THE PRINCIPAL SERIES OF p-ADIC GROUPS WITH DISCONNECTED CENTRE

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Abstract. Let $\mathcal{G}$ be a split connected reductive group over a local non-archimedean field. We classify all irreducible complex $\mathcal{G}$-representations in the principal series, irrespective of the (dis)connectedness of the centre of $\mathcal{G}$. This leads to a local Langlands correspondence for principal series representations of $\mathcal{G}$. It satisfies all expected properties, in particular it is functorial with respect to homomorphisms of reductive groups.

At the same time we show that every Bernstein component $\mathfrak{s}$ in the principal series has the structure of an extended quotient of Bernstein's torus by Bernstein's finite group (both attached to $\mathfrak{s}$).

Contents

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1. Introduction

Let F be a local non-Archimedean field and let $\mathcal{G}$ be the group of the F-rational points of an F-split connected reductive algebraic group, and let $\mathcal{T}$ be a maximal torus in $\mathcal{G}$. The principal series consists of all $\mathcal{G}$-representations that are constituents of parabolically induced representations from characters of $\mathcal{T}$. The first and most important subclass that was studied, was that of Iwahori-spherical representations. Borel proved in [Bor1] that the category of the representations of $\mathcal{G}$ that are generated by their Iwahori-fixed vectors is naturally equivalent with the category of modules over the Iwahori–Hecke algebra of $\mathcal{G}$. In 1987, in case the centre of $\mathcal{G}$ is connected, Kazhdan and Lusztig [KaLu] classified the representations of this algebra, in terms of data that immediately give rise to Langlands parameters. In 2002, Reeder [Ree2] removed the connectedness assumption of the centre of $\mathcal{G}$ (in the case of Iwahori-spherical representations).

In 1998, granted a mild restriction on the residual characteristic of F, Roche [Roc] generalized the Iwahori-type equivalence of categories to arbitrary Bernstein components in the principal series, proving that the entire principal series of $\mathcal{G}$ can be described in terms of module categories of suitable extended affine Hecke algebras. This was used by Reeder [Ree2] to find a local Langlands correspondence for all principal series representations of split groups $\mathcal{G}$, but then under the assumption that the centre of $\mathcal{G}$ is connected (an assumption which already excludes groups like $\mathrm{SL}_n(F)$).

In this paper, following a similar approach to that of Reeder, we use Roche's realization of types, and his equivalence of categories with Iwahori–Hecke algebras of (possibly disconnected) groups, to construct a local Langlands classification for all the principal series representations of $\mathcal{G}$, with $\mathcal{G}$ the F-points of an arbitrary F-split connected reductive algebraic group (up to the same restriction as in [Roc] on the residual characteristic). We explicitly verify the desiderata for the local Langlands correspondence proposed by Borel in [Bor2].

We further show that these representations are parametrized nicely by suitable extended quotients, in line with the ABPS conjecture, proving that every Bernstein component in the principal series of $\mathcal{G}$ has the structure of an extended quotient. (In the case of connected centre we already established that structure in [ABPS2].)

We will now describe our results in more detail. Let $\mathfrak{B}(\mathcal{G})$ denote the Bernstein spectrum of $\mathcal{G}$, and let $\mathfrak{B}(\mathcal{G},\mathcal{T})$ be the subset of $\mathfrak{B}(\mathcal{G})$ given by all cuspidal pairs $(\mathcal{T},\chi)$, where χ is a character of $\mathcal{T}$. For each $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$ we construct a commutative triangle of bijections

$$
\begin{tikzcd}
 & (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \arrow[rr] & & \Psi(G)^{\mathfrak{s}}_{\mathrm{en}}
\end{tikzcd}
\tag{1}
$$

Here $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is the Bernstein component of $\mathbf{Irr}(\mathcal{G})$ attached to $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$, $\Psi(G)^{\mathfrak{s}}_{\mathrm{en}}$ is the set of enhanced Langlands parameters associated to $\mathfrak{s}$, and G is the complex dual group of $\mathcal{G}$. Furthermore $T^{\mathfrak{s}}$ and $W^{\mathfrak{s}}$ are Bernstein's torus and finite group for $\mathfrak{s}$, and $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is the extended quotient of the second kind resulting from the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$. Equivalently, $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is the set of equivalence classes of irreducible representations of the crossed product algebra $\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$:

$$(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \simeq \mathbf{Irr}(\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}).$$

In examples, $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is much simpler to directly calculate than either $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ or $\Psi(G)^{\mathfrak{s}}_{\rm en}$.

The point $\mathfrak{s} \in \mathfrak{B}(\mathcal{G}, \mathcal{T})$ determines a certain complex reductive group $H^{\mathfrak{s}}$ in the dual group G. If $\mathcal{G}$ has connected centre, then:

- $H^{\mathfrak{s}}$ is connected
- Bernstein's finite group $W^{\mathfrak{s}}$ is the Weyl group of $H^{\mathfrak{s}}$
- Bernstein's torus $T^{\mathfrak{s}}$ is the maximal torus of $H^{\mathfrak{s}}$
- the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ is the standard action of the Weyl group of $H^{\mathfrak{s}}$ on the maximal torus of $H^{\mathfrak{s}}$.

If $\mathcal{G}$ does not have connected centre, then:

- $H^{\mathfrak{s}}$ can be non-connected
- $W^{\mathfrak{s}}$ is the semidirect product $W^{\mathfrak{s}} = \mathcal{W}^{H_0^{\mathfrak{s}}} \rtimes \pi_0(H^{\mathfrak{s}})$ where $H_0^{\mathfrak{s}}$ is the identity component of $H^{\mathfrak{s}}$, and $\mathcal{W}^{H_0^{\mathfrak{s}}}$ is the Weyl group of $H_0^{\mathfrak{s}}$
- $T^{\mathfrak{s}}$ is the maximal torus T of $H_0^{\mathfrak{s}}$
- $W^{\mathfrak{s}} = \mathrm{N}_{H^{\mathfrak{s}}}(T)/T$. The action on T is the evident conjugation action, and $\mathrm{N}_{H^{\mathfrak{s}}}(T)$ is the normalizer in $H^{\mathfrak{s}}$ of T.

See Lemma 3.2 and Eqn. (81).

Semidirect products by $\pi_0(H^{\mathfrak{s}})$ occur frequently in this paper, e.g.

$$\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(H_0^{\mathfrak{s}}) \rtimes \pi_0(H^{\mathfrak{s}})$$

Here $\mathcal{H}^{\mathfrak{s}}$ is a finite type algebra attached by Bernstein to $\mathfrak{s}$ and $\mathcal{H}(H_0^{\mathfrak{s}})$ is the affine Hecke algebra of $H_0^{\mathfrak{s}}$, with parameter q equal to the cardinality of the residue field. Thus $\mathcal{H}^{\mathfrak{s}}$ is an extended affine Hecke algebra.

Similarly, $\pi_0(H^{\mathfrak{s}})$ acts on Lusztig's asymptotic algebra $\mathcal{J}(H_0^{\mathfrak{s}})$. The crossed product algebra

$$\mathcal{J}(H_0^{\mathfrak{s}}) \rtimes \pi_0(H^{\mathfrak{s}})$$

features crucially in Section 13.

In the above commutative triangle, the right slanted arrow is constructed and proved to be a natural bijection by suitably generalising the Springer correspondence for finite and affine Weyl groups (Sections 4 and 8), and by comparing the involved parameters (Sections 6 and 7).

The left slanted arrow in (1) is defined and proved to be a bijection by applying the representation theory of affine Hecke algebras and, in particular, Lusztig's asymptotic algebra. However, in order to apply this theory, it is necessary to prove the equality of certain 2-cocycles, see §13. The technical issues that are confronted in this paper arise from Clifford theory and are very closely connected to the analysis of these 2-cocycles.

Similar 2-cocycles for connected non-split groups can be non-trivial. Hence, for connected non-split groups, a twisted extended quotient must be used in the statement of the ABPS geometric structure conjecture. The ABPS conjecture for connected non-split reductive p-adic groups is developed in [ABPS3].

The horizontal arrow in our main result (see the above commutative triangle and Theorem 15.1 and Proposition 16.1) generalises the Kazhdan–Lusztig parametrization of the irreducible representations of affine Hecke algebras with equal parameters (§9), and also generalises the Reeder–Roche parametrization of the irreducible $\mathcal{G}$-representations in the principal series for groups $\mathcal{G}$ with connected centre (cf. §11). We note that most of the representations considered by Roche–Reeder have positive depth.

We use the new input from $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ to prove that, although the horizontal arrow in (1) is in general not canonical, every element of $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ does canonically determine a Langlands parameter for $\mathcal{G}$ (§14). To establish the horizontal arrow as a local Langlands correspondence for these representations, we also show that it satisfies all the desiderata of Borel, see Sections 16 and 17. In particular we show our constructions are functorial with respect to homomorphisms of reductive groups that have commutative kernel and cokernel. Thus we prove the local Langlands conjectures for a class of representations which contains elements of arbitrarily high depth.

The union over all the $\mathfrak{s} \in \mathfrak{B}(\mathcal{G}, \mathcal{T})$ of the extended quotients of the second kind $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is the extended quotient of the second kind $(\mathbf{Irr}(\mathcal{T})/\!/\mathcal{W}^{\mathcal{G}})_2$, with $\mathcal{W}^{\mathcal{G}} = \mathrm{N}_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$, and the triangles (1) for different $\mathfrak{s}$ combine to a bijective commutative diagram

$$\begin{tikzcd}
 & (\mathbf{Irr}(\mathcal{T})/\!/\mathcal{W}^{\mathcal{G}})_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G}, \mathcal{T}) \arrow[rr] & & \Psi(G)^{\mathrm{prin}}_{\mathrm{en}}
\end{tikzcd}$$

where $\Psi(G)^{\mathrm{prin}}_{\mathrm{en}}$ denotes the collection of enhanced L-parameters for the principal series of $\mathcal{G}$, and $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$ denotes the collection of irreducible principal series representations of $\mathcal{G}$. This diagram shows that the space $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$ can be obtained by a remarkably simple procedure from two well-understood pieces of data, the characters of $\mathcal{T}$ and the Weyl group. All this holds under the restrictions on the residual characteristic stated in Condition 11.1.

2. Extended quotients

Let Γ be a finite group acting on a topological space X,

$$\Gamma \times X \to X.$$

The quotient space X/Γ is obtained by collapsing each orbit to a point. For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

The *extended quotient of the first kind* is obtained by replacing the orbit of x by the set of conjugacy classes of Γ_x. This is done as follows.

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}$, a subspace of $\Gamma \times X$. The group Γ acts on it:

$$\begin{aligned} \Gamma \times \widetilde{X} &\to \widetilde{X} \\ \alpha(\gamma, x) &= (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}. \end{aligned}$$

The extended quotient, denoted $X/\!/\Gamma$, is $\widetilde{X}/\Gamma$. Thus the extended quotient $X/\!/\Gamma$ is the usual quotient for the action of Γ on $\widetilde{X}$. The projection $\widetilde{X} \to X$, $(\gamma, x) \mapsto x$ is Γ-equivariant and gives a surjection of quotient spaces

$$X/\!/\Gamma \to X/\Gamma.$$

This pleasing geometric construction played a crucial role in the first versions of our conjectures [ABP, ABPS1]. However, it has gradually become clear that for purposes in representation theory it is often more appropriate to use another extension of the ordinary quotient. With Γ, X, Γ_x as above, let $\mathbf{Irr}(\Gamma_x)$ be the set of (equivalence classes of) irreducible representations of Γ_x. The *extended quotient of the second kind*, denoted $(X/\!/\Gamma)_2$, is constructed by replacing the orbit of x (for the given action of Γ on X) by $\mathbf{Irr}(\Gamma_x)$. This is done as follows:

Set $\widetilde{X}_2 = \{(x, \tau) \,|\, x \in X \text{ and } \tau \in \mathbf{Irr}(\Gamma_x)\}$. Then Γ acts on $\widetilde{X}_2$ by

$$\begin{aligned} &\Gamma \times \widetilde{X}_2 \to \widetilde{X}_2, \\ &\gamma(x, \tau) = (\gamma x, \gamma_* \tau), \end{aligned}$$

where $\gamma_* \colon \mathbf{Irr}(\Gamma_x) \to \mathbf{Irr}(\Gamma_{\gamma x})$. Now we define

$$(X/\!/\Gamma)_2 := \widetilde{X}_2/\Gamma,$$

i.e. $(X/\!/\Gamma)_2$ is the usual quotient for the action of Γ on $\widetilde{X}_2$. The projection

$$\widetilde{X}_2 \to X, \quad (x, \tau) \mapsto x$$

is Γ-equivariant and so passes to quotient spaces to give the projection

$$(X/\!/\Gamma)_2 \longrightarrow X/\Gamma.$$

Next we will define a twisted version of an extended quotient. Let $\natural$ be a given function which assigns to each $x \in X$ a 2-cocycle $\natural(x) : \Gamma_x \times \Gamma_x \to \mathbb{C}^\times$ where $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$. It is assumed that $\natural(\gamma x)$ and $\gamma_* \natural(x)$ define the same class in $H^2(\Gamma_x, \mathbb{C}^\times)$, where $\gamma_* : \Gamma_x \to \Gamma_{\gamma x}, \alpha \mapsto \gamma \alpha \gamma^{-1}$. Define

$$\widetilde{X}_2^\natural := \{(x, \rho) : x \in X, \rho \in \mathbf{Irr}\, \mathbb{C}[\Gamma_x, \natural(x)]\}.$$

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$\phi_{\gamma,x} : \mathbb{C}[\Gamma_x, \natural(x)] \to \mathbb{C}[\Gamma_{\gamma x}, \natural(\gamma x)]$$

such that:

- $\phi_{\gamma,x}$ is inner if $\gamma x = x$;
- $\phi_{\gamma', \gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma' \gamma, x}$ for all $\gamma', \gamma \in \Gamma, x \in X$.

We call these maps connecting homomorphisms, because they are reminiscent of a connection on a vector bundle. Then we can define Γ-action on $\widetilde{X}_2^\natural$ by

$$\gamma \cdot (x, \rho) = (\gamma x, \rho \circ \phi_{\gamma,x}^{-1}).$$

We form the *twisted extended quotient*

$$(X/\!/\Gamma)_2^\natural := \widetilde{X}_2^\natural/\Gamma.$$

Notice that this reduces to the extended quotient of the second kind if $\natural(x)$ is trivial for all $x \in X$. We will apply this construction in the following two special cases.

1. Given two finite groups Γ_1, Γ and a group homomorphism $\Gamma \to \mathrm{Aut}(\Gamma_1)$, we can form the semidirect product $\Gamma_1 \rtimes \Gamma$. Let $X = \mathbf{Irr}\,\Gamma_1$. Now Γ acts on $\mathbf{Irr}\,\Gamma_1$ and we get $\natural$ as follows. Given $x \in \mathbf{Irr}\,\Gamma_1$ choose an irreducible representation $\pi_x : \Gamma_1 \to \mathrm{GL}(V)$ whose isomorphism class is x. For each $\gamma \in \Gamma$ consider π_x twisted by γ *i.e.,* consider $\gamma \cdot \pi_x : \gamma_1 \mapsto \pi_x(\gamma^{-1}\gamma_1\gamma)$. Since $\gamma \cdot \pi_x$ is equivalent to $\pi_{\gamma x}$, there exists a nonzero intertwining operator

$$T_{\gamma,x} \in \mathrm{Hom}_{\Gamma_x}(\gamma \cdot \pi_x, \pi_{\gamma x}).$$

By Schur's lemma it is unique up to scalars, but in general there is no preferred choice. For $\gamma, \gamma' \in \Gamma_x$ there exists a unique $c \in \mathbb{C}^\times$ such that

$$T_{\gamma,x} \circ T_{\gamma',x} = cT_{\gamma\gamma',x}.$$

We define the 2-cocycle by $\natural(x)(\gamma,\gamma') = c$. Let $N_{\gamma,x}$ with $\gamma \in \Gamma_x$ be the standard basis of $\mathbb{C}[\Gamma_x, \natural(x)]$. The algebra homomorphism $\phi_{g,x}$ is essentially conjugation by $T_{g,x}$, but the precise definition is

$$(2) \qquad \phi_{g,x}(N_{\gamma,x}) = \lambda N_{g\gamma g^{-1},gx} \quad \text{if} \quad T_{g,x}T_{\gamma,x}T_{g,x}^{-1} = \lambda T_{g\gamma g^{-1},gx}, \lambda \in \mathbb{C}^\times.$$

Notice that (2) does not depend on the choice of $T_{g,x}$. This leads to a new formulation of a classical theorem of Clifford.

Lemma 2.1. *There is a bijection*

$$\mathbf{Irr}(\Gamma_1 \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\,\Gamma_1//\Gamma)_2^\natural.$$

Proof. The proof proceeds by comparing our construction with the classical theory of Clifford; for an exposition of Clifford theory, see [RaRa]. □

The above bijection is in general not canonical, it depends on the choice of the intertwining operators $T_{\gamma,x}$.

Lemma 2.2. *If Γ_1 is abelian, then we have a natural bijection*

$$\mathbf{Irr}(\Gamma_1 \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\,\Gamma_1//\Gamma)_2.$$

Proof. The irreducible representations of Γ_1 are 1-dimensional, and we have $\gamma \cdot \pi_x = \pi_x$ for $\gamma \in \Gamma_x$. In that case we take each $T_{\gamma,x}$ to be the identity, so that $\natural(x)$ is trivial. Then the projective representations of Γ_x which occur in the construction are all true representations and (2) simplifies to $\phi_{g,x}(T_{\gamma,x}) = T_{g\gamma g^{-1},gx}$. Thus we recover the extended quotient of the second kind in Lemma 2.1. □

2. Given a $\mathbb{C}$-algebra R, a finite group Γ and a group homomorphism $\Gamma \to \mathrm{Aut}(R)$, we can form the crossed product algebra

$$R \rtimes \Gamma := \{\sum_{\gamma \in \Gamma} r_\gamma \gamma \,:\, r_\gamma \in R\},$$

with multiplication given by the distributive law and the relation

$$\gamma r = \gamma(r)\gamma, \quad \text{for } \gamma \in \Gamma \text{ and } r \in R.$$

Now Γ acts on $X := \mathbf{Irr}\,R$. Assuming that all simple R-modules have countable dimension, so that Schur's lemma is valid, we construct $\natural(V)$ and $\phi_{\gamma,V}$ as above for group algebras. Here we have

$$\widetilde{X}_2^\natural = \{(V,\tau) \,:\, V \in \mathbf{Irr}\,R,\ \tau \in \mathbf{Irr}\,\mathbb{C}[\Gamma_V, \natural(V)]\}.$$

Lemma 2.3. *There is a bijection*

$$\mathbf{Irr}(R \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\, R//\Gamma)_2^{\natural}.$$

If all simple R-modules are one-dimensional, then it becomes a natural bijection

$$\mathbf{Irr}(R \rtimes \Gamma) \longleftrightarrow (\mathbf{Irr}\, R//\Gamma)_2.$$

Proof. The proof proceeds by comparing our construction with the theory of Clifford as stated in [RaRa, Theorem A.6]. The naturality part can be shown in the same way as Lemma 2.2. □

Notation 2.4. For (V, τ) as above, $V \otimes V_\tau^*$ is a simple $R \rtimes \Gamma_V$-module, in a way which depends on the choice of intertwining operators $T_{\gamma,V}$. The simple $R \rtimes \Gamma$-module associated to (V, τ) by the bijection of Lemma 2.3 is

$$V \rtimes \tau^* := \mathrm{Ind}_{R\rtimes\Gamma_V}^{R\rtimes V}(V \otimes V_\tau^*). \tag{3}$$

Similarly, we shall denote by $\tau_1 \rtimes \tau^*$ the element of $\mathbf{Irr}(\Gamma_1 \rtimes \Gamma)$ which corresponds to (τ_1, τ) by the bijection of Lemma 2.1.

3. Weyl groups of disconnected groups

Let M be a reductive complex algebraic group. Then M may have a finite number of connected components, M^0 is the identity component of M, and $\mathcal{W}^{M^0}$ is the Weyl group of M^0:

$$\mathcal{W}^{M^0} := \mathrm{N}_{M^0}(T)/T$$

where T is a maximal torus of M^0. We will need the analogue of the Weyl group for the possibly disconnected group M.

Lemma 3.1. *Let M, M^0, T be as defined above. Then we have*

$$\mathrm{N}_M(T)/T \cong \mathcal{W}^{M^0} \rtimes \pi_0(M).$$

Proof. The group $\mathcal{W}^{M^0}$ is a normal subgroup of $\mathrm{N}_M(T)/T$. Indeed, let $n \in \mathrm{N}_{M^0}(T)$ and let $n' \in \mathrm{N}_M(T)$, then $n'nn'^{-1}$ belongs to M^0 (since the latter is normal in M) and normalizes T, that is, $n'nn'^{-1} \in \mathrm{N}_{M^0}(T)$. On the other hand, $n'(nT)n'^{-1} = n'nn'^{-1}(n'Tn'^{-1}) = n'nn'^{-1}T$.

Let B be a Borel subgroup of M^0 containing T. Let $w \in \mathrm{N}_M(T)/T$. Then wBw^{-1} is a Borel subgroup of M^0 (since, by definition, the Borel subgroups of an algebraic group are the maximal closed connected solvable subgroups). Moreover, wBw^{-1} contains T. In a connected reductive algebraic group, the intersection of two Borel subgroups always contains a maximal torus and the two Borel subgroups are conjugate by a element of the normalizer of that torus. Hence B and wBw^{-1} are conjugate by an element w_1 of $\mathcal{W}^{M^0}$. It follows that $w_1^{-1}w$ normalises B. Hence

$$w_1^{-1}w \in \mathrm{N}_M(T)/T \cap \mathrm{N}_M(B) = \mathrm{N}_M(T, B)/T,$$

that is,

$$\mathrm{N}_M(T)/T = \mathcal{W}^{M^0} \cdot (\mathrm{N}_M(T, B)/T).$$

Finally, we have

$$\mathcal{W}^{M^0} \cap (\mathrm{N}_M(T, B)/T) = \mathrm{N}_{M^0}(T, B)/T = \{1\},$$

since $\mathrm{N}_{M^0}(B) = B$ and $B \cap \mathrm{N}_{M^0}(T) = T$. This proves that

$$\mathrm{N}_M(T) \cong \mathrm{N}_{M^\circ}(T) \rtimes \mathrm{N}_M(B,T).$$

Now consider the following map:

$$\mathrm{N}_M(T,B)/T \to M/M^0 \qquad mT \mapsto mM^0. \tag{4}$$

It is injective. Indeed, let $m, m' \in \mathrm{N}_M(T,B)$ such that $mM^0 = m'M^0$. Then $m^{-1}m' \in M^0 \cap \mathrm{N}_M(T,B) = \mathrm{N}_{M^0}(T,B) = T$ (as we have seen above). Hence $mT = m'T$.

On the other hand, let m be an element in M. Then $m^{-1}Bm$ is a Borel subgroup of M^0, hence there exists $m_1 \in M^0$ such that $m^{-1}Bm = m_1^{-1}Bm_1$. It follows that $m_1m^{-1} \in \mathrm{N}_M(B)$. Also $m_1m^{-1}Tmm_1^{-1}$ is a torus of M^0 which is contained in $m_1m^{-1}Bmm_1^{-1} = B$. Hence T and $m_1m^{-1}Tmm_1^{-1}$ are conjugate in B: there is $b \in B$ such that $m_1m^{-1}Tmm_1^{-1} = b^{-1}Tb$. Then $n := bm_1m^{-1} \in \mathrm{N}_M(T,B)$. It gives $m = n^{-1}bm_1$. Since $bm_1 \in M^0$, we obtain $mM^0 = n^{-1}M^0$. Hence the map (4) is surjective. □

Let G be a connected complex reductive group and let T be a maximal torus in G. The Weyl group of G is denoted $\mathcal{W}^G$.

Lemma 3.2. *Let A be a subgroup of T and write $M = \mathrm{Z}_G(A)$. Then the isotropy subgroup of A in $\mathcal{W}^G$ is*

$$\mathcal{W}^G_A = \mathrm{N}_M(T)/T \cong \mathcal{W}^{M^0} \rtimes \pi_0(M).$$

In case that the group M is connected, $\mathcal{W}^G_A$ is the Weyl group of M.

Proof. Let $R(G,T)$ denote the root system of G. According to [SpSt, § 4.1], the group $M = \mathrm{Z}_G(A)$ is the reductive subgroup of G generated by T and those root groups U_α for which $\alpha \in R(G,T)$ has trivial restriction to A together with those Weyl group representatives $n_w \in \mathrm{N}_G(T)$ ($w \in \mathcal{W}^G$) for which $w(t) = t$ for all $t \in A$. This shows that $\mathcal{W}^G_A = \mathrm{N}_M(T)/T$, which by Lemma 3.1 is isomorphic to $\mathcal{W}^{M^0} \rtimes \pi_0(M)$.

Also by [SpSt, § 4.1], the identity component of M is generated by T and those root groups U_α for which α has trivial restriction to A. Hence the Weyl group $\mathcal{W}^{M^\circ}$ is the normal subgroup of $\mathcal{W}^G_A$ generated by those reflections s_α and

$$\mathcal{W}^G_A/\mathcal{W}^{M^\circ} \cong M/M^\circ.$$

In particular, if M is connected then $\mathcal{W}^G_A$ is the Weyl group of M. □

Consequently, for $t \in T$ such that $M = \mathrm{Z}_G(t)$ we have

$$(T/\!/\mathcal{W}^G)_2 = \{(t,\sigma) : t \in T, \sigma \in \mathbf{Irr}(\mathcal{W}^G_t)\}/\mathcal{W}^G, \tag{5}$$

$$\mathbf{Irr}\, \mathcal{W}^G_t = (\mathbf{Irr}\, \mathcal{W}^{M^0} /\!/ \pi_0(M))^\natural_2. \tag{6}$$

We fix a Borel subgroup B_0 of M° containing T and let $\Delta(B_0,T)$ be the set of roots of (M°,T) that are simple with respect to B_0. We may and will assume that this agrees with the previously chosen simple reflections in $\mathcal{W}^{M^\circ}$. In every root subgroup U_α with $\alpha \in \Delta(B_0,T)$ we pick a nontrivial element u_α. The data $(M^\circ, T, (u_\alpha)_{\alpha\in\Delta(B_0,T)})$ are called a *pinning* of M°. This notion is useful in the following well-known result:

Lemma 3.3. *The short exact sequence*

$$1 \to M^\circ/\mathrm{Z}(M^\circ) \to M/\mathrm{Z}(M^\circ) \to \pi_0(M) \to 1$$

is split. A splitting can be obtained by sending $C \in \pi_0(M)$ to the unique element of $C/\mathrm{Z}(M^\circ) \subset M/\mathrm{Z}(M^\circ)$ that preserves the chosen pinning.

Proof. The connected reductive group M° acts transitively on the set of pairs (B', T') with B' a Borel subgroup containing a maximal torus T'. Since the different simple roots are independent functions on T, M° also acts transitively on the set of pinnings. The stabilizer of a given pinning is $\mathrm{Z}(M^\circ)$, so $M^\circ/\mathrm{Z}(M^\circ)$ acts simply transitively on the set of pinnings for M°. This shows that the given recipe is valid and produces a splitting. $\square$

4. An extended Springer correspondence

Let M° be a connected reductive complex group. We take $x \in M^\circ$ unipotent and we abbreviate

$$A_x := \pi_0(\mathrm{Z}_{M^0}(x)). \tag{7}$$

Let $x \in M^\circ$ be unipotent and let $\mathcal{B}^x = \mathcal{B}^x_{M^\circ}$ be the variety of Borel subgroups of M° containing x. All the irreducible components of $\mathcal{B}^x$ have the same dimension $d(x)$ over $\mathbb{R}$, see [ChGi, Corollary 3.3.24]. Let $H_{d(x)}(\mathcal{B}^x, \mathbb{C})$ be its top homology, let ρ be an irreducible representation of A_x and write

$$\tau(x,\rho) = \mathrm{Hom}_{A_x}\big(\rho, H_{d(x)}(\mathcal{B}^x, \mathbb{C})\big). \tag{8}$$

We call $\rho \in \mathbf{Irr}(A_x)$ geometric if $\tau(x,\rho) \neq 0$. The Springer correspondence yields a bijection

$$(x,\rho) \mapsto \tau(x,\rho) \tag{9}$$

between the set of M^0-conjugacy classes of pairs (x,ρ) formed by a unipotent element $x \in M^0$ and an irreducible geometric representation ρ of A_x, and the equivalence classes of irreducible representations of the Weyl group $\mathcal{W}^{M^0}$.

Remark 4.1. *The Springer correspondence which we employ here sends the trivial unipotent class to the trivial $\mathcal{W}^{M^\circ}$-representation and the regular unipotent class to the sign representation. The difference with Springer's construction via a reductive group over a field of positive characteristic consists of tensoring with the sign representation of $\mathcal{W}^{M^0}$, see* [Hot].

Choose a set of simple reflections for $\mathcal{W}^{M^\circ}$ and let Γ be a group of automorphisms of the Coxeter diagram of $\mathcal{W}^{M^\circ}$. Then Γ acts on $\mathcal{W}^{M^\circ}$ by group automorphisms, so we can form the semidirect product $\mathcal{W}^{M^\circ} \rtimes \Gamma$. Furthermore Γ acts on $\mathbf{Irr}(\mathcal{W}^{M^\circ})$, by $\gamma \cdot \tau = \tau \circ \gamma^{-1}$. The stabilizer of $\tau \in \mathbf{Irr}(\mathcal{W}^{M^\circ})$ is denoted Γ_τ. As described in Section 2, Clifford theory for $\mathcal{W}^{M^\circ} \rtimes \Gamma$ produces a 2-cocycle $\natural(\tau) : \Gamma_\tau \times \Gamma_\tau \to \mathbb{C}^\times$.

Since $M^\circ/Z(M^\circ)$ acts simply transitively on the set of pinnings of M° (see the proof of Lemma 3.3), the action of $\gamma \in \Gamma$ on the Coxeter diagram of $\mathcal{W}^{M^\circ}$ lifts uniquely to an action of γ on M° which preserves the pinning chosen in Section 3. In this way we construct the semidirect product $M := M^\circ \rtimes \Gamma$. By Lemma 3.2 we may identify $\mathcal{W}^M$ with $\mathcal{W}^{M^\circ} \rtimes \Gamma$. We want to generalize the Springer correspondence to this kind of group. First we need to prove a technical lemma, which in a sense extends Lemma 3.3.

Lemma 4.2. *Let* $\rho \in \mathbf{Irr}(\pi_0(\mathrm{Z}_{M^\circ}(x)))$ *and write*

$$\mathrm{Z}_M(x,\rho) = \{m \in \mathrm{Z}_M(x) | \rho \circ \mathrm{Ad}_m^{-1} \cong \rho\}.$$

Let $[x,\rho]_{M^\circ}$ *be the* M°*-orbit of* (x,ρ) *and* $\Gamma_{[x,\rho]_{M^\circ}}$ *its stabilizer in* Γ. *The following short exact sequence splits:*

$$1 \to \pi_0\big(\mathrm{Z}_{M^\circ}(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \Gamma_{[x,\rho]_{M^\circ}} \to 1.$$

Proof. First we ignore ρ. According to the classification of unipotent orbits in complex reductive groups [Car, Theorem 5.9.6] we may assume that x is distinguished unipotent in a Levi subgroup $L \subset M^\circ$ that contains T. Notice that the derived subgroup $\mathcal{D}(L)$ contains only the part of T generated by the coroots of (L,T). Then

$$L' := \mathrm{Z}_{M^\circ}(\mathcal{D}(L))(T \cap \mathcal{D}(L)) = \mathrm{Z}_{M^\circ}(\mathcal{D}(L))T.$$

is a reductive group with maximal torus T, whose roots are precisely those that are orthogonal to the coroots of (L,T). We choose Borel subgroups $B_L \subset L$ and $B'_L \subset L'$ such that $x \in B_L$ and $T \subset B_L \cap B'_L$.

Let $[x]_{M^\circ}$ be the M°-conjugacy class of x and $\Gamma_{[x]_{M^\circ}}$ its stabilizer in Γ. Any $\gamma \in \Gamma_{[x]_{M^\circ}}$ must also stabilize the M°-conjugacy class of L, and $T = \gamma(T) \subset \gamma(L)$, so there exists a $w_1 \in \mathcal{W}^{M^\circ}$ with $w_1\gamma(L) = L$. Adjusting w_1 by an element of $W(L,T) \subset \mathcal{W}^{M^\circ}$, we can achieve that moreover $w_1\gamma(B_L) = B_L$. Then $w_1\gamma(L') = L'$, so we can find a unique $w_2 \in W(L',T) \subset \mathcal{W}^{M^\circ}$ with $w_2 w_1 \gamma(B'_L) = B'_L$. Notice that the centralizer of $\Phi(B_L,T) \cup \Phi(B'_L,T)$ in $\mathcal{W}^{M^\circ}$ is trivial, because it is generated by reflections and no root in $\Phi(M^\circ,T)$ is orthogonal to this set of roots. Therefore the above conditions completely determine $w_2 w_1 \in \mathcal{W}^{M^\circ}$.

