \mathcal{Y}^+\}$.

3.5 Initial Conditions of Systems

The concepts of states and of initial conditions have an obvious significance in Lyapunov theory, which deals with equilibrium points of unforced systems with nonzero initial conditions, while the classical input-output theory considers forced systems with zero initial conditions. Although the notion of a system in Definition 3.2 is made without recourse to state, the concept of state is very useful in input-output theory when dealing with general nonlinear systems with nonzero initial conditions. In subsequent sections, we shall explore some appropriate notions of states and of initial conditions using past inputs and past outputs. In this sense, the state is a characterisation of input-output pasts, which captures the idea that the state at any time together with the future input completely determine the future output. Thus the initial conditions are defined as the

[13] mapping bounded inputs into bounded outputs

initial state spaces for given initial time $t = 0$, and the sizes of which are also defined below.

As discussed in intuitive terms in the control literature, see e.g., [Kalman et al., 1969, Zadeh and Desoer, 1963, Zames, 1963], the state is a classifier of input-output pasts and the state should contain all the information of past history of the system which at any time together with the future input completely determine the future output. The state at time 0 thus determines the initial conditions. In the following, we will give a precise way to define the state of an arbitrary input/output system. It is fundamental that the construction does not require a system representation, but we do show how the construction relates to the standard concepts of state for significant classes of system representations. The genesis of this approach lies in [French and Mueller, Section 7]. From the viewpoint of observability, for any observable nonlinear system represented by a state space model, the initial state can be reconstructed from observed output signals given some known input signals (see e.g., [Besançon, 2007, Gauthier and Kupka, 2001]). In [van der Schaft and Rapisarda, 2011] a canonical state construction for linear time-invariant systems described by higher-order ordinary differential equations is introduced based on integration by parts; and generalisation to infinite dimensional linear time-invariant systems (i.e., systems described by high-order linear partial differential equations) can be found in [Rapisarda and van der Schaft, 2012].

3.5.1 Definition of Initial Conditions

Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$, and consider the system Q (see Definition 3.2). We will now introduce an equivalence relation on $\mathfrak{B}_Q^- \triangleq R_- \mathfrak{B}_Q$ (see (3.7)) and show how this yields the state. Let $Q^{w_-}(u_+)$ denote the set (possibly empty) of all future output trajectories generated by the system past input-output trajectories $w_- \in \mathfrak{B}_Q^-$ and future input $u_+ \in \mathcal{U}_e^+$, i.e.,

$$Q^{w_-}(u_+) \triangleq \left\{ y_+ \in \mathcal{Y}_e^+ \mid w_- \wedge (u_+, y_+) \in \mathfrak{B}_Q \right\}. \tag{3.10}$$

where the concatenation $\wedge$ at time 0 is defined by (3.8).

Note that the set $Q^{w_-}(u_+)$ is possibly empty for some $u_+ \in \mathcal{U}_e^+$. However, if the system Q is well-posed, then there is a unique element in $Q^{w_-}(u_+)$ for every $w_- \in \mathfrak{B}_Q^-$ and every $u_+ \in \mathcal{U}_e^+$. In this case, $Q^{w_-}(\cdot)$ defines an input-output operator from future inputs to future outputs.

Next we define an equivalence relation $\sim$ on $\mathfrak{B}_Q^- \triangleq R_- \mathfrak{B}_Q$ (see (3.7)) by using (3.10) as follows: for any $w_-, \tilde{w}_- \in \mathfrak{B}_Q^-$, we say

$$w_- \ \sim \ \tilde{w}_- \ \Leftrightarrow \ Q^{w_-}(u_+) = Q^{\tilde{w}_-}(u_+), \forall u_+ \in \mathcal{U}_e^+. \tag{3.11}$$

That $\sim$ is an equivalence relation on $\mathfrak{B}_Q^-$ follows from the binary relation '$=$' being a reflexive, transitive and symmetric relation on $\mathcal{Y}_e^+$ (see Definition 2.15 on page 21).

Note that the definition of equivalence relation $\sim$ on $\mathfrak{B}_Q^-$ doesn't require the system Q to be well-posed; if so then $Q^{w_-}(\cdot)$ defines an operator from $\mathcal{U}_e^+$ to $\mathcal{Y}_e^+$. Given this equivalence relation $\sim$ on $\mathfrak{B}_Q^-$, the equivalence class of an element w_- in $\mathfrak{B}_Q^-$ is the subset of all elements in $\mathfrak{B}_Q^-$ which are equivalent to w_- denoted by $[w_-]$, defined as:

$$[w_-] \triangleq \left\{ \tilde{w}_- \in \mathfrak{B}_Q^- \mid \tilde{w}_- \sim w_- \right\}. \tag{3.12}$$

Definition 3.11. We define $\mathfrak{S}_Q$ the *initial state space of* Q *at initial time* 0 as the quotient set $\mathfrak{B}_Q^-/\sim$ which contains all equivalence classes in $\mathfrak{B}_Q^-$ related to the equivalence relation $\sim$, i.e.,

$$\mathfrak{S}_Q \triangleq \mathfrak{B}_Q^-/\sim \triangleq \left\{ [w_-] \mid w_- \in \mathfrak{B}_Q^- \right\}. \tag{3.13}$$

From the equivalence relation $\sim$, for any $x_0 \in \mathfrak{S}_Q$, we can define the set $Q^{x_0}(u_+)$ by:

$$Q^{x_0}(u_+) \triangleq Q^{w_-}(u_+), \quad \forall u_+ \in \mathcal{U}_e^+, \forall w_- \in x_0. \tag{3.14}$$

Remark 3.12. If we choose the separating time between past and future (or say initial time) as $t_0 \in \mathbb{R}$ not 0, we can similarly define the initial state space denoted by $\mathfrak{S}_Q^{t_0}$ of a system Q at initial time t_0 by the same procedure.

Remark 3.13. Note that the above definition of initial state space doesn't require the system to be well-posed; however, if so, then, there is a unique element in $Q^{w_-}(u_+)$ for every $w_- \in \mathfrak{B}_Q^-$ and every $u_+ \in \mathcal{U}_e^+$; and in this case, $Q^{w_-}(\cdot)$ can be regarded as an operator from $\mathcal{U}_e^+$ to $\mathcal{Y}_e^+$ for every $w_- \in \mathfrak{B}_Q^-$. In turn, this implies that for every $x_0 \in \mathfrak{S}_Q$, $Q^{x_0}(\cdot)$ is an operator from $\mathcal{U}_e^+$ to $\mathcal{Y}_e^+$.

The initial state of a system defined above contains information about the past history of the system which suffices to predict the effect of the past upon the future. It is a classifier of system pasts. This is the property of usual state in a state space model.

This equivalence class construction of the initial state space is not new; it is closely related to the construction of states in automata (or machine) theory and control theory via Nerode equivalence appearing in slightly different manner. This technique was introduced by [Nerode, 1958] when defining a state-equivalence relation in linear automata theory. The formal definition of Nerode equivalence can be found in [Sakarovitch, 2009, p. 114] in the general setting of automata theory including the nonlinear case; in [Arbib and Zeiger, 1969, Kailath, 1980] for discrete-time systems from an abstract algebraic point of view; in [Kalman et al., 1969, Chapters 7 and 10] including a discussion of connection between automata and control theory; and in [Sontag, 1998b, p. 309] for any time-invariant input/output behaviours including both discrete-time and continuous-time cases. A concrete approach to the Nerode equivalence construction for discrete time transfer functions was studied in [Kailath, 1980, Sections 5.1 and 6.6, pp. 315 and 470], as well as for continuous-time transfer functions in Section 2.5 on page 21. The equivalence relation considered in this work is slightly different from the one considered in standard texts (see e.g., [Sontag, 1998b, p. 309]), where equivalence classes only relate to input sequences, since we do not restrict ourself to input/output behaviours which can be associated with an input/output map, hence the equivalence class is constructed from both input and output pairs.

Within the behavioural approach, Willems constructs three canonical state representations by introducing three equivalence relations for a given system represented by a behaviour [Willems, 1989]. The construction of state in this work is similar to the *past-induced canonical state representation* in [Willems, 1989, Section 2]. Note that here we do not impose any requirements on the input/output partition for a system (see Definition 3.2). This construction of state enables us to define the well-posedness of a system and a closed-loop system in a unified way (see below). Notice that this is different from giving a definition of well-posedness for a system with Willems' input/output partition [Polderman and Willems, 1998, Definition 3.3.1]; since any systems with Willems' input/output partition already guarantee the existence property which is a very important property of a closed-loop system. Hence we relax the requirement that the input is *free* in [Polderman and Willems, 1998, Definition 3.3.1] in order to study closed-loop systems.

A functional χ assigns a notion of size to elements in the initial state space $\mathfrak{S}_Q$ of the system Q:

$$\chi: \quad \mathfrak{S}_Q \to [0,\infty], \quad x_0 \mapsto \chi(x_0) \triangleq \inf_{w_- \in x_0} \|w_-\|\,. \tag{3.15}$$

The norm on $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$ is defined in the usual way, $\|(u,y)\|_{\mathcal{W}} = (\|u\|_{\mathcal{U}}^q + \|y\|_{\mathcal{Y}}^q)^{\frac{1}{q}}$, $q \geq 1$. Also note that $\lim_{q\to\infty} \left(\|u\|_{\mathcal{U}}^q + \|y\|_{\mathcal{Y}}^q\right)^{\frac{1}{q}} = \max\left\{\|u\|_{\mathcal{U}}, \|y\|_{\mathcal{Y}}\right\}$.

This notion of size defined above related to finite energy reachability may be interpreted as the minimisation of energy of the past system trajectories that 'explain' the corresponding initial state. Notice that in Section 4.4 on page 90 we will give a detailed discussion about the concept of finite-time reachability which roughly means that any state can be reached from zero state by finite time. The notion of size defined above may also be interpreted as the required supply in the context of dissipative dynamical systems, see e.g., [Willems, 1972].

3.5.2 Additional Consideration of the Size of Initial Conditions

The determination of χ is a standard problem in optimal control theory, see e.g., [Anderson and Moore, 1971]. The following results Theorems 3.14 and 3.16 are from [Scherpen, 1993, 1994, Scherpen and Van der Schaft, 1994, Willems, 1971b].

Theorem 3.14. *Consider a smooth, i.e., C^∞, nonlinear system of the form*

$$\dot{x} = F(x) + G(x)u, \quad y = H(x) \tag{3.16}$$

where $u = (u_1, \cdots, u_m)^T \in \mathbb{R}^m$, $y = (y_1, \cdots, y_p)^T \in \mathbb{R}^p$ and $x = (x_1, \cdots, x_n)^T$ are local coordinates for a smooth state space manifold denoted by M. Functions $F(x), G(x), H(x)$ are smooth functions with $F(0) = 0$ and $H(0) = 0$. Assume that the system is zero-state observable (i.e., for any trajectories, $u(t) \equiv 0$, $y(t) \equiv 0$ implies $x(t) \equiv 0$). Define the past energy function by

$$E^-(x_0) = \inf_{\substack{u \in L^2(-\infty,0] \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \left(|u(t)|^2 + |y(t)|^2\right) dt \tag{3.17}$$

Then E^- (if exist, i.e., is finite) is the smooth non-negative solution to the following Hamilton-Jacobi-Bellman equation:

$$\frac{\partial E^-}{\partial x}(x)F(x) + \frac{1}{2}\frac{\partial E^-}{\partial x}(x)G(x)G^T(x)\frac{\partial^T E^-}{\partial x}(x) - \frac{1}{2}H(x)^T H(x) = 0, \quad E(0) = 0 \tag{3.18}$$

satisfying $-(F(x) + G(x)G(x)^T \frac{\partial^T E^-}{\partial x}(x))$ is asymptotically stable and E^- is minimised by an input $u = G(x)^T \frac{\partial^T E^-}{\partial x}(x)$.

Remark 3.15. If we assume that there is a smooth solution E of (3.18) such that $-(F(x) + G(x)G(x)^T \frac{\partial^T E}{\partial x}(x))$ is asymptotically stable, then the past energy function E^- in (3.17) exists [Scherpen and Van der Schaft, 1994].

For a linear-time-invariant system, above infimum (3.17) is simplified to the traditional linear quadratic optimal control problem and we have the following result:

Theorem 3.16. *Consider a linear-time-invariant system*

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{3.19}$$

where $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $x \in \mathbb{R}^n$. We assume that the system is minimum (i.e., (A, B) is controllable and (A, C) is observable). The past energy function $E^-(x_0)$ is defined as (3.17). Then $E^-(x_0) = \frac{1}{2}x_0^T P^{-1} x_0$, where P is the stabilising solutions (i.e., $A - PC^T C$ is asymptotically stable) of the following Algebraic Riccati equation:

$$AP + PA^T + BB^T - PC^T CP = 0 \tag{3.20}$$

We conclude this section by showing that for any system Q if it is linear (see Definition 3.6) then the corresponding initial state space $\mathfrak{S}_Q$ is a normed vector space equipped with the norm χ given by (3.15).

Proposition 3.17. *If the system Q is linear, then the corresponding initial state space $\mathfrak{S}_Q$ is a vector space. Moreover, the real-valued function χ given by (3.15) defines a norm on $\mathfrak{S}_Q$.*

Proof. Since the set $\mathfrak{B}_Q$ is a vector space, we have $0|_{(-\infty,+\infty)} \in \mathfrak{B}_Q$ which is the additive identity of $\mathfrak{B}_Q$ (where $0|_{(-\infty,+\infty)}$ is the zero function defined on $\mathbb{R}$). And it is not hard to see that $\mathfrak{B}_Q^- = R^-\mathfrak{B}_Q$ is also a vector space with the additive identity $0|_{(-\infty,0]}$. We now need to show that $\mathfrak{S}_Q = \mathfrak{B}_Q^-/\sim$ is also a vector space. To this end, we only need to prove the following claims:

$$\forall\, w_-, \tilde{w}_- \in \mathfrak{B}_Q^-, \forall\, \lambda \in \mathbb{R}: \quad [w_- \sim \tilde{w}_- \quad \Rightarrow \quad \lambda \cdot w_- \sim \lambda \cdot \tilde{w}_-]\,; \tag{3.21a}$$

$$\forall\, w_-, \tilde{w}_-, w_{1-} \in \mathfrak{B}_Q^-: \quad [w_- \sim \tilde{w}_- \quad \Rightarrow \quad (w_- + w_{1-}) \;\sim\; (\tilde{w}_- + w_{1-})]\,. \tag{3.21b}$$

Claim (3.21) follows from the definition of corresponding equivalence relation $\sim$ (see (3.11)) defined on $\mathfrak{B}_Q^-$ and the linear property of $\mathfrak{B}_Q^-$ and $\mathfrak{B}_Q$. The equivalence class $\left[(0|_{(-\infty,0]}\right]$ (simply denoted by $\mathbf{0}$) is the additive identity of $\mathfrak{S}_Q$, and from the definition of χ (see (3.15)) we have $\chi(\mathbf{0}) = 0$.

From claim (3.21) and (3.12)–(3.13), we can define addition "+" and scalar multiplication "·" on $\mathfrak{S}_Q$ as follows, for any $x_0, y_0 \in \mathfrak{S}_Q$ and any $\lambda \in \mathbb{R}$:

$$\lambda \cdot x_0 = [\lambda \cdot w_-], \quad \forall\, w_- \in x_0, \tag{3.22a}$$

$$x_0 + y_0 = [w_{1-} + w_{2-}], \quad \forall w_{1-} \in x_0,\ \forall\, w_{2-} \in y_0. \tag{3.22b}$$

Thus from the definition of vector space and claim (3.21) we obtain that $\mathfrak{S}_Q$ equipped with above addition "+" and scalar multiplication "·" is a vector space.

We have shown that the initial state space $\mathfrak{S}_Q$ is a vector space with $\mathbf{0} = \left[0|_{(-\infty,0]}\right]$ as its additive identity and satisfies $\chi(\mathbf{0}) = 0$. From the definition of χ (see (3.15)), it is easy to see that $\chi(x_0) \geq 0$ for any $x_0 \in \mathfrak{S}_Q$ and that if $\chi(x_0) = 0$, then we must have $0|_{(-\infty,0]} \in x_0$ (i.e., $x_0 = \mathbf{0}$). From (3.15) and (3.22) we obtain

$$\chi(\lambda \cdot x_0) = \chi([\lambda \cdot w_-]) = |\lambda|\, \chi([w_-]) = |\lambda|\, \chi(x_0), \quad \forall x_0 = [w_-] \in \mathfrak{S}_Q,\ \forall \lambda \in \mathbb{R}.$$

For any $w_{1-}, w_{2-} \in \mathfrak{B}_Q^-$ we have $\|w_{1-} + w_{2-}\| \leq \|w_{1-}\| + \|w_{2-}\|$. Thus from (3.15) and (3.22) we get

$$\chi(x_0 + y_0) \leq \chi(x_0) + \chi(y_0), \quad \forall x_0, y_0 \in \mathfrak{S}_Q.$$

This shows that $\chi(\cdot)$ (see (3.15)) defines a norm on the corresponding initial state space $\mathfrak{S}_Q$ for any linear set $\mathfrak{B}_Q$. □

3.5.3 Comparison with Classical Initial Condition Concepts

An interesting issue is the comparison of our notion of initial conditions with the classical ones. In this section, several examples are given to compare the initial conditions defined above with the classical initial condition concepts. The first example is a concrete delay line model borrowed from [Weiss, 1994] modelled as an abstract linear system there, which is a very interesting but simple example. The second example is about the classical finite dimensional nonlinear state-space models which directly use state variables to describe systems by some first-order differential equations. This is the framework used by Sontag to introduce the notions of input-to-state stability/input-to-output stability (ISS/IOS) (see e.g., [Sontag, 2008, Sontag and Wang, 1995, 1996, 2000])). The last example is about a class of delay-differential systems with pseudo-state space descriptions borrowed from [Rocha and Willems, 1997].

A Concrete Delay Line Model

We consider a time τ-delay line model ($\tau > 0$) which produces every output signal by time τ-delay from every input signal. Define input and output signal spaces $\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}, \mathbb{R})$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$. Then the input-output system of the time τ-delay line model is:

$$\mathfrak{B}_\tau \triangleq \{(u, y) \in \mathcal{W}_e \mid y(t) = u(t - \tau), \forall t \in \mathbb{R}\}. \tag{3.23}$$

According to (3.7), the set of past trajectories $\mathfrak{B}_\tau^-$ is defined by

$$\mathfrak{B}_\tau^- = \left\{(u_-, y_-) \in \mathcal{W}_e^- \mid y_-(t) = u_-(t - \tau), \forall t \leq 0\right\}. \tag{3.24}$$

According to Definition 3.11, the initial state space of $\mathfrak{B}_\tau$ is the quotient set $\mathfrak{B}_\tau^- / \sim$ with the equivalence relation $\sim$ on $\mathfrak{B}_\tau^-$ defined by

$$w_- \;\sim\; \tilde{w}_- \Leftrightarrow u_-(t) = \tilde{u}_-(t), \forall\, t \in [-\tau, 0), \tag{3.25}$$

where $w_- = (u_-, y_-) \in \mathfrak{B}_\tau^-$ and $\tilde{w}_- = (\tilde{u}_-, \tilde{y}_-) \in \mathfrak{B}_\tau^-$. And the equivalent class $[w_-]$ of any element $w_- = (u_-, y_-) \in \mathfrak{B}_\tau^-$ is

$$[w_-] = \left\{(\tilde{u}_-, \tilde{y}_-) \in \mathfrak{B}_\tau^- \;:\; \tilde{u}_-|_{[-\tau,0)} = u_-|_{[-\tau,0)}\right\}. \tag{3.26}$$

The real-valued function χ on $\mathfrak{B}_\tau^-/\sim$ is defined by

$$[w_-] \mapsto \chi([w_-]) \triangleq \inf\{\|\tilde{w}_-\| \,:\, \tilde{w}_- \in [w_-]\}.$$

According to (3.14) and (3.10), let $s_0 \in \mathfrak{B}_\tau^-/\sim$ be any initial state of $\mathfrak{B}_\tau$, and let $u_+ \in \mathcal{U}_e^+$ denote the future input signal of $\mathfrak{B}_\tau$, and let $y_+ \in \mathcal{Y}_e^+$ denote the future output signal of $\mathfrak{B}_\tau$, then we have

$$y_+(t) = (Q_\tau^{s_0}(u_+))(t) \triangleq \begin{cases} u_-(t-\tau), & \text{for } t \in [0,\tau), \\ u_+(t-\tau), & \text{for } t \geq \tau, \end{cases} \tag{3.27}$$

where $w_- = (u_-, y_-) \in \mathfrak{B}_\tau^-$ is any element in s_0.

We know that the time τ-delay line model is an abstract linear system, i.e., a quadruple $(\mathbb{T}, \Phi, \Psi, \mathbb{F})$ defined in Weiss [Weiss, 1994, p. 831]. Let the classical state space be $X = L^\infty([-\tau, 0), \mathbb{R})$, and let $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ be a family of bounded linear operators from X to X defined by, for any $t \geq 0$ and $\xi \in [-\tau, 0)$

$$(\mathbb{T}_t x)(\xi) = \begin{cases} x(\xi+t), & \text{for } \xi + t < 0; \\ 0, & \text{for } \xi + t \geq 0. \end{cases}$$

Let $\Phi = (\Phi_t)_{t\geq 0}$ be a family of bounded linear operators from $\mathcal{U}^+$ to X defined by, for any $t \geq 0$ and $\xi \in [-\tau, 0)$

$$(\Phi_t u_+)(\xi) = \begin{cases} u_+(\xi+t), & \text{for } \xi + t \geq 0; \\ 0, & \text{for } \xi + t < 0. \end{cases}$$

Let $\Psi = (\Psi_t)_{t\geq 0}$ be a family of bounded linear operators from X to $\mathcal{Y}^+$ defined by, for any $t \geq 0$ and $\theta \in [0, t)$

$$(\Psi_t x)(\theta) = \begin{cases} x(\theta-\tau), & \text{for } \theta - \tau < 0; \\ 0, & \text{for } \theta - \tau \geq 0. \end{cases}$$

For $\theta \geq t$ we put $(\Psi_t x)(\theta) = 0$. Finally, let $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ be a family of bounded linear operators from $\mathcal{U}^+$ to $\mathcal{Y}^+$ defined by, for any $t \geq 0$ and $\theta \in [0, t)$

$$(\mathbb{F}_t u_+)(\theta) = \begin{cases} u_+(\theta-\tau), & \text{for } \theta - \tau \geq 0; \\ 0, & \text{for } \theta - \tau < 0. \end{cases}$$

For $\theta \geq t$ we put $(\mathbb{F}_t u_+)(\theta) = 0$. Then $(\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is an abstract linear system, i.e., if $x_t \in X$ denotes the state at time $t \geq 0$, and $u_+ \in \mathcal{U}_e^+$ (note that $T_{[0,t)}u_+ \in \mathcal{U}^+$ in this example) and $y_+ \in \mathcal{Y}_e^+$ are the future input and output signals respectively, then

$$\begin{pmatrix} x_t \\ T_{[0,t)}y_+ \end{pmatrix} = \begin{pmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ T_{[0,t)}u_+ \end{pmatrix}$$

Thus, we obtain

$$y_+(t) = \begin{cases} x_0(t-\tau), & \text{for } t \in [0, \tau); \\ u_+(t-\tau), & \text{for } t \geq \tau. \end{cases} \tag{3.28}$$

We know by comparing (3.27) and (3.28) that the initial state space $\mathfrak{B}_\tau^- / \sim$ is actually equivalent to $X = L^\infty([-\tau, 0), \mathbb{R})$.

Classical Finite Dimensional Nonlinear State-Space Models

Before going into the next example, we introduce the notions of forward (resp., backward) completeness and (strongly) forward (resp., backward) observability. These notions will also be used in later Section 3.7.2 for the comparison with classical initial conditions for closed-loop systems.

Consider a system Σ described by the following finite dimensional state-space model:

$$\dot{x} = f(x, u), \quad y = h(x, u), \tag{3.29}$$

where $u(t) \in \mathbb{R}^m$ ($t \in \mathbb{R}$) is the input variable, and $x(t) \in M \subseteq \mathbb{R}^l$ ($t \in \mathbb{R}$) denotes the state variable (M is an open set), and $y(t) \in \mathbb{R}^p$ ($t \in \mathbb{R}$) represents the output variable, and both $f : M \times \mathbb{R}^m \to M$ and $g : M \times \mathbb{R}^m \to \mathbb{R}^p$ are continuous functions. Define signal spaces $\mathcal{U} \triangleq L^q(\mathbb{R}, \mathbb{R}^m)$ ($1 \leq q \leq \infty$), $\mathcal{Y} \triangleq L^q(\mathbb{R}, \mathbb{R}^p)$ ($1 \leq q \leq \infty$) and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$.

Definition 3.18. The state space model (3.29) is said to be *forward complete* [Angeli and Sontag, 1999], if for any $u_+ \in \mathcal{U}_e^+$ and any initial state $x_0 \in M$, there exists a unique $x(t) \in M$ (for all $t \geq 0$) satisfying (3.29). It is said to be *backward complete*, if for every $u_- \in \mathcal{U}_e^-$ and every initial state x_0, there exists a unique $x(t) \in M$ (for all $t \leq 0$) satisfying (3.29). It is said to be *complete* if it is both forward complete and backward complete.

Suppose that the state space model (3.29) is a complete representation. If the trajectories of (3.29) are required to satisfy the initial condition $x(0) = x_0$ ($x_0 \in M$), then the state space model defines a forward operator $\Sigma_+^{x_0}$ from $\mathcal{U}_e^+$ to $\mathcal{Y}_e^+$ as follows: each input $u_+ \in \mathcal{U}_e^+$ gives rise to a solution $x(t) \in M$ ($t \geq 0$) of $\dot{x} = f(x, u)$ satisfying the initial condition $x(0) = x_0$. This in turn defines an output $y_+ \in \mathcal{Y}_e^+$ by $y_+(t) = h(x(t), u_+(t))$ ($t \geq 0$), i.e.,

$$\Sigma_+^{x_0} : \mathcal{U}_e^+ \to \mathcal{Y}_e^+, \qquad u_+ \mapsto y_+. \tag{3.30}$$

Similarly, a backward operator $\Sigma_-^{x_0}$ can be defined by

$$\Sigma_-^{x_0} : \mathcal{U}_e^- \to \mathcal{Y}_e^-, \qquad u_- \mapsto y_-. \tag{3.31}$$

Definition 3.19. Suppose that the state space model (3.29) is complete. It is said to be *forward observable* if (see e.g., [Hermann and Krener, 1977]), for any initial states $x_0, x_0' \in M$ with $x_0 \neq x_0'$, there exists some $u_+ \in \mathcal{U}_e^+$ such that $\Sigma_+^{x_0}(u_+) \neq \Sigma_+^{x_0'}(u_+)$. It is said to be *strongly forward observable* if, for any initial states $x_0, x_0' \in M$ with $x_0 \neq x_0'$, for any $u_+ \in \mathcal{U}_e^+$, we have $\Sigma_+^{x_0}(u_+) \neq \Sigma_+^{x_0'}(u_+)$. It is said to be *backward observable* if, for any initial states $x_0, x_0' \in M$ with $x_0 \neq x_0'$, there exist some $u_- \in \mathcal{U}_e^-$ such that $\Sigma_-^{x_0}(u_-) \neq \Sigma_-^{x_0'}(u_-)$. It is said to be *strongly backward observable* if, for any initial states $x_0, x_0' \in M$ with $x_0 \neq x_0'$, for any $u_- \in \mathcal{U}_e^-$, we have $\Sigma_-^{x_0}(u_-) \neq \Sigma_-^{x_0'}(u_-)$.

Consider the system Σ described by the state-space model (3.29) with classical state space $M \subseteq \mathbb{R}^l$. We next discuss our notion of initial conditions for Σ. According to Definition 3.2, the system Σ is defined by the set:

$$\mathfrak{B}_\Sigma = \{w \in \mathcal{W}_e \mid w = (u, y) \text{ and (3.29) satisfies for some } x(t) \in M(t \in \mathbb{R})\}. \quad (3.32)$$

In $\mathfrak{B}_\Sigma$ we regard $u \in \mathcal{U}_e$ as the input and $y \in \mathcal{Y}_e$ as the output. By using Definition 3.11, we can define the initial state space $\mathfrak{S}_\Sigma$ for the set $\mathfrak{B}_\Sigma$ at initial time 0.

We use the notation $\mathfrak{B}_\Sigma^-(x_0)$ defined as follows to denote the set of all past input-output trajectories generated by initial state x_0 at initial time 0 ($x_0 \in M$):

$$\mathfrak{B}_\Sigma^-(x_0) \triangleq \left\{ \begin{pmatrix} u_- \\ y_- \end{pmatrix} \;\middle|\; \begin{array}{c} u_- \in \mathcal{U}_e^-, y_- \in \mathcal{Y}_e^- \text{ and (3.29) satisfies} \\ \text{for some } x(t) \in M(t \leq 0) \text{ with } x(0) = x_0 \end{array} \right\}. \quad (3.33)$$

It is easy to see that if the state space model (3.29) is complete and strongly backward observable, then we have

$$\mathfrak{B}_\Sigma^-(x_0) \cap \mathfrak{B}_\Sigma^-(x_0') = \emptyset, \quad \forall x_0, x_0' \in M \;\text{ with } x_0 \neq x_0'.$$

We now state the result concerning the relationship between the classical initial conditions and our notion of initial conditions.

Proposition 3.20. *Suppose that the state space model (3.29) is complete, forward observable and strongly backward observable. Then $F : x_0 \mapsto \mathfrak{B}_\Sigma^-(x_0)$ defines a bijection from M to $\mathfrak{S}_\Sigma$.*

Proof. From Definition 3.11, the initial state space at time 0 of $\mathfrak{B}_\Sigma$ (see (3.32)) is defined by $\mathfrak{S}_\Sigma \triangleq \mathfrak{B}_\Sigma^- / \sim$ with $\mathfrak{B}_\Sigma^- \triangleq R^- \mathfrak{B}_\Sigma$ (see (3.7)), and the corresponding equivalence relation $\sim$ on $\mathfrak{B}_\Sigma^-$ is defined as follows (see (3.10) and (3.11)): for any $w_-, \tilde{w}_- \in \mathfrak{B}_\Sigma^-$,

$$w_- \sim \tilde{w}_- \quad \Leftrightarrow \quad \Sigma^{w_-}(u_+) = \Sigma^{\tilde{w}_-}(u_+), \; \forall u_+ \in \mathcal{U}_e^+. \quad (3.34)$$

We obtain from (3.32) and (3.33) that $\mathfrak{B}_\Sigma^- = \bigcup_{x_0 \in M} \left\{\mathfrak{B}_\Sigma^-(x_0)\right\}$. Since the state space model (3.29) is complete and strongly backward observable, we have

$$\mathfrak{B}_\Sigma^-(x_0) \cap \mathfrak{B}_\Sigma^-(x_0') = \emptyset, \quad \forall x_0, x_0' \in M \text{ with } x_0 \neq x_0'.$$

In addition, for any $w_- \in \mathfrak{B}_\Sigma^-(x_0)$ and any $u_+ \in \mathcal{U}_e^+$, we have $\Sigma^{w_-}(u_+) = \Sigma_+^{x_0}(u_+)$ with $\Sigma_+^{x_0}(u_+)$ defined by (3.30). Thus, for any $x_0 \in M$, the set $\mathfrak{B}_\Sigma^-(x_0)$ is a subset of some equivalence class related to the equivalence relation $\sim$.

Since the state space model (3.29) is also forward observable, (i.e., $\Sigma_+^{x_0} \neq \Sigma_+^{x_0'}$ $\forall x_0, x_0' \in M$ with $x_0 \neq x_0'$), we get that $\mathfrak{B}_\Sigma^-(x_0)$ and $\mathfrak{B}_\Sigma^-(x_0')$ must be contained in two different equivalence classes related to the equivalence relation $\sim$. This, in turn, implies that $\left\{\mathfrak{B}_\Sigma^-(x_0) \mid x_0 \in M\right\}$ is the exact partition of $\mathfrak{B}_\Sigma^-$ related to the equivalence relation $\sim$. Therefore, we have $\mathfrak{S}_\Sigma = \left\{\mathfrak{B}_\Sigma^-(x_0) \mid x_0 \in M\right\}$ and the map $F : x_0 \mapsto \mathfrak{B}_\Sigma^-(x_0)$ is a bijection from M to $\mathfrak{S}_\Sigma$. This completes the proof. □

It is well-known that the state space model (3.29) is complete if f is continuous in t and u and Lipschitz continuous in x (see e.g., [Desoer and Chen, 1967]).

Corollary 3.21. *Consider the state space model (3.29). Suppose that f is continuous in t and u and Lipschitz continuous in x, and that Σ is forward observable and strongly backward observable, then there exists a bijective map from M to $\mathfrak{S}_\Sigma$.*

Proof. The assumption of f implies the completeness of (3.29). The rest of the proof follows from Proposition 3.20. □

For a linear time invariant (LTI) system we have the following result:

Corollary 3.22. *If the system Σ defined by (3.29) is a LTI system, i.e., $\dot{x} = f(x,u) = Ax + Bu$ and $y = h(x,u) = Cx + Du$, where $x(t) \in M = \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ for any $t \in \mathbb{R}$, and A, B, C, D are appropriate dimensional matrixes. Suppose that (A, C) is observable [Zhou et al., 1995], i.e., the $np \times n$ observability matrix $[C^T, (CA)^T, \cdots, (CA^{n-1})^T]^T$ are of full column rank n. Then there exists a bijective map from $M = \mathbb{R}^n$ to $\mathfrak{S}_\Sigma$.*

Proof. Since $f(x,u) = Ax + Bu$ is continuous in u and Lipschitz continuous in x, this implies that Σ is complete. While for linear time-invariant (LTI) system the observability matrix $[C^T, (CA)^T, \cdots, (CA^{n-1})^T]^T$ has full column rank n implies that the system is forward observable and strongly backward observable. Thus from Corollary 3.21 there exists a bijective map from $M = \mathbb{R}^n$ to $\mathfrak{S}_\Sigma$. □

A Class of Delay-Differential Systems

To give a further insight about our initial conditions for dynamical systems, consider the following class of delay-differential systems Σ which admit a pseudo-state description [Rocha and Willems, 1997] of the form

$$\dot{x} = A(\triangle_h)x + B(\triangle_h)u, \quad y = C(\triangle_h)x + D(\triangle_h)u \tag{3.35}$$

where $x(t) \in \mathbb{R}^n$ $(t \in \mathbb{R})$ is the n-dimensional pseudo-state, $u(t) \in \mathbb{R}^m$ $(t \in \mathbb{R})$ is the input, $y(t) \in \mathbb{R}^p$ $(t \in \mathbb{R})$ is the output, and $A(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$, $B(\lambda) \in \mathbb{R}^{n\times m}[\lambda]$, $C[\lambda] \in \mathbb{R}^{p\times n}[\lambda]$, $D(\lambda) \in \mathbb{R}^{p\times m}[\lambda]$ are polynomial matrixes in λ, and $\triangle_h$ with $h \in (0, \infty)$ denotes the h-time delay operator: $(\triangle_h\, f)(t) \triangleq f(t-h)$.

For the pseudo-state space model (3.35), the classical state at time t is defined [Rocha and Willems, 1997] as being $z(t) = (x(t), x_t)$, where $x_t \in L^2([-h, 0), \mathbb{R}^n)$ is given by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-h, 0)$. This produces the infinite-dimensional state space $Z \triangleq \mathbb{R}^n \times L^2([-h, 0), \mathbb{R}^n)$.

We will now define the input signal space $\mathcal{U} \triangleq L^q(\mathbb{R}, \mathbb{R}^m)$ $(1 \le q \le \infty)$, the output signal space $\mathcal{Y} \triangleq L^q(\mathbb{R}, \mathbb{R}^p)$ $(1 \le q \le \infty)$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$. According to Definition 3.2, the system Σ is defined by the following set:

$$\mathfrak{B}_\Sigma \triangleq \{(u, y) \in \mathcal{W}_e \mid \text{Eq. (3.35) satisfies for some } x(t) \in \mathbb{R}^n (t \in \mathbb{R})\} \tag{3.36}$$

In $\mathfrak{B}_\Sigma$ we regard $u \in \mathcal{U}_e$ as the input and $y \in \mathcal{Y}_e$ as the output. By using the same procedure in Section 3.5.1, we can define the initial state space $\mathfrak{S}_\Sigma$ (see (3.13)) for the above set $\mathfrak{B}_\Sigma$ at initial time 0.

It has been asserted in [Delfour and Mitter, 1975] that for any initial state (classical) $z_0 \in Z$ at initial time 0 and any $u_+ \in \mathcal{U}_e^+$ there exists a unique solution $x(\cdot, z_0, u_+) \in W_e^{1,q}(\mathbb{R}_+, \mathbb{R}^n)$ to (3.37) (note that $W^{1,q}(\mathbb{R}_+, \mathbb{R}^n)$ denotes the standard Sobolev space of all functions f in $L^q(\mathbb{R}_+, \mathbb{R}^n)$ such that $\dot{f} \in L^q(\mathbb{R}_+, \mathbb{R}^n)$). Therefore, for any $z_0 \in Z$, the system Σ (see (3.35)) defines an operator denoted by $\Sigma_+^{z_0}$ from future inputs to future outputs as follows:

$$\Sigma_+^{z_0} : \mathcal{U}_e^+ \to \mathcal{Y}_e^+, \quad u_+(\cdot) \mapsto y_+ = C(\triangle_h)x(\cdot, z_0, u_+) + D(\triangle_h)u_+ \tag{3.37}$$

In the following we give a definition for the pseudo-state space model (3.35) to be forward observable and strongly backward observable, which is a generalisation of the corresponding notions for the finite dimensional nonlinear state space model (3.29).

Definition 3.23. The pseudo-state space model (3.35) is said to be *forward observable* if, for any initial states (classical) $z_0, z_0' \in Z$ with $z_0 \neq z_0'$, we have $\Sigma_+^{z_0} \neq \Sigma_+^{z_0'}$, i.e., there exists some $u_+ \in \mathcal{U}_e^+$ such that $\Sigma_+^{z_0}(u_+) \neq \Sigma_+^{z_0'}(u_+)$.

Definition 3.24. The state space model (3.35) is said to be *strongly backward observable*, if

$$\mathfrak{B}_\Sigma^-(z_0) \cap \mathfrak{B}_\Sigma^-(z_0') = \emptyset, \quad \forall z_0, z_0' \in Z \text{ with } z_0 \neq z_0'$$

where the notation $\mathfrak{B}_\Sigma^-(z_0)$ defined as follows to denote the set of all past input-output trajectories generated by initial state (classical) z_0 at fixed initial time $0(z_0 \in Z)$, i.e.,

$$\mathfrak{B}_\Sigma^-(z_0) \triangleq \left\{ \begin{pmatrix} u_- \\ y_- \end{pmatrix} \;\middle|\; \begin{array}{c} u_- \in \mathcal{U}_e^-, y_- \in \mathcal{Y}_e^- \text{ and (3.35) satisfies for} \\ \text{some } x(t) \in \mathbb{R}^n (t \leq 0) \text{ with } (x(0), x_0(\cdot)) = z_0 \end{array} \right\} \tag{3.38}$$

Similar to Proposition 3.20, we have the following result for a class of delay-differential systems:

Proposition 3.25. *Suppose that the pseudo-state space model (3.35) is forward observable and strongly backward observable, then the map $F: z_0 \mapsto \mathfrak{B}_\Sigma^-(z_0)$ is a bijection from Z to $\mathfrak{S}_\Sigma$.*

Proof. Similar to the proof of Proposition 3.20. □

3.6 Notions of Stability with Initial Conditions

The concept of stability concerned with both initial conditions and input in state space model was first systematically studied in Sontag's works via the well-known input-to-state stability/input-to-output stability (ISS/IOS) theory introduced in [Sontag, 1989] and its many variants [Angeli et al., 2000, Sontag, 1998a, 2008, Sontag and Wang, 1997, 1999] etc. In this section, we give a notion of stability in our framework which is closely related to the ISS/IOS concepts of Sontag.

Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$, consider a system Q represented by the set $\mathfrak{B}_Q$ (see Definition 3.2) with initial state space $\mathfrak{S}_Q$ at initial time 0 (see Definition 3.11). Suppose that the system Q is well-posed. Then, from Remark 3.13, we know that Q^{x_0} is an operator from $\mathcal{U}_e^+$ to $\mathcal{Y}_e^+$ for any $x_0 \in \mathfrak{S}_Q$. It is easy to see that

$$\mathfrak{B}_Q = \bigcup_{x_0 \in \mathfrak{S}_Q} \{ w_- \wedge (u_+, Q^{x_0} u_+) \mid w_- \in x_0, \; u_+ \in \mathcal{U}_e^+ \}.$$

Thus we can regard the system Q as a family of operators $\{Q^{x_0} : x_0 \in \mathfrak{S}_Q\}$ indexed by initial states. For a well-posed system Q, if Q is causal, then we have Q^{x_0} is a causal operator from $\mathcal{U}_e^+$ to $\mathcal{Y}_e^+$.

Definition 3.26. The system Q is said to be *input to output stable* if and only if it is well-posed and causal, and there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that,

$\forall x_0 \in \mathfrak{S}_Q,\ \forall t > 0,\ \forall u_{0+} \in \mathcal{U}^+$,

$$|(Q^{x_0} u_{0+})(t)| \leq \beta\left(\chi(x_0), t\right) + \gamma(\|u_{0+}\|_{[0,t)}),$$

where the real-valued function $\chi(\cdot)$ is defined by (3.15).

The above ISS-like definition represents a generalisation of ISS for the system $\dot{x} = f(x,u)$, $y = x$ wherein the term $\beta\left(\chi(x_0), t\right)$ is replaced by $\beta\left(\|x_0\|, t\right)$ in Sontag's definition, and where x_0 is the initial state $x_0 = x(0) \in \mathbb{R}^n$ rather than the abstract initial condition developed here, which is appropriate for the more general system classes under consideration. More generally, the concept of input-to-output stability (IOS) [Sontag and Wang, 1999] permits the more general output map $y = h(x)$. The reason to adopt this ISS-like notion of stability is that we want to complement the successful robust stability theory of [Georgiou and Smith, 1997b] for purely input/output systems by introducing the abstract initial conditions in this work.

3.7 Closed-Loop Systems and their Initial Conditions

We recall the standard feedback configuration depicted in Figure 3.1 with equations (3.1) on page 38, i.e.,

$$[P, C]: \quad \begin{array}{l} w_i = (u_i, y_i) \text{ for } i = 0, 1, 2, \\ w_1 \in \mathfrak{B}_P, w_2 \in \mathfrak{B}_C, w_0 = w_1 + w_2, \end{array}$$

where (u_0, y_0) denote external disturbance; (u_1, y_1) are the input-output pairs of the plant P to be controlled; and (u_2, y_2) are the output-input pairs of the controller C.

