SPACES OF GRAPHS, BOUNDARY GROUPOIDS AND THE COARSE BAUM-CONNES CONJECTURE

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ABSTRACT. We introduce a new variant of the coarse Baum-Connes conjecture designed to tackle coarsely disconnected metric spaces called the boundary coarse Baum-Connes conjecture. We prove this conjecture for many coarsely disconnected spaces that are known to be counterexamples to the coarse Baum-Connes conjecture. In particular, we give a geometric proof of this conjecture for spaces of graphs that have large girth and bounded vertex degree. We then connect the boundary conjecture to the coarse Baum-Connes conjecture using homological methods, which allows us to exhibit all the current uniformly discrete counterexamples to the coarse Baum-Connes conjecture in an elementary way.

1. INTRODUCTION AND OUTLINE

The coarse Baum-Connes conjecture for metric spaces plays a central role in answering positively certain topological and group theoretic problems [HR95]; it implies a positive solution to the Novikov conjecture for finitely generated discrete groups [Hig00, FW95] and information about the existence of positive scalar curvature metrics on spin manifolds [FW95, HR00]. It is well known that this conjecture has counterexamples [HR00, Hig99, WY12] in the class of coarsely disconnected spaces.

In this paper we introduce a new conjecture associated to uniformly discrete bounded geometry metric spaces that is tailored to studying the geometry of coarsely disconnected spaces by utilising their asymptotic geometry. For a space X this conjecture is phrased in terms of a certain Baum-Connes conjecture for a naturally constructed groupoid called the *boundary groupoid* of X, which we denote by $G(X)|_{\partial \beta X}$.

Conjecture 1 (Boundary Coarse Baum-Connes Conjecture). *Let X be a uniformly discrete bounded geometry metric space. Then the assembly map:*

$$\mu_{bdry} : K^{top}_*(G(X)|_{\partial \beta X}, A_\partial) \to K_*(A_\partial \rtimes_r G(X)|_{\partial \beta X})$$

is an isomorphism.

The coefficient algebra A_∂ in the conjecture above is related to the asymptotic geometry of X and occurs in the work of Yu and Oyono-Oyono [OOY09]. The idea here is that this conjecture should be easier to prove in those instances that the coarse Baum-Connes conjecture fails, an idea that we confirm during the first half of this paper.

In particular we prove the following result:

Theorem 2. *The boundary coarse Baum-Connes conjecture holds for the following classes of coarsely disconnected spaces:*

(1) box spaces of finitely generated a-T-menable groups;
(2) coarse disjoint unions constructed from sequences of finite graphs with large girth and uniformly bounded above regularity.

The first point is an easy observation outlined in Section 2, however the second point is rather more interesting and relies on carefully studying the natural geometry of large girth sequences to produce a *partial action* of free group [KL04]. These ideas are covered in Section 3.3.

The second objective of this paper is to connect the boundary coarse Baum-Connes conjecture, via homological methods, to the coarse Baum-Connes conjecture. Using this machinery we give elementary proofs of many of the counterexample arguments from [HLS02, Hig99, WY12] as well as many results concerning classes of expander graphs present in the literature [CTWY08, GTY11, OOY09]; in particular in Section 4 we give an elementary proof of results of Willett and Yu [WY12] concerning large girth sequences:

Theorem 3. *Let X be a space of graphs with large girth constructed from a sequence of finite graphs with uniformly bounded vertex degree. Then:*

(1) *the coarse Baum-Connes assembly map is injective for X.*
(2) *If X comes from an expanding sequence then the coarse Baum-Connes assembly map for X is not surjective.*

Finally in Section 5 we give a counterexample to the boundary coarse Baum-Connes conjecture by considering a space introduced by Wang [Wan07] and adaptations of the counterexample arguments present in the literature:

Theorem 4. *There is a coarsely disconnected space Y constructed from a box space of $SL_2(\mathbb{Z})$ for which the boundary coarse Baum-Connes assembly map is injective but fails to be surjective.*

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1.2. **Groupoids, expanders and the coarse Baum-Connes conjecture.** In this section we recall some definitions and results from the literature that will appear within the text. We begin with groupoids by focusing on certain relevant examples.

A groupoid is *principal* if $(r, s) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is injective and *transitive* if (r, s) is surjective. A groupoid $\mathcal{G}$ is a *topological groupoid* if both $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are topological spaces, and the maps $r, s,^{-1}$ and the composition are all continuous. A Hausdorff, locally compact topological groupoid $\mathcal{G}$ is *proper* if (r, s) is a proper map and *étale* or *r-discrete* if the map r is a local homeomorphism. When $\mathcal{G}$ is étale, s and the product are also local homeomorphisms, and $\mathcal{G}^{(0)}$ is an open subset of $\mathcal{G}$.

Definition 1.1. Let $\mathcal{G}$ be a groupoid and let $x, y \in \mathcal{G}^{(0)}$ and $A, B \subset \mathcal{G}^{(0)}$. Set:

(1) $\mathcal{G}_x = s^{-1}(x)$
(2) $\mathcal{G}^y = r^{-1}(y)$
(3) $\mathcal{G}_x^y = \mathcal{G}^y \cap \mathcal{G}_x$

Denote by $\mathcal{G}|_A$ the subgroupoid $\mathcal{G}_A^A$, called the *reduction* of $\mathcal{G}$ to A.

Definition 1.2. Let $\mathcal{G}$ be a locally compact groupoid and let Z be a locally compact space. $\mathcal{G}$ acts on Z (or Z is a $\mathcal{G}$-space) if there is a continuous, open map $r_Z : Z \to \mathcal{G}^{(0)}$ and a continuous map $(\gamma, z) \mapsto \gamma.z$ from $\mathcal{G} * Z := \{(\gamma, z) \in \mathcal{G} \times Z | s_\mathcal{G}(\gamma) = r_Z(z)\}$ to Z such that $r_Z(z).z = z$ for all z and $(\eta\gamma).z = \eta.(\gamma.z)$ for all $\gamma, \eta \in \mathcal{G}^{(2)}$ with $s_\mathcal{G}(\gamma) = r_Z(z)$.

When it is clear we drop the subscripts on each map. Right actions are dealt with similarly, replacing each incidence of r_Z with s_Z.

Definition 1.3. Let $\mathcal{G}$ act on Z. The action is said to be *free* if $\gamma.z = z$ implies that $\gamma = r_Z(z)$.

We end this section with some useful examples.

Example 1.4. Let X be a topological Γ-space. Then the *transformation groupoid* associated to this action is given by the data $X \times G \rightrightarrows X$ with $s(x, g) = x$ and $r(x, g) = g.x$. We denote this by $X \rtimes G$. A basis $\{U_i\}$ for the topology of X lifts to a basis for the topology of $X \rtimes G$, given by sets $[U_i, g] := \{(u, g) | u \in U_i\}$.

Example 1.5. The construction in the example above can be generalized to actions of étale groupoids. We are concerned with the topology here: Given an étale groupoid $\mathcal{G}$ and a $\mathcal{G}$-space X as well as a with a basis $\{U_i\}$ for $\mathcal{G}^{(0)}$. We can pull this basis back to a basis for $X \rtimes \mathcal{G}$ given by $[r_z^{-1}(U_i), \gamma]$, where $U_i \subseteq s(\gamma)$.

Example 1.6. Let X be a coarse space with uniformly locally finite, weakly connected coarse structure $\mathcal{E}$. Define $G(X) := \cup_{E \in \mathcal{E}} \overline{E} \subseteq \beta(X \times X)$. Then $G(X)$ is a locally compact, Hausdorff topological space. To equip it with a product and inverse we would ideally have liked to be considering the natural extension of the pair groupoid product on $\beta X \times \beta X$. We remark that the map (r, s) from $X \times X$ extends first to an inclusion into $\beta X \times \beta X$ and universally to $\beta(X \times X)$, giving a map $(r, s) : \beta(X \times X) \to \beta X \times \beta X$. We can restrict this map to each entourage $E \in \mathcal{E}$ allowing us to map the set $G(X)$ to $\beta X \times \beta X$.

Lemma 1.7. [Roe03, Corollary 10.18] *Let X be a uniformly discrete bounded geometry metric space and let E be any entourage. Then the inclusion $E \to X \times X$ extends to an injective homeomorphism $\overline{E} \to \beta X \times \beta X$, where $\overline{E}$ denotes the closure of E in $\beta(X \times X)$.*

Using this Lemma, we can conclude that the groupoid $G(X)$ can be embedded topologically into $\beta X \times \beta X$ and so we can equip it with the induced product and inverse.

As we are considering the metric coarse structure we can reduce this to considering only generators:

$$G(X) := \bigcup_{R > 0} \overline{\Delta_R}$$

This groupoid plays an important role the groupoid formulation of the coarse Baum-Connes conjecture and the most general consequences of that conjecture. We recall some results concerning this groupoid from the literature:

Theorem 1.8. *Let X be a uniformly discrete bounded geometry metric space. Then following hold:*

> *(1) $G(X)$ is an étale locally compact Hausdorff principal topological groupoid with unit space $G(X)^{(0)} = \beta X$. [Roe03, Theorem 10.20][STY02, Proposition 3.2];*

(2) $C_r^*(G(X))$ *is isomorphic to the uniform Roe algebra* $C_u^*(X)$. [Roe03, Proposition 10.29];

(3) *The coarse Baum-Connes conjecture for* X *is equivalent to the Baum-Connes conjecture for* $G(X)$ *with coefficients in* $\ell^\infty(X, \mathcal{K})$. [STY02, Lemma 4.7].

Lastly, we recall the definition of an expander:

Definition 1.9. Let $\{X_i\}$ be a sequence of finite graphs and let X be the associated space of graphs. Then the space X (or the sequence $\{X_i\}$) is an *expander* if:

(1) There exists $k \in \mathbb{N}$ such that all the vertices of each X_i have degree at most k.
(2) $|X_i| \to \infty$ as $i \to \infty$.
(3) There exists $c > 0$ such that $spectrum(\Delta_i) \subseteq \{0\} \cup [c, 1]$ for all i.

Remark 1.10. Each Laplacian Δ_i has propagation 1, so we can form the product in the (algebraic) Roe algebra:

$$\Delta := \prod_i (\Delta_i \otimes q) \in C_u^*(X) \otimes \mathcal{K} \subset C^*X$$

Now we can consider projection p onto the kernel of Δ. For an expander X we have $spectrum(\Delta) \subseteq \{0\} \cup [c, 1]$ for some $c > 0$, and so by an application of the functional calculus we can conclude that $p \in C^*X$. As Δ breaks up as a product we observe that its $Ker(\Delta) = \oplus_i Ker(\Delta_i)$ and so the projection p decomposes as a product:

$$p = \prod_i p^{(i)}.$$

Additionally, it is easy to see that a function in $\ell^2(X_i)$ is an element of the kernel of Δ_i if and only if it is a constant function, and so each p^i has matrix entries $p_{x,y}^i = \frac{1}{|X_i|}$.

The following notion is due to Guoliang Yu (unpublished):

Definition 1.11. An operator $T \in C^*X$ is a *ghost operator* if $\forall \epsilon > 0$ there exists a bounded subset $B \subset X \times X$ such that the norm: $\|T_{xy}\| \leq \epsilon$ for all $(x, y) \in (X \times X) \setminus B$.

