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Minimal see-saw model predicting best fit lepton mixing angles

Stephen F. King

School of Physics and Astronomy, University of Southampton, Southampton, SO17 1BJ, United Kingdom

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ABSTRACT

We discuss a minimal predictive see-saw model in which the right-handed neutrino mainly responsible for the atmospheric neutrino mass has couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(0, 1, 1)$ and the right-handed neutrino mainly responsible for the solar neutrino mass has couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(1, 4, 2)$, with a relative phase $\eta = -2\pi/5$. We show how these patterns of couplings could arise from an A_4 family symmetry model of leptons, together with Z_3 and Z_5 symmetries which fix $\eta = -2\pi/5$ up to a discrete phase choice. The PMNS matrix is then completely determined by one remaining parameter which is used to fix the neutrino mass ratio m_2/m_3. The model predicts the lepton mixing angles $\theta_{12} \approx 34°, \theta_{23} \approx 41°, \theta_{13} \approx 9.5°$, which exactly coincide with the current best fit values for a normal neutrino mass hierarchy, together with the distinctive prediction for the CP violating oscillation phase $\delta \approx 106°$.

1. Introduction

Daya Bay [1] and RENO [2] have measured a non-zero reactor angle $\theta_{13} \approx 0.15$ which excludes Tri-Bimaximal (TB) mixing [3]. Recent global fits also hint at deviations of the atmospheric and solar angles from their TB values (for a recent review see e.g. [4]). Such deviations may be expressed in terms of the deviation parameters (s, a and r) from TB mixing [5] (for a related parametrisation see [6]):

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s), \qquad \sin\theta_{23} = \frac{1}{\sqrt{2}}(1+a),$$
$$\sin\theta_{13} = \frac{r}{\sqrt{2}}. \tag{1}$$

With zero solar and atmospheric deviations from TB mixing, $s = a = 0$, and Cabibbo-like reactor mixing described by $r = \lambda$, with $\lambda = 0.225$ being the Wolfenstein parameter, one is led to Tri-Bimaximal Cabibbo (TBC) mixing [7]. However, as mentioned above, current global fits prefer non-zero solar and atmospheric TB deviation parameters,

$$s = -\lambda^2/2, \qquad a = -\lambda/3, \qquad r = \lambda, \tag{2}$$

corresponding to the angles,

$$\theta_{12} = 34.2°, \qquad \theta_{23} = 40.8°, \qquad \theta_{13} = 9.15°. \tag{3}$$

These angles are close to the best fit values for all three global fits in the case of a normal neutrino mass ordering [4]. Assuming a normal neutrino mass *hierarchy* with $m_1 = 0$, one is led to [13],

$$\frac{m_2}{m_3} = \frac{3}{4}\lambda, \tag{4}$$

corresponding to $m_2/m_3 \approx 0.17$, close to the best fit value [4]. The deviation parameters in Eq. (2) have the feature that the atmospheric mixing angle is in the first octant and the solar mixing angle is somewhat less than its tri-maximal value, in agreement with the latest global fits for the case of a normal neutrino mass ordering. In particular it reproduces the best fit values of angles of all three global fits [4] to within one standard deviation.

There have been many attempts to describe the lepton mixing angles based on the type I see-saw model [8] combined with sequential dominance (SD) [9] in which the right-handed neutrinos contribute with sequential strength. Constrained sequential dominance (CSD) [10] involves the right-handed neutrino mainly responsible for the atmospheric neutrino mass having couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(0, 1, 1)$ and the right-handed neutrino mainly responsible for the solar neutrino mass having couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(1, 1, -1)$ and it led to TB mixing. CSD2 [11] was proposed to give a non-zero reactor angle and is based on the same atmospheric alignment but with right-handed neutrino mainly responsible for the solar neutrino mass having couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(1, 0, -2)$ or $(1, 2, 0)$ yielding a reactor angle $\theta_{13} \approx 6°$ which unfortunately is too small, although the situation can be rescued by invoking charged lepton corrections [12]. The CSD3 model in [13] involves the right-handed neutrino mainly responsible for the solar neutrino mass having

E-mail address: king@soton.ac.uk.

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Table 1
Lepton, Higgs and flavon superfields and how they transform under the symmetries relevant for the Yukawa sector of the model. The only non-trivial charged lepton charges are in the upper left of the table and the only non-trivial neutrino charges in the lower right of the table. Note that the only the lepton doublets L and A_4 symmetry, are common to both charged lepton and neutrino sectors and are given near the central column and row. The Standard Model gauge symmetries and $U(1)_R$ symmetry, under which all the leptons have a charge of unity while the Higgs and flavons have zero charge, are not shown in the table.

	θ	e^c	μ^c	τ^c	φ_e	φ_μ	φ_τ	H_1	L	H_2	ϕ_{atm}	ϕ_{sol}	N_{atm}	N_{sol}	ξ_{atm}	ξ_{sol}
Z_3^θ	ω	ω	ω^2	1	1	1	1	1	1	1	1	1	1	1	1	1
Z_3^e	1	ω^2	1	1	ω	1	1	1	1	1	1	1	1	1	1	1
Z_3^μ	1	1	ω^2	1	1	ω	1	1	1	1	1	1	1	1	1	1
Z_3^τ	1	1	1	ω^2	1	1	ω	1	1	1	1	1	1	1	1	1
A_4	**1**	**1**	**1**	**1**	**3**	**3**	**3**	**1**	**3**	**1**	**3**	**3**	**1**	**1**	**1**	**1**
Z_5^{atm}	1	1	1	1	1	1	1	1	1	1	ρ^3	1	ρ^2	1	ρ	1
Z_5^{sol}	1	1	1	1	1	1	1	1	1	1	1	ρ^3	1	ρ^2	1	ρ

couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(1,3,1)$ or $(1,1,3)$ with a relative phase $\mp\pi/3$ yielding a reactor angle $\theta_{13} \approx 8.5°$ close to the observed value. However CSD3 predicts approximate TBC mixing with an almost maximal atmospheric mixing angle disfavoured by the latest global fits, and so it may soon be challenged.

