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The local Langlands correspondence for inner forms of SL_n

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Abstract

Let F be a non-archimedean local field. We establish the local Langlands correspondence for all inner forms of the group $\mathrm{SL}_n(F)$. It takes the form of a bijection between, on the one hand, conjugacy classes of Langlands parameters for $\mathrm{SL}_n(F)$ enhanced with an irreducible representation of an S-group and, on the other hand, the union of the spaces of irreducible admissible representations of all inner forms of $\mathrm{SL}_n(F)$ up to equivalence. An analogous result is shown in the archimedean case. For p-adic fields, this is based on the work of Hiraga and Saito. To settle the case where F has positive characteristic, we employ the method of close fields. We prove that this method is compatible with the local Langlands correspondence for inner forms of $\mathrm{GL}_n(F)$, when the fields are close enough compared to the depth of the representations.

Keywords: Representation theory, Local Langlands conjecture, Division algebra, Close fields

Mathematics Subject Classification: 20G05, 22E50

Contents

1 Background

Let F be a local field and let D be a division algebra with centre F, of dimension $d^2 \geq 1$ over F. Then $G = \mathrm{GL}_m(D)$ is the group of F-rational points of an inner form of GL_{md}. We will say simply that G is an inner form of $\mathrm{GL}_n(F)$, where $n = md$. It is endowed with a reduced norm map Nrd: $\mathrm{GL}_m(D) \to F^\times$. The group $G^\sharp := \ker(\mathrm{Nrd} : G \to F^\times)$ is an inner form of $\mathrm{SL}_n(F)$. (The split case $D = F$ is allowed here.) In this paper, we will complete the local Langlands correspondence for $G^\sharp$.

We sketch how it goes and which part of it is new. For any reductive group over a local field, say H, let $\mathrm{Irr}(H)$ denote the set of (isomorphism classes of) irreducible admissi-

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ble H-representations. Let $\Phi(H)$ be the collection of (equivalence classes of) Langlands parameters for H, as defined in [13].

The local Langlands correspondence (LLC) for $\mathrm{GL}_n(F)$ was established in the important papers [26,28,37,38,44,52]. Together with the Jacquet–Langlands correspondence [7,19,43], this provides the LLC for inner forms $G = \mathrm{GL}_m(D)$ of $\mathrm{GL}_n(F)$, see [4,29]. Recall that $\Phi(\mathrm{GL}_m(D)) \subsetneq \Phi(\mathrm{GL}_n(F))$ if $\mathrm{GL}_m(D)$ is not split. For these groups, every L-packet $\Pi_\phi(G)$ is a singleton and the LLC is a canonical bijective map

$$\mathrm{rec}_{D,m} : \mathrm{Irr}(\mathrm{GL}_m(D)) \to \Phi(\mathrm{GL}_m(D)). \tag{1}$$

The LLC for inner forms of $\mathrm{SL}_n(F)$ is derived from the above, in the sense that every L-packet for $G^\sharp$ consists of the irreducible constituents of $\mathrm{Res}^G_{G^\sharp}(\Pi_\phi(G))$. Of course these L-packets have more than one element in general. To parametrize the members of $\Pi_{\phi^\sharp}(G^\sharp)$, one must enhance the Langlands parameter $\phi^\sharp$ with an irreducible representation of a suitable component group. This idea originated for unipotent representations of p-adic reductive groups in [40, § 1.5]. For $\mathrm{SL}_n(F)$, $\phi^\sharp$ is a map from the Weil–Deligne group of F to $\mathrm{PGL}_n(\mathbb{C})$ and a correct choice is the group of components of the centralizer of $\phi^\sharp$ in $\mathrm{PGL}_n(\mathbb{C})$. Indeed, using this component group the enhanced Langlands correspondence for $\mathrm{SL}_n(F)$ was already established by Gelbart and Knapp [22, § 4]—at that time still assuming that it could be done for $\mathrm{GL}_n(F)$.

In general, more subtle component groups $\mathcal{S}_{\phi^\sharp}$ are needed, see [3,32,48]. Our enhanced L-parameters will be pairs $(\phi^\sharp, \rho)$ consisting of a Langlands parameter $\phi^\sharp$ for $G^\sharp$, enhanced with a $\rho \in \mathrm{Irr}(\mathcal{S}_{\phi^\sharp})$. We consider such enhanced L-parameters for $G^\sharp$ modulo the equivalence relation coming from the natural $\mathrm{PGL}_n(\mathbb{C})$-action. When two L-parameters $\phi^\sharp_1$ and $\phi^\sharp_2$ are conjugate, there is a canonical bijection $\mathrm{Irr}(\mathcal{S}_{\phi^\sharp_1}) \to \mathrm{Irr}(\mathcal{S}_{\phi^\sharp_2})$, coming from conjugation by any $g \in \mathrm{PGL}_n(\mathbb{C})$ with $g^{-1}\phi^\sharp_1 g = \phi^\sharp_2$. With this in mind, it makes sense to speak about $\mathrm{Irr}(\mathcal{S}_{\phi^\sharp})$ for $\phi^\sharp \in \Phi(G^\sharp)$.

The LLC for $G^\sharp$ should be an injective map

$$\mathrm{Irr}(G^\sharp) \to \left\{(\phi^\sharp, \rho) : \phi^\sharp \in \Phi(G^\sharp), \quad \rho \in \mathrm{Irr}(\mathcal{S}_{\phi^\sharp})\right\} \tag{2}$$

which satisfies several natural properties. The map will almost never be surjective, but for every $\phi^\sharp$ which is relevant for $G^\sharp$ the image should contain at least one pair $(\phi^\sharp, \rho)$. The image should consist of all such pairs, which satisfy an additional relevance condition on ρ. This form of the LLC was proven for "GL_n-generic" representations of $G^\sharp$ in [29], under the assumption that the underlying local field has characteristic zero.

A remarkable aspect of Langlands' conjectures [48] is that it is better to consider not just one reductive group at a time, but all inner forms of a given group simultaneously. Inner forms share the same Langlands dual group, so in (2) the right-hand side is the same for all inner forms H of the given group. The hope is that one can turn (2) into a bijection by defining a suitable equivalence relation on the set of inner forms and taking the corresponding union of the sets $\mathrm{Irr}(H)$ on the left-hand side. Such a statement was proven for unipotent representations of simple p-adic groups in [41].

Let us make this explicit for inner forms of $\mathrm{GL}_n(F)$, respectively, $\mathrm{SL}_n(F)$. We define the equivalence classes of such inner forms to be in bijection with the isomorphism classes of central simple F-algebras of dimension n^2 via $\mathrm{M}_m(D) \mapsto \mathrm{GL}_m(D)$, respectively, $\mathrm{M}_m(D) \mapsto \mathrm{GL}_m(D)_{\mathrm{der}}$. This equivalence relation can also be motivated with Galois cohomology, see Sect. 2.

As Langlands dual group, we take $\mathrm{GL}_n(\mathbb{C})$, respectively, $\mathrm{PGL}_n(\mathbb{C})$. Langlands parameters for $\mathrm{GL}_n(F)$ (respectively, $\mathrm{SL}_n(F)$) take values in the dual group, and they must be considered up to conjugation. The group with which we want to conjugate Langlands parameters should be a central extension of the adjoint group $\mathrm{PGL}_n(\mathbb{C})$, but apart from that there is some choice. It does not matter for the equivalence classes of Langlands parameters, but it is important for the component group of centralizers that we will obtain. The interpretation of inner forms via Galois cohomology entails [3] that we must consider the conjugation action of the simply connected group $\mathrm{SL}_n(\mathbb{C})$ on the dual groups and on the collections of Langlands parameters for $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$.

For any Langlands parameter $\phi^\sharp$ for $\mathrm{SL}_n(F)$, we define the groups

$$\begin{aligned} C(\phi^\sharp) &= Z_{\mathrm{SL}_n(\mathbb{C})}(\mathrm{im}\,\phi^\sharp),\\ \mathcal{S}_{\phi^\sharp} &= C(\phi^\sharp)/C(\phi^\sharp)^\circ,\\ \mathcal{Z}_{\phi^\sharp} &= Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi^\sharp)^\circ \cong Z(\mathrm{SL}_n(\mathbb{C}))C(\phi^\sharp)^\circ/C(\phi^\sharp)^\circ. \end{aligned} \tag{3}$$

Notice that the centralizers are taken in $\mathrm{SL}_n(\mathbb{C})$ and not in the Langlands dual group $\mathrm{PGL}_n(\mathbb{C})$, where the image of $\phi^\sharp$ lies. It is worth noting that our group $C(\phi^\sharp)$ coincides with the group $S^+_{\phi^\sharp}$ defined by Kaletha in [32] with Z taken to be equal to the centre of $\mathrm{SL}_n(F)$.

More often one encounters the component group

$$S_{\phi^\sharp} := Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im}\,\phi^\sharp)/Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im}\,\phi^\sharp)^\circ.$$

It related to (3) by the short exact sequence

$$1 \to \mathcal{Z}_{\phi^\sharp} \to \mathcal{S}_{\phi^\sharp} \to S_{\phi^\sharp} \to 1.$$

Given a Langlands parameter ϕ for $\mathrm{GL}_n(F)$, we can define $C(\phi)$, $\mathcal{S}_\phi$ and $\mathcal{Z}_\phi$ by the same formulas as in (3). Again $C(\phi)$ coincides with the group that Kaletha considers. It is easily seen that $Z_{\mathrm{GL}_n(\mathbb{C})}(\mathrm{im}\,\phi)$ is connected, so $\mathcal{S}_\phi \cong \mathcal{Z}_\phi$. The usual component group S_ϕ is always trivial for $\mathrm{GL}_n(F)$.

Hence, $\mathcal{S}_\phi$ (resp. $\mathcal{S}_{\phi^\sharp}$) has more irreducible representations than S_ϕ (resp. $S_{\phi^\sharp}$). Via the Langlands correspondence, the additional ones are associated with irreducible representations of non-split inner forms of $\mathrm{GL}_n(F)$ (resp. $\mathrm{SL}_n(F)$). For example, consider a Langlands parameter ϕ for $\mathrm{GL}_2(F)$ which is elliptic, that is, whose image is not contained in any torus of $\mathrm{GL}_2(\mathbb{C})$. Then $\mathcal{S}_\phi = Z(\mathrm{SL}_2(\mathbb{C})) \cong \{\pm 1\}$. The pair $(\phi, \mathrm{triv}_{\mathcal{S}_\phi})$ parametrizes an essentially square-integrable representation of $\mathrm{GL}_2(F)$ and $(\phi, \mathrm{sgn}_{\mathcal{S}_\phi})$ parametrizes an irreducible representation of the non-split inner form $D^\times$, where D denotes a noncommutative division algebra of dimension 4 over F.

For general linear groups over local fields, we prove a result which was already known to experts, but which we could not find in the literature:

Theorem 1.1 (see Theorem 2.2) *There is a canonical bijection between:*

- *pairs (G, π) with $\pi \in \mathrm{Irr}(G)$ and G an inner form of $\mathrm{GL}_n(F)$, considered up to equivalence;*
- *pairs (ϕ, ρ) with $\phi \in \Phi(\mathrm{GL}_n(F))$ and $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$.*

For these Langlands parameters, $\mathcal{S}_\phi = \mathcal{Z}_\phi$ and a character of $\mathcal{Z}_\phi$ determines an inner form of $\mathrm{GL}_n(F)$ via the Kottwitz isomorphism [36]. In contrast to the usual LLC, our packets for general linear groups need not be singletons. To be precise, the packet Π_ϕ

contains the unique representation $\mathrm{rec}_{D,m}^{-1}(\phi)$ of $G = \mathrm{GL}_m(D)$ if ϕ is relevant for G, and no G-representations otherwise.

A similar result holds for special linear groups, but with a few modifications. Firstly, one loses canonicity, because in general there is no natural way to parametrize the members of an L-packet $\Pi_{\phi^\sharp}(G^\sharp)$ if there are more than one. Already for tempered representations of $\mathrm{SL}_2(F)$, the enhanced LLC is not canonical, see [6, Example 11.3]. It is possible to determine a unique parametrization of $\mathrm{Irr}(\mathrm{SL}_n(F))$ by fixing a Whittaker datum and of $\mathrm{Irr}(G^\sharp)$ by adding more data [32], but this involves non-canonical choices.

Secondly, the quaternion algebra $\mathbb{H}$ turns out to occupy an exceptional position. Our local Langlands correspondence for inner forms of the special linear group over a local field F can be stated as follows:

Theorem 1.2 (see Theorems 3.3 and 3.4)
There exists a correspondence between:

- *pairs* $(G^\sharp, \pi)$ *with* $\pi \in \mathrm{Irr}(G^\sharp)$ *and* $G^\sharp$ *an inner form of* $\mathrm{SL}_n(F)$*, considered up to equivalence;*
- *pairs* $(\phi^\sharp, \rho)$ *with* $\phi^\sharp \in \Phi(\mathrm{SL}_n(F))$ *and* $\rho \in \mathrm{Irr}(\mathcal{S}_{\phi^\sharp})$*,*

which is almost bijective, the only exception being that pairs $(\mathrm{SL}_{n/2}(\mathbb{H}), \pi)$ *correspond to two parameters* $(\phi^\sharp, \rho_1)$ *and* $(\phi^\sharp, \rho_2)$*.*

(a) *The group* $G^\sharp$ *determines* $\rho\big|_{\mathcal{Z}_{\phi^\sharp}}$ *and conversely.*
(b) *The correspondence satisfies the desired properties from* [13, § 10.3]*, with respect to restriction from inner forms of* $\mathrm{GL}_n(F)$*, temperedness and essential square integrability of representations.*

This theorem supports more general conjectures on L-packets and the LLC for non-split groups, cf. [3, § 3] and [32, § 5.4].

In the archimedean case, the classification of $\mathrm{Irr}(\mathrm{SL}_m(D))$ is well known, at least for $D \neq \mathbb{H}$. The main value of our result lies in the strong analogy with the non-archimedean case. The reason for the lack of bijectivity for the special linear groups over the quaternions is easily identified. Namely, the reduced norm map for $\mathbb{H}$ satisfies $\mathrm{Nrd}(\mathbb{H}^\times) = \mathbb{R}_{>0}$, whereas for all other local division algebras D with centre F the reduced norm map is surjective, that is, $\mathrm{Nrd}(D^\times) = F^\times$. Of course, there are various ad hoc ways to restore the bijectivity in Theorem 1.2, for example by decreeing that $\mathrm{SL}_m(\mathbb{H})$ appears twice among the equivalence classes of inner forms of $\mathrm{SL}_{2m}(\mathbb{R})$. This can be achieved in a natural way with strong inner forms, as in [1]. But one may also argue that for $\mathrm{SL}_m(\mathbb{H})$ one would actually be better off without any component groups.

For p-adic fields F, the above theorem can be derived rather quickly from the work of Hiraga and Saito [29].

By far the most difficult case of Theorem 1.2 is that where the local field F has positive characteristic. The paper [29] does not apply in this case, and it seems hard to generalize the techniques from [29] to fields of positive characteristic.

Our solution is to use the method of close fields to reduce it to the p-adic case. Let F be a local field of characteristic p, $\mathfrak{o}_F$ its ring of integers and $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$. There exist finite extensions $\widetilde{F}$ of $\mathbb{Q}_p$ which are l-close to F, which means that $\mathfrak{o}_F/\mathfrak{p}_F^l$ is isomorphic to the corresponding ring for $\widetilde{F}$. Let $\widetilde{D}$ be a division algebra with centre $\widetilde{F}$, such that D and $\widetilde{D}$

have the same Hasse invariant. Let K_r be the standard congruence subgroup of level $r \in \mathbb{N}$ in $\mathrm{GL}_m(\mathfrak{o}_D)$, and let $\mathrm{Irr}(G, K_r)$ be the set of irreducible representations of $G = \mathrm{GL}_m(D)$ with nonzero K_r-invariant vectors. Define $\widetilde{K}_r \subset \mathrm{GL}_m(\widetilde{D})$ and $\mathrm{Irr}(\mathrm{GL}_m(\widetilde{D}), \widetilde{K}_r)$ in the same way.

For l sufficiently large compared to r, the method of close fields provides a bijection

$$\mathrm{Irr}(\mathrm{GL}_m(D), K_r) \to \mathrm{Irr}(\mathrm{GL}_m(\widetilde{D}), \widetilde{K}_r) \tag{4}$$

which preserves almost all the available structure [7]. But this is not enough for Theorem 1.2; we also need to relate to the local Langlands correspondence. The l-closeness of F and $\widetilde{F}$ implies that the quotient of the Weil group of F by its l-th ramification subgroup is isomorphic to the analogous object for $\widetilde{F}$ [18]. This yields a natural bijection

$$\Phi_l(\mathrm{GL}_m(D)) \to \Phi_l(\mathrm{GL}_m(\widetilde{D})) \tag{5}$$

between Langlands parameters that are trivial on the respective l-th ramification groups. We show that:

Theorem 1.3 (see Theorems 6.1 and 6.2)
Suppose that F and $\widetilde{F}$ are l-close and that l is sufficiently large compared to r. Then the maps (1), (4) and (5) form a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}(\mathrm{GL}_m(D), K_r) & \to & \mathrm{Irr}(\mathrm{GL}_m(\widetilde{D}), \widetilde{K}_r) \\ \downarrow & & \downarrow \\ \Phi_l(\mathrm{GL}_m(D)) & \to & \Phi_l(\mathrm{GL}_m(\widetilde{D})). \end{array}$$

In the special case $D = F$ and $\widetilde{D} = \widetilde{F}$, this holds for all $l > 2^{n-1}r$.

The special case was also proven by Ganapathy [20,21], but without an explicit lower bound on l.

Theorem 1.3 says that the method of close fields essentially preserves Langlands parameters. The proof runs via the only accessible characterization of the LLC for general linear groups: by means of ϵ- and γ-factors of pairs of representations [27].

To apply Henniart's characterization with maximal effect, we establish a result with independent value. Given a Langlands parameter ϕ, we let $d(\phi) \in \mathbb{R}_{\geq 0}$ be the smallest number such that $\phi \notin \Phi_{d(\phi)}(\mathrm{GL}_n(F))$. That is, the smallest number such that ϕ is nontrivial on the $d(\phi)$-th ramification group of the Weil group of F with respect to the upper numbering. For a supercuspidal representation π of $\mathrm{GL}_n(F)$, let $d(\pi)$ be its normalized level, as in [14].

