TRAX: An approach for the time rational analysis of complex dynamic systems

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SUMMARY

This paper introduces a theoretical and algorithmic reduced model approach to efficiently evaluate time responses of complex dynamic systems. The proposed approach combines four main components: analytical expressions of the average of the system's transfer functions in the frequency domain, precise and convergent rational approximations of these exact expressions, exact evaluation of these approximations through model reduction in rational Krylov subspaces, and semi-analytical interpolation at just a few frequency points. The resulting algorithmic principles to evaluate the time response of a particular system are relatively straightforward: one first evaluates the response of the system with slight additional damping at a few frequencies and one then projects or reduces the system in the subspace spanned by these responses. The time response of the reduced model implicitly provides a precise evaluation of that of the original system. The properties of the reduced models and the precision of the proposed approach are studied and applications on complex matrix systems are presented and discussed. While the theory and numerical algorithms are presented in a matrix context, they are also transposable in a continuous functional context. Copyright © 2010 John Wiley & Sons, Ltd.

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1. INTRODUCTION

While, from a theoretical point of view, the study of the dynamic behaviour of complex structures such as cars, trains, and airplanes may appear to be a trivial matter if one wants to predict the noise or vibrations they generate, one can just analyse their modes of vibrations or propagating waves -, this approach actually becomes very quickly unfeasible when necessary complex details are taken into account. This is for example the case when seats, passengers, or carpets are considered in vehicles or more generally when one is concerned with a dynamic behaviour at relatively high frequency where propagating waves have very short wavelengths.

The models need more components and degrees of freedom, their dimensions increase and this leads to very large matrix problems with a huge number of modes and whose solution is then prohibitive. Paradoxically, the increased complexity and mixing of many modes leads to simplified patterns of the response, as individual details lose their importance. This is in some sense similar to the fact highlighted in the central limit theorem that the sum of many independent random variables converges to a standard normal distribution. There have been many efforts to extract and identify such simplified, important, patterns of the system's behaviour, either from deterministic or from statistical points of view. The consideration of fuzzy, complex, or reverberant components as for example in [1, 2, 3, 4, 5, 6] of a vibrating system falls within this realm where parts of the system made of a myriad of vibrating elements are replaced by a small number of components that provide equivalent properties.

The current work is in the same spirit in the sense that one can reduce the number of degrees of freedom or number of vibrating elements of an original linear system and still retain its time response. The smaller (reduced) model of the system with such equivalent properties is obtained through a combination of frequency averaging [7], rational approximation of some scalar special functions [8], and rational Krylov model reduction [9, 10, 11, 12, 13]. The focus is particularly on a Gaussian frequency average for which the impulse response of the system is preserved at smaller times.

The linear time invariant dynamic system and its time response are first introduced in section (2). The theory of the exact frequency average is then summarised in section (3) and its scalar and matrix rational approximations at individual frequencies are discussed in section (4). These matrix rational approximations can be evaluated by Krylov reduction methods which further allows to obtain a precise evaluation of the frequency average from using just a few Krylov subspaces built at chosen interpolation frequencies as described in section (5). The resulting small reduced model can be used and provide an accurate estimation of the system's time response. Several choices of actual projection approaches are finally proposed and their properties studied, in section (7). All the algorithmic steps presented along the way are tested and shown to perform very well in section (8).

2. DESCRIPTION OF THE DYNAMIC SYSTEM AND ITS TIME RESPONSE

In this section, the scene is set for the presentation of the proposed method. The description, assumptions, and notations of the dynamic systems are introduced and some typical techniques to evaluate their time response are discussed that will be reused later.

2.1. Linear time invariant system

The current work is applicable to and is presented for the evaluation of the time response of linear time invariant dynamic systems that can be written in the usual state-space matrix form as

$$\mathbf{B} \frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) - \mathbf{F}\mathbf{u}(t) \quad \text{and} \quad \mathbf{g}(t) = \mathbf{C}^T \mathbf{x}(t) + \mathbf{D}^T \mathbf{u}(t) \quad (1)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{F}$ are constant matrices. The theory and algorithms can generally be applied to a much wider range of situations, for example, in the case of systems with higher derivative orders in time and in that of continuous functional systems. Applications to second-order problems in time will notably be illustrated in section 8.

The vectors $\mathbf{x}(t)$, $\mathbf{u}(t)$ and $\mathbf{g}(t)$ are the state, input, and output vectors, generally of respective dimension N , M , and P . The underlined notation indicates functions of time. For simplicity, the single-input, single-output case, $M = P = 1$, is considered so that the input and output, $\mathbf{g}(t) = g(t)$, $\mathbf{u}(t) = u(t)$, are scalar functions, and $\mathbf{F} = \mathbf{f}$ and $\mathbf{C} = \mathbf{c}$ are vectors. One may also consider $\mathbf{D} = \mathbf{0}$ without loss of generality.

The interest is in the response function $g(t)$ to an input $u(t)$, particularly at smaller times, when the system is originally at rest. As more general responses can be derived once the impulse response to a delta Dirac function input, $u(t) = \delta(t)$, is known, the following particular system is considered,

$$\mathbf{B} \frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) - \mathbf{f}\delta(t) \quad \text{and} \quad g(t) = \mathbf{c}^T \mathbf{x}(t). \quad (2)$$

For simplicity of presentation and derivation, it is assumed that $\mathbf{B}$ is invertible and that the pair or pencil of matrices, $\mathbf{A}$ and $\mathbf{B}$, has a full set of eigenvectors, i.e. that there are N (non-zero) right eigenvectors ϕ_j and left eigenvectors ψ_j , such that

$$(\mathbf{A} - \lambda_j \mathbf{B})\phi_j = \mathbf{0} \quad \text{and} \quad \psi_j^T (\mathbf{A} - \lambda_j \mathbf{B}) = \mathbf{0}^T \quad (3)$$

for each of the possibly multiple eigenvalues λ_j , $j = 1, \dots, N$. Storing the eigenvectors and eigenvalues in matrices, $\Phi = [\phi_1 \dots \phi_N]$, $\Psi = [\psi_1 \dots \psi_N]$, and $\Lambda = \text{diag}_j(\lambda_j)$, the

system can then be diagonalised by using equation (3) in the form $\mathbf{A}\Phi = \mathbf{B}\Phi\Lambda$: considering the modal expansion $\underline{\mathbf{x}} = \Phi\underline{\mathbf{y}}$ and multiplying equation (2) to the left by $(\mathbf{B}\Phi)^{-1}$ gives scalar equations for each of the modal components,

$$\frac{dy_j(t)}{dt} = \lambda_j y_j(t) - \mathbf{e}_j^T (\mathbf{B}\Phi)^{-1} \mathbf{f} \delta(t) \quad \text{for } j = 1, \dots, N. \quad (4)$$