In this Letter we shall propose a model based on a new possibility called CSD4 which predicts the above best fit angles in Eq. (3) of the PMNS lepton mixing matrix and also makes predictions for the physical CP violating phases. Similar to all SD models, the CSD4 model involves effectively two right-handed neutrinos and a normal neutrino mass hierarchy, leading to $m_1 = 0$. As in CSD2 and CSD3, the CSD4 model only requires one input parameter, namely the ratio of neutrino masses which is selected to be $m_2/m_3 \approx 3\lambda/4$, which is a natural value that one would expect in such models. Also as in CSD2 and CSD3, once this value is chosen, the entire PMNS mixing matrix is then fixed by the theory (up to a discrete choice of phases) with no remaining free parameters. In the CSD4 model, the right-handed neutrino mainly responsible for the atmospheric neutrino mass has couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(0,1,1)$ and the right-handed neutrino mainly responsible for the solar neutrino mass has couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(1,4,2)$, with a relative phase $\eta = -2\pi/5$. These couplings and phase relation were first discovered in [13] and shown to lead to lepton mixing angles in good agreement with the latest global fits, but no model has been proposed based on CSD4. The goal of this Letter is to show how CSD4 can arise from an A_4 family symmetry, together with additional discrete Z_5 and Z_3 symmetries, and to present the first model of leptons along these lines. This is necessary since it is far from clear whether alignments such as $(1,4,2)$ are possible to achieve within a realistic model. The CSD4 model presented here predicts the best fit PMNS angles in Eq. (3) with the distinctive prediction for the oscillation phase $\delta \approx 106°$.

2. A minimal predictive A_4 model of leptons

In this section we outline a supersymmetric (SUSY) A_4 model of leptons with CSD4 along the lines of the A_4 models of leptons discussed in [11,14]. The basic idea is that the three families of lepton doublets L form a triplet of A_4 while the right-handed charged leptons e^c, μ^c, τ^c, right-handed neutrinos N_{atm}, N_{sol} and the two Higgs doublets H_1, H_2 required by SUSY are all singlets of A_4. In addition the model employs an additional Z_3^θ family symmetry in order to account for the charged lepton mass hierarchy.

The vacuum alignment that is required for the model is discussed in Appendix A. In Table 1 we have displayed the symmetries and superfields relevant for the Yukawa sector only. In Appendix A the transformation properties of the remaining superfields under $Z_3^l \times Z_5^{\nu_i}$ responsible for vacuum alignment is discussed and are consistent with the charges shown in Table 1, where we have written $\phi_{\text{atm}} \equiv \varphi_{\nu_3}$ and $\phi_{\text{sol}} \equiv \varphi_{\nu_4}$ and hence $Z_5^{\text{atm}} \equiv Z_5^{\nu_3}$ and $Z_5^{\text{sol}} \equiv Z_5^{\nu_4}$.

The charged lepton sector of the model employs the A_4 triplet flavons φ_e, φ_μ, φ_τ whose alignment is discussed in Appendix A. With the lepton symmetries in the upper left of Table 1 we may enforce the following charged lepton Yukawa superpotential at leading order

$$\mathcal{W}^e_{\text{Yuk}} \sim \frac{1}{\Lambda} H_1(\varphi_\tau \cdot L)\tau^c + \frac{1}{\Lambda^2}\theta H_1(\varphi_\mu \cdot L)\mu^c + \frac{1}{\Lambda^3}\theta^2 H_1(\varphi_e \cdot L)e^c, \tag{5}$$

which give the charged lepton Yukawa couplings after the flavons develop their vevs. Λ is a generic messenger mass scale, but in a renormalisable model the messengers scales may differ. The charged lepton symmetries include three lepton flavour symmetries $Z_3^{e,\mu,\tau}$ under which φ_e, φ_μ, φ_τ and e^c, μ^c, τ^c transform respectively as ω and ω^2, together with a lepton family symmetry Z_3^θ under which e^c, μ^c, τ^c transform as ω, ω^2, 1 respectively (where $\omega = e^{i2\pi/3}$) with the family symmetry breaking flavon θ transforming as ω and otherwise being a singlet under all other symmetries. H_1 and L and all other fields are singlets under $Z_3^{e,\mu,\tau}$ and Z_3^θ. With these charge assignments the higher order corrections are very suppressed.

The charged lepton Yukawa matrix is diagonal at leading order due to the alignment of the charged lepton-type flavons in Eq. (25) (where the driving fields responsible for the alignment in Eq. (24) absorb the charges under the newly introduced symmetries $Z_3^{e,\mu,\tau}$ and Z_3^θ) and has the form,

$$Y^e = \text{diag}(y_e, y_\mu, y_\tau) \sim \text{diag}\big(\epsilon^2, \epsilon, 1\big) y_\tau \tag{6}$$

where we choose $\epsilon \sim \langle\theta\rangle/\Lambda \sim \lambda^2$ in order to generate the correct order of magnitude charged lepton mass hierarchy, with precise charged lepton masses also dependent on order-one coefficients which we have suppressed here.