Proposition 1.4 (see Proposition 4.2) *The local Langlands correspondence for supercuspidal representations of $\mathrm{GL}_n(F)$ preserves depths, in the sense that*

$$d(\pi) = d(\mathrm{rec}_{F,n}(\pi)).$$

2 Inner forms of $\mathrm{GL}_n(F)$

Let F be a local field and let D be a division algebra with centre F, of dimension $\dim_F(D) = d^2$. The F-group $\mathrm{GL}_m(D)$ is an inner form of $\mathrm{GL}_{md}(F)$, and conversely every inner form of $\mathrm{GL}_n(F)$ is isomorphic to such a group.

In the archimedean case, there are only three possible division algebras: $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. The group $\mathrm{GL}_m(\mathbb{H})$ is an inner form of $\mathrm{GL}_{2m}(\mathbb{R})$, and (up to isomorphism) that already accounts for all the inner forms of the groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. One can parametrize these inner forms with characters of order at most two of $Z(\mathrm{SL}_n(\mathbb{C}))$, such that $\mathrm{GL}_n(F)$ is associated with the trivial character and

$$\mathrm{GL}_m(\mathbb{H}) \text{ corresponds to the character of order two of } Z(\mathrm{SL}_{2m}(\mathbb{C})). \tag{6}$$

Until further notice we assume that F is non-archimedean. Let us make our equivalence relation on the set of inner forms of $\mathrm{GL}_n(F)$ precise. Following [29], we consider inner twists of GL_n, that is, pairs $(\mathcal{G}, \xi)$ where $\mathcal{G}$ is an inner form of GL_n and $\xi : \mathrm{GL}_n \to \mathcal{G}$ is an isomorphism defined over $\bar{F}$, such that $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ is an inner automorphism of GL_n for every σ in the absolute Galois group of F. Equivalence classes of inner twists are classified by the first Galois cohomology group of F with values in the adjoint group of GL_n, $H^1(F, \mathrm{PGL}_n)$.

For a more explicit interpretation, we note that $H^1(F, \mathrm{PGL}_n)$ also parametrizes the isomorphism classes of central simple F-algebras of dimension n^2. In other words, we have defined that our equivalence classes of inner forms of $\mathrm{GL}_n(F)$ are in bijection with the isomorphism classes of central simple F-algebras of dimension n^2 via $\mathrm{M}_m(D) \mapsto \mathrm{GL}_m(D)$. By [36, Proposition 6.4], there exists a natural bijection

$$H^1(F, \mathrm{PGL}_n) \to \mathrm{Irr}\big(Z(\mathrm{SL}_n(\mathbb{C}))\big) = \mathrm{Irr}\big(\{z \in \mathbb{C}^\times : z^n = 1\}\big). \tag{7}$$

In fact, such a map exists on general grounds [36, § 6.5], in characteristic 0 it is bijective by [33, Satz 2] and in positive characteristic by [47]. Clearly, the map

$$\mathrm{Irr}\left(\left\{z \in \mathbb{C}^\times : z^n = 1\right\}\right) \to \left\{z \in \mathbb{C}^\times : z^n = 1\right\} : \chi \mapsto \chi(\exp(2\pi\sqrt{-1}/n))$$

is bijective. The composition of these two maps can also be interpreted in terms of classical number theory. For $\mathrm{M}_m(D)$ with $md = n$, the Hasse invariant $h(D)$ (in the sense of Brauer theory) is a primitive d-th root of unity. The element of $H^1(F, \mathrm{PGL}_n)$ associated with $\mathrm{M}_m(D)$ has the same image $h(D)$ in $\{z \in \mathbb{C}^\times : z^n = 1\}$. In particular, $1 \in \mathbb{C}^\times$ is associated with $\mathrm{M}_n(F)$ and the primitive n-th roots of unity correspond to division algebras of dimension n^2 over their centre F.

We warn the reader that our equivalence relation on the set of inner forms of $\mathrm{GL}_n(F)$ is rougher than isomorphism. Namely, if $h(D') = h(D)^{-1}$, then $\mathrm{M}_m(D')$ is isomorphic to the opposite algebra of $\mathrm{M}_m(D)$ and

$$\mathrm{GL}_m(D) \to \mathrm{GL}_m(D^{\mathrm{op}}) \cong \mathrm{GL}_m(D') : x \mapsto x^{-T}$$

is a group isomorphism. Since $x \mapsto x^{-T}$ is the only nontrivial outer automorphism of GL_n, all isomorphisms between groups $\mathrm{GL}_m(D)$ arise in this way (up to inner automorphisms).

Furthermore, there is a standard presentation of the division algebras D, which is especially useful in relation to the method of close fields. Let L be the unique unramified extension of F of degree d, and let χ be the character of $\mathrm{Gal}(L/F) \cong \mathbb{Z}/d\mathbb{Z}$, which sends the Frobenius automorphism to $h(D)$. If ϖ_F is a uniformizer of F, then D is isomorphic to the cyclic algebra $[L/F, \chi, \varpi_F]$, see Definition IX.4.6 and Corollary XII.2.3 of [50]. We will call a group of the form

$$\mathrm{GL}_m\big([L/F, \chi, \varpi_F]\big) \tag{8}$$

a standard inner form of $\mathrm{GL}_n(F)$.

The local Langlands correspondence for $G = \mathrm{GL}_m(D)$ has been known to experts for considerable time, although it did not appear in the literature until recently [4,29].

We need to understand it well for our later arguments, so we recall its construction. It generalizes and relies on the LLC for general linear groups:

$$\mathrm{rec}_{F,n} : \mathrm{Irr}(\mathrm{GL}_n(F)) \to \Phi(\mathrm{GL}_n(F)).$$

The latter was proven for supercuspidal representations in [26,28,38] and extended from there to $\mathrm{Irr}(\mathrm{GL}_n(F))$ in [52].

As G is an inner form of $\mathrm{GL}_n(F)$, $\check{G} = \mathrm{GL}_n(\mathbb{C})$ and the action of $\mathrm{Gal}(\overline{F}/F)$ on $\mathrm{GL}_n(\mathbb{C})$ determined by G is by inner automorphisms. Therefore, we may take as Langlands dual group ${}^L G = \check{G} = \mathrm{GL}_n(\mathbb{C})$.

Let us recall the notion of relevance of Langlands parameters for non-split groups. Let $\phi \in \Phi(\mathrm{GL}_n(F))$, and let $\check{M} \subset \mathrm{GL}_n(\mathbb{C})$ be a Levi subgroup that contains $\mathrm{im}(\phi)$ and is minimal for this property. As for all Levi subgroups,

$$\check{M} \cong \mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C}) \tag{9}$$

for some integers n_i with $\sum_{i=1}^k n_i = n$. Then ϕ is relevant for G if and only if $\check{M}$ corresponds to a Levi subgroup $M \subset G$. This is equivalent to $m_i := n_i/d$ being an integer for all i. Moreover, in that case

$$M \cong \mathrm{GL}_{m_i}(D) \times \cdots \times \mathrm{GL}_{m_k}(D). \tag{10}$$

By definition, $\Phi(G)$ consists of the G-relevant elements of $\Phi(\mathrm{GL}_n(F))$. Consider any $\phi \in \Phi(G)$. Conjugating by a suitable element of $\check{G}$, we can achieve that

- $\check{M} = \prod_{i=1}^l \mathrm{GL}_{n_i}(\mathbb{C})^{e_i}$ and $M = \prod_{i=1}^l \mathrm{GL}_{m_i}(D)^{e_i}$ are standard Levi subgroups of $\mathrm{GL}_n(C)$ and $\mathrm{GL}_m(D)$, respectively;
- $\phi = \prod_{i=1}^l \phi_i^{\otimes e_i}$ with $\phi_i \in \Phi(\mathrm{GL}_{m_i}(D))$ and $\mathrm{im}(\phi_i)$ not contained in any proper Levi subgroup of $\mathrm{GL}_{n_i}(\mathbb{C})$;
- ϕ_i and ϕ_j are not equivalent if $i \neq j$.

Then $\mathrm{rec}_{F,n_i}^{-1}(\phi_i) \in \mathrm{Irr}(\mathrm{GL}_{n_i}(F))$ is essentially square-integrable. Recall that the Jacquet–Langlands correspondence [7,19,43] is a natural bijection

$$\mathrm{JL} : \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D)) \to \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_n(F))$$

between essentially square-integrable irreducible representations of $G = \mathrm{GL}_m(D)$ and $\mathrm{GL}_n(F)$. It gives

$$\begin{aligned} \omega_i &:= \mathrm{JL}^{-1}\left(\mathrm{rec}_{F,n_i}^{-1}(\phi_i)\right) \in \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_{m_i}(D)), \\ \omega &:= \prod_{i=1}^{l} \omega_i^{\otimes e_i} \in \mathrm{Irr}_{\mathrm{ess}L^2}(M). \end{aligned} \tag{11}$$

We remark that ω is square-integrable modulo centre if and only all $\mathrm{rec}_{F,n_i}^{-1}(\phi_i)$ are so, because this property is preserved by the Jacquet–Langlands correspondence. The Zelevinsky classification for $\mathrm{Irr}(\mathrm{GL}_{n_i}(F))$ [52] (which is used for rec_{F,n_i}) shows that, in the given circumstances, this is equivalent to ϕ_i being bounded. Thus, ω is square-integrable modulo centre if and only ϕ is bounded.

The assignment $\phi \mapsto (M, \omega)$ sets up a bijection

$$\Phi(G) \longleftrightarrow \left\{(M, \omega) : M \text{ a Levi subgroup of } G, \omega \in \mathrm{Irr}_{\mathrm{ess}L^2}(M)\right\}/G. \tag{12}$$

It is known from [19, Theorem B.2.d] and [8] that for inner forms of $\mathrm{GL}_n(F)$ normalized parabolic induction sends irreducible square-integrable (modulo centre) representations to irreducible tempered representations and that every irreducible tempered representation can be obtained in that way. This shows that the tempered part of $\mathrm{Irr}(G)$ is naturally in bijection with the tempered part (i.e. ω tempered) of the right-hand side of (12).

Now we analyse the non-tempered part of $\mathrm{Irr}(G)$. Let M_1 be a Levi subgroup of G containing M such that ω is square-integrable modulo the centre of M_1, and such that M_1 is maximal for this property. Let P_1 be a parabolic subgroup of M_1 with Levi factor M. Then $\omega_1 = I_{P_1}^{M_1}(\omega)$ is irreducible and independent of P_1, while by the aforementioned results the restriction of ω_1 to the derived group of M_1 is tempered. Furthermore, the absolute value of the character of ω_1 on $Z(M_1)$ is regular in the sense that no root of $(G, Z(M_1))$ annihilates it. Hence, there exists a unique parabolic subgroup P_2 of G with Levi factor M_1, such that (P_2, ω_1) satisfies the hypothesis of the Langlands classification [34,37]. That result says that $I_{P_2}^{G}(\omega_1)$ has a unique irreducible quotient $L(P_2, \omega_1)$ and that every irreducible G-representation can be obtained in this way, from data that are unique up to G-conjugation. This provides a canonical bijection between $\mathrm{Irr}(G)$ and the right-hand side of (12).

To summarize the above constructions, let P be a parabolic subgroup of G with Levi factor M, such that $PM_1 = P_2$. By the transitivity of parabolic induction, $I_P^G(\omega)$ has a unique irreducible quotient, say $L(P, \omega)$, and it is isomorphic to $L(P_2, \omega_1)$. The composite map

$$\begin{array}{ccccc} \Phi(G) & & \to & & \mathrm{Irr}(G) \\ \phi & \mapsto & (M, \omega) & \mapsto & L(P, \omega) = L(P_2, \omega_1) \end{array} \tag{13}$$

is the local Langlands correspondence for $\mathrm{GL}_m(D)$. Since it is bijective, all the L-packets $\Pi_\phi(G) = \{L(P, \omega)\}$ are singletons.

By construction, $L(P, \omega)$ is essentially square-integrable if and only if $M = P = G$, which happens precisely when the image of ϕ is not contained in any proper Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$. By the uniqueness part of the Langlands classification [34, Theorem 3.5.ii] $L(P, \omega)$ is tempered if and only if ω is square-integrable modulo centre, which by the above is equivalent to boundedness of $\phi \in \Phi(G)$.

We denote the inverse of (13) by

$$\mathrm{rec}_{D,m} : \mathrm{Irr}(G) \to \Phi(G). \tag{14}$$

Because both the LLC for $\mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_{n_i}(F))$, the Jacquet–Langlands correspondence and I_P^G respect tensoring with unramified characters, $\mathrm{rec}_{D,m}(L(P, \omega \otimes \chi))$ and $\mathrm{rec}_{D,m}(L(P, \omega))$ differ only by the unramified Langlands parameter for M which corresponds to χ.

In the archimedean case, Langlands [37] himself established the correspondence between the irreducible admissible representations of $\mathrm{GL}_m(D)$ and Langlands parameters. The paper [37] applies to all real reductive groups, but it completes the classification only if parabolic induction of tempered representations of Levi subgroups preserves irreducibility. That is the case for $\mathrm{GL}_n(\mathbb{C})$ by the Borel–Weil theorem and for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_m(\mathbb{H})$ by [11, § 12].

The above method to go from essentially square-integrable to irreducible admissible representations is essentially the same over all local fields and stems from [37]. There also exists a Jacquet–Langlands correspondence over local archimedean fields [19, Appendix D]. Actually it is very simple, the only nontrivial cases are $\mathrm{GL}_2(\mathbb{R})$ and $\mathbb{H}$. Therefore, it is justified to say that (11)–(14) hold in the archimedean case.

With the S-groups from [3], we can build a more subtle version of (14). Since $Z_{\mathrm{GL}_n(\mathbb{C})}(\phi)$ is connected,

$$\begin{aligned}\mathcal{S}_\phi &= C(\phi)/C(\phi)^\circ = Z(\mathrm{SL}_n(\mathbb{C}))Z_{\mathrm{SL}_n(\mathbb{C})}(\phi)^\circ/Z_{\mathrm{SL}_n(\mathbb{C})}(\phi)^\circ \\ &\cong Z(\mathrm{SL}_n(\mathbb{C}))/\big(Z(\mathrm{SL}_n(\mathbb{C}))\cap Z_{\mathrm{SL}_n(\mathbb{C})}(\phi)^\circ\big).\end{aligned} \tag{15}$$

Let $\chi_G \in \mathrm{Irr}\big(Z(\mathrm{SL}_n(\mathbb{C}))\big)$ be the character associated with G via (7) or (6).

Lemma 2.1 *A Langlands parameter $\phi \in \Phi(\mathrm{GL}_n(F))$ is relevant for $G = \mathrm{GL}_m(D)$ if and only if* $\ker \chi_G \supset Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi)^\circ$.

Proof This can be derived with [2, Corollary 2.2] and [29, Lemma 9.1]. However, we prefer a more elementary proof.

Replacing ϕ by a suitable $\mathrm{GL}_n(\mathbb{C})$-conjugate L-parameter, we may assume that a Levi subgroup minimally containing ϕ is

$$L = \mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C}), \quad \text{where } n_1 + \cdots + n_k = n.$$

As explained in (10), ϕ is relevant for $\mathrm{GL}_m(D)$ if and only d divides every n_j.

Let us determine $Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi)^\circ$. The factor of ϕ in $\mathrm{GL}_{n_j}(\mathbb{C})$ is an irreducible n_j-dimensional representation of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$, so any element of $Z_{\mathrm{GL}_n(\mathbb{C})}(\mathrm{im}\,\phi)$ must also centralize $\mathrm{GL}_{n_j}(\mathbb{C}) \subset L$. As $Z_{\mathrm{GL}_n(\mathbb{C})}(L) = L$, we obtain

$$Z_{\mathrm{GL}_n(\mathbb{C})}(\mathrm{im}\,\phi)^\circ = Z_L(\mathrm{im}\,\phi)^\circ = Z(\mathrm{GL}_{n_1}(\mathbb{C})) \times \cdots \times Z(\mathrm{GL}_{n_k}(\mathbb{C})).$$

The determinant of a typical element $\big(e^{z_1}I_{n_1}, \ldots, e^{z_k}I_{n_k}\big)$ is $\exp(n_1z_1 + \cdots + n_kz_k)$. Hence, the Lie algebra of $Z_{\mathrm{SL}_n(\mathbb{C})}(\mathrm{im}\,\phi)$ is determined by the equation $n_1z_1 + \cdots + n_kz_k = 0$ and

$$C(\phi)^\circ = Z_{\mathrm{SL}_n(\mathbb{C})}(\mathrm{im}\,\phi)^\circ = \big\{\big(e^{z_1}I_{n_1}, \ldots, e^{z_k}I_{n_k}\big)\,|z_i \in \mathbb{C}, \quad n_1z_1 + \cdots + n_kz_k = 0\big\}.$$

For any integer l, we have $\big(e^{z_1}I_{n_1}, \ldots, e^{z_k}I_{n_k}\big) = e^{2\pi i l/n}I_n$ if and only if $\frac{z_j}{2\pi i} \in \mathbb{Z} + \frac{l}{n}$ for all j. This lies in $C(\phi)^\circ$ if and only if there are integers l_j such that

$$\sum_{j=1}^k n_j(l/n + l_j) = l + \sum_{j=1}^k n_jl_j$$

is zero. That is only possible if l is a multiple of the greatest common divisor g of the n_j. Hence, $Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi)^\circ$ is generated by $e^{2\pi i g/n} = \exp(2\pi i/n)^g$.

Reconsider $\mathrm{GL}_m(D)$ as above. As discussed after (7), the character $\chi_{\mathrm{GL}_m(D)}$ has order d. In particular, its kernel consists of the d-th powers in $Z(\mathrm{SL}_n(\mathbb{C}))$. Now we can conclude with some equivalences:

$$\begin{aligned}&Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi)^\circ \subset \ker \chi_{\mathrm{GL}_m(D)} \\ &\quad\Longleftrightarrow d \text{ divides } g \\ &\quad\Longleftrightarrow d\mathbb{Z} \supset g\mathbb{Z} = n_1\mathbb{Z} + \cdots + n_k\mathbb{Z} \\ &\quad\Longleftrightarrow d \text{ divides } n_j \text{ for all } j \\ &\quad\Longleftrightarrow \phi \text{ is relevant for } \mathrm{GL}_m(D).\end{aligned}$$

□

We regard

$$\Phi^e(\mathrm{inn}\,\mathrm{GL}_n(F)) := \big\{(\phi, \rho) : \phi \in \Phi(\mathrm{GL}_n(F)), \quad \rho \in \mathrm{Irr}(\mathcal{S}_\phi)\big\}$$

as the collection of enhanced Langlands parameters for all inner forms of $\mathrm{GL}_n(F)$. With this set, we can establish the local Langlands correspondence for all such inner forms simultaneously. To make it bijective, we must choose one group in each equivalence class of inner forms of $\mathrm{GL}_n(F)$. In the archimedean case it suffices to say that we use the quaternions, and in the non-archimedean case we take the standard inner forms (8).