Definition 3.27. Given normed signal spaces $\mathcal{U}, \mathcal{Y}$, $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$. Let the plant P and the controller C be represented by the sets $\mathfrak{B}_P$ and $\mathfrak{B}_C$, respectively [14]. We define the closed-loop system $[P, C]$ by the following set $\mathfrak{B}_{P//C}$ which is the interconnection of the *plant* P and *controller* C shown in Figure 3.1 that satisfies (3.1),

$$\mathfrak{B}_{P//C} \triangleq \{(w_0, w_1) \in \mathcal{W}_e \times \mathcal{W}_e \mid w_1 \in \mathfrak{B}_P, w_2 \triangleq w_0 - w_1 \in \mathfrak{B}_C\}. \tag{3.39}$$

In $\mathfrak{B}_{P//C}$ we view the external input w_0 as the (closed-loop) input and the internal signal w_1 as the (closed-loop) output. For the set $\mathfrak{B}_{P//C}$, we can define the initial state space at initial time 0 of $\mathfrak{B}_{P//C}$ in terms of Definition 3.11, i.e., let $\mathfrak{B}^-_{P//C} \triangleq R_- \mathfrak{B}_{P//C}$, we similarly define an equivalence relation $\sim$ on $\mathfrak{B}^-_{P//C}$ as (3.11), and the set of all equivalence classes $\mathfrak{B}^-_{P//C}/\sim$ is denoted as $\mathfrak{S}_{P//C}$ which we call initial state space of $\mathfrak{B}_{P//C}$ at initial time 0. The size of any initial state in $\mathfrak{S}_{P//C}$ is similarly defined as in

[14] Note that when considering the controller C, we need interchange the role of $\mathcal{U}_e$ and $\mathcal{Y}_e$ and think of $y_2 \in \mathcal{Y}_e$ as the input and $u_2 \in \mathcal{U}_e$ as the output.

(3.15). In order to study robustness of feedback stability for a closed-loop system with initial conditions as our setting, one of the key points is to understand the relationship between initial conditions of the interconnected system and that of the two subsystems.

3.7.1 Relationship between Initial Conditions of Open-Loop and Closed-Loop Systems

According to Definition 3.11 (see Section 3.5.1 on page 54), let $\mathfrak{S}_P$ and $\mathfrak{S}_C$ be the corresponding initial state spaces at initial time 0 of the plant P and the controller C, respectively. The size of any initial state is similarly defined by (3.15). For the classical state space model, it's very natural to define the initial state of the closed-loop system by combination of the initial states of corresponding subsystems. In this section, we will give some answer about the relation between $\mathfrak{S}_{P//C}$ and $\mathfrak{S}_P \times \mathfrak{S}_C$.

Suppose that the size of any $x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$ is defined in the usual way, e.g., for any given $q \in [1, \infty]$,

$$\begin{aligned}\chi(x_0) &\triangleq (\chi(x_{10})^q + \chi(x_{20})^q)^{\frac{1}{q}} \\ &= \inf\left\{(\|w_{1-}\|^q + \|w_{2-}\|^q)^{\frac{1}{q}} \;\middle|\; w_{1-} \in x_{10}, w_{2-} \in x_{20}\right\} \\ &= \inf\left\{\|(w_{1-}, w_{2-})\| \;\middle|\; w_{1-} \in x_{10}, w_{2-} \in x_{20}\right\}\end{aligned} \tag{3.40}$$

Note that for any $s_0 \in \mathfrak{S}_{P//C}$ and any $w_{0+} \in \mathcal{W}_e^+$, we have defined a set $\Pi^{s_0}_{P//C}(w_{0+})$ according to (3.14) and (3.10) (let $\mathfrak{B}_Q := \mathfrak{B}_{P//C}$ and $\Pi^{s_0}_{P//C}(w_{0+}) := Q^{s_0}(w_{0+})$), i.e.,

$$\Pi^{s_0}_{P//C}(w_{0+}) \triangleq \left\{ w_{1+} \in \mathcal{W}_e^+ \;\middle|\; \begin{array}{c}(w_{0-}, w_{1-}) \wedge (w_{0+}, w_{1+}) \in \mathfrak{B}_{P//C}, \\ \forall (w_{0-}, w_{1-}) \in s_0\end{array} \right\}. \tag{3.41}$$

To understand the relation between $\mathfrak{S}_{P//C}$ and $\mathfrak{S}_P \times \mathfrak{S}_C$, we need to define another set which is related to the product state in $\mathfrak{S}_P \times \mathfrak{S}_C$, denoted by $\overline{\Pi^{x_0}_{P//C}}(w_{0+})$, for any $x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$ and any $w_{0+} \in \mathcal{W}_e^+$, as follows:

$$\overline{\Pi^{x_0}_{P//C}}(w_{0+}) \triangleq \left\{ w_{1+} \in \mathcal{W}_e^+ \;\middle|\; \begin{array}{c}(w_{1-} + w_{2-}, w_{1-}) \wedge (w_{0+}, w_{1+}) \in \mathfrak{B}_{P//C}, \\ \forall\, (w_{1-},\, w_{2-}) \in x_0\end{array} \right\}. \tag{3.42}$$

The result is the following:

Theorem 3.28. *There exists a surjective and bounded*[15] *map $\pi : \mathfrak{S}_P \times \mathfrak{S}_C \to \mathfrak{S}_{P//C}$ such that for all $x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$ and all $w_{0+} \in \mathcal{W}_e^+$,*

$$\overline{\Pi^{x_0}_{P//C}}(w_{0+}) = \Pi^{\pi(x_0)}_{P//C}(w_{0+})$$

[15] Here bounded means that there exists a positive number $r \geq 0$ such that $\chi(\pi(x_0)) \leq r \cdot \chi(x_0)$ for any $x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$ with function χ defined by (3.15).

Moreover, if we define an equivalence relation on $\mathfrak{S}_P \times \mathfrak{S}_C$ *as follows*

$$x_0 \overset{\pi}{\sim} y_0 \quad \Leftrightarrow \quad \pi(x_0) = \pi(y_0),$$

and the equivalence class $[x_0] \triangleq \{\, y_0 \in \mathfrak{S}_P \times \mathfrak{S}_C |\ y_0 \overset{\pi}{\sim} x_0\}$*, and the size of the equivalence class* $[x_0]$*,*

$$\chi([x_0]) \triangleq \inf_{y_0 \in [x_0]} \{\chi(y_0)\}, \tag{3.43}$$

and we define another map $\bar{\pi}$ *induced by* π *as follows*

$$\bar{\pi} : (\mathfrak{S}_P \times \mathfrak{S}_C)/_{\overset{\pi}{\sim}} \to \mathfrak{S}_{P//C}, \qquad \bar{\pi}([x_0]) = \pi(x_0). \tag{3.44}$$

Then $\bar{\pi}$ *is a bijective and bounded map, and the inverse* $\bar{\pi}^{-1}$ *is also bounded.*

Proof. For any $x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$, choose any $w_{1-} \in x_{10}$, $w_{2-} \in x_{20}$ and define $w_{0-} \triangleq w_{1-} + w_{2-}$. According to the definition of initial conditions (see Definition 3.11 on page 54) and the definition of closed-loop systems (see Definition 3.27), we have $s_0 \triangleq [(w_{0-}, w_{1-})] \in \mathfrak{S}_{P//C}$. In the following, we show that s_0 is independent of the choice of $w_{1-} \in x_{10}$ and $w_{2-} \in x_{20}$.

Choose any other $w'_{1-} \in x_{10}$ and any other $w'_{2-} \in x_{20}$ and define $w'_{0-} = w'_{1-} + w'_{2-}$, thus we have $s'_0 \triangleq [(w'_{0-}, w'_{1-})] \in \mathfrak{S}_{P//C}$. We need to show $s'_0 = s_0$, according to (3.11) and (3.12) (or see Section 3.5.1), this is equivalent to say

$$\Pi_{P//C}^{(w_{0-},w_{1-})}(w_{0+}) = \Pi_{P//C}^{(w'_{0-},w'_{1-})}(w_{0+}),\ \forall w_{0+} \in \mathcal{W}_e^+. \tag{3.45}$$

In order to prove (3.45), by symmetry, we only need to show that

$$\Pi_{P//C}^{(w_{0-},w_{1-})}(w_{0+}) \subseteq \Pi_{P//C}^{(w'_{0-},w'_{1-})}(w_{0+}),\ \forall w_{0+} \in \mathcal{W}_e^+.$$

To this end, for any $w_{1+} \in \Pi_{P//C}^{(w_{0-},w_{1-})}(w_{0+})$, we define $w_{2+} = w_{0+} - w_{1+}$. Thus, from the definition of closed-loop systems (see Definition 3.27), we have

$$w_{1-} \wedge w_{1+} \in \mathfrak{B}_P \text{ and } w_{2-} \wedge w_{2+} \in \mathfrak{B}_C. \tag{3.46}$$

Since both w_{1-} and w'_{1-} belong to x_{10}, from the definition of initial conditions for P, we have

$$P^{w_{1-}}(u_{1+}) = P^{w'_{1-}}(u_{1+}),\ \forall u_{1+} \in \mathcal{U}_e^+. \tag{3.47}$$

From (3.46) and (3.47), this implies that $w'_{1-} \wedge w_{1+} \in \mathfrak{B}_P$. By similar argument, we also have $w'_{2-} \wedge w_{2+} \in \mathfrak{B}_C$. Thus, from the definition of closed-loop systems (see Definition

3.27), we obtain

$$(w'_{0-} \wedge w_{0+},\ w'_{1-} \wedge w_{1+}) \in \mathfrak{B}_{P//C}.$$

This, in turn, implies that $w_{1+} \in \Pi_{P//C}^{(w'_{0-}, w'_{1-})}(w_{0+})$ and thus (3.45) holds. Therefore, s_0 is independent of the choosing $w_{1-} \in x_{10}$ and $w_{2-} \in x_{20}$. We also have $\overline{\Pi_{P//C}^{x_0}}(w_{0+}) = \Pi_{P//C}^{s_0}(w_{0+})$ for any $w_{0+} \in \mathcal{W}_e^+$.

A natural map π from $\mathfrak{S}_P \times \mathfrak{S}_C$ to $\mathfrak{S}_{P//C}$ can be defined as follows

$$\pi : \mathfrak{S}_P \times \mathfrak{S}_C \to \mathfrak{S}_{P//C}, \quad x_0 \mapsto s_0$$

From (3.15) and $s_0 = [(w_{0-}, w_{1-})]$, we have

$$\chi(\pi(x_0)) = \chi(s_0) \leq \|(w_{0-}, w_{1-})\| = \|(w_{1-} + w_{2-},\ w_{1-})\| \tag{3.48}$$

Since w_{1-} and w_{2-} are arbitrarily chosen from x_{10} and x_{20}, respectively. We get from (3.48) and the inequality $(a+b)^n \leq 2^n \cdot \max\{a^n, b^n\}$ (with $a \geq 0$, $b \geq 0$) that

$$\begin{aligned} \chi(\pi(x_0)) &\leq \inf_{\substack{w_{1-} \in x_{10} \\ w_{2-} \in x_{20}}} \{\|(w_{1-} + w_{2-},\ w_{1-})\|\} \\ &= \inf_{\substack{w_{1-} \in x_{10} \\ w_{2-} \in x_{20}}} \left\{(\|w_{1-} + w_{2-}\|^q + \|w_{1-}\|^q)^{1/q}\right\} \\ &\leq \inf_{\substack{w_{1-} \in x_{10} \\ w_{2-} \in x_{20}}} \left\{(2^q \cdot \max\{\|w_{1-}\|^q, \|w_{2-}\|^q\} + \|w_{1-}\|^q)^{1/q}\right\} \\ &\leq (2^q + 1)^{1/q} \cdot \inf_{\substack{w_{1-} \in x_{10} \\ w_{2-} \in x_{20}}} \left\{(\|w_{1-}\|^q + \|w_{2-}\|^q)^{1/q}\right\} \\ &= (2^q + 1)^{1/q} \cdot \chi(x_0), \qquad \forall q \geq 1. \end{aligned} \tag{3.49}$$

This implies that the map π is bounded. Next we show that π is also a surjective map. To this end, for any $s''_0 \in \mathfrak{S}_{P//C}$, choose any $(w''_{0-}, w''_{1-}) \in s''_0$ and define $w''_{2-} \triangleq w''_{0-} - w''_{1-}$, thus, from (3.39) and (3.7), we have

$$w''_{1-} \in \mathfrak{B}_P^-, \quad w''_{2-} \in \mathfrak{B}_C^- \tag{3.50}$$

We then define $x''_{10} \triangleq [w''_{1-}]$, $x''_{20} \triangleq [w''_{2-}]$ and $x''_0 \triangleq (x''_{10}, x''_{20})$, thus we have

$$x''_0 \in \mathfrak{S}_P \times \mathfrak{S}_C \tag{3.51}$$

and $\pi(x''_0) = s''_0$. This implies that the map π is surjective.

Define a map $\bar{\pi}$ by (3.44). Since π is surjective, we obtain that $\bar{\pi}$ is a bijective map. It follows from $\chi(\bar{\pi}([x_0])) = \chi(\pi(x_0)) \leq (2^q + 1)^{1/q} \cdot \chi(x_0)$ for any $q \geq 1$ that the map $\bar{\pi}$ is also bounded.

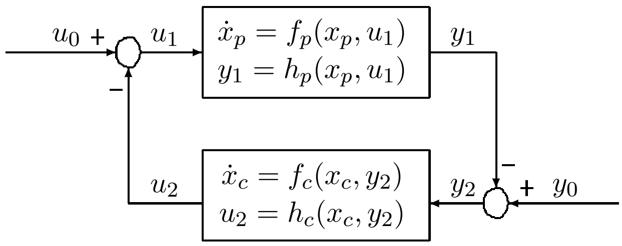

Figure 3.4: Classical state space model for closed-loop system

Finally, we show that the inverse map $\bar{\pi}^{-1} : \mathfrak{S}_{P//C} \to (\mathfrak{S}_P \times \mathfrak{S}_C)/_{\underset{\sim}{\pi}}$ is also bounded. To this end, for any $s_0'' \in \mathfrak{S}_{P//C}$, from the proof of map π being surjective (see (3.50) and (3.51)), we have $\bar{\pi}^{-1}(s_0'') = [x_0'']$. Thus by applying (3.43) and (3.40), we get $\chi(\bar{\pi}^{-1}(s_0'')) = \chi([x_0'']) \le \chi(x_0'') \le (\|w_{1-}''\|^q + \|w_{2-}''\|^q)^{1/q} = (\|w_{1-}''\|^q + \|w_{0-}'' - w_{1-}''\|^q)^{1/q} \le (2^q+1)^{1/q} \cdot (\|w_{0-}''\|^q + \|w_{1-}''\|^q)^{1/q}$. Since (w_{0-}'', w_{1-}'') is arbitrarily chosen from s_0'', we have

$$\begin{aligned} \chi(\bar{\pi}^{-1}(s_0'')) &\le (2^q+1)^{1/q} \cdot \inf_{(w_{0-}'', w_{1-}'') \in s_0''} \left\{ (\|w_{0-}''\|^q + \|w_{1-}''\|^q)^{1/q} \right\} \\ &= (2^q+1)^{1/q} \cdot \inf_{(w_{0-}'', w_{1-}'') \in s_0''} \left\{ (\|(w_{0-}'', w_{1-}''\| \right\} \\ &= (2^q+1)^{1/q} \cdot \chi(s_0''). \end{aligned} \tag{3.52}$$

This implies the inverse map $\bar{\pi}^{-1}$ is also bounded. □

Note that, in (3.49) and (3.52), we can get tighter bounds than previous ones for some particular choices of q. (e.g., when $q = 2$, the bound constant can be chosen as $\sqrt{3}$ by using the inequality $(a+b)^2 \le 2(a^2+b^2)$, while $(2^2+1)^{1/2} > \sqrt{3}$. When $q = 1$, by using the inequality $|a+b| \le |a| + |b|$, we can choose 2 not $(2^1+1)^{1/1} = 3$ as the bound constant.)

3.7.2 Comparison with Classical Initial Conditions for Closed-Loop Systems

Consider the closed-loop system shown in Figure 3.4. The forward and feedback loop represent the plant P and controller C, respectively. Both P and C with classical initial state spaces X_p and X_c, respectively, are defined like (3.29), i.e., $\dot{x}_p = f_p(x_p, u_1), y_1 = h_p(x_p, u_1)$ and $\dot{x}_c = f_c(x_c, y_2), u_2 = h_c(x_c, y_2)$. It is natural to consider the following closed-loop equations:

$$\dot{x}_p = f_p(x_p, u_1), \quad \dot{x}_c = f_c(x_c, y_0 - y_1), \tag{3.53a}$$

$$u_1 = u_0 - h_c(x_c, y_0 - y_1), \quad y_1 = h_p(x_p, u_1), \tag{3.53b}$$

with product state space $X_p \times X_c$ and (u_0, y_0) as inputs, and (u_1, y_1) as outputs.

The following notions of complete, forward observable and strongly backward observable in Theorems 3.29 and 3.30 are defined in Definitions 3.18 and 3.19 (see pages 60, 61).

Theorem 3.29. *Suppose that P, C, and the closed-loop (3.53) are complete. If both P and C are forward observable (resp., strongly backward observable), then the closed-loop (3.53) is forward observable (resp., strongly backward observable).*

Proof. We establish forward observability of the closed-loop (3.53) by contradiction. It is thus assumed that there exist $(x_{p0}, x_{c0}) \in X_p \times X_c$, $(x'_{p0}, x'_{c0}) \in X_p \times X_c$ with $(x_{p0}, x_{c0}) \neq (x'_{p0}, x'_{c0})$ such that

$$(u_1, y_1)|_{t\geq 0} = (u'_1, y'_1)|_{t\geq 0} \quad \text{for all } (u_0, y_0)|_{t\geq 0} = (u'_0, y'_0)|_{t\geq 0}. \tag{3.54}$$

This implies that

$$(y_1, u_2)|_{t\geq 0} \triangleq (h_p(x_p, u_1), h_c(x_c, y_2))|_{t\geq 0} = (h_p(x'_p, u'_1), h_c(x'_c, y'_2))|_{t\geq 0} \triangleq (y'_1, u'_2)|_{t\geq 0}$$

for any $(u_1, y_2)|_{t\geq 0} = (u'_1, y'_2)|_{t\geq 0}$ which satisfy,

$$\dot{x}_p = f_p(x_p, u_1), \quad \dot{x}_c = f_c(x_c, y_2), \quad (x_p(0), x_c(0)) = (x_{p0}, x_{c0}); \tag{3.55a}$$
$$\dot{x}'_p = f_p(x'_p, u'_1), \quad \dot{x}'_c = f_c(x'_c, y'_2), \quad (x'_p(0), x'_c(0)) = (x'_{p0}, x'_{c0}). \tag{3.55b}$$

To this end, let $u_0 = u_1 + u_2$, $u'_0 = u'_1 + u'_2$, $y_0 = y_1 + y_2$, and $y'_0 = y'_1 + y'_2$. It follows from the completeness of P that u_0 (resp., u'_0) is uniquely determined by u_1 and x_{p0} (resp., u'_1 and x'_{p0}). Similarly, y_0 (resp., y'_0) is uniquely determined by y_2 and x_{c0} (resp., y'_2 and x'_{c0}) by using the completeness of C. Since the closed-loop (3.53) is also complete, we know that for $(u''_0, y''_0) = (u_0, y_0)$ and $(x''_p(0), x''_c(0)) = (x'_{p0}, x'_{c0})$ there exist unique $x''_p, x''_c, u''_1, y''_1, u''_2, y''_2$ satisfying

$$\dot{x}''_p = f_p(x''_p, u''_1), \quad y''_1 = h_p(x''_p, u''_1), \quad u''_0 = u''_1 + u''_2;$$
$$\dot{x}''_c = f_c(x''_c, y''_2), \quad u''_2 = h_c(x''_c, y''_2), \quad y''_0 = y''_1 + y''_2.$$

From (3.54), we must have $(u''_i, y''_i)|_{t\geq 0} = (u_i, y_i)|_{t\geq 0}$ for $i = 0, 1, 2$; and thus $(u''_1, y''_2)|_{t\geq 0} = (u_1, y_2)|_{t\geq 0} = (u'_1, y'_2)|_{t\geq 0}$. Since (u'_0, y'_0) are uniquely determined by (u'_1, y'_2) and (x'_{p0}, x'_{c0}) (see above), we have $(u''_0, y''_0)|_{t\geq 0} = (u'_0, y'_0)|_{t\geq 0}$; and thus $(u''_i, y''_i)|_{t\geq 0} = (u'_i, y'_i)|_{t\geq 0}$ for $i = 0, 1, 2$. This in turn implies that $(u'_i, y'_i)|_{t\geq 0} = (u''_i, y''_i)|_{t\geq 0} = (u_i, y_i)|_{t\geq 0}$ for $i = 0, 1, 2$; and the required result $(y_1, u_2)|_{t\geq 0} = (y'_1, u'_2)|_{t\geq 0}$ follows.

Since $(u_1, y_2)|_{t\geq 0} = (u'_1, y'_2)|_{t\geq 0}$ in (3.55) can thus be taken as any element by choosing $u_0 = u_1 + h_c(x_c, y_2)$ and $y_0 = y_2 + h_p(x_p, u_1)$ with $\dot{x}_p = f_p(x_p, u_1)$, $\dot{x}_c = f_c(x_c, y_2)$ and $(x_p(0), x_x(0)) = (x_{p0}, x_{c0})$, we obtain that for the above given $(x_{p0}, x_{c0}) \neq (x'_{p0}, x'_{c0})$ we have $(y_1, u_2)|_{t\geq 0} = (y'_1, u'_2)|_{t\geq 0}$ for any $(u_1, y_2)|_{t\geq 0} = (u'_1, y'_2)|_{t\geq 0}$. This contradicts forward observability of P and C. Thus the closed-loop (3.53) is forward observable.

We also show strongly backward observability of the closed-loop (3.53) by contradiction. Assume therefore that there exist $(x_{p0}, x_{c0}) \in X_p \times X_c$, $(x'_{p0}, x'_{c0}) \in X_p \times X_c$ with $(x_{p0}, x_{c0}) \neq (x'_{p0}, x'_{c0})$ and $(u_0, y_0)|_{t\leq 0} = (u'_0, y'_0)|_{t\leq 0}$ such that $(u_1, y_1)|_{t\leq 0} = (u'_1, y'_1)|_{t\leq 0}$, and thus $(u_2, y_2)|_{t\leq 0} = (u'_2, y'_2)|_{t\leq 0}$. This implies that there exist $u_1|_{t\leq 0} = u'_1|_{t\leq 0}$ and $y_2|_{t\leq 0} = y'_2|_{t\leq 0}$ such that $y_1|_{t\leq 0} = y'_1|_{t\leq 0}$ and $u_2|_{t\leq 0} = u'_2|_{t\leq 0}$. This is a contradiction to that both P and C are strongly backward observable. Thus the closed-loop (3.53) is strongly backward observable. This completes the proof. □

We consider the set $\mathfrak{B}_{P//C}$ which consists of all input-output pairs $((u_0, y_0), (u_1, y_1))$ satisfying (3.53). Using the same procedure in Section 3.5.1, the initial state space $\mathfrak{S}_{P//C}$ (see (3.13)) for the set $\mathfrak{B}_{P//C}$ at initial time 0 is defined.

Theorem 3.30. *Suppose that P, C, and the closed-loop (3.53) are complete. If both P and C are forward observable and strongly backward observable. Then there exists a bijective map from $X_p \times X_c$ to $\mathfrak{S}_{P//C}$.*

Proof. This follows directly from Theorem 3.29 and Proposition 3.20 on page 61. □

3.7.3 Causality, Well-Posedness of Closed-Loop Systems

Since the closed-loop system $[P, C]$ represented by $\mathfrak{B}_{P//C}$ is still a system in terms of Definition 3.2, we can similarly define notions of causality, existence, uniqueness, well-posedness of $\mathfrak{B}_{P//C}$ according to Definitions 3.7, 3.8 on page 49.

Definition 3.31. The closed-loop system $[P, C]$ is said to be *causal* if, $\forall (w_0, w_1), (\bar{w}_0, \bar{w}_1) \in \mathfrak{B}_{P//C}, \forall t \in \mathbb{R}$:

$$\left[w_0|_{(-\infty,t]} = \bar{w}_0|_{(-\infty,t]} \Rightarrow \mathfrak{B}^{w_0}_{P//C}|_{(-\infty,t]} = \mathfrak{B}^{\bar{w}_0}_{P//C}|_{(-\infty,t]} \right],$$

where $\mathfrak{B}^{w_0}_{P//C} \triangleq \{(w_0, \tilde{w}_1) \in \mathcal{W}_e \times \mathcal{W}_e \mid \exists\, \tilde{w}_1 \text{ such that } (w_0, \tilde{w}_1) \in \mathfrak{B}_{P//C}\}$.

Definition 3.32. The closed-loop system $[P, C]$ is said to have the *existence property* if for all $(w_{0-}, w_{1-}) \in \mathfrak{B}^{-}_{P//C}$ and all $w_{0+} \in \mathcal{W}^{+}_e$ there exists a $w_{1+} \in \mathcal{W}^{+}_e$ such that $(w_{0-}, w_{1-}) \wedge (w_{0+}, w_{1+}) \in \mathfrak{B}_{P//C}$; and the *uniqueness property* if for all $w_{01-} \triangleq (w_{0-}, w_{1-}) \in \mathfrak{B}^{-}_{P//C}$ and all $w_{0+} \in \mathcal{W}^{+}_e$,

$$w_{01-} \wedge (w_{0+}, w_{1+}), w_{01-} \wedge (w_{0+}, \tilde{w}_{1+}) \in \mathfrak{B}_{P//C} \text{ with } w_{1+}, \tilde{w}_{1+} \in \mathcal{W}^{+}_e \;\Rightarrow\; w_{1+} = \tilde{w}_{1+}$$

and is *well-posed* if it has both the existence and uniqueness properties.

The following result follows directly from Definitions 3.11 and 3.32 and Theorem 3.28 (see pages 54, 71, and 66, respectively).

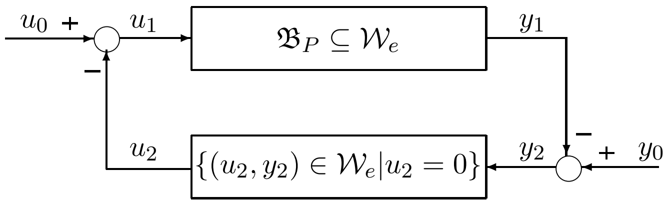

Figure 3.5: Closed-loop system $[P, 0]$

Theorem 3.33. *The closed-loop system* $[P, C]$ *has the existence property if for any* $s_0 \in \mathfrak{S}_{P//C}$ *and any* $w_{0+} \in \mathcal{W}_e^+$ *there exists a* $w_{1+} \in \mathcal{W}_e^+$ *such that* $w_{1+} \in \Pi_{P//C}^{s_0}(w_{0+})$ *with* $\Pi_{P//C}^{s_0}(w_{0+})$ *defined by (3.41); and the uniqueness property if for all* $s_0 \in \mathfrak{S}_{P//C}$ *and all* $w_{0+} \in \mathcal{W}_e^+$*,*

$$w_{1+}, \tilde{w}_{1+} \in \Pi_{P//C}^{s_0}(w_{0+}) \;\Rightarrow\; w_{1+} = \tilde{w}_{1+}$$

and is well-posed if it has both the existence and uniqueness properties.

By Theorem 3.28, we also know that $s_0 \in \mathfrak{S}_{P//C}$ and $\Pi_{P//C}^{s_0}$ can be replaced throughout in the above theorem by $s_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$ and $\overline{\Pi_{P//C}^{s_0}}$, respectively.

Note that from Theorem 3.33 if $[P, C]$ is well-posed then $\Pi_{P//C}^{s_0}$ in (3.41) (resp., (3.42)) actually defines an operator from $\mathcal{W}_e^+$ to $\mathcal{W}_e^+$ for any initial state $s_0 \in \mathfrak{S}_{P//C}$ (resp., $\mathfrak{S}_P \times \mathfrak{S}_C$). Moreover, we have a natural surjective map $\pi : \mathfrak{S}_P \times \mathfrak{S}_C \to \mathfrak{S}_{P//C}$ defined in Theorem 3.28 such that $\Pi_{P//C}^{\pi(x_0)} = \overline{\Pi_{P//C}^{x_0}}$ for any $x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$.

3.7.4 Relationship between the Well-Posedness of Open-Loop and Closed-Loop Systems

By considering the zero controller in closed loop with a plant (Figure 3.5), it is possible to relate well-posedness of the closed-loop system with well-posedness of the plant. This property is presented in the following theorem. Note that a zero controller is defined by the set $\{w_2 \in \mathcal{W}_e \mid w_2 \triangleq (u_2, y_2) \in \mathcal{U}_e \times \mathcal{Y}_e, u_2 = 0\}$.

Theorem 3.34. *The plant* P *is well-posed if and only if the closed-loop system* $[P, 0]$ *is well-posed.*

Proof. Since the controller $C = 0$ (i.e., $\mathfrak{B}_C = \{w_2 \in \mathcal{W}_e \mid w_2 \triangleq (u_2, y_2) \in \mathcal{U}_e \times \mathcal{Y}_e, u_2 = 0\}$), by using (3.39), we have

$$\mathfrak{B}_{P//0} = \{(w_0, w_1) \in \mathcal{W}_e \times \mathcal{W}_e \mid w_0 = (u_0, y_0) \text{ is input}, w_1 = (u_1, y_1) \in \mathfrak{B}_P, u_1 = u_0\}$$

According to Definition 3.32, the closed-loop system $[P, 0]$ is well-posed if and only if for any $(w_{0-}, w_{1-}) \in \mathfrak{B}_{P//0}^-$ and any $w_{0+} \in \mathcal{W}_e^+$, there exists a unique $w_{1+} \in \mathcal{W}_e^+$ such

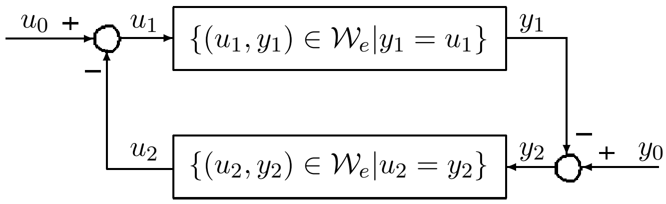

Figure 3.6: Ill-posed feedback system with well-posed plant and well posed controller

that $(w_{0-}, w_{1-}) \wedge (w_{0+}, w_{1+}) \in \mathfrak{B}_{P//0}$. Thus $[P, 0]$ is well-posed if and only if for any $w_{1-} \in \mathfrak{B}_P^-$ and any $w_{0-} \in \mathcal{W}_e^-$ with $u_{0-} = u_{1-}$ (note that $u_{2-} = 0$ since $C = 0$), and any $w_{0+} \in \mathcal{W}_e^+$, there exists a unique $w_{1+} \in \mathcal{W}_e^+$ such that $(w_{1-} \wedge w_{1+}) \in \mathfrak{B}_P$ and $u_{1+} = u_{0+}$ (note that $u_{2+} = 0$ since $C = 0$).

(well-posedness of $[P, 0] \Rightarrow$ well-posedness of P:) For any $w_{1-} \in \mathfrak{B}_P^-$ and any $u_{1+} \in \mathcal{U}_e^+$, choose $w_{0-} \triangleq (u_{0-}, y_{0-}) = (u_{1-}, 0)$ and $w_{0+} \triangleq (u_{0+}, y_{0+}) = (u_{1+}, 0)$, by well-posedness of $\mathfrak{B}_{P//0}$, there exists a unique $y_{1+} \in \mathcal{Y}_e^+$ such that $(w_{1-} \wedge w_{1+}) \in \mathfrak{B}_P$ with $w_{1+} = (u_{1+}, y_{1+})$. This implies that P is well-posed.

(well-posedness of $P \Rightarrow$ well-posedness of $[P, 0]$:) For any $w_{1-} \in \mathfrak{B}_P^-$ and any $w_{0-} \in \mathcal{W}_e^-$ with $u_{0-} = u_{1-}$, and any $w_{0+} \in \mathcal{W}_e^+$ (note that if choose $u_{1+} = u_{0+}$, then by well-posedness of $\mathfrak{B}_P$, there exists a unique $y_{1+} \in \mathcal{Y}_e^+$ such that $(w_{1-} \wedge w_{1+}) \in \mathfrak{B}_P$). Thus there exists a unique $w_{1+} \in \mathcal{W}_e^+$ such that $(w_{1-} \wedge w_{1+}) \in \mathfrak{B}_P$ and $u_{1+} = u_{0+}$. This implies that $[P, 0]$ is well-posed. □

Similarly, the controller C is well-posed if and only if the closed-loop system $[0, C]$ is well-posed. Note that two well-posed open subsystems (plant and controller) does not necessarily result in a well-posed closed-loop system, see e.g., Figure 3.6. A simple calculation shows that the feedback interconnection of Figure 3.6 implicitly requires $u_0 = y_0$. This means that for any closed-loop input $w_0 = (u_0, y_0)$ with $u_0 \neq y_0$ there exist no solutions $w_i = (u_i, y_i)$, $i = 1, 2$ with $y_1 = u_1$ and $u_2 = y_2$ for the closed-loop system depicted in Figure 3.6. This is a very simple example of ill-posed closed-loop systems given in [Willems, 1971a, Section 4.3.2].

3.7.5 Relationship between the Causality of Open and Closed-Loop Systems

The following counterexample is similar to the one given in [Willems, 1969], [Willems, 1971a, Section 4.3.2], which indicates that the causality of a closed-loop system doesn't follow from the causality of open-loop subsystems. Consider the feedback loop system shown in Figure 3.7. The plant P is simply a unit gain minus a time delay, and the controller C is simply a unit gain.

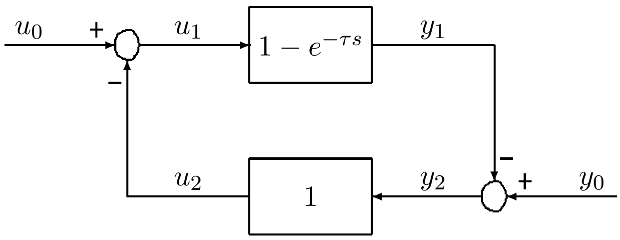

Figure 3.7: A predictor

More precisely, let $\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}, \mathbb{R})$. The plant P and the controller C are defined by $y_1(t) = (Pu_1)(t) = u_1(t) - u_1(t-\tau)$, $\tau > 0$ and $u_2(t) = (Cy_2)(t) = y_2(t)$. Clearly, P and C are causal. A simple calculation shows that a unique solution exists for any disturbance u_0, y_0, and

$$\begin{aligned} u_1(t) &= u_0(t+\tau) - y_0(t+\tau) \\ y_1(t) &= u_0(t+\tau) - y_0(t+\tau) - u_0(t) + y_0(t) \end{aligned}$$

The closed-loop system thus acts as a predictor which is not causal.

The above mathematic model of the closed-loop system does not represent a physical realisable system. The introduction of an infinitesimal time delay in the controller part, i.e., $u_2(t) = (Cy_2)(t) = y_2(t - \varepsilon)$, will yield a causal closed-loop system.

3.8 Notion of Stability for Closed-Loop Systems with Initial Conditions

Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$, consider the closed-loop system $[P, C]$ with the plant P and the controller C (Definition 3.27 on page 65). Let $\mathfrak{S}_P$, $\mathfrak{S}_C$, and $\mathfrak{S}_{P//C}$ defined according to Definition 3.11 on page 54 be the corresponding initial state spaces of P, C, and $[P, C]$ at initial time 0, respectively. According to (3.26) on page 64, we can similarly define the input to output stability for the closed-loop system $[P, C]$.

Definition 3.35. The closed-loop system $[P, C]$ with initial state space $\mathfrak{S}_{P//C}$ is said to be *input to output stable* if and only if it is well-posed and causal, and there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall s_0 \in \mathfrak{S}_{P//C}$, $\forall t > 0$, $\forall w_{0+} \in \mathcal{W}^+$,

$$\left| (\Pi^{s_0}_{P//C} w_{0+})(t) \right| \leq \beta\left(\chi(s_0), t\right) + \gamma(\|w_{0+}\|_{[0,t)}).$$

where the function χ is defined by (3.15) on page 55; and the set $\Pi^{s_0}_{P//C} w_{0+}$ consists of one element only because of well-posedness (see the discussion given below Theorem 3.33).

It is useful to remark that two well-posed open subsystems (plant and controller) does not necessarily result in a well-posed closed-loop system, and that the causality of a closed-loop system doesn't follow from the causality of open-loop subsystems (planter and controller), see previous Sections 3.7.4 and 3.7.5, or [Willems, 1971a, Section 4.3.2].

The following result gives an alternative characterisation of the property of input to output stable for a closed-loop system.

Theorem 3.36. *Suppose that the closed-loop system $[P, C]$ is well-posed and causal. The following four statements are equivalent:*

I. The closed-loop system $[P, C]$ is input to output stable.

II. There exist functions $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$ such that, $\forall s_0 \in \mathfrak{S}_{P//C}$, $\forall t > 0$, $\forall w_{0+} \in \mathcal{W}^+$,

$$|(\Pi^{s_0}_{P//C} w_{0+})(t)| \leq \beta_1 \left(\chi(s_0), t\right) + \gamma_1(\|w_{0+}\|_{[0,t)}). \tag{3.56}$$

III. There exist functions $\beta_2 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$ such that, $\forall x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$, $\forall t > 0$, $\forall w_{0+} \in \mathcal{W}^+$,

$$|(\overline{\Pi^{x_0}_{P//C}} w_{0+})(t)| \leq \beta_2 \left(\chi(x_0), t\right) + \gamma_2(\|w_{0+}\|_{[0,t)}). \tag{3.57}$$

IV. There exist functions $\beta_3 \in \mathcal{KL}$ and $\gamma_3 \in \mathcal{K}_\infty$ such that, $\forall x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$, $\forall t > 0$, $\forall w_{0+} \in \mathcal{W}^+$, $\forall w_{1-} \in x_{10}$, $\forall w_{2-} \in x_{20}$,

$$|(\overline{\Pi^{x_0}_{P//C}} w_{0+})(t)| \leq \beta_3 \left(\|(w_{1-}, w_{2-})\|, t\right) + \gamma_3(\|w_{0+}\|_{[0,t)}). \tag{3.58}$$

Moreover, we have $\gamma_1 = \gamma_2 = \gamma_3$ and $\beta_2 = \beta_3$.

Proof. I $\Leftrightarrow$ II: This follows from Definition 3.35.

II $\Rightarrow$ III: Suppose that (3.56) holds with given functions $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$. For any $x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$, by Theorem 3.28, we have $\pi(x_0) \in \mathfrak{S}_{P//C}$ and $\overline{\Pi^{x_0}_{P//C}} = \Pi^{\pi(x_0)}_{P//C}$, and $\chi(\pi(x_0)) \leq \|\pi\| \cdot \chi(x_0)$ (note that π is a bounded map). Define a function β_2 of class $\mathcal{KL}$ by $\beta_2(r, t) \triangleq \beta_1(\|\pi\| r, t)$ for all $r \geq 0$ and $t \geq 0$. We have (3.57) holds with $\gamma_2 = \gamma_1$.

III $\Rightarrow$ II: Suppose that (3.57) holds with given functions $\beta_2 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$. For any $s_0 \in \mathfrak{S}_{P//C}$, by Theorem 3.28, we have $\bar{\pi}^{-1}(s_0) \in (\mathfrak{S}_P \times \mathfrak{S}_C)/_{\underset{\sim}{\pi}}$ and $\chi(\bar{\pi}^{-1}(s_0)) \leq \|\bar{\pi}^{-1}\|\chi(s_0)$ (note that $\bar{\pi}^{-1}$ is a bounded bijective map). For any $\varepsilon > 0$, from (3.43), there exists an $x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$ such that $x_0 \in \bar{\pi}^{-1}(s_0)$ and $\chi(x_0) \leq \chi(\bar{\pi}^{-1}(s_0)) + \varepsilon$. Thus we have $|(\Pi^{s_0}_{P//C} w_{0+})(t)| = |(\overline{\Pi^{x_0}_{P//C}} w_{0+})(t)| \leq \beta_2 \left(\chi(x_0), t\right) + \gamma_2(\|w_{0+}\|_{[0,t)}) \leq \beta_2 \left(\|\bar{\pi}^{-1}\| \cdot \chi(s_0) + \varepsilon, t\right) + \gamma_2(\|w_{0+}\|_{[0,t)})$ for any $t > 0$ and any $w_{0+} \in \mathcal{W}^+$. Since ε is an arbitrarily chosen positive number, we have (3.56) holds with $\gamma_1 = \gamma_2$ and $\beta_1(r,t) = \beta_2(\|\bar{\pi}^{-1}\| \cdot r, t)$ for all $r \geq 0$ and $t \geq 0$.

III $\Rightarrow$ IV: Suppose that (3.57) holds with given functions $\beta_2 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$. From (3.40), we know that $\chi(x_0) \leq \|(w_{1-}, w_{2-})\|$ for any $w_{1-} \in x_{10}$ and any $w_{2-} \in x_{20}$. Thus, we have (3.58) holds with $\beta_3 = \beta_2$ and $\gamma_3 = \gamma_2$.

IV $\Rightarrow$ III: Suppose that (3.58) holds with given functions $\beta_3 \in \mathcal{KL}$ and $\gamma_3 \in \mathcal{K}_\infty$. For any $x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$, for any $\varepsilon > 0$, from (3.40), we know that there exist $w_{1-} \in x_{10}$ and $w_{2-} \in x_{20}$ such that $\|(w_{1-}, w_{2-})\| \leq \chi(x_0) + \varepsilon$. Thus we have $|(\overline{\Pi^{x_0}_{P//C}} w_{0+})(t)| \leq \beta_3(\|(w_{1-}, w_{2-})\|, t) + \gamma_3(\|w_{0+}\|_{[0,t)}) \leq \beta_3(\chi(x_0) + \varepsilon, t) + \gamma_3(\|w_{0+}\|_{[0,t)})$ for all $t \geq 0$ and all $w_{0+} \in \mathcal{W}^+$. Since ε is an arbitrarily chosen positive number, we have (3.57) holds with $\beta_2 = \beta_3$ and $\gamma_2 = \gamma_3$.

Thus we have I $\Leftrightarrow$ II $\Leftrightarrow$ III $\Leftrightarrow$ IV. This completes the proof. $\square$

3.9 Summary

In this chapter, a unified framework for the study of input-output systems with abstract initial conditions is introduced. We define a system by the set of all possible input-output pairs on a doubly infinite time axis corresponding to its description, such as a set of first-order differential equations. Properties of causality, existence, uniqueness for a system are defined and discussed in detail in this abstract framework. A general construction of the initial conditions is given in terms of an equivalence class of trajectories on the negative time axis. Comparison with classical initial concepts are addressed for several examples including a concrete delay line model, the classical finite dimensional nonlinear state space model, and a class of delay-differential systems. An ISS-like notion of input-to-output stability on the positive time axis with initial conditions is given. The chapter ends by several alternative characterisation of this notion of stability for a closed-loop system.

"Obvious" is the most dangerous word in mathematics.

Eric Temple Bell (1883-1960)

Chapter 4

Robust Stability Analysis of Feedback Systems with Initial Conditions

In Chapter 3 we have developed a general input/output framework which incorporates a general concept of initial conditions characterised by a purely input-output formalism drawn from [Willems, 1989]. The central result of the current chapter is to obtaine a generalisation of the robust stability results of [Georgiou and Smith, 1997b] whereby the initial conditions are reflected within the stability concept in an ISS-like manner (cf. [Sontag, 1989, Sontag and Ingalls, 2002, Sontag and Wang, 1995, 1996]) in this framework. The main result of this chapter is Theorem 4.8 which can also be viewed as a generalisation of the ISS approach to enable an explicit treatment of robust stability issues.

Two different versions of Theorem 4.8 are presented: one requires the well-posedness of the perturbed closed-loop system that is a typical assumption in the classical literature; while the other one requires only the uniqueness property of the perturbed closed-loop system which significantly eases the real-world application of the robust stability result. In the second case the existence property of the perturbed closed-loop system is established via the well-known Schauder fixed-point theorem. Several technical assumptions are imposed in order to use this fixed-point theorem, such as a compactness requirement for the plant perturbations and a relative continuity requirement for the nominal closed-loop system. These stronger technical requirements on the plant perturbations and the nominal closed-loop system in turn result in substantially weaker requirements on the perturbed closed-loop system, i.e., the uniqueness property of the perturbed closed-loop system, which is often far easier to be verified than the existence property. This strategy dealing with the existence issue in robust stability analysis first appeared in French and Bian [2012] to establish a bias version of robust stability result.