With the neutrino symmetries in the lower right part of Table 1 we may enforce the following leading order neutrino Yukawa superpotential

$$\mathcal{W}^\nu_{\text{Yuk}} \sim \frac{1}{\Lambda} H_2(\phi_{\text{atm}} \cdot L)N_{\text{atm}} + \frac{1}{\Lambda} H_2(\phi_{\text{sol}} \cdot L)N_{\text{sol}}. \tag{7}$$

Again the higher order corrections are completely negligible. The neutrino sector of the model exploits the A_4 triplet flavons $\phi_{\text{atm}} \equiv \varphi_{\nu_3}$, and $\phi_{\text{sol}} \equiv \varphi_{\nu_4}$ whose alignment is discussed in Appendix A.

As is typical in models of this kind [11,14], the RH neutrinos have no mass terms at the renormalisable level, but they become massive after some A_4 singlet flavons ξ_{atm} and ξ_{sol} develop their vevs due to the renormalisable superpotential,

$$\mathcal{W}_R \sim \xi_{\text{atm}} N_{\text{atm}}^2 + \xi_{\text{sol}} N_{\text{sol}}^2. \tag{8}$$

When the right-handed neutrino flavons develop their vevs $\langle\xi_{\text{atm}}\rangle \sim M_A$ together with $\langle\xi_{\text{sol}}\rangle \sim M_B$, then the RH neutrino mass matrix is diagonal as required,

$$M_R = \begin{pmatrix} M_A & 0 \\ 0 & M_B \end{pmatrix}. \tag{9}$$

To ensure that the mixed terms are absent at renormalisable order we have imposed a right-handed neutrino flavour symmetry Z_5^{atm} under which N_{atm} and ξ_{atm} transform as ρ^2 and ρ (where $\rho = e^{i2\pi/5}$) while ϕ_{atm} transforms as ρ^3 with all other fields being singlets. We have also imposed a similar symmetry Z_5^{sol} under which the "solar" fields transform in an analogous way. We remark that these charge assignments are consistent with the flavon superpotential in Eq. (29), where we identify $\phi_{\text{atm}} \equiv \varphi_{\nu_3}$, and $\phi_{\text{sol}} \equiv \varphi_{\nu_4}$, with suitable charges assigned to the driving fields.

With all masses, couplings and messenger scales set approximately equal, the driving flavon superpotentials would predict $|\langle\xi_{\text{atm}}\rangle| \approx |\langle\xi_{\text{sol}}\rangle|$ and hence approximately equal right-handed neutrino masses $M_A \approx M_B$. Similarly, from Appendix A with $\phi_{\text{atm}} \equiv \varphi_{\nu_3}$ and $\phi_{\text{sol}} \equiv \varphi_{\nu_4}$, we see that $\langle\phi_{\text{atm}}^2\rangle \approx \langle\phi_{\text{sol}}^2\rangle$ would lead to

$$\langle\phi_{\text{atm}}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} v_{\text{atm}}, \qquad \langle\phi_{\text{sol}}\rangle = \frac{1}{\sqrt{21}}\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} v_{\text{sol}}, \tag{10}$$

where $v_{\text{atm}} \approx v_{\text{sol}}$.

The above charge assignments allow higher order non-renormalisable mixed terms such as

$$\Delta\mathcal{W}_R \sim \frac{1}{\Lambda}(\phi_{\text{atm}} \cdot \phi_{\text{sol}}) N_{\text{atm}} N_{\text{sol}}, \tag{11}$$

which contribute off-diagonal terms to the right-handed neutrino mass matrix of a magnitude which depends on the absolute scale of the flavon vevs $\langle\phi_{\text{atm}}\rangle$ and $\langle\phi_{\text{sol}}\rangle$ compared to $\langle\xi_{\text{atm}}\rangle$ and $\langle\xi_{\text{sol}}\rangle$. If all flavon vevs and messenger scales are set equal then these terms are suppressed by $\epsilon \sim \lambda^2$ according to the estimate below Eq. (6), however they may be even more suppressed. We shall ignore the contribution of such off-diagonal mass terms in the following.

Implementing the see-saw mechanism, the effective neutrino mass matrix has the form,

$$m^\nu \sim \frac{v_2^2}{\Lambda^2}\left(\frac{\langle\phi_{\text{atm}}\rangle\langle\phi_{\text{atm}}\rangle^T}{\langle\xi_{\text{atm}}\rangle} + \frac{\langle\phi_{\text{sol}}\rangle\langle\phi_{\text{sol}}\rangle^T}{\langle\xi_{\text{sol}}\rangle}\right), \tag{12}$$

where $v_2 = \langle H_2\rangle$. Hence it can be parameterised, up to an overall irrelevant phase, as,

$$m^\nu = m_a\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_a e^{2i\eta}\epsilon_\nu\begin{pmatrix} 1 & 4 & 2 \\ 4 & 16 & 8 \\ 2 & 8 & 1 \end{pmatrix} \tag{13}$$

where m_a and ϵ_ν are real mass parameters which determine the physical neutrino masses m_3 and m_2 and we written the relative phase difference between the two terms as 2η. Using Eq. (10) the see-saw mechanism naturally leads to the neutrino mass matrix in Eq. (13) with $\epsilon_\nu \approx 2/21 \approx 0.1$. Hence the desired value of $\epsilon_\nu \approx 0.06$ assumed below is not unreasonable, and may be achieved for example by taking $M_B \approx 2M_A$. As discussed in [13], we shall also require a special phase relation $\eta = -2\pi/5$ in order to achieve our goal of predicting the best fit values of the lepton mixing angles.