Theorem 2.2 *Let F be a local field. There exists a canonical bijection*

$$\begin{aligned} \Phi^e(\mathrm{inn}\ \mathrm{GL}_n(F)) &\to \big\{(G,\pi) : G \text{ standard inner form of } \mathrm{GL}_n(F), \pi \in \mathrm{Irr}(G)\big\}, \\ (\phi, \chi_G) &\mapsto (G, \Pi_\phi(G)). \end{aligned}$$

It extends the local Langlands correspondence for $GL_n(F)$, which can be recovered by considering only pairs $(\phi, \mathrm{triv}_{\mathcal{S}_\phi})$.

Proof The elements of $\Phi^e(\mathrm{inn}\ \mathrm{GL}_n(F))$ with a fixed $\phi \in \Phi(\mathrm{GL}_n(F))$ are

$$\big\{(\phi, \chi) : \chi \in \mathrm{Irr}\big(Z(\mathrm{SL}_n(\mathbb{C}))\big), \ker \chi \supset Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi)^\circ\big\}. \tag{16}$$

First, we consider the non-archimedean case. By Lemma 2.1 and (7), (16) is in bijection with the equivalence classes of inner forms of $\mathrm{GL}_n(F)$ for which ϕ is relevant. In each such equivalence class, there is a unique standard inner form $G = \mathrm{GL}_m([L/F, \chi, \varpi_F])$. Hence, $\Phi^e(\mathrm{inn}\ \mathrm{GL}_n(F))$ contains precisely one element for every pair (G, ϕ) with G a standard inner form of $GL_n(F)$ and ϕ a Langlands parameter, which is relevant for G. Now we apply the LLC for G (13) and define that the map in the theorem sends (ϕ, χ) to $\Pi_\phi(G) := \mathrm{rec}^{-1}_{[L/F,\chi,\varpi_F],m}(\phi) \in \mathrm{Irr}(G)$. As $\mathrm{rec}_{[L/F,\chi,\varpi_F],m}$ is a canonical bijection, so is the thus obtained map.

In the archimedean case, the above argument does not suffice, because some characters of $Z(\mathrm{SL}_n(\mathbb{C}))$ do not parametrize an inner form of $\mathrm{GL}_n(F)$. We proceed by direct calculation, inspired by [37, § 3].

Suppose that $F = \mathbb{C}$. Then $\mathbf{W}_F = \mathbb{C}^\times$ and $\mathrm{im}(\phi)$ is just a real torus in $\mathrm{GL}_n(\mathbb{C})$. Hence, $Z_{\mathrm{GL}_n(\mathbb{C})}(\phi)$ is a Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$ and $C(\phi) = Z_{\mathrm{SL}_n(\mathbb{C})}(\phi)$ is the corresponding Levi subgroup of $\mathrm{SL}_n(\mathbb{C})$. All Levi subgroups of $\mathrm{SL}_n(\mathbb{C})$ are connected, so $\mathcal{S}_\phi = C(\phi)/C(\phi)^\circ = 1$. Consequently, $\Phi^e(\mathrm{inn}\ \mathrm{GL}_n(\mathbb{C})) = \Phi(\mathrm{GL}_n(\mathbb{C}))$, and the theorem for $F = \mathbb{C}$ reduces to the Langlands correspondence for $\mathrm{GL}_n(\mathbb{C})$.

Now we take $F = \mathbb{R}$. Recall that its Weil group is defined as

$$\mathbf{W}_\mathbb{R} = \mathbb{C}^\times \cup \mathbb{C}^\times \tau, \quad \text{where } \tau^2 = -1 \quad \text{and} \quad \tau z \tau^{-1} = \bar{z}.$$

Let M be a Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$ which contains the image of ϕ and is minimal for this property. Then $\phi(\mathbb{C}^\times)$ is contained in a unique maximal torus T of M. By replacing ϕ by a conjugate Langlands parameter, we can achieve that

$$M = \prod_{i=1}^n \mathrm{GL}_i(\mathbb{C})^{n_i}$$

is standard and that T is the torus of diagonal matrices. Then the projection of $\phi(\mathbf{W}_\mathbb{R})$ on each factor $\mathrm{GL}_i(\mathbb{C})$ of M has a centralizer in $\mathrm{GL}_i(\mathbb{C})$, which does not contain any torus larger than $Z(\mathrm{GL}_i(\mathbb{C}))$. On the other hand, $\phi(\tau)$ normalizes $Z_M(\phi(\mathbb{C}^\times)) = T$, so $Z_{GL_n(\mathbb{C})}(\phi) = Z_T(\phi(\tau))$. It follows that $n_i = 0$ for $i \geq 2$.

The projection of $\phi(\tau)$ on each factor $\mathrm{GL}_2(\mathbb{C})$ of M is either $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Hence, $Z_{\mathrm{GL}_n(\mathbb{C})}(\phi)$ contains the torus

$$T_\phi := (\mathbb{C}^\times)^{n_1} \times Z(\mathrm{GL}_2(\mathbb{C}))^{n_2}.$$

Suppose that $n_1 > 0$. Then the intersection $T_\phi \cap \mathrm{SL}_n(\mathbb{C})$ is connected, so

$$Z(\mathrm{SL}_n(\mathbb{C}))/\big(Z(\mathrm{SL}_n(\mathbb{C})) \cap Z_{\mathrm{SL}_n(\mathbb{C})}(\phi)^\circ\big) = 1.$$

Together with (15) this shows that $\mathcal{S}_\phi = 1$ if n is odd or if n is even and ϕ is not relevant for $\mathrm{GL}_{n/2}(\mathbb{H})$.

Now suppose $n_1 = 0$. Then $n = 2n_2$, ϕ is relevant for $\mathrm{GL}_{n_2}(\mathbb{H})$ and $T_\phi = \big\{(z_j I_2)_{j=1}^{n_2} : z_j \in \mathbb{C}^\times\big\}$. We see that $T_\phi \cap \mathrm{SL}_n(\mathbb{C})$ has two components, determined by whether $\prod_{j=1}^{n_2} z_j$ equals 1 or -1. Write $\phi = \prod_{j=1}^{n_2} \phi_j$ with $\phi_j \in \Phi(\mathrm{GL}_2(\mathbb{R}))$. We may assume that ϕ is normalized such that, whenever ϕ_j is $\mathrm{GL}_2(\mathbb{C})$-conjugate to $\phi_{j'}$, actually $\phi_{j'} = \phi_j = \phi_k$ for all k between j and j'. Then $Z_{\mathrm{M}_n(\mathbb{C})}(\phi)$ is isomorphic to a standard Levi subalgebra A of $\mathrm{M}_{n_2}(\mathbb{C})$, via the ring homomorphism

$$\mathrm{M}_{n_2}(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}) = \mathrm{M}_{n_2}(\mathrm{M}_2(\mathbb{C})) \text{ induced by } z \mapsto zI_2.$$

Hence, $Z_{\mathrm{SL}_n(\mathbb{C})}(\phi) \cong \{a \in A : \det(a)^2 = 1\}$, which clearly has two components. This shows that $|\mathcal{S}_\phi| = [C(\phi) : C(\phi)^\circ] = 2$ if ϕ is relevant for $\mathrm{GL}_{n/2}(\mathbb{H})$.

Thus, we checked that for every $\phi \in \Phi(\mathrm{GL}_n(\mathbb{R}))$, $\mathrm{Irr}(\mathcal{S}_\phi)$ parametrizes the equivalence classes of inner forms G of $\mathrm{GL}_n(\mathbb{R})$ for which ϕ is relevant. To conclude, we apply the LLC for G. □

3 Inner forms of $\mathrm{SL}_n(F)$

As in the previous section, let D be a division algebra over dimension d^2 over its centre F, with reduced norm Nrd: $D \to F$. We write

$$\mathrm{GL}_m(D)^\sharp := \big\{g \in \mathrm{GL}_m(D) : \mathrm{Nrd}(g) = 1\big\}$$

Notice that it equals the derived group of $\mathrm{GL}_m(D)$. It is an inner form of $\mathrm{SL}_{md}(F)$, and every inner form of $\mathrm{SL}_n(F)$ is isomorphic to such a group. We use the same equivalence relation and parametrization for inner forms of $\mathrm{SL}_n(F)$ as for $\mathrm{GL}_n(F)$, as described by (7) and (6).

As Langlands dual group of $G^\sharp = \mathrm{GL}_m(D)^\sharp$, we take

$${}^L G^\sharp = \check{G}^\sharp = \mathrm{PGL}_n(\mathbb{C}).$$

In particular, every Langlands parameter for $G = \mathrm{GL}_m(D)$ gives rise to one for $G^\sharp$. In line with [13, § 10], the L-packets for $G^\sharp$ are derived from those for G in the following way. It is known [49] that every $\phi^\sharp \in \Phi(G^\sharp)$ lifts to a $\phi \in \Phi(G)$. The L-packet $\Pi_\phi(G)$ from (13) consists of a single G-representation, which we will denote by the same symbol. Its restriction to $G^\sharp$ depends only on $\phi^\sharp$, because a different lift ϕ' of $\phi^\sharp$ would produce $\Pi_{\phi'}(G)$, which only differs from $\Pi_\phi(G)$ by a character of the form

$$g \mapsto |\mathrm{Nrd}(g)|_F^z \quad \text{with } z \in \mathbb{C}.$$

We call the restriction of $\Pi_\phi(G)$ to $G^\sharp$ $\pi_\phi(G)^\sharp$. In general it is reducible, and with it one associates the L-packet

$$\Pi_{\phi^\sharp}(G^\sharp) := \big\{\pi^\sharp \in \mathrm{Irr}(G^\sharp) : \pi^\sharp \text{ is a constituent of } \pi_\phi(G)^\sharp\big\}.$$

The goal of this section is an analogue of Theorem 2.2. First we note that every irreducible $G^\sharp$-representation (say π) is a member of an L-packet $\Pi_{\phi^\sharp}(G^\sharp)$, because it appears in a G-representation (for example in $\mathrm{Ind}_{G^\sharp}^{G}\pi$).

Lemma 3.1 [29, Lemma 12.1], see also [22, Theorem 4.1] when $G^\sharp = \mathrm{SL}_n(F)$.

Two L-packets $\Pi_{\phi_1^\sharp}(G^\sharp)$ and $\Pi_{\phi_2^\sharp}(G^\sharp)$ are either disjoint or equal, and the latter happens if and only if $\phi_1^\sharp$ and $\phi_2^\sharp$ are $\mathrm{PGL}_n(\mathbb{C})$-conjugate (i.e. equal in $\Phi(G^\sharp)$).

Thus, the main problem is the parametrization of the L-packets. Such a parametrization of $\Pi_{\phi^\sharp}(G^\sharp)$ was given in [29] in terms of S-groups, at least when F has characteristic zero and $\Pi_\phi(G)$ is "GL_n-generic". After recalling this method, we will generalize it. Put

$$X^G(\Pi_\phi(G)) = \{\gamma \in \mathrm{Irr}(G/G^\sharp) : \Pi_\phi(G) \otimes \gamma \cong \Pi_\phi(G)\}.$$

Notice that every element of $X^G(\Pi_\phi(G))$ is a character, which by Schur's lemma is trivial on $Z(G)$. Since $G/G^\sharp Z(G)$ is an abelian group and all its elements have order dividing n, the same goes for $X^G(\Pi_\phi(G))$. Moreover, $X^G(\Pi_\phi(G))$ is finite, as we will see in (21). On general grounds [29, Lemma 2.4], there exists a 2-cocycle $\kappa_{\phi^\sharp}$ such that

$$\mathbb{C}[X^G(\Pi_\phi(G)), \kappa_{\phi^\sharp}] \cong \mathrm{End}_{G^\sharp}(\Pi_\phi(G)). \tag{17}$$

More explicitly, by [29, Lemma 2.4] there are elements

$$I_\gamma \in \mathrm{Hom}_G(\Pi_\phi(G), \quad \Pi_\phi(G) \otimes \gamma) \quad \text{for } \gamma \in X^G(\Pi_\phi(G)), \tag{18}$$

which together form a basis of $\mathrm{End}_{G^\sharp}(\Pi_\phi(G))$. The map $\gamma \mapsto I_\gamma$ is in general not multiplicative, but it is multiplicative up to scalars, and that accounts for the cocycle $\kappa_{\phi^\sharp}$. By [29, Corollary 2.10], the decomposition of $\pi_\phi(G)^\sharp$ as a representation of $G^\sharp \times X^G(\Pi_\phi(G))$ is

$$\pi_\phi(G)^\sharp \cong \bigoplus_{\rho \in \mathrm{Irr}(\mathbb{C}[X^G(\Pi_\phi(G)), \kappa_{\phi^\sharp}])} \mathrm{Hom}_{\mathbb{C}[X^G(\Pi_\phi(G)), \kappa_{\phi^\sharp}]}(\rho, \pi_\phi(G)^\sharp) \otimes \rho. \tag{19}$$

Let us compare $C(\phi^\sharp)$ with $C(\phi) = Z_{\mathrm{SL}_n(\mathbb{C})}(\phi)$. Since the kernel of the projection $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ is central and $\phi^\sharp$ is the projection of ϕ, the subgroups $C(\phi) \subset C(\phi^\sharp)$ of $\mathrm{SL}_n(\mathbb{C})$ have the same Lie algebra. Consequently,

$$C(\phi) = Z(\mathrm{SL}_n(\mathbb{C}))C(\phi)^\circ = Z(\mathrm{SL}_n(\mathbb{C}))C(\phi^\sharp)^\circ.$$

We also note that

$$\begin{aligned} &C(\phi^\sharp)/C(\phi) \cong \mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}, \quad \text{where} \\ &\mathcal{Z}_{\phi^\sharp} = Z(\mathrm{SL}_n(\mathbb{C}))C(\phi^\sharp)^\circ/C(\phi^\sharp)^\circ \cong Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi^\sharp)^\circ. \end{aligned} \tag{20}$$

Assume for the moment that $D \not\cong \mathbb{H}$, so Nrd: $D \to F$ is surjective by [50, Proposition X.2.6]. Let $\hat\gamma : \mathbf{W}_F \to \mathbb{C}^\times \cong Z(\mathrm{GL}_n(\mathbb{C}))$ correspond to $\gamma \in \mathrm{Irr}(F^\times) \cong \mathrm{Irr}(G/G^\sharp)$ via local class field theory. By the LLC for G, ϕ is $\mathrm{GL}_n(\mathbb{C})$-conjugate to $\phi\hat\gamma$ for all $\gamma \in X^G(\Pi_\phi(G))$. As $(\phi\hat\gamma)^\sharp = \phi^\sharp$, ϕ and $\phi\hat\gamma$ are in fact conjugate by an element of $C(\phi^\sharp) \subset \mathrm{SL}_n(\mathbb{C})$. This gives an isomorphism

$$C(\phi^\sharp)/C(\phi) \cong X^G(\Pi_\phi(G)), \tag{21}$$

showing in particular that the left-hand side is abelian. Since $C(\phi^\sharp)/C(\phi)$ is the component group of the centralizer of the subset $\mathrm{im}(\phi^\sharp)$ of the algebraic group $\mathrm{PGL}_n(\mathbb{C})$, the groups in (21) are finite. Thus, we obtain a central extension of finite groups

$$1 \to \mathcal{Z}_{\phi^\sharp} \to \mathcal{S}_{\phi^\sharp} \to X^G(\Pi_\phi(G)) \to 1. \tag{22}$$

The algebra (17) can be described with the idempotent

$$e_{\chi_G} := |\mathcal{Z}_{\phi^\sharp}|^{-1} \sum_{z \in \mathcal{Z}_{\phi^\sharp}} \chi_G(z^{-1})z \ \in \ \mathbb{C}[\mathcal{Z}_{\phi^\sharp}].$$

Theorem 3.2 *Let $G = \mathrm{GL}_m(D)$ with $D \not\cong \mathbb{H}$. There exists an isomorphism*

$$\mathbb{C}\left[X^G(\Pi_\phi(G)), \kappa_{\phi^\sharp}\right] = \mathbb{C}[\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}, \kappa_{\phi^\sharp}] \cong e_{\chi_G}\mathbb{C}[\mathcal{S}_{\phi^\sharp}]$$

such that for any $s \in \mathcal{S}_{\phi^\sharp}$ the subspaces $\mathbb{C}s\mathcal{Z}_{\phi^\sharp}$ on both sides correspond. Moreover, any two such isomorphisms differ only by a character of $\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}$.

Proof First we determine the difference of two such isomorphisms. Their composition is an algebra automorphism of $\mathbb{C}[\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}, \kappa_{\phi^\sharp}]$, which preserves each of the subspaces $\mathbb{C}s\mathcal{Z}_{\phi^\sharp}$. The multiplication rules

$$s\mathcal{Z}_{\phi^\sharp} \cdot s'\mathcal{Z}_{\phi^\sharp} = \kappa_{\phi^\sharp}(s, s')ss'\mathcal{Z}_{\phi^\sharp}$$

in this algebra show that the automorphism is the $\mathbb{C}$-linear extension of $s\mathcal{Z}_{\phi^\sharp} \mapsto \lambda(s)s\mathcal{Z}_{\phi^\sharp}$ for a character λ of $\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}$.

Now we suppose that $\mathrm{char}(F) = 0$ and that the representation $\Pi_\phi(G)$ is tempered. In the archimedean case, the cocycle $\kappa_{\phi^\sharp}$ is trivial by [29, Lemma 3.1 and p. 69]. In the non-archimedean case, the theorem is a reformulation of [29, Lemma 12.5]. We remark that this is a deep result, and its proof makes use of endoscopic transfer and global arguments.

Next we consider the case where $\mathrm{char}(F) = 0$ and we have a possibly unbounded Langlands parameter $\phi^\sharp \in \Phi(G^\sharp)$, with a lift $\phi \in \Phi(G)$. Let Y be a connected set of unramified twists ϕ_χ of ϕ, such that $C(\phi_\chi) = C(\phi)$ and $C(\phi^\sharp_\chi) = C(\phi^\sharp)$ for all $\phi_\chi \in Y$. It is easily seen that we can always arrange that Y contains bounded Langlands parameters, confer [5, Proposition 3.2]. The reason is that for any element (here the image of a Frobenius element of $\mathbf{W}_F$ under ϕ) of a torus in a complex reductive group, there is an element of the maximal compact subtorus which has the same centralizer.