Theorem 4.8 can be regarded as a generalisation of the input-output operator robust stability theorem of Georgiou and Smith, to include the case of initial conditions. This is discussed in detail in Section 4.3 and summarised as Theorem 4.12. A notion of finite-time reachability for a system is defined in Section 4.4, and a more applicable robust stability result Theorem 4.18 than Theorem 4.8 in this framework is established. Applications of the main results of this chapter to linear time-invariant systems and a general nonlinear plants with input delay are given in Section 4.5 and Section 4.6, respectively. The chapter ends by a generalisation of this robust stability results to systems with potential for finite escape times.

4.1 Setting of the Problem

Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$ (such as $W = L^\infty(\mathbb{R}, \mathbb{R}^{m+p})$), consider the closed-loop system $[P, C]$ with the plant P and the controller C (Definition 3.27). Let the perturbed plant $\tilde{P}$ and the perturbed closed-loop system $[\tilde{P}, C]$ be represented by the sets $\mathfrak{B}_{\tilde{P}} \subseteq \mathcal{U}_e \times \mathcal{Y}_e$ and $\mathfrak{B}_{\tilde{P}//C} \subseteq \mathcal{W}_e \times \mathcal{W}_e$, respectively. Let $\mathfrak{S}_P$, $\mathfrak{S}_{\tilde{P}}$, $\mathfrak{S}_C$, $\mathfrak{S}_{P//C}$, and $\mathfrak{S}_{\tilde{P}//C}$, defined according to Definition 3.11, be the corresponding initial state spaces of $\mathfrak{B}_P$, $\mathfrak{B}_{\tilde{P}}$, $\mathfrak{B}_C$, $\mathfrak{B}_{P//C}$, and $\mathfrak{B}_{\tilde{P}//C}$ at initial time 0, respectively. Notice that, according to (3.9) on page 52, the graph $\mathcal{G}_P^{w_{1-}}$ of system P for a given past trajectory $w_{1-} \in \mathfrak{B}_P^-$ is defined by

$$\mathcal{G}_P^{w_{1-}} \triangleq \left\{ w_{1+} \in \mathcal{W}^+ \mid w_{1-} \wedge w_{1+} \in \mathfrak{B}_P \right\}$$

and $\mathcal{G}_{\tilde{P}}^{\tilde{w}_{1-}}$ for $\tilde{w}_{1-} \in \mathfrak{B}_{\tilde{P}}^-$ and $\mathcal{G}_C^{w_{2-}}$ for $w_{2-} \in \mathfrak{B}_C^-$ are similarly defined.

Before we come to our main result, we introduce the notions of truncation complete normed vector spaces and relatively continuous operators.

4.1.1 Truncation Complete Normed Vector Space

We first introduce the notion of *truncation complete* for a normed vector space:

Definition 4.1. A normed vector space $\mathcal{V}$ (not necessarily complete) is said to be *truncation complete* if $\mathcal{V}(J)$ is complete for all open subinterval $J \subseteq \mathbb{R}$ with finite length, i.e., $\mathcal{V}(t_1, t_2)$ is[1] complete for any $-\infty < t_1 < t_2 < \infty$. Similarly, the normed vector space $\mathcal{V}^+$ (not necessarily complete) is said to be *truncation complete* if $\mathcal{V}[0, \tau)$ is complete for any $0 < \tau < \infty$.

Note that the completeness is not specified for the normed vector space $\mathcal{V}$ (or $\mathcal{V}^+$). In fact we have the following results:

[1]To simplify notation we identify $\mathcal{V}(t_1, t_2)$ with $\mathcal{V}\big((t_1, t_2)\big)$ here and in what follows.

Proposition 4.2. *Let $\mathcal{V}$ be a complete normed vector space. Assume that $T_J x \in \mathcal{V}$ for any $J \triangleq (t_1, t_2)$ with $-\infty < t_1 < t_2 < \infty$ and any $x \in \mathcal{V}$ (i.e., $\mathcal{V}$ is* truncation closed*). Then $\mathcal{V}$ is truncation complete.*

Proof. For any $J \triangleq (t_1, t_2)$ with $-\infty < t_1 < t_2 < \infty$ and any Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{V}(J)$, i.e., $\|x_n - x_m\|_J \to 0$ as n, $m \to \infty$, we have to show that there exists an $x \in \mathcal{V}(t_1, t_2)$ such that $\|x_n - x\|_J \to 0$ as $n \to \infty$. From Assumption 3.1.(2) on page 41 we get $\|T_J(x_n - x_m)\| = \|x_n - x_m\|_J$ for any n and m, and hence $\|T_J x_n - T_J x_m)\| \to 0$ as n, $m \to \infty$. Since $T_J x_n \in \mathcal{V}$ for[2] all n and $\mathcal{V}$ is complete, there exists a $y \in \mathcal{V}$ such that $\|T_J x_n - y\| \to 0$ as $n \to \infty$. Define $x = R_J y$, then we have $x \in \mathcal{V}(J)$ since $y \in \mathcal{V}$. To conclude the proof we only need to show that $\|x_n - x\|_J \to 0$ as $n \to 0$. This is true since $\|x_n - x\|_J = \|T_J(x_n - x)\| = \|T_J x_n - T_J R_J y\| = \|T_J T_J x_n - T_J y)\| \leq \|T_J x_n - y\|$ and $\|T_J x_n - y\| \to 0$ as $n \to \infty$. □

It can be easily seen from the above proof that the assertion of Proposition 4.2 remains true if the assumption of truncation closedness of $\mathcal{V}$ is replaced by the following: for any $J \triangleq (t_1, t_2)$ with $-\infty < t_1 < t_2 < \infty$, there exists a uniformly continuous (or more general: Cauchy continuous) map $E_J : \mathcal{V}_J \to \mathcal{V}$ satisfying $R_J x = R_J(E_J x)$ for any $x \in \mathcal{V}_J$.

Proposition 4.3. *Let $\mathcal{V}$ (not necessarily complete) be a truncation complete normed vector space and let $\{x_n\}_{n=1}^{\infty}$ be any Cauchy sequence of $\mathcal{V}$. Then there exists an $x \in \mathcal{V}_e$ such that for any $J \triangleq (t_1, t_2)$ with $-\infty < t_1 < t_2 < \infty$ we have $\|R_J x_n - R_J x\|_J \to 0$ as $n \to \infty$.*

Proof. Since $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{V}$ is a Cauchy sequence, $\{R_J x_n\}_{n=1}^{\infty}$ is also a Cauchy sequence of $\mathcal{V}(J)$ for any $J \triangleq (t_1, t_2)$ with $-\infty < t_1 < t_2 < \infty$. From the completeness of $\mathcal{V}(J)$, we obtain that there is a $y^J \in \mathcal{V}(J)$ such that $\left\|R_J x_n - y^J\right\|_J \to 0$ as $n \to \infty$ for all J. Define a time function $x(t)$ on the infinite interval, $-\infty < t < \infty$ as follows: for any $t \in \mathbb{R}$, choose some open subinterval J of $\mathbb{R}$ with finite length and $t \in J$, let $x(t) \triangleq y^J(t)$. This function x is well-defined.[3] It follows from the definition of $\mathcal{V}_e$ that $x \in \mathcal{V}_e$, since $R_J x = R_J y^J$ with $y^J \in \mathcal{V}(J)$ for all J. To conclude the proof, we need to show $\|R_J x_n - R_J x\|_J \to 0$ as $n \to \infty$. This is obvious since $R_J y^J = y^J$ and $\left\|R_J x_n - y^J\right\|_J \to 0$ as $n \to \infty$ for all J. □

[2]Note that for any $x_n \in \mathcal{V}(J)$ there always exists a $z_n \in \mathcal{V}$ such that $x_n = R_J z_n$. Moreover, $T_J x_n = T_J R_J z_n = T_J z_n$.

[3]To see this, it suffices to show that $y^{J_1} = R_{J_1} y^{J_2}$ for any two open subintervals J_1, J_2 of $\mathbb{R}$ with finite length and $J_1 \subseteq J_2$. Since $R_{J_1} x_n = R_{J_1} R_{J_2} x_n$ for all n, we get $\left\|y^{J_1} - R_{J_1} y^{J_2}\right\|_{J_1} = \left\|(y^{J_1} - R_{J_1} x_n) + (R_{J_1} R_{J_2} x_n - R_{J_1} y^{J_2})\right\|_{J_1} \leq \left\|y^{J_1} - R_{J_1} x_n\right\|_{J_1} + \left\|R_{J_2} x_n - y^{J_2}\right\|_{J_2} \to 0$ as $n \to \infty$. This implies that $y^{J_1} = R_{J_1} y^{J_2}$.

4.1.2 Relatively Continuous Operators

The following definition of relative continuous is from [French and Bian, 2012, p. 1229].

Definition 4.4. An operator (possibly nonlinear) $\Psi : \mathcal{W}^+ \to \mathcal{W}^+$ is said to be *relatively continuous* if, for all operators (possibly nonlinear) $\Phi : \mathcal{W}^+ \to \mathcal{W}^+$ with $R_{[0,\tau)}\Phi$ compact for any $0 < \tau < \infty$, the operator $R_{[0,\tau)}\Phi \circ \Psi : \mathcal{W}^+ \to R_{[0,\tau)}\mathcal{W}^+$ is continuous.

If we are only concerned with linear operators in the above Definition 4.4, (i.e., a linear operator Ψ is relatively continuous if the linear operator $R_{[0,\tau)}\Phi \circ \Psi$ is continuous for any operator Φ with $R_{[0,\tau)}\Phi$ compact for any $0 < \tau < \infty$.) then every linear continuous operator in this case is also relatively continuous.[4]

Note that no compactness is specified for the operator Φ in the above Definition 4.4. In fact, we have the following result:

Proposition 4.5. *If the operator (possibly nonlinear) $\Phi : \mathcal{W}^+ \to \mathcal{W}^+$ is compact, then for any $0 < \tau < \infty$ the operator $R_{[0,\tau)}\Phi : \mathcal{W}^+ \to R_{[0,\tau)}\mathcal{W}^+$ is also compact.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be any bounded sequence of $\mathcal{W}^+$. We have to show that for any $0 < \tau < \infty$ the sequence $\{R_{[0,\tau)}\Phi x_n\}_{n=1}^{\infty}$ contains a Cauchy subsequence. Since the operator Φ is compact, there is a Cauchy subsequence $\{\Phi x_{n_k}\}_{k=1}^{\infty}$ of $\{\Phi x_n\}_{n=1}^{\infty}$, i.e., $\left\|\Phi x_{n_i} - \phi x_{n_j}\right\| \to 0$ as $i, j \to \infty$. From Assumption 3.1.(3) on page 41 we get $\left\|R_{[0,\tau)}(\Phi x_{n_i} - \Phi x_{n_j})\right\| \leq \left\|\Phi x_{n_i} - \Phi x_{n_j}\right\|$ for any i, j, and hence $\{R_{[0,\tau)}\Phi x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy subsequence of $\{R_{[0,\tau)}\Phi x_n\}_{n=1}^{\infty}$. This implies the compactness of the operator $R_{[0,\tau)}\Phi$ for any $0 < \tau < \infty$. □

The converse of above Proposition 4.5 is not necessarily true, since $\left\|R_{[0,\tau)}x_n\right\| \to 0$ as $n \to \infty$ for any $0 < \tau < \infty$ does not necessarily implies that $\|x_n\| \to 0$ as $n \to \infty$, for example:

Example 4.6. *For any $n = 1, 2, 3, \ldots$, define a function $x_n(t)$ of time t on the positive infinite interval $[0, \infty)$ as follows:*

$$x_n(t) = \begin{cases} 0\,, & \textit{if } t < n; \\ \frac{t^n}{n!}\,, & \textit{if } n \leq t \leq n+1; \\ 0\,, & \textit{if } t > n+1. \end{cases}$$

It can be easily verified that for any $0 < \tau < \infty$ we have $(R_{[0,\tau)}x_n)(t) = 0$ if $t < \tau \leq n$, and hence $\left\|R_{[0,\tau)}x_n\right\|_{L^\infty\left([0,\tau),\mathbb{R}\right)} \to 0$ as $n \to \infty$. However, $\|x_n\|_{L^\infty\left([0,\infty),\mathbb{R}\right)} = \frac{(n+1)^n}{n!} \to \infty$ as $n \to \infty$.

[4] Note that a linear operator is continuous if and only if it is bounded, and that every linear compact operator is also bounded and thus continuous (see Section 2.2 on page 15). In addition, from Lemma 2.13 on page 20, we know that the composition operator $C \circ B$ is always compact provided that C is compact and B is bounded.

4.2 General Systems: Theorem 4.8

We are finally in a position to state our main result in this chapter. The theorem below generalises Georgiou and Smith's input-output operator robust stability theorem to accommodate the initial conditions, including an appropriate generalisation of the nonlinear gap metric [Georgiou and Smith, 1997b]. The idea of looking at the abstract framework for studying the stability of interconnected systems is not new. In the paper [Sontag and Ingalls, 2002], the authors established an abstract small-gain theorem in an ISS sense including applications to purely input/output systems represented by input/output operators defined on the following kind of signal spaces:

$$L_0^\infty(\mathbb{R}, S) \triangleq \{u \in L^\infty(\mathbb{R}, S) \mid u(t) = 0, \forall t < t_0 \text{ for some } t_0 \in \mathbb{R}\}$$

with S being any normed linear space and $L^\infty(\mathbb{R}, S)$ consisting of all measurable locally essentially bounded maps from $\mathbb{R}$ to S. The IOS concept is still a doubly infinite time axis definition; but it precludes for example the uncontrollable stable linear case, since exponential functions do not lie in $L_0^\infty(\mathbb{R}, S)$. Note that the special representation of systems allows the authors to identify the 'state' only with the past input without using the past output; moreover, the well-posedness part of the small-gain theorem was not considered or just as a standing assumption, see [Sontag and Ingalls, 2002, Section 4.5.2] or [Ingalls et al., 1999].

The following assumptions on the normed vector space $\mathcal{W}^+$ are only required in the proof of Theorem 4.8 with condition II:

Assumption 4.7. *(1) For any $x \in \mathcal{W}_e^+$, if $\|x\| < \infty$, then $x \in \mathcal{W}^+$; (2) The normed vector space $\mathcal{W}^+$ (not necessarily complete) is truncation complete, i.e., $\mathcal{W}[0, \tau)$ is complete for any $0 < \tau < \infty$; (3) For any time interval $J \triangleq [0, \tau)$ with $0 < \tau < \infty$, there exists a continuous map $E_J : \mathcal{W}(J) \to \mathcal{W}^+$ such that $R_J x = R_J(E_J x)$ for any $x \in \mathcal{W}(J)$.*

Theorem 4.8. *Assume that P, $\tilde{P}$, and C are well-posed and causal systems, and that $[P, C]$ is time-invariant, well-posed and causal, and that $[\tilde{P}, C]$ is causal. Let $[P, C]$ be input to output stable, i.e., there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$, $\forall w_{0+} \in \mathcal{W}^+$, $\forall t > 0$,*

$$|(\overline{\Pi_{P//C}^{x_0}} w_{0+})(t)| \leq \beta\left(\chi(x_0), t\right) + \gamma(\|w_{0+}\|_{[0,t)}). \tag{4.1}$$

If there exist functions σ_0, $\sigma \in \mathcal{K}_\infty$ and $\beta_0 \in \mathcal{KL}$ such that for any $\tilde{w}_{1-} \in \mathcal{W}^- \cap \mathfrak{B}_{\tilde{P}}^-$ there exists a $w_{1-} \in \mathcal{W}^- \cap \mathfrak{B}_P^-$ with

$$\|w_{1-}\| \leq \sigma_0(\|\tilde{w}_{1-}\|) \tag{4.2}$$

and a causal surjective operator $\Phi : \mathrm{dom}(\Phi) \subseteq \mathcal{G}_P^{w_{1-}} \to \mathcal{G}_{\tilde{P}}^{\tilde{w}_{1-}}$ *satisfying,* $\forall t > h \geq 0,\ \forall w_{1+} \in \mathrm{dom}(\Phi)$,

$$|((\Phi - I)w_{1+})(t)| \leq \beta_0(\|w_{1-}\wedge w_{1+}\|_{(-\infty,h]}, t-h) + \sigma(\|w_{1+}\|_{[h,t)}). \tag{4.3}$$

In addition, if there exist two functions ρ, ε *of class* $\mathcal{K}_\infty$ *such that,* $\forall s \geq 0$,

$$\sigma \circ (I + \rho) \circ \gamma(s) \leq (I + \varepsilon)^{-1}(s); \tag{4.4}$$

and either of the following conditions is satisfied:

I. $[\tilde{P}, C]$ *is well-posed and* $\overline{\Pi_{\tilde{P}//C}^{\tilde{x}_0}}(\mathcal{W}^+) \subseteq \mathcal{W}^+$ *for any* $\tilde{x}_0 \in \mathfrak{S}_{\tilde{P}} \times \mathfrak{S}_C$;

II. Assumption 4.7 holds for $\mathcal{W}^+$, *and* $[\tilde{P}, C]$ *has the uniqueness property, and* $\overline{\Pi_{P//C}^{x_0}}$ *is relatively continuous for any* $x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$, *and* $R_{[0,\tau)}(\Phi - I)$ *is compact for any* $0 < \tau < \infty$.

Then the closed-loop system $[\tilde{P}, C]$ *is also input to output stable. More specifically, for any function* α *of class* $\mathcal{K}_\infty$, *there exists a function* $\tilde{\beta} \in \mathcal{KL}$ *such that,* $\forall \tilde{x}_0 \in \mathfrak{S}_{\tilde{P}} \times \mathfrak{S}_C,\ \forall \tilde{w}_{0+} \in \mathcal{W}^+,\ \forall t > 0$,

$$|(\overline{\Pi_{\tilde{P}//C}^{\tilde{x}_0}}\tilde{w}_{0+})(t)| \leq \tilde{\beta}\,(\chi(\tilde{x}_0), t) + (\alpha + \tilde{\gamma})(\|\tilde{w}_{0+}\|_{[0,t)}), \tag{4.5}$$

where $\tilde{\gamma} \in \mathcal{K}_\infty$ *is defined by*

$$\tilde{\gamma}(r) \triangleq (\sigma + I) \circ (I + \rho) \circ \gamma \circ (I + \varepsilon^{-1})^3(r),\ \forall r \geq 0. \tag{4.6}$$

Remark 4.9. if both σ and γ are linear functions, e.g., $\sigma(s) = r_1 \cdot s$ an $\gamma(s) = r_2 \cdot s$ for some $r_1 \geq 0, r_2 \geq 0$, then condition (4.4) is equivalent to $r_1 \cdot r_2 < 1$.

Note that the inequality (4.4) is equal to the following inequality

$$\gamma \circ (I + \varepsilon) \circ \sigma(r) \leq (I + \rho)^{-1}(r), \quad \forall r \geq 0. \tag{4.7}$$

This is easily to be seen by letting $s = (I + \varepsilon) \circ \sigma(r)$ for any $r \geq 0$ in (4.4). In fact, we have $\sigma \circ (I + \rho) \circ \gamma \circ (I + \varepsilon) \circ \sigma(r) \leq (I + \varepsilon)^{-1} \circ (I + \varepsilon) \circ \sigma(r)$, and then by applying $(I + \rho)^{-1} \circ \sigma^{-1}(\cdot)$ on both sides, we obtain (4.7).

Theorem 4.8 still holds when replacing the product state space $\mathfrak{S}_P \times \mathfrak{S}_C$ by $\mathfrak{S}_{P//C}$ by using Theorem 3.36.

The proof of Theorem 4.8 with Conditions I and II is organised into Section 4.2.1 and Section 4.2.2, repectively.

4.2.1 Proof of Theorem 4.8 with Condition I (well-posedness)

The proof of this part of Theorem 4.8 will make use a technical result borrowed from [Jiang et al., 1994] (see Lemma 2.25 on page 31).

Proof. For any $\tilde{w}_{0+} \in \mathcal{W}^+$ and any $\tilde{x}_0 \in \mathcal{S}_{\tilde{P}} \times \mathcal{S}_C$, choose any bounded $(\tilde{w}_{1-}, w_{2-}) \in \tilde{x}_0$ and let $\tilde{w}_{0-} = \tilde{w}_{1-} + w_{2-}$. Since $[\tilde{P}, C]$ is well-posed, causal and $\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\mathcal{W}^+) \subseteq \mathcal{W}^+$, there exists a unique $(\tilde{w}_{1+}, w_{2+}) \in \mathcal{W}^+ \times \mathcal{W}^+$ such that $\tilde{w}_{1+} \in \mathcal{G}^{\tilde{w}_{1-}}_{\tilde{P}}$, $w_{2+} \in \mathcal{G}^{w_{2-}}_C$ and $\tilde{w}_{0+} = \tilde{w}_{1+} + w_{2+}$, i.e., the operator $\Pi^{\tilde{x}_0}_{\tilde{P}//C} : \mathcal{W}^+ \to \mathcal{W}^+, \tilde{w}_{0+} \mapsto \tilde{w}_{1+}$ is well defined and causal.

Under conditions in Theorem 4.8, there exists a $w_{1-} \in \mathcal{W}^- \cap \mathfrak{B}^-_P$ for $\tilde{w}_{1-}$ such that $\|w_{1-}\| \leq \sigma_0(\|\tilde{w}_{1-}\|)$ (see (4.2)), and thus

$$\|(w_{1-}, w_{2-})\| \leq (\sigma_0 + I)(\|(\tilde{w}_{1-}, w_{2-})\|). \tag{4.8}$$

In addition, there exists a causal surjective operator $\Phi : \text{dom}(\Phi) \subseteq \mathcal{G}^{w_{1-}}_P \to \mathcal{G}^{\tilde{w}_{1-}}_{\tilde{P}}$. It follows from the surjection of Φ that there exists $w_{1+} \in \mathcal{G}^{w_{1-}}_P$ satisfying $\Phi(w_{1+}) = \tilde{w}_{1+}$. We choose $x_0 := ([w_{1-}], [w_{2-}]) \in \mathcal{S}_P \times \mathcal{S}_C$ and let $w_{0-} = w_{1-} + w_{2-}$ and $w_{0+} \triangleq w_{1+} + w_{2+}$. It follows from the well-posedness of $[P, C]$ that $\overline{\Pi^{x_0}_{P//C}}(w_{0+}) = w_{1+}$; and thus the following equations hold:

$$\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\tilde{w}_{0+}) = \tilde{w}_{1+} = \Phi \circ \overline{\Pi^{x_0}_{P//C}}(w_{0+}), \tag{4.9}$$

$$\tilde{w}_{0+} = \left(I + (\Phi - I) \circ \overline{\Pi^{x_0}_{P//C}}\right)(w_{0+}). \tag{4.10}$$

For ease of notation, we define

$$w_i \triangleq (w_{i-} \wedge w_{i+}) \ (i = 0, 1, 2), \quad \tilde{w}_j \triangleq (\tilde{w}_{j-} \wedge \tilde{w}_{j+}) \ (j = 0, 1).$$

We have, from (4.1) and Theorem 3.36, using time-invariance and causality of $[P, C]$,

$$|w_1(t)| \leq \beta(\|(w_1, w_2)\|_{(-\infty,h]}, t - h) + \gamma(\|w_0\|_{[h,t]}), \ \forall t \geq h \geq 0. \tag{4.11}$$

Next, we estimate the upper bound of $\|(w_1, w_2)\|$ by first giving the upper bound of $\|w_{0+}\|$. It follows from (4.10) that

$$\begin{aligned} \|w_{0+}\| &\leq \|\tilde{w}_{0+}\| + \|(I - \Phi)(\overline{\Pi^{x_0}_{P//C}} w_{0+})\| \qquad (4.12)\\ &\leq \|\tilde{w}_{0+}\| + \beta_0(\|w_{1-}\|, 0) + \sigma(\|\overline{\Pi^{x_0}_{P//C}} w_{0+}\|), \ \text{[by (4.3)]} \\ &\leq \|\tilde{w}_{0+}\| + \beta_0(\|w_{1-}\|, 0) + \sigma\left(\beta(\|(w_{1-}, w_{2-})\|, 0) + \gamma(\|w_{0+}\|)\right), \ \text{[by (4.11)]} \\ &\leq \|\tilde{w}_{0+}\| + \beta_0((\sigma_0 + I)(\|(\tilde{w}_{1-}, w_{2-})\|), 0) + \sigma \circ (I + \rho) \circ \gamma(\|w_{0+}\|) \\ &\quad + \sigma \circ (I + \rho^{-1}) \circ \beta((\sigma_0 + I)(\|(\tilde{w}_{1-}, w_{2-})\|), 0), \ \text{[by (4.8) and (2.19)]} \end{aligned}$$

Since $\sigma \circ (I+\rho) \circ \gamma(\cdot) < (I+\varepsilon)^{-1}(\cdot)$ (see (4.4)) and $(I - (I+\varepsilon)^{-1})^{-1} = (I+\varepsilon^{-1})$ (see (2.20)), we obtain from (4.12) that

$$\|w_{0+}\| \leq (I+\varepsilon^{-1})\left(\|\tilde{w}_{0+}\| + \Delta(\|(\tilde{w}_{1-}, w_{2-})\|)\right), \tag{4.13}$$

where function $\Delta \in \mathcal{K}$ is defined by,

$$\Delta(r) \triangleq \beta_0((\sigma_0+I)(r),0) + \sigma \circ (I+\rho^{-1}) \circ \beta((\sigma_0+I)(r),0),\ \forall r \geq 0. \tag{4.14}$$

Define three functions $\alpha_i \in \mathcal{K}_\infty,\ (i=1,2,3)$ by

$$\begin{aligned}
\alpha_1(s) &\triangleq (\sigma_0+I)(s) + 2\beta\big((\sigma_0+I)(s),0\big),\ \forall s \geq 0;\\
\alpha_2(s) &\triangleq \alpha_1(s) + (2\gamma+I) \circ (I+\varepsilon^{-1}) \circ (I+\varepsilon) \circ \Delta(s),\ \forall s \geq 0;\\
\alpha_3(s) &\triangleq (2\gamma+I) \circ (I+\varepsilon^{-1}) \circ (I+\varepsilon^{-1})(s),\ \forall s \geq 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|(w_1,w_2)\| &\leq \|(w_{1-},w_{2-})\| + 2\,\|w_{1+}\| + \|w_{0+}\|\\
&\leq (\sigma_0+I)(\|(\tilde{w}_{1-},w_{2-})\|) + 2\beta((\sigma_0+I)(\|(\tilde{w}_{1-},w_{2-})\|),0)\\
&\quad + 2\gamma(\|w_{0+}\|) + \|w_{0+}\| \ \ \text{[by (4.8) and (4.11)]}\\
&\leq \alpha_1(\|(\tilde{w}_{1-},w_{2-})\|) \quad \text{[by (4.13)]}\\
&\quad + (2\gamma+I) \circ (I+\varepsilon^{-1})\left(\|\tilde{w}_{0+}\| + \Delta(\|(\tilde{w}_{1-},w_{2-})\|)\right)\\
&\leq \alpha_2(\|(\tilde{w}_{1-},w_{2-})\|) + \alpha_3(\|\tilde{w}_{0+}\|) \triangleq s_\infty \quad \text{[by (2.19)]}
\end{aligned} \tag{4.15}$$

By using the equation (4.10), for any $t>0$, we have

$$\begin{aligned}
|w_{0+}(t)| &\leq |\tilde{w}_{0+}(t)| + \left|\big((\Phi - I) \circ \overline{\Pi^{x_0}_{P//C}}(w_{0+})\big)(t)\right|\\
&\leq \|\tilde{w}_0\|_{[0,t)} + \beta_0(\|w_1\|_{(-\infty,t/2]}, t-t/2) + \sigma(\|w_{1+}\|_{[t/2,t]}),\ \text{[by (4.3)]}\\
&\leq \|\tilde{w}_0\|_{[0,t)} + \beta_0(s_\infty,t/2) + \sigma\left(\beta(s_\infty,t/4) + \gamma(\|w_0\|_{[t/4,t)})\right),\ \text{[by (4.11)]}\\
&\leq \|\tilde{w}_0\|_{[0,t)} + \beta_0(s_\infty,t/2) + \sigma \circ (I+\rho^{-1}) \circ \beta(s_\infty,t/4)\\
&\quad + \sigma \circ (I+\rho) \circ \gamma(\|w_0\|_{[t/4,t)}),\ \text{[by (2.19)]}\\
&\leq \|\tilde{w}_0\|_{[0,t)} + \beta_1(s_\infty,t) + (I+\varepsilon)^{-1}(\|w_{0+}\|_{[t/4,t)}),
\end{aligned} \tag{4.16}$$

where s_∞ is defined by (4.15) and $\beta_1 \in \mathcal{KL}$ is defined by:

$$\beta_1(r,s) \triangleq \beta_0(r,s/2) + \sigma \circ (I+\rho^{-1}) \circ \beta(r,s/4),\ \forall r \geq 0,\ \forall s \geq 0. \tag{4.17}$$

By applying Lemma 2.25 to (4.16) with $\mu = \frac{1}{4}$ and $\lambda = (I+\varepsilon)^{-1}$ and $\delta = I + \varepsilon^{-1}$, it follows that a function β_2 of class $\mathcal{KL}$ exists such that, for all $t > 0$,

$$\begin{aligned} |w_{0+}(t)| &\le \beta_2(s_\infty, t) + (I-\lambda)^{-1} \circ \delta(\|\tilde{w}_{0+}\|_{[0,t)}) \\ &\le \beta_2(s_\infty, t) + (I+\varepsilon^{-1})^2(\|\tilde{w}_0\|_{[0,t)}), \text{ [by (2.20)]} \end{aligned} \tag{4.18}$$

Define functions $\beta_3 \in \mathcal{KL}$, $\hat{\beta} \in \mathcal{KL}$ and $\alpha_4 \in \mathcal{K}$ (without loss of generality, we could regard α_4 as a function of class $\mathcal{K}_\infty$) as follows, for all $r \ge 0$ and all $s \ge 0$,

$$\begin{aligned} \beta_3(r,s) &\triangleq \beta_0(r, s/2) + (\sigma + I) \circ (I + \rho^{-1}) \circ \beta(r, s/4); \\ \beta_4(r,s) &\triangleq \beta_3(r,s) + (\sigma + I) \circ (I+\rho) \circ \gamma \circ (I+\varepsilon) \circ \beta_2(r, s/4); \\ \alpha_4(r) &\triangleq \beta_3((\sigma_0 + I)(r), 0) + (\sigma+I) \circ (I+\rho) \circ \gamma \circ (I+\varepsilon^{-1}) \circ (I+\varepsilon) \circ \Delta(r). \end{aligned}$$

Hence, by using the equation (4.9), for any $t > 0$, we have

$$\begin{aligned} |\big(\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\tilde{w}_{0+})\big)(t)| &\le \|(\Phi - I) \circ \overline{\Pi^{x_0}_{P//C}}(w_{0+})\|_{[0,t]} + \|\overline{\Pi^{x_0}_{P//C}}(w_{0+})\|_{[0,t]} \\ &\le \beta_0(\|w_{1-}\|, 0) + (\sigma + I)(\|w_{1+}\|_{[0,t]}), \text{ [by (4.3)]} \\ &\le \beta_0(\|w_{1-}\|, 0) + (\sigma + I) \\ &\quad \circ \Big(\beta(\|(w_{1-}, w_{2-})\|, 0) + \gamma(\|w_0\|_{[0,t]})\Big), \text{ [by (4.11)]} \\ &\le \beta_3((\sigma_0 + I)(\|(\tilde{w}_{1-}, w_{2-})\|), 0) + (\sigma+I) \circ (I+\rho) \circ \gamma \\ &\quad \circ (I+\varepsilon^{-1})\,(\|\tilde{w}_{0+}\| + \Delta(\|(\tilde{w}_{1-}, w_{2-})\|)), \text{ [by (4.8) and (4.13)]} \\ &\le \alpha_4(\|(\tilde{w}_{1-}, w_{2-})\|) + \tilde{\gamma}(\|\tilde{w}_{0+}\|_{[0,t)}) \end{aligned} \tag{4.19}$$

with function $\tilde{\gamma} \in \mathcal{K}_\infty$ defined by (4.6) (note that $(I+\varepsilon^{-1})^2(\cdot) \le (I+\varepsilon^{-1})^3(\cdot)$). Moreover,

$$\begin{aligned} |\big(\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\tilde{w}_{0+})\big)(t)| &\le |\big((\Phi - I) \circ \overline{\Pi^{x_0}_{P//C}}(w_{0+})\big)(t)| + |\big(\overline{\Pi^{x_0}_{P//C}}(w_{0+})\big)(t)| \\ &\le \beta_0(\|w_1\|_{(-\infty, \frac{t}{2}]}, t - t/2) + (\sigma + I)(\|w_{1+}\|_{[\frac{t}{2}, t]}), \text{ [by (4.3)]} \\ &\le \beta_0(s_\infty, t/2) + (\sigma + I)\Big(\beta(s_\infty, t/4) + \gamma(\|w_0\|_{[\frac{t}{4}, t)})\Big), \text{ [by (4.11)]} \\ &\le \beta_3(s_\infty, t) + (\sigma+I) \circ (I+\rho) \circ \gamma(\|w_0\|_{[\frac{t}{4}, t)}) \text{ [by (2.19)]} \\ &\le \beta_3(s_\infty, t) + (\sigma+I) \circ (I+\rho) \circ \gamma \\ &\quad \circ \Big(\beta_2(s_\infty, t/4) + (I+\varepsilon^{-1})^2(\|\tilde{w}_{0+}\|_{[0,t)})\Big), \text{ [by (4.18)]} \\ &\le \hat{\beta}(s_\infty, t) + \tilde{\gamma}(\|\tilde{w}_{0+}\|_{[0,t)}) \end{aligned} \tag{4.20}$$

with function $\tilde{\gamma} \in \mathcal{K}_\infty$ defined by (4.6). Since $s_\infty = \alpha_2(\|(\tilde{w}_{1-}, w_{2-})\|) + \alpha_3(\|\tilde{w}_{0+}\|)$ (see (4.15)), from (4.19) and (4.20), we have for any $t \ge 0$,

$$\begin{aligned} |\big(\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\tilde{w}_{0+})\big)(t)| \le \tilde{\gamma}(\|\tilde{w}_{0+}\|) + \min\Big\{&\alpha_4(\|(\tilde{w}_{1-}, w_{2-})\|), \\ &\hat{\beta}\big(\alpha_2(\|(\tilde{w}_{1-}, w_{2-})\|) + \alpha_3(\|\tilde{w}_{0+}\|), t\big)\Big\}. \end{aligned} \tag{4.21}$$

Given any function α of $\mathcal{K}_\infty$, there are only two cases $\|(\tilde{w}_{1-}, w_{2-})\| \leq \alpha_4^{-1} \circ \alpha(\|\tilde{w}_{0+}\|)$ or $\|\tilde{w}_{0+}\| \leq \alpha^{-1} \circ \alpha_4(\|(\tilde{w}_{1-}, w_{2-})\|)$, thus from (4.21) and by considering the fact that for any fixed $t > 0$ the function $\hat{\beta}(\cdot, t) \in \mathcal{K}$, we have for any $t \geq 0$,

$$\begin{aligned}|\big(\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\tilde{w}_{0+})\big)(t)| &\leq \tilde{\gamma}(\|\tilde{w}_{0+}\|) + \alpha_4 \circ \alpha_4^{-1} \circ \alpha(\|\tilde{w}_{0+}\|) \\ &\quad + \hat{\beta}\big(\alpha_2(\|(\tilde{w}_{1-}, w_{2-})\|) + \alpha_3 \circ \alpha^{-1} \circ \alpha_4(\|(\tilde{w}_{1-}, w_{2-})\|), t\big).\end{aligned}$$

Since $[\tilde{P}, C]$ is causal, we have, for any $t > 0$,

$$|(\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}\tilde{w}_{0+})(t)| \leq \tilde{\beta}\left(\|(\tilde{w}_{1-}, w_{2-})\|, t\right) + (\alpha + \tilde{\gamma})(\|\tilde{w}_{0+}\|_{[0,t)}), \tag{4.22}$$

where the function $\tilde{\gamma} \in \mathcal{K}_\infty$ is defined by (4.6) and $\tilde{\beta} \in \mathcal{KL}$ is defined as follows

$$\tilde{\beta}(r, t) = \hat{\beta}(\alpha_2(r) + \alpha_3 \circ \alpha^{-1} \circ \alpha_4(r), t), \quad \forall r \geq 0, \forall t \geq 0. \tag{4.23}$$

Since $\tilde{x}_0$ and $\tilde{w}_{0+}$ are arbitrarily chosen from $\mathfrak{S}_{\tilde{P}} \times \mathfrak{S}_C$ and $\mathcal{W}^+$, respectively, we obtain that $[\tilde{P}, C]$ is input to output stable. Moreover, by Theorem 3.36, for any given function $\alpha \in \mathcal{K}_\infty$, from (4.22), we have (4.5) holds with $\tilde{\beta} \in \mathcal{KL}$ defined by (4.23). □

4.2.2 Proof of Theorem 4.8 with Condition II (only uniqueness)

The proof of this part of Theorem 4.8 will make use of the Schauder fixed-point theorem (see Lemma 2.11 on page 19).

Proof. For any $\tilde{w}_{0+} \in \mathcal{W}^+$ and any $\tilde{x}_0 \in \mathcal{S}_{\tilde{P}} \times \mathcal{S}_C$, choose any bounded $(\tilde{w}_{1-}, w_{2-}) \in \tilde{x}_0$ and let $\tilde{w}_{0-} = \tilde{w}_{1-} + w_{2-}$. Under conditions in Theorem 4.8, there exists a $w_{1-} \in \mathcal{W}^- \cap \mathfrak{B}^-_P$ for $\tilde{w}_{1-}$ such that $\|w_{1-}\| \leq \sigma_0(\|\tilde{w}_{1-}\|)$ (see (4.2)), and thus

$$\|(w_{1-}, w_{2-})\| \leq (\sigma_0 + I)(\|(\tilde{w}_{1-}, w_{2-})\|). \tag{4.24}$$

In addition, there exists a causal surjective operator $\Phi : \text{dom}(\Phi) \subseteq \mathcal{G}_P^{w_{1-}} \to \mathcal{G}_{\tilde{P}}^{\tilde{w}_{1-}}$ such that $R_J(\Phi - I)$ is compact with $J \triangleq [0, \tau)$ for any $0 < \tau < \infty$. We choose $x_0 := ([w_{1-}], [w_{2-}]) \in \mathcal{S}_P \times \mathcal{S}_C$ and let $w_{0-} = w_{1-} + w_{2-}$. Consider the equation

$$\begin{aligned}R_J\tilde{w}_{0+} &= R_J\left(I + (\Phi - I) \circ \overline{\Pi^{x_0}_{P//C}}\right)(z_{0+}) \\ &= R_J(I - \overline{\Pi^{x_0}_{P//C}})(z_{0+}) + R_J\Phi \circ \overline{\Pi^{x_0}_{P//C}}(z_{0+})\end{aligned} \tag{4.25}$$

and define a set M as follows,

$$M = \left\{\bar{z}_{0+} \in \mathcal{W}(J) \middle| \, \|\bar{z}_{0+}\|_J \leq (I + \varepsilon^{-1})\big(\|\tilde{w}_{0+}\| + \Delta(\|(\tilde{w}_{1-}, w_{2-})\|)\big)\right\} \tag{4.26}$$

with $\Delta \in \mathcal{K}$ defined by

$$\Delta(r) \triangleq \beta_0((\sigma_0 + I)(r), 0) + \sigma \circ (I + \rho^{-1}) \circ \beta((\sigma_0 + I)(r), 0), \ \forall r \geq 0, \tag{4.27}$$

and consider the operator

$$Q : M \to \mathcal{W}(J), \quad \bar{z}_{0+} \mapsto R_J \tilde{w}_{0+} + R_J(I - \Phi) \circ \overline{\Pi^{x_0}_{P//C}}(E_J \bar{z}_{0+}). \tag{4.28}$$

Theorem 3.36 tells us that (4.1) is equivalent to the following expression, for any $z_{0+} \in \mathcal{W}^+$:

$$|\overline{\Pi^{x_0}_{P//C}}(z_{0+})(t)| \leq \beta(\|(w_{1-}, w_{2-})\|, t) + \gamma(\|z_{0+}\|_{[0,t)}), \ \forall t > 0. \tag{4.29}$$

From (4.28), for any $\bar{z}_{0+} \in M$, define $z_{0+} \triangleq E_J \bar{z}_{0+}$, we have

$$\begin{aligned}
&\|Q(\bar{z}_{0+})\|_J \leq \|R_J \tilde{w}_{0+}\|_J + \|R_J(I - \Phi) \circ \overline{\Pi^{x_0}_{P//C}}(E_J \bar{z}_{0+})\|_J \\
&\leq \|\tilde{w}_{0+}\| + \beta_0(\|w_{1-}\|, 0) + \sigma(\|\overline{\Pi^{x_0}_{P//C}} z_{0+}\|), \ \text{[by (4.3)]} \\
&\leq \|\tilde{w}_{0+}\| + \beta_0(\|w_{1-}\|, 0) + \sigma \circ \big(\beta(\|(w_{1-}, w_{2-})\|, 0) + \gamma(\|z_{0+}\|)\big), \ \text{[by (4.29)]} \\
&\leq \|\tilde{w}_{0+}\| + \beta_0((\sigma_0 + I)(\|(\tilde{w}_{1-}, w_{2-})\|), 0) + \sigma \circ (I + \rho) \circ \gamma(\|z_{0+}\|) \\
&\quad + \sigma \circ (I + \rho^{-1}) \circ \beta\big((\sigma_0 + I)(\|(\tilde{w}_{1-}, w_{2-})\|), 0\big), \ \text{[by (4.8) and (2.19)]} \\
&\leq \|\tilde{w}_{0+}\| + \Delta(\|(\tilde{w}_{1-}, w_{2-})\|) + (I + \varepsilon)^{-1}(\|z_{0+}\|), \ \text{[by (4.14) and (4.4)]} \\
&\leq \big(I + (I + \varepsilon)^{-1} \circ (I + \varepsilon^{-1})\big)\big(\|\tilde{w}_{0+}\| + \Delta(\|(\tilde{w}_{1-}, w_{2-})\|)\big), \ \text{[by (4.26)]} \\
&= (I + \varepsilon^{-1})\left(\|\tilde{w}_{0+}\| + \Delta(\|(\tilde{w}_{1-}, w_{2-})\|)\right), \ \text{[by (2.21) and (2.20)]}.
\end{aligned}$$

Therefore $Q(M) \subseteq M \subseteq \mathcal{W}(J)$ with $\mathcal{W}(J)$ being a Banach space (note that $\mathcal{W}^+$ is truncation complete). Since $R_J(\Phi - I)$ is compact and $\overline{\Pi^{x_0}_{P//C}}$ is bounded, it follows from Lemma 2.13 that Q in (4.28) is also compact. From the relatively continuity of $\overline{\Pi^{x_0}_{P//C}}$, we know that Q is continuous. The set M is nonempty, closed, bounded and convex follows from Lemma 2.12. Thus by applying the Schauder fixed-point theorem (see Lemma 2.11) to the operator $Q : M \to \mathcal{W}(J)$, there exists some $\bar{w}_{0+} \in M \subseteq \mathcal{W}(J)$ such that $\bar{w}_{0+} = Q(\bar{w}_{0+}) \in \mathcal{W}(J)$. Hence equation (4.25) has a solution $z_{0+} = E_J \bar{w}_{0+}$.