2.2. Time response of the dynamic system

In principle, system (2) can be solved analytically in terms of the modal information as described in standard dynamics or control textbooks. Assuming that the system is stable, i.e. that the real part of its eigenvalues is strictly negative, $\Re(\lambda_j) < 0$ so that all of its modal components ultimately tend to zero after impulse, its impulse response can be expressed, in scalar form, as

$$\underline{g}(t) = - \sum_{j=1, \dots, N} [\mathbf{c}^T \phi_j \mathbf{e}_j^T (\mathbf{B}\Phi)^{-1} \mathbf{f} e^{\lambda_j t}] \quad \text{for } t > 0 \quad (5)$$

or, in matrix form, as $\underline{g}(t) = -\mathbf{c}^T \Phi \text{diag}_j(e^{\lambda_j t}) (\mathbf{B}\Phi)^{-1} \mathbf{f}$. This is demonstrated in appendix for completeness.

While this analytical approach is elegant and simple, and applicable to moderate size problems, there are multiple reasons that practically restrict its applicability to dimensionally large and complex systems. The assumed fact that the modal decomposition is available and can actually be used in a useful manner is far from trivial. One of the issues is that the evaluation itself of the entirety of the set of eigenvalues is, in general, a difficult if not impossible quest for such very large problems. Furthermore, the "exact" eigenvalues may be very sensitive to modification of the system properties or to numerical error in their evaluation. The knowledge of their estimated values may therefore be of dubious utility. Even if all the eigenvalues were known, additional practical difficulties are that the eigenvectors themselves would, in principle, need to be computed and stored in order to evaluate and recombine all the modal contributions and also that this recombination might be, itself, prone to general loss of accuracy. In most situations, a further argument for avoiding this approach is that not all modal components actually contribute individually to the time response at the times of interest. Some sets of modal components can be merged together while others can simply be omitted. The question of which combination of modes to retain is however a complicated problem. In a sense, the approach proposed here offers some understanding and steps towards a convenient answer.

Other commonly used approaches for evaluating time responses of the linear time invariant system (2) at hand consist in first discretising the time response $\underline{\mathbf{x}}(t)$ at discrete times such as $\underline{\mathbf{x}}^{(0)} = \underline{\mathbf{x}}(t = t_0 = 0)$, $\underline{\mathbf{x}}^{(1)} = \underline{\mathbf{x}}(t = t_1)$, $\underline{\mathbf{x}}^{(2)} = \underline{\mathbf{x}}(t = t_2)$, ... and then solving step by step for the estimated values at increasing times. For example, if time steps were equidistant, as at $t_k = k\Delta t$, for $k = 0, 1, 2, \dots$ and if the derivative was discretised as $\frac{d\underline{\mathbf{x}}(t_k)}{dt} \approx (\underline{\mathbf{x}}^{(k+1)} - \underline{\mathbf{x}}^{(k)})/\Delta t$, then the steps to generate the estimates could be

$$\underline{\mathbf{x}}^{(k+1)} = \mathbf{B}^{-1}(\Delta t \mathbf{A} + \mathbf{B})\underline{\mathbf{x}}^{(k)} \quad \text{and} \quad \underline{\mathbf{x}}^{(0)} = -\mathbf{B}^{-1}\mathbf{f}. \quad (6)$$

More sophisticated standard time-stepping approaches use information about several past steps or even refine a first estimate at future steps through so-called predictor-corrector methods. An advantage is that modal analysis can then be avoided. General issues with these methods however remain and turn around questions about the stability of the iterative steps, the progressive loss of precision for increasing times, and the computational cost, mostly due to having to evaluate operations in the dimensionally large space at each step.

There is thus a quest for alternative approaches, notably through model reduction. This latter category which the current method is a part of can be seen as the following: instead of working with the full state-space vector, $\underline{\mathbf{x}}(t)$, in the whole space of dimension, N , one restricts the problem to a subspace spanned by the few columns of a rectangular matrix, $\mathbf{V}$, so that the response is approximated by $\underline{\mathbf{x}}(t) \approx \mathbf{V}\underline{\mathbf{z}}(t)$ where the coefficients of the vector $\underline{\mathbf{z}}(t)$ are thus the contribution

from each column of $\mathbf{V}$ to the approximation. Further projecting the residue of system (2), i.e. the difference between its left- and right-hand sides, into a smaller subspace, spanned by the columns of another matrix, $\mathbf{W}$, gives an approximate smaller dimensional system

$$(\mathbf{W}^T \mathbf{B} \mathbf{V}) \frac{d\mathbf{z}(t)}{dt} = (\mathbf{W}^T \mathbf{A} \mathbf{V}) \mathbf{z}(t) - (\mathbf{W}^T \mathbf{f}) \delta(t) \quad \text{and} \quad \underline{g}(t) \approx \mathbf{c}^T \mathbf{V} \mathbf{z}(t) \quad (7)$$

whose reduced matrices are $(\mathbf{W}^T \mathbf{B} \mathbf{V})$ and $(\mathbf{W}^T \mathbf{A} \mathbf{V})$, and the input and output vectors are $(\mathbf{W}^T \mathbf{f})$ and $(\mathbf{V}^T \mathbf{c})$. Provided that the subspaces $\mathbf{V}$ and $\mathbf{W}$ have been well chosen and that the dimension of their column spaces is not too large, it may then be possible to evaluate a good approximate time response of the original system by applying the analytical techniques described at the beginning of this section to the moderately sized reduced model. The choice of projection matrices is therefore critical and leads to different characteristics of the reduced model and approximate time response.

The choice of projectors proposed here and its resulting properties, are based on an excursion into the frequency domain and on analytical frequency averaging as detailed in the next section. A two-step quasi-analytical rational approximation of the average is then applied through rational Krylov projection methods as will be seen in section 4.