It is clear, from the definition as a product given above, that the kernel p of the Laplacian Δ is a ghost operator.

2. The Boundary Coarse Baum-Connes Conjecture.

Throughout this section let G be an étale, Hausdorff locally compact topological groupoid. For what we outline below these are essentially unnecessary assumptions, however our focus on the coarse groupoid leads us to study this class. We outline the main concept introduced in [HLS02].

Definition 2.1. A subset of $F \subseteq G^{(0)}$ is said to be *saturated* if for every element of $\gamma \in G$ with $s(\gamma) \in F$ we have $r(\gamma) \in F$. For such a subset we can form subgroupoid of G, denoted by G_F which has unit space F and $G_F^{(2)} = \{\gamma \in G | s(\gamma) \text{ and } r(\gamma) \in F\}$.

We will be considering closed saturated subsets F. Any subset of this type gives rise to an algebraic decomposition:

$$G = G_{F^c} \sqcup G_F$$

This lets us construct maps on the $*-$algebras of compactly supported functions associated with G, G_F and G_{F^c}:

$$0 \to C_c(G_{F^c}) \to C_c(G) \to C_c(G_F) \to 0.$$

Where the quotient map $C_c(G) \to C_c(G_F)$ is given by restriction and the inclusion $C_c(G_{F^c}) \to C_c(G)$ is given by extension. By the functorial properties of the maximal C^*-norm this extends to the maximal groupoid C^*-algebras:

$$0 \to C^*_{max}(G_{F^c}) \to C^*_{max}(G) \to C^*_{max}(G_F) \to 0.$$

On the other hand this may fail to be an exact sequence when we complete in the norm that arises from the left regular representation λ_G. This is detected at the level of K-theory, as discussed in [HLS02], by considering the sequence:

(1) $$K_0(C^*_r(G_U)) \to K_0(C^*_r(G)) \to K_0(C^*_r(G_F))$$

This was used in [HLS02] to construct multiple different types of counterexample to the Baum-Connes conjecture for groupoids. We observe that whilst the sequence:

$$0 \longrightarrow C^*_r(G_{F^c}) \overset{\alpha}{\longrightarrow} C^*_r(G) \overset{q}{\longrightarrow} C^*_r(G_F) \longrightarrow 0$$

may fail to be exact in the middle term the maps α and q both exist and the map q is surjective by considering the following diagram.

$$
\begin{array}{ccc}
C^*_{max}(G) & \longrightarrow & C^*_{max}(G_F) \\
\downarrow & & \downarrow \\
C^*_r(G) & \longrightarrow & C^*_r(G_F)
\end{array}
$$

It is also clear that the image of α is contained in the kernel of q, whence we can make the sequence exact artificially by replacing $C^*_r(G_{F^c})$ by the ideal $I := \ker(q)$. We can then define a new assembly map in the first term to be the composition of the original assembly map μ_{F^c} and the K-theory map induced by inclusion $i_* : K_*(C^*_r(G_{F^c})) \to K_*(I)$. Then in terms of assembly maps this gives us a new commutative diagram:

$$
\begin{array}{ccccccccc}
\twoheadrightarrow K_1(C^*_r(G_F) & \longrightarrow & K_0(I) & \longrightarrow & K_0(C^*_r(G)) & \twoheadrightarrow & K_0(C^*_r(G_F) & \longrightarrow & K_1(I) \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow K_1^{top}(G_F) & \longrightarrow & K_0^{top}(G_{F^c}) & \longrightarrow & K_0^{top}(G) & \longrightarrow & K_0^{top}(G_F) & \longrightarrow & K_1^{top}(G_{F^c}) \twoheadrightarrow
\end{array}
$$

where the rows here are exact. As in [HLS02] we now consider $G = G(X)$, the coarse groupoid associated to some uniformly discrete bounded geometry metric space X.

2.1. The Coarse Groupoid Conjecture.

Let X be a uniformly discrete bounded geometry metric space. From what was described above we can associate to each closed saturated subset F of the unit space space βX a long exact sequence in K-theory. We consider the obvious

closed saturated subset: $\partial\beta X \subset G(X)^{(0)}$. This gives us the following commutative diagram (omitting coefficients):

$$K_1(C_r^*(G(X)|_{\partial\beta X})) \longrightarrow K_0(I) \longrightarrow K_0(C_r^*(G(X))) \longrightarrow K_0(C_r^*(G(X)|_{\partial\beta X})) \longrightarrow K_1(I)$$
$$K_1^{top}(G(X)|_{\partial\beta X}) \longrightarrow K_0^{top}(X \times X) \longrightarrow K_0^{top}(G(X)) \longrightarrow K_0^{top}(G(X)|_{\partial\beta X}) \longrightarrow K_1^{top}(X \times X)$$

with the vertical μ_{bdry} map.

We can now properly formulate the boundary conjecture by replacing the coefficients. Let $A_\partial := \frac{\ell^\infty(X,\mathcal{K})}{C_0(X,\mathcal{K})}$

Conjecture 2.2 (Boundary Coarse Baum-Connes Conjecture). *Let X be a uniformly discrete bounded geometry metric space. Then the assembly map:*

$$\mu_{bdry} : K_*^{top}(G(X)|_{\partial\beta X}, A_\partial) \to K_*(A_\partial \rtimes_r G(X)|_{\partial\beta X})$$

is an isomorphism.

2.2. **Tools to Prove the Boundary Conjecture.**
We consider the situation when a group Γ acts on a space of graphs X. We introduce a definition:

Definition 2.3. Let $\mathcal{E}$ be a coarse structure on X and let $\mathcal{S}$ be a family of subsets $\mathcal{E}$. We say that S generates $\mathcal{E}$ at infinity if for all $E \in \mathcal{E}$:

$$E \subseteq (\bigcup_{k=1}^n S_k) \cup F$$

Where each $S_k \in \mathcal{S}$ and F is a finite subset of $X \times X$.

Remark 2.4. The above definition is equivalent to asking that $\overline{E} \setminus E \subseteq \bigcup_{k=1}^n \overline{S_k} \setminus S_k$, where the closure takes place in $\beta X \times \beta X$.
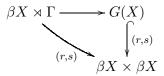
Recall that $\Delta_g := \{(x, x.g)|x \in X\}$ is the g-diagonal in X.

Proposition 2.5. *Let X be uniformly discrete bounded geometry metric space and let Γ be a finitely generated discrete group. If Γ acts on X such that the induced action on βX is free on $\partial\beta X$ and the action generates the metric coarse structure at infinity. Then $G(X)|_{\partial\beta X} \cong \partial\beta X \rtimes \Gamma$.*
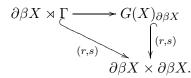
Proof. Recall that under the hypothesis that the Γ action generates the metric coarse structure at infinity we know that $G(X) = (\bigcup_g \overline{\Delta_g}) \cup (X \times X)$.

We consider a map from transformation groupoid $\beta X \rtimes \Gamma$ to $G(X)$. Observe that Δ_g is the bijective image of the set $\{(x,g)|x \in X\}$. We extend this map to the respective closures, giving a map for each g from the set of $\{(\omega, g)|\omega \in \beta X\}$ to $\overline{\Delta_g}$. Let $\{x_\lambda\}$ be a net of elements in X that converge to ω. Then clearly each pair (x_λ, g) is mapped to $(x_\lambda, x_\lambda.g)$ under the identification. Consider the element of the closure $\gamma_g = \lim_\lambda (x_\lambda, x_\lambda.g) \in \overline{\Delta_g}$ and define the extension of the bijection to be the map: $(\omega, g) \mapsto \gamma_g$.

This map is well-defined as $G(X)$ is principal and so γ_g is completely determined by its source and range - in particular any other net y_λ that converges to ω gives rise to the same element γ_g. We can then extend this over the entire groupoid $\beta X \rtimes \Gamma$ element wise, where it is certainly continuous but not in general injective or surjective (injectivity would require a free action and surjectivity a transitive one).

The map is a groupoid homomorphism because $G(X)$ is principal; this follows as principality implies the following diagram commutes:

$$\begin{array}{ccc} \beta X \rtimes \Gamma & \longrightarrow & G(X) \\ & \searrow_{(r,s)} & \big\uparrow{\scriptstyle (r,s)} \\ & & \beta X \times \beta X \end{array}$$

We now restrict this map to the boundary $\partial \beta X$. We prove that the coronas $\overline{\Delta_g} \setminus \Delta_g$ are disjoint: let $\gamma \in \overline{\Delta_g} \setminus \Delta_g$ and $\overline{\Delta_h} \setminus \Delta_h$ for $g \neq h$. Then we have that $s(\gamma) = \omega, r(\gamma) = \omega.g = \omega.h$, hence we have a fixed point on the boundary, which is a contradiction. Consider the restricted diagram:

$$\begin{array}{ccc} \partial\beta X \rtimes \Gamma & \longrightarrow & G(X)_{\partial\beta X} \\ & \searrow_{(r,s)} & \big\uparrow{\scriptstyle (r,s)} \\ & & \partial\beta X \times \partial\beta X. \end{array}$$

As the coronas are disjoint, for any $\gamma \in G(X)$ it is possible to find a unique g and ω such that $s(\gamma) = \omega, r(\gamma) = \omega.g$. It then follows that the pair (ω, g) map onto γ. Hence the map is a surjection. To see injectivity, we appeal again to freeness of the action. The action being free implies that the groupoid $\partial\beta X \rtimes \Gamma$ is principal. From a brief consideration of the diagram above injectivity follows.

It remains to consider the topology. For each $g \in \Gamma$, the map from $\partial\beta X \rtimes \Gamma$ to $G(X)_{\partial\beta X}$ on the piece $[\partial\beta X, g]$ is a homeomorphism as $\overline{\Delta_g} \setminus \Delta_g$ can be identified with $\partial\beta X$, induced by the source map or range map. Hence the map takes clopen sets to clopen sets; these form a basis for the topology of $G(X)|_{\partial\beta X}$ and so the map is a global homeomorphism. $\square$

Example 2.6. (Box spaces) Let Γ be a finitely generated residually finite group and let $\{N_i\}$ be a family of finite index normal subgroups such that $N_i \leq N_{i+1}$, $\bigcap_{i \in \mathbb{N}} N_i = 1$ and a fixed generating set S. Then the sequence of groups $\{\Gamma/N_i\}$ with generating sets $\pi_i(S)$ admits an action via quotient maps. Let $\square\Gamma = \sqcup_{i \in \mathbb{N}} \frac{\Gamma}{N_i}$, equipped with a metric that restricts to the metric induced from the generating sets $\pi_i(S)$ for each i, and has the property that $d(\frac{\Gamma}{N_i}, \frac{\Gamma}{N_j}) \to \infty$ as $i + j \to \infty$. This is called a *box space* for Γ. The Stone-Čech boundary admits a free action of the group and the metric structure is generated at infinity by the quotients maps and the right action of the group via these maps. Proposition 2.5 then provides a description of the boundary groupoid and converts the boundary conjecture into a case of the Baum-Connes conjecture with coefficients for Γ.