Since $\tilde{w}^J_{1+} \triangleq \Phi \circ \overline{\Pi^{x_0}_{P//C}}(E_J \bar{w}_{0+}) \in \mathcal{G}^{\tilde{w}_{1-}}_{\tilde{P}}$ and $w^J_{2+} \triangleq (I - \overline{\Pi^{x_0}_{P//C}})(E_J \bar{w}_{0+}) \in \mathcal{G}^{w_{2-}}_{C}$, it follows from (4.25) that $R_J \tilde{w}^J_{1+} + R_J w^J_{2+} = R_J \tilde{w}_{0+}$ and that $\tilde{w}^J_{1+}$, $\tilde{w}^J_{2+}$ are bounded independent of J. This in turn shows that $[\tilde{P}, C]$ has the existence property up to time τ (note that $J \triangleq [0, \tau)$). Since this holds for all $0 < \tau < \infty$, and $[\tilde{P}, C]$ is causal and has the uniqueness property, it follows from Corollary 3.10 on page 51 that $[\tilde{P}, C]$ is well-posed. Since both $\tilde{x}_0$ and $\tilde{w}_{0+}$ are arbitrarily chosen from $\mathfrak{S}_{\tilde{P}} \times \mathfrak{S}_C$ and $\mathcal{W}^+$, respectively, we obtain that $\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\mathcal{W}^+) \subseteq \mathcal{W}^+$ for any $\tilde{x}_0 \in \mathfrak{S}_{\tilde{P}} \times \mathfrak{S}_C$. The rest of the proof follows as per the proof of Theorem 4.8 with extra condition I (see Section 4.2.1). □

It is useful to remark that if the operator Φ used to define the operator Q is *locally Lipschitz continuous* (see e.g., [Willems, 1971a, p. 89]), i.e.,

$$\sup_{R_{[0,\tau)}x \neq R_{[0,\tau)}y} \frac{\left\|R_{[0,\tau)}(\Phi x - \Phi y)\right\|_{[0,\tau)}}{\left\|R_{[0,\tau)}(x-y)\right\|_{[0,\tau)}} < \infty \quad \text{for all} \quad \tau \in (0,\infty),$$

or more general *locally continuous*, i.e., $R_{[0,\tau)}\Phi$ is continuous for any $\tau \in (0,\infty)$, then the relative continuity requirement for the map $\overline{\Pi^{x_0}_{P//C}}$ can be replaced by the requirement that $\overline{\Pi^{x_0}_{P//C}}$ is continuous.

4.3 Relationship between [Georgiou and Smith, 1997b, Theorem 1] and Theorem 4.8

In this section, we show to some extent that our robust stability theorem represents a generalisation of the input-output operator robust stability theorem of Georgiou and Smith, to include the case of initial conditions. In terms of notations in this thesis, [Georgiou and Smith, 1997b, Theorem 1] can be expressed as follows:

Theorem 4.10. *Consider the feedback configuration in Figure 3.1 on page 38. Assume that P, $\tilde{P}$, C, $[P,C]$, and $[\tilde{P},C]$ are well-posed and causal systems with $\mathfrak{B}^-_P = \{0\}$, $\mathfrak{B}^-_{\tilde{P}} = \{0\}$, and $\mathfrak{B}^-_C = \{0\}$. Let $[P,C]$ be stable, i.e., $\|\Pi^0_{P//C}\| < \infty$. If there exists a casual bijective map Φ_0 from $\mathcal{G}^0_P$ to $\mathcal{G}^0_{\tilde{P}}$ with $\Phi(0) = 0$ such that*

$$\|(\Phi_0 - I)|_{\mathcal{G}^0_P}\| < \|\Pi^0_{P//C}\|^{-1}, \tag{4.30}$$

then $[\tilde{P},C]$ is stable and $\|\Pi^0_{\tilde{P}//C}\| \leq \|\Pi^0_{P//C}\| \frac{1+\|(\Phi_0-I)|_{\mathcal{G}^0_P}\|}{1-\|\Pi^0_{P//C}\|\cdot\|(\Phi_0-I)|_{\mathcal{G}^0_P}\|}$.

In Georgiou and Smith [1997b], the plant and controller are assumed to be casual mappings from signal spaces to signal spaces which are only defined on positive time axis. The properties of mapping zero input to zero output for the plant and controller implicity require that they have zero initial conditions. Thus we assume that P, $\tilde{P}$, C are well-posed and causal systems with $\mathfrak{B}^-_P = \{0\}$, $\mathfrak{B}^-_{\tilde{P}} = \{0\}$, and $\mathfrak{B}^-_C = \{0\}$ in terms of notations of this thesis for above theorem. That the nominal and perturbed closed-loop systems are casual and well-posed are also standing assumptions in Georgiou and Smith [1997b]. Also, notice that the condition (4.30) is equivalent to [Georgiou and Smith, 1997b, Theorem 1, Condition (2)].

Lemma 4.11. *Consider the following LTI system*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \quad x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ for all $t \geq 0$ and the matrices A, B, C, D are of appropriate dimensions. Suppose that (A, B) is stabilisable and (A, C) is detectable. Then the following three statements are equivalent:

I. the matrix A is stable;

II. the system with zero initial conditions is stable with L^∞-linear gain;[5]

III. the system with initial conditions is input-to-output stable with L^∞-linear gain.[6]

Moreover, the linear gain in III can be chosen as the same one in II.

Proof. For I $\Leftrightarrow$ II, see [Vidyasagar, 1993, Section 6.3, Theorem 4]. For I $\Rightarrow$ III, see Example 2.29. For III $\Rightarrow$ II, we get II by setting initial conditions to be zero in III. Thus we have I $\Leftrightarrow$ II $\Leftrightarrow$ III. That the linear gain in III can be chosen as the same one in II follows from the linearity of the system. This completes the proof. □

The relationship between Theorem 4.8 and [Georgiou and Smith, 1997b, Theorem 1] is now given as follows:

Theorem 4.12. *Under the conditions that P, $\tilde{P}$, C, $[P, C]$, and $[\tilde{P}, C]$ are LTI systems, and that P and $\tilde{P}$ are controllable and observable, and that $[P, C]$ and $[\tilde{P}, C]$ are stabilisable and detectable. The first part of Theorem 4.8 (i.e., with extra condition I) is equivalent to Theorem 4.10 (i.e., [Georgiou and Smith, 1997b, Theorem 1]).*

Remark 4.13. If the premises of the first part of Theorem 4.8 and Theorem 4.10 are A_1 and A_2, and the conclusions of the first part of Theorem 4.8 and Theorem 4.10 are B_1 and B_2, respectively. Then equivalence means that $(A_1 \Rightarrow B_1) \Leftrightarrow (A_2 \Rightarrow B_2)$.

Proof. Under the conditions in Theorem 4.12. From Lemma 4.11, we know that the LTI nominal closed-loop system $[P, C]$ with zero initial conditions is stable with L^∞-linear gain if and only if $[P, C]$ with initial conditions is input-to-output stable with the same L^∞-linear gain, i.e., gain function γ in (4.1) in Theorem 4.8 is a linear function such that $\gamma(s) = \|\Pi^0_{P//C}\| \cdot s$ for $s \geq 0$. From Section 4.5.2 (especially (4.55) in Proposition 4.21), the gap function σ in (4.3) in Theorem 4.8 is a linear function such that $\sigma(s) = \|(\Phi - I)|_{\mathcal{G}^0_P}\| \cdot s$ for $s \geq 0$. From Remark 4.9, we know that condition (4.4) in Theorem 4.8 is equivalent to the condition (4.30) in Theorem 4.10. By using the notation in Remark 4.13, this implies

$$A_1 \Leftrightarrow A_2 \tag{4.31}$$

[5] i.e., $\sup\left\{\frac{\|y\|_{L^\infty([0,t],\mathbb{R})}}{\|u\|_{L^\infty([0,t],\mathbb{R})}} : t > 0, \|u\|_{L^\infty([0,t],\mathbb{R})} \neq 0, x(0) = 0\right\} < \infty$.

[6] i.e., $|y(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_{L^\infty([0,t],\mathbb{R})})$ for all $t \geq 0$ with $\beta \in \mathcal{KL}$ and a linear function $\gamma \in \mathcal{K}_\infty$.

By using Lemma 4.11 for the LTI perturbed closed-loop system $[\tilde{P}, C]$. we know that $[\tilde{P}, C]$ with zero initial conditions is stable with L^∞-linear gain if and only if $[\tilde{P}, C]$ with initial conditions is input-to-output stable with the same L^∞-linear gain. By using the notation in Remark 4.13 again, this implies

$$B_1 \Leftrightarrow B_2 \tag{4.32}$$

From (4.31) and (4.32), we get $(A_1 \Rightarrow B_1) \Leftrightarrow (A_2 \Rightarrow B_2)$. By Remark 4.13, we know that the first part of Theorem 4.8 (i.e., with extra condition I) is equivalent to Theorem 4.10 (i.e., [Georgiou and Smith, 1997b, Theorem 1]). □

4.4 Finite-Time Reachable Systems: Theorem 4.18

We first introduce the notion of a finite-time reachable system:

Definition 4.14. Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$. Consider the system Q represented by the set $\mathfrak{B}_Q$ (see Definition 3.2) and the initial state space $\mathfrak{S}_Q$ of Q at initial time 0 defined by Definition 3.11. Let $\delta \in (0, \infty)$, then the system Q is called *finite-time δ-reachable* if for any $x_0 \in \mathfrak{S}_Q$ there exists a $w_- \in x_0$ such that $w_-(t) = 0$ for all $t \in (-\infty, -\delta)$. The system Q is called *finite-time reachable* if there exist a $\delta \in (0, \infty)$ such that Q is finite-time δ-reachable.

We will now let $t_0 > 0$ be the given initial time and $\mathfrak{S}_Q^{t_0}$ (see Remark 3.12) be the initial state space of Q at time t_0. Suppose that the system Q is finite-time t_0-reachable (i.e., for any $x_0 \in \mathfrak{S}_Q^{t_0}$ there exists a $w_- \in x_0$ such that $w_-(t) = 0$ for all $t < 0$). Let us define a map ι as follows:

$$\iota : x_0 \mapsto \left\{ w \in \mathcal{W}[0, t_0] \mid 0_{(-\infty,0)} \wedge w \in x_0 \right\}, \quad \forall x_0 \in \mathfrak{S}_Q^{t_0}. \tag{4.33}$$

Since Q is finite-time t_0-reachable, we know that $\iota(x_0) \neq \emptyset$ for any $x_0 \in \mathfrak{S}_Q^{t_0}$.

4.4.1 Preliminary Results

Denote by $\iota(\mathfrak{S}_Q^{t_0})$ the image of above map ι. The following Theorem 4.15 shows that $\iota : \mathfrak{S}_Q^{t_0} \to \iota(\mathfrak{S}_Q^{t_0})$ is a bijective map.

Theorem 4.15. *The map $\iota : \mathfrak{S}_Q^{t_0} \to \iota(\mathfrak{S}_Q^{t_0})$ is a bijection.*

Proof. We only need to prove ι is an injection. To this end, we have to show $x_1 = x_2$ for any $x_1, x_2 \in \mathfrak{S}_Q^{t_0}$ satisfying $\iota(x_1) = \iota(x_2)$. Choose any $w \in \iota(x_1) = \iota(x_2)$, from (4.33) we know $0_{(-\infty,0)} \wedge w$ belongs to both x_1 and x_2. Thus from the definition of initial state space $\mathfrak{S}_Q^{t_0}$ we get $x_1 = x_2$. This completes the proof. □

Recalling the definition of graph of a system for particular past trajectory 0 (see (3.9)), i.e.,

$$\mathcal{G}_Q^0 \triangleq \left\{ w_+ \in \mathcal{W}^+ \mid 0_{(-\infty,0)} \wedge w_+ \in \mathfrak{B}_Q \right\}.$$

The following Theorem 4.16 shows that the image of the map ι produces a partition for the restriction of graph $\mathcal{G}_Q^0$ to $[0, t_0]$.

Theorem 4.16. *The image $\iota(\mathfrak{S}_Q^{t_0})$ of the map ι is a partition of $\mathcal{G}_Q^0\big|_{[0,t_0]}$.*

Proof. Since Q is finite-time t_0-reachable, we have $\iota(x_0) \neq \emptyset$ for any $x_0 \in \mathfrak{S}_Q^{t_0}$ and thus $\emptyset \notin \iota(\mathfrak{S}_Q^{t_0})$. For any $w_{[0,t_0]} \in \mathcal{G}_Q^0\big|_{[0,t_0]}$, there must exist a $x_0 \in \mathfrak{S}_Q^{t_0}$ such that $0_{(-\infty,0)} \wedge w_{[0,t_0]} \in x_0$, and therefore $w_{[0,t_0]} \in \iota(x_0)$. This together with $\iota(\mathfrak{S}_Q^{t_0}) \subseteq \mathcal{G}_Q^0\big|_{[0,t_0]}$ shows that $\bigcup \iota(\mathfrak{S}_Q^{t_0}) = \mathcal{G}_Q^0\big|_{[0,t_0]}$. For any $x_1, x_2 \in \mathfrak{S}_Q^{t_0}$ with $\iota(x_1) \neq \iota(x_2)$ (i.e., $x_1 \neq x_2$ by Theorem 4.15), we have $\iota(x_1) \cap \iota(x_2) = \emptyset$ since any common element belongs to both $\iota(x_1)$ and $\iota(x_2)$ will imply $x_1 = x_2$. According to Definition 2.16, above claims show that $\iota(\mathfrak{S}_Q^{t_0})$ is a partition of $\mathcal{G}_Q^0\big|_{[0,t_0]}$. □

By definition of the map ι (see (4.33)) and Theorem 4.15, we know that, given initial time $t_0 > 0$, for finite-time t_0-reachable system, we can actually only use trajectories with zero past up to time 0 to define all our state at initial time $t_0 > 0$. In this case, we can slightly change the definition of the size of any state $x_{t_0} \in \mathfrak{S}_Q^{t_0}$ (i.e., $\chi(x_{t_0})$ see (3.15)) by another real-valued function $\tilde{\chi}$:

$$\tilde{\chi} : \mathfrak{S}_Q^{t_0} \to \mathbb{R}^+, \quad x_{t_0} \mapsto \tilde{\chi}(x_{t_0}) \triangleq \inf_{w \in x_{t_0}, w(t)=0 (\forall t<0)} \left\{ \|w\|_{(-\infty,\ t_0]} \right\}. \tag{4.34}$$

It is easy to see that $\tilde{\chi}(x_{t_0}) = \inf_{w \in \iota(x_{t_0})} \left\{ \|w\|_{[0,t_0]} \right\} \geq \chi(x_{t_0})$ for any $x_{t_0} \in \mathfrak{S}_Q^{t_0}$.

According to above discussions for finite-time reachable systems, by using a new size function (4.34) for initial states and the same procedure of proof for the main Theorem 4.8, we can obtain the following friendly applicable robust stability Theorem 4.18.

4.4.2 Theorem 4.18

The following assumptions on the normed vector space $\mathcal{W}[t_0, \infty)$ are only required in the proof of Theorem 4.18 with condition II:

Assumption 4.17. *(1) For any $x \in \mathcal{W}_e[t_0, \infty)$, if $\|x\| < \infty$, then $x \in \mathcal{W}[t_0, \infty)$; (2) The normed vector space $\mathcal{W}[t_0, \infty)$ (not necessarily complete) is truncation complete, i.e., $\mathcal{W}[t_0, \tau)$ is complete for any $t_0 < \tau < \infty$; (3) For any time interval $J \triangleq [t_0, \tau)$ with $t_0 < \tau < \infty$, there exists a continuous map $E_J : \mathcal{W}(J) \to \mathcal{W}^+$ such that $R_J x = R_J(E_J x)$ for any $x \in \mathcal{W}(J)$.*

Theorem 4.18. *Give initial time $t_0 > 0$, and assume that P, $\tilde{P}$, and C are well-posed, finite-time t_0-reachable and causal systems, and that $[P, C]$ is time-invariant, well-posed and causal, and that $[\tilde{P}, C]$ is causal. Let $[P, C]$ be input to output stable, i.e., there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall x_{t_0} = (x_{1t_0}, x_{2t_0}) \in \mathfrak{S}_P^{t_0} \times \mathfrak{S}_C^{t_0}$, $\forall w_{0+} \in \mathcal{W}[t_0, \infty)$, $\forall t > t_0$,*

$$|(\overline{\Pi_{P//C}^{x_{t_0}}} w_{0+})(t)| \leq \beta\left(\tilde{\chi}(x_{t_0}), t - t_0\right) + \gamma(\|w_{0+}\|_{[t_0,t)}). \tag{4.35}$$

If there exists a causal surjective mapping $\Phi : \mathrm{dom}(\Phi) \subseteq \mathcal{G}_P^0 \to \mathcal{G}_{\tilde{P}}^0$ and functions $\beta_0 \in \mathcal{KL}$, $\sigma \in \mathcal{K}_\infty$, $\sigma_0 \in \mathcal{K}_\infty$, such that, $\forall w \in \mathrm{dom}(\Phi) \subseteq \mathcal{G}_P^0$,

$$\|w\|_{[0,t_0]} \leq \sigma_0(\|\Phi w\|_{[0,t_0]}), \tag{4.36}$$

$$|((\Phi - I)w)(t)| \leq \beta_0(\|w\|_{[0,h]}, t - h) + \sigma(\|w\|_{[h,t)}), \quad \forall t > h \geq 0. \tag{4.37}$$

In addition, if there exist two functions ρ, ε of class $\mathcal{K}_\infty$ such that, $\forall s \geq 0$,

$$\sigma \circ (I + \rho) \circ \gamma(s) \leq (I + \varepsilon)^{-1}(s). \tag{4.38}$$

And either of the following conditions is satisfied:

I. $[\tilde{P}, C]$ is well-posed and $\overline{\Pi_{\tilde{P}//C}^{\tilde{x}_{t_0}}}(\mathcal{W}[t_0, \infty)) \subseteq \mathcal{W}[t_0, \infty)$ for any $\tilde{x}_{t_0} \in \mathfrak{S}_{\tilde{P}}^{t_0} \times \mathfrak{S}_C^{t_0}$;

II. Assumption 4.17 holds for $\mathcal{W}[t_0, \infty)$, and $[\tilde{P}, C]$ has the uniqueness property, and $\overline{\Pi_{P//C}^{x_{t_0}}}$ is relatively continuous for any $x_{t_0} \in \mathfrak{S}_P^{t_0} \times \mathfrak{S}_C^{t_0}$, and $R_{[t_0,\tau)}(\Phi - I)$ is compact for any $t_0 < \tau < \infty$.

Then the closed-loop system $[\tilde{P}, C]$ is also input to output stable. More specifically, for any function α of class $\mathcal{K}_\infty$, there exists a function $\tilde{\beta} \in \mathcal{KL}$ such that, $\forall \tilde{x}_{t_0} \in \mathfrak{S}_{\tilde{P}}^{t_0} \times \mathfrak{S}_C^{t_0}$, $\forall \tilde{w}_{0+} \in \mathcal{W}[t_0, \infty)$, $\forall t > t_0$,

$$|(\overline{\Pi_{\tilde{P}//C}^{\tilde{x}_{t_0}}} \tilde{w}_{0+})(t)| \leq \tilde{\beta}\left(\tilde{\chi}(\tilde{x}_{t_0}), t\right) + (\alpha + \tilde{\gamma})(\|\tilde{w}_{0+}\|_{[t_0,t)}), \tag{4.39}$$

where $\tilde{\gamma} \in \mathcal{K}_\infty$ is defined by

$$\tilde{\gamma}(r) \triangleq (\sigma + I) \circ (I + \rho) \circ \gamma \circ (I + \varepsilon^{-1})^3(r), \ \forall r \geq 0. \tag{4.40}$$

Proof. It follows directly from Theorems 4.8, 4.15 and 4.16. □

The assertion of Theorem 4.18 remains valid if the product state space $\mathfrak{S}_P^{t_0} \times \mathfrak{S}_C^{t_0}$ is replaced with $\mathfrak{S}_{P//C}^{t_0}$ by using Theorem 3.36.

4.5 Application to Linear Time-Invariant Systems

Let $\mathcal{U} \triangleq L^\infty(\mathbb{R}, \mathbb{R}^m)$, $\mathcal{Y} \triangleq L^\infty(\mathbb{R}, \mathbb{R}^p)$, and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$. Suppose that A, B, C, D are real matrices of dimensions $n \times n$, $n \times m$, $n \times p$, $m \times p$, respectively, with (A, B) controllable and (A, C) observable. The nominal plant P is defined by the set $\mathfrak{B}_P \triangleq \mathfrak{B}_{A,B,C,D}$ with

$$\mathfrak{B}_{A,B,C,D} \triangleq \left\{ (u, y) \in \mathcal{W}_e \;\middle|\; \begin{array}{c} \dot{x} = Ax + Bu, \;\; y = Cx + Du \\ \text{satisfies for some } x \in L_e^\infty(\mathbb{R}, \mathbb{R}^n) \end{array} \right\}.$$

Similarly, suppose that $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are real matrices of dimensions $\tilde{n} \times \tilde{n}$, $\tilde{n} \times m$, $\tilde{n} \times p$, $m \times p$, respectively, with $(\tilde{A}, \tilde{B})$ controllable and $(\tilde{A}, \tilde{C})$ observable. We define the perturbed plant $\tilde{P}$ by the set $\mathfrak{B}_{\tilde{P}} \triangleq \mathfrak{B}_{\tilde{A},\tilde{B},\tilde{C},\tilde{D}}$.

4.5.1 Finite-Time t_0-Reachable Situation

Consider the nominal and perturbed plants P and $\tilde{P}$ defined in Section 4.5. Let $t_0 > 0$ be the initial time. Since (A, B) is controllable, any initial state value $x(t_0)$ at time t_0 can be generated from state value $x(0) \equiv 0$ at time 0 by some input u on time domain $[0, t_0]$. Therefore, the nominal plant P is finite-time t_0-reachable and so is also the perturbed plant $\tilde{P}$.

For the nominal plant P, from the controllability of (A, B) and the observability of (A, C), we can choose real matrices F and H such that both $A + BF$ and $A + HC$ are stable (all eigenvalues in $Re\ s < 0$). Now we define two operators as follows:

$$\begin{aligned} \begin{pmatrix} \mathbb{M}_+ \\ \mathbb{N}_+ \end{pmatrix} : \mathcal{U}^+ \to \mathcal{W}^+, \quad & v \mapsto \left(t \mapsto \int_0^t \begin{pmatrix} M \\ N \end{pmatrix} (t-\tau) v(\tau) d\tau, \; t \geq 0 \right), \\ \mathbb{L}_+ : \mathcal{W}^+ \to \mathcal{U}^+, \quad & w \mapsto \left(t \mapsto \int_0^t L(t-\tau) w(\tau) d\tau, \; t \geq 0 \right), \end{aligned} \tag{4.41}$$

where the following δ denotes the unit delta distribution [Vidyasagar, 1993, Section 6.4.1] and for any $t \geq 0$,

$$\begin{aligned} \begin{pmatrix} M \\ N \end{pmatrix}(t) &\triangleq \begin{pmatrix} F\exp\{t(A+BF)\}B+\delta(t)I_{m\times m} \\ (C+DF)\exp\{t(A+BF)\}B+\delta(t)D \end{pmatrix}, \\ L(t) &\triangleq \begin{pmatrix} F & F \end{pmatrix} \exp\left\{ t \begin{pmatrix} A+HC & 0 \\ 0 & A+HC \end{pmatrix} \right\} \begin{pmatrix} H & 0 \\ 0 & -B-HD \end{pmatrix} + \delta(t) \begin{pmatrix} 0_{m\times m} & I_{m\times p} \end{pmatrix}. \end{aligned} \tag{4.42}$$

We have

$$\begin{aligned} \mathcal{G}_P^0 \triangleq \left\{ w_+ \in \mathcal{W}^+ \mid 0_{(-\infty,0)} \wedge w_+ \in \mathfrak{B}_P \right\} &= \left\{ \begin{pmatrix} \mathbb{M}_+ \\ \mathbb{N}_+ \end{pmatrix} v \,\middle|\, v \in \mathcal{U}^+ \right\}, \\ v = \mathbb{L}_+ \circ \begin{pmatrix} \mathbb{M}_+ \\ \mathbb{N}_+ \end{pmatrix} v, &\quad \forall v \in \mathcal{U}^+, \end{aligned} \tag{4.43}$$

where $\mathcal{G}_P^0$ is the graph of P for the particular past trajectory 0 (see (3.9)).

Similarly, for the perturbed plant $\tilde{P}$, we can choose real matrices $\tilde{F}$ and $\tilde{H}$ such that $\tilde{A}+\tilde{B}\tilde{F}$ and $\tilde{A}+\tilde{H}\tilde{C}$ are stable, and then define operators $\left(\begin{smallmatrix}\tilde{\mathbb{M}}_+\\ \tilde{\mathbb{N}}_+\end{smallmatrix}\right)$, $\tilde{\mathbb{L}}_+$ with $\left(\begin{smallmatrix}\tilde{M}\\ \tilde{N}\end{smallmatrix}\right)$, $\tilde{L}$ and the graph $\mathcal{G}^0_{\tilde{P}}$ like (4.41) and (4.43), respectively.

Proposition 4.19. *A map Φ_0 from $\mathcal{G}^0_P$ to $\mathcal{G}^0_{\tilde{P}}$ can be defined as follows*

$$\Phi_0 : \mathcal{G}^0_P \to \mathcal{G}^0_{\tilde{P}}, \quad \begin{pmatrix}\mathbb{M}_+\\ \mathbb{N}_+\end{pmatrix} v \mapsto \begin{pmatrix}\tilde{\mathbb{M}}_+\\ \tilde{\mathbb{N}}_+\end{pmatrix} v, \;\; \forall v \in \mathcal{U}^+. \tag{4.44}$$

Then Φ_0 is causal, surjective and time-invariant, and for all $w \in \mathcal{G}^0_P$, $t > h \geq 0$,

$$\|w\|_{[0,h]} \leq \left\| \left(\begin{smallmatrix}M\\ N\end{smallmatrix}\right) \right\|_A \cdot \|\tilde{L}\|_A \cdot \|\Phi_0 w\|_{[0,h]}\,, \tag{4.45}$$

$$|((\Phi_0 - I)w)(t)| \leq \beta_0(\|w\|_{[0,h]}\,, t-h) + \|(\Phi_0 - I)\| \cdot \|w\|_{[h,t)}\,, \tag{4.46}$$

where function $\beta_0 \in \mathcal{KL}$ and $\|\cdot\|_A$ is the norm for distribution [Vidyasagar, 1993, Section 6.4.1].

Proof. It is easy to see that Φ_0 is causal, surjective, and time-invariant. For any $w \in \mathcal{G}^0_P$ there exists a $v \in \mathcal{U}^+$ such that $w = \left(\begin{smallmatrix}\mathbb{M}_+\\ \mathbb{N}_+\end{smallmatrix}\right)v$. Since $v = \tilde{\mathbb{L}}_+ \circ \left(\begin{smallmatrix}\tilde{\mathbb{M}}_+\\ \tilde{\mathbb{N}}_+\end{smallmatrix}\right)v$ and $\Phi_0 w = \left(\begin{smallmatrix}\tilde{\mathbb{M}}_+\\ \tilde{\mathbb{N}}_+\end{smallmatrix}\right)v$, we get $w = \left(\begin{smallmatrix}\mathbb{M}_+\\ \mathbb{N}_+\end{smallmatrix}\right) \circ \tilde{\mathbb{L}}_+(\Phi_0 w)$ and thus this implies (4.45). Since $v = \mathbb{L}_+ \circ \left(\begin{smallmatrix}\mathbb{M}_+\\ \mathbb{N}_+\end{smallmatrix}\right)v$, we have

$$(\Phi_0 - I)w = \begin{pmatrix}\tilde{\mathbb{M}}_+ - \mathbb{M}_+\\ \tilde{\mathbb{N}}_+ - \mathbb{N}_+\end{pmatrix} \mathbb{L}_+ w = \begin{pmatrix}\tilde{\mathbb{M}}_+ - \mathbb{M}_+\\ \tilde{\mathbb{N}}_+ - \mathbb{N}_+\end{pmatrix} \mathbb{L}_+ w_h + \begin{pmatrix}\tilde{\mathbb{M}}_+ - \mathbb{M}_+\\ \tilde{\mathbb{N}}_+ - \mathbb{N}_+\end{pmatrix} \mathbb{L}_+ w^h$$

with $w_h(\tau) \triangleq \left\{\begin{smallmatrix} w(\tau), & \forall\tau\in[0,h),\\ 0, & \forall\tau\geq h\end{smallmatrix}\right.$ and $w^h(\tau) \triangleq \left\{\begin{smallmatrix} 0, & \forall\tau\in[0,h),\\ w(\tau), & \forall\tau\geq h\end{smallmatrix}\right.$. Thus we can find a function $\beta_0 \in \mathcal{KL}$ (for SISO system see Lemma 2.26) such that

$$\left|\left(\begin{pmatrix}\tilde{\mathbb{M}}_+ - \mathbb{M}_+\\ \tilde{\mathbb{N}}_+ - \mathbb{N}_+\end{pmatrix} \mathbb{L}_+ w_h\right)(t)\right| \leq \beta_0(\|w\|_{[0,h]}\,, t-h), \;\; \forall t > h \geq 0;$$

$$\left|\left(\begin{pmatrix}\tilde{\mathbb{M}}_+ - \mathbb{M}_+\\ \tilde{\mathbb{N}}_+ - \mathbb{N}_+\end{pmatrix} \mathbb{L}_+ w^h\right)(t)\right| \leq \|(\Phi_0 - I)\| \cdot \|w\|_{[h,t)}\,, \;\; \forall t > h \geq 0.$$

This implies (4.46) and completes the proof. □

4.5.2 General Situation

Consider the nominal and perturbed plants P and $\tilde{P}$ defined in Section 4.5. Let $t = 0$ be the initial time. In this section, we define operators

$$\begin{pmatrix}\mathbb{M}\\ \mathbb{N}\end{pmatrix} : \mathcal{U} \to \mathcal{W} \quad \text{and} \quad \mathbb{L} : \mathcal{W} \to \mathcal{W}$$

for the nominal plant P corresponding to operators $\left(\begin{smallmatrix}\mathbb{M}_+\\ \mathbb{N}_+\end{smallmatrix}\right)$ and $\mathbb{L}_+$ defined in Section 4.5.1 by replacing $\int_0^t$ with $\int_{-\infty}^t$ in (4.41). Note that operators $\left(\begin{smallmatrix}\tilde{\mathbb{M}}\\ \tilde{\mathbb{N}}\end{smallmatrix}\right)$ and $\tilde{\mathbb{L}}$ are similarly defined for the perturbed plant $\tilde{P}$.

Proposition 4.20. *For the perturbed plant $\tilde{P}$, define a functional π_1 as follows*

$$\pi_1 : \mathcal{U}^- \to \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}, \quad u \mapsto \begin{pmatrix} \tilde{\mathbb{M}}(u \wedge 0) \\ \tilde{\mathbb{N}}(u \wedge 0) \end{pmatrix} \Big|_{(-\infty,0]}. \tag{4.47}$$

Then, there exists a functional $\pi_2 : \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}} \to (\mathcal{U}^-)_0$ and a nonnegative number $\tilde{\rho} \geq 0$ such that for any $\tilde{w}_- \in \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}$,

$$\pi_1 \circ \pi_2(\tilde{w}_-) = \begin{pmatrix} \tilde{\mathbb{M}}(\pi_2(\tilde{w}_-) \wedge 0) \\ \tilde{\mathbb{N}}(\pi_2(\tilde{w}_-) \wedge 0) \end{pmatrix} \Big|_{(-\infty,0]}, \quad \|\pi_1 \circ \pi_2(\tilde{w}_-)\| \leq \tilde{\rho} \cdot \|\tilde{w}_-\|, \tag{4.48}$$

and for any $\tilde{w}_- \in \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}$, the graph $\mathcal{G}^{\tilde{w}_-}_{\tilde{P}}$ defined by (3.9) satisfies

$$\mathcal{G}^{\tilde{w}_-}_{\tilde{P}} = \left\{ \begin{pmatrix} \tilde{\mathbb{M}}(\pi_2(\tilde{w}_-) \wedge v) \\ \tilde{\mathbb{N}}(\pi_2(\tilde{w}_-) \wedge v) \end{pmatrix} \Big|_{[0,\infty)} \in \mathcal{W}^+ \;\Big|\; v \in \mathcal{U}^+ \right\} = \mathcal{G}^{\pi_1 \circ \pi_2(\tilde{w}_-)}_{\tilde{P}}, \tag{4.49}$$

where $(\mathcal{U}^-)_0 \triangleq \{u \in \mathcal{U}^- \mid \exists T_u \in [0,\infty),$ such that $u(t) \equiv 0, \forall t \leq -T_u\}$.

Proof. Since $(\tilde{A}, \tilde{C})$ is observable, we have that, for any $\tilde{w}_- \in \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}$, there exists a unique $\tilde{x}_0 \in \mathbb{R}^{\tilde{n}}$ such that the equations $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$ and $y = \tilde{C}\tilde{x} + \tilde{D}u$ hold with $(u(t), y(t)) = \tilde{w}_-(t)$ for $t \leq 0$ and $\tilde{x}(0) = \tilde{x}_0$. In addition, $|\tilde{x}_0| \leq r_1 \|\tilde{w}_-\|$ with $r_1 \geq 0$ independent of $\tilde{w}_-$. Since $(\tilde{A}, \tilde{B})$ is controllable, we obtain that $(\tilde{A} + \tilde{B}\tilde{F}, \tilde{B})$ is controllable, and thus, for this $\tilde{x}_0 \in \mathbb{R}^{\tilde{n}}$, there exists a $v_{\tilde{x}_0} \in (\mathcal{U}^-)_0$ such that

$$\tilde{x}_0 = \int_{-\infty}^{0} \exp\left\{(0 - \tau)(\tilde{A} + \tilde{B}\tilde{F})\right\} \tilde{B} v_{\tilde{x}_0}(\tau) d\tau. \tag{4.50}$$

Moreover, $\|v_{\tilde{x}_0}\| \leq r_2 |\tilde{x}_0|$ with $r_2 \geq 0$ independent of $\tilde{x}_0$. Thus a functional π_2 can be defined by

$$\pi_2 : \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}} \to (\mathcal{U}^-)_0, \quad \tilde{w}_- \mapsto v_{\tilde{x}_0}, \tag{4.51}$$

and we have $\|\pi_2(\tilde{w}_-)\| \leq r_2 r_1 \|\tilde{w}_-\|$. From similar techniques in [French et al., 2009, Section 4.4], we know that the graph $\mathcal{G}^{\tilde{w}_-}_{\tilde{P}}$ defined by (3.9) can be expressed as

$$\mathcal{G}^{\tilde{w}_-}_{\tilde{P}} = \left\{ \begin{pmatrix} \tilde{\mathbb{M}}_+ v + \tilde{F} \exp\{\cdot \tilde{A}_{\tilde{F}}\} \tilde{x}_0 \\ \tilde{\mathbb{N}}_+ v + \tilde{C}_{\tilde{F}} \exp\{\cdot \tilde{A}_{\tilde{F}}\} \tilde{x}_0 \end{pmatrix} \Big|_{[0,\infty)} \in \mathcal{W}^+ \;\Big|\; v \in \mathcal{U}^+ \right\}. \tag{4.52}$$

By using (4.50) and (4.51), we know that the right hand side of (4.52) equals to

$$\left\{ \begin{pmatrix} \tilde{\mathbb{M}}(\pi_2(\tilde{w}_-) \wedge v) \\ \tilde{\mathbb{N}}(\pi_2(\tilde{w}_-) \wedge v) \end{pmatrix} \Big|_{[0,\infty)} \in \mathcal{W}^+ \;\Big|\; v \in \mathcal{U}^+ \right\}. \tag{4.53}$$

From (4.47) and (4.51), we have (4.48) holds with $\tilde{\rho} \triangleq \|(\begin{smallmatrix} \tilde{M} \\ \tilde{N} \end{smallmatrix})\|_{A} \cdot r_2 \cdot r_1 \geq 0$, and thus $\mathcal{G}^{\pi_1 \circ \pi_2(\tilde{w}_-)}_{\tilde{P}}$ equals (4.53); this implies (4.49). □

Proposition 4.21. *For any* $\tilde{w}_- \in \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}$*, there exists a* $w_- \in \mathcal{W}^- \cap \mathfrak{B}^-_P$ *with*

$$\|w_-\| \le \tilde{\rho} \cdot \left\| \begin{pmatrix} M \\ N \end{pmatrix} \right\|_A \cdot \|\tilde{L}\|_A \cdot \|\tilde{w}_-\|, \tag{4.54}$$

and a causal surjective map $\Phi_{\tilde{w}_-} : \mathcal{G}_P^{w_-} \to \mathcal{G}_{\tilde{P}}^{\tilde{w}_-}$ *satisfying,* $\forall t > h \ge 0,\ \forall w_+ \in \mathcal{G}_P^{w_-}$*,*

$$\left|((\Phi_{\tilde{w}_-} - I)w_+)(t)\right| \le \beta_0(\|w_- \wedge w_+\|_{(-\infty,h]}, t-h) + \|(\Phi_0 - I)\| \cdot \|w_+\|_{[h,t)}, \tag{4.55}$$

where function $\beta_0 \in \mathcal{KL}$ *and* $\|\cdot\|_A$ *and* $\|(\Phi_0 - I)\|$ *are defined in Proposition 4.19 and* $\tilde{\rho} \ge 0$ *is the same as in Proposition 4.20.*

Proof. Let the functional π_1, π_2 be defined as in Proposition 4.20. For any $\tilde{w}_- \in \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}$, we have (4.48) and (4.49) hold. It is easy to see that $w_- \triangleq \begin{pmatrix} \mathbb{M}(\pi_2(\tilde{w}_-)\wedge 0) \\ \mathbb{N}(\pi_2(\tilde{w}_-)\wedge 0) \end{pmatrix}\Big|_{(-\infty,0]} \in \mathcal{W}^- \cap \mathfrak{B}^-_P$ and that the graph $\mathcal{G}_P^{w_-}$ of the nominal plant P is

$$\mathcal{G}_P^{w_-} = \left\{ \begin{pmatrix} \mathbb{M}(\pi_2(\tilde{w}_-)\wedge v) \\ \mathbb{N}(\pi_2(\tilde{w}_-)\wedge v) \end{pmatrix}\Big|_{[0,\infty)} \in \mathcal{W}^+ \ \Big|\ v \in \mathcal{U}^+ \right\}.$$

Thus, a natural causal surjective map $\Phi_{\tilde{w}_-} : \mathcal{G}_P^{w_-} \to \mathcal{G}_{\tilde{P}}^{\tilde{w}_-}$ can be defined as follows

$$\begin{pmatrix} \mathbb{M}(\pi_2(\tilde{w}_-)\wedge v) \\ \mathbb{N}(\pi_2(\tilde{w}_-)\wedge v) \end{pmatrix}\Big|_{[0,\infty)} \mapsto \begin{pmatrix} \tilde{\mathbb{M}}(\pi_2(\tilde{w}_-)\wedge v) \\ \tilde{\mathbb{N}}(\pi_2(\tilde{w}_-)\wedge v) \end{pmatrix}\Big|_{[0,\infty)}, \quad \forall v \in \mathcal{U}^+. \tag{4.56}$$

Since $\pi_2(\tilde{w}_-) \in (\mathcal{U}^-)_0$, there exists a $T_{\tilde{w}_-} \in [0,\infty)$ such that $\pi_2(\tilde{w}_-)(t) \equiv 0$ for all $t \le -T_{\tilde{w}_-}$. It follows from the time-invariance of Φ_0 in (4.44) and (4.45) that

$$\|w_-\| = \left\| \begin{pmatrix} \mathbb{M}(\pi_2(\tilde{w}_-)\wedge 0) \\ \mathbb{N}(\pi_2(\tilde{w}_-)\wedge 0) \end{pmatrix} \right\|_{(-T_{\tilde{w}_-},0]} \le \left\| \begin{pmatrix} M \\ N \end{pmatrix} \right\|_A \cdot \|\tilde{L}\|_A \cdot \left\| \begin{pmatrix} \tilde{\mathbb{M}}(\pi_2(\tilde{w}_-)\wedge 0) \\ \tilde{\mathbb{N}}(\pi_2(\tilde{w}_-)\wedge 0) \end{pmatrix} \right\|_{[-T_{\tilde{w}_-},0]},$$

and thus from (4.48), we have (4.54) holds.

For any $w_+ \in \mathcal{G}_P^{w_-}$, there exists a $v \in \mathcal{U}^+$ such that $w_+ = \begin{pmatrix} \mathbb{M}(\pi_2(\tilde{w}_-)\wedge v) \\ \mathbb{N}(\pi_2(\tilde{w}_-)\wedge v) \end{pmatrix}\Big|_{[0,\infty)}$. From (4.56) we get $((\Phi_{\tilde{w}_-} - I)w_+)(t) = \begin{pmatrix} \tilde{\mathbb{M}}-\mathbb{M} \\ \tilde{\mathbb{N}}-\mathbb{N} \end{pmatrix}(\pi_2(\tilde{w}_-) \wedge v)(t), \forall t \ge 0$, and thus from the time-invariance of Φ_0 in (4.44) and (4.46) we have for any $t \ge h > 0$ that

$$\begin{aligned} \left|((\Phi_{\tilde{w}_-} - I)w_+)(t)\right| \le \beta_0(&\left\| \begin{pmatrix} \mathbb{M} \\ \mathbb{N} \end{pmatrix} (\pi_2(\tilde{w}_-) \wedge v) \right\|_{[-T_{\tilde{w}_-},h]}, t-h) \\ &+ \|(\Phi_0 - I)\| \cdot \left\| \begin{pmatrix} \mathbb{M} \\ \mathbb{N} \end{pmatrix} (\pi_2(\tilde{w}_-) \wedge v) \right\|_{[h,t-h)}, \end{aligned}$$

where $\beta_0 \in \mathcal{KL}$ and $\|(\Phi_0 - I)\|$ are the same as in Proposition 4.19. Therefore, from $w_- \wedge w_+ = (\pi_2(\tilde{w}_-) \wedge v)$, we obtain that (4.55) holds. □

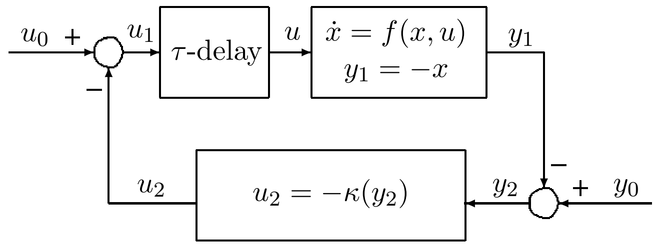

Figure 4.1: Nonlinear plant with input delay in closed-loop system

4.6 Application to General Nonlinear Plants with Input Delay

Consider the following closed-loop system which consists of a nonlinear plant with input delay and a nonlinear controller shown in Figure 4.1. Assume that both functions f and κ are continuous with $f(0,0) = \kappa(0) = 0$, and that the system $\dot{x} = f(x,u)$ is forward complete [Angeli and Sontag, 1999] [7], and that the system $\dot{x} = f(x, u_0 + \kappa(x + y_0))$ with input $w_0 = (u_0, y_0)$ and state x is input-to-state stable (in state space model) [Sontag, 1989].