The phase difference $\eta = -2\pi/5$ between flavon vevs can be obtained in the context of spontaneous CP violation from discrete symmetries as discussed in [12], and we shall follow the strategy outlined there. The basic idea is to impose CP conservation on the theory so that all couplings and masses are real. Note that the A_4 assignments in Table 1 do not involve the complex singlets $1'$, $1''$ or any complex Clebsch–Gordan coefficients so that the definition of CP is straightforward in this model and hence CP may be defined in different ways which are equivalent for our purposes (see [12] for a discussion of this point). The CP symmetry is broken in a discrete way by the form of the superpotential terms. We shall follow [12] and suppose that the flavon vevs $\langle\phi_{\text{atm}}\rangle$ and $\langle\phi_{\text{sol}}\rangle$ to be real with the phase η in Eq. (13) originating from the solar right-handed neutrino mass due to the flavon vev $\langle\xi_{\text{sol}}\rangle \sim M_B e^{4i\pi/5}$ having a complex phase of $4\pi/5$, while the flavon vev $\langle\xi_{\text{atm}}\rangle \sim M_A$ is real and positive. This can be arranged if the right-handed neutrino flavon vevs arise from the superpotential,

$$W_{A_4}^{\text{flavon},R} = gP\left(\frac{\xi_{\text{atm}}^5}{\Lambda^3} - M^2\right) + g'P'\left(\frac{\xi_{\text{sol}}^5}{\Lambda'^3} - M'^2\right), \tag{14}$$

where, as in [12], the driving singlet fields P, P' denote linear combinations of identical singlets and all couplings and masses are real due to CP conservation. The F-term conditions from Eq. (14) are,

$$\left|\frac{\langle\xi_{\text{atm}}\rangle^5}{\Lambda^3} - M^2\right|^2 = \left|\frac{\langle\xi_{\text{sol}}\rangle^5}{\Lambda'^3} - M'^2\right|^2 = 0. \tag{15}$$

These are satisfied by $\langle\xi_{\text{atm}}\rangle = |(\Lambda^3 M^2)^{1/5}|$ and $\langle\xi_{\text{sol}}\rangle = |(\Lambda'^3 M'^2)^{1/5}|e^{4i\pi/5}$ where we arbitrarily select the phases to be zero and $4\pi/5$ from amongst a discrete set of possible choices in each case. More generally we require a phase difference of $4\pi/5$ since the overall phase is not physically relevant, which would happen one in five times by chance. In the basis where the right-handed neutrino masses are real and positive this is equivalent to having a phase difference $\eta = -2\pi/5$ between flavon vevs in Eq. (10) according to the see-saw result in Eq. (12).

Similarly the flavons appearing in Eqs. (36) each have a discrete choice of phases. The charged lepton flavons φ_l may take any phases since such phases are unphysical. In fact the only physically significant flavon phases from the previous subsection are those of $\phi_{\text{atm}} \equiv \varphi_{\nu_3}$, and $\phi_{\text{sol}} \equiv \varphi_{\nu_4}$ whose phases are selected to be equal. As before, this would occur one in five times by chance.

We emphasise that, with the alignments including the phase η fixed, the neutrino mass matrix is completely determined by only two parameters, namely an overall mass scale m_a, which may be taken to fix the atmospheric neutrino mass $m_3 = 0.048$–0.051 eV, the ratio of input masses ϵ_ν, which may be taken to fix the solar to atmospheric neutrino mass ratio $m_2/m_3 = 0.17$–0.18. In particular the entire PMNS mixing matrix and all the parameters therein are then predicted as a function of m_2/m_3 controlled by the only remaining parameter ϵ_ν. In Table 2 we show the predictions for CSD4 as a function of ϵ_ν and hence m_2/m_3.

We remark that an accuracy of one degree in the angles is all that can be expected due to purely theoretical corrections in a realistic model due to renormalisation group running [16] and canonical normalisation corrections [17]. In addition, there may be small contributions from a heavy third right-handed neutrino [18] which can affect the results.

As in the case of CSD2, the neutrino mass matrix implies the TM_1 mixing form [20] where the first column of the PMNS matrix is proportional to $(2,-1,1)^T$. The reason is simply that $\langle\varphi_{\nu_1}\rangle \propto (2,-1,1)^T$ is an eigenvector of m^ν in Eq. (13) with a zero eigenvalue corresponding to the first neutrino mass m_1 being zero. The reason for this is that m^ν in Eq. (13) is a sum of two terms, the first being proportional to $AA^T \propto \langle\varphi_{\nu_3}\rangle\langle\varphi_{\nu_3}\rangle^T$ and the second being proportional to $BB^T \propto \langle\varphi_{\nu_4}\rangle\langle\varphi_{\nu_4}\rangle^T$. Since $\langle\varphi_{\nu_1}\rangle \propto (2,-1,1)^T$ is orthogonal to both $\langle\varphi_{\nu_3}\rangle$ and $\langle\varphi_{\nu_4}\rangle$ it is then clearly annihilated by the neutrino mass matrix, i.e. it is an eigenvector with zero eigenvalue. Therefore we immediately expect m^ν in Eq. (13) to be

Table 2
The predictions for PMNS parameters and m_2/m_3 arising from CSD4 as a function of ϵ_ν. Note that these predictions assume $\eta = -2\pi/5$. The predictions are obtained numerically using the Mixing Parameter Tools package based on [15]. The leading order analytic results are not reliable as discussed in Appendix B.