3. FREQUENCY AVERAGING

The interest now turns to the analytical frequency averaging of the original complex system. Working in the frequency domain is common practice, in the acoustics, vibro-acoustics, structural dynamics, control, and other fields: taking the (inverse) Fourier transform of equation (2), defined for a function $\underline{v}(t)$ by

$$\underline{v}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(\omega) e^{i\omega t} d\omega \quad \text{and the inverse} \quad v(\omega) = \int_{-\infty}^{\infty} \underline{v}(t) e^{-i\omega t} dt, \quad (8)$$

one then finds

$$[\mathbf{A} - \omega(\mathbf{iB})] \mathbf{x}(\omega) = \mathbf{f} \quad \text{and} \quad g(\omega) = \mathbf{c}^T \mathbf{x}(\omega). \quad (9)$$

Coming back to the modal decomposition of the pencil of matrices as described after equations (3), one finds

$$[\mathbf{A} - i\omega \mathbf{I}] \mathbf{y}(\omega) = (\mathbf{B} \Phi)^{-1} \mathbf{f}, \quad g(\omega) = \mathbf{c}^T \Phi \mathbf{y}(\omega), \quad (10)$$

and that the resonances of the system, expressed in terms of the pencil $(\mathbf{A}, \mathbf{iB})$, happen at its eigenvalues $\omega_j = -i\lambda_j$. For a stable system, all ω_j eigenvalues have therefore a strictly positive imaginary part, $\Im(\omega_j) > 0$. Solving for and combining all of the modal components gives

$$g(\omega) = (-i) \mathbf{c}^T \Phi \text{diag}(\omega_j - \omega) (\mathbf{B} \Phi)^{-1} \mathbf{f} = (-i) \sum_{j=1, \dots, N} \frac{\mathbf{c}^T \Phi_j \mathbf{e}_j^T (\mathbf{B} \Phi)^{-1} \mathbf{f}}{\omega_j - \omega}. \quad (11)$$

3.1. General averaging or filtering

The proposed technique proceeds by evaluating the average or weighted integral of the transfer function for a given frequency shift, $\Delta\omega$, with particular weighting function $p_a(\Delta\omega)$ and averaging width a . The focus is particularly on a Gaussian function

$$p_a^{(g)}(\Delta\omega) = \sqrt{\frac{1}{2\pi a^2}} e^{-\frac{\Delta\omega^2}{2a^2}} \quad (12)$$

but the same general principles presented here can be directly adapted to other weighting functions as, for example, those discussed in [7]. Thus, at a particular frequency ω , frequency averaging

consists, in general, in evaluating the following integral

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(\omega + \Delta\omega) p_a(\Delta\omega) d\Delta\omega \quad (13)$$

$$= (-i) \sum_{j=1, \dots, N} \left[\mathbf{c}^T \Phi_j \mathbf{e}_j^T (\mathbf{B} \Phi)^{-1} \mathbf{f} \int_{-\infty}^{\infty} \frac{1}{(\omega_j - \omega) - \Delta\omega} p_a(\Delta\omega) d\Delta\omega \right] \quad (14)$$

$$= (-i) \sum_{j=1, \dots, N} \left\{ [\mathbf{c}^T \Phi_j \mathbf{e}_j^T (\mathbf{B} \Phi)^{-1} \mathbf{f}] S(\omega, a, \omega_j) \right\} \quad (15)$$

where

$$S(\omega, a, \omega_j) = \int_{-\infty}^{\infty} \frac{1}{(\omega_j - \omega) - \Delta\omega} p_a(\Delta\omega) d\Delta\omega. \quad (16)$$

The frequency averaging width, a , can in general be frequency dependent but this is not explicitly denoted here to keep notations tight and since a constant value will be considered in the rest of the paper. The advantage of the modal form of last equations (15) and (16), is that the evaluation of the average can be expressed in the form of scalar integrals with a pole at $\Delta\omega = \omega_j - \omega$. This fact and a similar one for the evaluation of variances and covariances has been noted and used to evaluate the statistics of dynamic systems with uncertainty and for frequency averaging in the last decade or so. Detailed description can notably be found in [14, 15, 16, 17, 18, 19] and [20, 21, 7] and in the references therein.

3.2. Gaussian averaging or filtering

In the Gaussian case considered here, for which frequency averaging corresponds to applying a Gaussian filter to the transfer function $g(\omega)$, all integrals can be evaluated analytically, as discussed in detail in [17] and summarised in [7]. In the damped case, i.e. when the imaginary part of the eigenvalues, ω_j , is strictly positive, this corresponds to [22, 23]

$$S^{(g)}(\omega, a, \omega_j) = \int_{-\infty}^{\infty} \frac{1}{(\omega_j - \omega) - \Delta\omega} p_a^{(g)}(\Delta\omega) d\Delta\omega = \frac{-i}{a} \sqrt{\frac{\pi}{2}} w\left(\frac{\omega_j - \omega}{\sqrt{2}a}\right) \quad (17)$$

where $w(b)$ denotes the Faddeeva function [24, Eq.(7.1.4)], $e^{-b^2} \operatorname{erfc}(-ib)$, a scaled form of the complementary error function, which is itself defined for real argument as $\operatorname{erfc}(x) = (2/\sqrt{\pi}) \int_x^{\infty} e^{-t^2} dt$. Other forms of the integrals are available for undamped and unstable modal components of the systems, with the cautionary note that these integrals have to be understood in a principal value sense for undamped modes, i.e. for those whose imaginary part is zero, $\Im(\omega_j) = 0$. The focus is however here only on stable modes as it is a common situation in engineering. Furthermore, this readily permits the matrix expansion of the rational expressions introduced in the next section.

4. RATIONAL FORMS OF THE FREQUENCY AVERAGES AND THEIR MATRIX KRYLOV EXPRESSIONS

Accurate matrix expressions of the exact frequency averages are now derived. While a matrix form of the exact average, as those discussed in [19], could be used before deriving an approximation, the presentation here proceeds by first developing the scalar rational approximation of each of the modal functions $S(\omega, a, \omega_j)$, following the same approach introduced in [21]. The general principles are presented before being specialised to a particular category of rational expansions for the Gaussian average.