The element $w_1\gamma \in \mathcal{W}^{M^\circ} \rtimes \Gamma$ acts on $\Delta(B_L,T)$ by a diagram automorphism. So upon choosing $u_\alpha \in U_\alpha \setminus \{1\}$ for $\alpha \in \Delta(B_L,T)$, Lemma 3.3 shows that $w_1\gamma$ can be represented by a unique element

$$\overline{w_1\gamma} \in \mathrm{Aut}\big(\mathcal{D}(L), T, (u_\alpha)_{\alpha \in \Delta(B_L,T)}\big).$$

The distinguished unipotent class of $x \in L$ is determined by its Bala–Carter diagram. The classification of such diagrams [Car, §5.9] shows that there exists an element $\bar{x}$ in the same class as x, such that $\mathrm{Ad}_{\overline{w_1\gamma}}(\bar{x}) = \bar{x}$. We may just as well assume that we had $\bar{x}$ instead of x from the start, and that $\overline{w_1\gamma} \in \mathrm{Z}_M(x)$. Clearly we can find a representative $\overline{w_2}$ for w_2 in $\mathrm{Z}_M(x)$, so we obtain

$$\overline{w_2}\,\overline{w_1\gamma} \in \mathrm{Z}_M(x) \cap \mathrm{N}_M(T) \quad \text{and} \quad w_2 w_1 \gamma \in \frac{\mathrm{Z}_M(x) \cap \mathrm{N}_M(T)}{\mathrm{Z}(M^\circ)\, T}.$$

Since $w_2 w_1 \in \mathcal{W}^{M^\circ}$ is unique,

$$s : \Gamma_{[x]_{M^\circ}} \to \frac{\mathrm{Z}_M(x) \cap \mathrm{N}_M(T)}{\mathrm{Z}(M^\circ)\, T},\ \gamma \mapsto w_2 w_1 \gamma \tag{10}$$

is a group homomorphism.

We still have to analyse the effect of $\Gamma_{[x]_{M^\circ}}$ on $\rho \in \mathbf{Irr}(A_x)$. Obviously composing with Ad_m for $m \in \mathrm{Z}_{M^\circ}(x)$ does not change the equivalence class of any representation of $A_x = \pi_0(\mathrm{Z}_{M^\circ}(x))$. Hence $\gamma \in \Gamma_{[x]_{M^\circ}}$ stabilizes ρ if and only if any lift of γ in $\mathrm{Z}_M(x)$ does. This applies in particular to $\overline{w_2}\,\overline{w_1}\gamma$, and therefore

$$s(\Gamma_{[x,\rho]_{M^\circ}}) \subset \big(\mathrm{Z}_M(x,\rho) \cap \mathrm{N}_M(T)\big) / \big(\mathrm{Z}(M^\circ)\,T\big).$$

Since the torus T is connected, s determines a group homomorphism from $\Gamma_{[x,\rho]_{M^\circ}}$ to $\pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big)$, which is the required splitting. □

A further step towards a Springer correspondence for $\mathcal{W}^M$ is:

Proposition 4.3. *The class of $\natural(\tau)$ in $H^2(\Gamma_\tau, \mathbb{C}^\times)$ is trivial for all $\tau \in$ $\mathbf{Irr}(\mathcal{W}^{M^\circ})$. There is a bijection between*

$$\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})/\!/\Gamma\big)_2 \qquad \textit{and} \qquad \mathbf{Irr}(\mathcal{W}^{M^\circ} \rtimes \Gamma) = \mathbf{Irr}(\mathcal{W}^M).$$

Proof. There are various ways to construct the Springer correspondence for $\mathcal{W}^{M^\circ}$, for the current proof we use the method with Borel–Moore homology. Let Z_{M° be the Steinberg variety of M° and $H_{\mathrm{top}}(\mathrm{Z}_{M^\circ})$ its homology in the top degree

$$2\dim_{\mathbb{C}} \mathrm{Z}_{M^\circ} = 4\dim_{\mathbb{C}} \mathcal{B}_{M^\circ} = 4(\dim_{\mathbb{C}} M^\circ - \dim_{\mathbb{C}} B_0),$$

with rational coefficients. We define a natural algebra isomorphism

$$\mathbb{Q}[\mathcal{W}^{M^\circ}] \to H_{\mathrm{top}}(\mathrm{Z}_{M^\circ}) \tag{11}$$

as the composition of [ChGi, Theorem 3.4.1] and a twist by the sign representation of $\mathbb{Q}[\mathcal{W}^{M^\circ}]$. By [ChGi, Section 3.5] the action of $\mathcal{W}^{M^\circ}$ on $H_*(\mathcal{B}^x, \mathbb{C})$ (as defined by Lusztig) corresponds to the convolution product in Borel–Moore homology.

Since M° is normal in M, the groups Γ, M and $M/\mathrm{Z}(M)$ act on the Steinberg variety Z_{M° via conjugation. The induced action of the connected group M° on $H_{\mathrm{top}}(\mathrm{Z}_{M^\circ})$ is trivial, and it easily seen from [ChGi, Section 3.4] that the action of Γ on $H(\mathrm{Z}_{M^\circ})$ makes (11) Γ-equivariant.

The groups Γ, M and $M/\mathrm{Z}(M)$ also act on the pairs (x,ρ) and on the varieties of Borel subgroups, by

$$\begin{aligned} \mathrm{Ad}_m(x,\rho) &= (mxm^{-1}, \rho \circ \mathrm{Ad}_m^{-1}), \\ \mathrm{Ad}_m &: \mathcal{B}^x \to \mathcal{B}^{mxm^{-1}},\ B \mapsto mBm^{-1}. \end{aligned}$$

Given $m \in M$, this provides a linear bijection $H_*(\mathrm{Ad}_m)$:

$$\mathrm{Hom}_{A_x}(\rho, H_*(\mathcal{B}^x, \mathbb{C})) \to \mathrm{Hom}_{A_{mxm^{-1}}}(\rho \circ \mathrm{Ad}_m^{-1}, H_*(\mathcal{B}^{mxm^{-1}}, \mathbb{C})).$$

The convolution product in Borel–Moore homology is compatible with these M-actions so, as in [ChGi, Lemma 3.5.2], the following diagram commutes for all $h \in H_{\mathrm{top}}(\mathrm{Z}_{M^\circ})$:

$$\begin{array}{ccc} H_*(\mathcal{B}^x, \mathbb{C}) & \xrightarrow{h} & H_*(\mathcal{B}^x, \mathbb{C}) \\ \downarrow H_*(\mathrm{Ad}_m) & & \downarrow H_*(\mathrm{Ad}_m) \\ H_*(\mathcal{B}^{mxm^{-1}}, \mathbb{C}) & \xrightarrow{m\cdot h} & H_*(\mathcal{B}^{mxm^{-1}}, \mathbb{C}). \end{array} \tag{12}$$

In case $m \in M^\circ\gamma$ and $m \cdot h$ corresponds to $w \in \mathcal{W}^{M^\circ}$, the element $h \in H(\mathrm{Z}_{M^\circ})$ corresponds to $\gamma^{-1}(w)$, so (12) becomes

$$H_*(\mathrm{Ad}_m) \circ \tau(x,\rho)(\gamma^{-1}(w)) = \tau(mxm^{-1}, \rho \circ \mathrm{Ad}_m^{-1})(w) \circ H_*(\mathrm{Ad}_m). \tag{13}$$

Denoting the M°-conjugacy class of (x,ρ) by $[x,\rho]_{M^\circ}$, we can write

$$\begin{aligned}(14)\qquad \Gamma_{\tau(x,\rho)} &= \{\gamma \in \Gamma \mid \tau(x,\rho) \circ \gamma^{-1} \cong \tau(x,\rho)\} \\ &= \{\gamma \in \Gamma \mid [\mathrm{Ad}_\gamma (x,\rho)]_{M^\circ} = [x,\rho]_{M^\circ}\} =: \Gamma_{[x,\rho]_{M^\circ}}.\end{aligned}$$

This group fits in an exact sequence

$$(15)\quad 1 \to \pi_0\big(\mathrm{Z}_{M^\circ}(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \Gamma_{[x,\rho]_{M^\circ}} \to 1,$$

which by Lemma 4.2 admits a splitting

$$s : \Gamma_{[x,\rho]_{M^\circ}} \to \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big).$$

By homotopy invariance in Borel–Moore homology $H_*(\mathrm{Ad}_z) = \mathrm{id}_{H_*(\mathcal{B}^x,\mathbb{C})}$ for any $z \in \mathrm{Z}_{M^\circ}(x,\rho)^\circ \mathrm{Z}(M^\circ)$, so $H_*(\mathrm{Ad}_m)$ is well-defined for $m \in \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big)$. In particular we obtain for every $\gamma \in \Gamma_{\tau(x,\rho)} = \Gamma_{[x,\rho]_{M^\circ}}$ a linear bijection

$$H_*(\mathrm{Ad}_{s(\gamma)}) : \mathrm{Hom}_{A_x}(\rho, H_{d(x)}(\mathcal{B}_x,\mathbb{C})) \to \mathrm{Hom}_{A_x}(\rho, H_{d(x)}(\mathcal{B}_x,\mathbb{C})),$$

which by (13) intertwines the $\mathcal{W}^{M^\circ}$-representations $\tau(x,\rho)$ and $\tau(x,\rho)\circ\gamma^{-1}$. By construction

$$(16)\qquad H_*(\mathrm{Ad}_{s(\gamma)}) \circ H_*(\mathrm{Ad}_{s(\gamma')}) = H_*(\mathrm{Ad}_{s(\gamma\gamma')}).$$

This establishes the triviality of the 2-cocycle $\natural(\tau) = \natural(\tau(x,\rho))$.

Consider any $g \in \Gamma \setminus \Gamma_x$. Then $g\tau$ corresponds to

$$\mathrm{Ad}_g(x,\rho) = (gxg^{-1}, \rho \circ \mathrm{Ad}_g^{-1}).$$

For $\gamma \in \Gamma_x$ we define an intertwining operator in

$$\mathrm{End}_{\mathcal{W}^{M^\circ}}\big(\mathrm{Hom}_{A_{gxg^{-1}}}(\rho\circ\mathrm{Ad}_g^{-1}, H_{d(x)}(\mathcal{B}_{gxg^{-1}},\mathbb{C}))\big)$$

associated to $g\gamma g^{-1} \in \Gamma_{gxg^{-1}}$ as

$$(17)\qquad H_{d(x)}(\mathrm{Ad}_{gs(\gamma)g^{-1}}) = H_{d(x)}(\mathrm{Ad}_g) H_{d(x)}(\mathrm{Ad}_{s(\gamma)}) H_{d(x)}(g^{-1}).$$

We do the same for any other point in the Γ-orbit of (x,ρ). Then (16) shows that the resulting intertwining operators do not depend on the choices of the elements g.

We follow the same recipe for any other Γ-orbit of Springer parameters (x',ρ'). As connecting homomorphism $\phi_{g,(x',\rho')}$ we take conjugation by $H_{d(x')}(\mathrm{Ad}_g)$. From this construction and Lemma 2.3 we obtain a bijection between $\mathbf{Irr}(\mathcal{W}^{M^\circ} \rtimes \pi_0(M))$ and the extended quotient of the second kind $\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})//\Gamma\big)_2$. $\square$

We note that the bijection from Proposition 4.3 is in general not canonical, because the splitting from Lemma 4.2 is not. But with some additional effort we can extract a natural description of $\mathbf{Irr}(\mathcal{W}^M)$ from Proposition 4.3.

We say that an irreducible representation ρ_1 of $\mathrm{Z}_M(x)$ is geometric if every irreducible $\mathrm{Z}_{M^\circ}(x)$-subrepresentation of ρ_1 is geometric in the previously defined sense. Notice that this condition forces ρ_1 to factor through the component group $\pi_0(\mathrm{Z}_M(x))$.

We note that $\pi_0(\mathrm{Z}_M(x))$ acts naturally on $H_{d(x)}(\mathcal{B}^x)$ and on $\mathbb{C}[\Gamma]$, via the isomorphism

$$(18)\qquad \mathrm{Z}_M(x)/\mathrm{Z}_{M^\circ}(x) \cong \Gamma_{[x]_{M^\circ}}.$$

Theorem 4.4. *There is a natural bijection from*

$$\big\{(x,\rho_1) \mid x \in M^\circ \textit{ unipotent}, \rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x))\big) \textit{ geometric}\big\}/M$$

to $\mathbf{Irr}(\mathcal{W}^M)$*, which sends* (x,ρ_1) *to*

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_M(x))}\big(\rho_1, H_{d(x)}(\mathcal{B}^x) \otimes \mathbb{C}[\Gamma]\big).$$

Proof. Let us take another look at the geometric representations of $A_x = \mathrm{Z}_{M^\circ}(x)$. By construction they factor through $\pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ))$. From (10) we get a group isomorphism

$$\pi_0(\mathrm{Z}_M(x)/\mathrm{Z}(M^\circ)) \cong \pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ)) \rtimes s\big(\Gamma_{[x]_{M^\circ}}\big). \tag{19}$$

Suppose that $\rho \in \mathbf{Irr}(A_x)$ is geometric. Then the operators $H_{d(x)}(\mathrm{Ad}_{s(\gamma)})$ intertwine ρ with the $\pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ))$-representation $s(\gamma)\cdot\rho$ and they satisfy the multiplicativity relation (16). Now it follows from Lemma 2.1 that every irreducible geometric representation of $\pi_0(\mathrm{Z}_M(x))$ can be written in a unique way as $\rho \rtimes \sigma$, with $\rho \in \mathbf{Irr}(A_x)$ geometric and

$$\sigma \in \mathbf{Irr} s(\Gamma_{[x,\rho]_{M^\circ}}) = \mathbf{Irr}(\Gamma_{[x,\rho]_{M^\circ}}).$$

This enables us to rewrite $\widetilde{\mathbf{Irr}(\mathcal{W}^{M^\circ})}$ as a union of pairs $(x, \rho_1 = \rho \rtimes \sigma)$, with x in a finite union of chosen Γ-orbits of unipotent elements. Clearly M acts on the larger space

$$\big\{(x,\rho_1) \mid x \in M^\circ \text{ unipotent}, \rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x))\big) \text{ geometric}\big\}$$

by conjugation of the x-parameter and the action induced by $H_*(\mathrm{Ad}_m)$ on the ρ_1-parameter. By (17) and the construction of $s(\gamma)$ in Lemma 4.2, this extends the action of Γ on $\widetilde{\mathbf{Irr}(\mathcal{W}^{M^\circ})}$. That provides the bijection from $\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})/\!/\Gamma\big)_2$ to set of the M-association classes of pairs (x,ρ_1). Combining this with Proposition 4.3, we obtain a bijection between $\mathbf{Irr}(\mathcal{W}^M)$ and the latter set. If we work out the definitions and use (3), we see that it sends $(x, \rho_1 = \rho \rtimes \sigma)$ to

$$\tau(x,\rho) \rtimes \sigma^* = \mathrm{Ind}_{\mathcal{W}^{M^\circ} \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ} \rtimes \Gamma} \big(\tau(x,\rho) \otimes \sigma^*\big).$$

We can rewrite this as

$$\mathrm{Ind}_{\mathcal{W}^{M^\circ} \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ} \rtimes \Gamma} \big(\mathrm{Hom}_{A_x}(\rho, H_{d(x)}(\mathcal{B}^x)) \otimes \sigma^*\big) \cong$$
$$\mathrm{Ind}_{\mathcal{W}^{M^\circ} \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ} \rtimes \Gamma} \big(\mathrm{Hom}_{\Gamma_{[x,\rho]_{M^\circ}}}\big(\sigma, \mathrm{Hom}_{A_x}(\rho, H_{d(x)}(\mathcal{B}^x)) \otimes \mathbb{C}[\Gamma_{[x,\rho]_{M^\circ}}]\big)\big).$$

In view of Lemma 4.2, the previous line is isomorphic to

$$\mathrm{Ind}_{\mathcal{W}^{M^\circ} \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{\mathcal{W}^{M^\circ} \rtimes \Gamma} \big(\mathrm{Hom}_{\mathrm{Z}_M(x,\rho)}\big(\rho \otimes \sigma, H_{d(x)}(\mathcal{B}^x) \otimes \mathbb{C}[\Gamma_{[x,\rho]_{M^\circ}}]\big)\big) \cong$$
$$\mathrm{Ind}_{\mathcal{W}^{M^\circ} \rtimes \Gamma_{[x]_{M^\circ}}}^{\mathcal{W}^{M^\circ} \rtimes \Gamma} \big(\mathrm{Hom}_{\mathrm{Z}_M(x,\rho)}\big(\rho \otimes \sigma, H_{d(x)}(\mathcal{B}^x) \otimes \mathbb{C}[\Gamma_{[x]_{M^\circ}}]\big)\big).$$

With Frobenius reciprocity and (18) we simplify the above expression to

$$\mathrm{Ind}_{\mathcal{W}^{M^\circ} \rtimes \Gamma_{[x]_{M^\circ}}}^{\mathcal{W}^{M^\circ} \rtimes \Gamma} \big(\mathrm{Hom}_{\mathrm{Z}_M(x)}\big(\rho \rtimes \sigma, H_{d(x)}(\mathcal{B}^x) \otimes \mathbb{C}[\Gamma_{[x]_{M^\circ}}]\big)\big) \cong$$
$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_M(x))}\big(\rho \rtimes \sigma, H_{d(x)}(\mathcal{B}^x) \otimes \mathbb{C}[\Gamma]\big).$$

The last line is natural in $(x, \rho_1 = \rho \rtimes \sigma)$ because the $Z_M(x)$-representation $H_{d(x)}(\mathcal{B}^x)$ depends in a natural way on x, as we observed at the start of the proof of Proposition 4.3. $\square$

There is natural partial order on the unipotent classes in M:

$$\mathcal{O} < \mathcal{O}' \quad \text{when} \quad \overline{\mathcal{O}} \subsetneq \overline{\mathcal{O}'}.$$

Let $\mathcal{O}_x \subset M$ be the class containing x. We transfer this to a partial order on our extended Springer data by defining

$$(x, \rho_1) < (x', \rho_1') \quad \text{when} \quad \overline{\mathcal{O}_x} \subsetneq \overline{\mathcal{O}_{x'}}. \tag{20}$$

We will use it to formulate a property of the composition series of some $\mathcal{W}^M$-representations that will appear later on.

Lemma 4.5. *Let $x \in M$ be unipotent and let $\rho \rtimes \sigma$ be a geometric irreducible representation of $\pi_0(\mathrm{Z}_M(x))$. There exist multiplicities $m_{x,\rho\rtimes\sigma^*,x',\rho'\rtimes\sigma'^*} \in \mathbb{Z}_{\geq 0}$ such that*

$$\begin{aligned} \mathrm{Ind}_{W \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{W \rtimes \Gamma} \big(\mathrm{Hom}_{A_x}\big(\rho, H_*(\mathcal{B}^x, \mathbb{C})\big) \otimes \sigma^*\big) \cong \\ \tau(x,\rho) \rtimes \sigma^* \oplus \bigoplus_{(x',\rho'\rtimes\sigma') > (x,\rho\rtimes\sigma)} m_{x,\rho\rtimes\sigma,x',\rho'\rtimes\sigma'}\, \tau(x',\rho') \rtimes \sigma'^*. \end{aligned}$$

Proof. Consider the vector space $\mathrm{Hom}_{A_x}\big(\rho, H_*(\mathcal{B}^x, \mathbb{C})\big)$ with the $\mathcal{W}^{M^\circ}$-action coming from (11). The proof of Proposition 4.3 remains valid for these representations. The group $H^*(\mathcal{B}^x, \mathbb{C})$ is dual to $H_*(\mathcal{B}^x, \mathbb{C})$, and we will denote by $\tilde{\tau}(x, \rho)$ the corresponding Springer representation. Let $\mathrm{Unip}(G)$ be the set of G-conjugacy classes of pairs (x', ρ'), where x is a unipotent element in G and $\rho' \in A_x$. By [BoMa, Corollaire 1], we have

$$H^i(\mathcal{B}^x, \mathbb{C}) \simeq \bigoplus_{(x',\rho') \in \mathrm{Unip}(G)} \tilde{\tau}_{(x',\rho')} \otimes \mathcal{H}^{i-2d_x}\big(\mathrm{IC}(\overline{\mathcal{O}}_{x'}, \mathcal{F}_{\rho'})\big)_x, \tag{21}$$

where $d_x = \dim \mathcal{B}^x$, and $\mathcal{O}_{x'}$ and $\mathcal{F}_{\rho'}$ are the G-conjugacy class of x' and the G-equivariant irreducible local system on $\mathcal{O}_{x'}$ corresponding to ρ', respectively. In particular, if $H^i(\mathcal{B}^x, \mathbb{C}) \neq 0$, then necessarily $0 \leq i \leq 2d_x$. As noted in [BoMa, Corollaire 2], the right hand side of (21) is zero unless $\mathcal{O}_x \subset \overline{\mathcal{O}_{x'}}$. So for any i there exist multiplicities $m_{i,x,\rho,x',\rho'} \in \mathbb{Z}_{\geq 0}$ satisfying

$$\mathrm{Hom}_{A_x}\big(\rho, H^i(\mathcal{B}^x, \mathbb{C})\big) \cong \bigoplus_{(x',\rho') \geq (x,\rho)} m_{i,x,\rho,x',\rho'}\, \tilde{\tau}(x', \rho'). \tag{22}$$

Moreover, in [Lus1, Theorem 24.8], Lusztig has proved that for any (x, ρ) we have $\mathcal{H}^i\big(\mathrm{IC}(\overline{\mathcal{O}}_x, \mathcal{F}_\rho)\big) = 0$ if i is odd, and that the polynomial

$$\Pi_{(x,\rho),(x',\rho')} := \sum\nolimits_j \big(\mathcal{F}_{\rho'} : \mathcal{H}^{2j}\big(\mathrm{IC}(\overline{\mathcal{O}}_x, \mathcal{F}_\rho)\big)|_{\mathcal{O}_{x'}}\big)\, \mathbf{q}^{\mathbf{m}},$$

in the indeterminate $\mathbf{q}$, satisfies $\Pi_{(x,\rho),(x,\rho)} = 1$. It gives

$$\big(\mathcal{F}_\rho : \mathcal{H}^{2j-2d_x}\big(\mathrm{IC}(\overline{\mathcal{O}}_x, \mathcal{F}_\rho)\big)|_{\mathcal{O}_x}\big) = \begin{cases} 0 & \text{if } j \neq d_x \\ 1 & \text{if } j = d_x. \end{cases} \tag{23}$$

From (23), we get that

$$m_{i,x,\rho,x,\rho} = \begin{cases} 0 & \text{if } i \neq 2d_x \\ 1 & \text{if } i = 2d_x. \end{cases}$$

It follows that there exist multiplicities $m_{x,\rho,x',\rho'} \in \mathbb{Z}_{\geq 0}$ such that

$$\mathrm{Hom}_{A_x}\big(\rho, H_*(\mathcal{B}^x, \mathbb{C})\big) \cong \tau(x,\rho) \oplus \bigoplus_{(x',\rho') > (x,\rho)} m_{x,\rho,x',\rho'}\, \tau(x',\rho'). \tag{24}$$

By (14) and (13) $\Gamma_{[x,\rho]_{M^\circ}}$ also stabilizes the $\tau(x',\rho')$ with $m_{x,\rho,x',\rho'} > 0$, and by Proposition 4.3 the associated 2-cocycles are trivial. It follows that

$$\begin{aligned} \mathrm{Ind}_{W \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{W \rtimes \Gamma} &\big(\mathrm{Hom}_{A_x}\big(\rho, H_*(\mathcal{B}^x, \mathbb{C})\big) \otimes \sigma^*\big) \cong \\ &\tau(x,\rho) \rtimes \sigma^* \oplus \bigoplus_{(x',\rho') > (x,\rho)} m_{x,\rho,x',\rho'} \mathrm{Ind}_{W \rtimes \Gamma_{[x,\rho]_{M^\circ}}}^{W \rtimes \Gamma} \big(\tau(x',\rho') \otimes \sigma^*\big). \end{aligned} \tag{25}$$

Decomposing the right hand side into irreducible representations then gives the statement of the lemma. □

5. Langlands parameters for the principal series

Let $\mathbf{W}_F$ denote the Weil group of F, let $\mathbf{I}_F$ be the inertia subgroup of $\mathbf{W}_F$. Let $\mathbf{W}_F^{\mathrm{der}}$ denote the closure of the commutator subgroup of $\mathbf{W}_F$, and write $\mathbf{W}_F^{\mathrm{ab}} = \mathbf{W}_F / \mathbf{W}_F^{\mathrm{der}}$. The group of units in $\mathfrak{o}_F$ will be denoted $\mathfrak{o}_F^\times$.

We recall the Artin reciprocity map $\mathbf{a}_F : \mathbf{W}_F \to F^\times$ which has the following properties (local class field theory):

(1) The map $\mathbf{a}_F$ induces a topological isomorphism $\mathbf{W}_F^{\mathrm{ab}} \simeq F^\times$.
(2) An element $x \in \mathbf{W}_F$ is a geometric Frobenius if and only if $\mathbf{a}_F(x)$ is a prime element ϖ_F of F.
(3) We have $\mathbf{a}_F(\mathbf{I}_F) = \mathfrak{o}_F^\times$.

We now consider the principal series of $\mathcal{G}$. We recall that $\mathcal{G}$ denotes a connected reductive split p-adic group with maximal split torus $\mathcal{T}$, and that G, T denote the Langlands dual groups of $\mathcal{G}$, $\mathcal{T}$. Next, we consider conjugacy classes in G of continuous morphisms

$$\Phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$

which are rational on $\mathrm{SL}_2(\mathbb{C})$ and such that $\Phi(\mathbf{W}_F)$ consists of semisimple elements in G.

The (conjectural) local Langlands correspondence is supposed to be compatible with respect to inclusions of Levi subgroups. Therefore every Langlands parameter Φ for a principal series representation should have $\Phi(\mathbf{W}_F)$ contained in a maximal torus of G. As Φ is only determined up to G-conjugacy, it should suffice to consider Langlands parameters with $\Phi(\mathbf{W}_F) \subset T$.

In particular, for such parameters $\Phi\big|_{\mathbf{W}_F}$ factors through $\mathbf{W}_F^{ab} \cong F^\times$. We view the domain of Φ to be $F^\times \times \mathrm{SL}_2(\mathbb{C})$:

$$\Phi \colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G.$$

In this section we will build such a continuous morphism Φ from $\mathfrak{s}$ and data coming from the extended quotient of second kind. In Section 6 we show how such a Langlands parameter Φ can be enhanced with a parameter ρ.

Throughout this article, a Frobenius element Frob_F has been chosen and fixed. This determines a uniformizer ϖ_F via the equation $\mathbf{a}_F(\mathrm{Frob}_F) = \varpi_F$. That in turn gives rise to a group isomorphism $\mathfrak{o}_F^\times \times \mathbb{Z} \to F^\times$, which sends $1 \in \mathbb{Z}$ to ϖ_F. Let $\mathcal{T}_0$ denote the maximal compact subgroup of $\mathcal{T}$. As the latter is F-split,

$$\mathcal{T} \cong F^\times \otimes_{\mathbb{Z}} X_*(\mathcal{T}) \cong (\mathfrak{o}_F^\times \times \mathbb{Z}) \otimes_{\mathbb{Z}} X_*(\mathcal{T}) = \mathcal{T}_0 \times X_*(\mathcal{T}). \tag{26}$$

Because $\mathcal{W}$ does not act on $F^\times$, these isomorphisms are $\mathcal{W}$-equivariant if we endow the right hand side with the diagonal $\mathcal{W}$-action. Thus (26) determines a $\mathcal{W}$-equivariant isomorphism of character groups

$$\mathbf{Irr}(\mathcal{T}) \cong \mathbf{Irr}(\mathcal{T}_0) \times \mathbf{Irr}(X_*(\mathcal{T})) = \mathbf{Irr}(\mathcal{T}_0) \times X_{\mathrm{unr}}(\mathcal{T}). \tag{27}$$

The way $\mathbf{Irr}(\mathcal{T}_0)$ is embedded depends on the choice of ϖ_F. However, the isomorphisms

$$\mathbf{Irr}(\mathcal{T}_0) \cong \mathrm{Hom}(\mathfrak{o}_F^\times, T), \tag{28}$$

$$X_{\mathrm{unr}}(\mathcal{T}) \cong \mathrm{Hom}(\mathbb{Z}, T) = T. \tag{29}$$

are canonical.

Lemma 5.1. *Let χ be a character of $\mathcal{T}$, and let $[\mathcal{T}, \chi]_{\mathcal{G}}$ be the inertial class of the pair $(\mathcal{T}, \chi)$. Let*

$$\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}. \tag{30}$$

Then $\mathfrak{s}$ determines, and is determined by, the $\mathcal{W}$-orbit of a smooth morphism

$$c^{\mathfrak{s}} \colon \mathfrak{o}_F^\times \to T.$$

Proof. There is a natural isomorphism

$$\begin{aligned}\mathbf{Irr}(\mathcal{T}) &= \mathrm{Hom}(F^\times \otimes_{\mathbb{Z}} X_*(\mathcal{T}), \mathbb{C}^\times) \\ &\cong \mathrm{Hom}(F^\times, \mathbb{C}^\times \otimes_{\mathbb{Z}} X^*(\mathcal{T})) = \mathrm{Hom}(F^\times, T).\end{aligned}$$

Let $\hat{\chi} \in \mathrm{Hom}(F^\times, T)$ be the image of χ under these isomorphisms. By (28) the restriction of $\hat{\chi}$ to $\mathfrak{o}_F^\times$ is not disturbed by unramified twists, so we take that as $c^{\mathfrak{s}}$. Conversely, by (27) $c^{\mathfrak{s}}$ determines χ up to unramified twists. Two elements of $\mathbf{Irr}(\mathcal{T})$ are $\mathcal{G}$-conjugate if and only if they are $\mathcal{W}$-conjugate so, in view of (27), the $\mathcal{W}$-orbit of the $c^{\mathfrak{s}}$ contains the same amount of information as $\mathfrak{s}$. $\square$

Let $H = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}})$ and let $M = \mathrm{Z}_H(t)$ for some $t \in T$. Recall that a unipotent element $x \in M^0$ is said to be *distinguished* if the connected centre $\mathrm{Z}^0_{M^0}$ of M^0 is a maximal torus of $\mathrm{Z}_{M^0}(x)$. Let $x \in M^0$ unipotent. If x is not distinguished, then there is a Levi subgroup L of M^0 containing x and such that $x \in L$ is distinguished.

Let $X \in \mathrm{Lie}\, M^0$ such that $\exp(X) = x$. A cocharacter $h \colon \mathbb{C}^\times \to M^0$ is said to be *associated to x* if

$$\mathrm{Ad}(h(t))X = t^2 X \quad \text{for each } t \in \mathbb{C}^\times,$$

and if the image of h lies in the derived group of some Levi subgroup L for which $x \in L$ is distinguished (see [Jan, Rem. 5.5] or [FoRo, Rem.2.12]).

A cocharacter associated to a unipotent element $x \in M^0$ is not unique. However, any two cocharacters associated to a given $x \in M^0$ are conjugate under elements of $\mathrm{Z}_{M^0}(x)^0$ (see for instance [Jan, Lem. 5.3]).

We work with the Jacobson–Morozov theorem [ChGi, p. 183]. Let $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ be the standard unipotent matrix in $\mathrm{SL}_2(\mathbb{C})$ and let x be a unipotent element in M^0. There exist rational homomorphisms

$$\gamma \colon \mathrm{SL}_2(\mathbb{C}) \to M^0 \quad \text{with} \quad \gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = x, \tag{31}$$

see [ChGi, §3.7.4]. Any two such homomorphisms γ are conjugate by elements of $\mathrm{Z}_{M^\circ}(x)$.

For $\alpha \in \mathbb{C}^\times$ we define the following matrix in $\mathrm{SL}_2(\mathbb{C})$:

$$Y_\alpha = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right).$$

Then each γ as above determines a cocharacter $h \colon \mathbb{C}^\times \to M^0$ by setting

$$h(\alpha) := \gamma(Y_\alpha) \quad \text{for } \alpha \in \mathbb{C}^\times. \tag{32}$$

Each cocharacter h obtained in this way is associated to x, see [Jan, Rem. 5.5] or [FoRo, Rem.2.12]. Hence each two such cocharacters are conjugate under $\mathrm{Z}_{M^0}(x)^0$.

We set $\Phi(\varpi_F) = t \in T$. Define the Langlands parameter Φ as follows:

$$\Phi \colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G, \qquad (u\varpi_F^n, Y) \mapsto c^{\mathfrak{s}}(u) \cdot t^n \cdot \gamma(Y) \tag{33}$$

for all $u \in \mathfrak{o}_F^\times$, $n \in \mathbb{Z}$, $Y \in \mathrm{SL}_2(\mathbb{C})$.

Note that the definition of Φ uses the appropriate data: the semisimple element $t \in T$, the map $c^{\mathfrak{s}}$, and the homomorphism γ (which depends on the Springer parameter x).