Since both κ and f are continuous, there exist $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{K}_\infty$ such that

$$\kappa(x) \leq \rho_1(|x|), \quad |f(x,u)| \leq \rho_2(\max\{|x|, |u|\}).$$

Therefore, the nominal closed-loop system (i.e., closed-loop system shown in Figure 4.1 for nonlinear plant without input delay)

$$\dot{x} = f(x, u_0 + \kappa(x + y_0)), \tag{4.57a}$$

$$u_1 = u_0 + \kappa(x + y_0), \quad y_1 = -x, \tag{4.57b}$$

is input-to-output stable (in state space model) [Sontag and Wang, 1999], i.e.,

$$|w_1(t)| \leq \beta(|x_0|, t) + \gamma(\|w_0\|_{[0,t]}), \quad \forall t \geq 0, \forall w_0, \forall x(0) = x_0, \tag{4.58}$$

for some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ with $w_i \triangleq (u_i, y_i)$ for $i = 0, 1$.

The problem which follows is how much input delay can be tolerated in order to preserve the input-to-output stability of the closed-loop system shown in Figure 4.1. According to results in this chapter, we need to measure the distance between the nominal plant and the perturbed plant with input delay.

[7] The system $\dot{x} = f(x,u),\ x(0) = x_0$ is said to be forward complete if, for any initial condition x_0 and any locally measurable essentially bounded input u, the corresponding state trajectory is defined for all $t \geq 0$.

For the convenience of notation, let the nominal plant P be defined by the set:

$$\mathfrak{B}_P = \{w_1 \in \mathcal{W}_e \mid w_1 = (u_1, y_1) \text{ satisfies (4.60) for some } x\}, \tag{4.59}$$

$$\dot{x} = f(x, u_1), \quad y_1 = -x, \tag{4.60}$$

and let the perturbed plant $\tilde{P}$ be described by the set:

$$\mathfrak{B}_{\tilde{P}} = \{\tilde{w}_1 \in \mathcal{W}_e \mid \tilde{w}_1 = (\tilde{u}_1, \tilde{y}_1) \text{ satisfies (4.62) for some } x\}, \tag{4.61}$$

$$\dot{x}(t) = f(x(t), \tilde{u}_1(t-\tau)), \quad \tilde{y}_1 = -x, \quad \tau \in (0, \tau_0]. \tag{4.62}$$

For any $\tilde{w}_{1-} = (\tilde{u}_{1-}, \tilde{y}_{1-}) \in \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}$, choose $w_{1-} = (u_{1-}, y_{1-}) \in \mathcal{W}^- \cap \mathfrak{B}^-_P$ with $u_{1-} = \tilde{u}_{1-}$ and $y_{1-}(t-\tau) = \tilde{y}_{1-}(t)$ for $t \le 0$, we get

$$\|w_{1-}\| \le \max\left\{\|\tilde{w}_{1-}\|_{(-\infty,-\tau]}, \|(\tilde{u}_{1-}, y_{1-})\|_{[-\tau,0]}\right\} \le \max\left\{2\|\tilde{w}_{1-}\|, \|y_{1-}\|_{[-\tau,0]}\right\}.$$

Since $\dot{x} = f(x,u)$ with $f(0,0) = 0$ is forward complete, we have by using [Karafyllis, 2004, Lemma 3.5] that $\|y_{1-}\|_{[-\tau,0]} \le \mu(\tau)\nu(\|w_{1-}\|_{(-\infty,-\tau]} + \|u_{1-}\|_{[-\tau,0]}) \le \mu(\tau_0)\nu(2\|\tilde{w}_{1-}\|)$, and thus we obtain

$$\|w_{1-}\| \le 2\|\tilde{w}_{1-}\| + \mu(\tau_0)\nu(2\|\tilde{w}_{1-}\|), \tag{4.63}$$

where μ is a positive-valued continuous nondecreasing function and $\nu \in \mathcal{K}_\infty$.

Define a map $\Phi : \mathcal{G}_P^{w_{1-}} \to \mathcal{G}_{\tilde{P}}^{\tilde{w}_{1-}}$ by

$$w_{1+} \triangleq (u_{1+}, y_{1+}) \mapsto \tilde{w}_{1+} \triangleq (\tilde{u}_{1+}, \tilde{y}_{1+}) = (u_{1+}, \tilde{y}_{1+}),$$

and thus $\tilde{y}_{1+}(t) = (y_{1-} \wedge y_{1+})(t-\tau)$ for all $t \ge 0$.

For any $t > h \ge 0$, we have that

$$\sup\{|\dot{y}_{1+}(s)| : s \in [h,t]\} \le \sup\{|f(-y_{1+}(s), u_{1+}(s))| : s \in [h,t]\} \le \rho_2(\|w_{1+}\|_{[h,t]}),$$

and that if $t - \tau \ge h$ then

$$|(\tilde{y}_{1+} - y_{1+})(t)| = |y_{1+}(t-\tau) - y_{1+}(t)| \le \tau \cdot \sup\{|\dot{y}_{1+}(s)| : s \in [h,t]\},$$

and that if $t - \tau < h$ then

$$\begin{aligned} |(\tilde{y}_{1+} - y_{1+})(t)| &\le |(y_{1-} \wedge y_{1+})(t-\tau) - y_{1+}(h)| + |y_{1+}(h) - y_{1+}(t)| \\ &\le 2\|w_{1-} \wedge w_{1+}\|_{[-\infty,h]} + \tau \cdot \sup\{|\dot{y}_{1+}(s)| : s \in [h,t]\}. \end{aligned}$$

Hence, for any $t > h \ge 0$ and any $w_{1+} \in \mathcal{G}_P^{w_{1-}}$, we have

$$|((\Phi - I)w_{1+})(t)| \le \beta_0(\|w_{1-} \wedge w_{1+}\|_{(-\infty,h]}, t-h) + \tau \cdot \rho_2(\|w_{1+}\|_{[h,t]}) \tag{4.64}$$

with $\beta_0 \in \mathcal{KL}$ defined by

$$\beta_0(r,\xi) = \begin{cases} 2r + \frac{r}{1+\xi}, & \text{for } r \geq 0, \xi \in [0,\tau); \\ \frac{r}{1+\xi}, & \text{for } r \geq 0, \xi \geq \tau. \end{cases}$$

Theorem 4.8 now asserts that, by using (4.58) and (4.64), the perturbed closed-loop system shown in Figure 4.1 will remain input to output stable if the time delay τ satisfies:

$$\tau \cdot \rho_2 \circ (I+\rho) \circ \gamma(s) \leq (I+\varepsilon)^{-1}(s), \quad \forall s \geq 0, \tag{4.65}$$

for some functions ρ, ε of class $\mathcal{K}_\infty$.

In the following, we give a concrete nonlinear example to show that the closed-loop system remains input to output stable under the perturbation of sufficiently small time delay in the plant.

Example 4.22. *Consider the feedback configuration in Figure 4.1. Let* $\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R},\mathbb{R})$ *and* $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$, *and let*

$$f(x,u) = \phi(x) + u; \quad \kappa(y) = -ky.$$

where $k \in \mathbb{R}$ *and* $\phi : \mathbb{R} \to \mathbb{R}$ *is a memoryless nonlinear function satisfying the so-called sector condition* $\phi \in$ **Sector** (k_1, k_2) *with* $k_1, k_2 \in \mathbb{R}$ *and* $k_1 \leq k_2 < k$, *i.e.,*

$$[\phi(x) - k_1 x][\phi(x) - k_2 x] \leq 0, \quad \forall x \in \mathbb{R}. \tag{4.66}$$

This is equivalent to the following statement [Desoer and Vidyasagar, 1975]:

$$\phi(0) = 0 \quad \text{and} \quad k_1 x^2 \leq x\phi(x) \leq k_2 x^2, \ \forall x \in \mathbb{R}. \tag{4.67}$$

Thus, the nominal closed-loop equations in (4.57) is expressed as

$$\dot{x} = -(kx - \phi(x)) + u_0 - ky_0, \tag{4.68a}$$

$$u_1 = -kx + u_0 - ky_0, \quad y_1 = -x. \tag{4.68b}$$

Consider the Lyapunov function candidate $V(x) = x^2/2$, *the derivative of* V *along the trajectories of this system (4.68) is given by*

$$\dot{V} = -x(kx - \phi(x)) + x(u_0 - ky_0) \leq -(k-k_2)x^2 + x(u_0 - ky_0),$$

thus we get that for any $\varepsilon \in (0, k-k_2)$,

$$\dot{V} \leq -2\varepsilon V, \quad \forall\, |x| \geq \|u_0 - ky_0\| / (k - k_2 - \varepsilon).$$

Then, by using Theorem 2.28, we obtain that, for any $\varepsilon \in (0, k - k_2)$, *there exists a* $\beta_1 \in \mathcal{KL}$ *such that*

$$|x(t)| \leq \beta_1(|x(0)|, t) + \frac{1}{k - k_2 - \varepsilon} \|u_0 - ky_0\|_{[0,t)}, \quad \forall t \geq 0. \tag{4.69}$$

From (4.68)–(4.69), for any $\varepsilon \in (0, k - k_2)$ *we have (4.58) satisfies with gain function*

$$\gamma(r) = (1 + k + \frac{1 + k}{k - k_2 - \varepsilon}) \cdot r, \quad \forall r \geq 0, \tag{4.70}$$

where function $\beta \in \mathcal{KL}$ *in (4.58) also depends on* $\varepsilon \in (0, k - k_2)$.

Consider again $V(x) = x^2/2$, *the derivative of* V *along the trajectories of the system* $\dot{x} = f(x, u) = \phi(x) + u$ *is given by*

$$\dot{V} = x\phi(x) + xu \leq k_2 x^2 + (x^2 + u^2)/2 \leq (2k_2 + 1)V + u^2/2.$$

Thus, from [Angeli and Sontag, 1999, Corollary 2.11], we know that the system $\dot{x} = f(x, u) = \phi(x) + u$ *is forward complete. Therefore, (4.63) satisfies. Since* $|f(x, u)| \leq (1 + \max\{|k_1|, |k_2|\}) \cdot \max\{|x|, |u|\}$, *we have (4.64) satisfies with function* $\rho_2 \in \mathcal{K}_\infty$ *defined by*

$$\rho_2(r) = (1 + \max\{|k_1|, |k_2|\}) \cdot r, \quad \forall r \geq 0. \tag{4.71}$$

From (4.65), (4.70) and (4.71), and Remark 4.9, we obtain that the perturbed closed-loop system $[\tilde{P}, C]$ *will remain input to output stable if for any given* $\varepsilon \in (0, k - k_2)$ *the time delay* $\tau < 1/\omega$ *with* $\omega \triangleq (1 + \max\{|k_1|, |k_2|\})(1 + k + \frac{1+k}{k-k_2-\varepsilon})$.

4.7 Generalisation of Systems with Potential for Finite Escape Times

At the end of this chapter, we consider the generalisation of results given in previous sections to systems with potential for finite escape times. This is done by using a wider signal space (named ambient space) than the extended space, which is defined in Section 3.2 on page 39 in Chapter 3.

Consider the following state-space model

$$\dot{x}(t) = x^2(t) + u(t); \;\; y(t) = x(t); \;\; t \in \mathbb{R}.$$

It is easy to verify that $u(t) = k > 0$ and $y(t) = \sqrt{k}\tan(\sqrt{k} \cdot t)$ for $t \in (-\frac{\sqrt{k}\cdot\pi}{2k}, \frac{\sqrt{k}\cdot\pi}{2k})$ satisfy above equations. And the output escapes to infinity at time $t = -\frac{\sqrt{k}\cdot\pi}{2k}$ and $t = \frac{\sqrt{k}\cdot\pi}{2k}$. Clearly, this kind of input-output pairs cannot be considered in the definition of systems in previous sections as e.g., $y \notin \mathcal{W}_e$ for $\mathcal{W} = L^\infty(\mathbb{R}, \mathbb{R})$. In this section, we slightly modify the definition of systems in Chapter 3 by defining them on the ambient spaces to provide a robust stability theory for nonlinear systems including finite escape times phenomenon, and in particular to establish a generalisation of Theorem 4.8 in this context.

4.7.1 Systems, Closed-Loop Systems, and Initial Conditions

Definitions of systems, closed-loop systems, initial conditions, causality, existence and uniqueness properties are all slightly modified in this setting. For the definition and discussion of ambient spaces see Section 3.2 on page 39 in Chapter 3.

Definition 4.23. Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$, a system Q is defined to be the set:

$$\mathfrak{B}_Q \triangleq \left\{ w \in \mathcal{W}_a^- \oplus \mathcal{W}_a^+ \mid w = (u, y) \text{ is an input-output pair of } Q \right\} \tag{4.72}$$

which satisfies the assumption that any input-output pair $w \in \mathfrak{B}_Q$ is defined over a maximal interval $(-T_1, T_2)$ with both T_1 and T_2 belong to $(0, \infty]$, and that if T_1 (resp., T_2) is finite, then $\|w\|_{(\tau,0]} \to \infty$ (resp., $\|w\|_{[0,\tau)} \to \infty$) as τ tends to $-T_1$ (resp., T_2) from up (resp., below).

A system Q represented by the set $\mathfrak{B}_Q$ (see (4.72)) is said to be *time-invariant* if $w \in \mathfrak{B}_Q$ implies $\sigma_\tau w \in \mathfrak{B}_Q$ for all $\tau \in \mathbb{R}$ with $0 \in (a - \tau, b - \tau)$ (where $\text{dom}(w) = (a, b)$ and σ_τ is the shift operator defined by $(\sigma_\tau w)(\cdot) = w(\cdot + \tau)$. Otherwise, Q is said to be *time-variant*.

The following is the definition of causality for a system defined in the ambient space:

Definition 4.24. A system Q represented by the set $\mathfrak{B}_Q$ (see (4.72)) is said to be *causal* if, $\forall (u, y_u), (v, y_v) \in \mathfrak{B}_Q, \forall t \in \text{dom}(u, v)$,

$$u|_{(-\infty,t]\cap\text{dom}(u,v)} = v|_{(-\infty,t]\cap\text{dom}(u,v)} \Rightarrow \mathfrak{B}_Q^u|_{(-\infty,t]\cap\text{dom}(u,v)} = \mathfrak{B}_Q^v|_{(-\infty,t]\cap\text{dom}(u,v)}$$

where $\mathfrak{B}_Q^u = \{w \in \mathcal{W}_a \mid \exists y \in \mathcal{Y}_a \text{ s.t. } w = (u, y) \in \mathfrak{B}_Q\}$.

Note that any operator $\Phi : \mathcal{U}_a^+ \to \mathcal{Y}_a^+$ can be regarded as a special system in the sense of Definition 4.23, i.e., $\mathfrak{B}_\Phi = \{w = (u, y) \in \mathcal{W}_a^- \oplus \mathcal{W}_a^+ \mid y|_{(-\infty,0]} = u|_{(-\infty,0]} = 0, R_+ y = \Phi(R_+ u)\}$. We say the operator Φ is causal if and only if the corresponding system $\mathfrak{B}_\Phi$ is causal. For convenience, the special definition of a causal operator is stated below.

Given normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$, an operator $\Phi : \mathcal{U}_a^+ \to \mathcal{Y}_a^+$ is said to be *causal* if,

$$\left\{ \begin{array}{l} \forall\, u, v \in \mathcal{U}_a^+, \\ \forall t \in \text{dom}(u, v) \cap \text{dom}(\Phi u, \Phi v), \end{array} \right. : \left[\begin{array}{c} u|_{[0,t]} = v|_{[0,t]} \\ \Rightarrow\ (\Phi u)|_{[0,t]} = (\Phi v)|_{[0,t]} \end{array} \right]$$

Definition 4.25. Given a system Q represented by the set $\mathfrak{B}_Q$ (see (4.72)), its past trajectories is defined by

$$\mathfrak{B}_Q^- \triangleq R_- \mathfrak{B}_Q = \left\{ w_- \in \mathcal{W}_a^- \mid \exists\, w_+ \in \mathcal{W}_a^+, \text{ s.t. } w_- \wedge w_+ \in \mathfrak{B}_Q \right\}. \tag{4.73}$$

Here $\wedge$ denotes *concatenation at time* 0 (see (3.8) on page 50). The system Q is said to have the *existence property* if $\forall w_- \in \mathfrak{B}_Q^-, \forall u_+ \in \mathcal{U}_a^+$, $\exists y_+ \in \mathcal{Y}_a^+$ such that

$$\exists \hat{w}_+ \in \mathcal{W}_a^+, w_- \wedge \hat{w}_+ \in \mathfrak{B}_Q, (u_+, y_+)(t) = \hat{w}_+(t), \forall t \in \operatorname{dom}(u_+, y_+, \hat{w}_+)$$

and the *uniqueness property* if $\forall w_- \in \mathfrak{B}_Q^-, \forall w_+ = (u_+, y_+) \in \mathcal{W}_a^+, \forall \tilde{w}_+ = (\tilde{u}_+, \tilde{y}_+) \in \mathcal{W}_a^+$,

$$w_- \wedge w_+ \in \mathfrak{B}_Q, w_- \wedge \tilde{w}_+ \in \mathfrak{B}_Q, u_+ = \tilde{u}_+ \quad \Rightarrow \quad y_+ = \tilde{y}_+$$

and is *well-posed* if it has both the *existence* and *uniqueness* properties.

Definition 4.26. Given a system Q represented by the set $\mathfrak{B}_Q$ (see (4.72)), the graph $\mathcal{G}_Q^{w_-}$ for any given past trajectory $w_- \in \mathfrak{B}_Q^-$ is a subset of $\mathcal{W}_a^+$, which contains all of $w_+ \in \mathcal{W}_a^+$ defined over a maximal interval $[0, T)$ with $0 < T \leq \infty$ such that $w_- \wedge w_+ \in \mathfrak{B}_Q$, and if $T = \infty$ then $w_+ \in \mathcal{W}^+$, and if T is finite then $\|w_+\|_{[0,\tau)} \to \infty$ as τ tends to T from below.

Definition 4.27. Given a system Q represented by the set $\mathfrak{B}_Q$ (see (4.72)), we define $\mathfrak{S}_Q$ the *initial state space of* Q *at initial time* 0 as the quotient set $\mathfrak{B}_Q^-/\sim$ (i.e., $\mathfrak{S}_Q \triangleq \mathfrak{B}_Q^-/\sim$). While the equivalence relation $\sim$ on $\mathfrak{B}_Q^-$ (see (4.73)) is defined by

$$w_- \ \sim \ \tilde{w}_- \ \text{ if and only if } \ Q^{w_-}(u_+) = Q^{\tilde{w}_-}(u_+), \ \forall u_+ \in \mathcal{U}_a^+$$

where $w_-, \tilde{w}_- \in \mathfrak{B}_Q^-$ and $Q^{w_-}(u_+) \triangleq \left\{ y_+ \in \mathcal{Y}_a^+ \mid w_- \wedge (u_+, y_+) \in \mathfrak{B}_Q \right\}$ and the set $Q^{\tilde{w}_-}(u_+)$ is similarly defined.

The equivalence class of $w_- \in \mathfrak{B}_Q^-$ is $[w_-] \triangleq \left\{ \tilde{w}_- \in \mathfrak{B}_Q^- \mid \tilde{w}_- \sim w_- \right\} \in \mathfrak{S}_Q$. The size of $[w_-] \in \mathfrak{S}_Q$ is defined by $\chi([w_-]) \triangleq \inf_{\tilde{w}_- \in [w_-]} \{\|\tilde{w}_-\|\}$. (thus defined $\chi(\cdot)$ is a real-valued function on $\mathfrak{S}_Q$.)

From the equivalence relation $\sim$, for any initial state $x_0 \in \mathfrak{S}_Q$, we can define the set $Q^{x_0}(u_+)$ by:

$$Q^{x_0}(u_+) \triangleq Q^{w_-}(u_+), \quad \forall u_+ \in \mathcal{U}_a^+. \tag{4.74}$$

where $w_- \in \mathfrak{B}_Q^-$ is any element in x_0.

If the system Q is well-posed, then, for every $w_- \in \mathfrak{B}_Q^-$, $Q^{w_-}(\cdot)$ is an operator from $\mathcal{U}_a^+$ to $\mathcal{Y}_a^+$. This in turn implies that, for every $x_0 \in \mathfrak{S}_Q$, $Q^{x_0}(\cdot)$ is an operator from $\mathcal{U}_a^+$ to $\mathcal{Y}_a^+$.

For a well-posed system Q, if Q is causal, then we have Q^{x_0} is a causal operator from $\mathcal{U}_a^+$ to $\mathcal{Y}_a^+$.

The notion of locally input to output stability is defined as follows.

Definition 4.28. The system Q is said to be *locally input to output stable* if, and only if, it is well-posed and causal, and there exist $d > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall x_0 \in \mathfrak{S}_Q$, $\forall u_{0+} \in \mathcal{U}^+$, $\forall t \geq 0$

$$\max\{\chi(x_0), \|u_{0+}\|\} \leq d \quad \Rightarrow \quad |(Q^{x_0}u_{0+})(t)| \leq \beta\left(\chi(x_0), t\right) + \gamma(\|u_{0+}\|_{[0,t)})$$

where the real-valued function $\chi(\cdot)$ is defined in Definition 4.27.

Note that a potentially weaker definition might merely require that the above condition hold only for all $t \in [0, T_{x_0,u_{0+}})$, (where $[0, T_{x_0,u_{0+}})$ is the maximal interval over which $Q^{x_0}u_{0+}$ is defined). However, this definition turns out to be equivalent to the one given above. Indeed, by standard facts from differential equations (see e.g., [Sontag, 1998a], [Sontag, 1998b, Proposition C.3.6, p. 481]), since the right-hand side is bounded on a maximal interval, we have that the left-hand side is also bounded on the maximal interval and therefore that the maximal interval should be $[0, \infty)$.

The following is the definition of a closed-loop system:

Definition 4.29. Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$ (such as $W = L^\infty(\mathbb{R}, \mathbb{R}^{m+p})$). Let the sets $\mathfrak{B}_P$ and $\mathfrak{B}_C$ represent the subsystems P (plant) and C (controller), respectively. Consider the standard feedback configuration shown in Figure 3.1 on page 38 that satisfies equations (3.1). Then the closed-loop system $[P, C]$ represented by the set $\mathfrak{B}_{P//C}$ is defined by

$$\mathfrak{B}_{P//C} \triangleq \left\{(w_0, w_1) \in \mathcal{W}_a^2 \mid w_0 \text{ is input}, w_1 \in \mathfrak{B}_P \text{ is output}, w_0 - w_1 \in \mathfrak{B}_C\right\} \tag{4.75}$$

which satisfies the assumption that any input-output pair $(w_0, w_1) \in \mathfrak{B}_{P//C}$ is defined over a maximal interval $(-T_1, T_2)$ with both T_1 and T_2 belong to $(0, \infty]$, and that if T_1 (resp., T_2) is finite, then $\|(w_0, w_1)\|_{(\tau,0]} \to \infty$ (resp., $\|(w_0, w_1)\|_{[0,\tau)} \to \infty$) as τ tends to $-T_1$ (resp., T_2) from up (resp., below).

For the closed-loop system $[P, C]$ represented by the set $\mathfrak{B}_{P//C}$, we can similarly define the initial state space $\mathfrak{S}_{P//C}$ at initial time 0 in terms of Definition 4.27. And the closed-loop system $[P, C]$ has the existence property, the uniqueness property, and the well-posedness property if and only if the set $\mathfrak{B}_{P//C}$ has the existence property, the uniqueness property, and the well-posedness property, respectively, according to Definition 4.25.

Note that for any $s_0 \in \mathfrak{S}_{P//C}$ and $w_{0+} \in \mathcal{W}_a^+$, we have defined a set $\Pi_{P//C}^{s_0}(w_{0+})$ according to (4.74) and Definition 4.27 (let $\mathfrak{B}_Q = \mathfrak{B}_{P//C}$ and $\Pi_{P//C}^{s_0}(w_{0+}) = Q^{s_0}(w_{0+})$), i.e.,

$$\Pi^{s_0}_{P//C}(w_{0+}) = \{w_{1+} \in \mathcal{W}_a^+ \mid (w_{0-}, w_{1-}) \wedge (w_{0+}, w_{1+}) \in \mathfrak{B}_{P//C}, \forall (w_{0-}, w_{1-}) \in s_0\}$$

If the closed-loop system $[P, C]$ is well-posed, then $\Pi^{s_0}_{P//C}(\cdot)$ defines an operator from $\mathcal{W}_a^+$ to $\mathcal{W}_a^+$.

In the following we give the notion of stability for closed-loop system which is derived from the notion of stability for system in Definition 4.28.

Definition 4.30. The closed-loop system $[P, C]$ represented by the set $\mathfrak{B}_{P//C}$ with initial state space $\mathfrak{S}_{P//C}$ is said to be *locally input to output stable* if, and only if, it is well-posed and causal, and there exist $d > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall s_0 \in \mathfrak{S}_{P//C}$, $\forall w_{0+} \in \mathcal{W}^+$, $\forall t \geq 0$,

$$\max\{\chi(s_0), \|w_{0+}\|\} \leq d \quad \Rightarrow \quad |(\Pi^{s_0}_{P//C} w_{0+})(t)| \leq \beta\left(\chi(s_0), t\right) + \gamma(\|w_{0+}\|_{[0,t)})$$

where the real-valued function $\chi(\cdot)$ is defined in Definition 4.27.

Define another set which is related to the product state in $\mathfrak{S}_P \times \mathfrak{S}_C$, denoted by $\overline{\Pi^{x_0}_{P//C}}(w_{0+})$, for any $x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$ and any $w_{0+} \in \mathcal{W}_a^+$, as follows:

$$\overline{\Pi^{x_0}_{P//C}}(w_{0+}) \triangleq \left\{ w_{1+} \in \mathcal{W}_a^+ \;\middle|\; \begin{array}{c} (w_{0-}, w_{1-}) \wedge (w_{0+}, w_{1+}) \in \mathfrak{B}_{P//C}, \\ \forall\, (w_{1-},\, w_{0-} - w_{1-}) \in x_0 \end{array} \right\} \tag{4.76}$$

If the closed-loop system $[P, C]$ is well-posed, then $\overline{\Pi^{x_0}_{P//C}}(\cdot)$ defines an operator from $\mathcal{W}_a^+$ to $\mathcal{W}_a^+$.

We next present several equivalent characterisation of this notion of stability as follows.

Theorem 4.31. *Suppose that the closed-loop system $\mathfrak{B}_{P//C}$ is well-posed and causal. The following four statements are equivalent:*

I. The closed-loop system $\mathfrak{B}_{P//C}$ is locally input to output stable.

II. There exist $d_1 > 0$ and functions $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$ such that, $\forall s_0 \in \mathfrak{S}_{P//C}$, $\forall t > 0$, $\forall w_{0+} \in \mathcal{W}^+$,

$$\max\{\chi(s_0), \|w_{0+}\|\} \leq d_1 \Rightarrow |(\Pi^{s_0}_{P//C} w_{0+})(t)| \leq \beta_1\left(\chi(s_0), t\right) + \gamma_1(\|w_{0+}\|_{[0,t)})$$

III. There exist $d_2 > 0$ and functions $\beta_2 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$ such that, $\forall x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$, $\forall t > 0$, $\forall w_{0+} \in \mathcal{W}^+$,

$$\max\{\chi(x_0), \|w_{0+}\|\} \leq d_2 \Rightarrow |(\overline{\Pi^{x_0}_{P//C}} w_{0+})(t)| \leq \beta_2\left(\chi(x_0), t\right) + \gamma_2(\|w_{0+}\|_{[0,t)})$$

IV. There exist $d_3 > 0$ and functions $\beta_3 \in \mathcal{KL}$ and $\gamma_3 \in \mathcal{K}_\infty$ such that, $\forall x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$, $\forall t > 0$, $\forall w_{0+} \in \mathcal{W}^+$, $\forall w_{1-} \in x_{10}$, $\forall w_{2-} \in x_{20}$,

$$\max\{\chi(x_0), \|w_{0+}\|\} \leq d_3 \Rightarrow |(\overline{\Pi^{x_0}_{P//C}} w_{0+})(t)| \leq \beta_3 \left(\|(w_{1-}, w_{2-})\|, t\right) + \gamma_3(\|w_{0+}\|_{[0,t)})$$

Moreover, we have $\gamma_1 = \gamma_2 = \gamma_3$, $d_2 = d_3$ and $\beta_2 = \beta_3$.

Proof. Similar to the proof of Theorem 3.36 on page 75. □

4.7.2 Robust Stability Theorem

The main result of this section is given by Theorem 4.33, which is also presented in two different frameworks: one requires the well-posedness of the perturbed closed-loop system; while the other one requires only the uniqueness property of the perturbed closed-loop system.

The following assumptions on the normed vector space $\mathcal{W}^+$ are only required in the proof of Theorem 4.33 with condition II:

Assumption 4.32. *(1) For any $x \in \mathcal{W}^+_e$, if $\|x\| < \infty$, then $x \in \mathcal{W}^+$; (2) The normed vector space $\mathcal{W}^+$ (not necessarily complete) is truncation complete, i.e., $\mathcal{W}[0, \tau)$ is complete for any $0 < \tau < \infty$; (3) For any time interval $J \triangleq [0, \tau)$ with $0 < \tau < \infty$, there exists a continuous map $E_J : \mathcal{W}(J) \to \mathcal{W}^+$ such that $R_J x = R_J(E_J x)$ for any $x \in \mathcal{W}(J)$.*

Theorem 4.33. *Assume that P, $\tilde{P}$, and C are well-posed and causal systems, and that $[P, C]$ is time-invariant, well-posed and causal, and that $[\tilde{P}, C]$ is causal. Let $[P, C]$ be locally input to output stable, i.e., there exist $d > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall x_0 = (x_{10}, x_{20}) \in \mathfrak{S}_P \times \mathfrak{S}_C$, $\forall w_{0+} \in \mathcal{W}^+$, $\forall t \geq 0$,*

$$\max\{\chi(x_0), \|w_{0+}\|\} \leq d \quad \Rightarrow \quad |(\overline{\Pi^{x_0}_{P//C}} w_{0+})(t)| \leq \beta\left(\chi(x_0), t\right) + \gamma(\|w_{0+}\|_{[0,t)}), \tag{4.77}$$

If there exist functions $\sigma_0, \sigma \in \mathcal{K}_\infty$ and $\beta_0 \in \mathcal{KL}$ such that for any $\tilde{w}_{1-} \in \mathcal{W}^- \cap \mathfrak{B}^-_{\tilde{P}}$ there exists a $w_{1-} \in \mathcal{W}^- \cap \mathfrak{B}^-_P$ with

$$\|w_{1-}\| \leq \sigma_0(\|\tilde{w}_{1-}\|), \tag{4.78}$$

and a causal surjective operator $\Phi : \mathrm{dom}(\Phi) \subseteq \mathcal{G}^{w_{1-}}_P \to \mathcal{G}^{\tilde{w}_{1-}}_{\tilde{P}}$ satisfying, $\forall t > h \geq 0$, $\forall w_{1+} \in \mathrm{dom}(\Phi)$ with $\|w_{1+}\| \leq \beta(d, 0) + \gamma(d)$,

$$|((\Phi - I)w_{1+})(t)| \leq \beta_0(\|w_{1-} \wedge w_{1+}\|_{(-\infty,h]}, t - h) + \sigma(\|w_{1+}\|_{[h,t)}). \tag{4.79}$$

In addition, if there exist two functions ρ, ε of class $\mathcal{K}_\infty$ such that, $\forall s \geq 0$,

$$\sigma \circ (I+\rho) \circ \gamma(s) \leq (I+\varepsilon)^{-1}(s). \tag{4.80}$$

And either of the following conditions is satisfied:

I. *$[\tilde{P}, C]$ is well-posed and $\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}(\mathcal{W}^+) \subseteq \mathcal{W}^+$ for any $\tilde{x}_0 \in \mathfrak{S}_{\tilde{P}} \times \mathfrak{S}_C$;*

II. *Assumption 4.32 holds for $\mathcal{W}^+$, and $[\tilde{P}, C]$ has the uniqueness property, and $\overline{\Pi^{x_0}_{P//C}}$ is relatively continuous for any $x_0 \in \mathfrak{S}_P \times \mathfrak{S}_C$, and $R_{[0,\tau)}(\Phi - I)$ is compact for any $0 < \tau < \infty$.*

Then the closed-loop system $[\tilde{P}, C]$ is also locally input to output stable. More specifically, there exist $\tilde{d} > 0$, for any function α of class $\mathcal{K}_\infty$, there exists a function $\tilde{\beta} \in \mathcal{KL}$ such that, $\forall \tilde{x}_0 \in \mathfrak{S}_{\tilde{P}} \times \mathfrak{S}_C$, $\forall \tilde{w}_{0+} \in \mathcal{W}^+$, $\forall t > 0$,

$$\max\{\chi(\tilde{x}_0), \|\tilde{w}_{0+}\|\} \leq \tilde{d} \Rightarrow |(\overline{\Pi^{\tilde{x}_0}_{\tilde{P}//C}}\tilde{w}_{0+})(t)| \leq \tilde{\beta}\left(\chi(\tilde{x}_0), t\right) + (\alpha + \tilde{\gamma})(\|\tilde{w}_{0+}\|_{[0,t)}) \tag{4.81}$$

where $\tilde{d} = \min\{(I+\Delta)^{-1} \circ (I+\varepsilon^{-1})^{-1}(d),\ (\sigma_0 + I)^{-1}(d)\}$ with functions $\Delta \in \mathcal{K}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ defined by

$$\Delta(r) \triangleq \beta_0((\sigma_0+I)(r), 0) + \sigma \circ (I+\rho^{-1}) \circ \beta((\sigma_0+I)(r), 0),\ \forall r \geq 0, \tag{4.82a}$$

$$\tilde{\gamma}(r) \triangleq (\sigma + I) \circ (I+\rho) \circ \gamma \circ (I+\varepsilon^{-1})^3(r),\ \forall r \geq 0. \tag{4.82b}$$

Proof. To prove above theorem we need to change slightly the proof of Theorem 4.8 in Chapter 4. Choose $\tilde{d} = \min\{(I+\Delta)^{-1} \circ (I+\varepsilon^{-1})^{-1}(d),\ (\sigma_0+I)^{-1}(d)\}$. Note that the function Δ defined in (4.82) is the same as (4.14) (or (4.27)) in the proof of Theorem 4.8. For any $\max\{\chi(\tilde{x}_0), \|\tilde{w}_{0+}\|\} \leq \tilde{d}$, from (4.8) (or (4.24)) and $\tilde{d} \leq (\sigma_0+I)^{-1}(d)$ we have $\chi(x_0) \leq d$; and from $\tilde{d} \leq (I+\Delta)^{-1} \circ (I+\varepsilon^{-1})^{-1}(d)$ and (4.13) we have $\|w_{0+}\| \leq d$. The rest of proof follows from the proof of Theorem 4.8 on page 81. □

4.8 Summary

In Chapter 3 we have developed a unified construction of an underlying abstract state space applicable to input-output systems defined over a doubly infinite time axis. The current chapter is the main part of this thesis, which provides an input-output theory with an integrated treatment of initial conditions, culminating in a statement and proof of a robust stability result. The resulting gap distances take into account both the effect of the perturbation on the state space structure (and hence the initial condition) as well as the input-output response. This complements the robust stability theory of Georgiou and Smith [Georgiou and Smith, 1997b] by introducing initial conditions and applies the ideas of the ISS framework in a situation whereby the conventional state-space formalism of ISS is not directly applicable due to variation in the structure of the state space between the nominal and perturbed systems which arise naturally in a robust stability setting. Two different versions of the main results are presented. One requires the well-posedness, while the other one requires only the uniqueness property of the perturbed closed-loop system. In real-world applications, both well-posedness (i.e., existence and uniqueness) and stability are required to be verified for a feedback system. In general uniqueness conclusions are more easily obtained than existence conclusions. Establishing existence and stability simultaneously by only using uniqueness greatly eases the real-time application of the robust stability result (see also the discussions given in [French and Bian, 2012]). Generalisation of this robust stability result to systems with potential for finite escape times is discussed at the end of this chapter.

I think that only daring speculation can lead us further and not accumulation of facts.

Albert Einstein (1879-1955)

Chapter 5

Generalised Small-Gain Theorem for Systems with Initial Conditions

5.1 Introduction

The use of the small-gain theorem in control theory dates back to the 1960's by [Zames, 1966b,c] and [Sandberg, 1964]. The original version of the small-gain theorem involves systems with finite linear gains from input to output with or without a bias term (see e.g., [Desoer and Vidyasagar, 1975]). Extensions of the small-gain theorem to nonlinear gains have been studied by many researchers. The work on the small-gain theory involving nonlinear gain began with [Hill, 1991, Mareels and Hill, 1992], where the monotone gain was proposed for a nonlinear generalisation of the classical small-gain theorem. In [Jiang et al., 1994], the authors developed a nonlinear ISS-type small-gain theorem in the sense of [Sontag, 1989] for interconnection of nonlinear systems in state space representations, which led an extensive follow-up literature (e.g., [Chen and Huang, 2005, Jiang and Marcels, 1997, Jiang et al., 1996]). Several interesting extensions of the small-gain theorem were also obtained for systems with special structures such as Volterra systems [Zheng and Zafiriou, 1999], general networks [Dashkovskiy et al., 2007], large-scale complex systems [Jiang and Wang, 2008], stochastic systems [Lu and Skelton, 2002], hybrid systems [Liberzon and Nešić, 2006, Nešić and Teel, 2008], etc. In the present chapter, we present a nonlinear small-gain theorem on input to output stability for nonlinear feedback systems from an input-output point of view.

Note that the classical small-gain theorem obtained in the input-output framework has the benefit that the stability property is completely disconnected from the existence, uniqueness property, etc.; see e.g., [Desoer and Vidyasagar, 1975]. Most of the results of the ISS-type nonlinear small-gain theorem were obtained for nonlinear state space

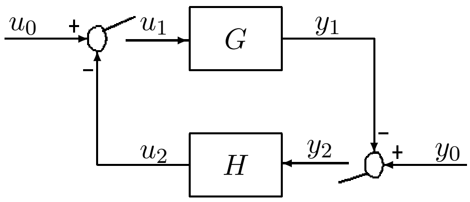

Figure 5.1: Nonlinear feedback configuration $[G, H]$

models, and a priori requirements of existence and uniqueness property of systems are imposed (e.g., requiring smoothness or Lipschitz continuity of dynamical functions), and extra "observability" conditions are imposed to guarantee that the state trajectories are bounded when the input and output are bounded. In [Ingalls et al., 1999, Sontag and Ingalls, 2002], the authors presented an abstract ISS-type small-gain theorem including applications to purely input/output systems represented by i/o operators defined on spaces of signals beginning at some *finite* time in the past. The special representation of systems allows the authors to identify the 'state' only with the past input without using the past output; but it precludes for example the uncontrollable stable linear case (see also the discussion related to Theorem 4.8 on page 81).

5.2 Setting of the Problem

Given normed signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} \triangleq \mathcal{U} \times \mathcal{Y}$ (such as $W = L^\infty(\mathbb{R}, \mathbb{R}^{m+p})$). Consider the form of feedback configuration shown in Figure 5.1. The signals u_i and y_i $(i = 0, 1, 2)$ belong to the extended signal spaces $\mathcal{U}_e$ and $\mathcal{Y}_e$, respectively. Define $w_i = (u_i, y_i)$ for $i = 0, 1, 2$, thus w_i for $i = 0, 1, 2$ belong to $\mathcal{W}_e$. The symbols G and H represent two subsystems which consist of all the input-output signal pairs $w_1 = (u_1, y_1) \in \mathcal{W}_e$ related by G and all the output-input signal pairs $w_2 = (u_2, y_2) \in \mathcal{W}_e$ related by H, respectively, when the switches are open. (Here G, H are relations (i.e., "multivalued functions").) When the switches are closed from some given initial time (say 0), the interconnection equation $w_0 = w_1 + w_2$ also holds.

The subsystems G and H are determined by the sets $\mathfrak{B}_G$ and $\mathfrak{B}_H$ (Definition 3.2 on page 47), respectively; and the corresponding initial state spaces $\mathfrak{S}_G$ and $\mathfrak{S}_H$ at given initial time are defined according to Definition 3.11 on page 54. Note that the definitions of corresponding initial state spaces are not related to the well-posedness of the systems (see Remark 3.13 on page 54). We define the interconnected system $[G, H]$ shown in Figure 5.1 by the following set $\mathfrak{B}_{[G,H]}$,

$$\begin{aligned}\mathfrak{B}_{[G,H]} \triangleq \Big\{(w_0, w_1, w_2) \in \mathcal{W}_e \times \mathcal{W}_e \times \mathcal{W}_e \;\Big|\; & \\ w_0 \text{ is input}, (w_1, w_2) \text{ is output}, & \\ w_1 \in \mathfrak{B}_G, w_2 \in \mathfrak{B}_H, w_0 = w_1 + w_2\Big\}. & \qquad (5.1)\end{aligned}$$

In $\mathfrak{B}_{[G,H]}$ we view the external input w_0 as the (closed-loop) input and the internal signals (w_1, w_2) as the (closed-loop) output.

We make the following notations to let the statement of the main result in this Chapter more concise. For any $x_0 \in \mathfrak{S}_G$ and any $u_{1+} \in \mathcal{U}_e^+$, we let $G_{x_0}u_{1+}$ denote any of $y_{1+} \in \mathcal{Y}_e^+$ (if exists) such that $w_{1-} \wedge (u_{1+}, y_{1+})$ (for any $w_{1-} \in x_0$) is an input-output signal pair of G, where y_{1+} is often called an "*image*" of G_{x_0} with respect to u_{1+}. Similarly, for any $z_0 \in \mathfrak{S}_H$ and any $y_{2+} \in \mathcal{Y}_e^+$, we let $H_{z_0}y_{2+}$ denote any of $u_{2+} \in \mathcal{U}_e^+$ (if exists) such that $w_{2-} \wedge (u_{2+}, y_{2+})$ (for any $w_{2-} \in z_0$) is an output-input signal pair of H. Note that both G_{x_0} and H_{z_0} are "multivalued functions". Denote by $[G_{x_0}, H_{z_0}]$ the closed-loop relation which consists of all positive time input-output signal pairs (w_{0+}, w_{1+}, w_{2+}) with $w_{0+} \in \mathcal{W}_e^+$ denoting inputs and $(w_{1+}, w_{2+}) \in \mathcal{W}_e^+ \times \mathcal{W}_e^+$ denoting outputs of $[G_{x_0}, H_{z_0}]$ such that

$$w_{0+} = w_{1+} + w_{2+}, \quad w_{1+} \triangleq (u_{1+}, G_{x_0}u_{1+}), \quad w_{2+} \triangleq (H_{z_0}y_{2+}, y_{2+}). \tag{5.2}$$

5.3 Generalised Small-Gain Theorem

Before giving the main result of this chapter we establish the following lemma:

Lemma 5.1. *Consider the feedback configuration shown in Figure 5.1 (i.e., with the switches closed). Let G, H be two causal time-invariant systems with above notations and $[G, H]$ be causal. Suppose that there are functions $\beta_1, \beta_2 \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for any $x_0 \in \mathfrak{S}_G$, $z_0 \in \mathfrak{S}_H$ and any $t > 0$, $u_{1+} \in \mathcal{U}_e^+$, $y_{2+} \in \mathcal{Y}_e^+$,*

$$\begin{aligned} |(G_{x_0}u_{1+})(t)| &\le \beta_1(\chi(x_0), t) + \gamma_1(\|u_{1+}\|_{[0,t)}), \\ |(H_{z_0}y_{2+})(t)| &\le \beta_2(\chi(z_0), t) + \gamma_2(\|y_{2+}\|_{[0,t)}), \end{aligned} \tag{5.3}$$

where (5.3) holds for all the "images" $G_{x_0}u_{1+}$ and $H_{z_0}y_{2+}$ of each $u_{1+} \in \mathcal{U}_e^+$ and $y_{2+} \in \mathcal{Y}_e^+$, and the real-valued function χ is defined in (3.15). Then there are class $\mathcal{K}_\infty$ functions $\eta_i, \theta_i, (i = 1, 2)$ independent of x_0, z_0, u_{1+}, y_{2+} such that for any $t \ge 0$,

$$\begin{aligned} \chi(x(t)) &\le \eta_1(\chi(x_0)) + \theta_1(\|(u_{1+}, G_{x_0}u_{1+})\|_{[0,t)}), \\ \chi(z(t)) &\le \eta_2(\chi(z_0)) + \theta_2(\|(H_{z_0}y_{2+}, y_{2+})\|_{[0,t)}), \end{aligned} \tag{5.4}$$

where $x(t) \in \mathfrak{S}_G^t$ and $z(t) \in \mathfrak{S}_H^t$ are the corresponding states of G and H related to initial states x_0 and z_0 at time $t \ge 0$ with $x(0) = x_0$ and $z(0) = z_0$, respectively.