The construction of the intertwining operators

$$I_\gamma \in \mathrm{Hom}_G(\Pi_\phi(G), \Pi_\phi(G) \otimes \gamma), \quad \gamma \in X^G(\Pi_\phi(G))$$

from (18) is similar to that for R-groups. It determines the 2-cocycle $\kappa_{\phi^\sharp}$ by

$$I_\gamma I_{\gamma'} = \kappa_{\phi^\sharp}(\gamma, \gamma')I_{\gamma\gamma'}.$$

The I_γ can be chosen independently of $\chi \in X_{nr}(M)$, so the $\kappa_{\phi^\sharp_\chi}$ do not depend on χ. For $\phi^\sharp_\chi$ tempered, we already have the required algebra isomorphisms, and now they extend by constancy to all $\phi^\sharp_\chi \in Y$. This concludes the proof in the case $\mathrm{char}(F) = 0$. □

The proof of the case $\mathrm{char}(F) > 0$ requires more techniques; we dedicate Sects. 4–6 to it.

For a character χ of $\mathcal{Z}_{\phi^\sharp}$ or of $Z(\mathrm{SL}_n(\mathbb{C}))$, we write

$$\mathrm{Irr}(\mathcal{S}_{\phi^\sharp}, \chi) := \mathrm{Irr}\big(e_\chi\mathbb{C}[\mathcal{S}_{\phi^\sharp}]\big) = \{(\pi, V) \in \mathrm{Irr}(\mathcal{S}_{\phi^\sharp}) : \mathcal{Z}_{\phi^\sharp} \text{ acts on } V \text{ as } \chi\}. \tag{23}$$

We will use this with the characters $\chi_G = \chi_{G^\sharp}$ from Lemma 2.1.

We still assume that $D \not\cong \mathbb{H}$. As shown in [29, Corollary 2.10], the isomorphism (17) and Theorem 3.2 imply that

$$\pi(\phi^\sharp, \rho) := \mathrm{Hom}_{\mathcal{S}_{\phi^\sharp}}(\rho, \Pi_\phi(G)) \tag{24}$$

defines an irreducible $G^\sharp$-representation for every $\rho \in \mathrm{Irr}(\mathcal{S}_{\phi^\sharp}, \chi_{G^\sharp})$. More precisely, (24) determines a bijection

$$\mathrm{Irr}(\mathcal{S}_{\phi^\sharp}, \chi_{G^\sharp}) \to \big\{[\pi] \in \mathrm{Irr}(G^\sharp) : \pi \text{ is a constituent of } \Pi_\phi(G)\big\}, \tag{25}$$

where $[\pi]$ denotes the isomorphism class of π. In general $\pi(\phi^\sharp, \rho)$ is not canonical, it depends on the choice of an algebra isomorphism as in Theorem 3.2. Hence, the map $\rho \mapsto \pi(\phi^\sharp, \rho)$ is canonical up to an action of

$$\mathrm{Irr}\big(\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}\big) \cong \mathrm{Irr}\big(X^G(\Pi_\phi(G))\big)$$

on $\mathrm{Irr}\big(e_{\chi_G}\mathbb{C}[\mathcal{S}_{\phi^\sharp}]\big)$. Via (19) and Theorem 3.2, this corresponds to an action of $\mathrm{Irr}\big(X^G(\Pi_\phi(G))\big)$ on $\Pi_{\phi^\sharp}(G)$, which can be described explicitly. Since $X^G(\Pi_\phi(G))$ is a subgroup of $\mathrm{Irr}\big(G/G^\sharp Z(G)\big)$, $\mathrm{Irr}\big(X^G(\Pi_\phi(G))\big)$ is a quotient of $G/G^\sharp Z(G)$, say G/H for some $H \supset G^\sharp Z(G)$. This means that every $c \in \mathrm{Irr}\big(X^G(\Pi_\phi(G))\big)$ determines a coset $g_c H$ in G. Now the formula

$$c \cdot \pi = g_c \cdot \pi, \quad \text{where } (g_c \cdot \pi)(g) = \pi\left(g_c^{-1} g g_c\right) \tag{26}$$

defines the action of $\mathrm{Irr}\big(X^G(\Pi_\phi(G))\big)$ on $\Pi_{\phi^\sharp}(G^\sharp)$. In other words, the representation $\pi(\phi^\sharp, \rho) \in \Pi_{\phi^\sharp}(G^\sharp)$ is canonical up to the action of G on $G^\sharp$-representations. Since $\Pi_\phi(G)$ is irreducible, the action of G on $\Pi_{\phi^\sharp}(G^\sharp)$ is in fact transitive, which means that with suitable choices one can arrange that $\pi(\phi^\sharp, \rho)$ is any element of this L-packet.

For $D = \mathbb{H}$, some modifications must be made. In that case, $G = G^\sharp Z(G)$, so $\mathrm{Res}^G_{G^\sharp}$ preserves irreducibility of representations and $X^G(\Pi_\phi(G)) = 1$. Moreover, $G/G^\sharp \cong \mathbb{R}^\times_{>0} \not\cong \mathbb{R}^\times$, which causes (21) and (22) to be invalid for $D = \mathbb{H}$. However, (23) still makes sense, so we define

$$\pi(\phi^\sharp, \rho) := \Pi_\phi(\mathrm{GL}_m(\mathbb{H})) \quad \text{for all} \quad \rho \in \mathrm{Irr}\big(\mathcal{S}_{\phi^\sharp}, \chi_{\mathbb{H}^\times}\big). \tag{27}$$

As mentioned before, Hiraga and Saito [29] have established the local Langlands correspondence for irreducible "GL_n-generic" representations of inner forms of $\mathrm{SL}_n(F)$, where F is a local field of characteristic zero. We will generalize this on the one hand to local fields F of arbitrary characteristic and on the other hand to all irreducible admissible representations. We will do so for all inner forms of $\mathrm{SL}_n(F)$ simultaneously, to obtain an analogue of Theorem 2.2.

Like for $\mathrm{GL}_n(F)$ we define

$$\Phi^e(\mathrm{inn}\ \mathrm{SL}_n(F)) = \big\{(\phi^\sharp, \rho) : \phi^\sharp \in \Phi(\mathrm{SL}_n(F)), \quad \rho \in \mathrm{Irr}(\mathcal{S}_{\phi^\sharp})\big\}.$$

Notice that the restriction of ρ to $\mathcal{Z}_{\phi^\sharp} \cong Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi^\sharp)^\circ$ determines an inner form G_ρ of $\mathrm{GL}_n(F)$ (up to equivalence) via (7) and Lemma 2.1. Its derived group $G_\rho^\sharp$ is the inner form of $\mathrm{SL}_n(F)$ associated with ρ.

We note that the actions of $\mathrm{PGL}_n(\mathbb{C})$ on the various $\Phi^e(G^\sharp)$ combine to an action on $\Phi^e(\mathrm{inn}\ \mathrm{SL}_n(F))$. With the collection of equivalence classes $\Phi^e(\mathrm{inn}\ \mathrm{SL}_n(F))$, we can formulate the local Langlands correspondence for all such inner forms simultaneously.

First we consider the non-archimedean case. As for $\mathrm{GL}_n(F)$, we fix one group in every equivalence class of inner forms. We choose the groups $\mathrm{GL}_m([L/F, \chi, \varpi_F])^\sharp$ with $[L/F, \chi, \varpi_F]$ as in (8) and call these the standard inner forms of $\mathrm{SL}_n(F)$.

Theorem 3.3 *Let F be a non-archimedean local field. There exists a bijection*

$$\begin{array}{lll} \Phi^e(\mathrm{inn}\ \mathrm{SL}_n(F)) & \to & \big\{(G^\sharp, \pi) : G^\sharp \textit{ standard inner form of } \mathrm{SL}_n(F), \pi \in \mathrm{Irr}(G^\sharp)\big\} \\ (\phi^\sharp, \rho) & \mapsto & \big(G_\rho^\sharp, \pi(\phi^\sharp, \rho)\big) \end{array}$$

with the following properties:

(a) *Suppose that* ρ *sends* $\exp(2\pi i/n) \in Z(\mathrm{SL}_n(\mathbb{C}))$ *to a primitive d-th root of unity z. Then* $G^\sharp_\rho = \mathrm{GL}_m([L/F, \chi, \varpi_F])^\sharp$*, where* $md = n$ *and* $\chi : \mathrm{Gal}(L/F) \to \mathbb{C}^\times$ *sends the Frobenius automorphism to z.*

(b) *Suppose that* $\phi^\sharp$ *is relevant for* $G^\sharp$ *and lifts to* $\phi \in \Phi(G)$*. Then the restriction of* $\Pi_\phi(G)$ *to* $G^\sharp$ *is* $\bigoplus_{\rho\in\mathrm{Irr}(\mathcal{S}_{\phi^\sharp},\chi_{G^\sharp})} \pi(\phi^\sharp,\rho)\otimes\rho$*.*

(c) $\pi(\phi^\sharp,\rho)$ *is essentially square-integrable if and only if* $\phi^\sharp(\mathbf{W}_F\times\mathrm{SL}_2(\mathbb{C}))$ *is not contained in any proper parabolic subgroup of* $\mathrm{PGL}_n(\mathbb{C})$*).*

(d) $\pi(\phi^\sharp,\rho)$ *is tempered if and only if* $\phi^\sharp$ *is bounded.*

Proof Let $\phi^\sharp \in \Phi(\mathrm{SL}_n(F))$ and lift it to $\phi \in \Phi(\mathrm{GL}_n(F))$. Then $C(\phi^\sharp)^\circ = C(\phi)^\circ$ and $\mathcal{Z}_{\phi^\sharp} = \mathcal{Z}_\phi$, so

$$\mathcal{Z}_{\phi^\sharp} = \mathcal{Z}_\phi = Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C}))\cap C(\phi)^\circ = Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C}))\cap C(\phi^\sharp)^\circ.$$

By this and Lemma 2.1, the set of standard inner forms of $\mathrm{SL}_n(F)$ for which $\phi^\sharp$ is relevant is in natural bijection with

$$\mathrm{Irr}(\mathcal{Z}_{\phi^\sharp}) = \mathrm{Irr}\big(Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C}))\cap C(\phi^\sharp)^\circ\big).$$

Hence, the collection of $(\phi^\sharp,\rho)\in\Phi^e(\mathrm{inn}\ \mathrm{SL}_n(F))$ with $\phi^\sharp$ fixed is

$$\big\{(\phi^\sharp,\rho) : \rho\in\mathrm{Irr}(\mathcal{S}_{\phi^\sharp},\chi_{G^\sharp}) \text{ with } \phi^\sharp \text{ relevant for } G^\sharp\big\}. \tag{28}$$

The description of $G^\sharp_\rho$ in part (a) is a reformulation of (7) and our choice of standard inner forms of $SL_n(F)$. By (25), the map of the theorem provides a bijection between (28) and the set of pairs $(G^\sharp,\pi)$ where $G^\sharp$ is a standard inner form of $SL_n(F)$ for which $\phi^\sharp$ is relevant and $\pi\in\mathrm{Irr}(G^\sharp)$ is a constituent of $\Pi_\phi(G)$. Now we see from Lemma 3.1 and Theorem 2.2 that the map of the theorem is bijective.

Part (b) is a consequence of (17) and Theorem 3.2, see [29, Corollary 2.10]. Parts (c) and (d) follow from the analogous statements for inner forms of $\mathrm{GL}_n(F)$ (which were discussed after (13)) in combination with [46, Proposition 2.7]. □

Let us formulate an archimedean analogue of Theorem 3.3, that is, for the groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_m(\mathbb{H})$. In view (27), we cannot expect a bijection, and part (b) has to be adjusted.

Theorem 3.4 *Let F be* $\mathbb{R}$ *or* $\mathbb{C}$*. There exists a canonical surjection*

$$\begin{array}{lll}\Phi^e(\mathrm{inn}\ \mathrm{SL}_n(F)) & \to & \big\{(G^\sharp,\pi) : G^\sharp \textit{ standard inner form of } \mathrm{SL}_n(F), \pi\in\mathrm{Irr}(G^\sharp)\big\}\\ (\phi^\sharp,\rho) & \mapsto & \big(G^\sharp_\rho,\pi(\phi^\sharp,\rho)\big)\end{array}$$

with the following properties:

(a) *The preimage of* $\mathrm{Irr}(\mathrm{SL}_n(F))$ *consists of the* $(\phi^\sharp,\rho)$ *with* $\mathcal{Z}_{\phi^\sharp}\subset\ker\rho$*, and the map is injective on this domain. The preimage of* $\mathrm{Irr}(\mathrm{SL}_{n/2}(\mathbb{H}))$ *consists of the* $(\phi^\sharp,\rho)$ *such that* ρ *is not trivial on* $\mathcal{Z}_{\phi^\sharp}$*, and the map is two-to-one on this domain.*

(b) *Suppose that* $\phi^\sharp$ *is relevant for* $G^\sharp = \mathrm{SL}_m(D)$ *and lifts to* $\phi\in\Phi(G)$*. Then the restriction of* $\Pi_\phi(G)$ *to* $G^\sharp$ *is irreducible if* $D=\mathbb{C}$ *or* $D=\mathbb{H}$ *and is isomorphic to* $\bigoplus_{\rho\in\mathrm{Irr}(\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp})}\pi(\phi^\sharp,\rho)\otimes\rho$ *in case* $D=\mathbb{R}$*.*

(c) $\pi(\phi^\sharp,\rho)$ *is essentially square-integrable if and only if* $\phi^\sharp(\mathbf{W}_F)$ *is not contained in any proper parabolic subgroup of* $\mathrm{PGL}_n(\mathbb{C})$*).*

(d) $\pi(\phi^\sharp,\rho)$ *is tempered if and only if* $\phi^\sharp$ *is bounded.*

Proof Theorem 2.2 and the start of the proof of Theorem 3.3 show that (28) is also valid in the archimedean case. To see that the map thus obtained is canonical, we will of course use that the LLC for $\mathrm{GL}_m(D)$ is so. For $\mathrm{SL}_n(F)$, the intertwining operators admit a canonical normalization in terms of Whittaker functionals [29, pp. 17 and 69], so the definition (24) of $\pi(\phi^\sharp, \rho)$ can be made canonical. For $\mathrm{SL}_m(\mathbb{H})$, the definition (27) clearly leaves no room for arbitrary choices.

Part (a) and part (b) for $D = \mathbb{R}$ follow as in the non-archimedean case, except that for $D = \mathbb{H}$ the preimage of $\pi(\phi^\sharp, \rho)$ is in bijection with $\mathrm{Irr}(\mathcal{S}_\phi, e_{\mathbb{H}^\times})$. To prove part (b) for $D = \mathbb{C}$ and $D = \mathbb{H}$, it suffices to remark that $\mathrm{Res}^G_{G^\sharp}$ preserves irreducibility, as $G = G^\sharp Z(G)$. The proof of parts (c) and (d) carries over from Theorem 3.3.

It remains to check that the map is two-to-one on $\mathrm{Irr}(\mathrm{SL}_m(\mathbb{H}))$. For this, we have to compute

$$\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp} = C(\phi^\sharp)/C(\phi^\sharp)^\circ = C(\phi^\sharp)/C(\phi). \tag{29}$$

Consider $\phi^\sharp \in \Phi(\mathrm{SL}_m(\mathbb{H}))$ with two lifts $\phi, \phi' \in \Phi(\mathrm{GL}_m(\mathbb{H}))$ that are conjugate under $\mathrm{GL}_{2m}(\mathbb{C})$. The restriction of $\phi^{-1}\phi'$ to $\mathbb{C}^\times \subset \mathbf{W}_\mathbb{R}$ is a group homomorphism $c : \mathbb{C}^\times \to Z(\mathrm{GL}_{2m}(\mathbb{C}))$. Clearly ϕ and ϕ' can only be conjugate if $c = 1$, so ϕ' can only differ from ϕ on $\tau \in \mathbf{W}_\mathbb{R}$. Since

$$\phi'(\tau)^2 = \phi'(-1) = \phi(-1) = \phi(\tau)^2,$$

either $\phi'(\tau) = -\phi(\tau)$ or $\phi' = \phi$. Recall the standard form of ϕ exhibited in the proof of Theorem 2.2, with image in the Levi subgroup $\mathrm{GL}_2(\mathbb{C})^m$ of $\mathrm{GL}_{2m}(\mathbb{C})$. It shows that the Langlands parameter ϕ' determined by $\phi'(\tau) = -\phi(\tau)$ is always conjugate to ϕ, for example by the element $\mathrm{diag}(1, -1, 1, \ldots, -1) \in \mathrm{GL}_{2m}(\mathbb{C})$. Therefore, (29) has precisely two elements. Now $e_{\mathbb{H}^\times}\mathbb{C}[\mathcal{S}_{\phi^\sharp}]$ is a two-dimensional semisimple $\mathbb{C}$-algebra, so it is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. We conclude that $\mathrm{Irr}(\mathcal{S}_{\phi^\sharp}, e_{\mathbb{H}^\times})$ has two elements, for every $\phi^\sharp \in \Phi(\mathrm{SL}_m(\mathbb{H}))$.

□

4 Characterization of the LLC for some representations of $\mathrm{GL}_n(F)$

In this section, F is any local non-archimedean field. It is known from [27] that generic representations of $\mathrm{GL}_n(F)$ can be characterized in terms of γ-factors of pairs, where the other part of the pair is a representation of a smaller general linear group. We will establish a more precise version for irreducible representations that have nonzero vectors fixed under a specific compact open subgroup.

Let F_s be a separable closure of F, and let $\mathrm{Gal}(F_s/F)^l$ be the l-th ramification group of $\mathrm{Gal}(F_s/F)$, with respect to the upper numbering. We define

$$\Phi_l(G) := \left\{\phi \in \Phi(G) : \mathrm{Gal}(F_s/F)^l \subset \ker(\phi)\right\}.$$

Notice that

$$\Phi_{l'}(G) \subset \Phi_l(G), \quad \text{if } l' \leq l.$$

It is known that the set of l's at which $\mathrm{Gal}(F_s/F)^l$ jumps consists of rational numbers and is discrete [45, Chap. IV, § 3]. In particular, there exists a unique rational number $d(\phi)$ such that

$$\phi \notin \Phi_{d(\phi)}(\mathrm{GL}_n(F)) \quad \text{and} \quad \phi \in \Phi_l(\mathrm{GL}_n(F)) \quad \text{for any } l > d(\phi). \tag{30}$$

We will say that $\phi \in \Phi(\mathrm{GL}_n(F))$ is *elliptic* if its image is not contained in any proper Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$. Recall the Swan conductor $\mathrm{swan}(\phi)$ from [25, § 2].