This process did not require a normal subgroup; box spaces can constructed using Schreier quotients. The conditions on the subgroups change to reflect the absence of normality. Let Γ be a residually finite group and let $\{H_i\}$ be a family of nested subgroups of finite index with trivial intersection, and additionally satisfying: each $g \in \Gamma$ belongs to only finitely many conjugates of the subgroups from the family $\{H_i\}$. Fixing a left invariant metric on Γ the box space can be constructed using the left quotients of Γ by the H_i. In this instance these spaces are graphs with no left action of Γ. However they do retain a right action of Γ that determines the metric at infinity and becomes free on the boundary (this is due to the additional constraint concerning conjugates of the H_i).

3. A Proof of the Boundary Conjecture for Spaces of Graphs with Large Girth.

The aim of this section is to prove the following result:

Theorem. *The boundary coarse Baum-Connes conjecture holds for spaces of graphs with large girth and uniformly bounded vertex degree.*

For this result we will need a weaker notion of action; We construct a partial action of F_k for some finite k. Using a stronger version of Proposition 2.5 we will show that the boundary groupoid $G(X)|_{\partial \beta X}$ is a transformation groupoid $\partial \beta X \rtimes \mathcal{G}_{\widehat{X}}$ for some étale groupoid $\mathcal{G}_{\widehat{X}}$. As before, this converts the boundary Baum-Connes conjecture into a special case of Baum-Connes with coefficients for the groupoid $\mathcal{G}_{\widehat{X}}$. Lastly, we construct a continuous, proper groupoid homomorphism from $\mathcal{G}_{\widehat{X}}$ to F_k, which we use to transfer the Haagerup property from F_k to the groupoid $\mathcal{G}_{\widehat{X}}$. This will complete the proof.

As noted above we want to consider partial actions of a group Γ on a space X. This means that the elements of Γ give rise to partial bijections of X, i.e bijections between subsets of X. These partial bijections are algebraically encoded within an inverse semigroup, and the partial action of Γ is a *dual prehomomorphism* from Γ into that semigroup.

3.1. **An Interlude into Inverse Semigroup Theory.**

Definition 3.1. Let S be a semigroup. We say S is *inverse* if there exists a unary operation $*: S \to S$ satisfying the following identities:

(1) $(s^*)^* = s$
(2) $ss^*s = s$ and $s^*ss^* = s^*$ for all $s \in S$
(3) $ef = fe$ for all idempotents $e, f \in S$

Recall that a semigroup with a unit element is called a *monoid*, and it makes sense to talk about inverse monoids in the obvious way. A very fundamental example is the *symmetric inverse monoid* on any set X; consider the collection of all partial bijections of X to itself, giving them them the natural composition law associated to functions, finding the largest possible domain.

When X is a metric space we will be considering a inverse submonoid of $I(X)$ in which every partial bijection that maps elements only a finite distance, that is a generalised (or partial) translation. We denote this by $I_b(X)$.

Definition 3.2. Let S be an inverse monoid. We denote by $E(S)$ the semilattice of idempotents (just by E if the context is clear). This is a meet semilattice, where the meet is given by the product of S restricted to E. In this situation, we can use the following partial order:

$$e \leq f \Leftrightarrow ef = e$$

This order can be extended naturally to the entire of S: $s \leq t$ if there exists $e \in E(S)$ such that $s = et$. In terms of partial bijections this order corresponds to restricting an element to a subset of its domain.

We remark that for a metric space X every idempotent element in $I(X)$ moves elements no distance, and hence $E(I(X)) = E(I_b(X))$.

We want to consider quotient structures of an inverse monoid. In general quotients are given by equivalence relations that preserve the semigroup structure.

Definition 3.3. An equivalence relation $\sim$ on S is called a *congruence* if for every $u, v, s, t \in S$ such that $s \sim t$, we know that $su \sim tu$ and $vs \sim vt$. This allows us to equip the quotient $\frac{S}{\sim}$ with a product, making it into an inverse monoid.

One such example of this arises from an *ideal* in S.

Definition 3.4. Let I be a subset of S. I is an ideal of S if $SI \cup IS \subset I$.

From an ideal we can get a quotient - at the cost of a *zero element*, that is an element 0 such that $0s = s0 = 0$ for all $s \in S$.

Definition 3.5. Let S be an inverse monoid and let I be an ideal of S. Then we can define $\frac{S}{I}$ to be the quotient of the set S by the congruence: $x \sim y$ if $x = y$ or x and y are elements of I.

Another specific congruence we will be interested in is called the minimum group congruence on S. This congruence, denoted by σ, is given by:

$$s\sigma t \Leftrightarrow (\exists e \in E)es = et$$

This congruence is *idempotent pure*, that is for $e \in E(S)$ and $s \in S$, $e \sim s$ implies $s \in E$.

Definition 3.6. We say S is 0-E-unitary if $\forall e \in E \setminus 0, s \in S \setminus 0$ $e \le s$ implies $s \in E$. We say it is 0-F-inverse if in addition there exists a subset $T \subset S$ such that for every $s \in S$ there exists a unique $t \in T$ such that $s \le t$ and if for all $u \in S$ with $s \le u$ then $u \le t$.

3.2. Groupoids from Inverse Monoids.

We take an inverse monoid S and produce a universal groupoid $\mathcal{G}_{\widehat{E}}$. One way to do this involves studying the actions of S on its semilattice E.

We outline the steps in the construction.

(1) Build an action of S on E.
(2) Build a dual space to E, which is compact and Hausdorff. This is a *Stone dual* to E. Show this admits an action of S.
(3) Build the groupoid $\mathcal{G}_{\widehat{E}}$ from this data.

Definition 3.7. (1) Let $D_e = \{f \in E | f \le e\}$. For $ss^* \in E$, we can define a map $\rho_s(ss^*) = s^*s$, extending to D_{ss^*} by $\rho_s(e) = s^*es$. This defines a partial bijection on E from D_{ss^*} to D_{s^*s}.

(2) We consider a subspace of $\mathbf{2}^E$ given by the functions ϕ such that $\phi(0) = 0$ and $\phi(ef) = \phi(e)\phi(f)$. This step is a generalisation of Stone duality [Law10]. We can topologise this as a subspace of $\mathbf{2}^E$, where it is closed. This makes it compact Hausdorff, with a base of topology given by $\widehat{D}_e = \{\phi \in \widehat{E} | \phi(e) = 1\}$. This admits a dual action induced from the action of S on E. This is given by the pointwise equation for every $\phi \in \widehat{D}_{s^*s}$:

$$\widehat{\rho}_s(\phi)(e) = \phi(\rho_s(e)) = \phi(s^*es)$$

The use of $\widehat{D}_e$ to denote these sets is not a coincidence, as we have the following map $D_e \to \widehat{D}_e$:

$$e \mapsto \phi_e, \phi_e(f) = 1 \text{ if } e \le f \text{ and } 0 \text{ otherwise }.$$

Remark 3.8. These character maps $\phi : E \to \{0, 1\}$ have an alternative interpretation, they can be considered as *filters* on E. A filter on E is given by a set $F \subset E$ with the following properties:

- for all $e, f \in F$ we have that $e \wedge f = ef \in F$
- for $e \in F$ with $e \leq f$ we have that $f \in F$ and
- $0 \notin F$

the relationship between characters and filters can be summarised as: To each character ψ there is a filter:

$$F_\psi = \{e \in E | \psi(e) = 1\}.$$

And every filter F provides a character by considering χ_F, its characteristic function.

(3) We take the set $S \times \widehat{E}$, topologise it as a product and consider subset $\Omega := \{(s, \phi) | \phi \in \widehat{D}_{s^*s}\}$ in the subspace topology. We then quotient this space by the relation:

$$(s, \phi) \sim (t, \phi') \Leftrightarrow \phi = \phi' \text{ and } (\exists e \in E) \text{ with } \phi \in \widehat{D}_e \text{ such that } es = et$$

We can give the quotient $\mathcal{G}_{\widehat{E}}$ a groupoid structure with the product set, unit space and range and source maps:

$$\mathcal{G}_{\widehat{E}}^{(2)} := \{([s, \phi], [t, \phi']) | \phi = \widehat{\rho}_t(\phi')\}$$

$$\mathcal{G}_{\widehat{E}}^{(0)} := \{[e, \phi] | e \in E\} \cong \widehat{E}$$

$$s([t, \phi]) = [t^*t, \phi], r([t, \phi]) = [tt^*, \phi],$$

and product and inverse:

$$[s, \phi].[t, \phi'] = [st, \phi'] \text{ if } ([s, \phi], [t, \phi']) \in \mathcal{G}_{\widehat{E}}^{(2)}, [s, \phi]^{-1} = [s^*, \widehat{\rho}_s(\phi)]$$

For all the details of the above, we refer to [Exe08, Section 4]. This is the *universal groupoid* associated to S. We collect some information about this groupoid from [Exe08, Pat99] in Theorem 3.9.

Theorem 3.9. *Let S be a countable 0-E-unitary inverse monoid, E its semilattice of idempotents and $\mathcal{G}_{\widehat{E}}$ its universal groupoid. Then the following hold for $\mathcal{G}_{\widehat{E}}$:*

- $\widehat{E}$ *is a compact, Hausdorff and second countable space.*
- $\mathcal{G}_{\widehat{E}}$ *is a Hausdorff groupoid* [Exe08, Corollary 10.9].
- *Every representation of S on Hilbert space gives rise to a covariant representation of $\mathcal{G}_{\widehat{E}}$ and vice versa* [Exe08, Corollary 10.16].
- *We have $C_r^*(S) \cong C_r^*(\mathcal{G}_{\widehat{E}})$* [Pat99, KS02]

We make use of the following technical property that arises from the presence of maximal elements:

Claim 3.10. *Let S be 0-F-inverse. Then every element $[s, \phi] \in \mathcal{G}_{\widehat{E}}$ has a representative $[t, \phi]$ where t is a maximal element.*

Proof. Take $t = t_s$ the unique maximal element above s. Then we know

$$s = t_s s^* s \text{ and } s^* s \leq t_s^* t_s$$

The second equation tells us that $t_s^* t_s \in F_\phi$ as filters are upwardly closed, thus (t_s, ϕ) is a valid element. Now to see $[t_s, \phi] = [s, \phi]$ we need to find an $e \in E$ such that $e \in F_\phi$ and $se = t_s e$. Take $e = s^* s$ and then use the first equation to see that $s(s^* s) = t_s(s^* s)$. $\qquad\square$

3.3. (Dual) Prehomomorphisms and General Partial Actions.

Definition 3.11. Let $\rho : S \to T$ be a map between inverse semigroups. This map is called a prehomomorphism if for every $s, t \in S$, $\rho(st) \leq \rho(s)\rho(t)$ and a dual prehomomorphism if for every $s, t \in S$ $\rho(s)\rho(t) \leq \rho(st)$.

We recall that a congruence is said to be *idempotent pure* for any $e \in E(S)$, $s \in S$ we have that s is related to e implies that $s \in E(S)$. We extend this definition to general maps in the following way.

Definition 3.12. A (dual) prehomomorphism ρ is called *idempotent pure* if $\rho(s)^2 = \rho(s)$ implies $s \in E$.

In addition we call a map $S \to T$ *0-restricted* if the preimage of $0 \in T$ is $0 \in S$.