Since x determines γ up to M°-conjugation, $c^{\mathfrak{s}}, x$ and t determine Φ up to conjugation by their common centralizer in G. Notice also that one can recover $c^{\mathfrak{s}}, x$ and t from Φ and that

$$h(\alpha) = \Phi(1, Y_\alpha). \tag{34}$$

6. Varieties of Borel subgroups

We clarify some issues with different varieties of Borel subgroups and different kinds of parameters arising from them. Let G be a connected reductive complex group and let

$$\Phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$

be as in (33). We write

$$\begin{aligned} H &= \mathrm{Z}_G(\Phi(\mathbf{I}_F)) = \mathrm{Z}_G(\operatorname{im} c^{\mathfrak{s}}), \\ M &= \mathrm{Z}_G(\Phi(\mathbf{W}_F)) = \mathrm{Z}_H(t). \end{aligned}$$

Although both H and M are in general disconnected, $\Phi(\mathbf{W}_F)$ is always contained in H° because it lies in the maximal torus T of G and H°. Hence $\Phi(\mathbf{I}_F) \subset \mathrm{Z}(H^\circ)$.

By construction t commutes with $\Phi(\mathrm{SL}_2(\mathbb{C})) \subset M$. For any $q^{1/2} \in \mathbb{C}^\times$ the element

$$t_q := t\Phi\big(Y_{q^{1/2}}\big) \tag{35}$$

satisfies the familiar relation $t_q x t_q^{-1} = x^q$. Indeed

$$
\begin{aligned}
t_q x t_q^{-1} &= t\Phi(Y_{q^{1/2}})\Phi\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\Phi(Y_{q^{1/2}}^{-1})t^{-1} \\
&= t\Phi\big(Y_{q^{1/2}}\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)Y_{q^{1/2}}^{-1}\big)t^{-1} \qquad (36)\\
&= t\Phi\left(\begin{smallmatrix}1 & q\\ 0 & 1\end{smallmatrix}\right)t^{-1} = x^q.
\end{aligned}
$$

Recall that B_2 denotes the upper triangular Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$. In the flag variety of M° we have the subvarieties $\mathcal{B}^x_{M^\circ}$ and $\mathcal{B}^{\Phi(B_2)}_{M^\circ}$ of Borel subgroups containing x and $\Phi(B_2)$, respectively. Similarly the flag variety of H° has subvarieties $\mathcal{B}^{t,x}_{H^\circ}$, $\mathcal{B}^{t_q,x}_{H^\circ}$ and

$$\mathcal{B}^{t,\Phi(B_2)}_{H^\circ} = \mathcal{B}^{t_q,\Phi(B_2)}_{H^\circ}.$$

Notice that $\Phi(\mathbf{I}_F)$ lies in every Borel subgroup of H°, because it is contained in $\mathrm{Z}(H^\circ)$. We abbreviate $\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})))$ and similarly for other groups.

Proposition 6.1. (1) *The inclusion maps*

$$
\begin{array}{ccccccc}
 & & \mathrm{Z}_{M^\circ}(\Phi) & \to & \mathrm{Z}_{M^\circ}(\Phi(B_2)) & \to & \mathrm{Z}_{M^\circ}(x), \\
\mathrm{Z}_H(t_q, x) & \leftarrow & \mathrm{Z}_H(\Phi) & \to & \mathrm{Z}_H(t, \Phi(B_2)) & \to & \mathrm{Z}_H(t, x),
\end{array}
$$

are homotopy equivalences. In particular they induce isomorphisms between the respective component groups.

(2) *The inclusions* $\mathcal{B}^{\Phi(B_2)}_{M^\circ} \to \mathcal{B}^x_{M^\circ}$ *and* $\mathcal{B}^{t_q,x}_{H^\circ} \leftarrow \mathcal{B}^{t,\Phi(B_2)}_{H^\circ} \to \mathcal{B}^{t,x}_{H^\circ}$ *are homotopy equivalences.*

Proof. It suffices to consider the statements for H and t_q, since the others can be proven in the same way.
(1) Our proof uses some elementary observations from [Ree2, §4.3]. There is a Levi decomposition

$$\mathrm{Z}_{H^\circ}(x) = \mathrm{Z}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C})))U_x$$

with $\mathrm{Z}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C}))) = \mathrm{Z}_{H^\circ}(\Phi(B_2))$ reductive and U_x unipotent. Since $t_q \in \mathrm{N}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C})))$ and $\mathrm{Z}_H(x^q) = \mathrm{Z}_H(x)$, conjugation by t_q preserves this decomposition. Therefore

$$\mathrm{Z}_{H^\circ}(t_q, x) = \mathrm{Z}_{H^\circ}(\Phi)\mathrm{Z}_{U_x}(t_q) = \mathrm{Z}_{H^\circ}(t_q, \Phi(B_2))\mathrm{Z}_{U_x}(t_q). \qquad (37)$$

We note that

$$\mathrm{Z}_{U_x}(t_q) \cap \mathrm{Z}_{H^\circ}(t_q, \Phi(B_2)) \subset U_x \cap \mathrm{Z}_{H^\circ}(\Phi(B_2)) = 1$$

and that $\mathrm{Z}_{U_x}(t_q) \subset U_x$ is contractible, because it is a unipotent complex group. It follows that

$$\mathrm{Z}_{H^\circ}(\Phi) = \mathrm{Z}_{H^\circ}(t_q, \Phi(B_2)) \to \mathrm{Z}_{H^\circ}(t_q, x) \qquad (38)$$

is a homotopy equivalence. If we want to replace H° by H, we find

$$\mathrm{Z}_H(\Phi)/\mathrm{Z}_{H^\circ}(\Phi) = \{hH^\circ \in \pi_0(H) \mid h\Phi h^{-1} \in \mathrm{Ad}(H^\circ)\Phi\},$$

and similarly with $(t_q, \Phi(B_2))$ or (t_q, x) instead of Φ.

Let us have a closer look at the H°-conjugacy classes of these objects. Given any Φ, we obviously know what t_q and x are. Conversely, suppose that t_q and x are given. We apply a refinement of the Jacobson–Morozov theorem due to Kazhdan and Lusztig. According to [KaLu, §2.3] there exist homomorphisms $\Phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$ as above, which return t_q and x in the prescribed way. Moreover all such homomorphisms are conjugate under $\mathrm{Z}_{H^\circ}(t_q, x)$, see [KaLu, §2.3.h] or Section 19. So from (t_q, x) we can reconstruct the $\mathrm{Ad}(H^\circ)$-orbit of Φ, and this gives bijections between H°-conjugacy classes of Φ, $(t_q, \Phi(B_2))$ and (t_q, x). Since these bijections clearly are $\pi_0(H)$-equivariant, we deduce

(39)
$$\mathrm{Z}_H(\Phi)/\mathrm{Z}_{H^\circ}(\Phi) = \mathrm{Z}_H(t_q, \Phi(B_2))/\mathrm{Z}_{H^\circ}(t_q, \Phi(B_2)) = \mathrm{Z}_H(t_q, x)/\mathrm{Z}_{H^\circ}(t_q, x).$$

Equations (38) and (39) imply that

$$\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(t_q, \Phi(B_2)) \to \mathrm{Z}_H(t_q, x)$$

is also a homotopy equivalence.
(2) By the aforementioned result [KaLu, §2.3.h]

$$\mathrm{Z}_{H^\circ}(t_q, x) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathcal{B}_{H^\circ}^{t_q, x}. \tag{40}$$

On the other hand, by (37)

(41)
$$\mathrm{Z}_{H^\circ}(t_q, x) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathrm{Z}_{U_x}(t_q)\mathrm{Z}_H(t_q, \Phi(B_2)) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathrm{Z}_{U_x}(t_q) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}.$$

For any $B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}$ and $u \in \mathrm{Z}_{U_x}(t_q)$ it is clear that

$$u \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \iff \Phi(B_2) \subset uBu^{-1} \iff u^{-1}\Phi(B_2)u \subset B.$$

Furthermore, since $\Phi(B_2) \subset B$ is generated by x and $\{\Phi \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \mid \alpha \in \mathbb{C}^\times\}$, the right hand side is equivalent to

$$u^{-1}\Phi \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) u \in B \quad \forall \alpha \in \mathbb{C}^\times.$$

In Lie algebra terms this can be reformulated as

$$\mathrm{Ad}_{u^{-1}}(d\Phi \left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)) \in \mathrm{Lie}\, B \quad \forall \alpha \in \mathbb{C}.$$

Because u is unipotent, this happens if and only if

$$\mathrm{Ad}_{u^\lambda}(d\Phi \left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)) \in \mathrm{Lie}\, B \quad \forall \lambda, \alpha \in \mathbb{C}.$$

By the reverse chain of arguments the last statement is equivalent with

$$u^\lambda \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \quad \forall \lambda \in \mathbb{C}.$$

Thus $\{u \in \mathrm{Z}_{U_x}(t_q) \mid u \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}\}$ is contractible for all $B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}$, and we already knew that $\mathrm{Z}_{U_x}(t_q)$ is contractible. Together with (40) and (41) these imply that $\mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \to \mathcal{B}_{H^\circ}^{t_q, x}$ is a homotopy equivalence. $\square$

For the affine Springer correspondence we will need more precise information on the relation between the varieties for G, for H and for M°.

Proposition 6.2. (1) *The variety $\mathcal{B}_{H^\circ}^{t,x}$ is isomorphic to $[\mathcal{W}^{H^\circ} : \mathcal{W}^{M^\circ}]$ copies of $\mathcal{B}_{M^\circ}^{x}$, and $\mathcal{B}_{H^\circ}^{t, \Phi(B_2)}$ is isomorphic to the same number of copies of $\mathcal{B}_{M^\circ}^{\Phi(B_2)}$.*

(2) *The group $\mathrm{Z}_{H^\circ}(t, x)/\mathrm{Z}_{M^\circ}(x)$ permutes these two sets of copies freely.*

(3) *The variety $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ is isomorphic to $[\mathcal{W}^G : \mathcal{W}^{H^\circ}]$ copies of $\mathcal{B}_{H^\circ}^{t, \Phi(B_2)}$. The group $\mathrm{Z}_G(\Phi)/\mathrm{Z}_{H^\circ}(\Phi)$ permutes these copies freely.*

Proof. (1) Let A be a subgroup of T such that $M^\circ = \mathrm{Z}_{H^\circ}(A)^\circ$ and let $\mathcal{B}^A_{H^\circ}$ denote the variety of all Borel subgroups of H° which contain A. With an adaptation of [ChGi, p.471] we will prove that, for any $B \in \mathcal{B}^A_{H^\circ}$, $B \cap M^0$ is a Borel subgroup of M^0.

Since $B \cap M^\circ \subset B$ is solvable, it suffices to show that its Lie algebra is a Borel subalgebra of Lie M°. Write Lie $T = \mathfrak{t}$ and let

$$\text{Lie } H^\circ = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_-$$

be the triangular decomposition, where Lie $B = \mathfrak{n} \oplus \mathfrak{t}$. Since $A \subset B$, it preserves this decomposition and

$$\begin{aligned} \text{Lie } M^\circ &= (\text{Lie } H)^A = \mathfrak{n}^A \oplus \mathfrak{t} \oplus \mathfrak{n}_-^A, \\ \text{Lie } B \cap M^\circ &= \text{Lie } B^A = \mathfrak{n}^A \oplus \mathfrak{t}. \end{aligned}$$

The latter is indeed a Borel subalgebra of Lie M°. Thus there is a canonical map

$$\mathcal{B}^A_{H^\circ} \to \text{Flag}\, M^0, \quad B \mapsto B \cap M^0. \tag{42}$$

The group M acts by conjugation on $\mathcal{B}^A_{H^\circ}$ and (42) clearly is M-equivariant. By [ChGi, p. 471] the M°-orbits form a partition

$$\mathcal{B}^A_{H^\circ} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_m. \tag{43}$$

At the same time these orbits are the connected components of $\mathcal{B}^A_{H^\circ}$ and the irreducible components of the projective variety $\mathcal{B}^A_{H^\circ}$. The argument from [ChGi, p. 471] also shows that (42), restricted to any one of these orbits, is a bijection from the M^0-orbit onto Flag M^0.

The number of components m can be determined as in the proof of [Ste1, Corollary 3.12.a]. The collection of Borel subgroups of M° that contain the maximal torus T is in bijection with the Weyl group $\mathcal{W}^{M^\circ}$. Retracting via (42), we find that every component $\mathcal{B}_i$ has precisely $|\mathcal{W}^{M^\circ}|$ elements that contain T. On the other hand, since $A \subset T$, $\mathcal{B}^A_{H^\circ}$ has $|\mathcal{W}^{H^\circ}|$ elements that contain T, so

$$m = [\mathcal{W}^{H^\circ} : \mathcal{W}^{M^\circ}].$$

To obtain our desired isomorphisms of varieties, we let A be the group generated by t and we restrict $\mathcal{B}_i \to$ Flag M° to Borel subgroups that contain t, x (respectively $t, \Phi(B_2)$).
(2) By Proposition 6.1

$$\mathrm{Z}_{H^\circ}(t,x)/\mathrm{Z}_{M^\circ}(x) \cong \mathrm{Z}_{H^\circ}(t,\Phi(B_2))/\mathrm{Z}_{M^\circ}(\Phi(B_2)).$$

Since the former is a subgroup of M/M° and the copies under consideration are in M-equivariant bijection with the components (43), it suffices to show that M/M° permutes these components freely. Pick B, B' in the same component $\mathcal{B}_i$ and assume that $B' = hBh^{-1}$ for some $h \in M$. Since $\mathcal{B}_i$ is M°-equivariantly isomorphic to the flag variety of M° we can find $m \in M^\circ$ such that $B' = m^{-1}Bm$. Then mh normalizes B, so $mh \in B$. As B is connected, this implies $mh \in M^\circ$ and $h \in M^\circ$.
(3) Apply the proofs of parts 1 and 2 with $A = \Phi(\mathbf{I}_F)$, G in the role of H°, H° in the role of M° and $t\Phi(B_2)$ in the role of x. □

7. Comparison of different parameters

In the following sections we will make use of several different but related kinds of parameters.

Kazhdan–Lusztig–Reeder parameters (KLR parameters)
For a Langlands parameter as in (33), the variety of Borel subgroups $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ is nonempty, and the centralizer $\mathrm{Z}_G(\Phi)$ of the image of Φ acts on it. Hence the group of components $\pi_0(\mathrm{Z}_G(\Phi))$ acts on the homology $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. We call an irreducible representation ρ of $\pi_0(\mathrm{Z}_G(\Phi))$ geometric if it appears in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. We define a Kazhdan–Lusztig–Reeder parameter for G to be a such pair (Φ, ρ). The group G acts on these parameters by

$$g \cdot (\Phi, \rho) = (g \Phi g^{-1}, \rho \circ \mathrm{Ad}_g^{-1}) \tag{44}$$

and we denote the corresponding equivalence class by $[\Phi, \rho]_G$.

Affine Springer parameters
As before, suppose that $t \in G$ is semisimple and that $x \in \mathrm{Z}_G(t)$ is unipotent. Then $\mathrm{Z}_G(t,x)$ acts on $\mathcal{B}_G^{t,x}$ and $\pi_0(\mathrm{Z}_G(t,x))$ acts on the homology of this variety. In this setting we say that $\rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t,x))\big)$ is geometric if it appears in $H_{\mathrm{top}}(\mathcal{B}_G^{t,x}, \mathbb{C})$, where top refers to highest degree in which the homology is nonzero, the real dimension of $\mathcal{B}_G^{t,x}$. We call such triples (t, x, ρ_1) affine Springer parameters for G, because they appear naturally in the representation theory of the affine Weyl group associated to G. The group G acts on such parameters by conjugation, and we denote the conjugacy classes by $[t, x, \rho_1]_G$.

Kazhdan–Lusztig triples
Next we consider a unipotent element $x \in G$ and a semisimple element $t_q \in G$ such that $t_q x t_q^{-1} = x^q$. As above, $\mathrm{Z}_G(t_q, x)$ acts on the variety $\mathcal{B}_G^{t_q,x}$ and we call $\rho_q \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t_q,x))\big)$ geometric if it appears in $H_*\big(\mathcal{B}_G^{t_q,x}, \mathbb{C}\big)$. We refer to triples (t_q, x, ρ_q) of this kind as Kazhdan–Lusztig triples for G. Again they are endowed with an obvious G-action and we denote the equivalence classes by $[t_q, x, \rho_q]_G$.

We note that in all cases the representations of the component groups stem from the action of G on a variety of Borel subgroups. The centre of G acts trivially on such a variety, so in all three above cases an irreducible representation of the appropriate component group can only be if all elements coming from $\mathrm{Z}(G)$ act trivially.

In [KaLu, Ree2] there are some indications that these three kinds of parameters are essentially equivalent. Proposition 6.1 allows us to make this precise in the necessary generality.

Lemma 7.1. *Let $\mathfrak{s}$ be a Bernstein component in the principal series, associate $c^{\mathfrak{s}} \colon \mathfrak{o}_F^\times \to T$ to it as in Lemma 5.1 and write $H = \mathrm{Z}_G(c^{\mathfrak{s}}(\mathfrak{o}_F^\times))$. There are natural bijections between H°-equivalence classes of:*

- *Kazhdan–Lusztig–Reeder parameters for G with $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}$ and $\Phi(\varpi_F) \in H^\circ$;*

- *affine Springer parameters for H°;*
- *Kazhdan–Lusztig triples for H°.*

Proof. Since $\mathrm{SL}_2(\mathbb{C})$ is connected and commutes with $\mathfrak{o}_F^\times$, its image under Φ must be contained in the connected component of H. Therefore KLR-parameters with these properties are in canonical bijection with KLR parameters for H° and it suffices to consider the case $H^\circ = G$.

As in (33) and (35), any KLR parameter gives rise to the ingredients t, x and t_q for the other two kinds of parameters. As we discussed after (33), the pair (t, x) is enough to recover the conjugacy class of Φ. A refined version of the Jacobson–Morozov theorem says that the same goes for the pair (t_q, x), see [KaLu, §2.4] or [Ree2, Section 4.2].

To complete $\Phi, (t, x)$ or (t_q, x) to a parameter of the appropriate kind, we must add an irreducible representation ρ, ρ_1 or ρ_q. For the affine Springer parameters it does not matter whether we consider the total homology or only the homology in top degree. Indeed, it follows from Propositions 6.1 and 6.2 and [Sho, bottom of page 296 and Remark 6.5] that any irreducible representation ρ_1 which appears in $H_*\big(\mathcal{B}_G^{t,x}, \mathbb{C}\big)$, already appears in the top homology of this variety.

This and Proposition 6.1 show that there is a natural correspondence between the possible ingredients ρ, ρ_1 and ρ_q. $\square$

8. The affine Springer correspondence

An interesting instance of Section 4 arises when M is the centralizer of a semisimple element t in a connected reductive complex group G. As before we assume that t lies in a maximal torus T of G and we write $\mathcal{W}^G = W(G, T)$. By Lemma 3.2

$$\mathcal{W}^M := \mathrm{N}_M(T)/\mathrm{Z}_M(T) \cong \mathcal{W}^{M^\circ} \rtimes \pi_0(M) \tag{45}$$

is the stabilizer of t in $\mathcal{W}^G$, so the role of Γ is played by the component group $\pi_0(M)$. In contrast to the setup in Section 4, it is possible that some elements of $\pi_0(M) \setminus \{1\}$ fix W pointwise. This poses no problems however, as such elements never act trivially on T. For later use we record the following consequence of (14):

$$\pi_0(M)_{\tau(x,\rho)} \cong \big(\mathrm{Z}_M(x)/\mathrm{Z}_{M^\circ}(x)\big)_\rho. \tag{46}$$

Recall from Section 2 that

$$\widetilde{T}_2 := \{(t, \sigma) \,:\, t \in T, \sigma \in \mathbf{Irr}(\mathcal{W}_t^G)\},$$
$$(T/\!/\mathcal{W}^G)_2 := \widetilde{T}_2/\mathcal{W}^G.$$

We note that the rational characters of the complex torus T span the regular functions on the complex variety T:

$$\mathcal{O}(T) = \mathbb{C}[X^*(T)].$$

From (5), (6), Lemma 2.2 and Proposition 4.3 we infer the following rough form of the extended Springer correspondence for the affine Weyl group $X^*(T) \rtimes \mathcal{W}^G$.

Theorem 8.1. *There are bijections*

$$(T//\mathcal{W}^G)_2 \simeq \mathbf{Irr}\,(X^*(T) \rtimes \mathcal{W}^G) \simeq \{(t, \tau(x,\varrho) \rtimes \psi)\}/\mathcal{W}^G$$

with $t \in T, \tau(x,\varrho) \in \mathbf{Irr}\,\mathcal{W}^{M^0}, \psi \in \mathbf{Irr}(\pi_0(M)_{\tau(x,\varrho)})$.

Now we recall the geometric realization of irreducible representations of $X^*(T) \rtimes \mathcal{W}^G$ by Kato [Kat]. For a unipotent element $x \in M^\circ$ let $\mathcal{B}_G^{t,x}$ be the variety of Borel subgroups of G containing t and x. Fix a Borel subgroup B of G containing T and let $\theta_{G,B} : \mathcal{B}_G^{t,x} \to T$ be the morphism defined by

$$\theta_{G,B}(B') = g^{-1}tg \text{ if } B' = gBg^{-1} \text{ and } t \in gTg^{-1}. \tag{47}$$

The image of $\theta_{G,B}$ is $\mathcal{W}^G t$, the map is constant on the irreducible components of $\mathcal{B}_G^{t,x}$ and it gives rise to an action of $X^*(T)$ on the homology of $\mathcal{B}_G^{t,x}$. Furthermore $\mathbb{Q}[\mathcal{W}^G] \cong H(\mathcal{Z}_G)$ acts on $H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})$ via the convolution product in Borel–Moore homology, as described in (11). Both actions commute with the action of $\mathrm{Z}_G(t,x)$ induced by conjugation of Borel subgroups. By homotopy invariance, the latter action factors through $\pi_0(\mathrm{Z}_G(t,x))$.

Let $\rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t,x))\big)$. By [Kat, Theorem 4.1] the $X^*(T) \rtimes \mathcal{W}^G$-module

$$\tau(t,x,\rho_1) := \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t,x))}\big(\rho_1, H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})\big) \tag{48}$$

is either irreducible or zero. Moreover every irreducible representation of $X^*(T) \rtimes \mathcal{W}^G$ is obtained is in this way, and the data (t, x, ρ_1) are unique up to G-conjugacy. This generalizes the Springer correspondence for finite Weyl groups, which can be recovered by considering the representations on which $X^*(T)$ acts trivially.

Propositions 4.3 and 6.2 shine some new light on this:

Theorem 8.2. (1) *There are bijections between the following sets:*
- $\mathrm{Irr}(X^*(T) \rtimes \mathcal{W}^G) = \mathrm{Irr}(\mathcal{O}(T) \rtimes \mathcal{W}^G)$;
- $(T//\mathcal{W}^G)_2 = \big\{(t, \tilde\tau) \mid t \in T, \tilde\tau \in \mathrm{Irr}(\mathcal{W}^M)\big\}/\mathcal{W}^G$;
- $\big\{(t, \tau, \sigma) \mid t \in T, \tau \in \mathrm{Irr}(\mathcal{W}^{M^\circ}), \sigma \in \mathrm{Irr}(\pi_0(M)_\tau)\big\}/\mathcal{W}^G$;
- $\big\{(t, x, \rho, \sigma) \mid t \in T, x \in M^\circ \text{ unipotent}, \rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_{M^\circ}(x))\big)$ geometric, $\sigma \in \mathrm{Irr}\big(\pi_0(M)_{\tau(x,\rho)}\big)\big\}/G$;
- $\big\{(t, x, \rho_1) \mid t \in T, x \in M^\circ \text{ unipotent}, \rho_1 \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t,x))\big)$ geometric$\big\}/G$.

Here a representation of $\pi_0(\mathrm{Z}_{M^\circ}(x))$ *(or* $\pi_0(\mathrm{Z}_G(t,x))$*) is called geometric if it appears in* $H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C})$ *(respectively* $H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})$*). Apart from the third and fourth sets, these bijections are natural.*

(2) *The* $X^*(T) \rtimes \mathcal{W}^G$*-representation corresponding to* (t, x, ρ_1) *via these bijections is Kato's module* (48).

We remark that in the fourth and fifth sets it would be more natural to allow t to be any semisimple element of G. In fact that would give the affine Springer parameters from Lemma 7.1. Clearly G acts on the set of such more general parameters (t, x, ρ, σ) or (t, x, ρ_1), which gives equivalence relations $/G$. The two above $/G$ refer to the restrictions of these equivalence relations to parameters with $t \in T$.

Proof. (1) Recall that the isotropy group of t in $\mathcal{W}^G$ is

$$\mathcal{W}^G_t = \mathcal{W}^M = \mathcal{W}^{M^\circ} \rtimes \pi_0(M).$$

Hence the bijection between the first two sets is an instance of Clifford theory, see Lemma 2.3. The second and third sets are in bijection by Proposition 4.3. The Springer correspondence for $\mathcal{W}^{M^\circ}$ provides the bijection with the fourth collection. To establish a bijection with the fifth collection, we first observe that

$$\begin{aligned} \pi_0(\mathrm{Z}_G(t,x)) = \pi_0(\mathrm{Z}_M(x)) &\cong \pi_0\big(\mathrm{Z}_{M^\circ}(x) \rtimes \pi_0(M)_{[x]_{M^\circ}}\big) \\ &= \pi_0(\mathrm{Z}_{M^\circ}(x)) \rtimes \pi_0(M)_{[x]_{M^\circ}}. \end{aligned} \tag{49}$$

Furthermore $\pi_0(M)_{\tau(x,\rho)} = \pi_0(M)_{[x,\rho]_{M^\circ}}$ by (14). From that and Proposition 4.3 it follows that every irreducible representation of (49) is of the form $\rho \rtimes \sigma$ (see Notation 2.4), with ρ and σ as in the fourth set. By Proposition 6.2

$$H_*(\mathcal{B}^{t,x}_G, \mathbb{C}) \cong H_*(\mathcal{B}^x_{M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_G(t,x)/\mathrm{Z}_{M^\circ}(x)] \otimes \mathbb{C}^{[\mathcal{W}^G:\mathcal{W}^G_t]} \tag{50}$$

as $\mathrm{Z}_G(t,x)$-representations. By [Ree2, §3.1]

$$\mathrm{Z}_G(t,x)/\mathrm{Z}_{M^\circ}(x) \cong \pi_0(M)_{[x]_{M^\circ}}$$

is abelian. Hence $\mathrm{Ind}_{\pi_0(M)_{[x,\rho]_{M^\circ}}}^{\pi_0(M)_{[x]_{M^\circ}}}(\sigma)$ appears exactly once in the regular representation of this group and

$$\begin{aligned} &\mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t,x))}\big(\rho \rtimes \sigma, H_{d(x)}(\mathcal{B}^{t,x}_G, \mathbb{C})\big) \cong \\ &\qquad \mathrm{Hom}_{\pi_0(\mathrm{Z}_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C})\big) \rtimes \sigma^* \otimes \mathbb{C}^{[\mathcal{W}^G:\mathcal{W}^G_t]}. \end{aligned} \tag{51}$$

In particular we see that ρ is geometric if and only if $\rho \rtimes \sigma$ is geometric, which establishes the final bijection. Now the resulting bijection between the second and fifth sets is natural by Theorem 4.4.
(2) The $X^*(T) \rtimes \mathcal{W}^G$-representation constructed from $(t, x, \rho \rtimes \sigma)$ by means of our bijections is

$$\mathrm{Ind}_{X^*(T)\rtimes\mathcal{W}^G_t}^{X^*(T)\rtimes\mathcal{W}^G}\big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C})\big) \rtimes \sigma^*\big). \tag{52}$$

On the other hand, by [Kat, Proposition 6.2]

$$\begin{aligned} H_*(\mathcal{B}^{t,x}_G, \mathbb{C}) &\cong \mathrm{Ind}_{X^*(T)\rtimes\mathcal{W}^{M^\circ}}^{X^*(T)\rtimes\mathcal{W}^G}(H_*(\mathcal{B}^x_{M^\circ}, \mathbb{C})) \\ &\cong \mathrm{Ind}_{X^*(T)\rtimes\mathcal{W}^G_t}^{X^*(T)\rtimes\mathcal{W}^G}\big(H_*(\mathcal{B}^x_{M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_G(t,x)/\mathrm{Z}_{M^\circ}(x)]\big) \end{aligned} \tag{53}$$

as $\mathrm{Z}_G(t,x) \times X^*(T) \rtimes \mathcal{W}^G$-representations. Together with the proof of part 1 this shows that $\tau(t, x, \rho \rtimes \sigma)$ is isomorphic to (52). □

We can extract a little more from the above proof. Recall that $\mathcal{O}_x$ denotes the conjugacy class of x in M. Let us agree that the affine Springer parameters with a fixed $t \in T$ are partially ordered by

$$(t, x, \rho_1) < (t, x', \rho'_1) \quad \text{when} \quad \overline{\mathcal{O}_x} \subsetneq \overline{\mathcal{O}_{x'}}.$$

Lemma 8.3. *There exist multiplicities* $m_{t,x,\rho_1,x',\rho_1'} \in \mathbb{Z}_{\geq 0}$ *such that*

$$\begin{aligned}\mathrm{Hom}_{\pi_0(Z_G(t,x))}\big(\rho_1, H_*(\mathcal{B}_G^{t,x},\mathbb{C})\big) \cong \\ \tau(t,x,\rho_1) \oplus \bigoplus_{(t,x',\rho_1') > (t,x,\rho_1)} m_{t,x,\rho_1,x',\rho_1'}\, \tau(t,x',\rho_1').\end{aligned}$$

Proof. It follows from (53), (50) and (51) that

$$\begin{aligned}(54)\quad & \mathrm{Hom}_{\pi_0(Z_G(t,x))}\big(\rho \rtimes \sigma, H_*(\mathcal{B}_G^{t,x},\mathbb{C})\big) \cong \\ & \mathrm{Ind}_{X^*(T)\rtimes \mathcal{W}_t^G}^{X^*(T)\rtimes \mathcal{W}^G} \mathrm{Ind}_{\mathcal{W}^{M^\circ}\rtimes \pi_0(M)_{[x,\rho]_{M^\circ}}}^{\mathcal{W}_t^G} \big(\mathrm{Hom}_{\pi_0(Z_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}^x_{M^\circ},\mathbb{C})\big) \otimes \sigma\big).\end{aligned}$$

The functor $\mathrm{Ind}_{X^*(T)\rtimes \mathcal{W}_t^G}^{X^*(T)\rtimes \mathcal{W}^G}$ provides an equivalence between the categories

- $X^*(T) \rtimes \mathcal{W}_t^G$-representations with $\mathcal{O}(T)^{\mathcal{W}_t^G}$-character t;
- $X^*(T) \rtimes \mathcal{W}^G$-representations with $\mathcal{O}(T)^{\mathcal{W}^G}$-character $\mathcal{W}^G t$.

Therefore we may apply Lemma 4.5 to the right hand side of (54), which produces the required formula. □

Let us have a look at the representations with an affine Springer parameter of the form $(t, x = 1, \rho_1 = \mathrm{triv})$. Equivalently, the fourth parameter in Theorem 8.2 is $(t, x = 1, \rho = \mathrm{triv}, \sigma = \mathrm{triv})$. The $\mathcal{W}^{M^\circ}$-representation with Springer parameter $(x = 1, \rho = \mathrm{triv})$ is the trivial representation, so $(x = 1, \rho = \mathrm{triv}, \sigma = \mathrm{triv})$ corresponds to the trivial representation of $\mathcal{W}_t^G$. With (52) we conclude that the $X^*(T) \rtimes \mathcal{W}^G$-representation with affine Springer parameter $(t, 1, \mathrm{triv})$ is

$$\tau(t,1,\mathrm{triv}) = \mathrm{Ind}_{X^*(T)\rtimes \mathcal{W}_t^G}^{X^*(T)\rtimes \mathcal{W}^G}\big(\mathrm{triv}_{\mathcal{W}_t^G}\big). \tag{55}$$

Notice that this is the only irreducible $X^*(T) \rtimes \mathcal{W}^G$-representation with an $X^*(T)$-weight t and nonzero $\mathcal{W}^G$-fixed vectors.

9. Geometric representations of affine Hecke algebras

Let G be a connected reductive complex group, B a Borel subgroup and T a maximal torus of G contained in B. Let $\mathcal{H}(G)$ be the affine Hecke algebra with the same based root datum as (G, B, T) and with a parameter $q \in \mathbb{C}^\times$ which is not a root of unity.

As we will have to deal with disconnected reductive groups, we include some additional automorphisms in the picture. In every root subgroup U_α with $\alpha \in \Delta(B, T)$ we pick a nontrivial element u_α. Let Γ be a finite group of automorphisms of $(G, T, (u_\alpha)_{\alpha\in\Delta(B,T)})$. Since G need not be semisimple, it is possible that some elements of Γ fix the entire root system of (G, T). Notice that Γ acts on the Weyl group $\mathcal{W}^G = W(G, T)$ and on $X^*(T)$ because it stabilizes T. Furthermore Γ acts on the standard basis of $\mathcal{H}(G)$ by

$$\gamma(T_w) = T_{\gamma(w)}, \text{ where } \gamma \in \Gamma, w \in X^*(T) \rtimes \mathcal{W}^G.$$

Since Γ stabilizes B, it determines an algebra automorphism of $\mathcal{H}(G)$. We form the crossed product algebra $\mathcal{H}(G) \rtimes \Gamma$ with respect to this Γ-action. It follows from the Bernstein presentation of $\mathcal{H}(G)$ [Lus5, §3] that $Z(\mathcal{H}(G) \rtimes \Gamma) \cong \mathcal{O}(T/\mathcal{W}^G \rtimes \Gamma)$.