ϵ_ν	m_2/m_3	θ_{12}	θ_{13}	θ_{23}	δ	β
0.057	0.166	34.2°	9.0°	40.8°	107°	−84°
0.058	0.170	34.2°	9.2°	40.9°	107°	−83°
0.059	0.174	34.1°	9.4°	41.0°	106°	−82°
0.060	0.177	34.1°	9.6°	41.1°	105°	−80°
0.061	0.181	34.1°	9.7°	41.3°	104°	−79°

diagonalised by the TM_1 mixing matrix [20] where the first column is proportional to $\langle\varphi_{\nu_1}\rangle \propto (2,-1,1)^T$. Therefore we already know that CSD4 must lead to TM_1 mixing exactly to all orders according to this general argument.

Exact TM_1 mixing angle and phase relations are obtained by equating moduli of PMNS elements to those of the first column of the TB mixing matrix (see also [20]):

$$c_{12}c_{13} = \sqrt{\frac{2}{3}}, \tag{16}$$

$$\left|c_{23}s_{12} + s_{13}s_{23}c_{12}e^{i\delta}\right| = \frac{1}{\sqrt{6}}, \tag{17}$$

$$\left|s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta}\right| = \frac{1}{\sqrt{6}}. \tag{18}$$

From Eq. (16) we see that TM_1 mixing approximately preserves the successful TB mixing for the solar mixing angle $\theta_{12} \approx 35°$ as the correction due to a non-zero but relatively small reactor angle is of second order. While general TM_1 mixing involves an undetermined reactor angle θ_{13}, we emphasise that CSD4 fixes this reactor angle. For $\eta = -2\pi/5$ the reactor angle is in the correct range as shown in Table 2.

In an approximate linear form, the relations in Eqs. (16)–(18) imply the atmospheric sum rule relation $a = r\cos\delta$ [11], hence,

$$\theta_{23} \approx 45° + \sqrt{2}\theta_{13}\cos\delta. \tag{19}$$

For $\eta = -2\pi/5$ the predictions shown in Table 2 for the small deviations of the atmospheric angle from maximality are well described by the sum rule in Eq. (19). In the present model this sum rule is satisfied by particular predicted values of angles and CP phase which only depend on the neutrino mass ratio m_2/m_3. Over the successful range of m_2/m_3 we predict CP violation with $\delta \approx 106°$ and $\theta_{23} \approx 41°$ which satisfy the sum rule. Note that according to this sum rule, non-maximal atmospheric mixing is linked to non-maximal CP violation.

3. Conclusions

There is long history of attempts to explain the neutrino mixing angles starting from the type I see-saw mechanism and using SD, first using CSD to account for TB mixing, then using CSD2 to obtain a small reactor angle before going to CSD3 where the correct reactor angle can be reproduced along with maximal atmospheric mixing. We have discussed a minimal predictive see-saw model based on CSD4 in which the right-handed neutrino mainly responsible for the atmospheric neutrino mass has couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(0,1,1)$ and the right-handed neutrino mainly responsible for the solar neutrino mass has couplings to $(\nu_e, \nu_\mu, \nu_\tau)$ proportional to $(1,4,2)$, with a relative phase $\eta = -2\pi/5$. We have shown how these patterns of couplings and phase could arise from an A_4 family symmetry model of leptons.

We remark that the type of model presented here is referred to as "indirect" according to the classification scheme of models in [4], meaning that the family symmetry is completely broken by flavons and its only purpose is to generate the desired vacuum alignments. By contrast, the "direct" models where the symmetries of the neutrino and charged lepton mass matrices is identified as a subgroup of the family symmetry, requires rather large family symmetry groups in order to account for the reactor angle [21]. It is possible to have "semi-direct" models, either at leading order or emerging due to higher order corrections [4], but these are inherently less predictive. In the light of the observed reactor angle, "indirect models" therefore offer the prospect of full predictivity at the leading order from a small family symmetry group. Spontaneous CP violation seems to be an important ingredient in the "indirect" approach since a particular phase relation between flavons a crucial requirement.

The particular indirect model presented here, in which CSD4 emerges from an A_4 family symmetry, offers a highly predictive framework involving only one free parameter which is used to fix the neutrino mass ratio m_2/m_3, together with an overall neutrino mass scale which is used to fix the atmospheric neutrino mass m_3. Remarkably, the model then predicts the PMNS angles $\theta_{12} \approx 34°$, $\theta_{23} \approx 41°$, $\theta_{13} \approx 9.5°$, which exactly coincide with the current best fit values for a normal neutrino mass hierarchy, together with the distinctive prediction for the CP violating oscillation phase $\delta \approx 106°$. These predictions will surely be tested by current and planned high precision neutrino oscillation experiments.

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Appendix A. Vacuum alignment

In this appendix we shall discuss how to achieve the following vacuum alignment,

$$\frac{\langle\phi_{\mathrm{atm}}\rangle}{\Lambda} \propto \begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \frac{\langle\phi_{\mathrm{sol}}\rangle}{\Lambda} \propto \begin{pmatrix}1\\4\\2\end{pmatrix}, \tag{20}$$

which we refer to as CSD4.