Lemma 4.1 *Let $\phi \in \Phi(\mathrm{GL}_n(F))$, such that ϕ is elliptic and $\mathrm{SL}_2(\mathbb{C}) \subset \ker(\phi)$. Then*

$$d(\phi) = \begin{cases} 0 & \text{if } \mathbf{I}_F \subset \ker(\phi), \\ \mathrm{swan}(\phi)/n & \text{otherwise.} \end{cases} \tag{31}$$

Proof Recall that the filtration of $\mathrm{Gal}(F_s/F)$ with the lower numbering only has jumps at integer values of the index. Let $c(\phi)$ denote the smallest integer such that

$$\mathrm{Gal}(F_s/F)_{c(\phi)+1} \subset \ker(\phi),$$

if $\mathbf{I}_F = \mathrm{Gal}(F_s/F)_0$ is not contained in $\ker(\phi)$, and -1 otherwise. Recall the Herbrand function $\varphi_{F_s/F}$ [45, Chap. IV, § 3] that allows us to pass from the lower number to the upper ones:

$$\mathrm{Gal}(F_s/F)_l = \mathrm{Gal}(F_s/F)^{\varphi_{F_s/F}(l)}.$$

Let $a(\phi)$ denote the Artin conductor of ϕ. Because ϕ is assumed to be elliptic, the restriction of ϕ to $\mathbf{W}_F$ is irreducible. The equality

$$a(\phi) = n\left(\varphi_{F_s/F}(c(\phi)) + 1\right)$$

was shown for $n = 1$ in [45, Chap. VI, § 2, Proposition 5]. The proof for arbitrary n is similar, see [25, § 2]. By the very definition of the Swan conductor

$$\varphi_{F_s/F}(c(\phi)) = \frac{a(\phi)}{n} - 1 = \frac{\mathrm{swan}(\phi)}{n}.$$

Then it follows from the definition of $c(\phi)$ that $d(\phi)$ is the largest rational number such that

$$\mathrm{Gal}(F_s/F)^{d(\phi)} \not\subset \ker(\phi).$$

□

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{M}_n(F)$. Let $\mathfrak{P}$ denote the Jacobson radical of $\mathfrak{A}$, and let $e(\mathfrak{A})$ denote the $\mathfrak{o}_F$-period of $\mathfrak{A}$, that is, the integer e defined by $\mathfrak{p}_F\mathfrak{A} = \mathfrak{P}^e$. Define a sequence of compact open subgroups of $\mathrm{GL}_n(F)$ by

$$U^0(\mathfrak{A}) = \mathfrak{A}^\times, \quad \text{and} \quad U^m(\mathfrak{A}) = 1 + \mathfrak{P}^m, \quad m \geq 1.$$

Let m, m' be integers satisfying $m > m' \geq \lfloor m/2 \rfloor$. There is a canonical isomorphism

$$U^{m'+1}(\mathfrak{A})/U^{m+1}(\mathfrak{A}) \to \mathfrak{P}^{m'+1}/\mathfrak{P}^{m+1},$$

given by $x \mapsto x - 1$. This leads to an isomorphism from $\mathfrak{p}^{-1}\mathfrak{P}^{-m}/\mathfrak{p}^{-1}\mathfrak{P}^{-m'}$ to the Pontryagin dual of $U^{m'+1}(\mathfrak{A})/U^{m+1}(\mathfrak{A})$, explicitly given by

$$\beta + \mathfrak{p}^{-1}\mathfrak{P}^{-m'} \mapsto \psi_\beta \qquad \beta \in \mathfrak{p}^{-1}\mathfrak{P}^{-m},$$

with $\psi_\beta(1+x) = (\psi \circ \mathrm{tr}_{\mathrm{M}_n(F)})(\beta x)$, for $x \in \mathfrak{P}^{-m'}$.

We recall from [16, (1.5)] that a *stratum* is a quadruple $[\mathfrak{A}, m, m', \beta]$ consisting of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{M}_n(F)$, integers $m > m' \geq 0$, and an element $\beta \in \mathrm{M}_n(F)$ with $\mathfrak{A}$-valuation $\nu_{\mathfrak{A}}(\beta) \geq -m$. A stratum of the form $[\mathfrak{A}, m, m-1, \beta]$ is called *fundamental* [16, (2.3)] if the coset $\beta + \mathfrak{p}^{-1}\mathfrak{P}^{1-m}$ does not contain a nilpotent element of $\mathrm{M}_n(F)$. We remark that the formulation in [14] is slightly different because the notion of a fundamental stratum there allows m to be 0.

Fix an irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$. According to [14, Theorem 2], there exists a hereditary order $\mathfrak{A}$ in $\mathrm{M}_n(F)$ such that either

(a) π contains the trivial character of $U^1(\mathfrak{A})$, or
(b) there is a fundamental stratum $[\mathfrak{A}, m, m-1, \beta]$ in $\mathrm{M}_n(F)$ such that π contains the character ψ_β of $U^m(\mathfrak{A})$.

Moreover, in case (b), if a stratum $[\mathfrak{A}_1, m_1, m_1-1, \beta_1]$ is such that β_1 occurs in the restriction of π to $U^{m_1}(\mathfrak{A}_1)$, then $m_1/e(\mathfrak{A}_1) \geq m/e(\mathfrak{A})$, and we have equality here if and only $[\mathfrak{A}_1, m_1, m_1-1, \beta_1]$ is fundamental [14, Theorem 2′].

The above provides a useful invariant of the representation, called the *depth* (or *normalized level*) of π. It is defined as

$$d(\pi) := \min\left\{m/e(\mathfrak{A})\right\}, \tag{32}$$

where $(m, \mathfrak{A})$ ranges over all pairs consisting of an integer $m \geq 0$ and a hereditary $\mathfrak{o}_F$-order in $\mathrm{M}_n(F)$ such that π contains the trivial character of $U^{m+1}(\mathfrak{A})$.

The following result was claimed in [51, Theorem 2.3.6.4]. Although Yu did not provide a proof, he indicated that an argument along similar lines as ours is possible.

Proposition 4.2 *Let $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F))$ be supercuspidal and $\phi := \mathrm{rec}_{F,n}(\pi)$. Then*

$$d(\phi) = d(\pi).$$

Proof We have

$$\epsilon(s, \phi, \psi) = \epsilon(0, \phi, \psi)\, q^{-a(\phi)s} \quad \text{with } \epsilon(0, \phi, \psi) \in \mathbb{C}^\times. \tag{33}$$

It is known that the LLC for $\mathrm{GL}_n(F)$ preserves the ϵ-factors:

$$\epsilon(s, \phi, \psi) = \epsilon(s, \pi, \psi),$$

where $\epsilon(s, \pi, \psi)$ is the Godement–Jacquet local constant [24]. It takes the form

$$\epsilon(s, \pi, \psi) = \epsilon(0, \pi, \psi)\, q^{-f(\pi)s}, \quad \text{where } \epsilon(0, \pi, \psi) \in \mathbb{C}^\times. \tag{34}$$

Recall that $f(\pi)$ is an integer, called the conductor of π. It follows from (33) and (34) that

$$a(\phi) = f(\pi). \tag{35}$$

In the case when π is an unramified representation of $F^\times$, the inertia subgroup $\mathbf{I}_F$ is contained in $\ker\phi$, with $\phi = \mathrm{rec}_{F,1}(\pi)$. Hence, (31) implies that $a(\phi) = 0$. On the other hand, π is trivial on $\mathfrak{o}_F^\times$, and a fortiori trivial on $1 + \mathfrak{p}_F = U^1(\mathfrak{A})$, with $\mathfrak{A} = \mathfrak{o}_F$. Then (32) implies that $d(\pi) = 0 = d(\phi)$.

From now on, we will assume that we are not in the above special case, that is, we assume that $n \neq 1$ or that π is ramified. Let $\mathfrak{A}$ be a principal $\mathfrak{o}_F$-order in $\mathrm{M}_n(F)$ such that $e(\mathfrak{A}) = n/\gcd(n, f(\pi))$, and let $\mathfrak{K}(\mathfrak{A})$ denote the normalizer in $\mathrm{GL}_n(F)$ of $\mathfrak{A}$. By [14, Theorem 3], the restriction of π to $\mathfrak{K}(\mathfrak{A})$ contains a *nondegenerate* (in the sense of [14, (1.21)]) representation ϱ of $\mathfrak{K}(\mathfrak{A})$, and we have [14, (3.7)]

$$d(\varrho) = e(\mathfrak{A})\left(\frac{f(\pi)}{n} - 1\right), \tag{36}$$

where $d(\varrho) \geq 0$ is the least integer such that

$$U^{d(\varrho)+1}(\mathfrak{A}) \subset \ker(\varrho).$$

Moreover, if the irreducible representation ϱ' of $\mathfrak{K}(\mathfrak{A})$ occurs in the restriction of π to $\mathfrak{K}(\mathfrak{A})$, then $d(\varrho') = d(\varrho)$ if and only if ϱ' is nondegenerate [14, (5.1) (iii)]. Hence, we obtain from (35) and (36) that

$$\frac{d(\varrho')}{e(\mathfrak{A})} = \frac{f(\pi)}{n} - 1 = \frac{a(\phi)}{n} - 1 = d(\phi) \tag{37}$$

for every nondegenerate irreducible representation ρ' of $\mathfrak{K}(\mathfrak{A})$ which occurs in the restriction of π to $\mathfrak{K}(\mathfrak{A})$.

It follows from the definition (32) of $d(\pi)$, that

$$d(\pi) \leq \frac{d(\varrho')}{e(\mathfrak{A})}, \tag{38}$$

for every nondegenerate irreducible representation ρ' of $\mathfrak{K}(\mathfrak{A})$ which occurs in the restriction of π to $\mathfrak{K}(\mathfrak{A})$.

We will check that (38) is actually an equality. The case where $d(\pi) = 0$ is easy, so we only consider $d(\pi) > 0$.

Let $\mathfrak{A}'$ be any hereditary $\mathfrak{o}_F$-order $\mathfrak{A}'$ in $\mathrm{M}_n(F)$, and define $m_{\mathfrak{A}'}(\pi)$ to be the least non-negative integer m such that the restriction of π to $U^{m+1}(\mathfrak{A}')$ contains the trivial character. Then choose $\mathfrak{A}'$ so that $m_{\mathfrak{A}'}(\pi)/e(\mathfrak{A}')$ is minimal, and let $[\mathfrak{A}', m_{\mathfrak{A}'}(\pi), m_{\mathfrak{A}'}(\pi) - 1, \beta]$ be a stratum occurring in π. By [14, Theorem 2′], this is a fundamental stratum. By [14, (3.4)], we may assume that the integers $e(\mathfrak{A}')$ and $m_{\mathfrak{A}'}(\pi)$ are relatively prime. Hence, we may apply [14, (3.13)]. We find that $\mathfrak{A}'$ is principal and that every irreducible representation ϱ of $\mathfrak{K}(\mathfrak{A}')$ which occurs in the restriction of π to $\mathfrak{K}(\mathfrak{A}')$, and such that the restriction of ϱ to $U^{m_{\mathfrak{A}'}(\pi)}(\mathfrak{A}')$ contains ψ_β, is nondegenerate. In particular, we have $d(\varrho') = m_{\mathfrak{A}'}(\pi)$.

It remains to check that the principal order $\mathfrak{A}'$ satisfies

$$e(\mathfrak{A}') = n/\gcd(n, f(\pi)). \tag{39}$$

Let $b = \gcd(n, f(\pi))$. Set $n = n'b$ and $f(\pi) = f'(\pi)b$. By using [14, (3.9)], we obtain that n' divides $e(\mathfrak{A}')$. Let $\mathfrak{P}'$ denote the Jacobson radical of $\mathfrak{A}'$. Then [15, (3.3.8)] and [14, (3.8)] assert that

$$q^{f(\pi)} = \left[\mathfrak{A}' : \mathfrak{p}_F(\mathfrak{P}')^{d(\varrho')}\right]^{1/n}.$$

That is, since $\mathfrak{p}_F\mathfrak{A}' = (\mathfrak{P}')^{e(\mathfrak{A}')}$,

$$q^{f(\pi)} = \left[\mathfrak{A}' : (\mathfrak{P}')^{d(\varrho')+e(\mathfrak{A}')}\right]^{1/n} = q^{n(d(\varrho')+e(\mathfrak{A}'))/e(\mathfrak{A}')} = q^{n(1+d(\varrho')/e(\mathfrak{A}'))}.$$

Hence, we get

$$f(\pi) = n\left(1 + \frac{d(\varrho')}{e(\mathfrak{A}')}\right),$$

that is,

$$d(\varrho') = \frac{e(\mathfrak{A}')f(\pi)}{n} - e(\mathfrak{A}') = \frac{e\left(\mathfrak{A}'\right)f'(\pi)}{n'} - e(\mathfrak{A}').$$

Hence, we have

$$n'd(\varrho') = e(\mathfrak{A}')f'(\pi) - e(\mathfrak{A}')n'.$$

Since $e(\mathfrak{A}')$ and $d(\rho') = m_{\mathfrak{A}'}(\pi)$ are relatively prime, we deduce that $e(\mathfrak{A}')$ divides n'. Thus, we have $e(\mathfrak{A}') = n'$, which means that (39) holds.

We conclude that (38) is indeed an equality, which together with (37) shows that $d(\varrho') = d(\pi)$. □

As congruence subgroups are the main examples of groups like $U^m(\mathfrak{A})$ above, they have a link with depths. This can be made precise. Let $K_0 = \mathrm{GL}_n(\mathfrak{o}_F)$ be the standard maximal compact subgroup of $\mathrm{GL}_n(F)$ and define, for $r \in \mathbb{Z}_{>0}$:

$$K_r = \ker\left(\mathrm{GL}_n(\mathfrak{o}_F) \to \mathrm{GL}_n(\mathfrak{o}_F/\mathfrak{p}_F^r)\right) = 1 + \mathrm{M}_n(\mathfrak{p}_F^r).$$

We denote the set of irreducible smooth $\mathrm{GL}_n(F)$-representations that are generated by their K_r-invariant vectors by $\mathrm{Irr}(\mathrm{GL}_n(F), K_r)$. To indicate the ambient group $\mathrm{GL}_n(F)$, we will sometimes denote K_r by $K_{r,n}$.

Lemma 4.3 *For $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F))$ and $r \in \mathbb{Z}_{>0}$, the following are equivalent:*

- $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F), K_r)$,
- $d(\pi) \leq r - 1$.

Proof For this result, it is convenient to use the equivalent definition of depth provided by Moy and Prasad [42]. In their notation, the group K_r is $P_{o,(r-1)+}$, where o denotes the origin in the standard apartment of the Bruhat–Tits building of $\mathrm{GL}_n(F)$. From the definition in [42, § 3.4], we read off that any $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F), K_r)$ has depth $\leq r - 1$.

Conversely, suppose that $d(\pi) \leq r - 1$. Then π has nonzero vectors fixed by the group $P_{x,(r-1)+}$, where x is some point of the Bruhat–Tits building. Since we may move x within its $\mathrm{GL}_n(F)$-orbit and there is only one orbit of vertices, we may assume that x lies in the star of o. As $r - 1 \in \mathbb{Z}_{\geq 0}$, there is an inclusion

$$P_{x,(r-1)+} \supset P_{o,(r-1)+} = K_r,$$

so π has nonzero vectors fixed by K_r. □

Let us recall some basic properties of generic representations, from [31, Section 2]. Let $\psi : F \to \mathbb{C}^\times$ be a character which is trivial on $\mathfrak{o}_F$ but not on $\varpi_F^{-1}\mathfrak{o}_F$. We note that ψ is unitary because $F/\mathfrak{o}_F$ is a union of finite subgroups. Let $U = U_n$ be the standard unipotent subgroup of $\mathrm{GL}_n(F)$, consisting of upper triangular matrices. We need a character θ of U, which does not vanish on any of the root subgroups associated with simple roots. Any choice is equally good, and it is common to take

$$\theta\left((u_{i,j})_{i,j=1}^n\right) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right).$$

Let $(\pi, V) \in \mathrm{Irr}(\mathrm{GL}_n(F))$. One calls π generic if there exists a nonzero linear form λ on V such that

$$\lambda(\pi(u)v) = \theta(u)\lambda(v) \quad \text{for all} \;\; u \in U, v \in V.$$

Such a linear form is called a Whittaker functional, and the space of those has dimension 1 (if they exist). Let $W(\pi, \theta)$ be the space of all functions $W : G \to \mathbb{C}$ of the form

$$W_v(g) = \lambda(\pi(g)v) \qquad g \in G, v \in V.$$

Then $W(\pi, \theta)$ is stable under right translations, and the representation thus obtained is isomorphic to π via $v \leftrightarrow W_v$. Most irreducible representations of $\mathrm{GL}_n(F)$, and in particular all the supercuspidal ones, are generic [23].

We consider one irreducible generic representation π of $\mathrm{GL}_n(F)$ and another one, π', of $\mathrm{GL}_{n-1}(F)$. For $W \in W(\pi, \theta)$ and $W' \in W(\pi', \theta)$ one defines the integral

$$\Psi(s, W, W') = \int_{U_{n-1}\backslash \mathrm{GL}_{n-1}(F)} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \overline{W'(g)}\, |\det(g)|_F^{s-1/2} \mathrm{d}\mu(g), \tag{40}$$

where μ denotes the quotient of Haar measures on $\mathrm{GL}_{n-1}(F)$ and on U_{n-1}. This integral is known to converge absolutely when $\mathrm{Re}(s)$ is large [31, Theorem 2.7.i]. The contragredient representations $\check{\pi}$ and $\check{\pi}'$ are also generic. We define $\check{W} \in W(\check{\pi}, \overline{\theta})$ by

$$\check{W}(g) = W(w_n g^{-T}) \qquad g \in \mathrm{GL}_n(F),$$

where g^{-T} is the transpose inverse of g and w_n is the permutation matrix with ones on the diagonal from the lower left to the upper right corner.

We denote the central character of π' by $\omega_{\pi'}$. With these notations the L-functions, ϵ-factors and γ-factors of the pair (π, π') are related by

$$\frac{\Psi(s, W, W')}{L(s, \pi \times \pi')}\epsilon(s, \pi \times \pi', \psi) = \omega_{\pi'}(-1)^{n-1}\frac{\Psi(1-s, \check{W}, \check{W}')}{L(1-s, \check{\pi} \times \check{\pi}')}, \tag{41}$$

$$\gamma(s, \pi \times \pi', \psi) = \epsilon(s, \pi \times \pi', \psi)\frac{L(1-s, \check{\pi} \times \check{\pi}')}{L(s, \pi \times \pi')}, \tag{42}$$

see [31, Theorem 2.7.iii]. We regard these equations as definitions of the ϵ- and γ-factors.