We define a Kazhdan–Lusztig triple for $\mathcal{H}(G) \rtimes \Gamma$ to be a triple (t_q, x, ρ) such that:

- $t_q \in G$ is semisimple, $x \in G$ is unipotent and $t_q x t_q^{-1} = x^q$;
- ρ is an irreducible representation of the component group $\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q, x))$, such that every irreducible subrepresentation of the restriction of ρ to $\pi_0(\mathrm{Z}_G(t_q, x))$ appears in $H_*(\mathcal{B}^{t_q,x}, \mathbb{C})$.

The group $G \rtimes \Gamma$ acts on such triples by conjugation, and we denote the conjugacy class of a triples by $[t_q, x, \rho]_{G \rtimes \Gamma}$. Now we generalize [KaLu, Theorem 7.12] and [Ree2, Theorem 3.5.4]:

Theorem 9.1. *There exists a natural bijection between* $\mathrm{Irr}(\mathcal{H}(G) \rtimes \Gamma)$ *and* $G \rtimes \Gamma$*-conjugacy classes of Kazhdan–Lusztig triples.*

The $\mathcal{H}(G) \rtimes \Gamma$*-module* $\pi(t_q, x, \rho)$ *has central character* $(\mathcal{W}^G \rtimes \Gamma) t_q$ *and is the unique irreducible quotient of the* $\mathcal{H}(G) \rtimes \Gamma$*-module*

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q,x))}\big(\rho, H_*(\mathcal{B}^{t_q,x}, \mathbb{C}) \otimes \mathbb{C}[\Gamma]\big).$$

Proof. First we recall the geometric constructions of $\mathcal{H}(G)$-modules by Kazhdan, Lusztig and Reeder, taking advantage of Lemma 4.2 to simplify the presentation somewhat. As in [Ree2, §1.5], let

$$1 \to C \to \tilde{G} \to G \to 1 \tag{56}$$

be a finite central extension such that $\tilde{G}$ is a connected reductive group with simply connected derived group. The kernel C acts naturally on $\mathcal{H}(\tilde{G})$ and

$$\mathcal{H}(\tilde{G})^C \cong \mathcal{H}(G). \tag{57}$$

The action of Γ on the based root datum of (G, B, T) lifts uniquely to an action on the corresponding based root datum for $\tilde{G}$, so the Γ-actions on G and on $\mathcal{H}(G)$ lift naturally to actions on $\tilde{G}$ and $\mathcal{H}(\tilde{G})$. Let $\mathcal{H}_{\mathbf{q}}(\tilde{G})$ be the variation on $\mathcal{H}(\tilde{G})$ with scalars $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$ where $\mathbf{q}$ is a formal variable (instead of scalars $\mathbb{C}$ and $q \in \mathbb{C}^\times$). In [KaLu, Theorem 3.5] an isomorphism

$$\mathcal{H}_{\mathbf{q}}(\tilde{G}) \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \tag{58}$$

is constructed, where the right hand side denotes the $\tilde{G} \times \mathbb{C}^\times$-equivariant K-theory of the Steinberg variety $\mathcal{Z}_{\tilde{G}}$ of $\tilde{G}$. Since $G \rtimes \Gamma$ acts via conjugation on $\tilde{G}$ and on $\mathcal{Z}_{\tilde{G}}$, it also acts on $K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}})$. However, the connected group G acts trivially, so the action factors via Γ. Now the definition of the generators in [KaLu, Theorem 3.5] shows that (58) is Γ-equivariant. In particular it specializes to Γ-equivariant isomorphisms

$$\mathcal{H}(\tilde{G}) \cong \mathcal{H}_{\mathbf{q}}(\tilde{G}) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_q \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_q. \tag{59}$$

Let $(\tilde{t}_q, \tilde{x}) \in (\tilde{G})^2$ be a lift of $(t_q, x) \in G^2$ with $\tilde{x}$ unipotent. The $\tilde{G}$-conjugacy class of $\tilde{t}_q$ defines a central character of $\mathcal{H}(\tilde{G})$ and

$$\mathcal{H}(\tilde{G}) \otimes_{\mathrm{Z}(\mathcal{H}(\tilde{G}))} \mathbb{C}_{\tilde{t}_q} \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{R(\tilde{G} \times \mathbb{C}^\times)} \mathbb{C}_{\tilde{t}_q,q}.$$

According to [ChGi, Proposition 8.1.5] there is an isomorphism

$$K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{R(\tilde{G} \times \mathbb{C}^\times)} \mathbb{C}_{\tilde{t}_q,q} \cong H_*(\mathcal{Z}_{\tilde{G}}^{\tilde{t}_q,q}, \mathbb{C}). \tag{60}$$

Moreover (60) is Γ-equivariant, because all the maps involved in the proof of [ChGi, Proposition 8.1.5] are functorial with respect to isomorphisms of algebraic varieties. To be precise, one should note that throughout [ChGi, Chapter 8] it is assumed that $\tilde G$ is simply connected. However, as we already have (59) at our disposal, [ChGi, §8.1] also applies whenever the derived group of $\tilde G$ is simply connected.

Any Borel subgroup of $\tilde G$ contains C, so $\mathcal{B}^{\tilde t_q,\tilde x} = \mathcal{B}_{\tilde G}^{\tilde t_q,\tilde x}$ and $\mathcal{B}^{t_q,x} = \mathcal{B}_G^{t_q,x}$ are isomorphic algebraic varieties. From [ChGi, p. 414] we see that the convolution product in Borel–Moore homology leads to an action of $H_*(\mathcal{Z}_{\tilde G}^{\tilde t_q,q},\mathbb{C})$ on $H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C})$. Notice that for $\tilde h \in H_*(\mathcal{Z}_{\tilde G}^{\tilde t_q,q},\mathbb{C})$ and $g \in G \rtimes \Gamma$ we have

$$g \cdot \tilde h \in H_*(\mathcal{Z}_{\tilde G}^{g\tilde t_q g^{-1},q},\mathbb{C}) \cong \mathcal{H}(\tilde G) \otimes_{\mathrm{Z}(\mathcal{H}(\tilde G))} \mathbb{C}_{g\tilde t_q g^{-1}}.$$

An obvious generalization of [ChGi, Lemma 8.1.8] says that all these constructions are compatible with the above actions of $G \rtimes \Gamma$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C}) & \xrightarrow{\tilde h} & H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C}) \\ \downarrow H_*(\mathrm{Ad}_g) & & \downarrow H_*(\mathrm{Ad}_g) \\ H_*(\mathcal{B}^{g\tilde t_q g^{-1},g\tilde x g^{-1}},\mathbb{C}) & \xrightarrow{g\cdot\tilde h} & H_*(\mathcal{B}^{g\tilde t_q g^{-1},g\tilde x g^{-1}},\mathbb{C}). \end{array} \tag{61}$$

In particular the component group $\pi_0(\mathrm{Z}_{\tilde G}(\tilde t_q,\tilde x))$ acts on $H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C})$ by $\mathcal{H}(\tilde G)$-intertwiners. Let $\tilde\rho$ be an irreducible representation of this component group, appearing in $H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C})$. In other words, $(\tilde t_q,\tilde x,\tilde\rho)$ is a Kazhdan–Lusztig triple for $\mathcal{H}(\tilde G)$. According to [KaLu, Theorem 7.12]

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde G}(\tilde t_q,\tilde x))}(\tilde\rho, H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C})) \tag{62}$$

is a $\mathcal{H}(\tilde G)$-module with a unique irreducible quotient, say $V_{\tilde t_q,\tilde x,\tilde\rho}$.

Following [Ree2, §3.3] we define a group $R_{\tilde t_q,\tilde x}$ by

$$1 \to \pi_0(\mathrm{Z}_{\tilde G}(\tilde t_q,\tilde x)) \to \pi_0(\mathrm{Z}_{\tilde G}(t_q,x)) \to R_{\tilde t_q,\tilde x} \to 1. \tag{63}$$

Obviously $\mathrm{Z}_{\tilde G}(\tilde t_q,\tilde x)$ contains $\mathrm{Z}(\tilde G)$, so the sequence

$$1 \to \pi_0(\mathrm{Z}_{\tilde G}(\tilde t_q,\tilde x)/\mathrm{Z}(\tilde G)) \to \pi_0(\mathrm{Z}_{\tilde G}(t_q,x)/\mathrm{Z}(\tilde G)) \to R_{\tilde t_q,\tilde x} \to 1 \tag{64}$$

is also exact. For the middle term we have

$$\mathrm{Z}_{\tilde G}(t_q,x)/\mathrm{Z}(\tilde G) \cong \mathrm{Z}_G(t_q,x)/\mathrm{Z}(G)$$

Since the derived group of $\tilde G$ is simply connected, $\mathrm{Z}_{\tilde G}(t_q)^\circ = \mathrm{Z}_{\tilde G}(\tilde t_q)$. In the second term of (64) we get

$$\mathrm{Z}_{\tilde G}(\tilde t_q,\tilde x)/\mathrm{Z}(\tilde G) \cong \mathrm{Z}_{\mathrm{Z}_{\tilde G}(t_q)^\circ}(\tilde x)/\mathrm{Z}(\tilde G) \cong \mathrm{Z}_{\mathrm{Z}_G(t_q)^\circ}(x)/\mathrm{Z}(G).$$

Let us abbreviate $M = \mathrm{Z}_G(t_q)$. Then (64) can be written as

$$1 \to \pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(G)) \to \pi_0(\mathrm{Z}_M(x)/\mathrm{Z}(G)) \to R_{\tilde t_q,\tilde x} \to 1.$$

Like in (64) we can derive another short exact sequence

$$1 \to \pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(M^\circ)) \to \pi_0(\mathrm{Z}_M(x)/\mathrm{Z}(M^\circ)) \to R_{\tilde t_q,\tilde x} \to 1. \tag{65}$$

It can also be obtained from (63) by the dividing the two appropriate groups by the inverse image of $Z(M^\circ)$ in $\tilde G$. From Lemma 4.2 (with the trivial representation of $\pi_0(Z_G(t_q)^\circ(x))$ in the role of ρ) we know that (65) splits. By Proposition 6.2 and (43) $Z(M^\circ)$ acts trivially on $H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C})$. Hence all the 2-cocycles of subgroups of $R_{\tilde t_q,\tilde x}$ appearing associated to (62) are trivial.

Let $\tilde\sigma$ be any irreducible representation of $R_{\tilde t_q,\tilde x,\tilde\rho}$, the stabilizer of the isomorphism class of $\tilde\rho$ in $R_{\tilde t_q,\tilde x}$. Clifford theory for (65) produces $\tilde\rho \rtimes \tilde\sigma \in \mathbf{Irr}\big(\pi_0(Z_M(x)/Z(M^\circ))\big)$, a representation which lifts to $\pi_0(Z_G(t_q,x))$. Moreover by [Ree2, Lemma 3.5.1] it appears in $H_*(\mathcal{B}^{t_q,x},\mathbb{C})$, and conversely every irreducible representation with the latter property is of the form $\tilde\rho\rtimes\tilde\sigma$.

With the above in mind, [Ree2, Lemma 3.5.2] says that the $\mathcal{H}(G)$-module

$$\begin{aligned} M(t_q,x,\tilde\rho\rtimes\tilde\sigma) &:= \mathrm{Hom}_{\pi_0(Z_G(t_q,x))}\big(\tilde\rho\rtimes\tilde\sigma, H_*(\mathcal{B}^{t_q,x},\mathbb{C})\big) \\ &= \mathrm{Hom}_{R_{\tilde t_q,\tilde x,\tilde\rho}}\big(\tilde\sigma, \mathrm{Hom}_{\pi_0(Z_{\tilde G}(\tilde t_q,\tilde x))}(\tilde\rho, H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C}))\big) \end{aligned} \tag{66}$$

has a unique irreducible quotient

$$\pi(t_q,x,\tilde\rho\rtimes\tilde\sigma) = \mathrm{Hom}_{R_{\tilde t_q,\tilde x,\tilde\rho}}(\tilde\sigma, V_{\tilde t_q,\tilde x,\tilde\rho}). \tag{67}$$

According [Ree2, Lemma 3.5.3] this sets up a bijection between $\mathrm{Irr}(\mathcal{H}(G))$ and G-conjugacy classes of Kazhdan–Lusztig triples for G. Since $H_*(\mathcal{B}^{\tilde t_q,\tilde x},\mathbb{C})$ has $Z(\mathcal{H}(\tilde G))$-character $\mathcal{W}^G\tilde t_q$, the $\mathcal{H}(G)$-modules $H_*(\mathcal{B}^{t_q,x},\mathbb{C})$ and (67) have $Z(\mathcal{H}(G))$-character $\mathcal{W}^G t_q$.

Remark 9.2. *The module* (66) *is well-defined for any* $q\in\mathbb{C}^\times$*, although for roots of unity it may have more than one irreducible quotient. For* $q=1$ *the algebra* $\mathcal{H}(G)$ *reduces to* $\mathbb{C}[X^*(T)\rtimes\mathcal{W}^G]$ *and* [ChGi, Section 8.2] *shows that Kato's module* (48) *is a direct summand of* $M(t_1,x,\rho_1)$.

Next we study what Γ does to all these objects. There is natural action of Γ on Kazhdan–Lusztig triples for G, namely

$$\gamma\cdot(t_q,x,\rho_q) = \big(\gamma t_q\gamma^{-1},\gamma x\gamma^{-1},\rho_q\circ\mathrm{Ad}_\gamma^{-1}\big).$$

From (61) and (66) we deduce that the diagram

$$\begin{array}{ccc} \pi(t_q,x,\rho_q) & \xrightarrow{h} & \pi(t_q,x,\rho_q) \\ \downarrow H_*(\mathrm{Ad}_g) & & \downarrow H_*(\mathrm{Ad}_g) \\ \pi\big(gt_qg^{-1},gxg^{-1},\rho_q\circ\mathrm{Ad}_g^{-1}\big) & \xrightarrow{\gamma(h)} & \pi\big(gt_qg^{-1},gxg^{-1},\rho_q\circ\mathrm{Ad}_g^{-1}\big) \end{array} \tag{68}$$

commutes for all $g\in G\gamma$ and $h\in\mathcal{H}(G)$. Hence

(69) Reeder's parametrization of $\mathbf{Irr}(\mathcal{H}(G))$ is Γ-equivariant.

Let $\pi\in\mathbf{Irr}(\mathcal{H}(G))$ and choose a Kazhdan–Lusztig triple such that π is equivalent with $\pi(t_q,x,\rho_q)$. Composition with γ^{-1} on π gives rise to a 2-cocycle $\natural(\pi)$ of Γ_π. Clifford theory tells us that every irreducible representation of $\mathcal{H}(G)\rtimes\Gamma$ is of the form $\pi\rtimes\rho_2^*$ for some $\pi\in\mathbf{Irr}(\mathcal{H}(G))$, unique up to Γ-equivalence, and a unique $\rho_2^*\in\mathbf{Irr}(\mathbb{C}[\Gamma_\pi,\natural(\pi)])$. By the above the stabilizer of π in Γ equals the stabilizer of the G-conjugacy class $[t_q,x,\rho_q]_G$. Thus we have parametrized $\mathbf{Irr}(\mathcal{H}(G)\rtimes\Gamma)$ in a natural way with $G\rtimes\Gamma$-conjugacy classes of quadruples (t_q,x,ρ_q,ρ_2^*), where (t_q,x,ρ_q) is a Kazhdan–Lusztig triple for G and $\rho_2^*\in\mathbf{Irr}\big(\mathbb{C}[\Gamma_{[t_q,x,\rho_q]_G},\natural(\pi(t_q,x,\rho_q))]\big)$.

The short exact sequence

$$1 \to \pi_0(\mathrm{Z}_G(t_q, x)) \to \pi_0(\mathrm{Z}_{G\rtimes\Gamma}(t_q, x)) \to \Gamma_{[t_q,x]_G} \to 1 \tag{70}$$

yields an action of $\Gamma_{[t_q,x]_G}$ on $\mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t_q, x))\big)$. Restricting this to the stabilizer of ρ_q, we obtain another 2-cocycle $\natural(t_q, x, \rho_q)$ of $\Gamma_{[t_q,x,\rho_q]_G}$, which we want to compare to $\natural(\pi(t_q, x, \rho_q))$. Let us decompose

$$H_*(\mathcal{B}^{t_q,x}, \mathbb{C}) \cong \bigoplus\nolimits_{\rho_q} \rho_q \otimes M(t_q, x, \rho_q)$$

as $\pi_0((\mathrm{Z}_G(t_q, x)) \times \mathcal{H}(G)$-modules. We sum over all $\rho_q \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t_q, x))\big)$ for which the contribution is nonzero, and we know that for such ρ_q the $\mathcal{H}(G)$-module $M(t_q, x, \rho_q)$ has a unique irreducible quotient $\pi(t_q, x, \rho_q)$. We know that $\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(t_q, x))$

- acts linearly on $H_*(\mathcal{B}^{t_q,x}, \mathbb{C})$, via conjugation of Borel subgroups;
- acts projectively on ρ_q, with 2-cocycle $\natural(t_q, x, \rho_q)$;
- acts projectively on $\pi(t_q, x, \rho_q)$ and $M(t_q, x, \rho_q)$, via its quotient $\Gamma_{[t_q,x]_G}$ and with 2-cocycle $\natural(\pi(t_q, x, \rho_q))$.

It follows that

$$\natural(t_q, x, \rho_q) = \natural(\pi(t_q, x, \rho_q))^{-1} \text{ as 2-cocycles of } \Gamma_{[t_q,x,\rho_q]_G}. \tag{71}$$

Hence every irreducible representation ρ of $\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(t_q, x))$ is of the form $\rho_q \rtimes \rho_2$ for ρ_q and ρ_2^* as above. Moreover ρ determines ρ_q up to $\Gamma_{[t_q,x]_G}$-equivalence and ρ_2 is unique if ρ_q has been chosen. Finally, if ρ_q appears in $H_{\mathrm{top}}(\mathcal{B}^{t_q,x}, \mathbb{C})$ then every irreducible $\pi_0(\mathrm{Z}_G(t_q, x))$-subrepresentation of ρ does, because $\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(t_q, x))$ acts naturally on $H_*(\mathcal{B}^{t_q,x}, \mathbb{C})$. Therefore we may replace the above quadruples $(t_q, x, \rho_q, \rho_2^*)$ by Kazhdan–Lusztig triples (t_q, x, ρ).

The module associated to $(t_q, x, \rho_q, \rho_2^*)$ via the above constructions is the unique irreducible quotient of the $\mathcal{H}(G) \rtimes \Gamma$-module

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t_q,x))}\big(\rho_q, H_*(\mathcal{B}^{t_q,x}, \mathbb{C})\big) \rtimes \rho_2^*. \tag{72}$$

The same reasoning as in the proof of Theorem 4.4 shows that, with $\rho = \rho_q \rtimes \rho_2$, (72) is isomorphic to

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(t_q,x))}\big(\rho, H_*(\mathcal{B}^{t_q,x}, \mathbb{C}) \otimes \mathbb{C}[\Gamma]\big). \tag{73}$$

Since the $\mathcal{H}(G)$-module $H_*(\mathcal{B}^{t_q,x}, \mathbb{C})$ depends in a natural way on (t_q, x), so does the unique irreducible quotient of (73). As $H^*(\mathcal{B}^{t_q,x}, \mathbb{C})$ has $Z(\mathcal{H}(G))$-character $\mathcal{W}^G t_q$, (73) has $Z(\mathcal{H}(G) \rtimes \Gamma)$-character $(\mathcal{W}^G \rtimes \Gamma)t_q$. □

For use in Section 15 we discuss some analytic properties of $\mathcal{H}(G) \rtimes \Gamma$-modules. Let $\{\theta_x T_w : x \in X^*(T), w \in \mathcal{W}^G\}$ be the Bernstein basis of $\mathcal{H}(G)$ [Lus5, §3]. Recall from [Opd, Lemma 2.20] that a finite-dimensional $\mathcal{H}(G)$-module (π, V) is tempered if and only if all eigenvalues of the $\pi(\theta_x)$ with x in the closed positive cone $X^*(T)^+ \subset X^*(T)$ have absolute value ≤ 1. Similarly, (π, V) is square-integrable if and only if the eigenvalues of the $\pi(\theta_x)$ with $x \in X^*(T) \setminus \{0\}$ have absolute value < 1 [Opd, Lemma 2.22]. We say that (π, V) is essentially square-integrable if its restriction to $\mathcal{H}(G/Z(G))$ is square-integrable.

We will treat these criteria as definitions of temperedness and (essential) square-integrability. We define that a finite-dimensional $\mathcal{H}(G) \rtimes \Gamma$-module is tempered or (essentially) square-integrable if and only if its restriction to $\mathcal{H}(G)$ is so. One can check that square-integrability is equivalent to temperedness plus essential square-integrability.

Proposition 9.3. *Let (t_q, x, ρ) be a Kazhdan–Lusztig triple for $\mathcal{H}(G) \rtimes \Gamma$.*

1. *Lift (t_q, x, ρ) to a KLR parameter (Φ, ρ) as in Lemma 7.1 and write $t = \Phi(\varpi_F) \in T$. The $\mathcal{H}(G) \rtimes \Gamma$-module $\pi(t_q, x, \rho)$ is tempered if and only if t is contained in a compact subgroup of T.*
2. *$\pi(t_q, x, \rho)$ is essentially square-integrable if and only if $\{t_q, x\}$ is not contained in any Levi subgroup of a proper parabolic subgroup of G.*

Proof. (1) Since $X^*(G/G_{\mathrm{der}}) + X^*(T/\mathrm{Z}(G))$ is a sublattice of finite index in $X^*(T)$, a finite dimensional $\mathcal{H}(G)$-module is tempered if and only its restrictions to $\mathcal{H}(G/G_{\mathrm{der}})$ and to $\mathcal{H}(T/\mathrm{Z}(G))$ are both tempered.

As G/G_{der} is a torus, $\pi_z := \pi(t_q, x, \rho)|_{\mathcal{H}(G/G_{\mathrm{der}})}$ is tempered if and only if all the eigenvalues of the $\pi_z(\theta_x)$ with $x \in X^*(G/G_{\mathrm{der}})$ have absolute value 1. By Theorem 9.1 the central character of $\pi(t_q, x, \rho)$ is $(\mathcal{W}^G \rtimes \Gamma)t_q$, so the eigenvalues of $\pi_z(\theta_x)$ belong to $x((\mathcal{W}^G \rtimes \Gamma)t_q)$. Because $\mathcal{W}^G \rtimes \Gamma$ acts on T by algebraic automorphisms, it preserves the unique maximal compact subgroup T_{cpt}. Therefore π_z is tempered if and only if $t_q \in T_{\mathrm{cpt}}/T_{\mathrm{cpt}} \cap G_{\mathrm{der}}$. Since $t_q \in t\Phi(\mathrm{SL}_2(\mathbb{C}))$ and $\Phi(\mathrm{SL}_2(\mathbb{C})) \subset G_{\mathrm{der}}$, this is equivalent to t being compact in $T/T \cap G_{\mathrm{der}}$.

We denote the restriction of $\pi(t_q, x, \rho)$ to $\mathcal{H}(G/\mathrm{Z}(G))$ by π_d. Since the varieties of Borel subgroups for G and for $G/\mathrm{Z}(G)$ are isomorphic and the relevant irreducible modules are made from the homologies of such varieties, π_d is a direct sum of representations $\pi(t_q, x, \rho_i)$ with $\rho_i \in \mathbf{Irr}\big(\pi_0(Z_{G/\mathrm{Z}(G)}(t_q, x))\big)$.

We would like to apply [KaLu, Theorem 8.2], but it is only proven for simply connected groups. Let $\tilde{H}$ be the simply connected cover of $G/\mathrm{Z}(G)$. Since $\mathrm{SL}_2(\mathbb{C})$ is simply connected, $\Phi|_{\mathrm{SL}_2(\mathbb{C})}$ lifts in unique way to a homomorphism $\tilde{\Phi} : \mathrm{SL}_2(\mathbb{C}) \to \tilde{H}$. Lift t to $\tilde{t} \in \tilde{H}$. Then $\tilde{t}$ centralizes $\tilde{\Phi}(\mathrm{SL}_2(\mathbb{C}))$, because it centralizes the Lie algebra of that group. Now $\tilde{t}_q := t\tilde{\Phi}\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \in \tilde{H}$ is a lift of t_q.

From (72), (73) and (66) we see that π_d is a $\mathcal{H}(G/\mathrm{Z}(G))$-summand of a sum of irreducible $\mathcal{H}(\tilde{H})$-modules $\pi(\tilde{t}_q, \tilde{x}, \tilde{\rho}_i)$ with $\tilde{\rho}_i \in \mathbf{Irr}\big(\pi_0(Z_{\tilde{H}}(\tilde{t}_q, \tilde{x}))\big)$. All irreducible $\mathcal{H}(G/\mathrm{Z}(G))$-submodules of $\pi(\tilde{t}_q, \tilde{x}, \tilde{\rho}_i)$ are conjugate by invertible elements of $\mathcal{H}(\tilde{H})$, so $\pi(\tilde{t}_q, \tilde{x}, \tilde{\rho}_i)$ is tempered if and only if any of its $\mathcal{H}(G/\mathrm{Z}(G))$-constituents is tempered. It follows that π_d is tempered if and only if $\pi(\tilde{t}_q, \tilde{x}, \tilde{\rho}_i)$ is tempered for every relevant $\tilde{\rho}_i$.

According to [KaLu, Theorem 8.2], the latter condition is equivalent to "$P = G$", which by [KaLu, §7.1] is the same as "all eigenvalues of $\mathrm{Ad}(\tilde{t})$ on $\mathrm{Lie}(\tilde{H})$ have absolute value ≤ 1". Since $\tilde{H} \to G/\mathrm{Z}(G)$ is a central extension, this statement is equivalent to the analogous one for $\mathrm{Ad}(t)$ acting on $\mathrm{Lie}(G/\mathrm{Z}(G))$. We showed that π_d is tempered if and only if all the eigenvalues of $\mathrm{Ad}(t)$ on $\mathrm{Lie}(G/\mathrm{Z}(G))$ have absolute value ≤ 1. The nontrivial eigenvalues come in pairs $\alpha(t), \alpha(t)^{-1}$ with $\alpha \in \Phi(G, T)$, so we may replace the condition ≤ 1 by $= 1$.

We conclude with a chain of equivalences.

$\pi(t_q, x, \rho)$ is tempered $\Longleftrightarrow$

π_z is tempered and π_d is tempered $\Longleftrightarrow$

t is compact in $T/T \cap G_{\mathrm{der}}$ and $|\alpha(t)| = 1\ \forall \alpha \in \Phi(G,T) \Longleftrightarrow$

$|x(t)| = 1\ \forall x \in X^*(G/G_{\mathrm{der}})$ and $|x(t)| = 1\ \forall x \in X^*(T/Z(G)) \Longleftrightarrow$

$|x(t)| = 1\ \forall x \in X^*(T) \Longleftrightarrow$

t belongs to the maximal compact subgroup of T.

(2) In the same way as in the proof of part (1), this can be reduced to irreducible modules of $\mathcal{H}(\tilde H)$. Since $\tilde H$ is simply connected, essential square-integrability is now the same as square-integrability, and [KaLu] applies. According to [KaLu, Theorem 8.3] a $\mathcal{H}(\tilde H)$-module $\pi(\tilde t_q, \tilde x, \tilde \rho)$ is square-integrable if and only if $\{\tilde t_q, \tilde x\}$ is not contained in any Levi subgroup of a proper parabolic subgroup of $\tilde H$. There is a canonical bijection between parabolic (respectively Levi) subgroups of $\tilde H$ and of G, so this statement is equivalent to the analogous one for $\{t_q, x\}$ and G. $\square$

10. Spherical representations

Let G, B, T and Γ be as in the previous section. Let $\mathcal{H}(\mathcal{W}^G)$ be the Iwahori–Hecke algebra of the Weyl group $\mathcal{W}^G$, with a parameter $q \in \mathbb{C}^\times$ which is not a root of unity. This is a deformation of the group algebra $\mathbb{C}[\mathcal{W}^G]$ and a subalgebra of the affine Hecke algebra $\mathcal{H}(G)$. The multiplication is defined in terms of the basis $\{T_w \mid w \in \mathcal{W}^G\}$ by

$$\begin{aligned} T_x T_y = T_{xy}, &\quad \text{if } \ell(xy) = \ell(x) + \ell(y), \text{ and} \\ (T_s - q)(T_s + 1) = 0, &\quad \text{if } s \text{ is a simple reflection.} \end{aligned} \tag{74}$$

Recall that $\mathcal{H}(G)$ also has a commutative subalgebra $\mathcal{O}(T)$, such that the multiplication maps

$$\mathcal{O}(T) \otimes \mathcal{H}(\mathcal{W}^G) \longrightarrow \mathcal{H}(G) \longleftarrow \mathcal{H}(\mathcal{W}^G) \otimes \mathcal{O}(T) \tag{75}$$

are bijective.

The trivial representation of $\mathcal{H}(\mathcal{W}^G) \rtimes \Gamma$ is defined as

$$\mathrm{triv}(T_w \gamma) = q^{\ell(w)} \quad w \in \mathcal{W}^G, \gamma \in \Gamma. \tag{76}$$

It is associated to the idempotent

$$p_{\mathrm{triv}}\, p_\Gamma := \sum_{w \in \mathcal{W}^G} T_w P_{\mathcal{W}^G}(q)^{-1} \sum_{\gamma \in \Gamma} \gamma |\Gamma|^{-1} \quad \in \mathcal{H}(\mathcal{W}^G) \rtimes \Gamma,$$

where $P_{\mathcal{W}^G}$ is the Poincaré polynomial

$$P_{\mathcal{W}^G}(q) = \sum\nolimits_{w \in \mathcal{W}^G} q^{\ell(w)}.$$

Notice that $P_{\mathcal{W}^G}(q) \neq 0$ because q is not a root of unity. The trivial representation appears precisely once in the regular representation of $\mathcal{H}(\mathcal{W}^G) \rtimes \Gamma$, just like for finite groups.

An $\mathcal{H}(G) \rtimes \Gamma$-module V is called spherical if it is generated by the subspace $p_{\mathrm{triv}} p_\Gamma V$ [HeOp, (2.5)]. This admits a nice interpretation for the unramified principal series representations. Recall that $\mathcal{H}(G) \cong \mathcal{H}(\mathcal{G}, \mathcal{I})$ for an Iwahori subgroup $\mathcal{I} \subset \mathcal{G}$. Let $\mathcal{K} \subset \mathcal{G}$ be a good maximal compact subgroup containing $\mathcal{I}$. Then p_{triv} corresponds to averaging over $\mathcal{K}$ and $p_{\mathrm{triv}}\mathcal{H}(\mathcal{G},\mathcal{I})p_{\mathrm{triv}} \cong \mathcal{H}(\mathcal{G},\mathcal{K})$, see [HeOp, Section 1]. Hence spherical $\mathcal{H}(\mathcal{G},\mathcal{I})$-modules correspond to smooth $\mathcal{G}$-representations that are generated by their $\mathcal{K}$-fixed vectors, also known as $\mathcal{K}$-spherical $\mathcal{G}$-representations. By the Satake transform

$$p_{\mathrm{triv}}\mathcal{H}(\mathcal{G},\mathcal{I})p_{\mathrm{triv}} \cong \mathcal{H}(\mathcal{G},\mathcal{K}) \cong \mathcal{O}(T/\mathcal{W}^G), \tag{77}$$

so the irreducible spherical modules of $\mathcal{H}(G) \cong \mathcal{H}(\mathcal{G},\mathcal{I})$ are parametrized by $T/\mathcal{W}^G$ via their central characters. We want to determine the Kazhdan–Lusztig triples (as in Theorem 9.1) of these representations.

Proposition 10.1. *For every central character $(\mathcal{W}^G \rtimes \Gamma)t \in T/(\mathcal{W}^G \rtimes \Gamma)$ there is a unique irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-module, and it is associated to the Kazhdan–Lusztig triple $(t, x = 1, \rho = \mathrm{triv})$.*

Proof. We will first prove the proposition for $\mathcal{H}(G)$, and only then consider Γ.

By the Satake isomorphism (77) there is a unique irreducible spherical $\mathcal{H}(G)$-module for every central character $\mathcal{W}^G t \in T/\mathcal{W}^G$. The equivalence classes of Kazhdan–Lusztig triples of the form $(t, x = 1, \rho = \mathrm{triv})$ are also in canonical bijection with $T/\mathcal{W}^G$. Therefore it suffices to show that $\pi(t, 1, \mathrm{triv})$ is spherical for all $t \in T$.

The principal series of $\mathcal{H}(G)$ consists of the modules $\mathrm{Ind}_{\mathcal{O}(T)}^{\mathcal{H}(G)}\mathbb{C}_t$ for $t \in T$. This module admits a central character, namely $\mathcal{W}^G t$. By (75) every such module is isomorphic to $\mathcal{H}(\mathcal{W}^G)$ as a $\mathcal{H}(\mathcal{W}^G)$-module. In particular it contains the trivial $\mathcal{H}(\mathcal{W}^G)$-representation once and has a unique irreducible spherical subquotient.