Proof. According to the definition of state in Definition 3.11, the inequalities (5.4) are immediately obtained by letting

$$\eta_i(s) = \beta_i(s,0) + s, \quad \theta_i(s) = \gamma_i(s) + s,$$

for any $i = 1, 2$ and any $s \geq 0$. □

The main result of this chapter is a small-gain theorem incorporating initial conditions given as follows:

Theorem 5.2. *Under the same conditions and notations in Lemma 5.1. If there exist two functions $\rho \in \mathcal{K}_\infty$ and $\varepsilon \in \mathcal{K}_\infty$ such that*

$$\gamma_1 \circ (I + \rho) \circ \gamma_2(s) \leq (I + \varepsilon)^{-1}(s), \quad \forall s \geq 0, \tag{5.5}$$

Then, for any function $\alpha \in \mathcal{K}_\infty$, there exists a function $\beta \in \mathcal{KL}$ such that for any $i = 1, 2$ and all $t > 0$, and all $w_{0+} \in \mathcal{U}_e^+ \times \mathcal{Y}_e^+$,

$$|w_{i+}(t)| \leq \beta(\chi(x_0, z_0), t) + (\alpha + \gamma)(\|w_{0+}\|_{[0,t)}), \tag{5.6}$$

where the real-valued function χ is defined in (3.15) and $\gamma \in \mathcal{K}_\infty$ is defined as follows, for any $r \geq 0$,

$$\begin{cases} \gamma(r) = \left(I + (I + \rho^{-1})^2 \circ \gamma_3 + (I + \varepsilon^{-1})^2 \circ \gamma_4\right)(r), \\ \gamma_3(r) = \left(I + \gamma_2 \circ (I + \varepsilon^{-1})^2\right)(r), \\ \gamma_4(r) = \left(I + \gamma_1 \circ (I + \rho^{-1})^2\right)(r). \end{cases} \tag{5.7}$$

Proof. Choose $s = (I+\varepsilon)\circ\gamma_1(\hat{s}), (\hat{s} \geq 0)$ in (5.5), we have $\gamma_1 \circ (I+\rho)\circ\gamma_2\circ(I+\varepsilon)\circ\gamma_1(\hat{s}) \leq \gamma_1(\hat{s}), (\hat{s} \geq 0)$. Hence, we get

$$\gamma_2 \circ (I + \varepsilon) \circ \gamma_1(\hat{s}) \leq (I + \rho)^{-1}(\hat{s}), \quad \forall \hat{s} \geq 0, \tag{5.8}$$

For any initial states $x_0 \in \mathfrak{S}_G$ and $z_0 \in \mathfrak{S}_H$ and any $w_{0+} = (u_{0+}, y_{0+}) \in \mathcal{U}^+ \times \mathcal{Y}^+$, we define two nonnegative constants $b_{10} = \beta_1(\chi(x_0), 0)$ and $b_{20} = \beta_2(\chi(z_0), 0)$. Then, from (5.2) and (5.3), we obtain that

$$\begin{aligned} \|u_{1+}\|_{[0,t)} &\leq \|u_{0+}\|_{[0,t)} + \|H_{z_0} y_{2+}\|_{[0,t)} \\ &\leq \|u_{0+}\|_{[0,t)} + b_{20} + \gamma_2(\|y_{2+}\|_{[0,t)}), \ \forall t > 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|y_{2+}\|_{[0,t)} &\leq \|y_{0+}\|_{[0,t)} + \|G_{x_0} u_{1+}\|_{[0,t)} \\ &\leq \|y_{0+}\|_{[0,t)} + b_{10} + \gamma_1(\|u_{1+}\|_{[0,t)}), \ \forall t > 0. \end{aligned}$$

Hence, we get

$$\begin{aligned}\|u_{1+}\|_{[0,t)} \leq \|u_{0+}\|_{[0,t)} &+ b_{20} \\ &+ \gamma_2 \circ (I+\varepsilon) \circ \gamma_1(\|u_{1+}\|_{[0,t)}) \\ &+ \gamma_2 \circ (I+\varepsilon^{-1})(\|y_{0+}\|_{[0,t)} + b_{10}). \end{aligned} \tag{5.9}$$

Since $\gamma_2 \circ (I+\varepsilon) \circ \gamma_1(s) \leq (I+\rho)^{-1}(s)$,$\forall s \geq 0$ (see (5.8)), and $(I-(I+\rho)^{-1})^{-1}(\cdot) = (I+\rho^{-1})(\cdot)$ (see (2.20)), we have, for all $t \geq 0$,

$$\begin{aligned}\|u_{1+}\|_{[0,t)} \leq (I+\rho^{-1}) \Big(&\|u_{0+}\|_{[0,t)} + b_{20} \\ &+\gamma_2 \circ (I+\varepsilon^{-1})(\|y_{0+}\|_{[0,t)} + b_{10})\Big). \end{aligned} \tag{5.10}$$

Similarly, we have, for all $t > 0$,

$$\begin{aligned}\|y_{2+}\|_{[0,t)} \leq (I+\varepsilon^{-1}) \Big(&\|y_{0+}\|_{[0,t)} + b_{10} \\ &+\gamma_1 \circ (I+\rho^{-1})(\|u_{0+}\|_{[0,t)} + b_{20})\Big). \end{aligned} \tag{5.11}$$

Note that, for all $t > 0$, $\|u_{2+}\|_{[0,t)} \leq \|u_{0+}\|_{[0,t)} + \|u_{1+}\|_{[0,t)}$ and $\|y_{1+}\|_{[0,t)} \leq \|y_{0+}\|_{[0,t)} + \|y_{2+}\|_{[0,t)}$. Hence, by applying Lemma 2.23 to (5.10) and (5.11), we obtain that there exists a class $\mathcal{K}_\infty$ function κ such that, for any $i = 1, 2$ and all $t > 0$,

$$\|w_{i+}\|_{[0,t)} \leq \gamma(\|w_{0+}\|_{[0,t)}) + \kappa(\chi(x_0, z_0)), \tag{5.12}$$

where $\gamma \in \mathcal{K}_\infty$ is defined in (5.7).

From (5.4) in Lemma 5.1 and (5.12), and by using Lemma 2.23, we know that, for any $t > 0$,

$$\begin{aligned}\chi(x(t), z(t)) &\leq (\eta_1 + \eta_2)(\chi(x_0, z_0)) \\ &\quad + (\theta_1 + \theta_2)(\max\{\|w_{1+}\|_{[0,t)}, \|w_{2+}\|_{[0,t)}\}) \\ &\leq \delta_1(\chi(x_0, z_0)) + \delta_2(\|w_{0+}\|_{[0,\infty)}) \\ &\triangleq s_\infty, \quad \forall t > 0, \end{aligned} \tag{5.13}$$

where $x(t)$ and $z(t)$ are the corresponding states at time $t > 0$ of G and H related to initial states x_0 and z_0, respectively; and $\delta_1(s) = (\eta_1 + \eta_2)(s) + (\theta_1 + \theta_2) \circ (I + \rho^{-1}) \circ \kappa(s)$ and $\delta_2(s) = (\theta_1 + \theta_2) \circ (I + \rho) \circ \gamma(s)$, $\forall s \geq 0$.

It's easy to see that both δ_1 and δ_2 are class $\mathcal{K}_\infty$ functions. Next we estimate the bound of $|w_i(t)|, i = 1, 2$ for any $t > 0$. Since both G and H are causal and time-invariant, by using (5.3) and (5.13), we have for any $t > 0$ and any $u_{1+} \in \mathcal{U}_e^+$, and any $y_{2+} \in \mathcal{Y}_e^+$,

$$\begin{aligned} |(G_{x_0}u_{1+})(t)| &\leq \beta_1(\chi(x(t/2)), t/2) + \gamma_1(\|u_{1+}\|_{[\frac{t}{2},t)}) \\ &\leq \beta_1(s_\infty, t/2) + \gamma_1(\|u_{1+}\|_{[\frac{t}{2},t)}), \\ |(H_{z_0}y_{2+})(t)| &\leq \beta_2(\chi(z(t/2)), t/2) + \gamma_2(\|y_{2+}\|_{[\frac{t}{2},t)}) \\ &\leq \beta_2(s_\infty, t/2) + \gamma_2(\|y_{2+}\|_{[\frac{t}{2},t)}). \end{aligned} \tag{5.14}$$

Thus, by applying (5.2) and (5.14), we have, for all $t > 0$,

$$\begin{aligned} |u_{1+}(t)| &\leq |u_{0+}(t)| + |(H_{z_0}y_{2+})(t)| \\ &\leq \|u_{0+}\|_{[0,t)} + \beta_2(s_\infty, t/2) + \gamma_2(\|y_{2+}\|_{[\frac{t}{2},t)}); \\ |y_{2+}(t)| &\leq |y_{0+}(t)| + \|(G_{x_0}u_{1+})(t)\| \\ &\leq \|y_{0+}\|_{[0,t)} + \beta_1(s_\infty, t/2) + \gamma_1(\|u_{1+}\|_{[\frac{t}{2},t)}). \end{aligned}$$

Hence, we get, for all $t > 0$,

$$\begin{aligned} |u_{1+}(t)| &\leq \|u_{0+}\|_{[0,t)} + \beta_2(s_\infty, t/2) \\ &\quad + \gamma_2 \circ (I + \varepsilon) \circ \gamma_1(\|u_{1+}\|_{[\frac{t}{2},t)}) \\ &\quad + \gamma_2 \circ (I + \varepsilon^{-1})(\|y_{0+}\|_{[0,t)} + \beta_1(s_\infty, t/2)) \\ &\leq \|u_{0+}\|_{[0,t)} + \beta_2(s_\infty, t/2) \\ &\quad + (I + \rho)^{-1}(\|u_{1+}\|_{[\frac{t}{2},t)}) \\ &\quad + \gamma_2 \circ (I + \varepsilon^{-1})(\|y_{0+}\|_{[0,t)} + \beta_1(s_\infty, t/2)) \\ &\leq \beta_3(s_\infty, t) + (I + \rho)^{-1}(\|u_{1+}\|_{[\frac{t}{2},t)}) \\ &\quad + \gamma_3(\|w_{0+}\|_{[0,t)}) \end{aligned} \tag{5.15}$$

with $\gamma_3 \in \mathcal{K}_\infty$ defined in (5.7) and $\beta_3 \in \mathcal{KL}$ defined by

$$\begin{aligned} \beta_3(r, s) \triangleq \beta_2(r, s/2) &+ \gamma_2 \circ (I + \varepsilon^{-1}) \\ &\circ (I + \varepsilon) \circ \beta_1(r, s/2)), \ \forall r \geq 0, \ \forall s \geq 0. \end{aligned}$$

Next we apply Lemma 2.25 to (5.15) (with $\mu := \frac{1}{2}$), it follows that a function β_4 of class $\mathcal{KL}$ exists such that, for all $t > 0$,

$$\begin{aligned} |u_{1+}(t)| &\leq \beta_4(s_\infty, t) + (I - (I + \rho)^{-1})^{-1} \\ &\quad \circ (I + \rho^{-1}) \circ \gamma_3(\|w_{0+}\|_{[0,\infty)}) \\ &= \beta_4(s_\infty, t) + (I + \rho^{-1})^2 \circ \gamma_3(\|w_{0+}\|_{[0,\infty)}), \end{aligned} \tag{5.16}$$

where we use the fact that $(I - (I + \rho)^{-1})^{-1}(s) = (I + \rho^{-1})(s)$ for any $s \geq 0$.

Similarly, there exist a function $\beta_5 \in \mathcal{KL}$ such that, for all $t > 0$,

$$|y_{2+}(t)| \leq \beta_5(s_\infty, t) + (I + \varepsilon^{-1})^2 \circ \gamma_4(\|w_{0+}\|_{[0,\infty)}) \tag{5.17}$$

with $\gamma_4 \in \mathcal{K}_\infty$ defined in (5.7).

Note that, for all $t > 0$, $|u_{2+}(t)| \leq |u_{0+}(t)| + |u_{1+}(t)|$ and $|y_{1+}(t)| \leq |y_{0+}(t)| + |y_{2+}(t)|$. Hence, we have, for all $t > 0$,

$$|w_{i+}(t)| \leq \beta_6(s_\infty, t) + \gamma(\|w_{0+}\|_{[0,\infty)}), \quad i = 1, 2, \tag{5.18}$$

with $\beta_6(r, s) \triangleq \max\{\beta_4(r, s), \beta_5(r, s)\}$, $\forall r \geq 0, \forall s \geq 0$, and $\gamma \in \mathcal{K}_\infty$ defined in (5.7).

Since $s_\infty = \delta_1(\chi(x_0, z_0)) + \delta_2(\|w_{0+}\|_{[0,\infty)})$ (see (5.13)), from (5.12) and (5.18), we have for any $t \geq 0$,

$$|w_{i+}(t)| \leq \gamma(\|w_{0+}\|_{[0,\infty)}) + \min\Big\{\kappa(\chi(x_0, z_0)), \beta_6\Big(\delta_1\big(\chi(x_0, z_0)\big) + \delta_2\big(\|w_{0+}\|_{[0,\infty)}\big), t\Big)\Big\}. \tag{5.19}$$

Given any function α of $\mathcal{K}_\infty$, there are only two cases $\chi(x_0, z_0) \leq \kappa^{-1} \circ \alpha(\|w_{0+}\|_{[0,\infty)})$ or $\|w_{0+}\|_{[0,\infty)} \leq \alpha^{-1} \circ \kappa(\chi(x_0, z_0))$, thus from (5.19) and by considering the fact that for any fixed $t > 0$ the function $\beta_6(\cdot, t) \in \mathcal{K}$, we have for any $t \geq 0$,

$$|w_{i+}(t)| \leq \gamma(\|w_{0+}\|_{[0,\infty)}) + \kappa \circ \kappa^{-1} \circ \alpha(\|w_{0+}\|_{[0,\infty)}) + \beta_6(\delta_1(\chi(x_0, z_0)) + \delta_2 \circ \alpha^{-1} \circ \kappa(\chi(x_0, z_0)), t).$$

Thus, by the causality of $[G, H]$ and the definition of extended space, for any $\alpha \in \mathcal{K}_\infty$ and any $i = 1, 2$ and all $t > 0$, and all $w_{0+} \in \mathcal{U}_e^+ \times \mathcal{Y}_e^+$, we have,

$$|w_{i+}(t)| \leq \beta(\chi(x_0, z_0), t) + (\alpha + \gamma)(\|w_{0+}\|_{[0,t)}),$$

with $\beta(r, s) \triangleq \beta_6((\delta_1 + \delta_2 \circ \alpha^{-1} \circ \kappa)(r), s)$, $\forall r \geq 0, \forall s \geq 0$, and $\gamma \in \mathcal{K}_\infty$ defined in (5.7). □

5.4 An Illustrative Example of Theorem 5.2

We next illustrate Theorem 5.2 by considering the following example for systems with time delay and nonzero initial conditions.

Example 5.3. *The subsystem G is defined by the set*

$$\mathfrak{B}_G = \{w_1 \in \mathcal{W}_e \mid w_1 = (u_1, y_1) \text{ satisfies (5.21)}\}, \tag{5.20}$$

$$\dot{y_1}(t) = -ay_1(t-\tau_1) + \varepsilon(e^{u_1(t)} - 1), \tag{5.21}$$

and the subsystem H is defined by the set

$$\mathfrak{B}_H = \{w_2 \in \mathcal{W}_e \mid w_2 = (u_2, y_2) \text{ satisfies (5.23)}\}, \tag{5.22}$$

$$\dot{u_2}(t) = \mathrm{sat}\{-bu_2(t-\tau_2) + \mathrm{sat}[y_2(t)]\}, \tag{5.23}$$

with the interconnection conditions $u_0 = u_1 + u_2$ and $y_0 = y_1 + y_2$, where $a > 0, b > 0$ are fixed real numbers, and $\varepsilon \in \mathbb{R}, \tau_1 > 0, \tau_2 > 0$ are small parameters, and the saturation function $\mathrm{sat} : \mathbb{R} \to \mathbb{R}$ satisfying $\mathrm{sat}(s) = s$ when $|s| \leq 1$ and $\mathrm{sat}(s) = 1$ when $s > 1$ and $\mathrm{sat}(s) = -1$ when $s < -1$.

The corresponding initial state spaces $\mathfrak{S}_G$ and $\mathfrak{S}_H$ at given initial time 0 are defined according to Definition 3.11. The interconnected system $[G, H]$ is defined as (5.1). Both G and H are causal and time-invariant, and $[G, H]$ is causal.

Note that, for any $\kappa_1 > 0$ and any $\varepsilon_1 \in (0, a)$, when $\dot{x}(t) = -ax(t-\tau_1) + f(t)$, the following inequality

$$|x(t)| \geq \max\left\{\frac{(1+\kappa_1)a^2\tau_1}{a-\varepsilon_1}\|x\|_{[t-2\tau_1,t]}, \frac{(1+1/\kappa_1)(a\tau_1+1)}{a-\varepsilon_1}\|f\|_{[t-\tau_1,t]}\right\}$$

implies[1] *that $\frac{d}{dt}x^2(t) \leq -2\varepsilon_1 |x(t)|^2$. Also note that, for any $\kappa_2 > 0$ and any $\varepsilon_2 \in (0, b)$, when $\dot{z}(t) = \mathrm{sat}[-bz(t-\tau_2) + g(t)]$, the following inequality*

$$|z(t)| \geq \max\left\{\frac{(1+\kappa_2)b^2\tau_2}{b-\varepsilon_2}\|z\|_{[t-2\tau_2,t]}, \frac{(1+1/\kappa_2)(b\tau_2+1)}{b-\varepsilon_2}\|g\|_{[t-\tau_2,t]}\right\}$$

implies[2] *that$\frac{d}{dt}z^2(t) \leq -2|z(t)|\,\mathrm{sat}(\varepsilon_2 |z(t)|)$.*

So, for the subsystems G and H, by applying the Razumikhin-type theorem (see [Teel, 1998, Theorem 2]), we have that, for any $\kappa_1 > 0$, $\kappa_2 > 0$ and any $\varepsilon_1 \in (0, a)$, $\varepsilon_2 \in (0, b)$,

[1]This follows from $\dot{x}(t) = -ax(t) + ax(t) - ax(t-\tau_1) + f(t) = -ax(t) + a\tau_1\dot{x}(\theta_1) + f(t)$ for some $\theta_1 \in (t-\tau_1, t)$ that $|\dot{x}(t) + ax(t)| \leq a^2\tau_1 \|x\|_{[t-2\tau_1,t]} + (a\tau_1+1)\|f\|_{[t-\tau_1,t]}$. By using the fact that $A + B \leq \max\{(1+\kappa_1)A, (1+1/\kappa_1)B\}$ for any $A \geq 0, B \geq 0$ and $\kappa_1 > 0$ in the previous inequality, we have $|\dot{x}(t) + ax(t)| \leq \max\{(1+\kappa_1)a^2\tau_1\|x\|_{[t-2\tau_1,t]}, (1+1/\kappa_1)(a\tau_1+1)\|f\|_{[t-\tau_1,t]}\} \leq (a-\varepsilon_1)|x(t)|$ and thus $x(t)\dot{x}(t) \leq -\varepsilon_1 |x(t)|^2$.

[2]Similarly, this follows from $\dot{z}(t) = \mathrm{sat}\big(-bz(t) + b\tau_2\dot{z}(\theta_2) + g(t)\big)$ for some $\theta_2 \in (t-\tau_2, t)$ and from $|b\tau_2\dot{z}(\theta_2) + g(t)| \leq b^2\tau_2\|z\|_{[t-2\tau_2,t]} + (b\tau_2+1)\|g\|_{[t-\tau_2,t]} \leq \max\{(1+\kappa_2)b^2\tau_2\|z\|_{[t-2\tau_2,t]}, (1+1/\kappa_2)(b\tau_2+1)\|g\|_{[t-\tau_2,t]}\} \leq (b-\varepsilon_2)|z(t)|$ that $z(t)\dot{z}(t) \leq z(t)\,\mathrm{sat}\big(-bz(t) + (b-\varepsilon_2)z(t)\big) = -|z(t)|\,\mathrm{sat}(\varepsilon_2|z(t)|)$.

if $\frac{(1+\kappa_1)a^2\tau_1}{a-\varepsilon_1} < 1$ *and* $\frac{(1+\kappa_2)b^2\tau_2}{b-\varepsilon_2} < 1$*, then there exist* $\beta_{\kappa_1,\varepsilon_1} \in \mathcal{KL}$*,* $\beta_{\kappa_2,\varepsilon_2} \in \mathcal{KL}$ *such that, for any* $x_0 \triangleq [(u_{1-}, y_{1-})] \in \mathfrak{S}_G$*,* $z_0 \triangleq [(u_{2-}, y_{2-})] \in \mathfrak{S}_H$*, and any* $u_{1+} \in \mathcal{U}_e^+$*,* $y_{2+} \in \mathcal{Y}_e^+$*, and any* $t > 0$*,*

$$\begin{aligned}|y_{1+}(t)| &\leq \beta_{\kappa_1,\varepsilon_1}(\|y_{1-}\|_{[-2\tau_1,0]}, t) + \gamma_1(\|u_{1+}\|_{[0,t)})\\ &\leq \beta_{\kappa_1,\varepsilon_1}(\chi(x_0), t) + \gamma_1(\|u_{1+}\|_{[0,t)}),\\ |u_{2+}(t)| &\leq \beta_{\kappa_2,\varepsilon_2}(\|u_{2-}\|_{[-2\tau_2,0]}, t) + \gamma_2(\|y_{2+}\|_{[0,t)})\\ &\leq \beta_{\kappa_2,\varepsilon_2}(\chi(z_0), t) + \gamma_2(\|y_{2+}\|_{[0,t)}),\end{aligned}$$

with the real-valued function χ *defined in (3.15) and two nonlinear gain function* γ_1 *and* γ_2 *defined as follows*

$$\begin{aligned}\gamma_1(s) &= \frac{(1+1/\kappa_1)(a\tau_1+1)}{a-\varepsilon_1}\,|\varepsilon|\,(e^s-1),\ \ \forall s \geq 0,\\ \gamma_2(s) &= \frac{(1+1/\kappa_2)(b\tau_2+1)}{b-\varepsilon_2}\,\mathrm{sat}(s),\ \ \forall s \geq 0.\end{aligned}$$

Theorem 5.2 now asserts that, for the interconnected system $[G, H]$*, the inequalities (5.6) will hold if there exist two functions* $\rho_1(s), \rho_2(s), s \geq 0$ *of class* $\mathcal{K}_\infty$ *such that*

$$\gamma_1 \circ (I + \rho_1) \circ \gamma_2(s) \leq (I + \rho_2)^{-1}(s), \quad \forall s \geq 0. \tag{5.24}$$

Graphically, the above inequality (5.24) is equivalent to say that the distance between the curves $(x, \gamma_2(x))$ *and* $(\gamma_1(y), y)$ *grows without bound in the first quadrant of Cartesian coordinate system* (x, y)*. So, if* $\gamma_1 \circ \gamma_2(1) < 1$*, then (5.24) will be satisfied for some functions* ρ_1, ρ_2 *of class* $\mathcal{K}_\infty$*.*

Hence, for the interconnected system $[G, H]$*, the inequalities (5.6) will hold if the parameters* $\varepsilon \in \mathbb{R}, \tau_1 > 0, \tau_2 > 0$ *satisfying*

$$\begin{cases}\tau_1 < \tau_1^* \triangleq \dfrac{a-\varepsilon_1}{(1+\kappa_1)a^2}, \quad \tau_2 < \tau_2^* \triangleq \dfrac{b-\varepsilon_2}{(1+\kappa_2)b^2},\\ |\varepsilon| < \dfrac{a-\varepsilon_1}{(1+\frac{1}{\kappa_1})(a\tau_1+1)\{\exp[\frac{(1+1/\kappa_2)(b\tau_2+1)}{b-\varepsilon_2}]-1\}},\end{cases}$$

for any $\kappa_1 > 0$*,* $\kappa_2 > 0$ *and any* $\varepsilon_1 \in (0, a)$*,* $\varepsilon_2 \in (0, b)$*. Note that for any* $\tau_1^* < 1/a$ *and any* $\tau_2^* < 1/b$*, we can always choose* κ_1, κ_2 *and* $\varepsilon_1, \varepsilon_2$ *so that the above inequalities are satisfied.*

5.5 Summary

In this chapter we consider the development of a general nonlinear ISS-type small-gain theorem based on the input/output framework set up in Chapter 3. One major contribution of contribution of this chapter is that we present a nonlinear ISS-type small-gain theorem without the extra "observability" conditions and with complete disconnection between the stability property and the existence, uniqueness properties. The main idea of the proof is motivated by [Jiang et al., 1994]. On one hand this small-gain result can be reviewed as a generalisation of the classical input/output operator type small-gain theorems to incorporate abstract initial conditions, and on the other hand a generalisation of the ISS/IOS framework type small-gain theorems to incorporate more general system classes. An illustrative example is given for systems with time delay and nonzero initial conditions to show the utility of Theorem 5.2 at the end of this chapter.

Intelligence consists of this: that we recognise the similarity of different things and the difference between similar ones.

Montesquieu (1689-1755)

Chapter 6

Connections between Georgiou and Smith's Robust Stability Type Theorems and the Nonlinear Small-Gain Theorems

The small-gain theorem was introduced into the control theory literature for studying the stability of general interconnected systems. It treats the stability problem of feedback systems from a functional analysis point of view, and has been well investigated due to the simplicity of the result that if the loop gain defined in an appropriate sense is less than unity, then the closed-loop system is stable in a suitable sense (see e.g., [Zames, 1966a], [Sandberg, 1964], [Desoer and Vidyasagar, 2009], [Mareels and Hill, 1992], [Hill, 1991]). For linear systems, the small-gain theorem has been used as a basis to derive the robust stability criteria for feedback systems under perturbations (see e.g., [Zhou and Doyle, 1998]).

The robust stability theorem is to the effect that a stabilising controller of the nominal plant provides stability for any plant close to the nominal one in an appropriate sense. For nonlinear systems, Georgiou and Smith [Georgiou and Smith, 1997b] developed an input-output approach using gap metric as an analysis tool for studying the robustness of stability of feedback systems under perturbations; but the derivation of Georgiou and Smith's robust stability theorem for nonlinear feedback systems does not make use of the nonlinear small-gain theorem.

Being inspired by the linear results, we discuss the connections between Georgiou and Smith's robust stability type theorems and the nonlinear small-gain theorems in this chapter. Three versions of the nonlinear small-gain theorems in this chapter are presented.

The first version is the usual one regarding systems as relations (one-to-many mapping) on signal spaces and using $\mathcal{K}_\infty$ functions, in which the stability property is stated without referring to the existence and uniqueness properties of the corresponding feedback systems. A special case of this result is shown to be equivalent to a fundamental robust stability theorem of Georgiou and Smith [Georgiou and Smith, 1997b, Theorem 6] with a slight modification. Note that Teel also developed an independent robust stability result similar to Georgiou and Smith's theorem by incorporating a nonlinear small-gain idea in [Teel, 1996].

The second version of the nonlinear small-gain theorem establishes the existence and boundedness properties simultaneously based on the Schauder's fixed point theorem which requires an extra compactness condition. Existence property is often the first requirement of a feedback system and is in general difficult to be obtained than the uniqueness property [Zeidler, 1986, p. 4]. Thus it is very useful to establish the existence and boundedness properties simultaneously by only using the uniqueness property. This technique has its origin in the classical ordinary differential equations (ODEs) theory with linkage of boundedness and existence to get global solutions (see e.g., [Hirsch et al., 2004, Chapter 7]). In [Willems, 1969, p. 655], the author also indicated the close relation between the questions of existence and boundedness for feedback systems. A type of Georgiou and Smith's robust stability theorem establishing boundedness and existence simultaneously is given by applying a special case of the second version of the nonlinear small-gain theorem. Such consideration can also be found in the paper [French and Bian, 2012] where an affine gain with bias property was adopted.

The local form of the nonlinear small-gain theorem was considered in the paper [Zheng and Zafiriou, 1999] by using the contracting mapping theorem. In this chapter, we give a different local version of the nonlinear small-gain theorem by still using the Schauder's fixed point theorem, and use a special case of the result to show a local version of Georgiou and Smith's robust stability theorem.

6.1 Mathematical Preliminaries

In this section we introduce some further notations used in the sequel. Signal spaces defined on doubly infinite time domain have been discussed in detain in Chapter 3 (see Section 3.2). In this chapter we restrict ourself to signal spaces defined on positive time domain; and for simplicity the superscript '$+$' in $\mathcal{V}$ (resp., $\mathcal{V}_e$) is dropped.

Let $\mathcal{S}_\omega$ ($\forall\, \omega \in (0, \infty]$) denote the set of all measurable maps from $[0, \omega)$ to some normed vector space X (e.g., $X = \mathbb{R}^n$). For any $\tau \in (0, \omega)$, the truncation operator $T_\tau : \mathcal{S}_\omega \to \mathcal{S}_\infty$

and the restriction operator $R_\tau : \mathcal{S}_\omega \to \mathcal{S}_\tau$ are defined as follows:

$$T_\tau : \mathcal{S}_\omega \to \mathcal{S}_\infty, \quad v \mapsto T_\tau v \triangleq \left(t \mapsto \begin{cases} v(t), & t \in [0,\tau] \\ 0, & \text{otherwise} \end{cases} \right).$$
$$R_\tau : \mathcal{S}_\omega \to \mathcal{S}_\tau, \quad v \mapsto R_\tau v \triangleq \Big(t \mapsto v(t), \quad t \in [0,\tau] \Big).$$

For ease of notation we use $v_\tau \triangleq T_\tau v$. Suppose that $\mathcal{V} \subseteq \mathcal{S}_\infty$ is a normed vector space with norm $\|\cdot\| = \|\cdot\|_{\mathcal{V}}$; we can define a norm $\|\cdot\|_\tau$ on $\mathcal{S}_\tau$ by $\|v\|_\tau = \|v_\tau\|$ for $v \in \mathcal{S}_\tau$ (define $\|v\|_\tau \triangleq \infty$ if $T_\tau v \in \mathcal{S}_\infty \setminus \mathcal{V}$). A *signal space* $\mathcal{V}_e$ is an extended vector space defined by

$$\mathcal{V}_e \triangleq \{ v \in \mathcal{S}_\infty \mid \forall \tau \in (0,\infty) : v_\tau \in \mathcal{V} \}.$$

We call the elements of $\mathcal{V}$ *bounded signals* and those of $\mathcal{V}_e$ *finite time bounded signals*.

To find the relationships between variations of small-gain theorem and Georgiou & Smith's robust stability theorem, it is necessary to define systems as relations rather than operators. A *relation* R between two nonempty sets S_1 and S_2 is a subset of the Cartesian product $S_1 \times S_2$; and we simply say that the relation is on S_1 if $S_1 = S_2$. Suppose that R is a relation between S_1 and S_2. Then we say that $x \in S_1$ is related to $y \in S_2$ if the ordered pair $(x, y) \in R$. The subset D_R of S_1 defined below

$$D_R \triangleq \{ x \in S_1 \mid \exists\, y \in S_2 \text{ s.t. } (x, y) \in R \}$$

is called the *domain* of R. The *image* $R(x)$ of an element $x \in S_1$ under R is defined by

$$R(x) \triangleq \{ y \in S_2 \mid (x, y) \in R \}.$$

Similarly, the *image* $R(A)$ of a subset $A \subseteq S_1$ under R is defined by

$$R(A) \triangleq \{ y \in S_2 \mid (x, y) \in R \text{ for some } x \in A \}.$$

Thus we have $R(x) = R(\{x\})$ for any $x \in S_1$. The *inverse* relation R^{-1} of R is defined by

$$R^{-1} \triangleq \{ (y, x) \mid (x, y) \in R \}.$$

Assume that $f : S_1 \to S_2$ is a map. Then f defines a relation R_f between S_1 and S_2, i.e.,

$$R_f = \{ (x, f(x)) \mid x \in S_1 \}$$

with domain S_1, and we have $f(x) = R_f(x), \forall x \in S_1$ and $f(A) = R_f(A), \forall A \subseteq S_1$. Note that not all relations are of the above form for maps. Consider a relation R between S_1 and S_2, then the image $R(x)$ might be an empty or a multivalued set for some $x \in S_1$. Thus relations can be viewed as a generalisation of maps defined in Section 2.1 to include multivalued maps, and whose domain need not be the whole of S_1.

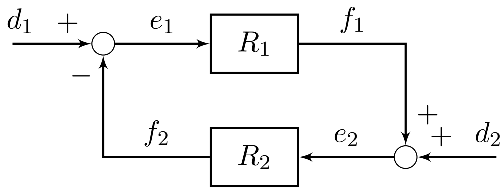

Figure 6.1: Feedback configuration of general small-gain theorem

We next introduce the notion of causal relations, which generalises the notion of causal operators and gives a key insight for the definition of causal systems defined on doubly infinite time domain in Chapter 3 (see Section 3.4.1). A relation R between two signal spaces $\mathcal{V}_{1e}$ and $\mathcal{V}_{2e}$ is said to be *causal* if

$$\forall (x_i, y_i) \in R, i = 1, 2, \forall \tau > 0 : [T_\tau x_1 = T_\tau x_2 \Rightarrow T_\tau(R(x_1)) = T_\tau(R(x_2))] \,.$$

Note that an operator $\Phi : \mathcal{V}_{1e} \to \mathcal{V}_{2e}$ is said to be causal if and only if

$$\forall u, v \in \mathcal{V}_{1e}, \forall \tau > 0 : [u_\tau = v_\tau \Rightarrow (\Phi u)_\tau = (\Phi v)_\tau] \,.$$

The following notion of stability is considered throughout this chapter, which is a generalisation initiated in [Mareels and Hill, 1992] of the classical finite-gain stability (i.e., linear gain [Zames, 1966b]).

A causal relation R between two signal spaces $\mathcal{V}_{1e}$ and $\mathcal{V}_{2e}$ is said to be *stable* if there exists a function $\gamma \in \mathcal{K}_\infty$ such that $\|y\|_\tau \leq \gamma(\|x\|_\tau), \forall (x, y) \in R, \forall \tau > 0$. Specifically, a causal operator $\Phi : \mathcal{V}_{1e} \to \mathcal{V}_{2e}$ is said to be *stable* if there exists a function $\gamma \in \mathcal{K}_\infty$ such that $\|\Phi u\|_\tau \leq \gamma(\|u\|_\tau), \forall u \in \mathcal{V}_{1e}, \forall \tau > 0$.

We call γ a *gain* function of the stable relation R. We remark that gain stability with bias (see [Desoer and Vidyasagar, 2009, Chapter III] or [Vidyasagar, 1993, Chapter 6]) can also be considered with slight modifications. This is a special case of the ISS-like notion of stability given in Section 3.6.

We next introduce some basic properties of feedback systems shown in Figure 6.1, which gives a key insight for their generalisations to input-output systems defined over a doubly infinite time axis in previous Chapter 3.

Consider the basic feedback system shown in Figure 6.1 with two signal spaces $\mathcal{V}_{1e}$ and $\mathcal{V}_{2e}$. Let $R_1 \subseteq \mathcal{V}_{1e} \times \mathcal{V}_{2e}$ and $R_2 \subseteq \mathcal{V}_{2e} \times \mathcal{V}_{1e}$ be two causal relations representing the two subsystems. Signals e_1, e_2 are inputs to the subsystems R_1, R_2 and f_1, f_2 are the corresponding output signals. The scheme of Figure 6.1 is just a symbolic description of the functional equations

$$\begin{cases} e_1 = d_1 - f_2, \; e_2 = d_2 + f_1, \\ \text{and } (e_i, f_i) \in R_i \text{ for } i = 1, 2, \end{cases}$$

where d_1, d_2 are the inputs and e_1, e_2, f_1, f_2 are the outputs with respect to the feedback system.

For $\Omega \subseteq \mathcal{V}_e \triangleq \mathcal{V}_{1e} \times \mathcal{V}_{2e}$ the feedback system shown in Figure 6.1 is said to

- have the *existence property on* Ω if for any $(d_1, d_2) \in \Omega$, there exists a $(e_1, e_2) \in \mathcal{V}_e$ such that $(e_1, e_2 - d_2) \in R_1$ and $(e_2, d_1 - e_1) \in R_2$.
- have the *uniqueness property on* Ω if for any $(d_1, d_2) \in \Omega$ and any $(e_1, e_2) \in \mathcal{V}_e$, $(\tilde{e}_1, \tilde{e}_2) \in \mathcal{V}_e$ with $(e_1, e_2 - d_2) \in R_1$, $(e_2, d_1 - e_1) \in R_2$ and $(\tilde{e}_1, \tilde{e}_2 - d_2) \in R_1$, $(\tilde{e}_2, d_1 - \tilde{e}_1) \in R_2$, we have $(e_1, e_2) = (\tilde{e}_1, \tilde{e}_2)$.
- be *causal on* Ω if the defined (feedback) relation R_f between Ω and $\mathcal{V}_e$ is causal, where $R_f \triangleq \{((d_1, d_2), (e_1, e_2)) \in \Omega \times \mathcal{V}_e \mid (e_1, e_2 - d_2) \in R_1, (e_2, d_1 - e_1) \in R_2\}$.
- be *stable on* Ω if it is causal on Ω and the corresponding (feedback) relation R_f between Ω and $\mathcal{V}_e$ is stable.
- be *well-posed on* Ω if it is locally causal on Ω and has both the existence and uniqueness properties on Ω.

6.2 Small-Gain Theorem and Georgiou & Smith's Robust Stability Theorem

This section contains variations of small-gain theorem (see e.g., [Desoer and Vidyasagar, 2009]) and Georgiou & Smith's robust stability theorem [Georgiou and Smith, 1997b].

6.2.1 Small-Gain Theorem

The small-gain theorem is a very general theorem, which gives an analysis tool for studying the stability of feedback systems. The following is a variation of the small-gain result from [Desoer and Vidyasagar, 2009, Chapter 3].

Theorem 6.1 (Small-Gain Theorem)**.** *Consider the feedback system shown in Figure 6.1 with two signal spaces $\mathcal{V}_{1e}$, $\mathcal{V}_{2e}$. Suppose that the two causal relations $R_1 \subseteq \mathcal{V}_{1e} \times \mathcal{V}_{2e}$, $R_2 \subseteq \mathcal{V}_{2e} \times \mathcal{V}_{1e}$ are stable with gains $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ respectively, and that the feedback system is causal on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$. Let $(e_i, f_i) \in R_i$, $i = 1, 2$ and* define *$d_1 = e_1 + f_2$, $d_2 = e_2 - f_1$. Suppose that there exist two functions ρ, $\varepsilon \in \mathcal{K}_\infty$ such that*

$$\gamma_2 \circ (I + \rho) \circ \gamma_1(s) \le (I + \varepsilon)^{-1}(s), \quad \forall s \ge 0. \tag{6.1}$$

Then the feedback system is stable on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$ *and for any* $\tau > 0$,

$$\|e_1\|_\tau \le (I + \varepsilon^{-1}) \left(\|d_1\|_\tau + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau)\right), \tag{6.2}$$

$$\|e_2\|_\tau \le (I + \rho^{-1}) \left(\|d_2\|_\tau + \gamma_1 \circ (I + \varepsilon^{-1})(\|d_1\|_\tau)\right). \tag{6.3}$$

Proof. The proof of this result is slightly modified from that of [Desoer and Vidyasagar, 2009, Chapter 3].

Since relations R_1, R_2 are stable with gains γ_1, $\gamma_2 \in \mathcal{K}_\infty$, respectively, it follows from $(e_i, f_i) \in R_i$, $i = 1, 2$ that $\|f_i\|_\tau \le \gamma_i(\|e_i\|_\tau)$, $\forall \tau > 0$. Since $d_1 = e_1 + f_2$ and $d_2 = e_2 - f_1$, we have

$$\|e_1\|_\tau \le \gamma_2 \left(\gamma_1(\|e_1\|_\tau) + \|d_2\|_\tau\right) + \|d_1\|_\tau, \quad \forall \tau > 0. \tag{6.4}$$

By using inequalities (2.19) on page 30, we get

$$\|e_1\|_\tau \le \gamma_2 \circ (I + \rho) \circ \gamma_1(\|e_1\|_\tau) + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau) + \|d_1\|_\tau, \quad \forall \tau > 0. \tag{6.5}$$

From (6.1) and (6.5), we have

$$\|e_1\|_\tau \le \left(I - (I + \varepsilon)^{-1}\right)^{-1} \left(\gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau) + \|d_1\|_\tau\right), \quad \forall \tau > 0.$$

This in turn implies (6.2). Similarly, we also obtain (6.3). □

Note that the inequality (6.1) is equivalent to the following inequality

$$\gamma_1 \circ (I + \varepsilon) \circ \gamma_2(r) \le (I + \rho)^{-1}(r), \quad \forall r \ge 0. \tag{6.6}$$

This is readily to be seen by letting $s = (I + \varepsilon) \circ \gamma_2(r)$ for any $r \ge 0$ in (6.1). In fact, we have $\gamma_2 \circ (I + \rho) \circ \gamma_1 \circ (I + \varepsilon) \circ \gamma_2(r) \le \gamma_2(r)$, and then by applying $(I + \rho)^{-1} \circ \gamma_2^{-1}(\cdot)$ on both sides, we obtain (6.6).

Weaker stability conditions can be formulated for feedback systems shown in Figure 6.1 with $d_2 \equiv 0$.

Theorem 6.2 (Small-Gain Theorem with $d_2 \equiv 0$)**.** *Consider the feedback structure of Figure 6.1 with* $d_2 \equiv 0$*. Let* $\mathcal{V}_{1e}$, $\mathcal{V}_{2e}$ *be two signal spaces. Suppose that the two causal relations* $R_1 \subseteq \mathcal{V}_{1e} \times \mathcal{V}_{2e}$, $R_2 \subseteq \mathcal{V}_{2e} \times \mathcal{V}_{1e}$ *are stable with gains* $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ *respectively, and that the feedback system is causal on* $\mathcal{V}_{1e} \times \{0\}$*. Let* $(e_i, f_i) \in R_i$, $i = 1, 2$ *and let* $d_1 = e_1 + f_2$, $e_2 = f_1$*. Suppose that there exists a function* $\varepsilon \in \mathcal{K}_\infty$ *such that*

$$\gamma_2 \circ \gamma_1(s) \le (I + \varepsilon)^{-1}(s), \quad \forall s \ge 0. \tag{6.7}$$

Then the feedback system is stable on $\mathcal{V}_{1e} \times \{0\}$*, and* $\|e_1\|_\tau \le (I + \varepsilon^{-1})(\|d_1\|_\tau)$ *and* $\|e_2\|_\tau \le \gamma_1 \circ (I + \varepsilon^{-1})(\|d_1\|_\tau)$ *for any* $\tau > 0$.