Definition 3.13. Let S be a 0-E-unitary inverse monoid. We say S is *strongly 0-E-unitary* if there exists an idempotent pure, 0-restricted prehomomorphism, Φ to a group G with a zero element adjoined, that is: $\Phi : S \to G^0$. We say it is *strongly 0-F-inverse* if it is 0-F-inverse and strongly 0-E-unitary. This is equivalent to the fact that the preimage of each group element under Φ contains a maximum element.

Example 3.14. In [BR84, LMS06] the authors introduce an inverse monoid that is universal for dual prehomomorphisms from a general inverse semigroup. In the context of a group G This is called the *prefix expansion*; its elements are given by pairs: (X, g) for $\{1, g\} \subset X$, where X is a finite subset of G. The set of such (X, g) is then equipped with a product and inverse:
$$(X, g)(Y, h) = (X \cup gY, gh) \ , \ (X, g)^{-1} = (g^{-1}X, g^{-1})$$
This has maximal group homomorphic image G, and it has the universal property that it is the largest such inverse monoid. We denote this by G^{Pr}. The partial order on G^{Pr} can be described by reverse inclusion, induced from reverse inclusion on finite subsets of G. It is F-inverse, with maximal elements: $\{(\{1, g\}, g) : g \in G\}$.

Definition 3.15. Let G be a finitely generated discrete group and let X be a (locally compact Hausdorff) topological space. A *partial action* of G on X is a dual prehomomorphism θ of G in the symmetric inverse monoid $\mathcal{I}(X)$ that has the following properties:

(1) The domain $D_{\theta_g^* \theta_g}$ is an open set for every g.
(2) θ_g is a continuous map.
(3) The union: $\bigcup_{g \in G} D_{\theta_g^* \theta_g}$ is X.

Given this data we can generate an inverse monoid S using the set of θ_g. This would then give a representation of S into $\mathcal{I}(X)$. If the space X is a coarse space, then it makes sense to ask if each θ_g is a close to the identity. In this case, we would get a representation into the bounded symmetric inverse monoid $\mathcal{I}_b(X)$. We call such a θ a *bounded partial action* of G.

Let $X = \sqcup X_i$ be a space of graphs admitting a bounded partial action of a discrete group G. We remark that in this setting partial bijections in the group can have the following form:

$$\theta_g = \theta_g^0 \sqcup \bigsqcup_{i > i_0} \theta_g^i.$$

Where i_0 is the first i for which the distances between the X_is is greater than the upper bound of the distance moved by θ_g, and the θ_g^i are componentwise partial bijections of the X_i. We collect all the additional pieces that act only between the first i_0 terms into θ_g^0, which could be the empty translation. We remark now that it is possible that there are partial bijections θ_g that could have finite support, that is only finitely many terms that are non-empty after i_0. To avoid this we observe the following:

Proposition 3.16. *Let $S = \langle \theta_g | g \in G \rangle$ and let $I_{fin} = \{\theta_g | supp(\theta_g) \text{ is finite}\}$. Then I_{fin} is an ideal and the Rees quotient $S_{inf} = \frac{S}{I}$ is an inverse monoid with 0.*

Proof. To be an ideal, it is enough to show that $I_{fin}S \subset I_{fin}, SI_{fin} \subset I_{fin}$. Using the description of the multiplication of partial bijections from section 3.1, it is clear that either combination si or is yields an element of finite support. Now we can form the Rees quotient, getting an inverse monoid with a zero - the zero element being the equivalence class of elements with finite support. $\qquad\square$

We want to utilise a partial action to construct a groupoid, so we apply the general construction outlined in section 3.1 to get an improved version of Proposition 2.5. To do this requires a notion of translation length:

Definition 3.17. The length of each θ_g is defined to be:

$$|\theta_g| = \sup\{d(x, \theta_g(x)) : x \in Dom(\theta_g)\}.$$

Definition 3.18. Recall that we say a bounded partial action θ generates the metric coarse structure at infinity if for all $R > 0$ there exists $S > 0$ such that $\overline{\Delta_R} \setminus \Delta_R \subseteq \bigcup_{|\theta_g| < S} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$. We say it finitely generates the metric coarse structure if the number of θ_g required for each R is finite.

Remark 3.19. Recall a groupoid G is said to be *principal* if the map $(s, r) : G \to G^{(0)} \times G^{(0)}$ is injective.

Proposition 3.20. *Let $\{X_i\}$ be a sequence of finite graphs and let X be the corresponding space of graphs. If $\theta : G \to \mathcal{I}(X)$ is a bounded partial action of G on X such that the induced action on βX is free on $\partial \beta X$, the inverse monoid S_{inf} is 0-F-inverse with maximal element set $\{\theta_g | g \in G\}$ and the partial action finitely generates the metric coarse structure at infinity then there is a second countable, étale topological groupoid $\mathcal{G}_{\widehat{X}}$ such that $G(X)|_{\partial \beta X} \cong \partial \beta X \rtimes \mathcal{G}_{\widehat{X}}$.*

Proof. Observe now that the finite θ_g play no role in the action on the boundary and so we work with S_{inf}. We build the groupoid from the bottom up, by first constructing the unit space using Proposition 10.6 and Theorem 10.16 from [Exe08].

We consider the representation of the inverse monoid S_{inf} on $\ell^2(X)$ induced by θ to get a representation $\pi_\theta : S \to \mathcal{B}(\ell^2(X))$. We can complete the semigroup ring in this representation to get an algebra $C^*_{\pi_\theta}S$, which has a unital commutative subalgebra $C^*_{\pi_\theta}E$. Proposition 10.6 [Exe08] then tells us that the spectrum of this algebra, which we will denote by $\widehat{X}$, is a

subspace of $\widehat{E}$ that is closed and invariant under the action of S on which the representation π_θ is supported.

As the space $\widehat{X}$ is closed and invariant we can reduce the universal groupoid $\mathcal{G}_{\widehat{E}}$ for S_{inf} to $\widehat{X}$. This we denote by $\mathcal{G}_{\widehat{X}}$.

We show this groupoid acts on βX; we make use of the assumption that θ_g is a bounded partial bijection for each $g \in G$ and again of the representation π_θ. Each θ_g bounded implies that the algebra $C^*_\theta(S)$ is a subalgebra of $C^*_u X$. We now remark that the representation π_X, when restricted to C^*E assigns each idempotent a projection in $C^*_u X$, that is $C^*_{\pi_X}(E) = \pi_X(C^*E) \subset \ell^\infty(X)$. Taking the spectra associated to this inclusion then gives us a map:

$$r_{\beta X} : \beta X \twoheadrightarrow \widehat{X}$$

which is continuous. In particular as both βX and $\widehat{X}$ are compact Hausdorff spaces; this map is closed (and open) and hence a quotient. We make use of this to define an action on βX. By Claim 3.10 we have that each element of our groupoid $\mathcal{G}_{\widehat{X}}$ can be represented by a pair $[\theta_g, \phi]$, for some $\phi \in \widehat{X}$. Observe also that as X is discrete so are all of it subspaces, hence the maps θ_g are continuous (open) for each $g \in G$. These then extend to βX, and so coupled with the map $r_{\beta X}$ we can act by:

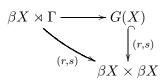$$[\theta_g, \phi].\omega = \theta_g(\omega)$$

for all $\omega \in D_{\theta_g^* \theta_g}$ with $r_Z(\omega) = \phi$. We see that $r_Z(\omega).\omega = [\theta_e, r_Z(\omega)].\omega = \omega$. and for all $([\theta_g, \phi], [\theta_h, \phi']) \in \mathcal{G}_{\widehat{X}}^{(2)}$ with $\phi' = r_Z(\omega)$ we have:

$$[\theta_g \theta_h, \phi'].\omega = \theta_g \theta_h(\omega) = \theta_g([\theta_h, \phi'].\omega) = [\theta_g, \phi].([\theta_h, \phi'].\omega)$$

as $r_Z([\theta_h, \phi'].\omega) = \theta_h(\phi') = \phi$.

It remains to prove the isomorphism of topological groupoids: $G(X)|_{\partial \beta X} \cong \partial \beta X \rtimes \mathcal{G}_{\widehat{X}}$. We follow the scheme of Proposition 2.5 and build a map from $\beta X \rtimes \mathcal{G}_{\widehat{X}}$ to $G(X)$. Recall that as the partial action of G generates the metric coarse structure at infinity $G(X) = (\bigcup_g \Delta_{\theta_g}) \cup (X \times X)$. We observe that each Δ_{θ_g} maps bijectively onto the domain of θ_g, a subset of X.

This map extends to the closure of the domain precisely as in Proposition 2.5, where here we map the pair $(\omega, [\theta_g, \phi])$ to the element $\gamma_{g,\phi}$ that is the limit $\lim_\lambda (x_\lambda, \theta_g(x_\lambda))$ for some net $\{x_\lambda\}$ that converges to ω (and also to ϕ). This map is well defined as the groupoid $G(X)$ is principal, and it fits into the following commutative diagram:

$$
\begin{array}{ccc}
\beta X \rtimes \Gamma & \longrightarrow & G(X) \\
& {\scriptstyle (r,s)} \searrow & \downarrow {\scriptstyle (r,s)} \\
& & \beta X \times \beta X
\end{array}
$$

Again by principality, we can deduce that the covering map is a groupoid homomorphism.

We now restrict this map to the boundary $\partial \beta X$. As we know that the group action generates the metric coarse structure at infinity and that the partial action of the group G is free on the boundary. Using these facts we can see that:
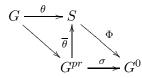
(1) $\partial \beta X \rtimes \mathcal{G}_{\widehat{X}}$ is principal.
(2) $G(X)|_{\partial \beta X} = \bigsqcup_g \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$.

From both (1) and (2) we can further deduce that the covering map is a bijection on the boundary. Both groupoids are also étale and so each component $\overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$ is mapped homeomorphically onto its image and is therefore clopen. It follows then that we get the desired isomorphism $\partial\beta X \rtimes \mathcal{G}_{\widehat{X}} \cong G(X)|_{\partial\beta X}$ of topological groupoids. $\square$

We are interested in understanding the analytic properties of the groupoid $\mathcal{G}_{\widehat{X}}$. In particular we are interested in showing that the groupoid has the Haagerup property. To do this we study the inverse monoid S associated to the partial action θ.

Proposition 3.21. *Let $S = \langle \theta_g | g \in G \rangle$, where $\theta : G \to S$ is a dual prehomomorphism. If S is 0-F-inverse with $Max(S) = \{\theta_g | g \in G\}$ where each nonzero θ_g is not idempotent when $g \neq e$ then S is strongly 0-F-inverse.*

Proof. We build a map Φ back onto G^0. Let $m : S \setminus \{0\} \to Max(S)$ be the map that sends each non-zero s to the maximal element $m(s)$ above s and consider the following diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\;\theta\;} & S \\
 & \searrow \overline{\theta} \quad \uparrow \overline{\theta} \quad \Phi \searrow & \\
 & G^{pr} \xrightarrow{\;\sigma\;} & G^0
\end{array}
$$

where G^{pr} is the prefix expansion of G. Define the map $\Phi : S \to G^0$ by:

$$\Phi(s) = \sigma(m(\overline{\theta}^{-1}(m(s)))), \Phi(0) = 0$$

For each maximal element the preimage under $\overline{\theta}$ is well defined as the map θ_g has the property that $\theta_g = \theta_h \Rightarrow g = h$ precisely when $\theta_g \neq 0 \in S$. Given the preimage is a subset of the F-inverse monoid G^{pr} we know that the maximal element in the preimage is the element $(\{1, g\}, g)$ for each $g \in G$, from where we can conclude that the map σ takes this onto $g \in G$.