As in Section 9, let $\tilde G$ be a finite central extension of G with simply connected derived group. Let $\tilde T, \tilde B$ be the corresponding extensions of T, B. We identify the roots and the Weyl groups of $\tilde G$ and G. Let $\tilde t \in \tilde T$ be a lift of $t \in T$. From the general theory of Weyl groups it is known that there is a unique $t^+ \in \mathcal{W}^G \tilde T$ such that $|\alpha(t^+)| \geq 1$ for all $\alpha \in R(\tilde B, \tilde T) = R(B, T)$. By (61)

$$H_*(\mathcal{B}_{\tilde G}^{\tilde t}, \mathbb{C}) \cong H_*(\mathcal{B}_{\tilde G}^{t^+}, \mathbb{C})$$

as $\mathcal{H}(\tilde G)$-modules. These $t^+, \tilde B$ fulfill [Ree2, Lemma 2.8.1], so by [Ree2, Proposition 2.8.2]

$$M_{\tilde t, \tilde x = 1, \tilde \rho = \mathrm{triv}} = H_*(\mathcal{B}_{\tilde G}^{t^+}, \mathbb{C}) \cong \mathrm{Ind}_{\mathcal{O}(\tilde T)}^{\mathcal{H}(\tilde G)}\mathbb{C}_{t^+}. \tag{78}$$

According to [Ree1, (1.5)], which applies to t^+, the spherical vector p_{triv} generates $M_{\tilde t, 1, \mathrm{triv}}$. Therefore it cannot lie in any proper $\mathcal{H}(\tilde G)$-submodule of $M_{\tilde t, 1, \mathrm{triv}}$ and represents a nonzero element of $\pi(\tilde t, 1, \mathrm{triv})$. We also note that the central character of $\pi(\tilde t, 1, \mathrm{triv})$ is that of $M_{\tilde t, 1, \mathrm{triv}}$, $\mathcal{W}^G \tilde t = \mathcal{W}^G t^+$.

Now we analyse this is an $\mathcal{H}(G)$-module. The group $R_{\tilde t, 1} = R_{\tilde t, \tilde x = 1, \tilde \rho = \mathrm{triv}}$ from (63) is just the component group $\pi_0(\mathrm{Z}_G(t))$, so by (67)

$$\pi(\tilde t, 1, \mathrm{triv}) \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t))}(\rho, \pi(\tilde t, 1, \mathrm{triv})) = \bigoplus\nolimits_\rho \pi(t, 1, \mathrm{triv}).$$

The sum runs over $\mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t))\big)$, all these representations ρ contribute nontrivially by [Ree2, Lemma 3.5.1]. Recall from Lemma 3.2 that $\pi_0(\mathrm{Z}_G(t))$

can be realized as a subgroup of $\mathcal{W}^G$ and from (77) that $p_{\text{triv}} \in \pi(\tilde{t}, 1, \text{triv})$ can be regarded as a function on $\tilde{\mathcal{G}}$ which is bi-invariant under a good maximal compact subgroup $\tilde{\mathcal{K}}$. This brings us in the setting of [Cas, Proposition 4.1], which says that $\pi_0(\mathrm{Z}_G(t))$ fixes $p_{\text{triv}} \in \pi(\tilde{t}, 1, \text{triv})$. Hence $\pi(t, 1, \text{triv})$ contains p_{triv} and is a spherical $\mathcal{H}(G)$-module. Its central character is the restriction of the central character of $\pi(\tilde{t}, 1, \text{triv})$, that is, $\mathcal{W}^G t \in T/\mathcal{W}^G$.

Now we include Γ. Suppose that V is a irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-module. By Clifford theory its restriction to $\mathcal{H}(G)$ is a direct sum of irreducible $\mathcal{H}(G)$-modules, each of which contains p_{triv}. Hence V is built from irreducible spherical $\mathcal{H}(G)$-modules. By (69)

$$\gamma \cdot \pi(t, 1, \text{triv}) = \pi(\gamma t, 1, \text{triv}),$$

so the stabilizer of $\pi(t, 1, \text{triv}) \in \mathbf{Irr}(\mathcal{H}(G))$ in Γ equals the stabilizer of $\mathcal{W}^G t \in T/\mathcal{W}^G$ in Γ. Any isomorphism of $\mathcal{H}(G)$-modules

$$\psi_\gamma : \pi(t, 1, \text{triv}) \to \pi(\gamma t, 1, \text{triv})$$

must restrict to a bijection between the onedimensional subspaces of spherical vectors in both modules. We normalize ψ_γ by $\psi_\gamma(p_{\text{triv}}) = p_{\text{triv}}$. Then $\gamma \mapsto \psi_\gamma$ is multiplicative, so the 2-cocycle of $\Gamma_{\mathcal{W}^G t}$ is trivial. With Theorem 9.1 this means that the irreducible $\mathcal{H}(G) \rtimes \Gamma$-modules whose restriction to $\mathcal{H}(G)$ is spherical are parametrized by equivalence classes of triples $(t, 1, \text{triv} \rtimes \sigma)$ with $\sigma \in \mathbf{Irr}(\Gamma_{\mathcal{W}^G t})$. The corresponding module is

$$\pi(t, 1, \text{triv} \rtimes \sigma) = \pi(t, 1, \text{triv}) \rtimes \sigma^* = \mathrm{Ind}_{\mathcal{H}(G) \rtimes \Gamma_{\mathcal{W}^G t}}^{\mathcal{H}(G) \rtimes \Gamma} \big(\pi(t, 1, \text{triv}) \otimes \sigma^* \big).$$

Clearly $\pi(t, 1, \text{triv} \rtimes \sigma)$ contains the spherical vector $p_{\text{triv}} p_\Gamma$ if and only if σ is the trivial representation. It follows that the irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-modules are parametrized by equivalence classes of triples $\big(t, 1, \text{triv}_{\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t))}\big)$, that is, by $T/(\mathcal{W}^G \rtimes \Gamma)$. $\square$

11. From the principal series to affine Hecke algebras

Let χ be a smooth character of the maximal torus $\mathcal{T} \subset \mathcal{G}$. We recall that

$$\begin{aligned} \mathfrak{s} &= [\mathcal{T}, \chi]_{\mathcal{G}}, \\ c^{\mathfrak{s}} &= \hat{\chi}\big|_{\mathfrak{o}_F^\times}, \\ H &= \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}}), \\ W^{\mathfrak{s}} &= \mathrm{Z}_{\mathcal{W}^G}(\mathrm{im}\, c^{\mathfrak{s}}). \end{aligned}$$

Let $\{\text{KLR parameters}\}^{\mathfrak{s}}$ be the collection of Kazhdan–Lusztig–Reeder parameters for G such that $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}$. Notice that the condition forces $\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})) \subset H$. This collection is not closed under conjugation by elements of G, only $H = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}})$ acts naturally on it.

Recall that $T^{\mathfrak{s}}$ and $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$ are Bernstein's torus and Bernstein's centre associated to $\mathfrak{s}$. Clearly T acts simply transitively on $T^{\mathfrak{s}}$, but we need a little more. Consider the bijections

$$T^{\mathfrak{s}} \longrightarrow \{\text{L-parameters } \Phi \text{ for } \mathcal{T} \text{ with } \Phi|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}\} \xrightarrow{\mathrm{ev}_{\varpi_F}} T, \tag{79}$$

where the first map is the restriction of the local Langlands correspondence for $\mathcal{T}$ to $T^{\mathfrak{s}}$ and the second map sends Φ to $\Phi(\varpi_F)$. The latter is not natural because it depends on our choice of ϖ_F, but since we use the same uniformizer everywhere this is not a problem.

As $T^{\mathfrak{s}}$ is a maximal torus in H, every semisimple element of H° is conjugate to one in T. By Lemma 3.2 $W^{\mathfrak{s}} \cong N_H(T)/T$, so we can identify $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ with the space $c(H)_{\mathrm{ss}}$ of semisimple conjugacy classes in H that consist of elements if H°.

In general H need not be connected. Recall from Lemma 3.3 that any choice of a pinning of H° determines a splitting of the short exact sequence

$$1 \to H^\circ/\mathrm{Z}(H^\circ) \to H/\mathrm{Z}(H^\circ) \to \pi_0(H) \to 1. \tag{80}$$

Lemma 3.2 shows that

$$W^{\mathfrak{s}} = \mathcal{W}^G_{\mathrm{im}\, c^{\mathfrak{s}}} \cong \mathcal{W}^{H^\circ} \rtimes \pi_0(H). \tag{81}$$

We fix a Borel subgroup $B \subset G$ containing T, and a pinning of H° with T as maximal torus and $B_H = B \cap H^\circ$ as Borel subgroup. This determines a conjugation action of $\pi_0(H)$ on H°, and hence on objects associated to H°. Like in Section 9, let $\mathcal{H}(H^\circ)$ be the affine Hecke algebra with the same based root datum as (H°, B), and with parameter q equal to the cardinailty of the residue field of F. By our conventions $\pi_0(H)$ normalizes B, so it acts on $\mathcal{H}(H^\circ)$ by algebra automorphisms. Following [Roc, Section 8] we define

$$\mathcal{H}(H) = \mathcal{H}(H^\circ) \rtimes \pi_0(H). \tag{82}$$

We denote the Hecke algebra of $\mathcal{G}$ by $\mathcal{H}(\mathcal{G})$. Recall that its consists of all locally constant compactly supported functions $\mathcal{G} \to \mathbb{C}$ and is endowed with the convolution product. The category $\mathrm{Rep}(\mathcal{G})$ of smooth $\mathcal{G}$-representations is naturally equivalent to the category of nondegenerate $\mathcal{H}(\mathcal{G})$-modules. Let $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}}$ be the block of $\mathrm{Rep}(\mathcal{G})$ associated to $\mathfrak{s}$.

The link between these representations and Section 9 is provided by results of Roche. In [Roc, p. 378–379] Roche imposes some conditions on the residual characteristic of the field.

Condition 11.1. *If the root system $R(H,T)$ is irreducible, then the restriction on the residual characteristic p of F is as follows:*

- *for type* $A_n \quad p > n+1$
- *for types* $B_n, C_n, D_n \quad p \neq 2$
- *for type* $F_4 \quad p \neq 2,3$
- *for types* $G_2, E_6 \quad p \neq 2,3,5$
- *for types* $E_7, E_8 \quad p \neq 2,3,5,7.$

If $R(H,T)$ is reducible, one excludes primes attached to each of its irreducible factors.

Since $R(H,T)$ is a subset of $R(G,T) \cong R(\mathcal{G},\mathcal{T})^\vee$, these conditions are fulfilled when they hold for $R(\mathcal{G},\mathcal{T})$.

Theorem 11.2. *Assume that Condition 11.1 holds. There exists an equivalence of categories*

$$\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \longleftrightarrow \mathrm{Mod}(\mathcal{H}(H))$$

such that:

(1) *The cuspidal support of an irreducible $\mathcal{G}$-representation corresponds to the central character of the associated $\mathcal{H}(H)$-module via the canonical bijection $T^{\mathfrak{s}}/W^{\mathfrak{s}} \to c(H)_{\mathrm{ss}}$.*
(2) *It does not depend on the choice of χ with $[\mathcal{T},\chi]_{\mathcal{G}} = \mathfrak{s}$.*

Proof. First we note that, although Roche [Roc] works with a p-adic field, it follows from [AdRo] that his arguments apply just as well over local fields of positive characteristic. By [Roc, Corollary 7.9] there exists a type (J,τ) for $\mathfrak{s} = [\mathcal{T},\chi]_{\mathcal{G}}$, where τ is a character. Then the τ-spherical Hecke algebra $\mathcal{H}(\mathcal{G},\tau)$ of $\mathcal{H}(\mathcal{G})$ (see [BuKu, §2]) equals $e_\tau \mathcal{H}(\mathcal{G}) e_\tau$, where $e_\tau \in \mathcal{H}(J)$ is the central idempotent corresponding to τ. According to [BuKu, Theorem 4.3] there exists an equivalence of categories

$$\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \to \mathrm{Mod}(\mathcal{H}(\mathcal{G},\tau)) : V \mapsto V^\tau, \tag{83}$$

where $V^\tau = e_\tau V$ is the τ-isotypical subspace of $V|_J$. From the proof of [BuKu, Proposition 3.3] we see that the inverse of (83) is given by

$$\mathrm{Mod}(\mathcal{H}(\mathcal{G},\tau)) \to \mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} : M \mapsto \mathcal{H}(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} M. \tag{84}$$

Theorem 8.2 of [Roc] says that there exists a support preserving algebra isomorphism

$$\mathcal{H}(H) \to \mathcal{H}(\mathcal{G},\tau). \tag{85}$$

The combination of (83) and (85) yields the desired equivalence of categories. It satisfies property (1) by [Roc, Theorem 9.4].

In [Roc, §9] it is shown that (J,τ) is a cover of the type $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$, in the sense of [BuKu, §8]. With [Roc, Theorem 9.4] one sees that the above equivalence of categories does not change if one twists χ by an unramified character of $\mathcal{T}$, basically because that does not effect $\chi|_{\mathcal{T}_0}$.

Every other character of $\mathcal{T}$ determining the same inertial equivalence class $\mathfrak{s}$ can be obtained from χ by an unramified twist and conjugation by an element of $W^{\mathfrak{s}}$. Reeder [Ree2, §6] checked that the latter operation does not change Roche's equivalence of categories. We note that in [Ree2] it is assumed that H is connected. Fortunately this does not play a role in [Ree2, §6], because all the underlying results from [Roc] and [Mor] are known irrespective of the connectedness. □

We emphasize that Theorem 11.2 is the only cause of our conditions on the residual characteristic. If one can prove Theorem 11.2 for a particular Bernstein component and a p which is excluded by Condition 11.1, then everything in our paper (except possibly Lemma 12.1) holds for that case.

For example, for unramified characters χ Theorem 11.2 is already classical, proven without any restrictions on p by Borel [Bor1]. As Roche remarks in [Roc, 4.14], all the main results of [Roc] (and hence Theorem 11.2) are valid without restrictions on p when $\mathcal{G} = \mathrm{GL}_n(F)$ or $\mathcal{G} = \mathrm{SL}_n(F)$. For $\mathrm{GL}_n(F)$ this is easily seen, for $\mathrm{SL}_n(F)$ one can use [GoRo].

Theorems 11.2 and 9.1 provide a bijection

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(H)) \to \{\text{KLR-parameters}\}^{\mathfrak{s}}/H. \tag{86}$$

Unfortunately this bijection is not entirely canonical in general.

Example 11.3. *Consider the unramified principal series representations of* $\mathrm{SL}_2(F)$*. Then the type is the trivial representation of an Iwahori subgroup* $\mathcal{I} \subset \mathrm{SL}_2(F)$ *and Theorem 11.2 reduces to* [Bor1]*. The functor sends a* $\mathrm{SL}_2(F)$*-representation to its space of* $\mathcal{I}$*-fixed vectors. The Iwahori subgroup is determined by the choice of a maximal compact subgroup and a Borel subgroup of* $\mathrm{SL}_2(F)$*, and these data also determine the isomorphism* $\mathcal{H}(\mathrm{SL}_2(F), \mathrm{triv}_{\mathcal{I}}) \cong \mathcal{H}(H)$*.*

However, there are two conjugacy classes of maximal compact subgroups in $\mathrm{SL}_2(F)$*. If we pick a maximal compact subgroup in the other class and perform the same operations, we obtain an alternative map* (86)*. The difference is not big, for almost all* $\mathrm{SL}_2(F)$*-representations the two maps have the same image. But look at the parabolically induced representation* $\pi = I_{\mathcal{B}}^{\mathrm{SL}_2(F)}(\chi_{-1})$*, where* χ_{-1} *denotes the unique unramified character of* $\mathcal{T}$ *of order 2. It is well-known that* π *is the direct sum of two inequivalent irreducible representations, say* π_+ *and* π_-*. It turns out that the difference between our two candidates for* (86) *is just interchanging* π_+ *and* π_-*.*

We will determine in Section 14 how canonical (86) is precisely.

12. Main result (special case)

In the current section we will study the relations between $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ and $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$, in the case that H is connected. Although in general H° is the connected centralizer of a semisimple element in G [Roc, p. 397], H need not be connected. Fortunately it is for most $\mathfrak{s}$, a sufficient condition is:

Lemma 12.1. *Suppose that G has simply connected derived group and that the residual characteristic p satisfies Condition 11.1 for $R(G,T)$. Then H is connected.*

Proof. We consider first the case where $\mathfrak{s} = [\mathcal{T}, 1]_{\mathcal{G}}$. Then we have $c^{\mathfrak{s}} = 1, H = G$ and $W^{\mathfrak{s}} = \mathcal{W}$.

We assume now that $c^{\mathfrak{s}} \neq 1$. Then $\mathrm{im}\, c^{\mathfrak{s}}$ is a finite abelian subgroup of T which has the following structure: the direct product of a finite abelian p-group A_p with a cyclic group B_{q-1} whose order divides $q-1$. This follows from the well-known structure theorem for the group $\mathfrak{o}_F^\times$, see [Iwa, §2.2]:

$$\mathrm{im}\, c^{\mathfrak{s}} = A_p \cdot B_{q-1}.$$

We have

$$H = \mathrm{Z}_{H_A}(B_{q-1}) \quad \text{where} \quad H_A := \mathrm{Z}_G(A_p).$$

Since G has simply connected derived group, A_p is a p-group and p is not a torsion prime for the root system $R(G,T)$, it follows from Steinberg's connectedness theorem [Ste2, 2.16.b] that the group H_A is connected. It was shown in [Roc, p. 397] that $H_A = \mathrm{Z}_G(x)$ for a well-chosen $x \in T$. Then [Ste2, 2.17] says that the derived group of $H_A^\circ = H_A$ is simply connected.

Now B_{q-1} is cyclic. Applying Steinberg's connectedness theorem to the group H_A, we get that H itself is connected. □

Remark 12.2. *Notice that H does not necessarily have a simply connected derived group in setting of Lemma 12.1. For instance, if G is the exceptional group of type G_2 and χ is the tensor square of a ramified quadratic character of $F^\times$, then $H = \mathrm{SO}_4(\mathbb{C})$.*

In the remainder of this section we will assume that H is connected. Then Lemma 3.2 shows that $W^{\mathfrak{s}}$ is the Weyl group of H.

Theorem 12.3. *Let $\mathcal{G}$ be a split reductive p-adic group and let $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$ be a point in the Bernstein spectrum of the principal series of $\mathcal{G}$. Assume that H is connected and that Condition 11.1 holds. Then there is a commutative diagram of bijections, in which the triangle is canonical:*

$$\begin{tikzcd} & & (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2 \arrow[dl] \arrow[dr] & \\ \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \arrow[r] & \mathbf{Irr}(\mathcal{H}(H)) \arrow[rr] & & \{\text{KLR parameters}\}^{\mathfrak{s}}/H \end{tikzcd}$$

In the triangle the right slanted map stems from Kato's affine Springer correspondence [Kat]*. The bottom horizontal map is the bijection established by Reeder* [Ree2] *and the left slanted map can be constructed via the asymptotic Hecke algebra of Lusztig.*

Proof. Roche's equivalence of categories in Theorem 11.2 provides the bijection $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(H))$.

The right slanted map is the composition of Theorem 8.2.1 (applied to H) and Lemma 7.1 (with the condition $\Phi(\varpi_F) = t$). We can take as the horizontal map the parametrization of irreducible $\mathcal{H}(H)$-modules by Kazhdan, Lusztig and Reeder as described in Section 9. These are both canonical bijections, so there is a unique left slanted map which makes the diagram commute, and it is also canonical. We want to identify it in terms of Hecke algebras.

Fix a KLR parameter (Φ, ρ) and recall from Theorem 8.2.2 that the corresponding $X^*(T) \rtimes \mathcal{W}^H$-representation is

$$\tau(t, x, \rho) = \mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho, H_{d(x)}(\mathcal{B}_H^{t,x}, \mathbb{C})\big). \tag{87}$$

Similarly, by Theorem 9.1 the corresponding $\mathcal{H}(H)$-module is the unique irreducible quotient of the $\mathcal{H}(H)$-module

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t_q,x))}\big(\rho_q, H_*(\mathcal{B}_H^{t_q,x}, \mathbb{C})\big). \tag{88}$$

In view of Proposition 6.1 both spaces are unchanged if we replace t by t_q and ρ by ρ_q, and the vector space (88) is also naturally isomorphic to

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(\Phi))}\big(\rho, H_*(\mathcal{B}_H^{t,\Phi(B_2)}, \mathbb{C})\big). \tag{89}$$

Recall the asymptotic Hecke algebra $\mathcal{J}(H)$ from [Lus3]. We remark that, although in [Lus3] the underlying reductive group H is supposed to be semisimple, this assumption is shown to be unnecessary in [Lus4]. Lusztig constructs canonical bijections

$$\mathbf{Irr}(\mathcal{H}(H)) \longleftrightarrow \mathbf{Irr}(\mathcal{J}(H)) \longleftrightarrow \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H) \tag{90}$$

which we will analyse with our terminology. According to [Lus4, Theorem 4.2] $\mathbf{Irr}(\mathcal{J}(H))$ is naturally parametrized by the set of H-conjugacy classes of Kazhdan–Lusztig triples for H. By Lemma 7.1 we can also use KLR parameters, so may call the $\mathcal{J}(H)$-module with parameters (t_q, x, ρ_q) $\tilde{\pi}(\Phi, \rho)$. Its retraction to $\mathcal{H}(H)$ via

$$\mathcal{H}(H) \xrightarrow{\phi_q} \mathcal{J}(H) \xleftarrow{\phi_1} X^*(T) \rtimes \mathcal{W}^H \tag{91}$$

is described in [Lus4, 2.5]. It is essentially the ρ_q-isotypical part of the $\langle t_q \rangle \times \mathbb{C}^\times$-equivariant K-theory of the variety $\mathcal{B}^{t_q,x}$. With [ChGi, Theorem 6.2.4] this can be translated to the terminology of Section 9, and one can see that it is none other than (88).

Recall that $\mathbf{q}$ is an indeterminate and let $\mathcal{H}_v(H) = \mathcal{H}_{\mathbf{q}}(H) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_v$ be the affine Hecke algebra with the same based root datum as H and with parameter $v \in \mathbb{C}^\times$. Thus

$$\mathcal{H}_q(H) = \mathcal{H}(H) \quad \text{and} \quad \mathcal{H}_1(H) = \mathbb{C}[X^*(T) \rtimes \mathcal{W}^H].$$

Like in (56), let $\tilde H$ be a central finite extension of H whose derived group is simply connected. By (57) and (59)

$$\mathcal{H}_v(H) \cong \big(K^{\tilde H \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde H}) \otimes_{\mathbb{C}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_v \big)^{\ker(\tilde H \to H)}. \tag{92}$$

The above, in particular (88), describes the retraction $\tilde\pi(\Phi,\rho) \in \mathbf{Irr}(\mathcal{J}(H))$ to $\mathcal{H}_v(H)$ for any $v \in \mathbb{C}^\times$.

In [Lus3, Corollary 3.6] the a-function is used to single out a particular irreducible quotient $\mathcal{H}_v(H)$-module of (88). This applies when $v = 1$ or v is not a root of unity. For $\mathcal{H}_q(H)$ we saw in (66) that there is only one such quotient, which by definition is $\pi(t_q, x, \rho_q)$. This is our description of the left hand side of (90).

For $v = 1$ we need a different argument. By the above and (89) we obtain the $\mathcal{H}_1(H)$-module

$$\mathrm{Hom}_{\pi_0(Z_H(t,x))}\big(\rho, H_*(\mathcal{B}_H^{t,x}, \mathbb{C})\big) \tag{93}$$

with the action coming from (92), (60) and the convolution product in Borel–Moore homology. Let us compare this with Kato's action [Kat], as described in Section 8. On the subalgebra $\mathbb{C}[\mathcal{W}^H]$ both are defined in terms of Borel–Moore homology, respectively with $K^{\tilde H \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde H})$ and with $H(\mathcal{Z}_{\tilde H})$. It follows from [ChGi, (7.2.12)] that they agree. An element $\lambda \in X^*(T)$ acts via (92) on K-theory as tensoring with a line bundle over $\mathcal{B}_{\tilde H}^{\tilde x}$ canonically associated to λ, see [ChGi, p. 395] or [KaLu, Theorem 3.5]. From the descriptions given in [ChGi, p. 420] and [Kat, §3] we see that on (93) this reduces to the action coming from (47). In other words, we checked that the $\mathcal{H}_1(H)$-module (93) contains Kato's module (48), as the homology in top degree.

We want to see what the right hand bijection in (90) does to $\tilde\pi(\Phi,\rho)$. By construction it produces a certain irreducible quotient of (93), namely the unique one with minimal a-weight. Unfortunately this is not so easy to analyse directly. Therefore we consider the opposite direction, starting with an irreducible $\mathcal{H}_1(H)$-module V with a-weight a_V. According to [Lus3, Corollary 3.6] the $\mathcal{J}(H)$-module

$$\tilde V := \mathcal{H}_1(H)^{a_V} \otimes_{\mathcal{H}_1(H)} V,$$

is irreducible and has a-weight a_V. See [Lus3, Lemma 1.9] for the precise definition of $\tilde V$.

Now we fix $t \in T$ and we will prove with induction to $\dim \mathcal{O}_x$ that $\widetilde{\tau(t,x,\rho)}$ is none other than $\tilde\pi(\Phi,\rho)$. Our main tool is Lemma 8.3, which says that the constituents of (93) are $\tau(t,x,\rho)$ and irreducible representations corresponding to larger affine Springer parameters (with respect to the partial order defined via the unipotent classes $\mathcal{O}_x \subset M$). For $\dim \mathcal{O}_{x_0} = 0$ we see immediately that only the $\mathcal{J}(H)$-module $\tilde{\pi}(t, x_0, \rho_0)$ can contain $\tau(t, x_0, \rho_0)$, so that must be $\widetilde{\tau(t, x_0, \rho_0)}$. For $\dim \mathcal{O}_{x_n} = n$ Lemma 8.3 says that (93) can only contain $\tau(t, x_n, \rho_n)$ if $x \in \overline{\mathcal{O}_{x_n}}$. But when $\dim \mathcal{O}_x < n$

$$\widetilde{\tau(t, x_n, \rho_n)} \not\cong \tilde{\pi}(\Phi, \rho),$$

because the right hand side already is $\widetilde{\tau(t, x, \rho)}$, by the induction hypothesis and the bijectivity of $V \mapsto \tilde{V}$. So the parameter of $\widetilde{\tau(t, x_n, \rho_n)}$ involves an x with $\dim \mathcal{O}_x = n$. Then another look at Lemma 8.3 shows that moreover (x, ρ) must be M-conjugate to (x_n, ρ_n). Hence $\widetilde{\tau(t, x, \rho)}$ is indeed (93).

We showed that the bijections (90) work out as

$$\begin{array}{ccccc} \mathbf{Irr}(\mathcal{H}(H)) & \leftrightarrow & \mathbf{Irr}(\mathcal{J}(H)) & \leftrightarrow & \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H) \\ \pi(t_q, x, \rho_q) & \leftrightarrow & \tilde{\pi}(\Phi, \rho) & \leftrightarrow & \tau(t, x, \rho), \end{array} \tag{94}$$

where all the objects in the bottom line are determined by the KLR parameter (Φ, ρ). $\square$

13. Main result (Hecke algebra version)

In this section $q \in \mathbb{C}^\times$ is allowed to be any element of infinite order. We study how Theorem 12.3 can be extended to the algebras and modules from Section 9. So let Γ be a group of automorphisms of G that preserves a chosen pinning, which involves T as maximal torus. With the disconnected group $G \rtimes \Gamma$ we associate three kinds of parameters:

- The extended quotient of the second kind $(T//\mathcal{W}^G \rtimes \Gamma)_2$.
- The space $\mathbf{Irr}(\mathcal{H}_q(G) \rtimes \Gamma)$ of equivalence classes of irreducible representations of the algebra $\mathcal{H}_q(G) \rtimes \Gamma$.
- Equivalence classes of unramified Kazhdan–Lusztig–Reeder parameters. Let $\Phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$ be a group homomorphism with $\Phi(\mathbf{I}_F) = 1$ and $\Phi(\mathbf{W}_F) \subset T$. As in Section 6, the component group

  $$\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(\Phi)) = \pi_0(\mathrm{Z}_{G \rtimes \Gamma}(\Phi(\mathbf{W}_F \times B_2)))$$

  acts on $H_*(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C})$. We take $\rho \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(\Phi))\big)$ such that every irreducible $\pi_0(\mathrm{Z}_G(\Phi))$-subrepresentation of ρ appears in $H_*(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C})$. The set $\{\text{KLR parameters for } G \rtimes \Gamma\}^{\mathrm{unr}}$ of pairs (Φ, ρ) carries an action of $G \rtimes \Gamma$ by conjugation. We consider the collection $\{\text{KLR parameters for } G \rtimes \Gamma\}^{\mathrm{unr}} / G \rtimes \Gamma$ of conjugacy classes $[\Phi, \rho]_{G \rtimes \Gamma}$.

As in the proof of Theorem 12.3, let $\mathcal{J}(G)$ be the asymptotic Hecke algebra of G. The group Γ acts on the extended affine Weyl group $X^*(T) \rtimes \mathcal{W}^G$ in a length-preserving way. Hence every $\gamma \in \Gamma$ naturally determines an automorphism of $\mathcal{J}(G)$, as described in [Lus4, §1]. This enables us to form the crossed product $\mathcal{J}(G) \rtimes \Gamma$.

Theorem 13.1. *There exists a commutative diagram of natural bijections*

$$\begin{tikzcd}
 & (T/\!/\mathcal{W}^G \rtimes \Gamma)_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{H}_q(G) \rtimes \Gamma) \arrow[rr] & & \{\text{KLR parameters for } G \rtimes \Gamma\}^{\mathrm{unr}}/G \rtimes \Gamma
\end{tikzcd}$$

It restricts to bijections between the following subsets:

- *the ordinary quotient* $T/(\mathcal{W}^G \rtimes \Gamma) \subset (T/\!/\mathcal{W}^G \rtimes \Gamma)_2$,
- *the collection of spherical representations in* $\mathbf{Irr}(\mathcal{H}_q(G) \rtimes \Gamma)$,
- *equivalence classes of KLR parameters* (Φ, ρ) *for* $G \rtimes \Gamma$ *with* $\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})) = 1$ *and* $\rho = \mathrm{triv}_{\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi))}$.

Moreover the left slanted map can be constructed via the (irreducible representations of) the algebra $\mathcal{J}(G) \rtimes \Gamma$.

Proof. The corresponding statement for G, proven in Theorem 12.3, is the existence of natural bijections

$$\begin{tikzcd}
\mathbf{Irr}(\mathcal{J}(G)) \arrow[rr] \arrow[d] & & (T/\!/\mathcal{W}^G)_2 \arrow[d] \arrow[dll] \\
\mathbf{Irr}(\mathcal{H}_q(G)) \arrow[r] & \{\text{KLR parameters for } G\}^{\mathrm{unr}}/G &
\end{tikzcd} \tag{95}$$

Although in Section 12 q was a prime power, we notice that among the objects in (95) only the algebra $\mathcal{H}_q(G)$ depends on q. Fortunately the bottom, slanted and left hand vertical maps in (95) are defined equally well for our more general $q \in \mathbb{C}^\times$, as can be seen from the proofs of Theorems 9.1 and 12.3. Thus we may use (95) as our starting point.

Step 1. The bijections in (95) *are* Γ*-equivariant.*
The action of Γ on $(T/\!/\mathcal{W}^G)_2$ can be written as

$$\gamma \cdot [t, \tilde\tau]_{\mathcal{W}^G} = [\gamma(t), \tilde\tau \circ \mathrm{Ad}_\gamma^{-1}]_{\mathcal{W}^G}. \tag{96}$$

In terms of the multiplication in $G \rtimes \Gamma$, the action on KLR parameters is

$$\gamma \cdot [\Phi, \rho_1]_G = [\gamma\Phi\gamma^{-1}, \rho_1 \circ \mathrm{Ad}_\gamma^{-1}]_G \tag{97}$$

We recall the right hand vertical map in (95) from Theorem 8.2. Write $M = \mathrm{Z}_G(t)$ and $\mathcal{W}_t^G = W(M^\circ, T) \rtimes \pi_0(M)$. Then the $\mathcal{W}_t^G$-representation $\tilde\tau$ can be written as $\tau(x, \rho_3) \rtimes \sigma$ for a unipotent element $x \in M^\circ$, a geometric $\rho_3 \in \mathrm{Irr}(\mathrm{Z}_{M^\circ}(x))$ and a $\sigma \in \mathrm{Irr}(\pi_0(M)_{\tau(x,\rho_3)})$. The associated KLR parameter is $[\Phi, \rho_3 \rtimes \sigma]_G$, where $\Phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = x$ and Φ maps a Frobenius element of $\mathbf{W}_F$ to t. From (13) we see that $\tau(x, \rho_3) \circ \mathrm{Ad}_\gamma^{-1}$ is equivalent with $\tau(\gamma x \gamma^{-1}, \rho_3 \circ \mathrm{Ad}_\gamma^{-1})$, so

$$\tilde\tau \circ \mathrm{Ad}_\gamma^{-1} \text{ is equivalent with } \tau(\gamma x \gamma^{-1}, \rho_3 \circ \mathrm{Ad}_\gamma^{-1}) \rtimes (\sigma \circ \mathrm{Ad}_\gamma^{-1}).$$

Hence (96) is sent to the KLR parameter (97), which means that the right hand vertical map in (95) is indeed Γ-equivariant.