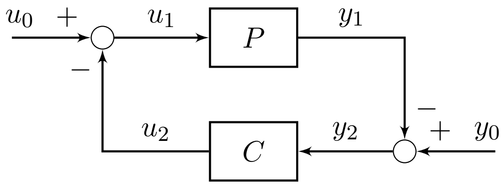

Figure 6.2: Closed-loop systems of Georgiou and Smith's theorem

Proof. Condition (6.4) simplifies to $\|e_1\|_\tau \le \gamma_2 \circ \gamma_1(\|e_1\|_\tau) + \|d_1\|_\tau$, $\forall \tau > 0$. Thus from (6.7) we obtain the conclusion. □

Although the condition (6.7) of Theorem 6.2 is weaker than the condition (6.1) of Theorem 6.1, Theorem 6.2 can be implied by Theorem 6.1 due to the extra condition $d_2 \equiv 0$.

Theorem 6.3. *If Theorem 6.1 is true, then Theorem 6.2 is true.*

The task is as follows: under the premise of Theorem 6.2, we need to use Theorem 6.1 to establish the conclusion of Theorem 6.2.

Proof. Given $(e_i, f_i) \in R_i$ with $i = 1, 2$ and $d_1 = e_1 + f_2$ and $0 = d_2 = e_2 - f_1$. For any positive real number $\delta > 0$, there exist a function $\varepsilon_\delta \in \mathcal{K}_\infty$ such that

$$(I+\varepsilon)^{-1}(s) < (I+\varepsilon_\delta)^{-1}(s) \le (I+\varepsilon)^{-1}(s) + \delta, \quad \forall s \ge 0.$$

From the condition (6.7), we have

$$\gamma_2 \circ \gamma_1(s) < (I+\varepsilon_\delta)^{-1}(s), \quad \forall s \ge 0.$$

Thus there exists a function $\rho_\delta \in \mathcal{K}_\infty$ such that

$$\gamma_2 \circ (I+\rho_\delta) \circ \gamma_1(s) \le (I+\varepsilon_\delta)^{-1}(s), \quad \forall s \ge 0.$$

By using Theorem 6.1, we have

$$\|e_1\|_\tau \le (I+\varepsilon_\delta^{-1}) \left(\|d_1\|_\tau + \gamma_2 \circ (I+\rho_\delta^{-1})(\|d_2\|_\tau)\right), \quad \forall \tau > 0.$$

It follows from $d_2 = 0$ that $\|e_1\|_\tau \le (I+\varepsilon_\delta^{-1})(\|d_1\|_\tau) \le (I+\varepsilon)^{-1}(\|d_1\|_\tau) + \delta$ for any $\tau > 0$. Since δ can be chosen to be any positive real number, we have $\|e_1\|_\tau \le (I+\varepsilon)^{-1}(\|d_1\|_\tau)$ and thus $\|e_2\|_\tau = \|f_1\|_\tau \le \gamma_1(\|e_1\|_\tau) \le \gamma_1 \circ (I+\varepsilon)^{-1}(\|d_1\|_\tau)$ for any $\tau > 0$. This completes the proof of Theorem 6.2. □

6.2.2 Georgiou & Smith's Robust Stability Theorem

Consider the closed-loop system shown in Figure 6.2 with input (u_0, y_0) and output (u_1, y_1). Signals u_1, y_2 are inputs to the plant P and controller C, and y_1, u_2 are the corresponding output signals.

Assumption 6.4. *Consider the feedback configuration of Figure 6.2 with two signal spaces $\mathcal{U}_e$ and $\mathcal{Y}_e$. The plant and controller are causal operators $P : \mathcal{U}_e \to \mathcal{Y}_e$ and $C : \mathcal{Y}_e \to \mathcal{U}_e$ which satisfy $P0 = 0$ and $C0 = 0$. The closed-loop system is well-posed on $\mathcal{U}_e \times \mathcal{Y}_e$.*

Note that by Assumption 6.4 the closed-loop system of Figure 6.2 is well-posed on $\mathcal{U}_e \times \mathcal{Y}_e$, that is to say that for any input $(u_0, y_0) \in \mathcal{W}_e \triangleq \mathcal{U}_e \times \mathcal{Y}_e$ there exist unique signals u_1, $u_2 \in \mathcal{U}_e$ and y_1, $y_2 \in \mathcal{Y}_e$ such that $w_0 = w_1 + w_2$ with $w_i \triangleq (u_i, y_i)$, $i = 0, 1, 2$, and $y_1 = Pu_1$, $u_2 = Cy_2$; moreover, if we define a closed-loop operator $\Pi_{P//C} : \mathcal{W}_e \to \mathcal{W}_e$ by $w_0 \mapsto w_1$, then $\Pi_{P//C}$ is causal.

The *graph* of P is defined by

$$\mathcal{G}_P \triangleq \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} : u \in \mathcal{U}_e,\, Pu \in \mathcal{Y}_e \right\} \subseteq \mathcal{W}_e \triangleq \mathcal{U}_e \times \mathcal{Y}_e$$

and the graph (or called the inverse graph) of C is defined by

$$\mathcal{G}_C \triangleq \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} : Cy \in \mathcal{U}_e,\, y \in \mathcal{Y}_e \right\} \subseteq \mathcal{W}_e.$$

Theorem 6.5 (Georgiou & Smith's Robust Stability Theorem)**.** *Consider the feedback configuration of Figure 6.2 under Assumption 6.4. Suppose that $\Pi_{P//C}$ is stable with gain $\gamma_1 \in \mathcal{K}_\infty$. Assume that the plant P is perturbed to be another plant $\tilde{P}$, and that Assumption 6.4 also holds for the interconnection of $\tilde{P}$ and C. If there exist a map $\psi : \mathcal{G}_{\tilde{P}} \to \mathcal{G}_P$ and a function $\gamma_2 \in \mathcal{K}_\infty$ such that the inverse relation R_ψ^{-1} of the relation R_ψ deduced from the map ψ satisfy*

$$\|\tilde{w}_1 - w_1\|_\tau \leq \gamma_2(\|w_1\|_\tau), \quad \forall (w_1, \tilde{w}_1) \in R_\psi^{-1}, \forall \tau > 0, \tag{6.8}$$

then $\Pi_{\tilde{P}//C}$ is stable on $\mathcal{W}_e$ and for any $\tilde{w}_0 \in \mathcal{W}_e$, $\tau > 0$,

$$\left\| \psi \circ \Pi_{\tilde{P}//C} \tilde{w}_0 \right\|_\tau \leq \gamma_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau), \tag{6.9}$$

$$\left\| \Pi_{\tilde{P}//C} \tilde{w}_0 \right\|_\tau \leq (I + \gamma_2) \circ \gamma_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau), \tag{6.10}$$

provided: the inequality $\gamma_2 \circ \gamma_1(s) \leq (I + \varepsilon)^{-1}(s)$, $\forall s \geq 0$ holds for some function $\varepsilon \in \mathcal{K}_\infty$.

We remark that Theorem 6.5 is based on Georgiou and Smith's robust stability theorem ([Georgiou and Smith, 1997b, Theorem 6]). The relation R_ψ^{-1} has a simpler form $R_\psi^{-1} =$

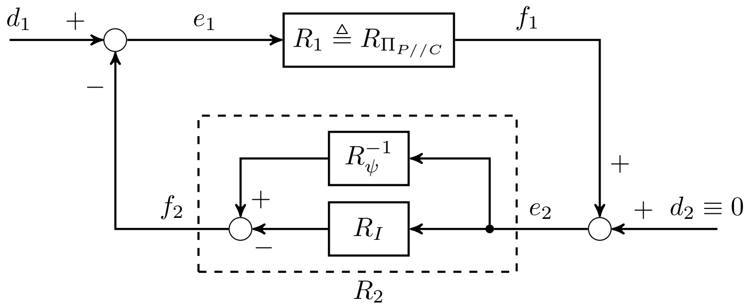

Figure 6.3: Deduction of Theorem 6.5 from Theorem 6.2

R_ϕ which is deduced from an onto map $\phi : D \subseteq \mathcal{G}_P \to \mathcal{G}_{\tilde{P}}$ in ([Georgiou and Smith, 1997b, Theorem 6]); and the condition (6.8) in this case is equivalent to $\phi - I$ being stable (on $D \subseteq \mathcal{G}_P$) with gain γ_2. Here ϕ is allowed to be a multivalued map. In the following we give an alternative proof for Theorem 6.5 by using small-gain theorem. The essential idea lies in Figure 6.3 (Note that R_I represents a relation deduced from an identity map I).

6.2.3 Equivalence of the Small-Gain Theorem and the Georgiou & Smith's Robust Stability Theorem

We first argue that the small-gain Theorem 6.2 implies the Georgiou & Smith's robust stability Theorem 6.5.

Theorem 6.6. *If Theorem 6.2 is true, then Theorem 6.5 is true.*

Proof. The task is as follows: under the premises of Theorem 6.5, we need to establish the conclusions of Theorem 6.5 by using the small-gain Theorem 6.2. The main idea lies in Figure 6.3.

Since $\Pi_{P//C}$ is stable with gain $\gamma_1 \in \mathcal{K}_\infty$, it follows that the relation $R_1 \triangleq R_{\Pi_{P//C}}$ is stable with gain γ_1. Define a relation R_2 on $\mathcal{W}_e$ by

$$R_2 \triangleq \{(w_1, \tilde{w}_1 - w_1) \in \mathcal{W}_e \times \mathcal{W}_e \mid (w_1, \tilde{w}_1) \in R_\psi^{-1}\}.$$

It follows from (6.8) that the above relation R_2 is stable with gain $\gamma_2 \in \mathcal{K}_\infty$.

For any $\tilde{w}_0 \in \mathcal{W}_e$, from Assumption 6.4 for the interconnection of $\tilde{P}$ and C, we have $\Pi_{\tilde{P}//C}\tilde{w}_0 \in \mathcal{G}_{\tilde{P}}$, and thus $\tilde{w}_0 - \Pi_{\tilde{P}//C}\tilde{w}_0 \in \mathcal{G}_C$ and $\psi \circ \Pi_{\tilde{P}//C}\tilde{w}_0 \in \mathcal{G}_P$. Define

$$d_1 \triangleq \tilde{w}_0, \quad d_2 \triangleq 0, \quad e_1 \triangleq (\tilde{w}_0 - \Pi_{\tilde{P}//C}\tilde{w}_0) + \psi \circ \Pi_{\tilde{P}//C}\tilde{w}_0,$$
$$e_2 = f_1 \triangleq \psi \circ \Pi_{\tilde{P}//C}\tilde{w}_0, \quad f_2 \triangleq \Pi_{\tilde{P}//C}\tilde{w}_0 - \psi \circ \Pi_{\tilde{P}//C}\tilde{w}_0.$$

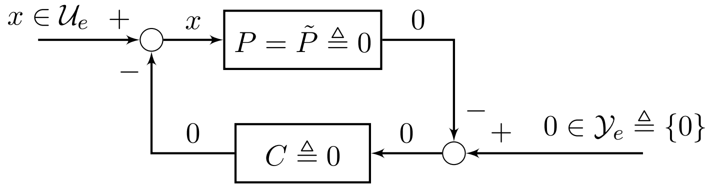

Figure 6.4: Deduction of Theorem 6.2 from Theorem 6.5

Since $(e_2, e_2+f_2) = (\psi\circ\Pi_{\tilde{P}//C}\tilde{w}_0, \Pi_{\tilde{P}//C}\tilde{w}_0) \in R_\psi^{-1}$, we have $(e_2, f_2) \in R_2$. By using Assumption 6.4 for the interconnection of P and C, we also have $\Pi_{P//C}e_1 = \psi\circ\Pi_{\tilde{P}//C}\tilde{w}_0 = f_1$ and this in turn implies $(e_1, f_1) \in R_1 \triangleq R_{\Pi_{P//C}}$.

Thus, for the feedback structure of Figure 6.3, the relations R_1, R_2 on $\mathcal{W}_e$ are stable with gains $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$, respectively, and $(e_i, f_i) \in R_i$, $i = 1, 2$ and $d_1 = e_1 + f_2$, $0 = d_2 = e_2 - f_1$. Since $\gamma_2 \circ \gamma_1(s) \leq (I+\varepsilon)^{-1}(s)$, $\forall s \geq 0$ with $\varepsilon \in \mathcal{K}_\infty$, by applying Theorem 6.2 with $\mathcal{V}_{1e} = \mathcal{V}_{2e} = \mathcal{W}_e$, we obtain

$$\left\|\psi \circ \Pi_{\tilde{P}//C}\tilde{w}_0\right\|_\tau = \|e_2\|_\tau \leq \gamma_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau) \tag{6.11}$$

for any $\tau > 0$. Since $(\psi \circ \Pi_{\tilde{P}//C}\tilde{w}_0, \Pi_{\tilde{P}//C}\tilde{w}_0) \in R_\psi^{-1}$, by using (6.8) and (6.11), we get

$$\left\|\Pi_{\tilde{P}//C}\tilde{w}_0\right\| \leq (I + \gamma_2) \circ \gamma_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau), \quad \forall \tau > 0.$$

Since $\|\tilde{w}_0\|_\tau$ is arbitrarily chosen from $\mathcal{W}_e$, we obtain that $\Pi_{\tilde{P}//C}$ is stable and (6.9), (6.10) holds. □

Next we show that the small-gain Theorem 6.2 can also be derived from the Georgiou & Smith's robust stability Theorem 6.5.

Theorem 6.7. *If Theorem 6.5 is true, then Theorem 6.2 is true.*

Proof. The task is as follows: under the premises of Theorem 6.2, we need to establish the conclusions of Theorem 6.2 by using the Georgiou & Smith's robust stability Theorem 6.5. The main idea lies in Figure 6.4.

Consider any fixed $(e_i, f_i) \in R_i$, $i = 1, 2$ and $d_1 \in \mathcal{V}_{1e}$ with $d_1 = e_1 + f_2$ and $e_2 = f_1$. Note that $e_1, f_2 \in \mathcal{V}_{1e}$ and $e_2, f_1 \in \mathcal{V}_{2e}$. Let $\mathcal{U}_e \triangleq \mathcal{V}_{1e}, \mathcal{Y}_e \triangleq \{0\}$ and $\mathcal{W}_e \triangleq \mathcal{U}_e \times \mathcal{Y}_e$. Define the nominal plant $P : \mathcal{U}_e \to \mathcal{Y}_e$, the perturbed plant $\tilde{P} : \mathcal{U}_e \to \mathcal{Y}_e$ and the controller $C : \mathcal{Y}_e \to \mathcal{U}_e$ as follows: $P(x) = \tilde{P}(x) = 0$, $\forall x \in \mathcal{U}_e$ and $C(0) = 0$ (see Figure 6.4). It's trivial to see that Assumption 6.4 holds for the interconnection of P (also $\tilde{P}$) and C. The corresponding operator $\Pi_{P//C} : \mathcal{W}_e \to \mathcal{W}_e$ is defined by $\Pi_{P//C}(w_0) = w_0$, $\forall w_0 \in \mathcal{W}_e \triangleq \mathcal{U}_e \times \{0\}$. It is easy to see that $\Pi_{P//C}$ is stable with gain $\tilde{\gamma}_1 \in \mathcal{K}_\infty$, where $\tilde{\gamma}_1(s) = s$, $\forall s \geq 0$. Note that $\mathcal{G}_P = \mathcal{G}_{\tilde{P}} = \mathcal{W}_e = \mathcal{U}_e \times \{0\}$. Next we define a map $\psi : \mathcal{G}_{\tilde{P}} \to \mathcal{G}_P$ as follows: $\psi\left(\binom{x}{0}\right) \triangleq \binom{x}{0}$ for any $x \in \mathcal{U}_e$ with $x \neq d_1$, and

$\psi\left(\left(\begin{smallmatrix} d_1 \\ 0 \end{smallmatrix}\right)\right) \triangleq \left(\begin{smallmatrix} e_1 \\ 0 \end{smallmatrix}\right)$. Note that $\left\|\left(\begin{smallmatrix} d_1 \\ 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} e_1 \\ 0 \end{smallmatrix}\right)\right\|_\tau = \left\|\left(\begin{smallmatrix} f_2 \\ 0 \end{smallmatrix}\right)\right\|_\tau \leq \gamma_2(\|e_2\|_\tau) = \gamma_2(\|f_1\|_\tau) \leq \gamma_2 \circ \gamma_1(\|e_1\|_\tau) = \gamma_2 \circ \gamma_1\left(\left\|\left(\begin{smallmatrix} e_1 \\ 0 \end{smallmatrix}\right)\right\|_\tau\right)$. It is not hard to check that the inverse relation R_ψ^{-1} of the relation R_ψ deduced from the map ψ satisfies

$$\|\tilde{w}_1 - w_1\|_\tau \leq \gamma_2 \circ \gamma_1(\|w_1\|_\tau), \quad \forall (w_1, \tilde{w}_1) \in R_\psi^{-1}, \forall \tau > 0,$$

Define $\tilde{\gamma}_2 \triangleq \gamma_2 \circ \gamma_1$. From the condition of Theorem 6.2:

$$\gamma_2 \circ \gamma_1(s) \leq (I + \varepsilon)^{-1}(s), \forall s \geq 0,$$

for some function $\varepsilon \in \mathcal{K}_\infty$, we have $\tilde{\gamma}_2 \circ \tilde{\gamma}_1(s) \leq (I + \varepsilon)^{-1}(s)$, $\forall s \geq 0$. By Theorem 6.5, we have

$$\left\|\psi \circ \Pi_{\tilde{P}//C} \tilde{w}_0\right\|_\tau \leq \tilde{\gamma}_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau)$$

for any $\tilde{w}_0 \in \mathcal{W}_e$, $\tau > 0$. Choose $\tilde{w}_0 = \begin{pmatrix} d_1 \\ 0 \end{pmatrix}$, thus from above inequality and $\Pi_{\tilde{P}//C}\begin{pmatrix} d_1 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1 \\ 0 \end{pmatrix}$ and $\psi\left(\begin{pmatrix} d_1 \\ 0 \end{pmatrix}\right) = \left(\begin{smallmatrix} e_1 \\ 0 \end{smallmatrix}\right)$ we have

$$\left\|\begin{pmatrix} e_1 \\ 0 \end{pmatrix}\right\|_\tau \leq \tilde{\gamma}_1 \circ (I + \varepsilon^{-1})\left(\left\|\begin{pmatrix} d_1 \\ 0 \end{pmatrix}\right\|_\tau\right)$$

for any $\tau > 0$, i.e.,

$$\|e_1\|_\tau \leq (I + \varepsilon^{-1})(\|d_1\|_\tau), \forall \tau > 0.$$

This completes the proof of Theorem 6.7. □

We have thus shown the equivalence between a version of small-gain theorem and a slight variation of Georgiou & Smith's robust stability theorem.

6.3 Establishing Existence and Boundedness Simultaneously

To establish the well-posedness (i.e., existence and uniqueness) of a closed-loop system, for simplicity we often restrict ourself to the case that the corresponding open-loop subsystems are all well-posed themselves. Thus in the remainder of this chapter, we only consider (open-loop) systems which are defined by operators rather than relations on extended signal spaces.

6.3.1 Small-Gain Theorem–Existence and Boundedness

Traditionally, the small-gain theorem was formulated in a way as Theorem 6.1 that the stability property is completely disconnected from the properties of existence, uniqueness, etc [Desoer and Vidyasagar, 2009, Chapter III]. However, it is of critical importance to give a version of small-gain theorem establishing stability and existence simultaneously [French and Bian, 2012], since both existence and stability are important properties of a feedback system.

For any signal space $\mathcal{V}_e$ and any $\tau \in (0, \infty)$, define an *interval space* $\mathcal{V}[0, \tau)$ by

$$\mathcal{V}[0, \tau) \triangleq \{x \mid \exists y \in \mathcal{V}_e \text{ such that } x = R_\tau y\},$$

where R_τ is the restriction operator defined before.

An operator $Q : \mathcal{V}_{1e} \to \mathcal{V}_{2e}$ is said to be *relatively continuous* [French and Bian, 2012] if for any $\tau > 0$ and any operators $\Phi : \mathcal{V}_{2e} \to \mathcal{V}_{1e}$ with $R_\tau \phi$ compact, the operator $R_\tau \Phi Q : \mathcal{V}_{1e} \to \mathcal{V}_1[0, \tau)$ is continuous. Note that if the operator $R_\tau \phi$ for any $\tau > 0$ is also incrementally stable, i.e.,

$$\|R_\tau \phi x - R_\tau \phi y\|_\tau \leq \gamma(\|R_\tau x - R_\tau y\|_\tau), \quad \forall x, y \in \mathcal{V}_{2e},$$

for some function $\gamma \in \mathcal{K}_\infty$ (related to τ), then the operator Q is relatively continuous if $R_\tau Q$ is continuous for any $\tau > 0$.

We give a version of small-gain theorem which establishes existence and boundedness simultaneously as follows.

Theorem 6.8. *Consider the feedback system shown in Figure 6.1 with two signal spaces $\mathcal{V}_{1e}$ and $\mathcal{V}_{2e}$. Suppose that, for any $\tau \in (0, \infty)$, $\mathcal{V}_1[0, \tau)$ is complete and a continuous extension map $E_{1\tau} : \mathcal{V}_1[0, \tau) \to \mathcal{V}_{1e}$ exists such that $R_\tau x = R_\tau(E_{1\tau} x)$, $\forall x \in \mathcal{V}_1[0, \tau)$. Let $R_1 = H_1 : \mathcal{V}_{1e} \to \mathcal{V}_{2e}$ and $R_2 = H_2 : \mathcal{V}_{2e} \to \mathcal{V}_{1e}$ be two causal operators, which are stable with gains $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ respectively. Suppose that the operator $H_1(\cdot) + x$ is relatively continuous for any fixed $x \in \mathcal{V}_{2e}$, and that $R_\tau H_2 : \mathcal{V}_{2e} \to \mathcal{V}_1[0, \tau)$ is compact for any $\tau \in (0, \infty)$. Assume that the feedback system is causal and satisfies the uniqueness property on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$. If there exist two functions $\rho, \varepsilon \in \mathcal{K}_\infty$ such that*

$$\gamma_2 \circ (I + \rho) \circ \gamma_1(s) \leq (I + \varepsilon)^{-1}(s), \quad \forall s \geq 0. \tag{6.12}$$

Then the feedback system is well-posed on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$. Moreover, it is stable and for any $\tau > 0$,

$$\|e_1\|_\tau \leq (I + \varepsilon^{-1}) \left(\|d_1\|_\tau + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau)\right), \tag{6.13}$$

$$\|e_2\|_\tau \leq \|d_2\|_\tau + \gamma_1(\|e_1\|_\tau). \tag{6.14}$$

Proof. For any input $d = (d_1, d_2) \in \mathcal{V}_{1e} \times \mathcal{V}_{2e}$, consider the equation

$$e_1 = d_1 - H_2(H_1 e_1 + d_2) \tag{6.15}$$

For any $\tau \in (0, \infty)$, define a set M_τ^d by

$$M_\tau^d \triangleq \{x \in \mathcal{V}_1[0,\tau) \mid \|x\|_\tau \le (I + \varepsilon^{-1})\left(\|d_1\|_\tau + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau)\right)\}, \tag{6.16}$$

and an operator Q_τ^d as

$$Q_\tau^d : M_\tau^d \to \mathcal{V}_1[0,\tau), \quad x \mapsto R_\tau d_1 - R_\tau H_2(H_1 E_{1\tau} x + d_2),$$

By our assumptions, Q_τ^d is well defined and continuous in $\mathcal{V}_1[0,\tau)$. Let $x \in M_\tau^d$ and $r_0 \triangleq \|d_1\|_\tau + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau)$, then $\|E_{1\tau} x\|_\tau = \|x\|_\tau \le (I + \varepsilon^{-1})(r_0)$, and so

$$\begin{aligned}
\left\|Q_\tau^d x\right\|_\tau &\le \|R_\tau d_1\|_\tau + \|R_\tau H_2(H_1 E_{1\tau} x + d_2)\|_\tau \\
&\le \|d_1\|_\tau + \gamma_2\left(\gamma_1(\|E_{1\tau} x\|_\tau) + \|d_2\|_\tau\right) \\
&\le \|d_1\|_\tau + \gamma_2 \circ (I + \rho) \circ \gamma_1(\|E_{1\tau} x\|_\tau) + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau) \\
&\le r_0 + \gamma_2 \circ (I + \rho) \circ \gamma_1 \circ (I + \varepsilon^{-1})(r_0).
\end{aligned} \tag{6.17}$$

By using condition (6.12) and the inequality $(I+\varepsilon)^{-1} \circ (I+\varepsilon^{-1})(s) + s = (I+\varepsilon^{-1})(s)$, $\forall s \ge 0$, we have

$$\left\|Q_\tau^d x\right\|_\tau \le (I + \varepsilon^{-1})(r_0) = (I + \varepsilon^{-1})\left(\|d_1\|_\tau + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|_\tau\right).$$

Therefore, $Q_\tau^d(M_\tau^d) \subseteq M_\tau^d$. It follows from the compactness of $R_\tau H_2$ and boundedness of H_1 that Q_τ^d is compact. Since $\mathcal{V}_1[0,\tau)$ is complete, it follows by Schauder's fixed point theorem that Q_τ^d has a fixed point in M_τ^d, i.e., there exists a $x \in M_\tau^d \subseteq \mathcal{V}_1[0,\tau)$ such that $x = Q_\tau^d x = R_\tau d_1 - R_\tau H_2(H_1 E_{1\tau} x + d_2)$. Define $e_1^\tau \triangleq E_{1\tau} x$, we have

$$R_\tau e_1^\tau = R_\tau\left(d_1 - H_2(H_1 e_1^\tau + d_2)\right)$$

Since this holds for all $\tau \in (0, \infty)$, we have (6.15) holds with $e_1 \triangleq \lim_{\tau\to\infty} e_1^\tau \in \mathcal{V}_{1e}$, (note that e_1 is causally related to the input (d_1, d_2) by our assumption). Define $e_2 = d_2 + H_1 e_1 \in \mathcal{V}_{2e}$. We have from (6.15) that $d_1 = e_1 + H_2 e_2$. This shows the existence property on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$ for the feedback system (Figure 6.1). Since it also satisfies the uniqueness property on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$ by our assumption, it follows that it is well-posed on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$. Since $\|R_\tau e_1\|_\tau \le (I + \varepsilon^{-1})\left(\|d_1\| + \gamma_2 \circ (I + \rho^{-1})(\|d_2\|)\right)$, we have (6.13) and thus (6.14) hold. □

The following special case of Theorem 6.8 will be used later to show its corresponding robust stability theorem (see Theorem 6.12).

Theorem 6.9 ($d_2 \equiv 0$)**.** *Consider the feedback structure of Figure 6.1 with* $d_2 \equiv 0$. *Let* $\mathcal{V}_{1e}$, $\mathcal{V}_{2e}$ *be two signal spaces. Suppose that, for any* $\tau \in (0, \infty)$, $\mathcal{V}_1[0, \tau)$ *is complete and a continuous extension map* $E_{1\tau} : \mathcal{V}_1[0, \tau) \to \mathcal{V}_{1e}$ *exists such that* $R_\tau x = R_\tau(E_{1\tau}x)$, $\forall x \in \mathcal{V}_1[0, \tau)$. *Let* $R_1 = H_1 : \mathcal{V}_{1e} \to \mathcal{V}_{2e}$ *and* $R_2 = H_2 : D_{H_2} \subseteq \mathcal{V}_{2e} \to \mathcal{V}_{1e}$ *with* $H_1(\mathcal{V}_{1e}) \subseteq D_{H_2}$ *be two causal operators, which are stable with gains* $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ *respectively. Suppose that* H_1 *is relatively continuous and that* $R_\tau H_2 : D_{H_2} \subseteq \mathcal{V}_{2e} \to \mathcal{V}_1[0, \tau)$ *is compact for any* $\tau \in (0, \infty)$. *Assume that the feedback system is stable and satisfies the uniqueness property on* $\mathcal{V}_{1e} \times \{0\}$. *If there exists a function* $\varepsilon \in \mathcal{K}_\infty$ *such that*

$$\gamma_2 \circ \gamma_1(s) \leq (I + \varepsilon)^{-1}(s), \quad \forall s \geq 0. \tag{6.18}$$

Then the feedback system is well-posed on $\mathcal{V}_{1e} \times \{0\}$. *Moreover, it is stable on* $\mathcal{V}_{1e} \times \{0\}$ *and* $\|e_1\|_\tau \leq (I + \varepsilon^{-1})(\|d_1\|_\tau)$ *and* $\|e_2\|_\tau \leq \gamma_1 \circ (I + \varepsilon^{-1})(\|d_1\|_\tau)$ *for any* $\tau > 0$.

Proof. It follows directly from the proof of Theorem 6.8 and the same approach as in the proof of Theorem 6.2. □

6.3.2 Robust Stability Type Theorem–Existence and Boundedness

The corresponding version of the robust stability theorem is as follows.

Assumption 6.10. *Consider the feedback configuration of Figure 6.2 with two signal spaces* $\mathcal{U}_e$ *and* $\mathcal{Y}_e$. *The plant and controller are causal operators* $P : \mathcal{U}_e \to \mathcal{Y}_e$ *and* $C : \mathcal{Y}_e \to \mathcal{U}_e$ *which satisfy* $P0 = 0$ *and* $C0 = 0$. *The closed-loop system is causal and satisfies the uniqueness property on* $\mathcal{U}_e \times \mathcal{Y}_e$.[1]

Theorem 6.11. *Consider the feedback configuration of Figure 6.2. Let* $\mathcal{U}_e$, $\mathcal{Y}_e$ *be two signal spaces. Define* $\mathcal{W}_e \triangleq \mathcal{U}_e \times \mathcal{U}_e$. *Suppose that, for any* $\tau \in (0, \infty)$, $\mathcal{W}[0, \tau)$ *is complete and a continuous extension map* $E_\tau : \mathcal{W}[0, \tau) \to \mathcal{W}_e$ *exists such that* $R_\tau x = R_\tau(E_\tau x)$, $\forall x \in \mathcal{W}[0, \tau)$. *Suppose the Assumption 6.4 holds for the interconnection of* P *and* C. *Suppose that* $\Pi_{P//C}$ *is stable with gain* $\gamma_1 \in \mathcal{K}_\infty$ *and that* $\Pi_{P//C}$ *is relatively continuous. Assume that the plant* P *is perturbed to be another plant* $\tilde{P}$, *and that the weak Assumption 6.10 holds for the interconnection of* $\tilde{P}$ *and* C *(i.e., replacing* P *by* $\tilde{P}$ *in Assumption 6.10). If there exists a one-to-one map* $\Phi : \mathcal{G}_P \to \mathcal{G}_{\tilde{P}}$ *with* $R_\tau(\Phi - I)$, $\forall \tau \in (0, \infty)$ *compact and a function* $\gamma_2 \in \mathcal{K}_\infty$ *such that*

$$\|(\Phi - I)w_1\|_\tau \leq \gamma_2(\|w_1\|_\tau), \quad \forall w_1 \in \mathcal{G}_P, \forall \tau > 0.$$

Then the feedback interconnection of $\tilde{P}$ *and* C *is well-posed on* $\mathcal{W}_e$, *(i.e., for any* $\tilde{w}_0 \in \mathcal{W}_e$, *there exists a unique* $\tilde{w}_1 \in \mathcal{G}_{\tilde{P}}$ *such that* $\tilde{w}_0 - \tilde{w}_1 \in \mathcal{G}_C$*). Moreover,* $\Pi_{\tilde{P}//C}$ *is stable*

[1] That is to say that for any input $w_0 \in \mathcal{W}_e \triangleq \mathcal{U}_e \times \mathcal{Y}_e$, if there exist signals $w_1, \tilde{w}_1 \in \mathcal{G}_P$ such that $w_0 - w_1, w_0 - \tilde{w}_1 \in \mathcal{G}_C$, then $w_1 = \tilde{w}_1$; moreover, the thus defined closed-loop operator $\Pi_{P//C} : D \subseteq \mathcal{W}_e \to \mathcal{W}_e$, $w_0 \mapsto w_1$ is causal, where the domain of $\Pi_{P//C}$ is denoted by D which contains all of $w_0 \in \mathcal{W}_e$ such that there exists a $w_1 \in \mathcal{G}_P$ with $w_0 - w_1 \in \mathcal{G}_C$.

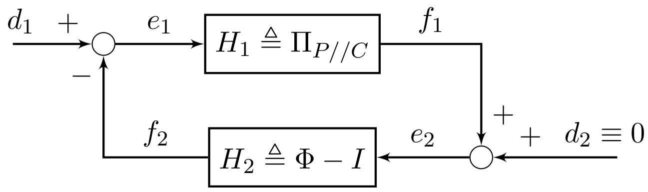

Figure 6.5: Deduction of Theorem 6.11 from Theorem 6.9

on $\mathcal{W}_e$ and

$$\left\|\Pi_{\tilde{P}//C}\tilde{w}_0\right\|_\tau \leq (I+\gamma_2)\circ\gamma_1\circ(I+\varepsilon^{-1})(\|\tilde{w}_0\|_\tau), \quad \forall \tilde{w}_0 \in \mathcal{W}_e, \forall \tau > 0, \tag{6.19}$$

provided: the inequality $\gamma_2\circ\gamma_1(s) \leq (I+\varepsilon)^{-1}(s)$, $\forall s \geq 0$ holds for some function $\varepsilon \in \mathcal{K}_\infty$.

We now come to the relation between these versions of small-gain theorem and robust stability theorem.

Theorem 6.12. *If Theorem 6.9 is true, then Theorem 6.11 is true.*

Proof. The task is as follows: under the premises of Theorem 6.11, we need to establish the conclusions of Theorem 6.11 by using Theorem 6.9. The essential idea lies in Figure 6.5.

From our assumptions in Theorem 6.11, we know that the operator $H_1 \triangleq \Pi_{P//C} : \mathcal{W}_e \to \mathcal{W}_e$ is stable with gain $\gamma_1 \in \mathcal{K}_\infty$, and that H_1 is relatively continuous with $H_1(\mathcal{W}_e) \subseteq \mathcal{G}_P$. Moreover, the operator $H_2 \triangleq \Phi - I : \mathcal{G}_P \subseteq \mathcal{W}_e \to \mathcal{W}_e$ is stable with gain $\gamma_2 \in \mathcal{K}_\infty$; and $R_\tau H_2 = R_\tau(\Phi - I)$ is compact for any $\tau \in (0,\infty)$.

Next we show that the feedback system of H_1 and H_2 of Figure 6.5 with $d_2 \equiv 0$ satisfies the uniqueness property on $\mathcal{W}_e \times \{0\}$, (i.e., for any input $d_1 \in \mathcal{W}_e$ and for any $e = (e_1, e_2) \in \mathcal{W}_e \times \mathcal{W}_e$, $\tilde{e} = (\tilde{e}_1, \tilde{e}_2) \in \mathcal{W}_e \times \mathcal{W}_e$ with $d_1 = e_1 + H_2e_2$, $e_2 = H_1e_1$ and $d_1 = \tilde{e}_1 + H_2\tilde{e}_2$, $\tilde{e}_2 = H_1\tilde{e}_1$, we need to show $e = \tilde{e}$). Since $e_2 = H_1e_1 \in \mathcal{G}_P$ and $\tilde{e}_2 = H_1\tilde{e}_1 \in \mathcal{G}_P$, it follows from the well-posedness on $\mathcal{W}_e$ of the feedback interconnection of P and C (see Assumption 6.4) that $d_1-(H_2+I)e_2 = e_1-e_2 \in \mathcal{G}_C$ and $d_1-(H_2+I)\tilde{e}_2 = \tilde{e}_1 - \tilde{e}_2 \in \mathcal{G}_C$. In addition, $(H_2 + I)e_2 = \Phi e_2 \in \mathcal{G}_{\tilde{P}}$ and $(H_2 + I)\tilde{e}_2 = \Phi\tilde{e}_2 \in \mathcal{G}_{\tilde{P}}$. Thus, from the uniqueness property on $\mathcal{W}_e$ for the interconnection of $\tilde{P}$ and C (replacing P by $\tilde{P}$ in Assumption 6.10), we obtain $(H_2 + I)e_2 = (H_2 + I)\tilde{e}_2$, i.e., $\Phi e_2 = \Phi\tilde{e}_2$. Since Φ is one-to-one, we have $e_2 = \tilde{e}_2$, and so $e_1 = d_1 - H_2e_2 = d_1 - H_2\tilde{e}_2 = \tilde{e}_1$. This shows $e = \tilde{e}$.

Since $\gamma_2 \circ \gamma_1(s) \leq (I + \varepsilon)^{-1}(s)$, $\forall s \geq 0$ with $\varepsilon \in \mathcal{K}_\infty$, by applying Theorem 6.9 with $\mathcal{V}_{1e} = \mathcal{V}_{2e} = \mathcal{W}_e$ and $E_{1\tau} = E_\tau$ for the feedback configuration of H_1 and H_2 (Figure 6.5) with $d_2 \equiv 0$, we obtain that the feedback system of H_1 and H_2 is well-posed on $\mathcal{W}_e \times \{0\}$, (i.e., for any input $d_1 \in \mathcal{W}_e$, there exists a unique $e = (e_1, e_2) \in \mathcal{W}_e \times \mathcal{W}_e$ such that

$d_1 = e_1 + H_2 e_2$ and $e_2 = H_1 e_1$). Moreover, for any $\tau > 0$,

$$\|e_1\|_\tau \leq (I + \varepsilon^{-1})(\|d_1\|_\tau), \quad \|e_2\|_\tau \leq \gamma_1 \circ (I + \varepsilon^{-1})(\|d_1\|_\tau). \tag{6.20}$$

Thus, for any $\tilde{w}_0 = d_1 \in \mathcal{W}_e$, it follows from $(H_2 + I)e_2 \in \mathcal{G}_{\tilde{P}}$, $d_1 - (H_2 + I)e_2 \in \mathcal{G}_C$ and the uniqueness property on $\mathcal{W}_e$ of the interconnection of $\tilde{P}$ and C that the feedback interconnection of $\tilde{P}$ and C is well-posed on $\mathcal{W}_e$. Moreover, $\Pi_{\tilde{P}//C}\tilde{w}_0 = (H_2 + I)e_2$, and from (6.20) we have $\left\|\Pi_{\tilde{P}//C}\tilde{w}_0\right\|_\tau \leq (I + \gamma_2) \circ \gamma_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau)$, $\forall \tau > 0$. Since $\tilde{w}_0$ is arbitrarily chosen from $\mathcal{W}_e$, we obtain that $\Pi_{\tilde{P}//C}$ is stable on $\mathcal{W}_e$ and (6.19) holds. $\square$

Theorem 6.12 shows that the small-gain theorem implies the robust stability theorem in a global setting with both of them establishing existence and boundedness simultaneously. However, for the converse part of Theorem 6.12, the existence property for the small-gain theorem cannot be established by using the robust stability theorem. This can be seen from the proof of Theorem 6.7 in which the starting point is to fix the closed-loop inputs and their corresponding outputs when showing the boundedness for the small-gain theorem from the boundedness for the robust stability theorem.

6.4 Local versions of Small-Gain Theorem and Georgiou & Smith's Robust Stability Theorem

In this section, we consider the relation between small-gain theorem and Georgiou and Smith's robust stability theorem in the local setting. In [Zheng and Zafiriou, 1999] the authors presented a local form of small-gain theorem obtained by using the contracting mapping theorem. In this section, we give a different local version of the nonlinear small-gain theorem by using the Schauder's fixed point theorem, which is used to show a variation of Georgiou and Smith's robust stability theorem in the local setting.

6.4.1 Local Version of Small-Gain Theorem

Let $\mathcal{V}_e$ be a signal space. The open ball of radius $d \geq 0$ in $\mathcal{V}_e$ is defined by

$$B_d(\mathcal{V}_e) = \{v \in \mathcal{V}_e : \|v\|_\tau \leq d, \forall \tau \in (0, \infty)\}$$

The small-gain theorem in the local setting is given as follows:

Theorem 6.13. *Consider the feedback system shown in Figure 6.1 with two signal spaces $\mathcal{V}_{1e}$ and $\mathcal{V}_{2e}$. Suppose that, for any $\tau \in (0, \infty)$, $\mathcal{V}_1[0, \tau)$ is complete and a continuous extension map $E_{1\tau} : \mathcal{V}_1[0, \tau) \to \mathcal{V}_{1e}$ exists such that $R_\tau x = R_\tau(E_{1\tau}x)$, $\forall x \in \mathcal{V}_1[0, \tau)$. Let $R_1 = H_1 : \mathcal{V}_{1e} \to \mathcal{V}_{2e}$ and $R_2 = H_2 : \mathcal{V}_{2e} \to \mathcal{V}_{1e}$ be two causal operators. Suppose that H_i is stable on $B_{h_i}(\mathcal{V}_{ie}) \subseteq \mathcal{V}_{ie}$ with gain $\gamma_i \in \mathcal{K}_\infty$ for $i = 1, 2$ with $0 \leq \gamma_1(h_1) \leq h_2 \leq \infty$.*

Suppose that the operator $H_1(\cdot) + d_2$ is relatively continuous for any $d_2 \in \mathcal{V}_{2e}$, and that $R_\tau H_2 : \mathcal{V}_{2e} \to \mathcal{V}_1[0,\tau)$ is compact for any $\tau \in (0,\infty)$. Assume that the feedback system is causal and satisfies the uniqueness *property on $\mathcal{V}_{1e} \times \mathcal{V}_{2e}$. If there exist two functions ρ, $\varepsilon \in \mathcal{K}_\infty$ such that*

$$\gamma_2 \circ (I+\rho) \circ \gamma_1(s) \le (I+\varepsilon)^{-1}(s), \quad \forall s \ge 0. \tag{6.21}$$

Then the feedback system is well-posed on $B_{k_1}(\mathcal{V}_{1e}) \times B_{k_2}(\mathcal{V}_{2e})$ with $k_1 \triangleq (I+\varepsilon^{-1})^{-1}(h_1) - \gamma_2 \circ (I+\rho^{-1})(k_2)$ for any $k_2 \le \min\{(I+\rho^{-1})^{-1} \circ \gamma_2^{-1} \circ (I+\varepsilon^{-1})^{-1}(h_1),\ h_2 - \gamma_1(h_1)\}$.[2] *Moreover, it is stable on $B_{k_1}(\mathcal{V}_{1e}) \times B_{k_2}(\mathcal{V}_{2e})$ and for any $\tau > 0$,*

$$\|e_1\|_\tau \le (I+\varepsilon^{-1})\left(\|d_1\|_\tau + \gamma_2 \circ (I+\rho^{-1})(\|d_2\|_\tau)\right), \tag{6.22}$$

$$\|e_2\|_\tau \le \|d_2\|_\tau + \gamma_1(\|e_1\|_\tau). \tag{6.23}$$

Proof. It follows from minor modifications of the proof of Theorem 6.8. For any $k_2 \le \min\{(I+\rho^{-1})^{-1} \circ \gamma_2^{-1} \circ (I+\varepsilon^{-1})^{-1}(h_1),\ h_2 - \gamma_1(h_1)\}$ and $k_1 \triangleq (I+\varepsilon^{-1})^{-1}(h_1) - \gamma_2 \circ (I+\rho^{-1})(k_2)$, consider any input $d = (d_1, d_2) \in B_{k_1}(\mathcal{V}_{1e}) \times B_{k_2}(\mathcal{V}_{2e})$, we know that the inequality (6.17) still holds, since from (6.16), $\|E_{1\tau}x\|_\tau = \|x\|_\tau \le (I+\varepsilon^{-1})\big(\|d_1\|_\tau + \gamma_2 \circ (I+\rho^{-1})(\|d_2\|_\tau)\big) \le (I+\varepsilon^{-1})\big(k_1 + \gamma_2 \circ (I+\rho^{-1})(k_2)\big) = h_1$ and $\|H_1 E_{1\tau} x + d_2\|_\tau \le \gamma_1(h_1) + k_2 \le h_2$ for any $\tau \in (0,\infty)$. The rest of proof is the same as that of Theorem 6.8. □

It is the following special case of Theorem 6.13 that we shall use to show a local version of Georgiou & Smith's robust stability theorem.