Theorem 4.4 *Let π be a supercuspidal representation in $\mathrm{Irr}(\mathrm{GL}_n(F), K_{r,n})$, with $r \in \mathbb{Z}_{>0}$. Let $\phi \in \Phi(\mathrm{GL}_n(F))$ be an elliptic parameter such that $\mathrm{SL}_2(\mathbb{C}) \subset \ker(\phi)$ and $d(\phi) \leq r-1$. Suppose that $\det \phi$ corresponds to the central character of π via local class field theory and that*

$$\epsilon(s, \pi \times \pi', \psi) = \epsilon(s, \phi \otimes \mathrm{rec}_{F,n'}(\pi'), \psi)$$

holds in one of the following cases:

- (a) *for $n' = n-1$ and every generic $\pi' \in \mathrm{Irr}(\mathrm{GL}_{n'}(F), K_{2r-1,n'})$;*
- (b) *for every n' such that $1 \leq n' < n$, and for every supercuspidal representation π' in $\mathrm{Irr}(\mathrm{GL}_{n'}(F), K_{2r-1,n'})$.*

Then $\phi = \mathrm{rec}_{F,n}(\pi)$.

Proof (b) By Proposition 4.2 $d(\mathrm{rec}_{F,n}(\pi)) \leq r-1$. By Lemma 4.3, the assumption applies to every supercuspidal $\pi' \in \mathrm{Irr}(\mathrm{GL}_{n'}(F))$ of depth $\leq 2r-2$. The point is that

$$2r-2 \geq 2\max\left\{d(\pi), d(\mathrm{rec}_{F,n}^{-1}(\phi))\right\},$$

which is a condition needed for [21, Theorem 7.5]. Its other conditions are among our assumptions, so from [21, Theorem 7.5] we see that indeed $\phi = \mathrm{rec}_{F,n}(\pi)$.

(a) We would like to show that

$$\epsilon(s, \pi \times \pi', \psi) = \epsilon\left(s, \mathrm{rec}_{F,n}^{-1}(\phi) \otimes \pi', \psi\right) \tag{43}$$

for every generic representation π' of $\mathrm{GL}_{n-1}(F)$. Since π and $\mathrm{rec}_{F,n}^{-1}(\phi)$ are supercuspidal

$$L(s, \pi \times \sigma) = L\left(s, \mathrm{rec}_{F,n}^{-1}(\phi) \times \sigma\right) = L(s, \check{\pi} \times \check{\sigma}) = L\left(s, \mathrm{rec}_{F,n}^{-1}(\check{\phi}) \times \check{\sigma}\right) = 1 \tag{44}$$

for any generic representation σ of a general linear group of smaller size [31, Theorem 8.1]. So we might just as well check (43) with γ-factors instead of ϵ-factors. We proceed as in the proof of [27, (3.3.4)]. First we write $\gamma(s, \pi \times \tau, \psi)$ as a product of $\gamma(s, \pi \times \langle\Delta_i\rangle^t, \psi)$, where $\langle\Delta_i\rangle^t$ is a Zelevinsky segment. Next we write $\gamma(s, \pi \times \langle\Delta_i\rangle^t, \psi)$ itself as a product

$$\prod_h \gamma(s, \pi \times \pi_i' |\ |^h, \psi),$$

with π_i' supercuspidal. The multiplicativity of γ-factors also gives the equality

$$\gamma(s, \mathrm{rec}_{F,n}^{-1}(\phi) \otimes \pi', \psi) = \prod_{i,h} \gamma\left(s, \mathrm{rec}_{F,n}^{-1}(\phi) \times \pi_i' |\ |^h, \psi\right).$$

Hence, to establish (43) it suffices to show that

$$\gamma(s, \pi \times \sigma, \psi) = \gamma\left(s, \mathrm{rec}_{F,n}^{-1}(\phi) \times \sigma, \psi\right) \tag{45}$$

whenever σ is a supercuspidal. In the case $d(\sigma) > 2r - 2$, this is the content of [21, Theorem 7.4]. We note that this result uses both the assumption on the central character of π and Proposition 4.2.

Consider a supercuspidal $\sigma \in \mathrm{GL}_{n'}(F)$ of depth $\leq 2r - 2$. By Lemma 4.3 has nonzero $K_{2r-1,n'}$-fixed vectors, so any constituent π' of

$$I_{\mathrm{GL}_{n'}(F)\times \mathrm{GL}_{n-1-n'}}^{\mathrm{GL}_n(F)}(\sigma \times \mathrm{triv})$$

lies in $\mathrm{Irr}(GL_{n-1}(F), K_{2r-1,n-1})$. One of these subquotients π' is generic, and then

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \pi \times \pi', \psi).$$

By the assumption of the theorem, the right-hand side equals

$$\gamma\left(s, \mathrm{rec}_{F,n}^{-1}(\phi) \times \pi', \psi\right) = \gamma\left(s, \mathrm{rec}_{F,n}^{-1}(\phi) \times \sigma, \psi\right).$$

This finishes the proof of (45) and of (43). Now we can apply [27, Théorème 1.1], which says that $\pi \cong \mathrm{rec}_{F,n}^{-1}(\phi)$. □

5 The method of close fields

Kazhdan's method of close fields [18,35] has proven useful to generalize results that are known for groups over p-adic fields to groups over local fields of positive characteristic. It was worked out for inner forms of $\mathrm{GL}_n(F)$ by Badulescu [7].

Let F and $\widetilde{F}$ be two local non-archimedean fields, which we think of as being similar in a way that will be made precise below. Let $G = \mathrm{GL}_m(D)$ be a standard inner form of $\mathrm{GL}_n(F)$ and let $\widetilde{G} = \mathrm{GL}_m(\widetilde{D})$ be the standard inner form of $\mathrm{GL}_n(\widetilde{F})$ with the same Hasse invariant as G.

In this section, an object with a tilde will always be the counterpart over $\widetilde{F}$ of an object (without tilde) over F, and a superscript $\sharp$ means the subgroup of elements with reduced norm 1. Then $\widetilde{G}^\sharp = \widetilde{G}_{\mathrm{der}}$ is an inner form of $\mathrm{SL}_n(\widetilde{F})$ with the same Hasse invariant as $G^\sharp$ and

$$\chi_{\widetilde{G}} = \chi_{\widetilde{G}^\sharp} = \chi_{G^\sharp} = \chi_G.$$

Let $\mathfrak{o}_D$ be the ring of integers of D, ϖ_D a uniformizer and $\mathfrak{p}_D = \varpi_D \mathfrak{o}_D$ its unique maximal ideal. The explicit multiplication rules in D [50, Proposition IX.4.11] show that we may assume that a power of ϖ_D equals ϖ_F, a uniformizer of F.

Generalizing the notation for $\mathrm{GL}_n(F)$, let $K_0 = \mathrm{GL}_m(\mathfrak{o}_D)$ be the standard maximal compact subgroup of G and define, for $r \in \mathbb{Z}_{>0}$:

$$K_r = \ker\left(\mathrm{GL}_m(\mathfrak{o}_D) \to \mathrm{GL}_m(\mathfrak{o}_D/\mathfrak{p}_D^r)\right) = 1 + \mathrm{M}_m(\mathfrak{p}_D^r).$$

We denote the category of smooth G-representations that are generated by their K_r-invariant vectors by $\mathrm{Mod}(G, K_r)$. Let $\mathcal{H}(G, K_r)$ be the convolution algebra of compactly supported K_r-biinvariant functions $G \to \mathbb{C}$. According to [12, Corollaire 3.9]

$$\begin{array}{ccc} \mathrm{Mod}(G, K_r) & \to & \mathrm{Mod}(\mathcal{H}(G, K_r)), \\ V & \mapsto & V^{K_r} \end{array} \tag{46}$$

is an equivalence of categories. The same holds for $(\widetilde{G}, \widetilde{K}_r)$.

From now on, we suppose that F and $\widetilde{F}$ are l-close for some $l \geq r$, that is,

$$\mathfrak{o}_F/\mathfrak{p}_F^l \cong \mathfrak{o}_{\widetilde{F}}/\mathfrak{p}_{\widetilde{F}}^l \tag{47}$$

as rings. As remarked in [18], for every local field of characteristic $p > 0$ and every $l \in \mathbb{N}$ there exists a finite extension of $\mathbb{Q}_p$ which is l-close to F.

Notice that (47) induces a group isomorphism $\mathfrak{o}_F^\times/1+\mathfrak{p}_F^l \cong \mathfrak{o}_{\widetilde{F}}^\times/1+\mathfrak{p}_{\widetilde{F}}^l$. A choice of uniformizers ϖ_F and $\varpi_{\widetilde{F}}$ then leads to

$$F^\times/1+\mathfrak{p}_F^l \cong \mathbb{Z} \times \mathfrak{o}_F^\times/1+\mathfrak{p}_F^l \cong \mathbb{Z} \times \mathfrak{o}_{\widetilde{F}}^l/1+\mathfrak{p}_{\widetilde{F}}^l \cong \widetilde{F}^\times/1+\mathfrak{p}_{\widetilde{F}}^l. \tag{48}$$

With [7, Théorème 2.4], (47) also gives rise to a ring isomorphism

$$\lambda_r : \mathfrak{o}_D/\mathfrak{p}_D^r \to \mathfrak{o}_{\widetilde{D}}/\mathfrak{p}_{\widetilde{D}}^r, \tag{49}$$

which in turn induces a group isomorphism

$$\mathrm{GL}_m(\lambda_r) : K_0/K_r = \mathrm{GL}_m(\mathfrak{o}_D/\mathfrak{p}_D^r) \;\to\; \widetilde{K}_0/\widetilde{K}_r = \mathrm{GL}_m\left(\mathfrak{o}_{\widetilde{D}}/\mathfrak{p}_{\widetilde{D}}^r\right).$$

We note that, whenever $r \leq r' \leq l$, one choose $\lambda_{r'}$ such that it induces λ_r. Then

$$\mathrm{GL}_m(\lambda_{r'}) : K_0/K_{r'} \to \widetilde{K}_0/\widetilde{K}_{r'} \tag{50}$$

refines $\mathrm{GL}_m(\lambda_r)$. Recall that the Cartan decomposition for G says that $K_0\backslash G/K_0$ can be represented by

$$A^+ := \left\{\mathrm{diag}(\varpi_D^{a_1}, \ldots, \varpi_D^{a_m}) \in \mathrm{GL}_m(D) : a_1 \leq \cdots \leq a_m\right\}.$$

Clearly A^+ is canonically in bijection with the analogous set $\widetilde{A}^+$ of representatives for $\widetilde{K}_0\backslash\widetilde{G}/\widetilde{K}_0$ (which of course depends on the choice of a uniformizer $\varpi_{\widetilde{D}}$). Since $K_r\backslash G/K_r$ can be identified with $K_r\backslash K_0 \times A^+ \times K_0/K_r$, that and $\mathrm{GL}_m(\lambda_r)$ determine a bijection

$$\zeta_r : K_r\backslash G/K_r \to \widetilde{K}_r\backslash\widetilde{G}/\widetilde{K}_r. \tag{51}$$

Most of the next result can be found in [7,10].

Theorem 5.1 *Suppose that F and $\widetilde{F}$ are sufficiently close, in the sense that the l in (47) is large. Then the map $1_{K_r gK_r} \mapsto 1_{\zeta_r(K_r gK_r)}$ extends to a $\mathbb{C}$-algebra isomorphism*

$$\zeta_r^G : \mathcal{H}(G, K_r) \to \mathcal{H}(\widetilde{G}, \widetilde{K}_r).$$

This induces an equivalence of categories

$$\overline{\zeta_r^G} : \mathrm{Mod}(G, K_r) \to \mathrm{Mod}(\widetilde{G}, \widetilde{K}_r)$$

such that:

(a) *$\overline{\zeta_r^G}$ respects twists by unramified characters and its effect on central characters is that of (48).*

(b) *For irreducible representations, $\overline{\zeta_r^G}$ preserves temperedness, essential square integrability and cuspidality.*

(c) *Let be P a parabolic subgroup of G with a Levi factor M which is standard, and let $\widetilde{P}$ and $\widetilde{M}$ be the corresponding subgroups of $\widetilde{G}$. Then*

$$\begin{array}{ccc} \mathrm{Mod}(G, K_r) & \xrightarrow{\overline{\zeta_r^G}} & \mathrm{Mod}(\widetilde{G}, \widetilde{K}_r) \\ \uparrow I_P^G & & \uparrow I_{\widetilde{P}}^{\widetilde{G}} \\ \mathrm{Mod}(M, K_r \cap M) & \xrightarrow{\overline{\zeta_r^M}} & \mathrm{Mod}(\widetilde{M}, \widetilde{K}_r \cap \widetilde{M}) \end{array}$$

commutes.

(d) $\overline{\zeta_r^G}$ *commutes with the formation of contragredient representations.*
(e) $\overline{\zeta_r^G}$ *preserves the L-functions, ϵ-factors and γ-factors.*

Proof The existence of the isomorphism ζ_r^G is [7, Théorème 2.13]. The equivalence of categories follows from that and (46).
(a) Let G^1 be the subgroup of G generated by all compact subgroups of G, that is, the intersection of the kernels of all unramified characters of G. Since K_r and $\widetilde{K}_r$ are compact, ζ_r restricts to a bijection

$$K_r \backslash G^1 / K_r \to \widetilde{K}_r \backslash \widetilde{G}^1 / \widetilde{K}_r.$$

Moreover, because $A^+ \to \widetilde{A}^+$ respects the group multiplication whenever it is defined, the induced bijection $G/G^1 \to \widetilde{G}/\widetilde{G}^1$ is in fact a group isomorphism. Hence, ζ_r induces an isomorphism

$$\overline{\zeta_r^{G/G^1}} : X_{nr}(G) = \mathrm{Irr}(G/G^1) \to \mathrm{Irr}(\widetilde{G}/\widetilde{G}^1) = X_{nr}(\widetilde{G}),$$

which clearly satisfies, for $\pi \in \mathrm{Mod}(G, K_r)$ and $\chi \in X_{nr}(G)$:

$$\overline{\zeta_r^G}(\pi \otimes \chi) = \overline{\zeta_r^G}(\pi) \otimes \overline{\zeta_r^{G/G^1}}(\chi).$$

The central characters can be dealt with similarly. The characters of $Z(G)$ appearing in $\mathrm{Mod}(G, K_r)$ are those of

$$Z(G)/Z(G) \cap K_r = F^\times / 1 + \mathfrak{p}_F^r.$$

Now we note that ζ_r^G and (48) have the same restriction to the above group.
(b) By [7, Théorème 2.17], $\overline{\zeta_r^G}$ preserves cuspidality and square integrability modulo centre. Combining the latter with part (a), we find that it also preserves essential square integrability. A variation on the proof of [7, Théorème 2.17.b] shows that temperedness is preserved as well. Alternatively, one can note that every irreducible tempered representation in $\mathrm{Mod}(G, K_r)$ is obtained with parabolic induction from a square-integrable modulo centre representation in $\mathrm{Mod}(M, M \cap K_r)$, and then apply part (c).
(c) This property, and its analogue for Jacquet restriction, are proven in [10, Proposition 3.15]. We prefer a more direct argument. The constructions in [7, § 2] apply equally well to $(M, K_r \cap M)$, so ζ_r induces an algebra isomorphism ζ_r^M and an equivalence of categories $\overline{\zeta_r^M}$. By [17, Corollary 7.12] the parabolic subgroup P determines an injective algebra homomorphism

$$t_P : \mathcal{H}(M, K_r \cap M) \to \mathcal{H}(G, K_r).$$

This in turn gives a functor

$$\begin{array}{rcl} (t_P)_* : \mathrm{Mod}(\mathcal{H}(M, K_r \cap M)) & \to & \mathrm{Mod}(\mathcal{H}(G, K_r)), \\ V & \mapsto & \mathrm{Hom}_{\mathcal{H}(M,K_r \cap M)}(\mathcal{H}(G, K_r), V), \end{array}$$

where $\mathcal{H}(G, K_r)$ and V are regarded as $\mathcal{H}(M, K_r \cap M)$-modules via t_P. This is a counterpart of parabolic induction, in the sense that

$$\begin{array}{ccc} \mathrm{Mod}(G, K_r) & \to & \mathrm{Mod}(\mathcal{H}(G, K_r)) \\ \uparrow I_P^G & & \uparrow (t_P)_* \\ \mathrm{Mod}(M, K_r \cap M) & \to & \mathrm{Mod}(\mathcal{H}(M, K_r \cap M)) \end{array} \qquad (52)$$

commutes [17, Corollary 8.4]. The construction of t_P in [17, § 7] depends only on properties that are preserved by ζ_r^G (and its counterparts for other groups), so

$$\begin{array}{ccc} \mathcal{H}(G,K_r) & \to & \mathcal{H}(\widetilde{G},\widetilde{K}_r) \\ \uparrow (t_P)_* & & \uparrow (t_{\widetilde{P}})_* \\ \mathcal{H}(M,K_r\cap M) & \to & \mathcal{H}(\widetilde{M},\widetilde{K}_r\cap\widetilde{M}) \end{array} \tag{53}$$

commutes. Now we combine (53) with (52) for G and $\widetilde{G}$.

(d) The contragredient of a G-representation (π, V) is the representation $\check{\pi}$, on the smooth part of the dual vector space of V, defined by $\check{\pi}(g)(\lambda) = \lambda \circ \pi(g^{-1})$. Similarly, the contragredient of a $\mathcal{H}(G,K_r)$-module W is W^* with the action

$$f \cdot \lambda = \lambda \circ f^* \qquad \lambda \in W^*, f \in \mathcal{H}(G,K_r),$$

where the involution on $\mathcal{H}(G,K_r)$ is given by $f^*(g) = f(g^{-1})$. Furthermore, $(V^*)^{K_r} \cong (V^{K_r})^*$, so the equivalence of categories (46) commutes with the formation of contragredients. The map $\overline{\zeta_r^G}$ does so because ζ_r^G commutes with the involution *.

(e) For the γ-factors, see [7, Théorème 2.19].