In view of Proposition 6.1 and (97), we already showed in (69) that the lower horizontal map in (95) is Γ-equivariant. By the commutativity of the triangle, so is the slanted map.

As we checked in the proof of Theorem 12.3, the left hand vertical map is retraction along $\phi_q : \mathcal{H}(G) \to \mathcal{J}(G)$ followed by taking the unique irreducible quotient. The algebra homomorphism ϕ_q is Γ-equivariant because Γ respects the entire setup in [Lus4, §1]. Therefore the left hand vertical map is also Γ-equivariant.

Step 2. Suppose that $\tilde\pi(\Phi,\rho), [t,\tilde\tau]_{\mathcal{W}^G}, \pi$ *and* $[\Phi,\rho_1]_G$ *are four corresponding objects in* (95). *Then their stabilizers in* Γ *coincide:*

$$\Gamma_{\tilde\pi(\Phi,\rho)} = \Gamma_{[t,\tilde\tau]_{\mathcal{W}^G}} = \Gamma_\pi = \Gamma_{[\Phi,\rho_1]_G}.$$

This follows immediately from step 1.

Step 3. Clifford theory produces 2-cocycles $\natural\big(\tilde\pi(\Phi,\rho)\big)$, $\natural\big([t,\tilde\tau]_{\mathcal{W}^G}\big), \natural(\pi)$ *and* $\natural\big([\Phi,\rho_1]_G\big)$ *of* Γ_x. *We can choose the same cocycle for all four of them.* For $\natural(\pi)$ and $\natural\big([\Phi,\rho_1]_G\big)$ this was already checked in (71), where we use Proposition 6.1 to translate between Φ and (t_q, x).

From (94), and Theorems 8.2 and 9.1 we see that $\tilde\pi(\Phi,\rho), [t,\tilde\tau]_{\mathcal{W}^G}$ and π come from three rather similar representations. The difference is that $\tilde\pi(\Phi,\rho)$ is built from the entire homology of a variety, whereas the other two are quotients thereof. The Γ_π-actions on these three modules are defined in the same way, so the two cocycles can be chosen equal.

We remark that $\natural\big([t,\tilde\tau]_{\mathcal{W}^G}\big)$ is trivial by Proposition 4.3, so the other 2-cocycles are also trivial.

Step 4. Upon applying $X \mapsto (X/\!/\Gamma)_2^\natural$ *to the commutative diagram* (95) *we obtain the corresponding diagram for* $G \rtimes \Gamma$.

Here $\natural$ denotes the family of 2-cocycles constructed in steps 2 and 3. For $\mathbf{Irr}(\mathcal{J}(G))$, $(T/\!/\mathcal{W}^G)_2$ and $\mathbf{Irr}(\mathcal{H}_q(G))$ we know from Lemmas 2.1 and 2.3 that this procedure yields the correct parameters. That it works for Kazhdan–Lusztig–Reeder parameters was checked in the last part of the proof of Theorem 9.1. By steps 1 and 3 the construction used in (2) yields the same homomorphisms between the twisted group algebras (called $\phi_{\gamma,x}$ in Section 2) in all four settings. Hence the maps from (95) can be lifted in a natural way to the diagram for $G \rtimes \Gamma$.

The ordinary quotient is embedded in $(T/\!/\mathcal{W}^G \rtimes \Gamma)_2$ as the collection of pairs $\big(t, \mathrm{triv}_{(\mathcal{W}^G \rtimes \Gamma)_t}\big)$. By an obvious generalization of (55) these correspond to the affine Springer parameters $(t, x = 1, \rho = \mathrm{triv})$. It is clear from the above construction that they are mapped to KLR parameters (Φ, triv) with $\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})) = 1$ and $\Phi(\varpi_F) = t$. By Proposition 10.1 the latter correspond to the spherical irreducible $\mathcal{H}(G) \rtimes \Gamma$-modules. □

14. Canonicity

We return to the notation from Section 11. We would like to combine Theorems 12.3 and 13.1 to a version that applies to $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ irrespective of the (dis)connectedness of $H = \mathrm{Z}_G(\mathrm{im}c^{\mathfrak{s}})$. We have observed already that everything in Theorem 13.1 is canonical, but we do not know yet how canonical Theorem 11.2 is. Unfortunately a discussion of this issue is avoided in the sources [Roc] and [Ree2].

For this purpose we need some technical results about the extended affine Hecke algebra $\mathcal{H}(H)$. Let us denote the elements of the Bernstein basis of $\mathcal{H}(H)$ by $\theta_\lambda T_w$, where $\lambda \in X^*(T)$ and $w \in \mathcal{W}^H$. The algebra $\mathcal{H}(T)$ is canonically isomorphic to $\mathcal{O}(T) = \mathbb{C}[X^*(T)]$, so it has a basis $\{[\lambda] : \lambda \in X^*(T)\}$. The assignment $[\lambda] \mapsto \theta_\lambda$ determines an algebra injection

$$t_U : \mathcal{H}(T) \cong \mathcal{O}(T) \to \mathcal{H}(H).$$

It is canonical in the sense that it depends only on the based root datum of (H, T), which was fixed by the choice of a Borel subgroup $B_H = B \cap H$. Via t_U we regard $\mathcal{O}(T)$ as a subalgebra of $\mathcal{H}(H)$. It is well-known from [Lus5, §3] that the centre of $\mathcal{H}(H)$ is $\mathcal{O}(T)^{\mathcal{W}^H}$. Let $\mathbb{C}(T)$ be the field of rational functions on T, the quotient field of $\mathcal{O}(T)$. Then $\mathcal{H}(H) \otimes_{\mathrm{Z}(\mathcal{H}(H))} \mathbb{C}(T)^{\mathcal{W}^H}$ carries a natural algebra structure, and as a vector space it is simply

$$\mathcal{H}(H) \otimes_{\mathcal{O}(T)} \mathbb{C}(T) \cong \mathbb{C}(T) \rtimes \mathcal{W}^H.$$

By [Lus5, §6] or [Sol, Proposition 1.5.1] there is an algebra isomorphism

$$\mathcal{H}(H) \otimes_{\mathrm{Z}(\mathcal{H}(H))} \mathbb{C}(T)^{\mathcal{W}^H} \cong \mathbb{C}(T) \rtimes \mathcal{W}^H, \tag{98}$$

which is the identity on $\mathcal{O}(T)$.

Proposition 14.1. *Let ϕ be an automorphism of $\mathcal{H}(H)$ which is the identity on $\mathcal{O}(T)$.*

1. *It induces an automorphism (also denoted by ϕ) of $\mathbb{C}(T) \rtimes \mathcal{W}^H$.*
2. *There exist $z_w \in \mathbb{C}^\times$ and $\lambda_w \in X^*(T)$ such that $\phi(w) = z_w \theta_{\lambda_w} w$ for all $w \in \mathcal{W}^H$.*
3. *For every reflection s_α with $\alpha \in R(H^\circ, T)$ we have $\lambda_{s_\alpha} \in \mathbb{Z}\alpha$.*
4. *$z_w = 1$ for $w \in \mathcal{W}^{H^\circ}$, and $w \mapsto z_w$ is a character of $\pi_0(H) \cong \mathcal{W}^H / \mathcal{W}^{H^\circ}$.*

Proof. (1) is a direct consequence of (98). By assumption ϕ is the identity on the quotient field $\mathbb{C}(T)$ of $\mathcal{O}(T)$. Hence it is of the form

$$\phi : \sum_{w \in \mathcal{W}^H} f_w w \mapsto \sum_{w \in \mathcal{W}^H} f_w \Phi_w w \tag{99}$$

for suitable $\Phi_w \in \mathbb{C}(T)$. Let $\imath_w^\circ \in \mathcal{H}(H) \otimes_{\mathrm{Z}(\mathcal{H}(H))} \mathbb{C}(T)^{\mathcal{W}^H}$ be the image of $w \in \mathcal{W}^H$ under (98). An explicit formula in the case of a simple reflection s_α is given in [Sol, (1.25)]:

$$1 + T_{s_\alpha} = q \frac{\theta_\alpha - q^{-1}}{\theta_\alpha - 1} (1 + \imath_{s_\alpha}^\circ). \tag{100}$$

Since ϕ preserves $\mathcal{H}(H)$, we see from (99) and (100) that $\Phi_{s_\alpha} \in \mathcal{O}(T)^\times = \mathbb{C}^\times X^*(T)$. Say $\Phi_{s_\alpha} = z\theta_\lambda$. Then we calculate in $\mathbb{C}(T) \rtimes \mathcal{W}^H$:

$$1 = s_\alpha^2 = \phi(s_\alpha)^2 = z\theta_\lambda s_\alpha z \theta_\lambda s_\alpha = z^2 \theta_\lambda \theta_{s_\alpha(\lambda)} s_\alpha^2 = z^2 \theta_{\lambda + s_\alpha(\lambda)}.$$

Therefore $z = \pm 1$ and $s_\alpha(\lambda) = -\lambda$, which means that $\lambda \in \mathbb{Q}\alpha \cap X^*(T)$. Now

$$\begin{aligned}\phi(1 + T_{s_\alpha}) &= \phi\Big(\frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 + \imath^\circ_{s_\alpha})\Big)\\ &= \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 + z\theta_\lambda \imath^\circ_{s_\alpha})\\ &= \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 - z\theta_\lambda) + z\theta_\lambda \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 + \imath^\circ_{s_\alpha})\\ &= \frac{q\theta_\alpha - 1}{\theta_\alpha - 1}(1 - z\theta_\lambda) + z\theta_\lambda (1 + T_{s_\alpha}).\end{aligned}$$

This is an element of $\mathcal{H}(H)$ and $q > 1$, so $\theta_\alpha - 1$ divides $1 - z\theta_\lambda$ in $\mathcal{O}(T)$. We deduce that $z = +1$ and $\lambda = \lambda_{s_\alpha} \in \mathbb{Z}\alpha$. In particular

$$\phi(\imath^\circ_{s_\alpha}) = \theta_{\lambda_{s_\alpha}} \imath^\circ_{s_\alpha},$$

which directly implies that for every $w \in \mathcal{W}^{H^\circ}$ there exists a $\lambda_w \in X^*(T)$ with $\phi(\imath^\circ_w) = \theta_{\lambda_w} \imath^\circ_w$.

If $w \in \mathcal{W}^{H^\circ}$ is any reflection, then $w = s_\beta$ for some $\beta \in R(H^\circ, T)$ and w is conjugate to some simple reflection s_α, say by $v \in \mathcal{W}^{H^\circ}$. Then

$$\begin{aligned}\theta_{\lambda_{s_\beta}} s_\beta = \phi(s_\beta) = \phi(v s_\alpha v^{-1}) &= \theta_{\lambda_v} v \theta_{\lambda_{s_\alpha}} s_\alpha v^{-1} \theta_{-\lambda_v}\\ = \theta_{\lambda_v} \theta_{v(\lambda_{s_\alpha})} \theta_{v s_\alpha v^{-1}(-\lambda_v)} v s_\alpha v^{-1} &= \theta_{v(\lambda_{s_\alpha}) + \lambda_v - s_\beta(\lambda_v)} s_\beta\\ &= \theta_{v(\lambda_{s_\alpha}) + \langle \beta^\vee, \lambda_v \rangle \beta} s_\beta.\end{aligned} \tag{101}$$

As $v(\lambda_{s_\alpha}) \in v(\mathbb{Z}\alpha) = \mathbb{Z}\beta$, we see that $\theta_{\lambda_{s_\beta}} \in \mathbb{Z}\beta$. This proves (ii), (iii) and (iv) on $\mathcal{W}^{H^\circ}$.

Recall from Lemma 3.1 that $\mathcal{W}^H \cong \mathcal{W}^{H^\circ} \rtimes \pi_0(H)$, with $\pi_0(H)$ preserving the simple roots. For $w \in \pi_0(H)$ we have $\imath^\circ_w = T_w$ by [Sol, Proposition 1.5.1], so the argument from (99) and (100) shows that $\Phi_w \in \mathcal{O}(T)^\times$. Therefore (ii) holds on $\mathcal{W}^H$. Knowing this, the multiplication rules in $\mathbb{C}(T) \rtimes \mathcal{W}^H$ entail that $w \mapsto z_w$ must be a character of $\mathcal{W}^H$. $\square$

To investigate the effect of automorphisms as in Proposition 14.1 on $\mathcal{H}(H)$-modules, we take a closer look at (91). Let $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$ be the affine Hecke algebra with the same data as $\mathcal{H}(H)$, but with a formal parameter $\mathbf{q}$ and over the ground ring $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$. This algebra also has a Bernstein presentation and a Bernstein basis like $\mathcal{H}(H)$, only over $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$. Any ϕ as in Proposition 14.1 lifts to a $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-linear automorphism of $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$, just use the same formula as in part (ii).

Like in (82) we define

$$\mathcal{J}(H) = \mathcal{J}(H^\circ) \rtimes \pi_0(H).$$

In [Lus2, §2.4] a homomorphism of $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebras

$$\mathcal{H}_{\sqrt{\mathbf{q}}}(H^\circ) \to \mathcal{J}(H^\circ) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}]$$

is constructed, which induces (91) by specialization of $\mathbf{q}^{1/2}$ at $q^{1/2}$ or at 1. Because the actions of $\pi_0(H)$ preserve all the data used to construct these algebras, it induces a homomorphism of $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebras

$$\tilde{\phi} : \mathcal{H}_{\sqrt{\mathbf{q}}}(H) \to \mathcal{J}(H) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}].$$

From the $\mathcal{J}(H)$-module $\tilde\pi(\Phi,\rho)$ and $\tilde\phi$ we obtain the $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module

$$\tilde\pi(\Phi,\rho)\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{q}^{\pm 1/2}]. \tag{102}$$

We call modules of this form, for any KLR parameter (Φ,ρ) with $\Phi\big|_{\mathfrak{o}_F^\times}=c^{\mathfrak{s}}$, standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-modules.

Lemma 14.2. *Let ϕ be any automorphism of the $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebra $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$. The induced map ϕ^* on $\mathrm{Mod}(\mathcal{H}_{\sqrt{\mathbf{q}}}(H))$ sends standard modules to standard modules.*

Proof. We saw in the proof of Theorem 12.3 that for every generic $v\in\mathbb{C}^\times$ the specialization of (102) at $\mathbf{q}^{1/2}$ at $v^{1/2}$ is an irreducible $\mathcal{H}_v(H)$-module, namely $\pi(t_v,x,\rho_v)$. All these modules have the same underlying vector space

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(\Phi))}\big(\rho, H_*(\mathcal{B}_H^{t,\Phi(B_2)},\mathbb{C})\big),$$

and the action of $\mathcal{H}_v(H)$ depends algebraically on $v^{\pm 1/2}$. It follows from Section 9 that, for generic v, there is only one way to embed $\pi(t_v,x,\rho_v)$ in a family of irreducible $\mathcal{H}_v(H)$-modules that depends algebraically on $v^{\pm 1/2}$ (varying in this generic set of parameters). Since ϕ is a $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebra automorphism,

$$\phi^*\big(\tilde\pi(\Phi,\rho)\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{q}^{\pm 1/2}]\big) \tag{103}$$

has irreducible specializations at all generic $v\in\mathbb{C}^\times$, and these still depend algebraically on $v^{\pm 1/2}$. So (103) looks like a standard module as long as only generic parameters are considered, say like $\tilde\pi(\Phi',\rho')\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{q}^{\pm 1/2}]$. But the set of generic parameters in dense in $\mathbb{C}^\times$, so

$$\phi^*\big(\tilde\pi(\Phi,\rho)\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{q}^{\pm 1/2}]\big)=\tilde\pi(\Phi',\rho')\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{q}^{\pm 1/2}]. \qquad\square$$

Recall the parametrization of irreducible $\mathcal{H}(H)$-modules in Theorem 9.1.

Lemma 14.3. *Let ϕ be an automorphism of $\mathcal{H}(H)$ which is the identity on $\mathcal{O}(T)$. For every Kazhdan–Lusztig triple (t_q,x,ρ_q) there exists a geometric $\rho'_q\in\mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(t_q,x))\big)$ such that $\phi^*(\pi(t_q,x,\rho_q))=\pi(t_q,x,\rho'_q)$.*

Proof. Consider the standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module (102), where (Φ,ρ) is associated to (t_q,x,ρ_q) via Lemma 7.1. Its specialization at $\mathbf{q}^{1/2}=q^{1/2}$ is a $\mathcal{H}_q(H)$-module with $\pi(t_q,x,\rho_q)$ as unique irreducible quotient. On the other hand, the specialization at $\mathbf{q}^{1/2}=1$ is the $\mathbb{C}[X^*(T)\rtimes\mathcal{W}^H]$-module

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho_1, H_*(\mathcal{B}_H^{t,x},\mathbb{C})\big).$$

By Theorem 8.2 its component in top homological degree $d(x)$ is

$$\tau(t,x,\rho_1)=\mathrm{ind}_{X^*(T)\rtimes\mathcal{W}^M}^{X^*(T)\rtimes\mathcal{W}^H}\tau^\circ(t,x,\rho_1))\in\mathbf{Irr}(X^*(T)\rtimes\mathcal{W}^H),$$
$$\tau^\circ(t,x,\rho_1)=\mathrm{Hom}_{\pi_0(\mathrm{Z}_M(x))}\big(\rho_1,H_{d(x)}(\mathcal{B}_{M^\circ}^x,\mathbb{C})\otimes\mathbb{C}[\pi_0(M)]\big),$$

where $M=\mathrm{Z}_H(t)$. Via Proposition 14.1 ϕ determines an automorphism of $\mathcal{H}_1(H)=\mathbb{C}[X^*(T)\rtimes\mathcal{W}^H]$, and we want to know $\phi^*(\tau(t,x,\rho_1))$. Since ϕ is the identity on $\mathcal{O}(T)$, composition with it does not change the parameter $t\in T$.

In $\tau^\circ(t,x,\rho_1) \in \mathbf{Irr}(\mathcal{W}^M)$ the unipotent class of $x \in M^\circ$ is already determined by the action of $\mathcal{W}^{M^\circ}$, see Proposition 4.3 and Theorem 4.4. Recall from [SpSt, §4.1] that M° is generated by the reflections s_α with $\alpha(t) = 1$. By Propostion 14.1.(4) $\phi(s_\alpha) = \theta_{\lambda_{s_\alpha}} s_\alpha$ for some $\lambda_{s_\alpha} \in \mathbb{Z}\alpha$. We calculate

$$\begin{aligned}\tau^\circ(t,x,\rho_1) \circ \phi^{-1}(s_\alpha) &= \tau^\circ(t,x,\rho_1)(s_\alpha \theta_{\lambda_{s_\alpha}}) \\ &= \tau^\circ(t,x,\rho_1)(s_\alpha)\theta_{\lambda_{s_\alpha}}(t) = \tau^\circ(t,x,\rho_1)(s_\alpha).\end{aligned}$$

Thus $\tau^\circ(t,x,\rho_1) \circ \phi^{-1}\big|_{\mathcal{W}^{H^\circ}} = \tau^\circ(t,x,\rho_1)\big|_{\mathcal{W}^{H^\circ}}$ and

$$\tau^\circ(t,x,\rho_1) \circ \phi^{-1} = \tau^\circ(t,x,\rho_1')$$

for a geometric $\rho_1' \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x))\big)$. It follows that

$$\phi^*\big(\tau(t,x,\rho_1)\big) = \tau(t,x,\rho_1'). \tag{104}$$

Lift ϕ to automorphism of $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$, using Proposition 14.1. By Lemma 14.2

$$\phi^*\big(\tilde\pi(\Phi,\rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}]\big) \tag{105}$$

is again a standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module. But there is only one standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module whose specialization at $\mathbf{q}^{1/2} = 1$ has $\tau(t,x,\rho_1')$ as component in top homological degree, namely the one with parameter (Φ,ρ'). By (105) the module (104) must be isomorphic to

$$\tilde\pi(\Phi,\rho') \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}].$$

In particular its specialization at $\mathbf{q}^{1/2} = q^{1/2}$ is

$$\phi^*\Big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t_q,x))}\big(\rho_q, H_*(\mathcal{B}_H^{t_q,x},\mathbb{C})\big)\Big) \cong \mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t_q,x))}\big(\rho_q', H_*(\mathcal{B}_H^{t_q,x},\mathbb{C})\big),$$

which has $\pi(t_q,x,\rho_q')$ as unique irreducible quotient. Consequently
$\phi^*(\pi(t_q,x,\rho_q)) \cong \pi(t_q,x,\rho_q')$. $\square$

We shall apply Lemma 14.3 to Theorem 11.2. The main role in the proof of that Theorem is played by a cover (J,τ) of $(\mathcal{T}_0,\chi|\mathcal{T}_0)$. Let $I_{\mathcal{B}}^{\mathcal{G}}$ denote the normalized parabolic induction functor, starting from the Borel subgroup $\mathcal{B} \subset \mathcal{G}$ corresponding to $B \subset G$. As shown in [Roc, §9], there exists an algebra injection

$$t_B : \mathcal{H}(\mathcal{T},\chi|_{\mathcal{T}_0}) \to \mathcal{H}(\mathcal{G},\tau)$$

such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \arrow[r, "\sim"] & \mathrm{Mod}(\mathcal{H}(\mathcal{G},\tau)) \arrow[r, "\sim"] & \mathrm{Mod}(\mathcal{H}(H)) \\ \mathrm{Rep}(\mathcal{T})^{[\mathcal{T},\chi]_{\mathcal{T}}} \arrow[r, "\sim"] \arrow[u, "I_{\mathcal{B}}^{\mathcal{G}}"] & \mathrm{Mod}(\mathcal{H}(\mathcal{T},\chi|_{\mathcal{T}_0})) \arrow[r, "\sim"] \arrow[u, "t_{B*}"] & \mathrm{Mod}(\mathcal{H}(T)) \arrow[u, "t_{U*}"] \end{tikzcd} \tag{106}$$

Here $t_{U*}(V) = \mathrm{Hom}_{\mathcal{H}T)}(\mathcal{H}(H),V)$, and similarly for t_{B*}.

Lemma 14.4. *Let B' be a Borel subgroup of G such that $B' \cap H^\circ = B \cap H^\circ$. Suppose that (J',τ') is another $\mathfrak{s}$-type which covers $(\mathcal{T}_0,\chi|_{\mathcal{T}_0})$, and that there exists an isomorphism $\mathcal{H}(\mathcal{G},\tau') \cong \mathcal{H}(H)$ that makes the diagram analogous to* (106)*, but with primes, commute. Then the map*

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(\mathcal{G},\tau')) \to \mathbf{Irr}(\mathcal{H}(H)) \to \{\text{KLR } \textit{parameters}\}^{\mathfrak{s}}/H$$

can only differ from its counterpart for (J,τ) in third ingredient ρ of a KLR parameter.

Proof. Let us denote the copy of $\mathcal{H}(H)$ obtained from $\mathcal{H}(\mathcal{G},\tau')$ by $\mathcal{H}'(H)$, to distinguish it from the earlier $\mathcal{H}(H)$. The assumptions of the lemma entail an equivalence of categories

$$\mathrm{Mod}(\mathcal{H}(H)) \longleftrightarrow \mathrm{Mod}(\mathcal{H}'(H)), \tag{107}$$

which sends any $\mathcal{H}(H)$-module induced from $\mathcal{H}(T)$ to an isomorphic $\mathcal{H}'(H)$-module. In particular the regular representation of $\mathcal{H}(H)$ is mapped to an $\mathcal{H}'(H)$-module isomorphic to the regular representation. Let $\mathcal{M}$ be a Morita $\mathcal{H}'(H) - \mathcal{H}(H)$-bimodule that implements (107). Then $\mathcal{H}(H) \cong \mathrm{End}_{\mathcal{H}'(H)}(\mathcal{M})$ as algebras and

$$\mathcal{M} \cong \mathcal{M} \otimes_{\mathcal{H}(H)} \mathcal{H}(H) \cong \mathcal{H}'(H)$$

as $\mathcal{H}'(H)$-modules. Hence

$$\mathcal{H}(H) \cong \mathrm{End}_{\mathcal{H}'(H)}(\mathcal{H}'(H)) \cong \mathcal{H}'(H), \tag{108}$$

providing an algebra isomorphism that has the same effect as (107). The map $t_U : \mathcal{H}(T) \to \mathcal{H}(H)$ depends only the choice of a positive system in $R(H^\circ, T)$, so it is the same for B' and B. From that and (106) we see that (108) is the identity on $\mathcal{O}(T) = t_U(\mathcal{H}(T))$. By Lemma 14.3 composition with this isomorphism sends an irreducible $\mathcal{H}(H)$-module $\pi(t_q, x, \rho_q)$ to $\pi(t_q, x, \rho'_q)$ for some ρ'_q. This statement is just another way to formulate the lemma. $\square$

Now we can answer the questions raised by (86) and Example 11.3.

Proposition 14.5. *Assume that Condition 11.1 holds. Consider the bijections*

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(H)) \to \{\text{KLR } parameters\}^{\mathfrak{s}}/H$$

from Theorems 11.2 and 9.1. Suppose that $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is mapped to $[t_q, x, \rho_q]_H$. Then the H-conjugacy class of (t_q, x) is uniquely determined by the condition that the equivalence of categories $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H}(H))$ comes from an $\mathfrak{s}$-type which is a cover of $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$ for some $\chi \in \mathbf{Irr}(\mathcal{T})$ with $[\mathcal{T}, \chi]_{\mathcal{G}} = \mathfrak{s}$.

Proof. Most of the work was done in 14.4 and Theorem 11.2. We only need to show that it does not depend on the choice of a Borel subgroup $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$, or equivalently of a Borel subgroup $T \subset B \subset G$. Any other Borel subgroup of $\mathcal{G}$ containing $\mathcal{T}$ is of the form $\mathcal{B}' = w\mathcal{B}w^{-1}$ for a unique $w \in \mathcal{W}^G$. If we would use $\mathcal{B}'$ instead of $\mathcal{B}$, we would end up studying the induced representation $I_{\mathcal{B}'}^{\mathcal{G}}(\chi)$ with the extended affine Hecke algebra $\mathcal{H}(H, wBw^{-1} \cap H^\circ)$, whose based root datum is that of $(H, wBw^{-1} \cap H^\circ)$. An irreducible constituent π' of $I_{\mathcal{B}'}^{\mathcal{G}}(\chi)$ would then produce a KLR parameter $[t'_q, x', \rho'_q]_H$. In this setting $wBw^{-1} \cap H^\circ$ is conjugate to $B_H = B \cap H^\circ$ by an element $h \in N_{H^\circ}(T)$, unique up to T. Let γ be its image in $\mathcal{W}^{H^\circ} \subset \mathcal{W}^G$. The map

$$\theta_\lambda T_u \mapsto \theta_{\gamma(\lambda)} T_{\gamma u \gamma^{-1}}$$

on the Bernstein bases determines an algebra isomorphism

$$\mathcal{H}(H, wBw^{-1} \cap H^\circ) \to \mathcal{H}(H) = \mathcal{H}(H, B_H).$$

Conjugating the entire situation by h, we obtain a constituent

$$\gamma \cdot \pi' \cong \pi' \quad \text{of} \quad I^{\mathcal{G}}_{\gamma\mathcal{B}'\gamma^{-1}}(\gamma \cdot \chi) \cong I^{\mathcal{G}}_{\mathcal{B}'}(\chi),$$

and an $\mathcal{H}(H)$-module with KLR parameter

$$[ht'_q h^{-1}, hx'h^{-1}, h \cdot \rho'_q]_H = [t'_q, x', \rho'_q]_H.$$

Notice that the Borel subgroup $B'' := hB'h^{-1} = hwBw^{-1}h^{-1}$ satisfies $B'' \cap H^\circ = B \cap H^\circ$. Any type which we used the produce the KLR parameters can also be conjugated by a lift of γ in $N_{\mathcal{G}}(\mathcal{T})$, and that yield a type which covers $(\mathcal{T}_0, \gamma \cdot \chi|_{\mathcal{T}_0})$. By Theorem 11.2.(2) that is just as good as a cover of $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$. Thus we have reduced to the situation of Lemma 14.4. $\square$

We remark that the conditions imposed in Proposition 14.5 seem quite reasonable. We need the algebra $\mathcal{H}(H)$ to arrive at the right parameter space, and we use the covering of $(\mathcal{T}_0, \chi|_{\mathcal{T}_0})$ to get a relation between $\mathcal{H}(T) \to \mathcal{H}(H)$ and the normalized parabolic induction functor $I^{\mathcal{G}}_{\mathcal{B}}$.

15. Main result (general case)

The preparations for our main theorem are now complete.

Theorem 15.1. *Let $\mathcal{G}$ be a split reductive p-adic group and let $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$ be a point in the Bernstein spectrum of the principal series of $\mathcal{G}$. Assume that Condition 11.1 holds. Then there is a commutative triangle of bijections*

$$\begin{tikzcd} & (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \arrow[dl] \arrow[dr] & \\ \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \arrow[rr] & & \{\text{KLR parameters}\}^{\mathfrak{s}}/H \end{tikzcd}$$

The slanted maps are generalizations of the slanted maps in Theorem 12.3 and the horizontal map stems from Theorem 9.1. The right slanted map is natural. If $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ corresponds to a KLR parameter (Φ, ρ), then the Langlands parameter Φ is determined canonically by π.

We denote the irreducible $\mathcal{G}$-representation associated to a KLR parameter (Φ, ρ) by $\pi(\Phi, \rho)$.

1. *The infinitesimal central character of $\pi(\Phi, \rho)$ is the H-conjugacy class*
$$\Phi\big(\varpi_F, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\big) \in c(H)_{\mathrm{ss}} \cong T^{\mathfrak{s}}/W^{\mathfrak{s}}.$$
2. *$\pi(\Phi, \rho)$ is tempered if and only if $\Phi(\mathbf{W}_F)$ is bounded, which is the case if and only if $\Phi(\varpi_F)$ lies in a compact subgroup of H.*

Proof. By Proposition 14.5 any $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ canonically determines a Langlands parameter Φ. The larger part of the commutative triangle was already discussed in (81), (82) and Theorem 13.1. It remains to show that the set $\{\text{KLR parameters}\}^{\mathfrak{s}}/H$ (as defined on page 21) is naturally in bijection with $\{\text{KLR parameters for } H^\circ \rtimes \pi_0(H)\}^{\mathrm{unr}}/H^\circ \rtimes \pi_0(H)$.

By (80) we are taking conjugacy classes with respect to the group $H/\mathrm{Z}(H^\circ)$ in both cases. It is clear from the definitions that that in both sets the ingredients Φ are determined by the semisimple element $\Phi(\varpi_F) \in H$. This provides the desired bijection between the Φ's in the two collections, so let us focus on the ingredients ρ.

For $(\Phi, \rho) \in \{\text{KLR parameters}\}^{\mathfrak s}$ the irreducible representation ρ of the component group $\pi_0(\mathrm{Z}_H(\Phi)) = \pi_0(\mathrm{Z}_G(\Phi))$ must appear in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. By Proposition 6.2.3 this space is isomorphic, as a $\pi_0(\mathrm{Z}_G(\Phi))$-representation, to a number of copies of

$$\mathrm{Ind}_{\pi_0(\mathrm{Z}_{H^\circ}(\Phi))}^{\pi_0(\mathrm{Z}_G(\Phi))} H_*\big(\mathcal{B}_{H^\circ}^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big).$$

Hence the condition on ρ is equivalent to requiring that every irreducible $\pi_0(\mathrm{Z}_{H^\circ}(\Phi))$-subrepresentation of ρ appears in $H_*\big(\mathcal{B}_{H^\circ}^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. That is exactly the condition on ρ in an unramified KLR parameter for $H^\circ \rtimes \pi_0(H)$. This establishes the properties of the commutative diagram.
(1) By Theorem 9.1 the $\mathcal{H}(H^\circ)$-module with Kazhdan–Lusztig triple (t_q, x, ρ_q) has central character

$$t_q = \Phi\big(\varpi_F, \left(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\right)\big) \in c(H^\circ)_{\mathrm{ss}} \cong T/\mathcal{W}^{H^\circ}.$$

It follows that the $\mathcal{H}(H)$-module with parameter (t_q, x, ρ_q) or (Φ, ρ) has central character $t_q \in c(H)_{\mathrm{ss}} \cong T/\mathcal{W}^H$. Via (79) we can also consider it as an element of $T^{\mathfrak s}/W^{\mathfrak s}$. In view of (85) and (84) the corresponding $\mathcal{G}$-representation is

$$\pi(\Phi, \rho) = \mathcal{H}(\mathcal{G}) \otimes_{\mathcal{H}(H)} \pi(t_q, x, \rho_q). \tag{109}$$

This tensor product defines an equivalence between $\mathrm{Mod}(\mathcal{H}(H))$ and $\mathrm{Rep}(\mathcal{G})^{\mathfrak s}$, which by definition transforms the central character into the infinitesimal character.
(2) Recall that Roche's equivalence in Theorem 11.2 comes from a type with associated idempotent $e_\tau \in \mathcal{H}(\mathcal{G})$. It was checked in [BHK, Theorem B] that $\pi(\Phi, \rho) \in \mathbf{Irr}(\mathcal{G})^{\mathfrak s}$ is tempered if and only if the corresponding $e_\tau \mathcal{H}(\mathcal{G}) e_\tau$-module is tempered, with respect to the canonical Hilbert algebra structure on $\mathcal{H}(\mathcal{G})$. By [Roc, Theorem 8.2] the isomorphism $e_\tau \mathcal{H}(\mathcal{G}) e_\tau \cong \mathcal{H}(H)$ transfers the *-operation of $\mathcal{H}(\mathcal{G})$ to the canonical *-operation on the extended affine Hecke algebra $\mathcal{H}(H)$. Hence temperedness in the sense of [BHK] is equivalent to temperedness in terms of the Plancherel measure for affine Hecke algebras, which by [DeOp, Corollary 4.4] is equivalent to temperedness in the sense of [Opd, §2]. Thus $\pi(\Phi, \rho) \in \mathbf{Irr}(\mathcal{G})^{\mathfrak s}$ is tempered if and only if $\pi(t_q, x, \rho_q) \in \mathbf{Irr}(\mathcal{H}(H))$ is tempered in the way used in Section 9. By Proposition 9.3.a. the latter holds if and only if $\Phi(\varpi_F) \in T$ is compact. Since $\Phi(\mathbf{W}_F)$ is generated by the finite group $\Phi(\mathbf{I}_F)$ and $\Phi(\varpi_F)$, the above condition on $\Phi(\varpi_F)$ is equivalent to boundedness of $\Phi(\mathbf{W}_F)$. $\square$

16. A local Langlands correspondence

As in the introduction, $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$ denotes the space of all irreducible $\mathcal{G}$-representations in the principal series. Considering Theorem 15.1 for all Bernstein components in the principal series simultaneously, we will parametrize $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$.