We now prove it is a prehomomorphism. Let $\theta_g, \theta_h \in S$, then:

$$\Phi(\theta_g) = \sigma(m(\overline{\theta}^{-1}(\theta_g))) = \sigma(\{1, g\}, g) = g$$
$$\Phi(\theta_h) = \sigma(m(\overline{\theta}^{-1}(\theta_h))) = \sigma(\{1, h\}, h) = h$$
$$\Phi(\theta_{gh}) = \sigma(m(\overline{\theta}^{-1}(\theta_{gh}))) = \sigma(\{1, gh\}, gh) = gh$$

Hence whenever θ_g, θ_h and θ_{gh} are defined we know that $\Phi(\theta_g \theta_h) = \Phi(\theta_g)\Phi(\theta_h)$. They fail to be defined if:

(1) If $\theta_{gh} = 0$ in S but θ_g and $\theta_h \neq 0$ in S, then $0 = \Phi(\theta_g \theta_h) \leq \Phi(\theta_g)\Phi(\theta_h)$
(2) If (without loss of generality) $\theta_g = 0$ then $0 = \Phi(0.\theta_h) = 0.\Phi(\theta_h) = 0$

So prove that the inverse monoid S is strongly 0-F-inverse it is enough to prove then that the map Φ is idempotent pure, and without loss of generality it is enough to consider maps of only the maximal elements - as the dual prehomomorphism property implies that in studying any word that is non-zero we will be less than some θ_g for some $g \in G$.

So consider the map Φ applied to a θ_g:

$$\Phi(\theta_g) = \sigma(m(\overline{\theta}^{-1}(\theta_g))) = \sigma(\{1, g\}, g) = g$$

Now assume that $\Phi(\theta_g) = e_G$. Then it follows that $\sigma(m(\overline{\theta}^{-1}(\theta_g))) = e_G$. As σ is idempotent pure, it follows then that $m(\overline{\theta}^{-1}(\theta_g)) = 1$, hence for any preimage $t \in \theta^{-1}(\theta_g)$ we know that $t \leq 1$, and by the property of being 0-E-unitary it then follows that $t \in E(G^{pr})$. Mapping this back onto θ_g we can conclude that θ_g is idempotent, but by assumption this only occurs if $g = e$. $\qquad\square$

Proposition 3.22. *Let $S = \langle \theta_g | g \in G \rangle$ be a strongly 0-F-inverse monoid with maximal elements $Max(S) = \{\theta_g : g \in G\}$, where $\theta : G \to S$ is a dual prehomomorphism. Then the groupoid $G_{\widehat{E}}$ is Hausdorff, second countable with compact unit space. Also it admits a continuous proper groupoid homomorphism onto the group G.*

Proof. We record the topological facts about this groupoid here for reference.

Using the map Φ we construct a map $\rho : \mathcal{G}_{\widehat{E}} \to G$ as follows:

$$\rho([m, \phi]) = \Phi(m)$$

A simple check proves this is a groupoid homomorphism. This map sends units to units as the map Φ is idempotent pure. We prove continuity by considering preimage of an open set in G:

$$\rho^{-1}(U) = \bigcup_{g \in U} [\theta_g, \widehat{D}_{\theta_g^* \theta_g}]$$

This is certainly open as each $[\theta_g, \widehat{D}_{\theta_g^* \theta_g}]$ are elements of the basis of topology of $\mathcal{G}_{\widehat{E}}$. We check it is proper by observing that for groups G compact sets are finite, and they have preimage:

$$\rho^{-1}(F) = \bigcup_{g \in F} [\theta_g, \widehat{D}_{\theta_g^* \theta_g}], \; |F| < \infty$$

This is certainly compact as these are open and closed sets in the basis of topology for the groupoid $\mathcal{G}_{\widehat{E}}$. $\qquad\square$

As $\mathcal{G}_{\widehat{X}} \subseteq \mathcal{G}_{\widehat{E}}$ we also get a continuous proper groupoid homomorphism from $\mathcal{G}_{\widehat{X}}$ onto a group. We recall a special case of [Tu99, Lemme 3.12].

Lemma 3.23. *Let G and H be locally compact, Hausdorff, étale topological groupoids and let $\varphi : G \to H$ be a continuous proper groupoid homomorphism. If H has the Haagerup property then so does G.* $\qquad\square$

This lets us conclude the following:

Corollary 3.24. *Let θ be a partial action of G on X such that all the conditions of Proposition 3.20 are satisfied and such that the inverse monoid S_{inf} is strongly 0-F-inverse. If G has the Haagerup property then so does $\mathcal{G}_{\widehat{X}}$.*

Proof. The map induced by the idempotent pure 0-restricted prehomomorphism from S_{inf} to G induces a continuous proper groupoid homomorphism from $G_{\widehat{X}}$ to G. This then follows from Lemma 3.23. $\qquad\square$

3.4. **Partial Actions on Sequences of Graphs.** Let $\{X_i\}$ be a sequence of finite graphs with degree $\leq 2k$ and large girth. We begin by considering Pedersens Lemma. The following is [Kön90, Theorem 6, Chapter XI]:

Lemma 3.25. *Let X be a finite graph. If at most $2k$ edges go into any vertex then all the edges of X can be divided into k classes such that at most two edges from the same class go into any vertex.* □

We remark that we can always assume that the $2k$ here is minimal; there is a smallest even integer that bounds above the degree of all graphs in the sequence. This in particular stops us from doing something unnatural like embedding the 4-regular tree into a 6-regular tree.

Lemma 3.26. *Such a sequence can be almost k-oriented and this defines a bounded partial action of F_k on X*

Proof. We work on just the X_i. Using Lemma 3.25, we partition the edges $E(X_i)$ into at most k sets E_j such that every vertex appears in at most 2 edges from each subset. Pick a generating set $S = \{a_j | j \in \{1, ..., k\}\}$ for F_k and assign them to the edge sets E_j, and label the edges that appear in each E_j by the corresponding generator. Pedersens Lemma ensures that no more than 2 edges at each point have the same label. This defines a map from the edges to the wedge $\bigvee_{j=i}^{k} S^1$. Choose an orientation of each circle and pull this back to the finite graph X_i - this provides the partial k-orientation. Now define for each generator the partial bijection $\theta^i_{a_j}$ that maps any vertex appearing as the source of any edge in E_j to the range of that edge. i.e:

$$\theta^i_{a_j}(v) := \left\{ \begin{array}{l} r(e) \text{ if } \exists e \in E_j : s(e) = v \\ \quad \text{undefined otherwise} \end{array} \right.$$

For $g = a_l^{e_l}...a_m^{e_m}$ we define θ^i_g as the product $\theta_{a_l^{e_l}}...\theta_{a_m^{e_m}}$; i.e θ^i_g moves vertices along any path that is labelled by the word g in the graph X_i. We observe that for $i \neq i'$ the domain $D^i_{\theta^*_g \theta_g} \cap D^{i'}_{\theta^*_g \theta_g}$ is empty hence we can add these partial bijections in $I(X)$ to form $\theta_g = \sqcup \theta^i_g$. It is a remark that as the topology of X is discrete these maps are all continuous and open. It is clear that as each X_i is connected that the partial bijections have the property that $\cup_g D_{\theta^*_g \theta_g} = X$. Lastly, this map is a dual prehomomorphism as for each $g, h \in G$ we have that $\theta_g \theta_h = \theta_{gh}$ precisely when both θ_h and θ_{gh} are defined and moreover if $\theta_g \theta_h$ is defined then so is θ_{gh}. Hence this collection forms a partial action of G on X. We also remark that as each bijection is given translation along a labelling in the free group it is clear that these move elements only a bounded distance and are therefore elements of $I_b(X)$. □

We need to show that the partial action generates the metric coarse structure at infinity, we recall the length of a partial bijection:

Definition 3.27. The length of each θ_g is defined to be:

$$|\theta_g| = \sup\{d(x, \theta_g(x)) : x \in Dom(\theta_g)\}.$$

Remark 3.28. As we have a concrete description of each θ_g, given on each X_i, we can see that the length on each X_i is given by:

$$|\theta^i_g| = max\{|p| : p \in \{ \text{ paths in } X_i \text{ labelled by g}\}.$$

Then $|\theta_g| = \sup_i |\theta^i_g|$.

In this situation we require that the partial action contains plenty of infinitely supported elements.

Proposition 3.29. *Let $\theta : F_k \to I(X)$ be the dual prehomomorphism corresponding to the bounded partial actions on each X_i. Then for each $R > 0$ there exist finitely many infinite θ_g with $|\theta_g| = |g| < R$.*

Proof. In the general case we know the following for each $R > 0$ and $i \in \mathbb{N}$: $|\theta_g^i| \leq |g| \leq R$. From Lemma 3.26 we know that the partial action is defined by moving along paths inside each individual X_i. So for each R we count the number of words in F_k with length less than R; this is finite (consider the Cayley graph, which has bounded geometry). Now we observe that on the other hand there are infinitely many simple paths of length less than R, thus we must repeat some labellings infinitely many times. These labellings will be contained in words in F_k of length less than R hence when we take the supremum we observe that $|\theta_g| = |g| < R$. $\square$

Corollary 3.30. *The bounded partial action θ of F_k on X finitely generates the metric coarse structure at infinity, that is the set $\overline{\Delta_R} \setminus \Delta_R = \bigcup_{|g|<R} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$ where the index set is finite.*

Proof. We proceed by decomposing Δ_R as we did in the proof of Proposition 2.5.

$$\Delta_R = (\bigcup_{|\theta_g|<R} \Delta_{\theta_g}) \cup F_R$$

Where F_R is the finitely many elements of Δ_R who move between components. We now consider the following decomposition of the set $A := \{\theta_g | |\theta_g| < R\}$ into:

$$A_\infty = \{\theta_g | |\theta_g| < R \text{ and } |supp(\theta_g)| = \infty\}$$
$$A_{fin} = \{\theta_g | |\theta_g| < R \text{ and } |supp(\theta_g)| < \infty\}$$

The first of these is in bijection with the words in F_k that have $|g| < R$ and define an infinite θ_g from Proposition 3.29. Then:

$$\Delta_R = (\bigcup_{g \in A_\infty} \Delta_{\theta_g}) \cup (\bigcup_{g \in A_{fin}} \Delta_{\theta_g}) \cup F_R$$

We complete the proof by observing that for each $\theta_g \in A_{fin}$ the set Δ_{θ_g} is finite. Therefore:

$$\overline{\Delta_R} \setminus \Delta_R = \bigcup_{g \in A_\infty} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g} = \bigcup_{|g|<R} \overline{\Delta_{\theta_g}} \setminus \Delta_{\theta_g}$$

$\square$

Lemma 3.31. *The partial action of F_k defined above extends to βX and is free on the boundary $\partial \beta X$.*

Proof. Fix $g \in F_k$. As the sequence of graphs has large girth there exists an $i_g \in \mathbb{N}$ such that for all $i \geq i_g$ θ_g^i has no fixed points. Let $\omega \in \widehat{D_{\theta_g^* \theta_g}}$ and assume for a contradiction that $\theta_g(\omega) = \omega$. Consider the graph G with vertex set $V = \sqcup_{i \geq i_g} X_i$, where two vertices x and y are joined by an edge if and only if $y = \theta_g(x)$. This graph will have degree at most 2. Additionally, the set of non-isolated vertices inside V contains $D_{\theta_g^* \theta_g}$ and so the subgraph with vertex set consisting of the non-isolated vertices and the same edge set is chosen by ω and has degree at most 2. Any such graph can be at most 3 coloured, subsequently the vertex set breaks into three disjoint pieces V_1, V_2 and V_3 and the ultrafilter ω will pick precisely one

of the V_i. The action of θ_g sends V_i into V_{i+1} modulo 3 by construction. Lastly, $V_i \in \omega$ implies $V_{i+1} \in \theta_g(\omega) = \omega$, which is a contradiction.   □

This freeness gives us a tool to understand the structure of $S_{\inf}$.