The vacuum alignments associated with TB mixing have been very well studied. Here we shall focus on the family symmetry A_4 as it is the smallest non-Abelian finite group with an irreducible triplet representation. The generators of the A_4 group, can be written as S and T with $S^2 = T^3 = (ST)^3 = \mathcal{I}$. A_4 has four irreducible representations, three singlets 1, $1'$ and $1''$ and one triplet. The products of singlets are:

$$\begin{aligned} &1\otimes 1 = 1, \qquad 1'\otimes 1'' = 1, \\ &1'\otimes 1' = 1'', \qquad 1''\otimes 1'' = 1'. \end{aligned} \tag{21}$$

We work in the basis [19],

$$S = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}, \qquad T = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}. \tag{22}$$

In this basis one has the following Clebsch rules for the multiplication of two triplets,

$$(ab)_1 = a_1b_1 + a_2b_2 + a_3b_3,$$

$$
\begin{aligned}
(ab)_{1'} &= a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3,\\
(ab)_{1''} &= a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3,\\
(ab)_{3_1} &= (a_2b_3, a_3b_1, a_1b_2),\\
(ab)_{3_2} &= (a_3b_2, a_1b_3, a_2b_1),
\end{aligned}
\tag{23}
$$

where $\omega^3 = 1$, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

Following the methods of [14] it is straightforward to obtain the vacuum alignments for charged lepton flavon alignments suitable for a diagonal charged lepton mass matrix. The charged lepton flavon alignments used to generate a diagonal charged lepton mass matrix are obtained from the renormalisable superpotential [14],

$$
\begin{aligned}
W_{A_4}^{\text{flavon},\ell} \sim{}& A_e\varphi_e\varphi_e + A_\mu\varphi_\mu\varphi_\mu + A_\tau\varphi_\tau\varphi_\tau + O_{e\mu}\varphi_e\varphi_\mu\\
&+ O_{e\tau}\varphi_e\varphi_\tau + O_{\mu\tau}\varphi_\mu\varphi_\tau.
\end{aligned}
\tag{24}
$$

The triplet driving fields $A_{e,\mu,\tau}$ give rise to flavon alignments $\langle\varphi_{e,\mu,\tau}\rangle$ with two zero components, and the singlet driving fields O_{ij} require orthogonality among the three flavon vevs so that we arrive at the vacuum structure [14],

$$
\langle\varphi_e\rangle = v_e\begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \langle\varphi_\mu\rangle = v_\mu\begin{pmatrix}0\\1\\0\end{pmatrix},
\qquad
\langle\varphi_\tau\rangle = v_\tau\begin{pmatrix}0\\0\\1\end{pmatrix}.
\tag{25}
$$

Of more interest to us in this Letter are the new neutrino flavon alignments. The starting point for the discussion is the usual standard TB neutrino flavon alignments proportional to the respective columns of the TB mixing matrix,

$$
\langle\varphi_{\nu_1}\rangle = v_{\nu_1}\begin{pmatrix}2\\-1\\1\end{pmatrix}, \qquad \langle\varphi_{\nu_2}\rangle = v_{\nu_2}\begin{pmatrix}1\\1\\-1\end{pmatrix},
\qquad
\langle\varphi_{\nu_3}\rangle = v_{\nu_3}\begin{pmatrix}0\\1\\1\end{pmatrix}.
\tag{26}
$$

We will also employ the alternative TB alignments which are related by phase redefinitions,

$$
\langle\varphi_{\nu_1'}\rangle = v'_{\nu_1}\begin{pmatrix}2\\1\\-1\end{pmatrix}, \qquad \langle\varphi_{\nu_2'}\rangle = v_{\nu_2'}\begin{pmatrix}1\\-1\\1\end{pmatrix},
\qquad
\langle\varphi_{\nu_3'}\rangle = v_{\nu_3'}\begin{pmatrix}0\\-1\\-1\end{pmatrix}.
\tag{27}
$$

In the remainder of this subsection we shall show how to obtain the neutrino flavon alignments including the new alignment,

$$
\langle\varphi_{\nu_4}\rangle = v_{\nu_4}\begin{pmatrix}1\\4\\2\end{pmatrix},
\tag{28}
$$

which corresponds to the CSD4 solar flavon alignment in Eq. (20). We shall identify $\phi_{\text{atm}} \equiv \varphi_{\nu_3}$, and $\phi_{\text{sol}} \equiv \varphi_{\nu_4}$. The renormalisable superpotential involving the driving fields necessary for aligning the neutrino-type flavons is given as

$$
\begin{aligned}
W_{A_4}^{\text{flavon},\nu} ={}& A_{\nu_2}(g_1\varphi_{\nu_2}\varphi_{\nu_2} + g_2\varphi_{\nu_2}\xi_{\nu_2})\\
&+ A_{\nu_2'}\big(g'_1\varphi_{\nu_2'}\varphi_{\nu_2'} + g'_2\varphi_{\nu_2'}\xi_{\nu_2'}\big)\\
&+ O_{e\nu_3}g_3\varphi_e\varphi_{\nu_3} + O_{\nu_2\nu_3}g_4\varphi_{\nu_2}\varphi_{\nu_3} + O_{\nu_1\nu_2}g_5\varphi_{\nu_1}\varphi_{\nu_2}\\
&+ O_{\nu_1\nu_3}g_6\varphi_{\nu_1}\varphi_{\nu_3} + O_{e\nu_3'}g'_3\varphi_e\varphi_{\nu_3'} + O_{\nu_2'\nu_3'}g'_4\varphi_{\nu_2'}\varphi_{\nu_3'}\\
&+ O_{\nu_1'\nu_2'}g'_5\varphi_{\nu_1'}\varphi_{\nu_2'} + O_{\nu_1'\nu_3'}g'_6\varphi_{\nu_1'}\varphi_{\nu_3'}\\
&+ O_{\mu\nu_5}g_7\varphi_\mu\varphi_{\nu_5} + O_{\nu_1'\nu_5}g_8\varphi_{\nu_1'}\varphi_{\nu_5} + O_{\mu\nu_6}g_9\varphi_\mu\varphi_{\nu_6}\\
&+ O_{\nu_5\nu_6}g_{10}\varphi_{\nu_5}\varphi_{\nu_6} + O_{\nu_6\nu_4}g_{11}\varphi_{\nu_6}\varphi_{\nu_4}\\
&+ O_{\nu_1\nu_4}g_{12}\varphi_{\nu_1}\varphi_{\nu_4},
\end{aligned}
\tag{29}
$$