Proposition 16.1. *Let $\mathcal{G}$ be a split reductive p-adic group, with restrictions on the residual characteristic as in Condition 11.1. There exists a commutative, bijective triangle*

$$\begin{tikzcd}
 & (\mathbf{Irr}\,\mathcal{T}//\mathcal{W}^G)_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G},\mathcal{T}) \arrow[rr] & & \{\text{KLR } \textit{parameters for } G\}/G
\end{tikzcd}$$

The right slanted map is natural, and via the bottom map any $\pi \in \mathbf{Irr}(\mathcal{G},\mathcal{T})$ canonically determines a Langlands parameter Φ for $\mathcal{G}$.

The restriction of this diagram to a single Bernstein component recovers Theorem 15.1. In particular the bottom arrow generalizes the Kazhdan–Lusztig parametrization of the irreducible $\mathcal{G}$-representations in the unramified principal series.

Proof. Let us work out what happens if in Theorem 15.1 we take the union over all Bernstein components $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$.

On the left we obtain (by definition) the space $\mathbf{Irr}(\mathcal{G},\mathcal{T})$. Notice that in Theorem 15.1, instead of $\{\text{KLR parameters}\}^{\mathfrak{s}}/H$ we could just as well take G-conjugacy classes of KLR parameters (Φ,ρ) such that $\Phi\big|_{\mathbf{I}_F}$ is G-conjugate to $c^{\mathfrak{s}}$. The union of those clearly is the space of all G-conjugacy classes of KLR parameters for G. For the space at the top of the diagram, choose a smooth character $\chi_{\mathfrak{s}}$ of $\mathcal{T}$ such that $(\mathcal{T},\chi_{\mathfrak{s}}) \in \mathfrak{s}$. By definition the $T^{\mathfrak{s}}$ in $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$ equals

$$T^{\mathfrak{s}} := \{\chi_{\mathfrak{s}} \otimes t \mid t \in T\},$$

where t is considered as an unramified character of $\mathcal{T}$. On the other hand, $\mathbf{Irr}\,\mathcal{T}$ can be obtained by picking representatives $\chi_{\mathfrak{s}}$ for $\mathbf{Irr}(\mathcal{T}_0) = (\mathbf{Irr}\,\mathcal{T})/T$ and taking the union of the corresponding $T^{\mathfrak{s}}$. Two such spaces $T^{\mathfrak{s}}$ give rise to the same Bernstein component for $\mathcal{G}$ if and only if they are conjugate by an element of $N_{\mathcal{G}}(\mathcal{T})$, or equivalently by an element of $\mathcal{W}^G$. Therefore

$$(\mathbf{Irr}\,\mathcal{T}//\mathcal{W}^G)_2 = \Big(\bigcup_{\mathfrak{s}\in\mathfrak{B}(\mathcal{G},\mathcal{T})} \mathcal{W}^G \cdot T^{\mathfrak{s}}//\mathcal{W}^G\Big)_2 = \bigcup_{\mathfrak{s}\in\mathfrak{B}(\mathcal{G},\mathcal{T})} \big(T^{\mathfrak{s}}//W^{\mathfrak{s}}\big)_2.$$

Hence the union of the spaces in the commutative triangles from Theorem 15.1 is as desired. The right slanted arrows in these triangles combine to a natural bijection

$$(\mathbf{Irr}\,\mathcal{T}//\mathcal{W}^G)_2 \to \{\text{KLR parameters for } G\}/G,$$

because the $\mathcal{W}^G$-action is compatible with the G-action. Suppose that $(\mathcal{T},\chi'_{\mathfrak{s}})$ is another base point for $\mathfrak{s}$. Up to an unramified twist, we may assume that $\chi'_{\mathfrak{s}} = w\chi_{\mathfrak{s}}$ for some $w \in \mathcal{W}^G$. Then the Hecke algebras $\mathcal{H}(H)$, and $\mathcal{H}(H')$ are isomorphic by a map that reflects conjugation by w and by Theorem 11.2.(2) this is compatible with the bijections between $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}, \mathbf{Irr}(\mathcal{H}(H))$ and $\mathbf{Irr}(\mathcal{H}(H'))$. It follows that the bottom maps in the triangles from Theorem 15.1 paste to a bijection

$$\mathbf{Irr}(\mathcal{G},\mathcal{T}) \to \{\text{KLR parameters for } G\}/G.$$

Finally, the map

$$(\mathbf{Irr}\,\mathcal{T}//\mathcal{W}^G)_2 \to \mathbf{Irr}(\mathcal{G},\mathcal{T})$$

can be defined as the composition of the other two bijections in the above triangle. Then it is the combination the left slanted maps from Theorem 15.1 because the triangles over there are commutative. $\square$

The bottom rows of Theorem 15.1 and Proposition 16.1 can be considered as a Langlands correspondence for $\mathbf{Irr}(\mathcal{G},\mathcal{T})$. In other words, for a Langlands parameter Φ as in (33) we define the principal series part of the L-packet $\Pi_\Phi(\mathcal{G})$ as

$$\{\pi(\Phi,\rho) \mid \rho \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_G(\Phi))\big) \text{ geometric } \}. \tag{110}$$

It is expected that $\Pi_\Phi(\mathcal{G})$ contains one $\mathcal{G}$-representation for every irreducible representation of $\pi_0(\mathrm{Z}_G(\Phi))$. Therefore we believe that (110) exhausts $\Pi_\Phi(\mathcal{G})$ if and only if every irreducible representation of $\pi_0(\mathrm{Z}_G(\Phi))$ appears in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F\times B_2)},\mathbb{C}\big)$.

To support our partial Langlands correspondence, we will show that it satisfies Borel's "desiderata" [Bor2, Section 10]. Let us recall them here:

Condition 16.2 (Borel's desiderata)**.**

(1) *Let χ_Φ be the character of $Z(\mathcal{G})$ canonically associated to Φ. Then any $\pi \in \Pi_\Phi(\mathcal{G})$ has central character χ_Φ.*
(2) *Let c be a one-cocycle of $\mathbf{W}_F$ with values in $Z(G)$ and let χ_c be the associated character of $\mathcal{G}$. Then $\Pi_{c\Phi}(\mathcal{G}) = \{\pi\otimes\chi_c \mid \pi\in\Pi_\Phi(\mathcal{G})\}$.*
(3) *If one element of $\Pi_\Phi(\mathcal{G})$ is essentially square-integrable, then all elements are. This happens if and only if the image of Φ is not contained in any proper Levi subgroup of the Langlands dual group ${}^L\mathcal{G}$.*
(4) *If one element of $\Pi_\Phi(\mathcal{G})$ is tempered, then all elements are. This is equivalent to $\Phi(\mathbf{W}_F)$ being bounded in G.*
(5) *Suppose that $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ is a morphism of connected reductive F-groups with commutative kernel and cokernel. Let ${}^L\eta : {}^L\mathcal{G} \to {}^L\tilde{\mathcal{G}}$ be the dual morphism and let $\pi\in\Pi_\Phi(\mathcal{G})$. Then $\pi\circ\eta$ is a direct sum of some $\tilde\pi \in \Pi_{{}^L\eta\circ\Phi}(\tilde{\mathcal{G}})$.*

Part (4) of Condition 16.2 has already been proved in part (2) of Theorem 15.1. Here we check the desiderata 1, 2 and 3.

Lemma 16.3. *Borel's desideratum (1) holds for our Langlands correspondence for $\mathbf{Irr}(\mathcal{G},\mathcal{T})$.*

Proof. The infinitesimal central character of $\pi(\Phi,\rho)$, as described in Theorem 15.1.(1), is its cuspidal support. For representations in the principal series this boils down to a character of $\mathcal{T}$, uniquely determined up to $\mathcal{W}^G$. Since $\mathrm{Z}(\mathcal{G})\subset\mathcal{T}$, the central character of $\pi(\Phi,\rho)$ is just the restriction of

$$\Phi\big(\varpi_F, \big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2}\end{smallmatrix}\big)\big) \in T/W^{\mathfrak{s}} \cong T^{\mathfrak{s}}/W^{\mathfrak{s}} \tag{111}$$

to $\mathrm{Z}(\mathcal{G})$. Recall that $\Phi\big(\big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2}\end{smallmatrix}\big)\big)$ comes from a homomorphism $\mathrm{SL}_2(\mathbb{C})\to G$. The image of $\Phi|_{\mathrm{SL}_2(\mathbb{C})}$ is generated by unipotent elements, so it is contained in the derived group G_{der}. That is the complex dual group of $\mathcal{G}/\mathrm{Z}(\mathcal{G})$, so $\Phi\big(\big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2}\end{smallmatrix}\big)\big)$ does not effect the $\mathrm{Z}(\mathcal{G})$-character of $\pi(\Phi,\rho)$ and we may consider $\Phi(\varpi_F)$ instead of (111). In view of our identification $T\cong T^{\mathfrak{s}}$ from

(79), this means that the central character of $\pi(\Phi,\rho)$ is just the restriction to $\mathrm{Z}(\mathcal{G})$ of the $\mathcal{T}$-character determined by $\Phi|_{\mathbf{W}_F}$ via the local Langlands correspondence for (split) tori. This agrees with Borel's construction given in [Bor2, §10.1]. $\square$

Lemma 16.4. *Desideratum (2) holds for* $\mathbf{Irr}(\mathcal{G},\mathcal{T})$*. More precisely, if* c *and* χ_c *are as in (2) and* (Φ,ρ) *is a KLR parameter for* $\mathcal{G}$*, then* $\pi(c\Phi,\rho)\cong\pi(\Phi,\rho)\otimes\chi_c$*.*

Proof. Since c takes values in $\mathrm{Z}(G)\subset T$, we can multiply any Langlands parameter for $\mathcal{T}$ with c and obtain another Langlands parameter for $\mathcal{T}$. If we transfer this map to $\mathbf{Irr}(\mathcal{T})$ via the local Langlands correspondence we get $\chi\mapsto\chi\otimes\chi_c$, see [Bor2, §10.2]. The composition

$$T\to T^{\mathfrak{s}}\xrightarrow{\otimes\chi_c}T^{\mathfrak{s}\chi_c}\to T, \tag{112}$$

where both outer maps come from (79), is of the form $t\mapsto c_T t$ for a unique $c_T\in\mathrm{Z}(G)\subset T$.

Because the image of c is contained in $\mathrm{Z}(G)$, the Langlands parameter $c\Phi$ has the same centralizer in G as Φ. Hence $(c\Phi,\rho)$ is a well-defined KLR parameter and the groups H and H_c, associated respectively to $\Phi|_{\mathbf{W}_F}$ and to $c\Phi|_{\mathbf{W}_F}$, coincide. Then (112) gives rise to an isomorphism

$$\phi_c:\mathcal{H}(H_c)\to\mathcal{H}(H)\quad\text{with}\quad\phi_c(T_w\theta_\lambda)=\lambda(c_T)T_w\theta_\lambda \tag{113}$$

for $w\in\mathcal{W}^H$ and $\lambda\in X^*(T)$. The induced map on irreducible representations is

$$\begin{aligned}\phi_c^*:\ &\mathbf{Irr}(\mathcal{H}(H))\to\mathbf{Irr}(\mathcal{H}(H_c)),\\ &\pi(t_q,x,\rho_q)\mapsto\pi(t_q,x,\rho_q)\otimes c_T=\pi(t_q c_T,x,\rho_q).\end{aligned} \tag{114}$$

Since χ_c is a character of $\mathrm{Z}(\mathcal{G})$, the $\mathfrak{s}\chi_c$-type (J_c,τ_c) from [Roc] equals to $(J,\tau\otimes\chi_c)$. Therefore the composition of ϕ_c with the two instances of (85) is

$$\mathcal{H}(\mathcal{G},\tau\otimes\chi_c)\to\mathcal{H}(\mathcal{G},\tau):f\mapsto\chi_c f.$$

(Here $\chi_c f$ denotes pointwise multiplication of functions $\mathcal{G}\to\mathbb{C}$, not a convolution product.) It follows that the composition of ϕ_c^* with the two instances of Theorem 11.2 is just

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}\xrightarrow{\otimes\chi_c}\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}\chi_c}.$$

This and (114) show that $\pi(c\Phi,\rho)\cong\pi(\Phi,\rho)\otimes\chi_c$. $\square$

Recall that $\mathcal{G}$ only has irreducible square-integrable representations if $\mathrm{Z}(\mathcal{G})$ is compact. A $\mathcal{G}$-representation is called essentially square-integrable if its restriction to the derived group of $\mathcal{G}$ is square-integrable. This is more general than square-integrable modulo centre, because for that notion $\mathrm{Z}(\mathcal{G})$ needs to act by a unitary character. Also recall that (essential) square-integrability for $\mathcal{H}(H)$-modules was defined just before Proposition 9.3.

To show that Roche's equivalence of categories Theorem 11.2 preserves essential square-integrability, we will first characterize in another way. Let $\mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$ be the set of tempered representations in $\mathbf{Irr}(\mathcal{G})$ or, equivalently the support of the Plancherel measure in $\mathbf{Irr}(\mathcal{G})$ or the dual of the reduced C^*-algebra of $\mathcal{G}$. The space of tempered irreducible $\mathcal{H}(H)$-modules $\mathbf{Irr}(\mathcal{H}(H))_{\mathrm{temp}}$ (denoted $\hat{\mathfrak{C}}$ in [Opd]) admits the analogous descriptions.

Lemma 16.5. (1) *A tempered irreducible $\mathcal{G}$-representation is essentially square-integrable if and only if it is contained in a connected component of* $\mathbf{Irr}(\mathcal{G})_{\text{temp}}$ *of minimal dimension, namely* $\dim(\mathrm{Z}(G))$.

(2) *A tempered irreducible $\mathcal{H}(H)$-module is essentially square-integrable if and only if it is contained in a connected component of* $\mathbf{Irr}(\mathcal{H}(H))_{\text{temp}}$ *of minimal dimension, namely* $\dim(H/H_{\text{der}})$.

Proof. (1) This follows from Harish-Chandra's the description of $\mathbf{Irr}(\mathcal{G})_{\text{temp}}$ in terms of essentially square-integrable representations of Levi subgroups of $\mathcal{G}$ [Wal, Proposition III.4.1]. Let us make it concrete.

If $\pi \in \mathbf{Irr}(\mathcal{G})_{\text{temp}}$ is essentially square-integrable then its central character is unitary, so it is "carré intégrable" in the sense of [Wal]. Its connected component in $\mathbf{Irr}(\mathcal{G})_{\text{temp}}$ consists of the twists of π by unitary unramified characters of $\mathcal{G}$, so it has dimension $\dim(\mathrm{Z}(G)) = \dim(\mathrm{Z}(\mathcal{G}))$.

Suppose now that $\pi \in \mathbf{Irr}(\mathcal{G})_{\text{temp}}$ is not essentially square-integrable. By [Wal, Proposition III.4.1] there exist a proper parabolic subgroup $\mathcal{P} \subset \mathcal{G}$ with Levi factor $\mathcal{M}$ and an essentially square-integrable representation $\omega \in \mathbf{Irr}(\mathcal{M})_{\text{temp}}$ such that π is a quotient of $I_{\mathcal{P}}^{\mathcal{G}}(\omega)$. Moreover (M, ω) is unique up to $\mathcal{G}$-conjugation. The connected component of π in $\mathbf{Irr}(\mathcal{G})_{\text{temp}}$ consists of all the subquotients of $I_{\mathcal{P}}^{\mathcal{G}}(\omega \otimes \chi_{\mathcal{M}})$, where $\chi_{\mathcal{M}}$ runs through the unitary unramified characters of $\mathcal{M}$. The space of such characters has dimension $\dim(\mathrm{Z}(\mathcal{M}))$, which is larger then $\dim(\mathrm{Z}(\mathcal{G}))$ because $\mathcal{P} \subsetneq \mathcal{G}$.

(2) Recall that $\mathcal{H}(H) = \mathcal{H}(H^\circ) \rtimes \pi_0(H)$ and that essential square-integrability of $\mathcal{H}(H)$-modules depends only on their restriction to $\mathcal{H}(H^\circ)$. For the affine Hecke algebra $\mathcal{H}(H^\circ)$ the claim can be proven in the same way as part (1), using the Plancherel theorem and the ensuing description of $\mathbf{Irr}_{\text{temp}}(\mathcal{H}(H^\circ))$ [DeOp, Corollary 5.7]. From there it can be generalized to $\mathcal{H}(H)$ with the comparison between $\mathbf{Irr}(\mathcal{H}(H))$ and $\mathbf{Irr}(\mathcal{H}(H^\circ))$ provided by Clifford theory for crossed products with finite groups [RaRa, Appendix]. $\square$

Proposition 16.6. *Let (Φ, ρ) be a KLR-parameter for G. Then $\pi(\Phi, \rho) \in$* $\mathbf{Irr}(\mathcal{G})$ *is essentially square-integrable if and only of the image of Φ is not contained in any Levi subgroup of a proper parabolic subgroup of G.*

In other words, Borel's desideratum (3) holds for our Langlands correspondence for $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$.

Proof. Let (Φ, ρ) be a KLR-parameter for G and let χ_Φ is the central character of $\pi(\Phi, \rho) \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$. There is a unique unramified character $\chi_c : \mathcal{G} \to \mathbb{R}_{>0}$ such that

$$\chi_c(z) = |\chi_\Phi(z)|^{-1} \text{ for all } z \in \mathrm{Z}(G).$$

Let $c : \mathbf{W} \to \mathrm{Z}(G)$ be the associated homomorphism. By Lemma 16.4

$$\pi(c\Phi, \rho) = \chi_c \otimes \pi(\Phi, \rho) \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}},$$

and by construction this representation has a unitary central character. Since χ_c is trivial on $\mathcal{G}_{\text{der}}, \pi(c\Phi, \rho)$ is essentially square-integrable if and only if $\pi(\Phi, \rho)$ is so. As $c(\mathbf{W}_F) \subset \mathrm{Z}(G)$, Borel's conditions on the Levi subgroups for Φ and $c\Phi$ are equivalent. Therefore it suffices to prove the proposition in case $\pi(\Phi, \rho)$ has unitary central character. We assume this from now on.

Suppose that $\pi(\Phi, \rho)$ is essentially square-integrable. For every matrix coefficient f of $\pi(\Phi, \rho), |f|$ is a square-integrable function on $\mathcal{G}/\mathrm{Z}(\mathcal{G})$. In particular $|f|$ is tempered, so $\pi(\Phi, \rho)$ is also tempered. By Lemma 16.5 $\pi(\Phi, \rho)$ is contained in a component of $\mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$ of minimal dimension $\dim(\mathrm{Z}(G))$.

According to [BHK, Theorem B] the equivalence of categories $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}} \to \mathrm{Mod}(\mathcal{H}(H))$ from Theorem 11.2 induces a homeomorphism

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}_{\mathrm{temp}} \to \mathbf{Irr}(\mathcal{H}(H))_{\mathrm{temp}}.$$

As checked in the proof of Theorem 15.1.2, all the relevant notions of temperedness for $\mathcal{H}(H)$-modules are equivalent. Therefore $\pi(t_q, x, \rho_q)$ lies in a $\dim(\mathrm{Z}(G))$-dimensional component of $\mathbf{Irr}(\mathcal{H}(H))_{\mathrm{temp}}$, and this is a component of minimal dimension in $\mathbf{Irr}(\mathcal{H}(H))_{\mathrm{temp}}$. Now Lemma 16.5.2 says that $\pi(t_q, x, \rho_q) \in \mathbf{Irr}(\mathcal{H}(G))$ is essentially square-integrable and that $\dim(\mathrm{Z}(G))$ equals $\dim(H/H_{\mathrm{der}})$. Since $H = \mathrm{Z}_G(\Phi(\mathbf{I}_F))$, this implies that $\dim H = \dim G$, which by the connectedness of G means that $H = G$.

Now Proposition 9.3.2 tells us that $\{t_q, x\}$ is not contained in any Levi subgroup of a proper parabolic subgroup of G. Obviously the same goes for the image of Φ.

For the opposite direction, suppose that the image of Φ is not contained in any proper Levi subgroup of G. The Levi subgroup $H^\circ \subset G$ contains the image of Φ, so $H^\circ = H = G$. If $\{t_q, x\}$ were contained in a proper Levi subgroup $L \subset G$, we could extend them to a Langlands parameter Φ' with image in L. By Lemma 7.1 Φ' would be conjugate to Φ, which would imply that the image of Φ is contained in a conjugate of L. That would violate our assumption, so $\{t_q, x\}$ cannot be contained in any proper Levi subgroup of G.

Proposition 9.3.2 says that $\pi(t_q, x, \rho_q) \in \mathbf{Irr}(\mathcal{H}(G))$ is essentially square-integrable. Recall that we assumed that $\pi(\Phi, \rho) \in \mathbf{Irr}(\mathcal{G})$ has unitary central character. With the description of its infinitesimal central character given in Theorem 15.1 we see that

$$|\lambda(t_q)| = 1 \text{ for all } \lambda \in X^*(G/G_{\mathrm{der}}) = X_*(\mathrm{Z}(G)).$$

Hence the restriction of $\pi(t_q, x, \rho_q)$ to $\mathcal{H}(G/G_{\mathrm{der}})$ is tempered. The restriction of $\pi(t_q, x, \rho_q)$ to $\mathcal{H}(G/\mathrm{Z}(G))$ is square-integrable, so in particular tempered. As noted in the proof of Proposition 9.3.1, these two facts imply that $\pi(t_q, x, \rho_q)$ is also tempered as $\mathcal{H}(G)$-module. Now the above arguments using Lemma 16.5 can be applied in the reverse order, and they show that $\pi(\Phi, \rho)$ is essentially square-integrable. □

17. Functoriality

The fifth of Borel's desiderata in Condition 16.2 says that the LLC should be functorial with respect to some specific morphisms of reductive groups. To show that this holds in our setting, we need substantial technical preparations. The first results of this section are valid without any restriction on the residual characteristic.

Let $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ be a morphism of connected reductive split F-groups, with commutative kernel and cokernel. Let $\check{\eta} : G \to \tilde{G}$ be the dual homomorphism, as in [Bor2, §1.2].

Lemma 17.1. *Define* $\tilde{\mathcal{T}} = \eta^{-1}(\mathcal{T})$.

(1) $\tilde{\mathcal{T}}$ *is a split maximal torus of* $\tilde{\mathcal{G}}$ *and* $\ker(\eta : \tilde{\mathcal{T}} \to \mathcal{T}) = \ker(\eta : \tilde{\mathcal{G}} \to \mathcal{G})$.
(2) $\ker \eta \subset \mathrm{Z}(\tilde{\mathcal{G}})$ *and* $\eta^{-1}(\mathrm{Z}(\mathcal{G})) = \mathrm{Z}(\tilde{\mathcal{G}})$.
(3) *The map* $X^*(\mathcal{T}) \to X^*(\tilde{\mathcal{T}}) : \alpha \mapsto \alpha \circ \eta$ *induces a bijection* $R(\mathcal{G}, \mathcal{T}) \to R(\tilde{\mathcal{G}}, \tilde{\mathcal{T}})$. *Similarly* $X^*(\tilde{T}) \to X^*(T) : \beta \mapsto \beta \circ \check{\eta}$ *induces a bijection* $R(\tilde{G}, \tilde{T}) \to R(G, T)$.
(4) $\mathrm{coker}(\eta : \tilde{\mathcal{T}} \to \mathcal{T}) \cong \mathrm{coker}(\eta : \tilde{\mathcal{G}} \to \mathcal{G})$.

Proof. (1) Decompose the Lie algebras of $\tilde{\mathcal{G}}$ and $\mathcal{G}$ as

$$\begin{aligned} \mathrm{Lie}(\tilde{\mathcal{G}}) &= \mathrm{Lie}(\tilde{\mathcal{G}}_{\mathrm{der}}) \oplus \mathrm{Lie}(\mathrm{Z}(\tilde{\mathcal{G}})), \\ \mathrm{Lie}(\mathcal{G}) &= \mathrm{Lie}(\mathcal{G}_{\mathrm{der}}) \oplus \mathrm{Lie}(\mathrm{Z}(\mathcal{G})). \end{aligned} \tag{115}$$

The assumptions on the kernel and cokernel of η imply that $\mathrm{Lie}(\eta)$ restricts to an isomorphism

$$\mathrm{Lie}(\eta) : \mathrm{Lie}(\tilde{\mathcal{G}}_{\mathrm{der}}) \to \mathrm{Lie}(\mathcal{G}_{\mathrm{der}}) \tag{116}$$

and to a F-linear map $\mathrm{Lie}(\mathrm{Z}(\tilde{\mathcal{G}})) \to \mathrm{Lie}(\mathrm{Z}(\mathcal{G}))$. Hence $\mathrm{Lie}(\tilde{\mathcal{T}})$ is the Lie algebra of $\mathrm{Z}(\tilde{\mathcal{G}})$ times a maximal split torus of $\tilde{\mathcal{G}}_{\mathrm{der}}$. It follows that the unit component $\tilde{\mathcal{T}}^\circ$ of $\tilde{\mathcal{T}}$ is a split maximal torus of $\tilde{\mathcal{G}}$. Now $\ker(\eta)\mathrm{Z}(\tilde{\mathcal{G}})^\circ / \mathrm{Z}(\tilde{\mathcal{G}})^\circ$ is a finite normal subgroup of $\tilde{\mathcal{G}}/\mathrm{Z}(\tilde{\mathcal{G}})^\circ$, so it is central and contained in the maximal torus $\tilde{\mathcal{T}}^\circ/\mathrm{Z}(\tilde{\mathcal{G}})^\circ$ of $\tilde{\mathcal{G}}/\mathrm{Z}(\tilde{\mathcal{G}})^\circ$. Consequently $\ker \eta$ is contained in $\tilde{\mathcal{T}}^\circ$. As $\mathcal{T}$ is connected and $\tilde{\mathcal{T}} = \eta^{-1}(\mathcal{T})$, $\tilde{\mathcal{T}}$ is also connected.
(2) We just saw that $\ker \eta$ is contained in the maximal split torus $\tilde{\mathcal{T}}$ of $\tilde{\mathcal{G}}$. But $\mathcal{T}$ was arbitrary, so $\ker \eta$ lies in every split maximal torus of $\tilde{\mathcal{G}}$. The intersection of all such tori is contained in the centre of $\tilde{\mathcal{G}}$, so $\ker \eta$ as well.

Since $\mathcal{G}$ is connected, $\mathrm{Z}(\mathcal{G})$ is the kernel of the adjoint representation of $\mathcal{G}$. As $\mathrm{Lie}(\mathrm{Z}(\mathcal{G}))$ is a trivial summand of the adjoint representation, $\mathrm{Z}(\mathcal{G})$ is also the kernel of $\mathrm{Ad} : \mathcal{G} \to \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(\mathcal{G}_{\mathrm{der}}))$. In view of (116) there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{G}} & \xrightarrow{\eta} & \mathcal{G} \\ \downarrow \tilde{\mathrm{Ad}} & & \downarrow \mathrm{Ad} \\ \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(\tilde{\mathcal{G}}_{\mathrm{der}})) & \longrightarrow & \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(\mathcal{G}_{\mathrm{der}})). \end{array}$$

Now we see that

$$\eta^{-1}(\mathrm{Z}(\mathcal{G})) = \eta^{-1}(\ker \mathrm{Ad}) = \ker \tilde{\mathrm{Ad}} = \mathrm{Z}(\tilde{\mathcal{G}}).$$

(3) From part (1) and the isomorphism (116) we deduce that η maps any root subgroup of $(\tilde{\mathcal{G}}, \tilde{\mathcal{T}})$ bijectively to a root subgroup of $(\mathcal{G}, \mathcal{T})$. This yields the bijection $R(\mathcal{G}, \mathcal{T}) \to R(\tilde{\mathcal{G}}, \tilde{\mathcal{T}})$, which is given explicitly by $\alpha \mapsto \alpha \circ \eta$. The second claim follows by dualizing the root data.
(4) In view of (116) η restricts to an isomorphism of root subgroups $\tilde{\mathcal{G}}_{\alpha \circ \eta} \to \mathcal{G}_\alpha$, for every $\alpha \in R(\mathcal{G}, \mathcal{T})$. With part (1) we see that the preimage $\tilde{\mathcal{B}} = \eta^{-1}(\mathcal{B})$ of our Borel subgroup $\mathcal{B} \subset \mathcal{G}$ is a Borel subgroup of $\tilde{\mathcal{G}}$. Let $\mathcal{U}$ (resp. $\tilde{\mathcal{U}}$) be the unipotent radical of $\mathcal{B}$ (resp. $\tilde{\mathcal{B}}$). The Bruhat decomposition says that

$$\mathcal{G} = \mathcal{U} N_{\mathcal{G}}(\mathcal{T}) \mathcal{U} \text{ and } \tilde{\mathcal{G}} = \tilde{\mathcal{U}} N_{\tilde{\mathcal{G}}}(\tilde{\mathcal{T}}) \tilde{\mathcal{U}}.$$

As $\eta : \tilde{\mathcal{U}} \to \mathcal{U}$ is an isomorphism and

$$N_{\mathcal{G}}(\mathcal{T})/\mathcal{T} \cong \mathcal{W}^{\mathcal{G}} \cong \mathcal{W}^{\tilde{\mathcal{G}}} \cong N_{\tilde{\mathcal{G}}}(\tilde{\mathcal{T}})/\tilde{\mathcal{T}},$$

the inclusions $\mathcal{T} \to \mathcal{G}$ and $\tilde{\mathcal{T}} \to \tilde{\mathcal{G}}$ give an isomorphism $\mathrm{coker}(\eta : \tilde{\mathcal{T}} \to \mathcal{T}) \to \mathrm{coker}(\eta : \tilde{\mathcal{G}} \to \mathcal{G})$. $\Box$

Recall that $\chi \in \mathbf{Irr}(\mathcal{T})$ and $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$. Let $c_{\mathfrak{s}} : \mathfrak{o}_F^{\times} \to T$ be the restriction of the Langlands parameter of χ.

Lemma 17.2. *Define* $\tilde{\mathfrak{s}} = [\tilde{\mathcal{T}}, \chi \circ \eta]_{\tilde{\mathcal{G}}}$ *and*

$$c_{\tilde{\mathfrak{s}}} = \check{\eta} \circ c_{\mathfrak{s}} : \mathfrak{o}_F^{\times} \to \tilde{T}.$$

Then $c_{\tilde{\mathfrak{s}}}$ *is the restriction of the Langlands parameter of* $\chi \circ \eta$ *to* $\mathfrak{o}_F^{\times}$ *and* η^* *sends* $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}}$ *to* $\mathrm{Rep}(\tilde{\mathcal{G}})^{\tilde{\mathfrak{s}}}$.

Proof. It follows from the construction of the local Langlands correspondence for split tori that the diagram

$$\begin{array}{ccc} \mathbf{Irr}(\mathcal{T}) & \xrightarrow{\eta^*} & \mathbf{Irr}(\tilde{\mathcal{T}}) \\ \downarrow & & \downarrow \\ \{\text{L-parameters for } \mathcal{T}\} & \xrightarrow{\check{\eta}} & \{\text{L-parameters for } \tilde{\mathcal{T}}\} \end{array}$$

commutes. With Lemma 5.1 this proves the first claim.