Lemma 3.32. *Let $\{X_i\}$ be a sequence of graphs and let G be a group which acts partially on each X_i. If G fixes any sequence in $\{X_i\}$ then the partial action is not free on $\partial \beta X$.*

Proof. Let θ_g denote the disjoint union of the θ_g^i arising from the partial action of G on each X_i. To prove this it is enough to show that there is a single $\omega \in \partial \beta X$ that is fixed by the action of some $g \in G$. The hypothesis that G fixes a sequence gives us $\mathbf{x} := \{x_n\}_I$ with I infinite and $\theta_g(\{x_n\}) = \{\theta_g^n(x_n)\}_I = \{x_n\}_I$.

Now consider an ultrafilter $\omega \in \partial \beta X$ that picks $\mathbf{x}$. Then this ultrafilter ω is an element of D_{g^*g} as $\mathbf{x} \subset D_{g^*g}$. Now for any $A \in \omega$ and consider the intersection $A \cap \mathbf{x}$. This is fixed by the action of g, as it is a subset of $\mathbf{x}$. Hence we have: $\theta_g(A \cap \mathbf{x}) \in \theta_g(\omega)$ for every $A \in \omega$. As $\theta_g(\omega)$ is an ultrafilter $A \in \theta_g(\omega)$, so in particular $\omega \subseteq \theta_g(\omega)$, whence $\theta_g(\omega) = \omega$.   □

Recall that the inverse monoid S_{inf} is represented geometrically by partial bijections on $I(X)$. This representation gives us access to the geometry of X, which we can utilise, in addition to Lemma 3.32, to understand the structure of S_{inf}.

Lemma 3.33. *Consider the inverse monoid S_{inf} as a submonoid of $I(X)$. Then the following hold:*

(1) S_{inf} has the property that $(g \neq e_G$ and $\theta_g \neq 0)$ implies θ_g is not an idempotent;
(2) S_{inf} is 0-E-unitary;
(3) S_{inf} has maximal element set $\{\theta_g : g \in F_k\}$.

Proof.

(1) We prove that no non-zero θ_g are idempotent. To do this we pass to the induced action on βX. We observe that if θ_g is idempotent on X then it extends to an idempotent on βX, hence on the boundary $\partial \beta X$. θ_g is non-zero implies that there is a non-principal ultrafilter ω in the domain $\widehat{D}_{\theta_g}$. The result then follows from the observation that $\theta_g \circ \theta_g(\omega) = \theta_g(\omega)$ implies that θ_g must now fix the ultrafilter $\theta_g(\omega)$, which by Lemma 3.31 cannot happen.

(2) For 0-E-unitary it is enough to prove that $f \leq \theta_g$ implies $\theta_g \in E(S)$. Again, we extend the action to βX. We observe that if θ_g contains an idempotent, then we can build a sequence of elements of $x_i \in f \cap \widehat{D}_{\theta_g^* \theta_g} \cap X_i$ such that θ_g fixes the sequence, and hence fixes any ultrafilter ω that picks this sequence by Lemma 3.32. This is a contradiction, from where we deduce that the only situation for which $f \leq \theta_g$ is precisely when $g = e_G$ hence trivially $e \leq \theta_g$ implies $\theta_g \in E(S)$. For the general case, we remark that by the above statement coupled with the dual prehomomorphism property shows that $f \leq s$ implies $s \leq \theta_{e_G}$, hence is an idempotent.

(3) We construct the maximal elements. Observe that using the dual prehomomorphism it is clear that every non-zero word $s \in S$ lives below a non-zero θ_g. So it is enough to prove that for $\theta_g, \theta_h \neq 0$, $\theta_g \leq \theta_h \Rightarrow \theta_g = \theta_h$. Let $\theta_g \leq \theta_h$. This translates to $\theta_h \theta_g^* \theta_g = \theta_g$, hence for all $x \in \widehat{D}_{\theta_g^* \theta_g} : \theta_h(x) = \theta_g(x)$. Hence $\theta_g^* \theta_h \in E(S)$. From here

we see that $\theta_g^* \theta_h \leq \theta_e$. From (2) we can deduce: $\theta_g^* \theta_h \leq \theta_{g^{-1}h}$ implies $\theta_{g^{-1}h} \in E(S_{inf})$. By (1) this implies $\theta_{g^{-1}h} = \theta_e$, and this happens if and only if $g^{-1}h = e$, i.e $g = h$.

$\square$

Appealing to the machinery we developed earlier in Propositions 3.21 and 3.22 we get the following corollary immediately.

Corollary 3.34. *The inverse monoid S_{inf} is strongly 0-F-inverse.*

We now have enough tools to prove the following result:

Theorem 3.35. *Let $\{X_i\}$ be a sequence of finite graphs of large girth and vertex degree uniformly bounded above by $2k$ and let X be the corresponding space of graphs. Then the boundary coarse Baum-Connes conjecture holds for X.*

Proof. Combining the results in the previous section we know that a free group of rank k acts partially freely on the boundary $\partial\beta X$ in such a way as to give us a second countable locally compact Hausdorff étale topological groupoid $\mathcal{G}_{\widehat{X}}$. This groupoid implements an isomorphism $G(X)|_{\partial\beta X} \cong \partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$. We also know using Corollary 3.34 that the inverse monoid generated by the infinite support elements S_{inf} is a strongly 0-F-inverse monoid, admitting a 0-restricted idempotent pure prehomomorphism onto F_k. Hence the groupoid $\mathcal{G}_{\widehat{X}}$ admits a proper continuous groupoid homomorphism onto F_k, and so has the Haagerup property by Corollary 3.24.
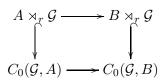
We can now conclude the Theorem by remarking that the isomorphism of groupoids $G(X)|_{\partial\beta X} \cong \partial\beta X \rtimes \mathcal{G}_{\widehat{X}}$ turns the Baum-Connes conjecture for $G(X)|_{\partial\beta X}$ into a specific case of the Baum-Connes conjecture for $\mathcal{G}_{\widehat{X}}$. As $\mathcal{G}_{\widehat{X}}$ has the Haagerup property we can conclude that the Baum-Connes assembly map with any coefficients is an isomorphism [Tu00] and so the assembly map required for the boundary conjecture is also an isomorphism. $\square$

4. Applications to the coarse Baum-Connes conjecture.

In this section we capture, via homological methods, all the known counterexample arguments present in the literature using the boundary conjecture. We begin by confirming that the kernel I constructed in section 2 is the ghost ideal I_G. In order to prove this we need some technology of [HLS02]:

Lemma 4.1. *[HLS02, Lemma 9] If an étale topological groupoid $\mathcal{G}$ acts on a C^*-algebra A, then the map $C_c(\mathcal{G}, A) \to C_0(\mathcal{G}, A)$ extends to an injection (functorial in A) from $A \rtimes_r \mathcal{G}$ to $C_0(\mathcal{G}, A)$.* $\square$

Remark 4.2. The phrase "functorial in A" allows us, given a map: $A \to B$ of $\mathcal{G} - C^*$-algebras, to build the following square:

$$
\begin{array}{ccc}
A \rtimes_r \mathcal{G} & \longrightarrow & B \rtimes_r \mathcal{G} \\
\uparrow\downarrow & & \downarrow \\
C_0(\mathcal{G}, A) & \longrightarrow & C_0(\mathcal{G}, B)
\end{array}
$$

Remark 4.3. The map provided is not a $*-$homomorphism as it takes convolution in $A \rtimes_r \mathcal{G}$ to point wise multiplication in $C_0(\mathcal{G}, A)$. However it suffices for applications as the map is continuous.

Proposition 4.4. *Let X be a uniformly discrete bounded geometry metric space. Then the kernel of the map*

$$l^\infty(X, \mathcal{K}) \rtimes_r G(X) \to (l^\infty(X, \mathcal{K})/C_0(X, \mathcal{K})) \rtimes_r G(X)|_{\partial \beta X}$$

is the ghost ideal I_G.

Proof. Lemma 4.1 implies that the following diagram commutes:

$$
\begin{array}{ccc}
\ell^\infty(X, \mathcal{K}) \rtimes G(X) & \xrightarrow{\ q\ } & \frac{\ell^\infty(X,\mathcal{K})}{C_0(X,\mathcal{K})} \rtimes G(X) \\
\Big\downarrow{\scriptstyle i'} & & \Big\downarrow{\scriptstyle i} \\
C_0(G(X), \ell^\infty(X, \mathcal{K})) & \xrightarrow{\ q'\ } & C_0(G(X), \frac{\ell^\infty(X,\mathcal{K})}{C_0(X,\mathcal{K})})
\end{array}
$$

The downward maps being injective implies that the kernel is precisely the kernel of induced map:

$$q : C^* X = \ell^\infty(X, \mathcal{K}) \rtimes_r G(X) \to C_0(G(X), l^\infty(X, \mathcal{K})/C_0(X, \mathcal{K})).$$

Using the definitions we can compute this kernel:

$$
\begin{aligned}
I &= \{f | i'(f) \in C_0(G(X), C_0(X, \mathcal{K}))\} \\
&= \{f | \forall \epsilon > 0 \exists K \subset X \times X \text{ compact } : |f_{xy}| \le \epsilon \forall (x, y) \in X \times X \setminus K\}.
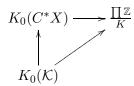\end{aligned}
$$

As X is uniformly discrete with bounded geometry and $X \times X$ is equipped with the product topology we can replace compact by bounded, which using Definition 1.11 is the ghost ideal I_G. $\qquad\square$

Recall that the assembly map μ_{I_G} associated to the open saturated subset X is given by the composition of $\mu_{X \times X} : K_*^{top}(X \times X) \to K_*(\mathcal{K}(\ell^2(X)))$ with the inclusion $i_* : K_*(\mathcal{K}(\ell^2(X))) \to K_*(I_G)$. So to understand how the assembly map μ_{I_G} behaves, it is enough to consider the behaviour of the inclusion i_* as the map $\mu_{X \times X}$ is an isomorphism.