where A_{ν_2} is a triplet driving field and O_{ij} are singlet driving fields whose F-terms lead to orthogonality relations between the accompanying flavon fields. Here g_i are dimensionless coupling constants. The first line of Eq. (29) produces the vacuum alignment $\langle\varphi_{\nu_2}\rangle \propto (1,1,-1)^T$ of Eq. (26) and $\langle\varphi_{\nu_2'}\rangle \propto (1,-1,1)^T$ of Eq. (27) as can be seen from the F-term conditions[1]

$$
2g_1\begin{pmatrix}\langle\varphi_{\nu_2}\rangle_2\langle\varphi_{\nu_2}\rangle_3\\ \langle\varphi_{\nu_2}\rangle_3\langle\varphi_{\nu_2}\rangle_1\\ \langle\varphi_{\nu_2}\rangle_1\langle\varphi_{\nu_2}\rangle_2\end{pmatrix} + g_2\langle\xi_{\nu_2}\rangle\begin{pmatrix}\langle\varphi_{\nu_2}\rangle_1\\ \langle\varphi_{\nu_2}\rangle_2\\ \langle\varphi_{\nu_2}\rangle_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}
\tag{30}
$$

plus similar conditions involving the primed flavons. The first two terms in the second line of Eq. (29) give rise to orthogonality conditions which uniquely fix the alignment $\langle\varphi_{\nu_3}\rangle \propto (0,1,1)^T$ of Eq. (26),

$$
\langle\varphi_e\rangle^T\cdot\langle\varphi_{\nu_3}\rangle = \langle\varphi_{\nu_2}\rangle^T\cdot\langle\varphi_{\nu_3}\rangle = 0 \quad\rightarrow\quad \langle\varphi_{\nu_3}\rangle \propto \begin{pmatrix}0\\1\\1\end{pmatrix}.
\tag{31}
$$

The last two terms in the second line of Eq. (29) give rise to orthogonality conditions which uniquely fix the alignment $\langle\varphi_{\nu_1}\rangle \propto (2,-1,1)^T$ of Eq. (26),

$$
\langle\varphi_{\nu_1}\rangle^T\cdot\langle\varphi_{\nu_2}\rangle = \langle\varphi_{\nu_1}\rangle^T\cdot\langle\varphi_{\nu_3}\rangle = 0 \quad\rightarrow\quad \langle\varphi_{\nu_1}\rangle \propto \begin{pmatrix}2\\-1\\1\end{pmatrix}.
\tag{32}
$$

Similarly the terms in the third line of Eq. (29) give rise to orthogonality conditions which fix the alternative TB alignments in Eq. (27) corresponding to a different choice of phases.

The terms in the fourth line of Eq. (29) give rise to orthogonality conditions which fix the alignments of the auxiliary flavon fields φ_{ν_5} and φ_{ν_6},

$$
\langle\varphi_\mu\rangle^T\cdot\langle\varphi_{\nu_5}\rangle = \langle\varphi_{\nu_1'}\rangle^T\cdot\langle\varphi_{\nu_5}\rangle = 0 \quad\rightarrow\quad \langle\varphi_{\nu_5}\rangle \propto \begin{pmatrix}1\\0\\2\end{pmatrix},
\tag{33}
$$

$$
\langle\varphi_\mu\rangle^T\cdot\langle\varphi_{\nu_6}\rangle = \langle\varphi_{\nu_5}\rangle^T\cdot\langle\varphi_{\nu_6}\rangle = 0 \quad\rightarrow\quad \langle\varphi_{\nu_6}\rangle \propto \begin{pmatrix}2\\0\\-1\end{pmatrix}.
\tag{34}
$$

The neutrino-type flavon of interest labelled as φ_{ν_4} gets aligned by the remaining terms in the fifth line of Eq. (29), leading to the desired alignment in Eq. (28),

$$
\langle\varphi_{\nu_1}\rangle^T\cdot\langle\varphi_{\nu_4}\rangle = \langle\varphi_{\nu_6}\rangle^T\cdot\langle\varphi_{\nu_4}\rangle = 0 \quad\rightarrow\quad \langle\varphi_{\nu_4}\rangle \propto \begin{pmatrix}1\\4\\2\end{pmatrix}.
\tag{35}
$$

So far we have only shown how to align the flavon vevs and have not enforced them to be non-zero. In order to do this we shall introduce the additional non-renormalisable superpotential terms which include,

[1] We remark that the general alignment derived from these F-term conditions is $\langle\varphi_{\nu_2}\rangle \propto (\pm 1, \pm 1, \pm 1)^T$. One can, however, show that all of them are equivalent up to phase redefinitions. Note that $(1,1,-1)$ is related to permutations of the minus sign as well as to $(-1,-1,-1)$ by A_4 transformations. The other four choices can be obtained from these by simply multiplying an overall phase (which would also change the sign of the ξ_{ν_2} vev).

$$\Delta W^{\text{flavon},\ell}_{A_4} \sim \sum_{l=e,\mu,\tau} \frac{P}{\Lambda}\big((\varphi_l \cdot \varphi_l)\rho_l - M^3\big) + P\left(\frac{\rho_l^3}{\Lambda} - M^2\right), \quad (36)$$

$$\Delta W^{\text{flavon},\nu}_{A_4} \sim \sum_{i=1}^{6} \frac{P}{\Lambda}\big((\varphi_{\nu_i} \cdot \varphi_{\nu_i})\rho_{\nu_i} - M^3\big) + P\left(\frac{\rho_{\nu_i}^5}{\Lambda^3} - M^2\right), \quad (37)$$

where, as in [11], the driving singlet fields P denote linear combinations of identical singlets and we have introduced explicit masses M to drive the non-zero vevs, as well as the messenger scales denoted as Λ. We have also introduced A_4 singlets ρ_l and ρ_{ν_i} whose vevs are driven by the F-terms of the singlets P in the second terms in Eqs. (36) and (37). These singlet vevs enter the first terms in Eqs. (36) and (37) which drive the vevs of the triplet flavons.