Consider the L-function of a supercuspidal $\sigma \in \mathrm{Irr}(G,K_r)$. By [24, Propositions 4.4 and 5.11] $L(s,\sigma) = 1$ unless $m = 1$ and $\sigma = \chi\circ\mathrm{Nrd}$ with $\chi : F^\times \to \mathbb{C}^\times$ unramified. This property is preserved by ζ_r^G, so $L\big(s, \overline{\zeta_r^G}(\sigma)\big) = 1$ if the condition is fulfilled. In the remaining case

$$L(s,\sigma) = L(s + (d-1)/2, \chi) = \big(1 - q^{-s+(1-d)/2}\chi(\varpi_F)\big)^{-1}.$$

The proof of part (a) shows that $\overline{\zeta_r^G}(\sigma) = \chi \circ \zeta_r^{F^\times} \circ \mathrm{Nrd}$, so

$$L\big(s, \overline{\zeta_r^G}(\sigma)\big) = \big(1 - q^{-s+(1-d)/2}\chi(\zeta_r^{F^\times}(\varpi_{\widetilde{F}}))\big)^{-1} = \big(1 - q^{-s+(1-d)/2}\chi(\varpi_F)\big)^{-1}.$$

Thus, $\overline{\zeta_r^G}$ preserve the L-functions of supercuspidal representations. By [30, § 3], the L-functions of general $\pi \in \mathrm{Irr}(G,K_r)$ are determined by the L-functions of supercuspidal representations of Levi subgroups of G, in combination with parabolic induction and twisting with unramified characters. In view of parts (a),(c) and the above, this implies that $\overline{\zeta_r^G}$ always preserves L-functions.

Now the relation

$$\epsilon(s,\pi,\psi) = \gamma(s,\pi,\psi)\frac{L(s,\pi)}{L(1-s,\check{\pi})}$$

and part (d) show that $\overline{\zeta_r^G}$ preserves ϵ-factors. □

For $r \le r' \le l$, $\mathrm{Mod}(G,K_r)$ is a subcategory of $\mathrm{Mod}(G,K_{r'})$ and it follows from (50) that

$$\overline{\zeta_{r'}^G} = \overline{\zeta_r^G} \text{ on } \mathrm{Mod}(G,K_r). \tag{54}$$

In [9], Badulescu showed that Theorem 5.1 has an analogue for $G^\sharp$ and $\widetilde{G}^\sharp$, which can easily be deduced from Theorem 5.1. We quickly recall how this works. Note that M is a central extension of $M^\sharp = \{m \in M : \mathrm{Nrd}(m) = 1\}$. A few properties of the reduced norm [50, § IX.2 and equation IX.4.9] entail

$$\begin{aligned} \mathrm{Nrd}(K_r \cap M) &= \mathrm{Nrd}(1 + \mathfrak{p}_D^r) = 1 + \mathfrak{p}_F^r, \\ M^\sharp(K_r \cap M) &= \big\{m \in M : \mathrm{Nrd}(m) \in 1 + \mathfrak{p}_F^r\big\}. \end{aligned} \tag{55}$$

Choose the Haar measures on M and $M^\sharp$ so that $\mathrm{vol}(K_r \cap M) = \mathrm{vol}(K_r \cap M^\sharp)$. The inclusion $M^\sharp \to M$ induces an algebra isomorphism

$$\mathcal{H}(M^\sharp, K_r \cap M^\sharp) \to \mathcal{H}(M^\sharp(K_r \cap M), K_r \cap M)$$
$$:= \{f \in \mathcal{H}(M, K_r \cap M) : \text{supp}(f) \subset M^\sharp(K_r \cap M)\}.$$

In view of (55) and the isomorphism $\mathfrak{o}_F/\mathfrak{p}_F^r \cong \mathfrak{o}_{\widetilde{F}}/\mathfrak{p}_{\widetilde{F}}^r$, ζ_r^M yields a bijection

$$\mathcal{H}(M^\sharp(K_r \cap M), K_r \cap M) \to \mathcal{H}(\widetilde{M}^\sharp(\widetilde{K}_r \cap \widetilde{M}), \widetilde{K}_r \cap \widetilde{M}).$$

Hence, it induces an algebra isomorphism

$$\zeta_r^{M^\sharp} : \mathcal{H}(M^\sharp, K_r \cap M^\sharp) \to \mathcal{H}(\widetilde{M}^\sharp, \widetilde{K}_r \cap \widetilde{M}^\sharp).$$

Corollary 5.2 *Theorem 5.1 (except part e) also holds for the corresponding subgroups of elements with reduced norm 1.*

Proof Using the isomorphisms $\zeta_r^{M^\sharp}$, this can be proven in the same way as Theorem 5.1 itself. For part (b) one can use that an irreducible G-representation is tempered (resp. essentially square-integrable or cuspidal) if and only if all its $G^\sharp$-constituents are so [46, Proposition 2.7].

As preparation for the next section, we will show that in certain special cases the functors $\overline{\zeta_r^G}$ preserve the L-functions, ϵ-factors and γ-factors of pairs of representations, as defined in [31].

Suppose that $\widetilde{F}$ is l-close to F and that $\widetilde{\psi} : \widetilde{F} \to \mathbb{C}^\times$ is a character which is trivial on $\mathfrak{o}_{\widetilde{F}}$. We say that $\widetilde{\psi}$ is l-close to ψ if $\widetilde{\psi}\big|_{\varpi_{\widetilde{F}}^{-l}\mathfrak{o}_{\widetilde{F}}/\mathfrak{o}_{\widetilde{F}}}$ corresponds to $\psi\big|_{\varpi_F^{-l}\mathfrak{o}_F/\mathfrak{o}_F}$ under the isomorphisms

$$\varpi_{\widetilde{F}}^{-l}\mathfrak{o}_{\widetilde{F}}/\mathfrak{o}_{\widetilde{F}} \cong \mathfrak{o}_{\widetilde{F}}/\varpi_{\widetilde{F}}^{l}\mathfrak{o}_{\widetilde{F}} \cong \mathfrak{o}_F/\varpi_F^l\mathfrak{o}_F \cong \varpi_F^{-l}\mathfrak{o}_F/\mathfrak{o}_F.$$

Theorem 5.3 *Assume that F and $\widetilde{F}$ are l-close for some $l > r$ and that $\widetilde{\psi}$ is l-close to ψ. Let $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F), K_{r,n})$ be supercuspidal and let $\pi' \in \mathrm{Irr}(\mathrm{GL}_{n-1}(F), K_{r,n-1})$ be generic. Then*

$$L\left(s, \overline{\zeta_r^{\mathrm{GL}_n(F)}}(\pi) \times \overline{\zeta_r^{\mathrm{GL}_{n-1}(F)}}(\pi')\right) = L(s, \pi \times \pi') = 1,$$
$$\epsilon\left(s, \overline{\zeta_r^{\mathrm{GL}_n(F)}}(\pi) \times \overline{\zeta_r^{\mathrm{GL}_{n-1}(F)}}(\pi'), \widetilde{\psi}\right) = \epsilon(s, \pi \times \pi', \psi),$$
$$\gamma\left(s, \overline{\zeta_r^{\mathrm{GL}_n(F)}}(\pi) \times \overline{\zeta_r^{\mathrm{GL}_{n-1}(F)}}(\pi'), \widetilde{\psi}\right) = \gamma(s, \pi \times \pi', \psi).$$

Remark It will follow from Theorem 6.1 that the above remains valid with any natural number instead of $n-1$ (except that the L-functions need not equal 1).

After the first version of this paper was put on the arXiv, the authors were informed that a similar result was proved in [20, Theorem 2.3.10]. See also [21, Theorem 7.6]. Our proof differs from Ganapathy's and yields a better bound on l, namely $l > r$ compared to $l \geq n^2 r + 4$.

Proof Since π and $\check{\pi}$ are supercuspidal, whereas π' and $\check{\pi}'$ are representations of a general linear group of lower rank, [31, Theorem 8.1] assures that all the L-functions appearing here are 1. By (42) this implies that the relevant γ-factors are equal to the ϵ-factors of the same pairs. Hence, it suffices to prove the claim for the ϵ-factors. We note that by Theorem 5.1

$$\omega_{\pi'}(-1)^{n-1} = \omega_{\overline{\zeta_r^{\mathrm{GL}_{n-1}(F)}}\pi'}(-1)^{n-1}, \tag{56}$$

so from (41) we see that it boils down to comparing the integrals $\Psi(s, W, W')$ and $\Psi(1-s, \check{W}, \check{W}')$ with their versions for $\widetilde{F}$.

Fix a Whittaker functional λ' for (π', V') and a vector $v' \in V^{K_{r,n-1}}$. Then $W' := W_{v'} \in W(\pi', \theta)$ is right $K_{r,n-1}$-invariant. Similarly we pick $W = W_v \in W(\pi, \theta)$, but now we have to require only that W is right invariant under $K_{r,n-1}$ on $\mathrm{GL}_{n-1}(F) \subset \mathrm{GL}_n(F)$. Because θ is unitary, the function

$$\mathrm{GL}_{n-1}(F) \to \mathbb{C}\colon g \mapsto W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \overline{W'(g)}$$

is constant on sets of the form $U_{n-1} g K_{r,n-1}$. Since the subgroup $K_{r,n-1}$ is stable under the automorphism $g \mapsto g^{-T}$, the functions $\check{W}$ and $\check{W}'$ are also right $K_{r,n-1}$-invariant. Both transform under left translations by U_{n-1} as $\overline{\theta}$, so

$$\mathrm{GL}_{n-1}(F) \to \mathbb{C}\colon g \mapsto \check{W}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \overline{\check{W}'(g)}$$

defines a function $U_{n-1}\backslash \mathrm{GL}_{n-1}(F)/K_{r,n-1} \to \mathbb{C}$. Since $\det(K_{r,n-1}) \subset \mathfrak{o}_F^\times$ and $\det(U_{n-1}) = 1$, the function $|\det|_F$ can also be regarded as a map
$U_{n-1}\backslash \mathrm{GL}_{n-1}(F)/K_{r,n-1} \to \mathbb{C}$.

Now the idea is to transfer these functions to objects over $\widetilde{F}$ by means of the Iwasawa decomposition as in [39, § 3], and to show that neither side of (41) changes.

Let $A_{\varpi_F} \subset \mathrm{GL}_{n'}(F)$ be the group of diagonal matrices all whose entries are powers of ϖ_F. The Iwasawa decomposition states that

$$\mathrm{GL}_n(F) = \bigsqcup_{a \in A_{\varpi_F}} U_n a K_{0,n}. \tag{57}$$

This, the canonical bijection $A_{\varpi_F} \to A_{\varpi_{\widetilde{F}}}\colon a \mapsto \widetilde{a}$ and the isomorphism $\mathrm{GL}_n(\lambda_r)$ from (49) combine to a bijection

$$\begin{aligned} \zeta_r'\colon U_n\backslash \mathrm{GL}_n(F)/K_{r,n} &\to \widetilde{U}_n\backslash \mathrm{GL}_n(\widetilde{F})/\widetilde{K}_{r,n}, \\ U_n\, ak\, K_{r,n} &\mapsto \widetilde{U}_n\, \widetilde{a} \mathrm{GL}_n(\lambda_r)(k)\, \widetilde{K}_{r,n}. \end{aligned} \tag{58}$$

Because $\widetilde{\psi}$ is l-close to ψ we may apply [39, Lemme 3.2.1], which says that there is a unique linear bijection

$$\rho_n\colon W(\pi, \theta)^{K_{r,n}} \to W\left(\zeta_r^{\mathrm{GL}_n(F)}(\pi), \widetilde{\theta}\right)^{\widetilde{K}_{r,n}} \tag{59}$$

which transforms the restriction of functions to $A_{\varpi_F} K_{0,n}$ according to ζ_r'. We will use (58) and (59) also with $n-1$ instead of n.

Put $\widetilde{W} = \rho_n(W)$ and $\widetilde{W'} = \rho_{n-1}(W')$. As (58) commutes with $g \mapsto g^{-T}$,

$$\widetilde{\check{W}} = \rho_n(\check{W}) \text{ and } \widetilde{\check{W}'} = \rho_{n-1}(\check{W}'). \tag{60}$$

These constructions entail that

$$\mathrm{GL}_{n-1}(\widetilde{F}) \to \mathbb{C}\colon \widetilde{g} \mapsto \widetilde{W}\begin{pmatrix} \widetilde{g} & 0 \\ 0 & 1 \end{pmatrix} \overline{\widetilde{W'}(\widetilde{g})}$$

defines a function $\widetilde{U}_{n-1}\backslash \mathrm{GL}_{n-1}(\widetilde{F})/\widetilde{K}_{r,n-1} \to \mathbb{C}$, and that

$$W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \overline{W'(g)} = \widetilde{W}\begin{pmatrix} \zeta_r'(g) & 0 \\ 0 & 1 \end{pmatrix} \overline{\widetilde{W'}(\zeta_r'(g))}. \tag{61}$$

It follows immediately from the definition of ζ_r' that

$$|\det(\zeta_r'(g))|_{\widetilde{F}} = |\det(g)|_F. \tag{62}$$

For the computation of $\Psi(s, W, W')$, we may normalize the measure μ such that every double coset $U_{n-1}\backslash U_{n-1}gK_{r,n-1}$ has volume 1, and similarly for the measure on $\widetilde{U}_{n-1}\backslash \mathrm{GL}_{n-1}(\widetilde{F})$. The equalities (61) and (62) imply

$$\begin{aligned}\Psi(s, W, W') &= \sum_{g\in A_{\varpi_F}K_{0,n-1}/K_{r,n-1}} W\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\overline{W'(g)}|\det(g)|_F^{s-1/2}\\ &= \sum_{\widetilde{g}\in A_{\varpi_{\widetilde{F}}}\widetilde{K}_{0,n-1}/\widetilde{K}_{r,n-1}} \widetilde{W}\begin{pmatrix} \widetilde{g} & 0\\ 0 & 1\end{pmatrix}\overline{\widetilde{W'}(\widetilde{g})}|\det(\widetilde{g})|_{\widetilde{F}}^{s-1/2} = \Psi(s, \widetilde{W}, \widetilde{W}').\end{aligned}$$

An analogous computation, additionally using (60), shows that

$$\Psi(s, \check{W}, \check{W}') = \Psi(s, \check{\widetilde{W}}, \check{\widetilde{W}}').$$

The previous two equalities and (56) prove that all terms in (41), expect possibly the ϵ-factors, have the same values as the corresponding terms defined over $\widetilde{F}$. To establish the desired equality of ϵ-factors, it remains to check that $\Psi(s, W, W')$ is nonzero for a suitable choice of right $K_{r,n-1}$-invariant functions W and W'.

Take v' as above, but nonzero. Then $W' = W_{v'}$ is nonzero because $V' \cong W(\pi', \theta)$. Choose $g_0 \in \mathrm{GL}_{n-1}(F)$ with $W'(g_0) \neq 0$ and define $H : \mathrm{GL}_{n-1}(F) \to \mathbb{C}$ by $H(g) = W'(g)$ if $g \in U_{n-1}g_0K_{r,n-1}$ and $H(g) = 0$ otherwise. According to [27, Lemme 2.4.1], there exists $W \in W(\pi, \psi)$ such that $W\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix} = H(g)$ for all $g \in \mathrm{GL}_{n-1}(F)$. Notice that such a W is automatically right invariant under $K_{r,n-1}$ on $\mathrm{GL}_{n-1}(F) \subset \mathrm{GL}_n(F)$. Now we can easily compute the required integral:

$$\begin{aligned}\Psi(s, W, W') &= \int_{U_{n-1}\backslash \mathrm{GL}_{n-1}(F)} |H(g)|^2|\det(g)|_F^{s-1/2}\mathrm{d}\mu(g)\\ &= \int_{U_{n-1}\backslash U_{n-1}g_0K_{r,n-1}} |W'(g)|^2|\det(g)|_F^{s-1/2}\mathrm{d}\mu(g)\\ &= \mu(U_{n-1}\backslash U_{n-1}g_0K_{r,n-1})|W'(g_0)|^2|\det(g_0)|_F^{s-1/2} \neq 0.\end{aligned}$$

□

6 Close fields and Langlands parameters

This section is based on Deligne's comparison of the Galois groups of close fields. According to [18, (3.5.1)], the isomorphism (47) gives rise to an isomorphism of profinite groups

$$\mathrm{Gal}(F_s/F)/\mathrm{Gal}(F_s/F)^l \cong \mathrm{Gal}(\widetilde{F}_s/\widetilde{F})/\mathrm{Gal}(\widetilde{F}_s/\widetilde{F})^l, \tag{63}$$

which is unique up to inner automorphisms. Since both $\mathbf{W}_F$ and $\mathbf{W}_{\widetilde{F}}$ can be described in terms of automorphisms of the residue field $\mathfrak{o}_F/\mathfrak{p}_F \cong \mathfrak{o}_{\widetilde{F}}/\mathfrak{p}_{\widetilde{F}}$, (63) restricts to an isomorphism

$$\mathbf{W}_F/\mathrm{Gal}(F_s/F)^l \cong \mathbf{W}_{\widetilde{F}}/\mathrm{Gal}(\widetilde{F}_s/\widetilde{F})^l. \tag{64}$$

We fix such an isomorphism (63), and hence (64) as well. Another choice would correspond to another separable closure of F, so that is harmless when it comes to Langlands parameters. Take $r < l$ and recall the map $\mathbf{W}_F/\mathrm{Gal}(F_s/F)^l \to F^\times/1+\mathfrak{p}_F^r$ from local class field theory. By [18, Proposition 3.6.1], the following diagram commutes:

$$\begin{array}{ccc} F^\times/1+\mathfrak{p}_F^r & \xrightarrow{\zeta_r} & \widetilde{F}^\times/1+\mathfrak{p}_{\widetilde{F}}^r \\ \uparrow & & \uparrow \\ \mathbf{W}_F/\mathrm{Gal}(F_s/F)^l & \to & \mathbf{W}_{\widetilde{F}}/\mathrm{Gal}(\widetilde{F}_s/\widetilde{F})^l \end{array}. \tag{65}$$

Notice that G and $\widetilde{G}$ have the same Langlands dual group, namely $\mathrm{GL}_n(\mathbb{C})$. Hence, (64) induces a bijection

$$\Phi_l^{\zeta} : \Phi_l(G) \to \Phi_l(\widetilde{G}). \tag{66}$$

In fact, Φ_l^{ζ} is already defined on the level of Langlands parameters without conjugation-equivalence, and in that sense $\Phi_l^{\zeta}(\phi)$ and ϕ always have the same image in $\mathrm{GL}_n(\mathbb{C})$. We remark that Φ_l^{ζ} can be defined in the same way for $G^\sharp$ and $\widetilde{G}^\sharp$, because these groups have the common Langlands dual group $\mathrm{PGL}_n(\mathbb{C})$.