Theorem 6.14 ($d_2 \equiv 0$)**.** *Consider the feedback structure of Figure 6.1 with $d_2 \equiv 0$. Let $\mathcal{V}_{1e}$, $\mathcal{V}_{2e}$ be two signal spaces. Suppose that, for any $\tau \in (0,\infty)$, $\mathcal{V}_1[0,\tau)$ is complete and a continuous extension map $E_{1\tau} : \mathcal{V}_1[0,\tau) \to \mathcal{V}_{1e}$ exists such that $R_\tau x = R_\tau(E_{1\tau}x)$, $\forall x \in \mathcal{V}_1[0,\tau)$. Let $R_1 = H_1 : \mathcal{V}_{1e} \to \mathcal{V}_{2e}$ and $R_2 = H_2 : D_{H_2} \subseteq \mathcal{V}_{2e} \to \mathcal{V}_{1e}$ with $H_1(\mathcal{V}_{1e}) \subseteq D_{H_2}$ be two causal operators. Suppose that H_i is stable on $B_{h_i}(\mathcal{V}_{ie}) \subseteq \mathcal{V}_{ie}$ with gain $\gamma_i \in \mathcal{K}_\infty$ for $i = 1,2$ with $0 \le \gamma_1(h_1) \le h_2 \le \infty$. Suppose that H_1 is relatively continuous and that $R_\tau H_2 : D_{H_2} \subseteq \mathcal{V}_{2e} \to \mathcal{V}_1[0,\tau)$ is compact for any $\tau \in (0,\infty)$. Assume that the feedback system is causal and satisfies the* uniqueness *property on $\mathcal{V}_{1e} \times \{0\}$. If there exists a function $\varepsilon \in \mathcal{K}_\infty$ such that*

$$\gamma_2 \circ \gamma_1(s) \le (I+\varepsilon)^{-1}(s), \quad \forall s \ge 0. \tag{6.24}$$

Then the feedback system with $d_2 \equiv 0$ is well-posed on $B_{k_1}(\mathcal{V}_{1e}) \times \{0\}$ with $k_1 \triangleq (I+\varepsilon^{-1})^{-1}(h_1)$.[3] *Moreover, it is stable on $B_{k_1}(\mathcal{V}_{1e}) \times \{0\}$ and $\|e_1\|_\tau \le (I+\varepsilon^{-1})(\|d_1\|_\tau)$ and $\|e_2\|_\tau \le \gamma_1 \circ (I+\varepsilon^{-1})(\|d_1\|_\tau)$ for any $d_1 \in B_{k_1}(\mathcal{V}_{1e})$ and any $\tau > 0$.*

[2] That is, for any input $d = (d_1, d_2) \in B_{k_1}(\mathcal{V}_{1e}) \times B_{k_2}(\mathcal{V}_{2e})$, there exists a unique $e = (e_1, e_2) \in \mathcal{V}_{1e} \times \mathcal{V}_{2e}$ such that $d_1 = e_1 + H_2 e_2$ and $d_2 = e_2 - H_1 e_1$.

[3] That is, for any input $d_1 \in B_{k_1}(\mathcal{V}_{1e})$, there exists a unique $e = (e_1, e_2) \in \mathcal{V}_{1e} \times \mathcal{V}_{2e}$ such that $d_1 = e_1 + H_2 e_2$ and $e_2 = H_1 e_1$.

Proof. It follows directly from the proof of Theorem 6.13 and the same approach as in the proof of Theorem 6.2. □

6.4.2 Local Version of Georgiou & Smith's Robust Stability Theorem

The corresponding version of Georgiou & Smith's robust stability theorem in the local setting is as follows.

Theorem 6.15. *Consider the feedback configuration of Figure 6.2. Let $\mathcal{U}_e$, $\mathcal{Y}_e$ be two signal spaces. Define $\mathcal{W}_e \triangleq \mathcal{U}_e \times \mathcal{U}_e$. Suppose that, for any $\tau \in (0,\infty)$, $\mathcal{W}[0,\tau)$ is complete and a continuous extension map $E_\tau : \mathcal{W}[0,\tau) \to \mathcal{W}_e$ exists such that $R_\tau x = R_\tau(E_\tau x)$, $\forall x \in \mathcal{W}[0,\tau)$. Suppose the Assumption 6.4 holds for the interconnection of P and C. Suppose that $\Pi_{P//C}$ is stable with gain $\gamma_1 \in \mathcal{K}_\infty$ on $B_{h_1}(\mathcal{W}_e) \subseteq \mathcal{W}_e$ with $h_1 \geq 0$ and that $\Pi_{P//C}$ is relatively continuous. Assume that the plant P is perturbed to be another plant $\tilde{P}$, and that the weak Assumption 6.10 holds for the interconnection of $\tilde{P}$ and C (i.e., replacing P by $\tilde{P}$ in Assumption 6.10). If there exists a one-to-one map $\Phi : \mathcal{G}_P \to \mathcal{G}_{\tilde{P}}$ with $R_\tau(\Phi - I)$, $\forall \tau \in (0,\infty)$ compact and a function $\gamma_2 \in \mathcal{K}_\infty$ with $0 \leq \gamma_1(h_1) \leq h_2 \leq \infty$ such that*

$$\|(\Phi - I)w_1\|_\tau \leq \gamma_2(\|w_1\|_\tau), \quad \forall w_1 \in \mathcal{G}_P \cap B_{h_2}(\mathcal{W}_e), \forall \tau > 0.$$

Then the feedback interconnection of $\tilde{P}$ and C is well-posed on $B_{k_1}(\mathcal{W}_e)$ with $k_1 \triangleq (I + \varepsilon^{-1})^{-1}(h_1)$.[4] *Moreover, $\Pi_{\tilde{P}//C}$ is stable on $B(k_1) \subseteq \mathcal{W}_e$ and*

$$\left\|\Pi_{\tilde{P}//C}\tilde{w}_0\right\|_\tau \leq (I + \gamma_2) \circ \gamma_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau), \quad \forall \tilde{w}_0 \in B_{k_1}(\mathcal{W}_e), \forall \tau > 0, \tag{6.25}$$

provided: the inequality $\gamma_2 \circ \gamma_1(s) \leq (I+\varepsilon)^{-1}(s)$, $\forall s \geq 0$ holds for some function $\varepsilon \in \mathcal{K}_\infty$.

Theorem 6.16. *If Theorem 6.14 is true, then Theorem 6.15 is true.*

Proof. The task is as follows: under the premises of Theorem 6.15, we need to establish the conclusions of Theorem 6.15 by using Theorem 6.14. The proof is similar to the proof of Theorem 6.12.

From our assumptions in Theorem 6.15, we know that the operator $H_1 \triangleq \Pi_{P//C} : \mathcal{W}_e \to \mathcal{W}_e$ is stable with gain $\gamma_1 \in \mathcal{K}_\infty$ on $B_{h_1}(\mathcal{W}_e)$, and that H_1 is relatively continuous with $H_1(\mathcal{W}_e) \subseteq \mathcal{G}_P$. Moreover, the operator $H_2 \triangleq \Phi - I : \mathcal{G}_P \subseteq \mathcal{W}_e \to \mathcal{W}_e$ is stable with gain $\gamma_2 \in \mathcal{K}_\infty$; and $R_\tau H_2 = R_\tau(\Phi - I)$ is compact for any $\tau \in (0,\infty)$.

Next we show that the feedback system of H_1 and H_2 of Figure 6.5 with $d_2 \equiv 0$ satisfies the uniqueness property on $\mathcal{W}_e \times \{0\}$, (i.e., for any input $d_1 \in \mathcal{W}_e$ and for any $e = (e_1, e_2) \in \mathcal{W}_e \times \mathcal{W}_e$, $\tilde{e} = (\tilde{e}_1, \tilde{e}_2) \in \mathcal{W}_e \times \mathcal{W}_e$ with $d_1 = e_1 + H_2 e_2$, $e_2 = H_1 e_1$ and

[4] That is, for any $\tilde{w}_0 \in B_{k_1}(\mathcal{W}_e)$, there exists a unique $\tilde{w}_1 \in \mathcal{G}_{\tilde{P}}$ such that $\tilde{w}_0 - \tilde{w}_1 \in \mathcal{G}_C$.

$d_1 = \tilde{e}_1 + H_2\tilde{e}_2$, $\tilde{e}_2 = H_1\tilde{e}_1$, we need to show $e = \tilde{e}$). Since $e_2 = H_1e_1 \in \mathcal{G}_P$ and $\tilde{e}_2 = H_1\tilde{e}_1 \in \mathcal{G}_P$, it follows from the well-posedness of the feedback interconnection of P and C (see Assumption 6.4) that $d_1 - (H_2 + I)e_2 = e_1 - e_2 \in \mathcal{G}_C$ and $d_1 - (H_2 + I)\tilde{e}_2 = \tilde{e}_1 - \tilde{e}_2 \in \mathcal{G}_C$. In addition, $(H_2 + I)e_2 = \Phi e_2 \in \mathcal{G}_{\tilde{P}}$ and $(H_2 + I)\tilde{e}_2 = \Phi\tilde{e}_2 \in \mathcal{G}_{\tilde{P}}$. Thus, from the uniqueness property on $\mathcal{W}_e$ for the interconnection of $\tilde{P}$ and C (replacing P by $\tilde{P}$ in Assumption 6.10), we obtain $(H_2 + I)e_2 = (H_2 + I)\tilde{e}_2$, i.e., $\Phi e_2 = \Phi\tilde{e}_2$. Since Φ is one-to-one, we have $e_2 = \tilde{e}_2$, and so $e_1 = d_1 - H_2e_2 = d_1 - H_2\tilde{e}_2 = \tilde{e}_1$. This shows $e = \tilde{e}$.

Since $\gamma_2 \circ \gamma_1(s) \leq (I + \varepsilon)^{-1}(s)$, $\forall s \geq 0$ with $\varepsilon \in \mathcal{K}_\infty$, and $0 \leq \gamma_1(h_1) \leq h_2 \leq \infty$, by applying Theorem 6.9 with $\mathcal{V}_{1e} = \mathcal{V}_{2e} \triangleq \mathcal{W}_e$, $D_{H_2} \triangleq \mathcal{G}_P$ and $E_{1\tau} \triangleq E_\tau$ for the feedback configuration of H_1 and H_2 (Figure 6.5) with $d_2 \equiv 0$, we obtain that the feedback system of H_1 and H_2 is well-posed on $B_{k_1}(\mathcal{V}_{1e}) \times \{0\}$ with $k_1 \triangleq (I + \varepsilon^{-1})^{-1}(h_1)$, i.e., for any input $d_1 \in B_{k_1}(\mathcal{W}_e)$, there exists a unique $e = (e_1, e_2) \in \mathcal{W}_e \times \mathcal{W}_e$ such that $d_1 = e_1 + H_2e_2$ and $e_2 = H_1e_1$. Moreover, for any $d_1 \in B_{k_1}(\mathcal{W}_e)$ and any $\tau > 0$,

$$\|e_1\|_\tau \leq (I + \varepsilon^{-1})(\|d_1\|_\tau), \quad \|e_2\|_\tau \leq \gamma_1 \circ (I + \varepsilon^{-1})(\|d_1\|_\tau). \tag{6.26}$$

Thus, for any $\tilde{w}_0 = d_1 \in B_{k_1}(\mathcal{W}_e)$, it follows from $(H_2 + I)e_2 \in \mathcal{G}_{\tilde{P}}$, $d_1 - (H_2 + I)e_2 \in \mathcal{G}_C$ and the uniqueness property on $\mathcal{W}_e$ of the interconnection of $\tilde{P}$ and C that the feedback interconnection of $\tilde{P}$ and C is well-posed on $B_{k_1}(\mathcal{W}_e)$ with $k_1 \triangleq (I+\varepsilon^{-1})^{-1}(h_1)$. Moreover, $\Pi_{\tilde{P}//C}\tilde{w}_0 = (H_2 + I)e_2$, and from (6.26) we have $\left\|\Pi_{\tilde{P}//C}\tilde{w}_0\right\|_\tau \leq (I + \gamma_2) \circ \gamma_1 \circ (I + \varepsilon^{-1})(\|\tilde{w}_0\|_\tau)$, $\forall \tau > 0$. Since $\tilde{w}_0$ is arbitrarily chosen from $\mathcal{W}_e$, we obtain that $\Pi_{\tilde{P}//C}$ is stable on $B_{k_1}(\mathcal{W}_e)$ and (6.25) holds. □

Here we have shown that the small-gain theorem implies the robust stability theorem in a local setting with both of them establishing existence and boundedness simultaneously. The converse part of Theorem 6.16 cannot be established because of the same reason given at the end of Section 6.3. However, if we only consider the boundedness for both theorems in the local setting, the equivalence between them can be similarly established as Theorems 6.6 and 6.7 in the global setting.

6.5 Summary

In this chapter, we consider the connections between Georgiou and Smith's robust stability type theorems and the nonlinear small-gain theorems. A fundamental robust stability theorem of Georgiou and Smith [Georgiou and Smith, 1997b, Theorem 6] is shown to be equivalent to a special case of the usual nonlinear small-gain theorem. Moreover, both the global and local versions of the nonlinear small-gain theorem which establishes simultaneously the existence and boundedness properties are presented to show the corresponding types of Georgiou and Smith's robust stability theorem.

In mathematics the art of proposing a question must be held of higher value than solving it.

Georg Cantor (1845-1918)

Chapter 7

Conclusions

In this chapter we will summarise the main contribution of this thesis and outline some directions for future research.

7.1 Summary of Contributions

The main contributions of this thesis are as follows:

- Appropriate signal spaces (i.e., interval spaces, extended spaces and ambient spaces) are introduced with some fundamental assumptions to constitute the basic framework for the study of input-output systems with abstract initial conditions.
- A unified construction of an underlying abstract state space is provided, which is applicable to input-output systems defined in a set theoretic manner from input-output pairs on a doubly infinite time axis. Fundamental properties (such as existence, uniqueness, well-posedness and causality) of both systems and closed-loop systems are defined and discussed from a very natural point of view.
- A fundamental robust stability result (Theorem 4.8 on page 81) is given based on the input-output framework set up in this work, which generalises the operator based robust stability theorem of [Georgiou and Smith, 1997b] to include the case of a general initial condition. This also includes a suitable generalisation of the nonlinear gap metric which takes into account both the effect of the perturbation on the state space structure (and hence the initial condition) as well as the input-output response. Theorem 4.8 can also be viewed as a generalisation of the ISS approach to enable a realistic treatment of robust stability in the context of perturbations which fundamentally change the structure of the state space. The proof of Theorem 4.8 is given in two different versions: one requires the well-posedness

of the perturbed closed-loop system; and another one requires only the uniqueness property of the perturbed closed-loop system.

- A notion of finite-time reachability for a system is defined, and a more applicable robust stability result than Theorem 4.8 is established in this framework (see Theorem 4.18 on page 92).

- Theorem 4.8 is also generalised to systems with potential for finite-time escape by extending signals on extended spaces to a wider space named ambient space.

- A general nonlinear ISS-type small-gain result (Theorem 5.2 on page 112) is developed based on the input-output framework set up in this thesis, which is established without extra "observability" conditions and with complete disconnection between the stability property and the existence, uniqueness properties of systems.

- Connections between Georgiou and Smith's robust stability type theorems and the nonlinear small-gain theorems are also discussed. An equivalence between a small-gain theorem and a slight variation on the fundamental robust stability result of Georgiou and Smith (i.e., [Georgiou and Smith, 1997b, Theorem 6]) is shown.

7.2 Directions for Future Research

In this section we give some further potential areas of research.

1. The applications of input-output theory for nonlinear systems are often restricted by the ability to compute those gain functions. Such difficulties remain for the application of Theorems 4.8 and 4.18 in this thesis. Further research into relevant issues of practical significance (such as computational issues, real applications, etc.) should be very useful for the application of Theorems 4.8 and 4.18.

2. Theorems 4.8 and 4.18 include a suitable generalisation of the nonlinear gap metric of [Georgiou and Smith, 1997b] by incorporating initial conditions (see (4.2–4.3) on page 81 and (4.36–4.37) on page 92). We have discussed the case of linear time-invariant systems (Section 4.5 on page 93) and a class of nonlinear systems with input delay (Section 4.6 on page 97). It will be very useful to develop a fuller description of the types of nonlinear systems within a gap ball, e.g., singular perturbation.

3. The tightness of the small-gain like condition (4.4) for the robust stability Theorem 4.8 on page 82 is not considered in this thesis. For systems defined by input-output operators, necessity results are available for linear systems (see e.g., [Dahleh and Ohta, 1988, Doyle and Stein, 1981, Shamma and Dahleh, 1991]) or nonlinear systems with fading memory (see e.g., Gonçalves and Dahleh [1998], Shamma

[1991], Shamma and Zhao [1993]); for general nonlinear systems, a weaker notion of gain (i.e., *conditional gain*) is required to recover the necessity results (see e.g., [Chen et al., 2004, Freeman, 2001]). A type of necessity result seems to be very useful to understand the degree of conservatism of Theorem 4.8.

4. In Chapter 6, we have discussed the connections between Georgiou and Smith's robust stability type theorems and the classical nonlinear small-gain theorems. It looks quite possible to extend these results to input-output systems with abstract initial conditions set up in this thesis. In particular, the generalised nonlinear small-gain theorem for systems with initial conditions developed in Chapter 5 may be used to establish the robust stability result (the first part of Theorem 4.8) given in Chapter 4 but of course with a looser bound for the gain of the perturbed closed-loop systems.

References

R. A. Adams and J. J. F. Fournier. *Sobolev Spaces*, volume 140. Academic Press, New York, 2nd edition, 2003. (Cited on pages 14, 15, 16, 19, 44, 45, 46, and 47.)

F. Albertini and E. D. Sontag. Continuous control-Lyapunov functions for asymptotically controllable time-varying systems. *International Journal of Control*, 72(18): 1630–1641, 1999. (Cited on page 30.)

B. D. O. Anderson and J. B. Moore. *Linear Optimal Control*. Prentice-Hall Electrical Engineering Series. Prentice-Hall, 1971. (Cited on page 56.)

D. Angeli and E. D. Sontag. Forward completeness, unboundedness observability, and their lyapunov characterizations. *Systems & Control Letters*, 38(4-5):209–217, 1999. (Cited on pages 60, 97, and 100.)

D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *Automatic Control, IEEE Transactions on*, 45(6):1082–1097, 2000. (Cited on page 64.)

M. A. Arbib and H. P. Zeiger. On the relevance of abstract algebra to control theory. *Automatica*, 5(5):589–606, 1969. (Cited on page 54.)

G. Besançon. *Nonlinear observers and applications*, volume 363. Springer Verlag, 2007. (Cited on page 53.)

W. Bian, M. French, and H. K. Pillai. An intrinsic behavioral approach to the gap metric. *SIAM Journal on Control and Optimization*, 47(4):1939–1960, 2008. (Cited on page 49.)

H. S. Black. Stabilized feedback amplifiers. *Bell Syst. Tech. J.*, 13(1):1–18, 1934. (Cited on pages 2 and 3.)

H. S. Black. Inventing the negative feedback amplifier. *IEEE Spectrum*, 14:54–60, Dec. 1977. (Cited on page 2.)

D. Buchstaller, J. Liu, and M. French. The deterministic interpretation of the kalman filter. *Systems & Control Letters (Submitted)*, pages 1–15, 2014. (Cited on page xi.)

R. H. Cannon. *Dynamics of Physical Systems*. Civil, Mechanical and Other Engineering Series. Dover Publications (originally published by McGraw-Hill, Inc., New York, in 1967), 2003. (Cited on page 27.)

A. L. Chen, G.-Q. Chen, and R. A. Freeman. Conditional gain is necessary and sufficient for the robust stabilization of nonlinear systems. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 2, pages 1347–1351, 2004. (Cited on page 141.)

A. L. Chen, G.-Q. Chen, and R. A. Freeman. Stability of nonlinear feedback systems: a new small-gain theorem. *SIAM J. on Control and Optimization*, 46:1995–2012, 2007. (Cited on page 50.)

Z. Chen and J. Huang. A simplified small gain theorem for time-varying nonlinear systems. *Automatic Control, IEEE Transactions on*, 50(11):1904–1908, 2005. (Cited on page 109.)

R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*. Springer-Verlag New York, Inc., NY, USA, 1995. (Cited on pages 14, 18, 23, and 24.)

M. A. Dahleh and Y. Ohta. A necessary and sufficient condition for robust BIBO stability. *Systems & Control Letters*, 11(4):271–275, 1988. (Cited on page 140.)

S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19:93–122, 2007. (Cited on page 109.)

K. Deimling. *Nonlinear Functional Analysis*. Springer-Verlag Berlin Heidelberg, Berlin, 1985. (Cited on pages 15, 16, 17, and 18.)

M. C. Delfour and S. K. Mitter. Hereditary differential systems with constant delays. I. general case. *Journal of Differential Equations*, 12(2):213–235, 1972. (Cited on page 42.)

M. C. Delfour and S. K. Mitter. Hereditary differential systems with constant delays. II. a class of affine systems and the adjoint problem. *Journal of Differential Equations*, 18(1):18–28, 1975. (Cited on page 63.)

C. Desoer and C.-T. Chen. Controllability and observability of feedback systems. *Automatic Control, IEEE Transactions on*, 12(4):474–475, 1967. (Cited on page 62.)

C. A. Desoer and M. Vidyasagar. *Feedback systems: input-output properties*. Academic Press (New York), 1975. (Cited on pages 99 and 109.)

C. A. Desoer and M. Vidyasagar. *Feedback systems: input-output properties*, volume 55. Society for Industrial and Applied Mathematics (1st published by Academic Press, Inc. in 1975), Philadelphia, 2009. (Cited on pages 32, 119, 122, 123, 124, and 130.)

J. Doyle, B. Francis, and A. Tannenbaum. *Feedback Control Theory*. Macmillan Publishing Co., 1990. (Cited on page 2.)

J. Doyle and G. Stein. Multivariable feedback design: Concepts for a classical/modern synthesis. *Automatic Control, IEEE Transactions on*, 26(1):4–16, 1981. (Cited on page 140.)

J. C. Doyle, T. T. Georgiou, and M. C. Smith. The parallel projection operators of a nonlinear feedback system. *Systems & Control Letters*, 20(2):79–85, 1993. (Cited on page 52.)

C. Foias, T. T. Georgiou, and M. C. Smith. Robust stability of feedback systems: A geometric approach using the gap metric. *SIAM Journal on Control and Optimization*, 31(6):1518–1537, 1993. (Cited on page 37.)

E. Fornasini and G. Marchesini. State-space realization theory of two-dimensional filters. *Automatic Control, IEEE Transactions on*, 21(4):484–492, 1976. (Cited on page 22.)

B. A. Francis. *A Course in H_∞ Control Theory*, volume 88 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag Berlin Heidelberg, 1987. (Cited on page 22.)

R. A. Freeman. On the necessity of the small-gain theorem in the performance analysis of nonlinear systems. In *Decision and Control, 2001. Proceedings of the 40th IEEE Conference on*, volume 1, pages 51–56, 2001. (Cited on page 141.)

M. French. Adaptive control and robustness in the gap metric. *Automatic Control, IEEE Transactions on*, 53(2):461–478, 2008. (Cited on page 5.)

M. French and W. Bian. A biased approach to nonlinear robust stability with applications in adaptive control. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 1393–1398, dec. 2009. (Cited on page 5.)

M. French and W. Bian. A biased approach to nonlinear robust stability and performance with applications to adaptive control. *SIAM Journal on Control and Optimization*, 50(3):1220–1243, 2012. (Cited on pages 5, 9, 40, 77, 80, 107, 120, and 130.)

M. French, A. Ilchmann, and M. Mueller. Robust stabilization by linear output delay feedback. *SIAM Journal on Control and Optimization*, 48(4):2533–2561, 2009. (Cited on pages 5 and 95.)

M. French and M. Mueller. Nonlinear high gain separation principles and fast sampling results ensuring robust stability. *Automatic Control, IEEE Transactions on (submitted)*. (Cited on pages 5 and 53.)

P. A. Fuhrmann. *Linear Systems and Operators in Hilbert Space*. Advanced book program. McGraw-Hill, Inc., 1981. (Cited on page 21.)

P. A. Fuhrmann, P. Rapisarda, and Y. Yamamoto. On the state of behaviors. *Linear Algebra and Its Applications*, 424(2-3):570–614, September 2007. (Cited on page 21.)

J. P. Gauthier and I. Kupka. *Deterministic observation theory and applications*. Cambridge University Press, Cambridge, UK, 2001. (Cited on page 53.)

B. R. Gelbaum and J. M. H. Olmsted. *Counterexamples in Analysis*. Dover Books on Mathematics. Dover Publications, Inc. (1st published in 1964 by Holden Day, Inc., San Francisco), Mineola, New York, 2003. (Cited on pages 15 and 17.)

T. T. Georgiou and M. C. Smith. Optimal robustness in the gap metric. *Automatic Control, IEEE Transactions on*, 35(6):673–686, 1990. (Cited on page 37.)

T. T. Georgiou and M. C. Smith. Biased norms and robustness analysis for nonlinear feedback systems. In *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, volume 1, pages 642–643, 1997a. (Cited on page 5.)

T. T. Georgiou and M. C. Smith. Robustness analysis of nonlinear feedback systems: an input-output approach. *Automatic Control, IEEE Transactions on*, 42(9):1200–1221, 1997b. (Cited on pages vi, 5, 8, 9, 10, 19, 37, 39, 52, 65, 77, 81, 88, 89, 90, 106, 119, 120, 123, 126, 127, 137, 139, and 140.)

J. M. Gonçalves and M. A. Dahleh. Necessary conditions for robust stability of a class of nonlinear systems. *Automatica*, 34(6):705–714, 1998. (Cited on page 140.)

R. Hermann and A. Krener. Nonlinear controllability and observability. *Automatic Control, IEEE Transactions on*, 22(5):728–740, 1977. (Cited on page 61.)

D. J. Hill. A generalization of the small-gain theorem for nonlinear feedback systems. *Automatica*, 27(6):1043–1045, 1991. (Cited on pages 109 and 119.)

M. W. Hirsch, S. Smale, and R. L. Devaney. *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, volume 60 of *Pure and Applied Mathematics*. Academic Press, Elsevier (USA), 2nd edition, 2004. (Cited on page 120.)

A. Hurwitz. Ueber die bedingungen, unter welchen eine gleichung nur wurzeln mit negativen reellen theilen besitzt. *Mathematische Annalen (*English translation *On the conditions under which an equation has only roots with negative real parts by H. G. Bergmann in Selected Papers on Mathematical Trends in Control Theory, R. Bellman and R. Kalaba Eds., New York: Dover, 1964, pp. 70-82.)*, 46(2):273–284, 1895. (Cited on page 2.)

B. Ingalls, E. D. Sontag, and Y. Wang. Remarks on input to output stability. In *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on*, volume 2, pages 1226–1231, 1999. (Cited on pages 81 and 110.)

A. Isidori. *Nonlinear Control Systems II*. Springer-Verlag, London, UK, 1999. (Cited on pages 28, 30, 33, and 34.)

Z.-P. Jiang and I. M. Y. Marcels. A small-gain control method for nonlinear cascaded systems with dynamic uncertainties. *Automatic Control, IEEE Transactions on*, 42 (3):292–308, 1997. (Cited on page 109.)

Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211–1215, 1996. (Cited on page 109.)

Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems (MCSS)*, 7:95–120, 1994. (Cited on pages 9, 31, 83, 109, and 118.)

Z.-P. Jiang and Y. Wang. A generalization of the nonlinear small-gain theorem for large-scale complex systems. In *Intelligent Control and Automation, 2008. WCICA 2008. 7th World Congress on*, pages 1188–1193, june 2008. (Cited on page 109.)

T. Kailath. *Linear systems*. Prentice-Hall information and system sciences series. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980. (Cited on pages 21, 26, 54, and 55.)

R. E. Kalman. Irreducible realizations and the degree of a rational matrix. *Journal of the Society for Industrial and Applied Mathematics*, 13(2):520–544, 1965. (Cited on page 28.)

R. E. Kalman, P. L. Falb, and M. A. Arbib. *Topics in mathematical system theory*. International series in pure and applied mathematics. New York: McGraw-Hill, 1969. (Cited on pages 21, 53, and 54.)

I. Karafyllis. The non-uniform in time small-gain theorem for a wide class of control systems with outputs. *European Journal of Control*, 10(4):307–323, 2004. (Cited on page 98.)

I. Karafyllis and Z.-P. Jiang. *Stability and Stabilization of Nonlinear Systems*. Communications and Control Engineering. Springer-Verlag London Limited, 2011. (Cited on page 30.)

T. Kato. *Perturbation Theory for Linear Operators*. Classics in mathematics. Springer-Verlag Berlin Heidelberg, reprint of the 1980 edition, 1995. (Cited on pages 15, 17, 19, and 20.)

H. K. Khalil. *Nonlinear systems*, volume 122. Prentice hall New Jersey, 3rd edition, 2002. (Cited on pages 3, 33, and 34.)

R. Kline. Harold Black and the negative-feedback amplifier. *Control Systems, IEEE*, 13 (4):82–85, 1993. (Cited on page 2.)

S. G. Krantz. *The Integral: A Crux for Analysis*. Synthesis Lectures on Mathematics and Statistics. Morgan & Claypool Publishers, 2011. (Cited on page 32.)

M. Krichman, E. D. Sontag, and Y. Wang. Input-output-to-state stability. *SIAM Journal on Control and Optimization*, 39(6):1874–1928, 2001. (Cited on page 5.)

F. L. Lewis. *Applied Optimal Control and Estimation: Digital Design and Implementation*. Digital signal processing series. Prentice Hall, 1992. (Cited on page 2.)

D. Liberzon and D. Nešić. Stability analysis of hybrid systems via small-gain theorems. In *Hybrid Systems: Computation and Control*, volume 3927 of *Lecture Notes in Computer Science*, pages 421–435. Springer-Verlag Berlin Heidelberg, 2006. (Cited on page 109.)

Y. Lin, E. D. Sontag, and Y. Wang. Recent results on lyapunov-theoretic techniques for nonlinear stability. Technical report, Report SYCON-93-09, 1993. (Cited on page 30.)

Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, 34(1):124–160, 1996. (Cited on page 30.)

J. Liu and M. French. A generalisation of the nonlinear small-gain theorem for systems with abstract initial conditions. In *Control Conference (ECC), 2013 European*, pages 1699–1704, 2013. (Cited on pages xi, 8, and 9.)

J. Liu and M. French. Connections between georgiou and smith's robust stability type theorems and the nonlinear small-gain theorems. *SIAM Journal on Control and Optimization (under submission)*, pages 1–18, 2014a. (Cited on pages xi and 10.)

J. Liu and M. French. Efficient minimal disturbance estimation in estimation based multiple model switched adaptive control. *Automatic Control, IEEE Transactions on (under submission)*, pages 1–24, 2014b. (Cited on page xi.)

J. Liu and M. French. Nerode equivalence for continuous-time transfer function matrices. *(under submission)*, pages 1–20, 2014c. (Cited on pages xi, 7, and 28.)

J. Liu and M. French. Robust stability of input-output systems with initial conditions. *SIAM Journal on Control and Optimization (accepted)*, pages 1–31, 2014d. (Cited on pages xi, 8, and 9.)

J. Lu and R. E. Skelton. Mean-square small gain theorem for stochastic control: discrete-time case. *Automatic Control, IEEE Transactions on*, 47(3):490–494, 2002. (Cited on page 109.)

A. M. Lyapunov. *The General Problem of the Stability Of Motion (English translation by A. T. Fullter)*, volume 55 of *Control Theory and Applications Series*. Taylor & Francis (International Journal of Control), first published in 1892 (in Russian), 1992. (Cited on page 3.)

I. M. Y. Mareels and D. J. Hill. Monotone stability of nonlinear feedback systems. *J. Math. Systems Estim. Control*, 2:275–291, 1992. (Cited on pages 109, 119, and 122.)

H. J. Marquez. *Nonlinear control systems: analysis and design*. John Wiley & Sons, INC., Hoboken, New Jersey, 2003. (Cited on page 49.)

J. C. Maxwell. On governors. *Proc. Royal Soc. London*, 16:270–283, 1868. (Cited on page 1.)

A. N. Michel. Stability: the common thread in the evolution of feedback control. *Control Systems, IEEE*, 16(3):50–60, 1996. (Cited on page 3.)

A. Nerode. Linear automaton transformations. *Proceedings of the American Mathematical Society*, 9(4):541–544, 1958. (Cited on pages 21 and 54.)

D. Nešić and A. R. Teel. A lyapunov-based small-gain theorem for hybrid ISS systems. In *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pages 3380–3385, dec. 2008. (Cited on page 109.)

N. K. Nikol'skiĭ. *Treatise on the Shift Operator: Spectral Function Theory*. Springer Verlag (translated from the Russian by J. Peetre), 1986. (Cited on pages 23, 24, and 25.)

K. Ogata. *Modern Control Engineering*. Instrumentation and controls series. Prentice-Hall, Inc., 4th edition, 2002. (Cited on page 27.)

J. R. Partington. *An Introduction to Hankel Operators*. London Mathematical Society Student Texts. Cambridge University Press, 1988. (Cited on pages 23, 25, and 26.)

J. R. Partington. *Linear Operators and Linear Systems: An Analytical Approach to Control Theory*, volume 60 of *London Mathematical Society Student Texts*. Cambridge University Press, 2004. (Cited on page 26.)

J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioral Approach*. Springer-Verlag New York, 1998. (Cited on pages 2, 6, 22, 48, and 55.)

P. Rapisarda and A. J. van der Schaft. Trajectory concatenability for systems described by partial differential equations. In *Proc. of the 20th International Symposium on Mathematical Theory of Networks and Systems*, Melbourne, Australia, 2012. (Cited on page 53.)

P. Rapisarda and J. C. Willems. State maps for linear systems. *SIAM Journal on Control and Optimization*, 35(3):1053–1091, May 1997. (Cited on page 21.)

P. Rocha and J. C. Willems. Behavioral controllability of delay-differential systems. *SIAM Journal on Control and Optimization*, 35(1):254–264, 1997. (Cited on pages 58 and 63.)

E. J. Routh. *A Treatise on the Stability of a Given State of Motion: Particularly Steady Motion*. Macmillan and Company, London, 1877. (Cited on page 2.)

B. P. Rynne and M. A. Youngson. *Linear Functional Analysis*. Springer Undergraduate Mathematics Series. Springer-Verlag London Limited, 2nd edition, 2008. (Cited on pages 18 and 19.)

J. Sakarovitch. *Elements of Automata Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009. (Cited on page 54.)

I. W. Sandberg. On the L_2-boundedness of solutions of nonlinear functional equations. *Bell System Technical Journal*, 43:1581–1599, 1964. (Cited on pages 4, 37, 109, and 119.)

I. W. Sandberg. On the boundedness of solutions of non-linear integral equations. *Bell Syst. Tech. J.*, 44(3):439–453, 1965a. (Cited on pages 4 and 37.)

I. W. Sandberg. Some results on the theory of physical systems governed by nonlinear functional equations. *Bell System Technical Journal*, 44:871–898, 1965b. (Cited on page 42.)

S. Sastry. *Nonlinear systems: analysis, stability, and control*. Springer Verlag, 1999. (Cited on page 3.)

J. M. A. Scherpen. Balancing for nonlinear systems. *Systems & Control Letters*, 21(2): 143–153, 1993. (Cited on page 56.)

J. M. A. Scherpen. *Balancing for Nonlinear Systems*. PhD thesis, University of Twente, 1994. (Cited on page 56.)

J. M. A. Scherpen and A. J. Van der Schaft. Normalized coprime factorizations and balancing for unstable nonlinear systems. *International Journal of Control*, 60(6): 1193–1222, 1994. (Cited on page 56.)

J. S. Shamma. The necessity of the small-gain theorem for time-varying and nonlinear systems. *Automatic Control, IEEE Transactions on*, 36(10):1138–1147, 1991. (Cited on page 140.)

J. S. Shamma and M. A. Dahleh. Time-varying versus time-invariant compensation for rejection of persistent bounded disturbances and robust stabilization. *Automatic Control, IEEE Transactions on*, 36(7):838–847, 1991. (Cited on page 140.)

J. S. Shamma and R. Z. Zhao. Fading-memory feedback systems and robust stability. *Automatica*, 29(1):191–200, 1993. (Cited on page 141.)

E. D. Sontag. Smooth stabilization implies coprime factorization. *Automatic Control, IEEE Transactions on*, 34(4):435–443, 1989. (Cited on pages 4, 8, 33, 38, 64, 77, 97, and 109.)

E. D. Sontag. Comments on integral variants of iss. *Systems & Control Letters*, 34(1-2): 93–100, 1998a. (Cited on pages 30, 64, and 103.)

E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Number 6 in Textbooks in Applied Mathematics. Springer, New York, second edition, 1998b. (Cited on pages 34, 54, 55, and 103.)

E. D. Sontag. Input to state stability: Basic concepts and results. In P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory*, volume 1932, pages 163–220. Springer-Verlag Berlin Heidelberg, 2008. (Cited on pages 5, 58, and 64.)

E. D. Sontag and B. Ingalls. A small-gain theorem with applications to input/output systems, incremental stability, detectability, and interconnections. *Journal of the Franklin Institute*, 339(2):211–229, 2002. (Cited on pages 30, 77, 81, and 110.)

E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995. (Cited on pages 5, 34, 58, and 77.)

E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *Automatic Control, IEEE Transactions on*, 41(9):1283–1294, 1996. (Cited on pages 5, 58, and 77.)

E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Systems & Control Letters*, 29(5):279–290, 1997. (Cited on page 64.)

E. D. Sontag and Y. Wang. Notions of input to output stability. *Systems & Control Letters*, 38(4-5):235–248, 1999. (Cited on pages 35, 64, 65, and 97.)

E. D. Sontag and Y. Wang. Lyapunov characterizations of input to output stability. *SIAM Journal on Control and Optimization*, 39(1):226–249, 2000. (Cited on pages 5, 35, and 58.)

T. Tao. *An Introduction to Measure Theory*, volume 126 of *Graduate studies in mathematics*. American Mathematical Soc., 2011. (Cited on page 19.)

A. R. Teel. On graphs, conic relations, and input-output stability of nonlinear feedback systems. *Automatic Control, IEEE Transactions on*, 41(5):702–709, 1996. (Cited on page 120.)

A. R. Teel. Connections between razumikhin-type theorems and the ISS nonlinear small gain theorem. *Automatic Control, IEEE Transactions on*, 43(7):960–964, 1998. (Cited on page 116.)

H. L. Trentelman, S. Fiaz, and K. Takaba. Optimal robust stabilization and dissipativity synthesis by behavioral interconnection. *SIAM Journal on Control and Optimization*, 49(1):288–314, 2011. (Cited on page 6.)

A. J. van der Schaft and P. Rapisarda. State maps from integration by parts. *SIAM Journal on Control and Optimization*, 49(6):2415–2439, 2011. (Cited on page 53.)

M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, 2nd edition, 1993. (Cited on pages 3, 20, 28, 89, 93, 94, and 122.)

M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. Synthesis Lectures on Control and Mechatronics. Morgan & Claypool Publishers (originally published by M.I.T. Press in 1985), 2011. (Cited on page 37.)

G. Vinnicombe. *Uncertainty and Feedback: H_∞ Loop-shaping and the ν-gap Metric*. World Scientific Publishing Company, 2001. (Cited on page 22.)

J. L. Walsh. *Interpolation and Approximation by Rational Functions in the Complex Domain*. American Mathematical Society: Colloquium publications. American Mathematical Society (1st published in 1935), 1969. (Cited on page 25.)

G. Weiss. Transfer functions of regular linear systems. part I: Characterizations of regularity. *Transactions of the American Mathematical Society*, 342(2):827–854, 1994. (Cited on pages 28, 58, and 59.)

J. C. Willems. Stability, instability, invertibility and causality. *SIAM Journal on Control*, 7:645–671, 1969. (Cited on pages 73 and 120.)

J. C. Willems. *The Analysis of Feedback Systems*. The MIT Press, Cambridge, Massachusetts, USA, 1971a. (Cited on pages 73, 75, and 88.)

J. C. Willems. Least squares stationary optimal control and the algebraic riccati equation. *Automatic Control, IEEE Transactions on*, 16(6):621–634, 1971b. (Cited on page 56.)

J. C. Willems. Dissipative dynamical systems - part I: General theory; part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45:321–351 and 352–393, 1972. (Cited on page 55.)

J. C. Willems. Models for dynamics. *Dynamics reported*, 2:171–269, 1989. (Cited on pages 6, 7, 55, and 77.)

J. C. Willems. Paradigms and puzzles in the theory of dynamical systems. *Automatic Control, IEEE Transactions on*, 36(3):259–294, 1991. (Cited on pages 48, 49, and 50.)

L. A. Zadeh and C. A. Desoer. *Linear System Theory: The State Space Approach*. New York: McGraw-Hill, 1963. (Cited on page 53.)

G. Zames. *Nonlinear Operators for System Analysis*. PhD thesis, Massachusetts Institute of Technology, 1960. (Cited on page 48.)

G. Zames. Functional analysis applied to nonlinear feedback systems. *Circuit Theory, IEEE Transactions on*, 10(3):392–404, 1963. (Cited on pages 4, 6, 37, 47, and 53.)

G. Zames. On the input-output stability of time-varying nonlinear feedback systems–part I and II. *Automatic Control, IEEE Transactions on*, 11:228–238 and 465–476, 1966a. (Cited on page 119.)

G. Zames. On the input-output stability of time-varying nonlinear feedback systems–part I: Conditions derived using concepts of loop gain, conicity, and positivity. *Automatic Control, IEEE Transactions on*, 11(2):228–238, 1966b. (Cited on pages 4, 37, 42, 109, and 122.)

G. Zames. On the input-output stability of time-varying nonlinear feedback systems—part II: Conditions involving circles in the frequency plane and sector nonlinearities. *Automatic Control, IEEE Transactions on*, 11(3):465–476, 1966c. (Cited on pages 4, 37, 42, and 109.)

G. Zames and A. K. El-Sakkary. Unstable systems and feedback: The gap metric. In *In Proc. of the Allerton Conf.*, pages 380–385, December 1980. (Cited on page 37.)

E. Zeidler. *Nonlinear functional analysis and its applications I: Fixed-Point Theorems*. Springer-Verlag New York, Inc., 1986. (Cited on pages 14, 15, 16, 17, 18, 19, 20, and 120.)

E. Zeidler. *Applied Functional Analysis: Applications to Mathematical Physics*. Applied Mathematical Sciences. Springer-Verlag, 1995. (Cited on pages 15, 19, and 20.)

Q. Zheng and E. Zafiriou. A local form of small gain theorem and analysis of feedback volterra systems. *Automatic Control, IEEE Transactions on*, 44(3):635–640, 1999. (Cited on pages 109, 120, and 134.)

K. Zhou and J. C. Doyle. *Essentials of Robust Control*. Prentice Hall, 1998. (Cited on page 119.)

K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, USA, 1995. (Cited on page 62.)

S. Q. Zhu and A. A. Stoorvogel. A representation and the norm of Toeplitz operators. *Systems & Control Letters*, 12(5):409–413, 1989. (Cited on page 24.)