The flavons and driving fields introduced in this appendix transform under $Z_3^l \times Z_5^{\nu_i}$ symmetries whose purpose is to allow only the terms in Eqs. (24), (29) and (36) and forbid all other terms. The superfields φ_{ν_i} transform under $Z_5^{\nu_i}$ as ρ^3 (where $\rho = e^{i2\pi/5}$) and are singlets under all other discrete symmetries. The superfields ρ_{ν_i} transform under $Z_5^{\nu_i}$ as ρ^4 and are singlets under all other discrete symmetries. Any superfield with a single subscript l transforms under Z_3^l as ω (where $\omega = e^{i2\pi/3}$) and is a singlet under all other discrete symmetries. The orthogonality driving superfields O_{ij} with two subscripts transform under $Z_3^l \times Z_5^{\nu_i}$ in such a way as to allow the terms in Eqs. (24), (29). For example the $O_{l\nu_i}$ driving fields transform under $Z_3^l \times Z_5^{\nu_i}$ as (ω^2, ρ^2). In addition driving superfields are assigned a charge of two while flavon superfields have zero charge under a $U(1)_R$ symmetry.

Appendix B. Leading order analytic results

For the case of atmospheric alignments of the form $(0, z_1, 1)$ and solar alignments of the form $(1, z_2, z_3)$, the leading order analytic results in [9,13] give,

$$\tan\theta_{23} \approx |z_1|, \quad (38)$$

$$\cot\theta_{12} \approx c_{23}|z_2|\cos\left(\eta_2 - \frac{\beta}{2}\right) - s_{23}|z_3|\cos\left(\eta_3 - \frac{\beta}{2}\right), \quad (39)$$

$$\theta_{13} \approx \frac{m_2}{m_3} s_{12}^2 c_{23}\big||z_3| + |z_2|\tan\theta_{23} e^{i(\eta_2-\eta_3)}\big|, \quad (40)$$

where $\eta_2 = \arg(z_2/z_1)$ and $\eta_3 = \arg(z_3)$, while β is a Majorana phase. With $z_1 = 1$ and arbitrarily assuming $\beta = 0$ and real phases ± 1 associated with η_2 and η_3 one finds the relations

$$\tan\theta_{23} \approx 1, \quad (41)$$

$$\cot\theta_{12} \approx \frac{1}{\sqrt{2}}|z_2 - z_3|, \quad (42)$$

$$\theta_{13} \approx \frac{m_2}{m_3}\frac{1}{3\sqrt{2}}|z_3 + z_2|. \quad (43)$$

We should say immediately that the assumption $\beta = 0$ is not justified so these results can be at best suggestive. With this caveat, we note that approximately tri-maximal solar mixing $\cot\theta_{12} \approx \sqrt{2}$ results from the general condition $|z_2 - z_3| = 2$ which is satisfied by all the proposed forms of CSD.[2] Moreover, CSD with $z_2 = 1$, $z_3 = -1$ leads to $\theta_{13} \approx 0$, CSD2 with $z_2 = 2$, $z_3 = 0$ leads to $\theta_{13} \approx \frac{m_2}{m_3}\frac{\sqrt{2}}{3}$, CSD3 with $z_2 = 3$, $z_3 = 1$ leads to $\theta_{13} \approx \frac{m_2}{m_3}\frac{4}{3\sqrt{2}}$, CSD4 with $z_2 = 4$, $z_3 = 2$ leads to $\theta_{13} \approx \frac{m_2}{m_3}\sqrt{2}$.

Although the above leading order results provide a qualitative understanding of the results obtained for CSD, CSD2, CSD3 and CSD4, they have large corrections of order m_2/m_3, much larger than the errors in the global fits and so do not give reliable predictions. In addition there is a strong dependence on the phase difference between the solar and atmospheric alignments which these results ignore. Moreover, the phase η does not appear in the leading order formula for θ_{13}, but in practise the reactor angle depends strongly on η, as discussed in [13]. On the other hand, while the phases do appear in the solar angle formula, we have arbitrarily and incorrectly assumed $\beta = 0$.

In summary, the leading order results, while providing a qualitative understanding, are quantitatively unreliable and cannot be used to estimate the mixing angles to the required accuracy. The general analysis as performed in [13] did not rely on the leading order results in any way and was not inspired by them. Starting from an exact master formula, the analysis [13] determined from first principles not only the moduli $|z_i|$ but also the phases η_i which are required for a proper definition of any new type of CSD. For example CSD4 with solar alignment $(1, 4, 2)$ is only properly defined once the phases $\eta_2 = \eta_3 = -2\pi/5$ are specified. In retrospect, it is rather fortuitous that the condition $|z_2 - z_3| = 2$ is satisfied for all the proposed forms of CSD. Indeed other more complicated but equally successful examples were found that violated the condition $|z_2 - z_3| = 2$ and these were also tabulated in [13].

[2] I would like to thank Stefan Antusch (private communications) for emphasising the condition $|z_2 - z_3| = 2$.

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