By definition $\mathrm{Rep}(\mathcal{G})^{\mathfrak{s}}$ is the category of all smooth $\mathcal{G}$-representations π with the property that every irreducible subquotient of π occurs as a subquotient of $I_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)$ for some unramified character $t \in X_{\mathrm{unr}}(\mathcal{T})$. So for the second claim it suffices to show that $\eta^*\big(I_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)\big) \in \mathrm{Rep}(\tilde{\mathcal{G}})^{\tilde{\mathfrak{s}}}$ for all $t \in X_{\mathrm{unr}}(\mathcal{T})$.

Clearly $\eta(\tilde{\mathcal{T}}_0) \subset \mathcal{T}_0$, so $\eta^*(t) \in \mathbf{Irr}(\tilde{\mathcal{T}})$ is unramified. Hence $\eta^*(\chi \otimes t)$ is an unramified twist of $\eta^*(\chi)$ and $\eta^*(\chi \otimes t) \in [\tilde{\mathcal{T}}, \chi \circ \eta]_{\tilde{\mathcal{T}}}$. By Lemma 17.1.(4) η induces a homeomorphism $\tilde{\mathcal{G}}/\tilde{\mathcal{B}} \to \mathcal{G}/\mathcal{B}$, which implies that

$$\eta^*\big(\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)\big) = \mathrm{Ind}_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{G}}}(\eta^*(\chi \otimes t)).$$

It follows from Lemma 17.1.(3) that the difference between parabolic induction and normalized parabolic induction consists of twisting by essentially the same unramified character on both sides, so we get

$$\eta^*\big(I_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes t)\big) = I_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{G}}}(\eta^*(\chi \otimes t)) \in \mathrm{Rep}(\tilde{\mathcal{G}})^{\tilde{\mathfrak{s}}}. \qquad \Box$$

As before, we define $H = \mathrm{Z}_G(c_{\mathfrak{s}})$ and $\tilde{H} = \mathrm{Z}_{\tilde{G}}(c_{\tilde{\mathfrak{s}}})$. By Lemma 17.1.(3) there is a bijection

$$\begin{aligned} (117) \quad R(H,T) = \{\alpha \in R(G,T) \mid \alpha(c_{\mathfrak{s}}(\mathfrak{o}_F^{\times})) = 1\} \longleftrightarrow \\ \{\tilde{\alpha} \in R(\tilde{G},\tilde{T}) \mid \tilde{\alpha}(\eta \circ c_{\mathfrak{s}}(\mathfrak{o}_F^{\times})) = 1\} = R(\tilde{H},\tilde{T}). \end{aligned}$$

Hence $\mathcal{W}^{H^\circ} \cong \mathcal{W}^{\tilde{H}^\circ}$. As $\check{\eta}(H) \subset \tilde{H}$, we have a canonical inclusion

$$\mathcal{W}^{\eta} : \mathcal{W}^{H} \to \mathcal{W}^{\tilde{H}}. \tag{118}$$

However, in general it is not a bijection.

Example 17.3. *Consider the canonical homomorphism $\eta : \tilde{\mathcal{G}} = \mathrm{SL}_3(F) \to \mathrm{PGL}_3(F) = \mathcal{G}$. Let ζ be a character of order 3 of $\mathfrak{o}_F^\times$ and put $c_\mathfrak{s} = (\zeta, \zeta^2, 1)$. Since $c_\mathfrak{s}$ is trivial on $Z(\mathrm{GL}_3(F))$, it defines a Bernstein component for the standard maximal torus $\tilde{\mathcal{T}}$ of $\mathrm{PGL}_3(F)$. In this case $\mathcal{W}^H = \{1\}$. Similarly $\check{\eta} \circ c_\mathfrak{s}$ defines a Bernstein component for the standard maximal torus $\mathcal{T}$ of $\mathrm{SL}_3(F)$. For $(a,b,c) \in \mathcal{T}(\mathfrak{o}_F)$:*

$$c_\mathfrak{s}(a,b,c) = \zeta(a)\zeta(b^2) = \zeta(bc^{-1}) = c_\mathfrak{s}(b,c,a),$$

from which it follows easily that $\mathcal{W}^{\tilde{H}} \cong \mathbb{Z}/3\mathbb{Z}$.

Let $\tilde{H}' \subset \tilde{H}$ be the subgroup generated by $\tilde{H}^\circ$ and $W^\mathfrak{s} = \mathcal{W}^H$, via (118). Then $\mathcal{H}(\tilde{H})$ contains a subalgebra

$$\mathcal{H}(\tilde{H}') \cong \mathcal{H}(\tilde{H}^\circ) \rtimes \pi_0(H).$$

The advantage of this algebra over $\mathcal{H}(\tilde{H})$ is that η induces an algebra homomorphism

$$\begin{aligned} &\phi_\eta : \mathcal{H}(\tilde{H}') \to \mathcal{H}(H), \\ &\phi_\eta(T_w \theta_\mu) = T_w \theta_{\eta(\mu)} \text{ for } w \in W^\mathfrak{s}, \mu \in X_*(\tilde{\mathcal{T}}) = X^*(\tilde{T}). \end{aligned} \tag{119}$$

Recall that Φ is a Langlands parameter for $\mathcal{G}$ as in (33). As remarked in Section 7, all the geometric representations ρ of $\pi_0\big(\mathrm{Z}_G(\Phi)$, which can be used to enhance Φ to a KLR parameter, factor through $\pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G)\big)$.

Lemma 17.4. *$\check{\eta}$ induces injective group homomorphisms*

$$\pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G)\big) \to \pi_0\big(\mathrm{Z}_{\tilde{G}}(\check{\eta} \circ \Phi)/\mathrm{Z}(\tilde{G})\big),$$
$$\pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(H)\big) \to \pi_0\big(\mathrm{Z}_{\tilde{G}}(\check{\eta} \circ \Phi)/\mathrm{Z}(\tilde{H}')\big).$$

Proof. Clearly $\check{\eta}$ restricts to a group homomorphism

$$\mathrm{Z}_H(\Phi) = \mathrm{Z}_G(\Phi) \to \mathrm{Z}_{\tilde{G}}(\check{\eta} \circ \Phi) = \mathrm{Z}_{\tilde{H}}(\eta \circ \Phi).$$

By Lemma 17.1.(4) it induces an injective homomorphism

$$\check{\eta} : \mathrm{Z}_H(\Phi)/\mathrm{Z}(G) \to \mathrm{Z}_{\tilde{H}}(\check{\eta} \circ \Phi)/\mathrm{Z}(\tilde{G}). \tag{120}$$

By (118) $\mathrm{Lie}(\eta)$ maps $\mathrm{Lie}(Z(H))$ to $\mathrm{Lie}(Z(\tilde{H}'))$, so by Lemma 17.1.(4) also

$$\check{\eta} : \mathrm{Z}_H(\Phi)/\mathrm{Z}(H) \to \mathrm{Z}_{\tilde{H}}(\check{\eta} \circ \Phi)/\mathrm{Z}(\tilde{H}') \tag{121}$$

is injective. We will show that (120) and (121) are isogenies.

The properties of η imply that

$$\begin{aligned} \mathrm{Lie}(\check{\eta}) : \mathrm{Lie}(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G)) = \mathrm{Lie}(H)/\mathrm{Z}(\mathrm{Lie}(G)) \to \\ \mathrm{Lie}(\mathrm{Z}_{\tilde{H}}(\check{\eta} \circ \Phi)/Z(\tilde{G})) = \mathrm{Lie}(\mathrm{Z}_{\tilde{H}}(\check{\eta} \circ \Phi))/\mathrm{Z}\mathrm{Lie}(\tilde{G}) \end{aligned} \tag{122}$$

is an isomorphism of reductive Lie algebras. The group $\Phi(\mathrm{SL}_2(\mathbb{C}))$ is contained in H_{der}, so by (122) $\mathrm{Lie}(\check{\eta})$ maps $\mathrm{Lie}\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big)$ bijectively to $\mathrm{Lie}\big(\check{\eta} \circ \Phi(\mathrm{SL}_2(\mathbb{C}))\big)$. Therefore

$$\mathrm{Lie}\big(\mathrm{Z}_H\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big)/\mathrm{Z}(G)\big) \to \mathrm{Lie}\big(\mathrm{Z}_{\tilde{H}}\big(\check{\eta} \circ \Phi(\mathrm{SL}_2(\mathbb{C}))\big)/\mathrm{Z}(\tilde{G})\big) \tag{123}$$

is another isomorphism of reductive Lie algebras. To reach the Lie algebras of (120), it remains to restrict to elements that commute with $\Phi(\varpi_F)$, respectively $\check{\eta} \circ \Phi(\varpi_F)$. For simplicity we assume that T is a maximal torus of $\mathrm{Z}_H(\Phi)$, this can always be achieved by replacing Φ with a conjugate Langlands parameter. Then we have a bijection

$$\begin{array}{c} R\big(\mathrm{Z}_H(\Phi),T\big) = \big\{\alpha \in R\big(\mathrm{Z}_H\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big),T\big) \mid \alpha(\Phi(\varpi_F)) = 1\big\} \longleftrightarrow \\ \big\{\tilde\alpha \in R\big(\mathrm{Z}_{\tilde H}\big(\check\eta \circ \Phi(\mathrm{SL}_2(\mathbb{C}))\big),\tilde T\big) \mid \tilde\alpha(\eta \circ \Phi(\varpi_F)) = 1\big\} = R\big(\mathrm{Z}_{\tilde H}(\check\eta \circ \Phi),\tilde T\big). \end{array}$$

With (123) this this implies that

$$\mathrm{Lie}(\check\eta) : \mathrm{Lie}(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G)) \to \mathrm{Lie}\big(\mathrm{Z}_{\tilde H}(\check\eta \circ \Phi)/\mathrm{Z}(\tilde G)\big).$$

is an isomorphism, so (120) is indeed an isogeny. The same argument shows that (121) is an isogeny.

As (120) and (121) are also injective, they embed the left hand side in the right hand side as a number of connected components. Hence the induced maps on the component groups are injective. $\square$

Now we reinstate Condition 11.1. Let $(\tilde J,\tilde\tau)$ be Roche's type for $\tilde{\mathfrak s}$. The explicit construction in [Roc, §3] shows that $\tilde J = \eta^{-1}(J)$ and $\tilde\tau = \rho \circ \eta$. By [Roc, Theorem 4.15] the support of $\mathcal H(\tilde{\mathcal G},\tilde\tau)$ is $\tilde J(W^{\tilde{\mathfrak s}} \rtimes X_*(\tilde{\mathcal T}))\tilde J$, where we embed $X_*(\tilde{\mathcal T})$ in $\mathcal T$ via $\mu \mapsto \mu(\varpi_F)$. By Theorem 11.2

$$\mathcal H(\tilde{\mathcal G},\tilde\tau) \cong \mathcal H(\tilde H) = \mathcal H(\tilde H^\circ) \rtimes \pi_0(\tilde H). \tag{124}$$

Let $\mathcal H(\tilde{\mathcal G},\tilde\tau)'$ be the subalgebra of $\mathcal H(\tilde{\mathcal G},\tilde\tau)$ isomorphic to $\mathcal H(\tilde H')$ and with support $\tilde J(W^{\mathfrak s} \rtimes X_*(\tilde{\mathcal T}))\tilde J$. We obtain an algebra homomorphism $\mathcal H(\eta,\tau) : \mathcal H(\tilde{\mathcal G},\tilde\tau)' \to \mathcal H(\mathcal G,\tau)$ with

$$\mathcal H(\eta,\rho)(f)(g) = \left\{ \begin{array}{ll} f(\eta^{-1}(g)) & g \in J(W^{\mathfrak s} \rtimes \eta(X_*(\mathcal T)))J \\ 0 & g \in G \setminus J(W^{\mathfrak s} \rtimes \eta(X_*(\mathcal T)))J. \end{array} \right. \tag{125}$$

This is well-defined because any $f \in \mathcal H(\tilde{\mathcal G},\tilde\tau)'$ takes one common value on the entire preimage $\eta^{-1}(g)$. Taking (84) and Lemma 17.2 into account, we consider the diagram

$$\begin{tikzcd} \mathrm{Rep}(\mathcal G)^{\mathfrak s} \arrow[d, "\eta^*"] & \mathrm{Mod}(\mathcal H(\mathcal G,\tau)) \arrow[l, "\mathrm{ind}_{\mathcal H(\mathcal G,\tau)}^{\mathcal H(\mathcal G)}"'] \arrow[d, "\mathcal H(\eta,\tau)^*"] & \mathrm{Mod}(\mathcal H(H)) \arrow[l, "\sim"'] \arrow[d, "\phi_\eta^*"] \\ \mathrm{Rep}(\tilde{\mathcal G})^{\tilde{\mathfrak s}} & \mathrm{Mod}(\mathcal H(\tilde{\mathcal G},\tilde\tau)') \arrow[l, "\mathrm{ind}_{\mathcal H(\tilde{\mathcal G},\tilde\tau)'}^{\mathcal H(\tilde{\mathcal G})}"'] & \mathrm{Mod}(\mathcal H(\tilde H')) \arrow[l, "\sim"'] \end{tikzcd} \tag{126}$$

The left upper horizontal arrow is invertible with as inverse the map that sends any $\mathcal G$-representation to its τ-isotypical component, as in (83).

Lemma 17.5. *The diagram* (126) *commutes up to a natural isomorphism.*

Proof. The right hand square commutes by definition, so consider only the left hand square. Take $V \in \mathrm{Mod}(\mathcal H(\mathcal G,\tau))$. We must compare

$$\mathcal H(\tilde{\mathcal G}) \otimes_{\mathcal H(\tilde{\mathcal G},\tilde\tau)'} \mathcal H(\eta,\tau)^*(V) \quad \text{with} \quad \eta^*\big(\mathcal H(\mathcal G) \otimes_{\mathcal H(\mathcal G,\tau)} V\big). \tag{127}$$

One problem that we encounter is the lack of a reasonable map $\mathcal H(\tilde{\mathcal G}) \to \mathcal H(\mathcal G)$. To overcome this we make use of the algebra $\mathcal H^\vee(\mathcal G)$ of essentially left-compact distributions on $\mathcal G$, which was introduced in [BeDe]. It naturally contains both $\mathcal H(\mathcal G)$ and a copy of $\mathcal G$. From [BeDe, §1.2] it is known that

$\mathrm{Mod}(\mathcal{H}^\vee(\mathcal{G}))$ is naturally equivalent with $\mathrm{Rep}(\mathcal{G})$. Moreover $\mathcal{H}(\mathcal{G})$ is a two-sided ideal of $\mathcal{H}^\vee(\mathcal{G})$, so the modules (127) are canonically isomorphic with

$$\mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)'} \mathcal{H}(\eta,\tau)^*(V), \text{ respectively } \eta^*\big(\mathcal{H}^\vee(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} V\big).$$

An advantage of $\mathcal{H}^\vee(\mathcal{G})$ over $\mathcal{H}(\mathcal{G})$ is that it is functorial in $\mathcal{G}$, see [Moy, Theorem 3.1]. The algebra homomorphism

$$\mathcal{H}^\vee(\eta) : \mathcal{H}^\vee(\tilde{\mathcal{G}}) \to \mathcal{H}^\vee(\mathcal{G}) \quad \text{extends} \quad \mathcal{H}(\eta,\tau) : \mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)' \to \mathcal{H}(\mathcal{G},\tau).$$

This yields a canonical map

$$\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V : \mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)'} \mathcal{H}(\eta,\tau)^*(V) \to \mathcal{H}^\vee(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} V.$$

By Lemma 17.1.(4) $\mathcal{G} = \eta(\tilde{\mathcal{G}})\mathcal{T}$, and the action of $\mathcal{T}$ on V is already given $\mathcal{H}(\mathcal{G},\tau)$ since $\mathcal{T} \subset JX_*(\mathcal{T})J$. Therefore $\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V$ is surjective.

It is also $\tilde{\mathcal{G}}$-equivariant if we regard its target as $\eta^*\big(\mathcal{H}^\vee(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} V\big)$. In particular its kernel is a $\tilde{\mathcal{G}}$-subrepresentation of $\mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)'} \mathcal{H}(\eta,\tau)^*(V)$. As $(\tilde J,\tilde\tau)$ is a $\tilde{\mathfrak{s}}$-type, $\ker(\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V)$ is of the form $\mathcal{H}^\vee(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)} N$ for some $\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)$-module

$$N \subset \mathrm{ind}_{\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)'}^{\mathcal{H}(\tilde{\mathcal{G}},\tilde\tau)} \mathcal{H}(\eta,\tau)^*(V).$$

Let E be a set of representatives for $\mathcal{W}^{\tilde H}/\mathcal{W}^\eta(\mathcal{W}^H)$. Then any element of N can be written as $n = \sum_{w \in E} T_w v_w$. We have

$$0 = \mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V(n) = \sum\nolimits_{w \in E} \mathcal{H}^\vee(\eta)(T_w) \otimes_{\mathcal{H}(\mathcal{G},\tau)} v_w.$$

The elements $\mathcal{H}^\vee(\eta)(T_w)$ with $w \in E$ are linearly independent over $\mathcal{H}(\mathcal{G}.\tau)$, because the support of $\mathcal{H}^\vee(\eta)(T_w)$ is $\eta(\tilde J w \tilde J) = JwJ$. It follows that $v_w = 0$ for all $w \in E$.

Hence $N = 0$ and $\mathcal{H}^\vee(\eta) \otimes \mathrm{id}_V$ is injective. This shows that (126) commutes up to the canonical isomorphism between the $\tilde{\mathcal{G}}$-representations (127). □

It is clear that the formula (119) also defines an algebra homomorphism $\phi_\eta : \mathcal{H}_v(\tilde H') \to \mathcal{H}_v(H)$ for any $v \in \mathbb{C}^\times$, and that these maps lift to a homomorphism of $\mathbb{C}[\mathbf{q}^{\pm 1/2}]$-algebras

$$\phi_\eta : \mathcal{H}_{\sqrt{\mathbf{q}}}(\tilde H') \to \mathcal{H}_{\sqrt{\mathbf{q}}}(H).$$

Denote the category of finite length semisimple modules of an algebra A by $\mathrm{Mod}_{\mathrm{fss}}(A)$.

Lemma 17.6. (1) $\phi_\eta^* : \mathrm{Mod}(\mathcal{H}_v(\tilde H')) \to \mathrm{Mod}(\mathcal{H}_v(H))$ *preserves finite length and complete reducibility.*

(2) *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{fss}}(\mathcal{H}_q(H)) & \longleftrightarrow & \mathrm{Mod}_{\mathrm{fss}}(\mathcal{W}^H \ltimes X^*(T)) \\ \downarrow \phi_\eta^* & & \downarrow \eta^* \\ \mathrm{Mod}_{\mathrm{fss}}(\mathcal{H}_q(\tilde H')) & \longleftrightarrow & \mathrm{Mod}_{\mathrm{fss}}(\mathcal{W}^H \ltimes X^*(\tilde T)) \end{array}$$

in which the horizontal arrows extend the left slanted map in Theorem 13.1 additively.

Proof. (1) For these considerations the kernel of ϕ_η plays no role, we need only look at the subalgebra $\phi_\eta(\mathcal{H}_v(\tilde H'))$ of $\mathcal{H}_v(H)$. It has a basis $\{T_w \theta_\lambda \mid w \in \mathcal{W}^H, \lambda \in \eta(X^*(\tilde T))\}$. Since η has commutative cokernel, $\mathcal{W}^H \ltimes \eta(X^*(\tilde T)) + X^*(T)^{\mathcal{W}^H}$ is of finite index in $\mathcal{W}^H \ltimes X^*(T)$, and it contains $\mathcal{W}^H \ltimes \mathbb{Z}R(H^\circ, T) + X^*(T)^{\mathcal{W}^H}$. The group extension

$$\mathcal{W}^H \ltimes \mathbb{Z}R(H^\circ, T) + X^*(T)^{\mathcal{W}^H} \subset \mathcal{W}^H \ltimes X^*(T) \tag{128}$$

is of the form $X \subset X \rtimes \Gamma$, where $\Gamma \subset \mathcal{W}^H \ltimes X^*(T)$ is the finite group of elements that preserve the fundamental alcove in the Coxeter complex of $\mathcal{W}^H \ltimes \mathbb{Z}R(H^\circ, T) + X^*(T)^{\mathcal{W}^H}$. Hence the inclusion of affine Hecke algebras corresponding to (128) is of the form

$$\mathcal{H}_v(H'') \subset \mathcal{H}_v(H'') \rtimes \Gamma = \mathcal{H}_v(H).$$

It is well-known from Clifford theory [RaRa, Appendix A] that the restriction map $\mathrm{Mod}(\mathcal{H}_v(H'') \rtimes \Gamma) \to \mathrm{Mod}(\mathcal{H}_v(H''))$ preserves finite length and complete reducibility. Since $\phi_\eta(\mathcal{H}_v(\tilde H'))$ lies between these two algebras, the same holds for

$$\mathrm{Mod}(\mathcal{H}_v(H)) \to \mathrm{Mod}\big(\phi_\eta(\mathcal{H}_v(\tilde H'))\big).$$

(2) Consider the standard $\mathcal{H}_{\sqrt{\mathbf{q}}}(H)$-module $\tilde\pi(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}]$, as in (102). Its specialization at a generic $v \in \mathbb{C}^\times$ is irreducible and it equals $\pi(t_v, x, \rho_v)$. Recall from 9 that this is a $\mathcal{H}_v(H)$-submodule of $\mathrm{H}_*(\mathcal{B}_H^{t_v,x}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(H)]$. By Lemma 17.1 H and $\tilde H$ have isomorphic varieties of Borel subgroups, and the description of $\mathcal{H}_v(H)$-action entails that

$$\phi_\eta^*\big(\mathrm{H}_*(\mathcal{B}_H^{t_v,x}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(H)]\big) \cong \mathrm{H}_*(\mathcal{B}_{\tilde H}^{\check\eta(t_v),\check\eta(x)}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(H)].$$

By part (1) $\phi_\eta^*(\pi(t_v, x, \rho_v))$ is completely reducible. Hence there is a unique representation $\tilde\rho = \oplus_i \tilde\rho_i$ of

$$\pi_0\big(Z_{\tilde H'}(\check\eta(t_v), \check\eta(x))\big) = \pi_0\big(Z_{\tilde H'}(\check\eta \circ \Phi)\big)$$

such that

$$\pi(\check\eta(t_v), \check\eta(x), \tilde\rho) := \oplus_i \pi(\check\eta(t_v), \check\eta(x), \tilde\rho_i).$$

We need to identify $\tilde\rho$. Like in the proof of Lemma 14.2, the family of modules $\phi_\eta^*(\pi(t_v, x, \rho_v))$ depends algebraically on v, so $\tilde\rho$ does not depend on $v \in \mathbb{C}^\times$ as long as v is generic. As the set of generic parameters is dense in $\mathbb{C}^\times$, we must have

$$\phi_\eta^*\big(\tilde\pi(\Phi, \rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}]\big) \cong \tilde\pi(\Phi, \tilde\rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}] = \oplus_i \tilde\pi(\Phi, \tilde\rho_i) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{q}^{\pm 1/2}].$$

In particular this holds for $v = q$ and for $v = 1$. Looking at the unique irreducible quotients (for $v = q$) or at the subrepresentations in top homological degree (for $v = 1$), we find

$$\begin{aligned}
\phi_\eta^*(\pi(t_q, x, \rho_q)) &\cong \pi(\check\eta(t_q), \check\eta(x), \tilde\rho) = \oplus_i \pi(\check\eta(t_q), \check\eta(x), \tilde\rho_i), \\
\eta^*(\tau(t_1, x, \rho_1)) &\cong \tau(\check\eta(t_1), \check\eta(x), \tilde\rho) = \oplus_i \tau(\check\eta(t_1), \check\eta(x), \tilde\rho_i).
\end{aligned}$$

Thus the diagram in statement commutes for irreducible representations and, being additive, for all semisimple modules of finite length. □

Now we can determine the effect of η^* on irreducible representations. Let (Φ, ρ) be a KLR parameter for $\mathcal{G}$, with (t_q, x, ρ_q) as in Lemma 7.1. Recall that ρ is trivial on the image of $Z(G)$ in $\pi_0(Z_G(\Phi))$.

Proposition 17.7. *Let $\eta : \tilde{\mathcal{G}} \to \mathcal{G}, \check{\eta} : G \to \tilde{G}$ and (Φ, ρ) be as above. Identify $\pi_0\big(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G)\big)$ with a subgroup of $\pi_0\big(\mathrm{Z}_{\tilde{G}}(\check{\eta}\circ\Phi)/\mathrm{Z}(\tilde{G})\big)$ via Lemma 17.4. Then*

$$\eta^*\big(\pi(\Phi, \rho)\big) = \pi\big(\check{\eta} \circ \Phi, \mathrm{ind}_{\pi_0(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde{G}}(\check{\eta}\circ\Phi)/\mathrm{Z}(\tilde{G}))} \rho\big).$$

Here we use the convention $\pi(\check{\eta} \circ \Phi, \oplus_i \rho_i) = \oplus_i \pi(\check{\eta} \circ \Phi, \rho_i)$ for $\rho_i \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_{\tilde{G}}(\check{\eta} \circ \Phi)/\mathrm{Z}(\tilde{G})\big)$.

Proof. By Lemma 17.5 and (109)

$$\begin{aligned} \eta^*(\pi(\Phi, \rho)) &= \eta^*\big(\mathcal{H}(\mathcal{G}) \otimes_{\mathcal{H}(\mathcal{G},\tau)} \pi(t_q, x, \rho_q)\big) \\ &\cong \mathcal{H}(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})'} \mathcal{H}(\eta, \tau)^* \pi(t_q, x, \rho_q) \\ &\cong \mathcal{H}(\tilde{\mathcal{G}}) \otimes_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})} \mathrm{ind}_{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})'}^{\mathcal{H}(\tilde{\mathcal{G}},\tilde{\tau})} \mathcal{H}(\eta, \tau)^* \pi(t_q, x, \rho_q). \end{aligned} \tag{129}$$

By (124) it suffices to analyse the module $\mathrm{ind}_{\mathcal{H}(\tilde{H}')}^{\mathcal{H}(\tilde{H})} \phi_\eta^* \pi(t_q, x, \rho_q)$. By Lemma 17.6 the module $\phi_\eta^* \pi(t_q, x, \rho_q)$ has finite length and is semisimple, and its parameters can be read off from the $X^*(\tilde{T}) \rtimes \mathcal{W}^H$-module $\eta^*(\tau(t_1, x, \rho_1))$.

For simplicity we drop the subscripts 1. Recall from (53) that $\tau(t, x, \rho)$ is isomorphic to

$$\mathrm{Ind}_{X^*(T)\rtimes\mathcal{W}_t^H}^{X^*(T)\rtimes\mathcal{W}^H} \big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x)/Z(G))}\big(\rho, H_*(\mathcal{B}_{M^\circ}^x, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_H(t, x)/\mathrm{Z}_{M^\circ}(x)]\big)\big).$$

By Lemma 17.1 $\check{\eta}$ induces an isomorphism

$$M^\circ/\mathrm{Z}(G) = \mathrm{Z}_H(t)^\circ/\mathrm{Z}(G) \to \mathrm{Z}_{\tilde{H}'}(\check{\eta}(t))^\circ/\mathrm{Z}(\tilde{G}) =: \tilde{M}^\circ/\mathrm{Z}(\tilde{G}).$$

As $\check{\eta}$ also provides an isomorphism between the respective unipotent varieties of M and $\tilde{M} = \mathrm{Z}_{\tilde{H}}(\check{\eta}(t))$,

$$\pi_0(\mathrm{Z}_{M^\circ}(x)/\mathrm{Z}(H)) \cong \pi_0(\mathrm{Z}_{\tilde{M}^\circ}(\check{\eta}(x))/\mathrm{Z}(\tilde{H}')).$$

Steinberg's description of the centralizer of a semisimple element [Ste1, 2.8] and again Lemma 17.1 show that the inclusion $N_{\tilde{H}'}(\tilde{T}) \to \tilde{H}'$ induces a group isomorphism

$$\mathcal{W}_{\check{\eta}(t)}^H / \mathcal{W}_t^H \xrightarrow{\sim} \mathrm{Z}_{\tilde{H}'}(\check{\eta}(t), \check{\eta}(x))/\check{\eta}(\mathrm{Z}_H(t, x))\mathrm{Z}(\tilde{G}).$$

It follows that $\tau(t, x, \rho)$ is also isomorphic to

$$\begin{aligned} \mathrm{Ind}_{X^*(T)\rtimes\mathcal{W}_{\check{\eta}(t)}^H}^{X^*(T)\rtimes\mathcal{W}^H} \big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x)/Z(G))}\big(\rho,& \\ H_{d(x)}(\mathcal{B}_{M^\circ}^x, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_{\tilde{H}'}(\check{\eta}(t), \check{\eta}(x))/\mathrm{Z}_{\tilde{M}^\circ}(\check{\eta}(x))]\big)\big).& \end{aligned}$$

By Lemma 17.4 the composition of this representation with η is

$$\begin{aligned}
&\mathrm{Ind}_{X^*(\tilde T)\rtimes \mathcal{W}^H_{\check\eta(t)}}^{X^*(\tilde T)\rtimes \mathcal{W}^H} \Big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_H(t,x)/\mathrm{Z}(G))} \big(\rho, \\
&\qquad\qquad H_{d(x)}(\mathcal{B}^{\check\eta(x)}_{\tilde M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}_{\tilde M^\circ}(\check\eta(x))] \big) \Big) \cong \\
&\mathrm{Ind}_{X^*(\tilde T)\rtimes \mathcal{W}^H_{\check\eta(t)}}^{X^*(\tilde T)\rtimes \mathcal{W}^H} \Big(\mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}(\tilde G))} \big(\mathrm{Ind}_{\pi_0(\mathrm{Z}_H(t,x)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}(\tilde G))} \rho, \\
&\qquad\qquad H_{d(x)}(\mathcal{B}^{\check\eta(x)}_{\tilde M^\circ}, \mathbb{C}) \otimes \mathbb{C}[\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}_{\tilde M^\circ}(\check\eta(x))] \big) \Big) \cong \\
&\tau \big(\check\eta(t), \check\eta(x), \mathrm{Ind}_{\pi_0(\mathrm{Z}_H(t,x)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta(t),\check\eta(x))/\mathrm{Z}(\tilde G))} \rho \big).
\end{aligned}$$

Now it follows from Lemma 17.6 and Lemma 7.1 that

$$\phi_\eta^* \pi(t_q, x, \rho_q) \cong \pi \big(\check\eta(t_q), \check\eta(x), \mathrm{Ind}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \rho \big).$$

Next we induce this $\mathcal{H}(\tilde H')$-module to $\mathcal{H}(\tilde H)$:

$$\begin{aligned}
&\mathrm{ind}_{\mathcal{H}(\tilde H')}^{\mathcal{H}(\tilde H)} \pi \big(\check\eta(t_q), \check\eta(x), \mathrm{Ind}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \rho \big) = \\
&\mathrm{ind}_{\mathcal{H}(\tilde H')}^{\mathcal{H}(\tilde H)} \mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta\circ\Phi))} \big(\mathrm{Ind}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \rho, H_*(\mathcal{B}^{\check\eta(x)}_{\tilde H}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(\tilde H')] \big) \cong \\
&\mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \big(\mathrm{Ind}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H'}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \rho, H_*(\mathcal{B}^{\check\eta(x)}_{\tilde H}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(\tilde H)] \big) \cong \\
&\mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde H}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \big(\mathrm{Ind}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \rho, H_*(\mathcal{B}^{\check\eta(x)}_{\tilde H}, \mathbb{C}) \otimes \mathbb{C}[\pi_0(\tilde H)] \big) \cong \\
&\pi \big(\check\eta(t_q), \check\eta(x), \mathrm{Ind}_{\pi_0(\mathrm{Z}_H(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde H}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \rho \big).
\end{aligned}$$

From this we get to the $\tilde{\mathcal{G}}$-representation

$$\pi \big(\check\eta \circ \Phi, \mathrm{ind}_{\pi_0(\mathrm{Z}_G(\Phi)/\mathrm{Z}(G))}^{\pi_0(\mathrm{Z}_{\tilde G}(\check\eta\circ\Phi)/\mathrm{Z}(\tilde G))} \rho \big)$$

via (124) and (129). $\square$

Remark 17.8. *The above result establishes a precise version of Condition 16.2.(5) for* $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$. *It says that our local Langlands correspondence for principal series representations is functorial in the best sense that could be expected. Namely, if a L-parameter* Φ *for the principal series of a split group* $\mathcal{G}$ *is mapped to a L-parameter* $\tilde\Phi$ *for the principal series of another split group* $\tilde{\mathcal{G}}$ *by a homomorphism of reductive groups, then the L-packet* $\Pi_\Phi(\mathcal{G})$ *(intersected with the principal series of) can be transfered explicitly to the L-packet* $\Pi_{\tilde\Phi}(\tilde{\mathcal{G}}$ *(intersected with the principal series). Every member of* $\Pi_{\tilde\Phi}(\tilde{\mathcal{G}}$ *appears as an irreducible component of the transfer of at least one member of* $\Pi_{\tilde\Phi}(\tilde{\mathcal{G}}$. *The multiplicities for this can be given in terms of enhanced L-parameters, by simple expressions which should fit well with trace formulas.*

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