Proposition 4.5. *Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of finite graphs. The the following hold for the space of graphs X:*

(1) The induced map $i_* : K_*(\mathcal{K}(\ell^2(X, \mathcal{K})) \to K_*(C^* X)$ is injective.
(2) If X is an expander then $K_*^{top}(X \times X) \cong K_*(\mathcal{K}(\ell^2(X))) \to K_*(I_G)$ is not surjective.

Proof. Consider the coarse map $X = \sqcup_i X_i \to P := \sqcup_i *$ given by projecting each factor to a point and P carries the coarse disjoint union metric. This induces a tracelike map on the K-theory of $C^* X$, which we denote by Tr. Observe also that $C^* P \cong \ell^\infty(P, \mathcal{K}) + \mathcal{K}(\ell^2(P, \mathcal{K}))$, hence has $K_0(C^* P) \cong \frac{\prod \mathbb{Z}}{K}$, where K is the subgroup of $\bigoplus \mathbb{Z}$ of elements with sum equal to 0. Now:

To prove (1), Observe that the following diagram commutes:

$$K_0(C^*X) \longrightarrow \frac{\prod \mathbb{Z}}{K}$$
$$\uparrow \qquad \nearrow$$
$$K_0(\mathcal{K})$$

Where the image of a rank one projection in $K_0(\mathcal{K})$ is given by a vector $(1, 0, ...)$, which is certainly an injection. It now follows that the upward map is injective.

For (2) consider a nontrivial ghost projection $p \in I_G$. It is well known that $Tr([p]) \notin \frac{\oplus \mathbb{Z}}{K}$, whilst $Tr([k]) \in \frac{\oplus \mathbb{Z}}{K}$ for any compact operator [WY12]. As these differ under Tr, they cannot possibly be equal in $K_0(I_G)$. □

The boundary coarse Baum-Connes conjecture has applications to the coarse Baum-Connes conjecture for spaces of graphs:

Theorem 4.6. *Let X be a space of graphs arising from a sequence of finite graphs $\{X_i\}$. If boundary coarse Baum-Connes conjecture is injective then the following hold:*

(1) *The coarse Baum-Connes assembly map for X is injective.*
(2) *If X is an expander then the coarse Baum-Connes assembly map for X fails to be surjective.*

Proof. Consider the diagram:

$$\frac{\prod_{i\in\mathbb{N}} K_0(C^*X_i)}{\bigoplus_{i\in\mathbb{N}} K_0(C^*X_i)}$$
$$\nearrow \quad d_* \uparrow$$

$$\rightarrow K_1(C^*_r(G(X)|_{\partial\beta X}) \rightarrow K_0(I_G) \rightarrow K_0(C^*_r(G(X))) \rightarrow K_0(C^*_r(G(X)|_{\partial\beta X}) \rightarrow K_1(I_G) \rightarrow$$
$$\updownarrow \qquad\qquad \updownarrow \quad\nearrow \quad \mu\uparrow \qquad\qquad\qquad \updownarrow \qquad\qquad 0\uparrow$$
$$\rightarrow K_1^{top}(G(X)|_{\partial\beta X}) \longrightarrow \mathbb{Z} \longrightarrow K_0^{top}(G(X)) \longrightarrow K_0^{top}(G(X)|_{\partial\beta X}) \longrightarrow 0 \longrightarrow$$

We prove (1) by considering an element $x \in K_0^{top}(G(X))$ such that $\mu(x) = 0$. Then x maps to $0 \in K_0^{top}(G(X)|_{\partial\beta X})$ and so comes from an element $y \in \mathbb{Z}$. Each square commutes hence y maps to $0 \in K_0(C^*_r(G(X)))$. As the composition up and left (as indicated in the diagram) is injective by Proposition 4.5, we know that $y \in \mathbb{Z}$ is in fact $0 \in \mathbb{Z}$. Hence $x = 0$.

To see (2): take any non-compact ghost projection $p \in K_0(I_G)$, which does not lie in the image of $\mathbb{Z}$ as it does not vanish under the trace d_*. Push this element to $q \in K_0(C^*_r(G(X)))$. Assume for a contradiction that μ is surjective. Then there is an element $x \in \mathbb{Z}$ that maps to q, and so $d_*(q) = 0$, as the image of compact operators lies in the kernel of the map d_*. However, we know that $d_*(q) = d_*(p)$ is certainly non-zero. □

4.1. **Some Remarks on the Max Conjecture.**

Proposition 4.7. *Let X be the space of graphs arising from a sequence of finite graphs $\{X_i\}$. Then*

(1) *the maximal coarse Baum-Connes assembly map is an isomorphism if and only if the maximal Boundary coarse Baum-Connes map is an isomorphism.*

(2) the maximal coarse assembly map is injective if and only if the maximal boundary assembly map is injective.

Proof. As before we consider a diagram, this time of maximal algebras:

$$
\begin{array}{ccccccccc}
K_1(C^*(G(X)|_{\partial\beta X}) & \longrightarrow & K_0(\mathcal{K}) & \longrightarrow & K_0(C^*(G(X))) & \succ & K_0(C^*(G(X)|_{\partial\beta X}) & \longrightarrow & K_1(\mathcal{K}) \\
\uparrow & & \wr\| \uparrow & & \uparrow & & {}_{\mu_{bdry}}\uparrow & & \wr\| \uparrow \\
K_1^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & K_0^{top}(X \times X) & \longrightarrow & K_0^{top}(G(X)) & \longrightarrow & K_0^{top}(G(X)|_{\partial\beta X}) & \longrightarrow & K_1^{top}(X \times X)
\end{array}
$$

Both parts follow from a diagram chase. $\qquad\square$

This result captures completely [OOY09, Corollary 4.18], but the proof is very much more elementary. It is also clear that the argument above also works in the case of injectivity, which is related to the content of [GWY08, Theorem 5.1].

5. A COUNTEREXAMPLE TO THE BOUNDARY CONJECTURE

In this section we develop the ideas of Higson, Lafforgue and Skandalis concerning the counterexamples to the coarse Baum-Connes conjecture further, to construct a space of graphs Y that has exceptional properties at infinity. The main idea is to decompose the boundary groupoid further, giving a new short exact sequence at infinity similar to the sequences considered in Chapter 4. From this, we then construct an operator that is not a ghost operator, but is ghostly on certain parts of the boundary. A tracelike argument, similar to those of [Hig99, WY12] then allows us to conclude that the boundary coarse Baum-Connes conjecture fails to be surjective for the space Y.

5.1. **The space and its non-ghosts.** The space we are going to consider first appeared in [Wan07].

Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of finite graphs. Then we construct a space of graphs in the following manner: Let $Y_{i,j} = X_i$ for all $j \in \mathbb{N}$ and consider $Y := \sqcup_{i,j} Y_{i,j}$. We metrize this space using a box metric - that is with the property that $d(Y_{i,j}, Y_{k,l}) \to \infty$ as $i+j+k+l \to \infty$.

Now let $\{X_i\}_i$ be an expander sequence. As discussed in Section 1.2 of Chapter 4, we can construct a ghost operator $p = \prod_i p_i$ on X, the space of graphs of $\{X_i\}_i$. Similarly, we can construct this operator on Y. In this situation we get a projection $q := \prod_{i,j} p_i \in C_u^*Y$, which is a constant operator in the j direction. This was precisely the operator of interest in [Wan07], as it can be seen that q is not a ghost operator, as its matrix entries do not vanish in the j direction - a fact proved below in Lemma 5.1.

Recall that associated to Y we have a short exact sequence of C^*-algebras:

$$
0 \longrightarrow ker(\pi) \longrightarrow C_r^*(G(Y)) \overset{\pi}{\longrightarrow} C_r^*(G(Y)|_{\partial\beta Y}) \longrightarrow 0.
$$

We remark the kernel, $ker(\pi)$ consists of all the ghost operators in $C_u^*(Y)$, that is those operators with matrix coefficients that tend to 0 in all directions on the boundary.

Lemma 5.1. *The projection $\pi(q) := \pi(\prod_{i,j} p_i) \neq 0 \in C_r^*(G(Y)|_{\partial\beta Y})$. That is $q \notin ker(\pi)$.*

Proof. We first observe that every bounded subset B of Y is contained in some rectangle of the form $R_{i_B,j_B} := \sqcup_{i \leq i_B, j \leq j_B} Y_{i,j}$. So to prove that q is not a ghost operator it suffices to show that there exists an $\epsilon > 0$ such that for all rectangles $R_{i,j}$ there is a pair of points x, y in the compliment of the rectangle such that the norm $\|q_{x,y}\| \geq \epsilon$. To prove this, recall that the projection q is a product of projections p_i on each X_i and fixing j, these projections form a ghost operator.

Fix $\epsilon = \frac{1}{2}$. Then there exists an i_ϵ with the property that $\forall i > i_\epsilon$ and for every $x, y \in \sqcup_i X_i$ we know $\|p_{i,x,y}\| < \epsilon$. We remark that this i_ϵ can be taken to be the smallest such. So for $i \leq i_\epsilon - 1$, we have that $\|p_{i,x,y}\| \geq \frac{1}{2}$. Now let $R_{i_\epsilon-1,\infty}$ be the vertical rectangle $\sqcup_{i \leq i_\epsilon-1,j} Y_{i,j}$.

To finish the proof, consider an arbitrary finite rectangle $R_{i,j}$. This intersects the infinite rectangle $R_{i_\epsilon-1,\infty}$ in a bounded piece. Now pick any pair of points in a fixed box $x, y \in Y_{k,l} \subset R_{i_\epsilon-1,\infty} \setminus R_{i,j}$. Then for those points x, y it is clear that $\|q_{x,y}\| = \|p_{k,x,y}\| \geq \frac{1}{2}$. $\square$

We now describe the boundary $\partial \beta Y$. We are aiming at a decomposition into saturated pieces and with that in mind we construct a map to βX.

Consider the map $\beta Y \twoheadrightarrow \beta X \times \beta \mathbb{N}$ induced by the bijection of Y with $X \times \mathbb{N}$ and the universal property of βY. Now define:
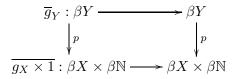
$$f : \beta Y \to \beta X \times \beta \mathbb{N} \to \beta X$$

The map f is continuous, hence the preimage of X under projection onto the first factor is an open subset of βY, which intersects the boundary $\partial \beta Y$. In fact, what we can see is that $f^{-1}(X) = \sqcup f^{-1}(X_i)$, where each $f^{-1}(X_i)$ is closed, and therefore homeomorphic to $X_i \times \beta \mathbb{N}$. We can define $U = f^{-1}(X) \cap \partial \beta Y$.