4.1. General matrix expressions

The idea is to first expand the function $S(\omega, a, \omega_j)$, that is $-\frac{i}{a}\sqrt{\frac{\pi}{2}}w(\frac{\omega_j-\omega}{\sqrt{2a}})$ in the real Gaussian case, as a rational function of its argument, such as, mostly for illustrative purpose,

$$S(\omega, a, \omega_j) = \lim_{K \rightarrow \infty} \sum_{k=0, \dots, K} \left[c_k \left(\frac{(\omega_j - \omega) - n_{k,K}}{(\omega_j - \omega) - d_{k,K}} \right)^{j_k} \right] \quad (18)$$

where the limit might be evaluated for a subset of the K terms. Substituting this expression in equation (15) then gives

$$\hat{g}(\omega) = (-i) \lim_{K \rightarrow \infty} \sum_{k=0, \dots, K} \left\{ c_k \mathbf{c}^T \left[\Phi \text{diag}_{j=1, \dots, N} \left(\frac{(\omega_j - \omega) - n_{k,K}}{(\omega_j - \omega) - d_{k,K}} \right)^{j_k} (\mathbf{B}\Phi)^{-1} \right] \mathbf{f} \right\}. \quad (19)$$

If the terms $d_{k,K}$ and $n_{k,K}$ are independent of the ω_j poles, one can express the average in matrix form as now explained. From section (2), one knows that $\mathbf{A} = (\mathbf{B}\Phi)\Lambda(\Phi)^{-1}$, and, evidently, that $\mathbf{B} = (\mathbf{B}\Phi)\mathbf{I}(\Phi)^{-1}$. It follows, inversely, that $(\Phi)\Lambda^{-1}(\mathbf{B}\Phi)^{-1} = \mathbf{A}^{-1}$, $(\Phi)(\mathbf{B}\Phi)^{-1} = \mathbf{B}^{-1}$, and that, for any d , $(\Phi)(\mathbf{A} - d\mathbf{I})^{-1}(\mathbf{B}\Phi)^{-1} = (\mathbf{A} - d\mathbf{B})^{-1}$. Considering the individual factors in the numerator and denominator individually, and denoting them here $N_{k,j}$ and $D_{k,j}$ for concision, one can expand the diagonal matrix of ratios of equation (19) so that $\text{diag}_j (N_{k,j}/D_{k,j})^{j_k} = [(\mathbf{B}\Phi)^{-1}(\mathbf{B}\Phi)\text{diag}_j (N_{k,j}) (\Phi)^{-1}(\Phi)\text{diag}_j (1/D_{k,j})]^{j_k}$ and find that

$$\hat{g}(\omega) = (-i) \lim_{K \rightarrow \infty} \sum_{k=0, \dots, K} \left(c_k \mathbf{c}^T \mathbf{B}^{-1} \left\{ [\mathbf{A} + (i\omega + in_{k,K})\mathbf{B}] [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \right\}^{j_k} \mathbf{f} \right). \quad (20)$$

In practice, the infinite sum must be evaluated analytically, approximated, or truncated. In the latter case, the resulting approximation of the exact average, say, up to maximum truncated order J , that is

$$\hat{g}_{J+1}(\omega) = (-i) \sum_{\{k | j_k \leq J\}} \left(c_k \mathbf{c}^T \mathbf{B}^{-1} \left\{ [\mathbf{A} + (i\omega + in_{k,K})\mathbf{B}] [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \right\}^{j_k} \mathbf{f} \right), \quad (21)$$

will only be precise for small values of J if the rational approximations are such that they converge quickly.

In summary, two general conditions for being able to derive an effective rational matrix expression of the average are that

1. the same scalar rational expression can be used for any argument $\omega_j - \omega$;
2. this expression is precise for relatively small maximum orders, J , of the rational expansion, independently of the argument, ω_j .

Approximations with such properties exist, as will be explicitly shown in section (4.3) for the Gaussian case.

4.2. Exact Krylov evaluation

Matrix expressions such as those of equation (21) can be evaluated exactly by making use of (Padé) interpolation properties of rational Krylov subspaces [9, 10, 11, 12, 13]. The idea is that instead of evaluating all the solutions and matrix-vector products in turn, such as the solution $\mathbf{p} = [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \mathbf{f}$ followed by the product $\mathbf{q} = [\mathbf{A} + (i\omega + in_{k,K})\mathbf{B}] \mathbf{p}$, one first evaluates a set of *directions* through similar solutions and products. A sufficient set of directions for matching is described in theorem (1). As long as the subspace spanned by these directions is included in the column space of left- and right-projection matrices, $\mathbf{W}$ and $\mathbf{V}$, that are used to reduce the system as explained in section . The average of the reduced model then matches *exactly* that of the original system. Note that slightly more general properties, also a direct consequence of [9,

Theorem 3.1], exist that include higher orders of the Krylov subspaces $\bigcup_{k=1}^{K_f} \mathcal{K}_k \{ \mathbf{B}^{-1} \mathbf{A}, \mathbf{B}^{-1} \mathbf{f} \}$ and $\bigcup_{k=1}^{K_c} \mathcal{K}_k \{ \mathbf{B}^{-T} \mathbf{A}^T, \mathbf{B}^{-T} \mathbf{c} \}$. They are formally described in [13, theorem 7.4.3].

While using a reduced subspace might look like an unnecessary detour, this provides two critical advantages: first, the evaluation remains more robust than the direct evaluation, even for moderately large system dimension [10], and, second and most importantly, this provides a framework in which to interpolate the approximations over a frequency range, as will be discussed in section (5).

Theorem 1

corollary of [9, Theorem 3.1] If

$$\mathbf{B}^{-1} \mathbf{f} \subseteq \text{colsp}(\mathbf{V}) \quad \text{or} \quad \mathbf{B}^{-T} \mathbf{c} \subseteq \text{colsp}(\mathbf{W}) \quad (22)$$

and, for each $k | j_k \leq J$, both

$$\bigcup_{j=1}^{J_{f,k}} \mathcal{K}_j \left\{ [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \mathbf{B}, [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}) \quad \text{and} \quad (23)$$

$$\bigcup_{j=1}^{J_{c,k}} \mathcal{K}_j \left\{ [\mathbf{A}^T + (i\omega + id_{k,K})\mathbf{B}^T]^{-1} \mathbf{B}^T, [\mathbf{A}^T + (i\omega + id_{k,K})\mathbf{B}^T]^{-1} \mathbf{c} \right\} \subseteq \text{colsp}(\mathbf{W}) \quad (24)$$

such that $J_{f,k} + J_{c,k} \geq j_k$, then the following function defined in equation (21) is exactly equal to its corresponding expression in the reduced subspaces, i.e.

$$\hat{g}_{J+1}(\omega) = (-i) \sum_{\{k | j_k \leq J\}} \left(c_k \mathbf{c}^T \mathbf{B}^{-1} \left\{ [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}] [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \right\}^{j_k} \mathbf{f} \right) \quad (25)$$

$$= (-i) \sum_{\{k | j_k \leq J\}} \left(c_k \hat{\mathbf{c}}^T \hat{\mathbf{B}}^{-1} \left\{ [\hat{\mathbf{A}} + (i\omega + id_{k,K})\hat{\mathbf{B}}] [\hat{\mathbf{A}} + (i\omega + id_{k,K})\hat{\mathbf{B}}]^{-1} \right\}^{j_k} \hat{\mathbf{f}} \right) \quad (26)$$

where $\hat{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \mathbf{V}$, $\hat{\mathbf{f}} = \mathbf{W}^T \mathbf{f}$, and $\hat{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$.