We will prove that Φ_l^{ζ} describes the effect that

$$\overline{\zeta_r^G} : \mathrm{Irr}(G, K_r) \to \mathrm{Irr}(\widetilde{G}, \widetilde{K}_r)$$

has on Langlands parameters, when l is large enough compared to $r \in \mathbb{Z}_{>0}$. First we do so for general linear groups over fields. The next result improves on [21, Corollary 7.7] and [20, Theorem 2.3.11] in the sense that it gives an explicit and better lower bound $(2^{n-1}r + 1)$ on the l for which the statement holds. Indeed, the inductive definition of the bound given in [21] shows that it is in $O(2^{n-2}n^2 r)$.

We remark that the obtained bound $l > 2^{n-1}r$ appears nevertheless to be much larger than necessary. We expect that the result is valid whenever $l > r$, but we did not manage to prove that.

Theorem 6.1 *Suppose that $r \in \mathbb{Z}_{>0}$, and that F and $\widetilde{F}$ are l-close for some $l > 2^{n-1}r$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Irr}(\mathrm{GL}_n(F), K_r) & \xrightarrow{\overline{\zeta_r^{\mathrm{GL}_n(F)}}} & \mathrm{Irr}\big(\mathrm{GL}_n(\widetilde{F}), \widetilde{K}_r\big) \\ \downarrow \mathrm{rec}_{F,n} & & \downarrow \mathrm{rec}_{\widetilde{F},n} \\ \Phi_l(\mathrm{GL}_n(F)) & \xrightarrow{\Phi_l^{\zeta}} & \Phi_l\big(\mathrm{GL}_n(\widetilde{F})\big) \end{array}$$

Proof The proof will be conducted with induction to n. For $n = 1$, the diagram becomes

$$\begin{array}{ccc} \mathrm{Irr}(F^\times/1 + \mathfrak{p}_F^r) & \xrightarrow{\overline{\zeta_r^{F^\times}}} & \mathrm{Irr}\big(\widetilde{F}^\times/1 + \mathfrak{p}_{\widetilde{F}}^r\big) \\ \downarrow \mathrm{rec}_F & & \downarrow \mathrm{rec}_{\widetilde{F}} \\ \mathrm{Irr}\big(\mathbf{W}_F/\mathrm{Gal}(F_s/F)^l\big) & \xrightarrow{\Phi_l^{\zeta}} & \mathrm{Irr}\big(\mathbf{W}_{\widetilde{F}}/\mathrm{Gal}(\widetilde{F}_s/\widetilde{F})^l\big) \end{array}, \tag{67}$$

which commutes by Deligne's result (65).

Now we fix $n > 1$ and we assume the theorem for all $n' < n$. Consider a supercuspidal $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F), K_r)$ with Langlands parameter $\phi = \mathrm{rec}_{F,n}(\pi) \in \Phi_l(\mathrm{GL}_n(F))$. By the construction of the local Langlands correspondence for general linear groups, $\mathrm{SL}_2(\mathbb{C}) \subset \ker \phi$ and ϕ is elliptic. By Theorem 5.1 $\overline{\zeta_r^{\mathrm{GL}_n(F)}}(\pi) \in \mathrm{Irr}\big(\mathrm{GL}_n(\widetilde{F}), \widetilde{K}_r\big)$ is also supercuspidal and its central character is related to that of π via (48).

Let $\widetilde{\phi}_l \in \Phi_l(\mathrm{GL}_n(\widetilde{F}))$ be the Langlands parameter of $\overline{\zeta_r^{\mathrm{GL}_n(F)}}(\pi)$ and write $\phi_l = (\Phi_l^{\zeta})^{-1}(\widetilde{\phi}_l)$. Clearly $\mathrm{SL}_2(\mathbb{C}) \subset \ker \phi_l$ and ϕ_l is elliptic, so $\mathrm{rec}_{F,n}^{-1}(\phi_l)$ is supercuspidal. The commutative diagram (67) says that $\mathrm{rec}_{F,n}^{-1}(\phi_l)$ has the same central character as π. By Theorem 5.1.e

$$\epsilon(s, \pi, \psi) = \epsilon\left(s, \overline{\zeta_r^{\mathrm{GL}_n(F)}}(\pi), \widetilde{\psi}\right) = \epsilon\left(s, \widetilde{\phi}_l, \widetilde{\psi}\right).$$

By [18, Proposition 3.7.1], the right-hand side equals

$$\epsilon\left(s, \widetilde{\phi}_l, \widetilde{\psi}\right) = \epsilon(s, \phi_l, \psi) = \epsilon\left(s, \mathrm{rec}_{F,n}^{-1}(\phi_l), \psi\right),$$

so $\mathrm{rec}_{F,n}^{-1}(\phi_l)$ has the same ϵ-factor as π. Now we consider any generic $\pi' \in \mathrm{Irr}(\mathrm{GL}_{n-1}(F), K_{2r,n-1})$ with Langlands parameter ϕ'. The induction hypothesis and Theorem 5.3 apply to π' because $2^{n-2}2r < l$. By Theorem 5.3, (54), the induction hypothesis and [18, Proposition 3.7.1]:

$$\begin{aligned}\epsilon(s, \pi \times \pi', \psi) &= \epsilon\left(s, \overline{\zeta_{2r}^{\mathrm{GL}_n(F)}}(\pi) \times \overline{\zeta_{2r}^{\mathrm{GL}_{n-1}(F)}}(\pi'), \widetilde{\psi}\right)\\ &= \epsilon\left(s, \overline{\zeta_r^{\mathrm{GL}_n(F)}}(\pi) \times \mathrm{rec}_{\widetilde{F},n-1}^{-1}(\Phi_l^{\zeta}(\phi')), \widetilde{\psi}\right)\\ &= \epsilon\left(s, \widetilde{\phi}_l \otimes \Phi_l^{\zeta}(\phi'), \widetilde{\psi}\right)\\ &= \epsilon(s, \phi_l \otimes \phi', \psi) = \epsilon\left(s, \mathrm{rec}_{F,n}^{-1}(\phi_l) \times \pi', \psi\right).\end{aligned} \tag{68}$$

Together with Theorem 4.4 this implies $\pi \cong \mathrm{rec}_{F,n}^{-1}(\phi_l)$. Hence, the diagram of the theorem commutes for supercuspidal $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F), K_r)$.

For non-supercuspidal representations in $\mathrm{Irr}(\mathrm{GL}_n(F), K_r)$, it is easier. As already discussed in Section 2, the extension of the LLC from supercuspidal representations to $\mathrm{Irr}(\mathrm{GL}_n(F))$ is based on the Zelevinsky classification [52]. More precisely, according to [27, § 2] the LLC is determined by:

- the parameters of supercuspidal representations;
- the parameters of generalized Steinberg representations;
- compatibility with unramified twists;
- compatibility with parabolic induction followed by forming Langlands quotients.

By Theorem 5.1 the functor $\overline{\zeta_r^{\mathrm{GL}_n(F)}}$ and its versions for groups of lower rank respect unramified twists and parabolic induction. As a Langlands quotient is the unique irreducible quotient of a parabolically induced representation, this operation is respected as well.

Let us recall the construction of a generalized Steinberg representation. Start with a supercuspidal representation π of $\mathrm{GL}_d(F)$, where $dm = n$. Let ν be the absolute value of the determinant character of $\mathrm{GL}_d(F)$, let P be the standard parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi factor $\mathrm{GL}_d(F)^m$ and consider

$$I_P^{\mathrm{GL}_n(F)}(\pi \otimes \nu\pi \otimes \cdots \nu^{m-1}\pi). \tag{69}$$

By [27, § 2.6] it has a unique irreducible quotient, called $\mathrm{St}_m(\pi)$. Every generalized Steinberg representation is of this form. By Theorem 5.1 $\overline{\zeta_r^{\mathrm{GL}_n(F)}}$ sends (69) to

$$I_{\tilde{P}}^{\mathrm{GL}_n(\tilde{F})}\left(\overline{\zeta_r^{\mathrm{GL}_d(F)}}\pi \otimes \nu\overline{\zeta_r^{\mathrm{GL}_n(F)}}\pi \otimes \cdots \nu^{m-1}\overline{\zeta_r^{\mathrm{GL}_n(F)}}\pi\right).$$

Hence, $\overline{\zeta_r^{\mathrm{GL}_n(F)}}(\mathrm{St}_m(\pi)) = \mathrm{St}_m\left(\overline{\zeta_r^{\mathrm{GL}_d(F)}}\pi\right)$. By [27, § 2.7], the Langlands parameter of $\mathrm{St}_m(\pi)$ is $\mathrm{rec}_{F,d}(\pi) \otimes R_m$, where R_m denotes the unique irreducible m-dimensional representation of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$ which is trivial on $\mathbf{W}_F$. Since we already know the theorem for the supercuspidal representation π, we deduce that

$$\Phi_l^{\zeta}(\mathrm{rec}_{F,d}(\pi) \otimes R_m) = \mathrm{rec}_{F,d}\left(\overline{\zeta_r^{\mathrm{GL}_d(F)}}\pi\right) \otimes R_m,$$

which is the Langlands parameter of $\mathrm{St}_m\left(\overline{\zeta_r^{\mathrm{GL}_d(F)}}\pi\right)$. That is, the diagram of the theorem commutes for generalized Steinberg representations.

To determine the Langlands parameters of elements of $\mathrm{Irr}(\mathrm{GL}_n(F), K_r)$ via the above method, one needs only representations (possibly of groups of lower rank) that have nonzero K_r-invariant vectors. We checked that in every step of this method the effect of $\overline{\zeta_r^{\mathrm{GL}_n(F)}}$ on the Langlands parameters is given by Φ_l^{ζ}. Hence, the diagram of the theorem commutes for all representations in $\mathrm{Irr}(\mathrm{GL}_n(F), K_r)$. □

Because the LLC for inner forms of $\mathrm{GL}_n(F)$ is closely related to that for $\mathrm{GL}_n(F)$ itself, we can generalize Theorem 6.1 to inner forms.

Theorem 6.2 *Let $G = \mathrm{GL}_m(D)$ and $\widetilde{G} = \mathrm{GL}_m(\widetilde{D})$, with the same Hasse invariant. For any $r \in \mathbb{N}$ there exists $l > r$ such that, whenever F and $\widetilde{F}$ are l-close, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Irr}(G, K_r) & \xrightarrow{\overline{\zeta_r^G}} & \mathrm{Irr}(\widetilde{G}, \widetilde{K}_r) \\ \downarrow \mathrm{rec}_{D,m} & & \downarrow \mathrm{rec}_{\widetilde{D},m} \\ \Phi_l(G) & \xrightarrow{\Phi_l^{\zeta}} & \Phi_l(\widetilde{G}) \end{array}$$

In other words, Theorem 6.1 also holds for inner forms of $\mathrm{GL}_n(F)$, but without an explicit lower bound for l.

Proof The bijection (13) shows that we can write any $\pi \in \mathrm{Irr}(G, K_r)$ as the Langlands quotient $L(P, \omega)$ of $I_P^G(\omega)$, where P is a standard parabolic subgroup, M is Levi factor of P and $\omega \in \mathrm{Irr}_{\mathrm{ess}L^2}(M)$. Moreover, we may assume that $M = \prod_j \mathrm{GL}_{m_j}(D)$ and $\omega = \otimes_j \omega_j$. The fact that π has nonzero K_r-invariant vectors implies $\omega_j \in \mathrm{Irr}(\mathrm{GL}_{m_j}(D), K_r)$. By construction (11)

$$\mathrm{rec}_{D,m}(\pi) = \prod_j \mathrm{rec}_{D,m_j}(\omega_j) = \prod_j \mathrm{rec}_{F,dm_j}(\mathrm{JL}(\omega_j)). \tag{70}$$

The right-hand side forces us to compare the Jacquet–Langlands correspondence with the method of close fields. In fact, this is how Badulescu proved this correspondence over local fields of positive characteristic. It follows from [7, p. 742–744] that there exist $l > r' \geq r$ such that, whenever F and $\widetilde{F}$ are l-close, the following diagram commutes for all $k \leq m$:

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_k(D), K_r) & \xrightarrow{\overline{\zeta_r^{\mathrm{GL}_k(D)}}} & \mathrm{Irr}(\mathrm{GL}_k(\widetilde{D}), \widetilde{K}_r) \\ \downarrow \mathrm{JL} & & \downarrow \mathrm{JL} \\ \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_{kd}(F), K_{r'}) & \xrightarrow{\overline{\zeta_r^{\mathrm{GL}_{kd}(F)}}} & \mathrm{Irr}(\mathrm{GL}_{kd}(\widetilde{F}), \widetilde{K}_{r'}) \end{array} \tag{71}$$

Enlarge l so that Theorem 6.1 applies to $\mathrm{Irr}(\mathrm{GL}_{kd}(F), K_{r'})$ for all $k \leq m$. By Theorem 5.1.c

$$\overline{\zeta_r^G}(\pi) = L\left(\widetilde{P}, \overline{\zeta_r^M}(\omega)\right) = L\left(\widetilde{P}, \otimes_j \overline{\zeta_r^{\mathrm{GL}_{m_j}(D)}}(\omega_j)\right).$$

Now (71) shows that

$$\mathrm{JL}\left(\overline{\zeta_r^M}(\omega)\right) = \otimes_j \mathrm{JL}\left(\overline{\zeta_r^{\mathrm{GL}_{m_j}(D)}}(\omega_j)\right) = \otimes_j \overline{\zeta_{r'}^{\mathrm{GL}_{dm_j}(F)}}(\mathrm{JL}(\omega_j)). \tag{72}$$

By (11) and Theorem 6.1

$$\mathrm{rec}_{\widetilde{D},m}\left(\overline{\zeta_r^G}(\pi)\right) = \prod_j \mathrm{rec}_{\widetilde{F},dm_j}\left(\overline{\zeta_{r'}^{\mathrm{GL}_{dm_j}(F)}}(\mathrm{JL}(\omega_j))\right) = \prod_j \Phi_l^{\zeta}\left(\mathrm{rec}_{F,dm_j}(\mathrm{JL}(\omega_j))\right).$$

Comparing this with (70) concludes the proof. □

Now we are ready to complete the proof of Theorem 3.2, and hence of our main result Theorem 3.3.

Proof of Theorem 3.2 *when* $\mathrm{char}(F) = p > 0$.

Choose $r \in \mathbb{N}$ such that $\Pi_\phi(G) \in \mathrm{Irr}(G, K_r)$ and choose $l \in \mathbb{N}$ such that Theorem 6.2 applies. Find a p-adic field $\widetilde{F}$ which is l-close to F, fix a representative for ϕ and define $\widetilde{\phi}$ as the map $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ obtained from ϕ via (64). Thus, $\widetilde{\phi}$ is a particular representative for $\Phi_l^\zeta(\phi) \in \Phi_l(G)$. By Theorem 6.2 $\Pi_{\widetilde{\phi}}(\widetilde{G}) = \overline{\zeta_r^G}(\Pi_\phi(G))$ and by Theorem 5.1

$$\mathrm{End}_{\widetilde{G}}\big(\Pi_{\widetilde{\phi}}(\widetilde{G})\big) \cong \mathrm{End}_G\big(\Pi_\phi(G)\big).$$

Let $\phi^\sharp \in \Phi(G^\sharp)$ and $\widetilde{\phi}^\sharp \in \Phi(\widetilde{G}^\sharp)$ be the Langlands parameters obtained from ϕ and $\widetilde{\phi}$ via the quotient map $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$. By construction, $\phi^\sharp$ and $\widetilde{\phi}^\sharp$ have the same image in $\mathrm{PGL}_n(\mathbb{C})$, so

$$\mathcal{S}_{\widetilde{\phi}^\sharp} = \mathcal{S}_{\phi^\sharp} \quad \text{and} \quad \mathcal{Z}_{\widetilde{\phi}^\sharp} = \mathcal{Z}_{\phi^\sharp}. \tag{73}$$

With (22), this provides natural isomorphisms

$$X^G(\Pi_\phi(G)) \cong \mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp} = \mathcal{S}_{\widetilde{\phi}^\sharp}/\mathcal{Z}_{\widetilde{\phi}^\sharp} \cong X^{\widetilde{G}}(\Pi_{\widetilde{\phi}}(\widetilde{G})).$$

In view of (67), the composite isomorphism $X^{\widetilde{G}}(\Pi_{\widetilde{\phi}}(\widetilde{G})) \cong X^G(\Pi_\phi(G))$ comes from $F^\times/1 + \mathfrak{p}_F^r \cong \widetilde{F}^\times/1 + \mathfrak{p}_{\widetilde{F}}^r$. For $\widetilde{\gamma} \in X^{\widetilde{G}}(\Pi_{\widetilde{\phi}}(\widetilde{G}))$, choose

$$I_{\widetilde{\gamma}} \in \mathrm{Hom}_{\widetilde{G}}\big(\Pi_{\widetilde{\phi}}(\widetilde{G}), \Pi_{\widetilde{\phi}}(\widetilde{G}) \otimes \widetilde{\gamma}\big)$$

as in [29, § 12]. Then Theorem 5.1 yields intertwining operators
$I_\gamma \in \mathrm{Hom}_G\big(\Pi_\phi(G), \Pi_\phi(G) \otimes \gamma\big)$. Consequently

$$\kappa_{\phi^\sharp}(\gamma, \gamma') = I_\gamma I_{\gamma'} I_{\gamma\gamma'}^{-1} = I_{\widetilde{\gamma}} I_{\widetilde{\gamma}'} I_{\widetilde{\gamma}\widetilde{\gamma}'}^{-1} = \kappa_{\widetilde{\phi}^\sharp}(\widetilde{\gamma}, \widetilde{\gamma}'). \tag{74}$$

Because we already proved Theorem 3.2 for $\widetilde{F}$, this gives

$$\mathbb{C}[\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}, \kappa_{\phi^\sharp}] = \mathbb{C}[\mathcal{S}_{\widetilde{\phi}^\sharp}/\mathcal{Z}_{\widetilde{\phi}^\sharp}, \kappa_{\widetilde{\phi}^\sharp}] \cong e_{\chi_{\widetilde{G}}}\mathbb{C}[\mathcal{S}_{\phi^\sharp}] = e_{\chi_G}\mathbb{C}[\mathcal{S}_{\phi^\sharp}]. \tag{75}$$

That the isomorphism $\mathbb{C}[\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}, \kappa_{\phi^\sharp}] \cong e_{\chi_G}\mathbb{C}[\mathcal{S}_{\phi^\sharp}]$ is of the required form and that it is unique up to twists by characters of $\mathcal{S}_{\phi^\sharp}/\mathcal{Z}_{\phi^\sharp}$ follows from the corresponding statements over $\widetilde{F}$ and (73). □

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