5.2. **The boundary groupoid associated to the box space of a discrete group with the Haagerup property.** Let Γ be a finitely generated residually finite discrete group with the Haagerup property, and let $\{N_i\}$ be a family of nested finite index subgroups with trivial intersection. Let $X_i := Cay(\frac{\Gamma}{N_i})$. In this context, the boundary groupoid is generated by the action of the group Γ extended to the boundary (see Proposition 2.5). In this context we can show that U defined above is saturated:

Lemma 5.2. *U is an open, saturated subset of the boundary $\partial \beta Y$.*

Proof. We have already shown above that U is open. To see it is saturated we prove that U^c is saturated. Let g_Y and g_X be the continuous extensions of the map obtained by acting through g on Y and X respectively. We observe that the following diagram commutes:

$$
\begin{array}{ccc}
\overline{g}_Y : \beta Y & \longrightarrow & \beta Y \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\overline{g_X \times 1} : \beta X \times \beta \mathbb{N} & \longrightarrow & \beta X \times \beta \mathbb{N}
\end{array}
$$

The projection onto βX is also equivariant under this action. Assume for a contradiction that U^c is not saturated; there exists γ in U^c such that $\overline{g}_Y(\gamma) \in U$. It follows that $\overline{g_X \times 1}(p(\gamma))$ is in $p(U)$, whilst $p(\gamma) \in p(U^c)$, hence $\overline{g_X}(f(\gamma)) \in U$ whilst $f(\gamma) \in U^c$. This is a contradiction as $f(U^c) = \partial \beta X$ is saturated. $\square$

This gives us two natural complimentary restrictions of $G(Y)|_{\partial\beta Y}$ and a short exact sequence of function algebras as in Chapter 4:

$$0 \longrightarrow C_c(G(Y)|_U) \longrightarrow C_c(G(Y)|_{\partial\beta Y}) \longrightarrow C_c(G(Y)|_{U^c}) \longrightarrow 0.$$

We will now show that the corresponding sequence:

$$0 \longrightarrow C_r^*(G(Y)|_U) \longrightarrow C_r^*(G(Y)|_{\partial\beta Y}) \xrightarrow{\ h\ } C_r^*(G(Y)|_{U^c}) \longrightarrow 0$$

fails to be exact in the middle. We proceed as in [WY12, HLS02] by using the element $\pi(q)$, which certainly vanishes under the quotient map from $C_r^*(G(Y)|_{\partial\beta Y}) \to C_r^*(G(Y)|_{U^c})$. To show the failure we will show this sequence fails to be exact in the middle at the level of K-theory and for this we will require a firm understanding of the structure of $G(Y)|_U$.

We observe the following facts:

(1) Γ acts on the space $Y := \sqcup_{i,j} Y_{i,j}$ built from $\{X_i\}$.
(2) This action becomes free on piece of the boundary that arises as $i \to \infty$, that is Γ acts freely on U^c.
(3) The group action generates the metric coarse structure on the boundary; the finite sets associated to each $R > 0$ in the decomposition are now finite rectangles. This follows from considerations of the metric on Y.

It follows from the proof of Proposition 2.5 that the groupoid $G(Y)|_{U^c}$ is isomorphic to $U^c \rtimes \Gamma$ and under the assumption that Γ has the Haagerup property we can conclude that the Baum-Connes assembly map for the groupoid $G(Y)|_{U^c}$ is an isomorphism (with any coefficients). We now concern ourselves with $G(Y)|_U$.

Lemma 5.3. *The groupoid $G(Y)|_U$ is isomorphic to $\sqcup_i (X_i \times X_i) \times G(\mathbb{N})|_{\partial\beta\mathbb{N}}$.*

Proof. We consider the preimages $f^{-1}(X_i)$. These are clearly invariant subsets of βY that when intersected with the boundary $\partial\beta Y$ are contained within U. The restriction of $(G(Y)|_{\partial\beta Y})|_{f^{-1}(X_i)}$ for each i isomorphic to the closed subgroupoid $G(X_i \times \mathbb{N})|_{\partial\beta\mathbb{N}}$ of $G(Y)|_U$. These groupoids are disjoint by construction and therefore the inclusion $\sqcup_i G(X_i \times \mathbb{N})|_{\partial\beta\mathbb{N}}$ is an open subgroupoid of $G(Y)|_U$. We now prove that:

(1) each $G(X_i \times \mathbb{N})|_{\partial\beta\mathbb{N}}$ is isomorphic to $(X_i \times X_i) \times G(\mathbb{N})|_{\partial\beta\mathbb{N}}$, where $\mathbb{N}$ has the well-spaced metric;
(2) the union $\sqcup_i G(X_i \times \mathbb{N})|_{\partial\beta\mathbb{N}}$ is the entire of $G(Y)|_U$.

To prove (1), observe that the groupoid decomposes as $G(X_i \times \mathbb{N}) = \bigcup_{R>0} \overline{\Delta_R(X_i \times \mathbb{N})}$. For each $R > 0$ we can find a j_R such that $\Delta_R(X_i \times \mathbb{N}) = F_R \cup \bigcup_{j>j_R} \Delta_R(X_i \times \{j\})$, hence for the boundary part of this groupoid it is enough to understand what happens in each piece $Y_{i,j}$, which is constant for each j. Secondly, observe that in the induced metric on a column, the pieces $Y_{i,j}$ separate as $j \to \infty$. This, coupled with the fact that for large enough R, we know that $\Delta_R^j(X_i \times \mathbb{N}) = X_i \times X_i$ allow us to deduce that any behaviour at infinity of this groupoid is a product of $X_i \times X_i$ and the boundary groupoid $G(\mathbb{N})|_{\partial\beta\mathbb{N}}$ where $\mathbb{N}$ has the coarsely disconnected metric. This groupoid is isomorphic to $\partial\beta\mathbb{N}$, from which we can deduce that $G(X_i \times \mathbb{N})|_{\partial\beta\mathbb{N}} = (X_i \times X_i) \times \partial\beta\mathbb{N}$ for each i.

To prove (2) we assume for a contradiction that there is a partial translation t, such that $\bar{t}$ is not an element of the disjoint union. Such an element maps some (x_i, ω) to (x_k, ω), where $i \neq k$. Without loss of generality assume also t has translation length at most R. Then the domain and range of t are both infinite (as the closure is defined in $G(Y)|_U$), and must be contained within a strip of width at most $R > 0$. From the definition of the metric, there are only finitely many $Y_{i,j}$ within such a rectangle, hence $t \in F_R$ and hence $\bar{t}$ is not defined in $G(Y)|_U$, which yields a contradiction. $\qquad\square$

Remark 5.4. Lemma 5.3 allows us to conclude that $C_r^*(G(Y)|_U) \cong \bigoplus_i M_{|X_i|} \otimes C(\partial\beta Y)$

To conclude that $[\pi(q)]$ is not an element of $K_0(C_r^*(G(Y)|_U))$ we construct a trace-like map.

Theorem 5.5. *The element $\pi(q)$ maps to 0 in $C_r^*(G(Y)|_{U^c})$, but does not belong to the image of $K_0(C_r^*(G(Y)|_U))$ in $K_0(C_r^*(G(Y)|_{\partial\beta Y})$.*

Proof. The first part follows from the definition of $\pi(q)$; the quotient map h kills all operators that are ghostly in the direction $i \to \infty$ and $\pi(q)$ is such an operator.

To prove the second component we remark that each $U_i := f^{-1}(X_i) \cap U$ is a closed saturated subset of $\partial\beta Y$, hence we can consider the reduction to U_i for each i. We consider the product, and the following map:

$$\phi : C_r^*(G(Y)|_{\partial\beta Y}) \quad \to \quad \prod_i C_r^*(G(Y)|_{U_i}) = \prod_i C_r^*(G(X_i \times \mathbb{N})$$
$$T \quad \mapsto \quad \prod_i T|_{U_i}$$

Under the map ϕ, the ideal $C_r^*(G(Y)|_U) = \bigoplus_i M_{|X_i|} \otimes C(\partial\beta Y)$ maps to the ideal $\bigoplus_i C_r^*(G(X_i \times \mathbb{N}))$. So, we can define a tracelike map, in analogy to [WY12, Section 6], by composing with the quotient map τ onto $\frac{\prod_i C_r^*(G(X_i \times \mathbb{N}))}{\bigoplus_i C_r^*(G(X_i \times \mathbb{N}))}$. This gives us a map at the level of K-theory:

$$Tr_* = \phi \circ \tau : K_0(C_r^*(G(Y)|_{\partial\beta Y})) \to \frac{\prod_i K_0(C_r^*(G(X_i \times \mathbb{N})))}{\bigoplus_i K_0(C_r^*(G(X_i \times \mathbb{N})))} = \frac{\prod_i K_0(C(\partial\beta\mathbb{N}))}{\bigoplus_i K_0(C(\partial\beta\mathbb{N}))}$$

By construction, $K_0(C_r^*(G(Y)|_U))$ vanishes under Tr_*. We now consider $[\pi(q)]$ under Tr_*. Recall that $q = \prod_{i,j} p_i$. We define $q_i = \prod_j p_i$ and observe that the operation of reducing to $G(Y)|_{U_i}$ can be performed in two commuting ways: restricting to U then $f^{-1}(X_i)$ or by restricting to $f^{-1}(X_i)$ then U. The second tells us that $q_i = p_i \otimes 1_{\beta\mathbb{N}}$ is constant in the j direction and when restricted to the boundary is $\pi(q_i) = p_i \otimes 1_{\partial\beta\mathbb{N}}$. Hence, $Tr_*([\pi(q)]) = [1_{\partial\beta\mathbb{N}}, 1_{\partial\beta\mathbb{N}}, ...] \neq 0$ and so $[\pi(q)] \notin K_0(C_r^*(G(Y)|_U))$. $\qquad\square$

So in this case we have the following diagram:

$$K_1(C(U^c) \rtimes \Gamma) \longrightarrow K_0(\ker(\pi)) \longrightarrow K_0(C_r^*(G(Y)|_{\partial\beta Y})) \twoheadrightarrow K_0(C(U^c) \rtimes \Gamma) \longrightarrow K_1(\ker(\pi))$$

$$\wr\| \big\uparrow \qquad\qquad \big\uparrow \qquad\qquad \overset{\mu_{bdry}}{\longleftrightarrow}\big\uparrow \qquad\qquad \wr\|\big\uparrow \qquad\qquad \big\uparrow$$

$$K_1^{top}(U^c \rtimes \Gamma) \longrightarrow K_0^{top}(G(Y)|_U) \longrightarrow K_0^{top}(G(Y)|_{\partial\beta Y}) \longrightarrow K_0^{top}(U^c \rtimes \Gamma) \longrightarrow K_1^{top}(X \times X)$$

Remark 5.6. We justify the diagonal inclusion of $K_0^{top}(G(Y)|_U)$ into $K_0(C_r^*(G(Y)|_{\partial\beta Y})$. This follows as the groupoid $G(Y)|_U$ is amenable, and hence the assembly map is an isomorphism. The algebra $C_r^*(G(Y)_U) = \bigoplus_i M_{|X_i|} \otimes C(\partial\beta Y)$ injects into the product $\prod_i M_{|X_i|} \otimes C(\partial\beta Y)$

at the level of K-theory and this inclusion factors through in the inclusion into the kernel of π and into $C_r^*(G(Y))|_{\partial \beta Y}$). These maps provide enough information to conclude injectivity of the assembly map μ_{bdry}.

A diagram chase under the assumption that the map μ_{bdry} is surjective quickly yields a contradiction, whence we have:

Corollary 5.7. *The assembly map μ_{bdry} associated to $Y = \sqcup_{i,j} Cay(\frac{\Gamma}{N_i})$ is not surjective but is injective.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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