Proof

The proof is trivial once one realises that the alternance of products of shifted matrices and their inverses is a sum of moments of the matrix pencil, $(\mathbf{A}, \mathbf{B})$, in the sense that

$$\mathbf{B}^{-1} \left\{ [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}] [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \right\}^J = d_B \mathbf{B}^{-1} + \sum_{j=0, \dots, J-1} \left(d_{A,j} \left\{ [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \mathbf{B} \right\}^j [\mathbf{A} + (i\omega + id_{k,K})\mathbf{B}]^{-1} \right) \quad (27)$$

for some scalars d_B and $d_{A,j}$. Matching is then proved by using this expression for both the original and the reduced model and considering [9, theorem 3.1]. $\square$

4.3. Gaussian matrix expressions and FRADS-G algorithm

The general principles of sections (4.1)-(4.2) are now specialised to a particular rational approximation of the real Gaussian case integral (17) that has been proposed by Weideman [8]. The system is assumed damped, i.e. all the eigenvalues ω_j have strictly positive imaginary part. The terms related to this particular Gaussian case are denoted by a superscript $^{(g)}$.

Rational expression of Faddeeva, Gaussian and Dawson's functions. The starting point is the following series from [8, Equation (38)(I)]:

$$w(z) \approx w_R(z) = \frac{1}{\sqrt{\pi}(L - iz)} + \frac{2}{(L - iz)^2} \sum_{r=2}^R \left[A_{r-1} \left(\frac{L + iz}{L - iz} \right)^{r-2} \right] \quad \text{for} \quad \Im(z) > 0. \quad (28)$$

It provides a rational approximation similar to that of equation (18), as will be discussed in detail in the next section. Here, R is the rational order of the approximation of $w(z)$, i.e. the order of the denominator's polynomial in z and $L = 2^{-1/4}(R-1)^{1/2}$ is the pole of an expansion of the Gaussian function whose value has been chosen in such a way that it *approaches optimal* accuracy for a given order R . The terms A_n are the estimated coefficients of the expansion that can be evaluated through a discrete fast Fourier transform, so that [8, Equation (39)],

$$A_n = \frac{1}{2M} \sum_{j=-M+1}^{M-1} \left[(L^2 + t_j^2) e^{-t_j^2} e^{-in\theta_j} \right] \quad \text{for } n = 1, \dots, R-1 \quad (29)$$

and

$$t_j = L \tan\left(\frac{1}{2}\theta_j\right), \quad \theta_j = \frac{\pi j}{M}, \quad \text{for } j = -M+1, \dots, M-1. \quad (30)$$

Weideman expounds that these coefficients decrease very quickly and are good estimates of the actual coefficients for the value $M = 2(R-1)$, that is also chosen here. Some of the other comments in [8] about the approximation of equation (28) that are also of some utility here are that

- independently of the location of the argument in the complex plane, it returns six significant digits or more of the function $w(\cdot)$ for $R = 17$ and twelve digits or more for $R = 33$;
- for real argument, $z = x$, it returns approximations of the Gaussian function and of Dawson's function [24] $F(x) = e^{-x^2} \int_0^x e^{t^2} dt$: $\Re(w(x)) = e^{-x^2}$ and $\Im(w(x)) = \frac{2}{\pi} F(x)$;
- the approximations converge to the exact functions for increasing rational order $R \rightarrow \infty$.

Rational modal and matrix expression of the Gaussian frequency average. In the present case of a damped system and of a real normal disturbance, one can thus substitute the approximation (28) of the Faddeeva function into equations (17) and (15) to obtain the approximation of the average, i.e.

$$S_R^{(g)}(\omega, a, \omega_j) = \frac{1}{(\omega_j + L\sqrt{2}ai) - \omega} + \frac{i2^{3/2}a\sqrt{\pi}}{[(\omega_j + L\sqrt{2}ai) - \omega]^2} \sum_{r=2}^R \left[A_{r-1} \left(\frac{(-\omega_j + L\sqrt{2}ai) + \omega}{(\omega_j + L\sqrt{2}ai) - \omega} \right)^{r-2} \right] \quad (31)$$

for $\Im(\omega_j) > 0$ and therefore

$$\begin{aligned} \hat{g}_R^{(g)}(\omega) \approx \hat{g}_R^{(g)}(\omega) &= \sum_{j=1, \dots, N} \left(\left\{ \frac{-i}{(\omega_j + L\sqrt{2}ai) - \omega} + \frac{2^{3/2}a\sqrt{\pi}}{[(\omega_j + L\sqrt{2}ai) - \omega]^2} \times \right. \right. \\ &\quad \left. \left. \sum_{r=2}^R \left[A_{r-1} \left(\frac{(-\omega_j + L\sqrt{2}ai) + \omega}{(\omega_j + L\sqrt{2}ai) - \omega} \right)^{r-2} \right] \right\} [\mathbf{c}^T \Phi_j \mathbf{e}_j^T (\mathbf{B}\Phi)^{-1} \mathbf{f}] \right) \quad (32) \end{aligned}$$

Expanding this expression in matrix form, as explained in section (4.1), then gives the very simple matrix expression,

$$\hat{g}_R^{(g)}(\omega) = \mathbf{c}^T \left(\mathbf{I} - 2^{3/2}a\sqrt{\pi} \mathbf{A}_L^{-1} \mathbf{B} \sum_{r=2}^R \left\{ A_{r-1} \left[-\mathbf{A}_L^{-1} (\mathbf{A}_L + 2^{3/2}La\mathbf{B}) \right]^{r-2} \right\} \right) \mathbf{A}_L^{-1} \mathbf{f} \quad (33)$$

where $\mathbf{A}_L = [(\mathbf{A} - L\sqrt{2}a\mathbf{B}) - (i\omega)\mathbf{B}]$. Note that the solutions with this $\mathbf{A}_L$ matrix correspond to solving the original system with increased damping in the sense that the eigenvalues, ω_j , of the original system are shifted in the imaginary direction so that they become $\omega_j + iL\sqrt{2}a$.

Rational Krylov expression of the Gaussian frequency average. One can take advantage of the exact evaluation of $\hat{g}_R^{(g)}(\omega)$. For this particular rational expansion of the Faddeeva function, one can also use the sufficient matching properties of rational Krylov projections as expressed in the following theorem.

Theorem 2

If

$$\bigcup_{r=1}^{R_f} \mathcal{K}_r \left\{ \left[(\mathbf{A} - L\sqrt{2}a\mathbf{B}) - (i\omega)\mathbf{B} \right]^{-1} \mathbf{B}, \left[(\mathbf{A} - L\sqrt{2}a\mathbf{B}) - (i\omega)\mathbf{B} \right]^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}) \quad \text{and} \quad (34)$$

$$\bigcup_{r=1}^{R_c} \mathcal{K}_r \left\{ \left[(\mathbf{A} - L\sqrt{2}a\mathbf{B}) - (i\omega)\mathbf{B} \right]^{-T} \mathbf{B}^T, \left[(\mathbf{A} - L\sqrt{2}a\mathbf{B}) - (i\omega)\mathbf{B} \right]^{-T} \mathbf{c} \right\} \subseteq \text{colsp}(\mathbf{W}) \quad (35)$$

for $R_f + R_c \geq R$, then the rational approximations of the average can be evaluated exactly, using their corresponding expressions in the reduced subspaces, i.e.

$$\hat{g}_R^{(g)}(\omega) = \hat{\mathbf{c}}^T \left(\mathbf{I} - 2^{3/2}a\sqrt{\pi}\hat{\mathbf{A}}_L^{-1}\hat{\mathbf{B}} \sum_{r=2}^R \left\{ A_{r-1} \left[-\hat{\mathbf{A}}_L^{-1} \left(\hat{\mathbf{A}}_L + 2^{3/2}La\hat{\mathbf{B}} \right) \right]^{r-2} \right\} \right) \hat{\mathbf{A}}_L^{-1}\hat{\mathbf{f}} \quad (36)$$

where $\hat{\mathbf{A}}_L = \mathbf{W}^T \mathbf{A}_L \mathbf{V} = \left[(\hat{\mathbf{A}} - L\sqrt{2}a\hat{\mathbf{B}}) - (i\omega)\hat{\mathbf{B}} \right]$, $\hat{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \mathbf{V}$, $\hat{\mathbf{f}} = \mathbf{W}^T \mathbf{f}$, and $\hat{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$.

Proof

This is a direct corollary of [9, Theorem 3.1] (or of theorem (1)). □

Algorithmic steps for rational Gaussian frequency averaging The operations to obtain the rational approximation through Krylov projection at a given frequency are then summarised in the, so-called, *FRADS-G* algorithm (1). The steps of this pseudo-algorithm are indicative of the actual operations that can be followed to evaluate the Gaussian average. For example, the location of the poles, as specified by the value L in step (1), that has been determined in such a way that they are close to optimality for a given order, R , could be chosen differently. Regarding the evaluation of the coefficients of the Fourier expansion in steps (2)-(3), considering the independence of the matrix solutions to them, and the limited cost of their evaluation, any precise alternative for their evaluation or increased value of M could be considered. Finally, the model reduction steps are presented sequentially while, in practice, all these operations could be combined and updated in an iterative Krylov method for increasing order, R .

Algorithm 1: *FRADS-G*: Frequency rational average of dynamic systems, in the Gaussian case using the approximation of equation (28).

Data: the system matrices and vectors, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{f}$, as defined in equation (2). A frequency of interest, ω , as described in equation (8), an averaging width, a , and a rational order, R .

Result: the rational estimation, $\hat{g}_R^{(g)}(\omega)$, of the Gaussian average.

- 1 choose the poles' shift by $L = 2^{-1/4}(R-1)^{1/2}$, so that $\mathbf{A}_L = \left[(\mathbf{A} - L\sqrt{2}a\mathbf{B}) - (i\omega)\mathbf{B} \right]$;
- 2 define $M = 2(R-1)$, $\theta_j = \pi j/M$, $t_j = L \tan(\theta_j/2)$ for $j = -M+1, \dots, M-1$;
- 3 evaluate the FFT $A_n = \frac{1}{2M} \sum_{j=-M+1}^{M-1} \left[(L^2 + t_j^2) e^{-t_j^2} e^{-in\theta_j} \right]$ for $n = 1, \dots, R-1$;
- 4 produce right- and left-projection matrices, $\mathbf{V}$ and $\mathbf{W}$, for $R_f + R_c \geq R$, so that,
 $\bigcup_{r=1}^{R_f} \mathcal{K}_r \{ \mathbf{A}_L^{-1} \mathbf{B}, \mathbf{A}_L^{-1} \mathbf{f} \} \subseteq \text{colsp}(\mathbf{V})$, $\bigcup_{r=1}^{R_c} \mathcal{K}_r \{ \mathbf{A}_L^{-T} \mathbf{B}^T, \mathbf{A}_L^{-T} \mathbf{c} \} \subseteq \text{colsp}(\mathbf{W})$;
- 5 define $\hat{\mathbf{A}}_L = \mathbf{W}^T \mathbf{A}_L \mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \mathbf{V}$, $\hat{\mathbf{f}} = \mathbf{W}^T \mathbf{f}$, and $\hat{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$;
// Alternatively, use other projection methods based on the information in $\mathbf{V}$ and $\mathbf{W}$, e.g. interpolatory methods if the system of equation (2) originates from a structured system
- 6 evaluate

$$\hat{g}_R^{(g)}(\omega) = \hat{\mathbf{c}}^T \left(\mathbf{I} - 2^{3/2}a\sqrt{\pi}\hat{\mathbf{A}}_L^{-1}\hat{\mathbf{B}} \sum_{r=2}^R \left\{ A_{r-1} \left[-\hat{\mathbf{A}}_L^{-1} \left(\hat{\mathbf{A}}_L + 2^{3/2}La\hat{\mathbf{B}} \right) \right]^{r-2} \right\} \right) \hat{\mathbf{A}}_L^{-1}\hat{\mathbf{f}}$$

One of the main properties of the *FRADS* algorithm is that, for the single frequency ω , the average can be evaluated without recurring to any modal analysis: the main cost is only a few solutions evaluated at a single shifted frequency, $\omega - iL\sqrt{2}a$. Also, thanks to the fact that the sufficient

condition for the matching property of theorem (2) involves *the sum* of the orders of both left- and right-Krylov subspaces, one can assure a rational order R with just half of the number of solutions, i.e. $R_f = R_c = \text{ceil}(R/2)$ where $\text{ceil}(\cdot)$ indicates the rounded up integer. In practice, several decimals are correctly evaluated with just a few solutions. It was demonstrated in [21] that even with *a single solution* at each frequency, the average could be evaluated with a relative error that was less than one percent. The alternative interpolatory projection methods mentioned after step (5) of the algorithm are further discussed in section (7).

5. SEMI-ANALYTICAL INTERPOLATION OF THE FREQUENCY AVERAGES

5.1. General expressions

Considering now the average in a frequency range, as suggested and presented without details in [21], the rational expressions, $\hat{g}_{J+1}(\omega)$, can be interpolated at only a few interpolation frequencies, $\omega = \sigma^{(p)}$, and still provide excellent approximation of the exact average at other frequencies. This operation is particularly interesting when the system is complex, i.e. when it has many frequencies of resonance in a band. These can then all be smoothed by the frequency averaging. For the general expression of section (4.1), the interpolation is a direct consequence of the sufficiency conditions of theorem (1). This fact is formalised in theorem (3). Note that both the maximum order and the rational expansion may vary from frequency to frequency. Each of the sets of orders, roots, and poles is therefore implicitly a function of ω and their particular values at the interpolation points $\omega = \sigma^{(p)}$ for $p = 1, \dots, P$ are distinguished by a superscript, $\cdot^{(p)}$, so that, for example, $J^{(p)} = J(\sigma_p)$.

Theorem 3

If

$$\mathbf{B}^{-1}\mathbf{f} \subseteq \text{colsp}(\mathbf{V}) \quad \text{or} \quad \mathbf{B}^{-T}\mathbf{c} \subseteq \text{colsp}(\mathbf{W}) \quad (37)$$

and, for each p and each $k | j_k^{(p)} \leq J^{(p)}$, both

$$\bigcup_{j=1}^{J_{f,k}^{(p)}} \mathcal{K}_j \left\{ \left[\mathbf{A} + (i\sigma^{(p)} + id_{k,K}^{(p)})\mathbf{B} \right]^{-1} \mathbf{B}, \left[\mathbf{A} + (i\sigma^{(p)} + id_{k,K}^{(p)})\mathbf{B} \right]^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}) \quad \text{and} \quad (38)$$

$$\bigcup_{j=1}^{J_{c,k}^{(p)}} \mathcal{K}_j \left\{ \left[\mathbf{A}^T + (i\sigma^{(p)} + id_{k,K}^{(p)})\mathbf{B}^T \right]^{-1} \mathbf{B}^T, \left[\mathbf{A}^T + (i\sigma^{(p)} + id_{k,K}^{(p)})\mathbf{B}^T \right]^{-1} \mathbf{c} \right\} \subseteq \text{colsp}(\mathbf{W}) \quad (39)$$

such that $J_{f,k}^{(p)} + J_{c,k}^{(p)} \geq j_k^{(p)}$, then the following function defined in equation (21) is exactly equal to its corresponding expression in the reduced subspaces at each of the interpolation points, i.e.

$$\hat{g}_{J+1}(\omega = \sigma^{(p)}) = \check{g}_{J+1}(\omega = \sigma^{(p)}) \quad (40)$$

$$\check{g}_{J+1}(\omega) = (-i) \sum_{\{k | j_k \leq J\}} \left(c_k \check{\mathbf{c}}^T \check{\mathbf{B}}^{-1} \left\{ \left[\check{\mathbf{A}} + (i\omega + in_{k,K})\check{\mathbf{B}} \right] \left[\check{\mathbf{A}} + (i\omega + id_{k,K})\check{\mathbf{B}} \right]^{-1} \right\}^{j_k} \check{\mathbf{f}} \right) \quad (41)$$

where $\check{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{V}$, $\check{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \mathbf{V}$, $\check{\mathbf{f}} = \mathbf{W}^T \mathbf{f}$, and $\check{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$.

An important advantage of this approach discussed in [21, 7] is that one can use the same reduced model to estimate precisely the frequency average of a complex dynamic system over a whole frequency range, even when the averaging width is frequency dependent.

5.2. Gaussian expressions and FRAX-G algorithm

The same rational interpolation of the approximations can be applied in the Gaussian case, which gives the results of theorem (4). Note that the values of the order, R , and averaging width, a , can generally depend on frequency as, consequently, the shifting value, L . The values at the p -th interpolation point, $\sigma^{(p)}$, are denoted by a superscript, $\cdot^{(p)}$.

Theorem 4

If, for each $p = 1, \dots, P$, where $\sigma^{(p)}$ is an interpolation point,

$$\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}) \quad \text{and} \quad (42)$$

$$\bigcup_{r=1}^{R_c^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-T} \mathbf{B}^T, \left(\mathbf{A}_L^{(p)} \right)^{-T} \mathbf{c} \right\} \subseteq \text{colsp}(\mathbf{W}) \quad (43)$$

for $\mathbf{A}_L^{(p)} = (\mathbf{A} - L^{(p)} \sqrt{2} a^{(p)} \mathbf{B}) - (i\sigma^{(p)}) \mathbf{B}$ and $R_f^{(p)} + R_c^{(p)} \geq R^{(p)}$, then the rational approximations of the average can be evaluated exactly, using their corresponding expressions in the reduced subspaces, i.e.

$$\check{g}_R^{(g)}(\omega = \sigma^{(p)}) = \hat{g}_R^{(g)}(\omega = \sigma^{(p)}) \quad (44)$$

where $\hat{g}_R^{(g)}$ was introduced in equation (33) and $\check{g}_R^{(g)}$ is defined as

$$\check{g}_R^{(g)}(\omega) = \check{\mathbf{c}}^T \left(\check{\mathbf{A}}_L^{-1} - 2^{3/2} a \sqrt{\pi} \check{\mathbf{A}}_L^{-1} \check{\mathbf{B}} \check{\mathbf{A}}_L^{-1} \sum_{r=2}^R \left\{ A_{r-1} \left[\left(-\check{\mathbf{A}}_L - 2^{3/2} L a \check{\mathbf{B}} \right) \check{\mathbf{A}}_L^{-1} \right]^{r-2} \right\} \right) \check{\mathbf{f}} \quad (45)$$

with $\check{\mathbf{A}}_L = \left[(\check{\mathbf{A}} - L \sqrt{2} a \check{\mathbf{B}}) - (i\omega) \check{\mathbf{B}} \right]$, $\check{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \mathbf{V}$, $\check{\mathbf{f}} = \mathbf{W}^T \mathbf{f}$, $\check{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$. All a , L , and A_{r-1} can be functions of ω .

Algorithmic steps for rational interpolation of Gaussian frequency averaging The operations to obtain the rational semi-analytical interpolation of the frequency average through Krylov projection, as described in theorem (4), are summarised in the *FRAX-G* algorithm (2).

Algorithm 2: *FRAX-G*: Frequency rational average of complex systems, in the Gaussian case, using the interpolated approximation of equation (45).

Data: the system matrices and vectors, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{f}$, as defined in equation (2). A frequency range of interest, $\omega \in [\omega_{\min}, \omega_{\max}]$, for ω as described in equation (8), a desired frequency dependent averaging width, $a(\omega)$, a rational order, $R(\omega)$, and a set of interpolation points, $\sigma^{(p)}$, for $p = 1, \dots, P$.

Result: the interpolated rational estimation, $\check{g}_R^{(g)}(\omega)$, of the Gaussian average, so that

$$\check{g}_R^{(g)}(\sigma^{(p)}) = \hat{g}_R^{(g)}(\sigma^{(p)}).$$

```

1 denote the values of the function  $R$  at  $\sigma^{(p)}$  by a superscript,  $\cdot^{(p)}$ :  $R^{(p)} = R(\sigma^{(p)})$ ;
2 for  $p \leftarrow 1$  to  $P$  do
3   choose the poles' shift by  $L^{(p)} = 2^{-1/4}(R^{(p)} - 1)^{1/2}$ , so that
      $\mathbf{A}_L^{(p)} = [(\mathbf{A} - L^{(p)}\sqrt{2}a^{(p)}\mathbf{B}) - (i\sigma^{(p)})\mathbf{B}]$ ;
4   produce right- and left-projection matrices,  $\mathbf{V}^{(p)}$  and  $\mathbf{W}^{(p)}$ , for  $R_f^{(p)} + R_c^{(p)} \geq R^{(p)}$ , so
     that,  $\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left( \mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left( \mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}^{(p)})$ ,
      $\bigcup_{r=1}^{R_c^{(p)}} \mathcal{K}_r \left\{ \left( \mathbf{A}_L^{(p)} \right)^{-T} \mathbf{B}^T, \left( \mathbf{A}_L^{(p)} \right)^{-T} \mathbf{c} \right\} \subseteq \text{colsp}(\mathbf{W}^{(p)})$ ;
5 end
6 produce right- and left-projection matrices,  $\mathbf{V}$  and  $\mathbf{W}$ , so that,  $\text{colsp}(\mathbf{V}^{(p)}) \subseteq \text{colsp}(\mathbf{V})$  and
    $\text{colsp}(\mathbf{W}^{(p)}) \subseteq \text{colsp}(\mathbf{W})$ , for  $p = 1, \dots, P$ ;
7 define  $\check{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{V}$ ,  $\check{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \mathbf{V}$ ,  $\check{\mathbf{f}} = \mathbf{W}^T \mathbf{f}$ , and  $\check{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$ ;
   // Alternatively, use other projection methods based on the
   // information in  $\mathbf{V}$  and  $\mathbf{W}$ , e.g. interpolatory methods if the
   // system of equation (2) originates from a structured system
8 for each frequency,  $\omega$ , of interest do
9   define  $L = 2^{-1/4}(R - 1)^{1/2}$ ,  $\check{\mathbf{A}}_L = [(\check{\mathbf{A}} - L\sqrt{2}a\check{\mathbf{B}}) - (i\omega)\check{\mathbf{B}}]$ ;
10  define  $M = 2(R - 1)$ ,  $\theta_j = \pi j/M$ ,  $t_j = L \tan(\theta_j/2)$  for  $j = -M + 1, \dots, M - 1$ ;
11  evaluate the FFT  $A_n = \frac{1}{2M} \sum_{j=-M+1}^{M-1} \left\{ [(L)^2 + t_j^2] e^{-t_j^2} e^{-in\theta_j} \right\}$  for  $n = 1, \dots, R - 1$ ;
12  evaluate  $\check{g}_R^{(g)}(\omega) =$ 
      $\check{\mathbf{c}}^T \left( \check{\mathbf{A}}_L^{-1} - 2^{3/2} a \sqrt{\pi} \check{\mathbf{A}}_L^{-1} \check{\mathbf{B}} \check{\mathbf{A}}_L^{-1} \sum_{r=2}^R \left\{ A_{r-1} \left[ (-\check{\mathbf{A}}_L - 2^{3/2} L a \check{\mathbf{B}}) \check{\mathbf{A}}_L^{-1} \right]^{r-2} \right\} \right) \check{\mathbf{f}}$ 
13 end
```

Again, the steps of this pseudo-algorithm are only indicative of the actual operations that can be followed to evaluate the Gaussian average. Additionally to the remarks identical to those made about the *FRADS-G* algorithm (1), one should note that the chosen order of presentation of the interpolation points is purely arbitrary and that one could as well build the $\mathbf{V}$ and $\mathbf{W}$ matrices by considering them and their multiplicities in any order [11]. In practice, attention must also be paid to cases such as those where linear combination of the directions occur within or between individual matrices, $\mathbf{V}_j$ and $\mathbf{W}_j$, when constructing the projection matrices, $\mathbf{V}$ and $\mathbf{W}$ as, for example, explained in [9]. The steps (10)-(11) and the definition of L are independent of the particular system and frequency considered. They can thus be computed and stored only once for each different value of the order R .

A main advantage of the proposed theory and algorithm is that a single reduced model may include information useful through the whole frequency range, thanks to the frequency dependance of the averaging width function. Even with interpolation, several decimals may be correctly evaluated with just a few solutions, *for the whole frequency range* as shown in [21].

6. TIME RESPONSES FROM FREQUENCY AVERAGED FUNCTIONS

The time responses can be evaluated directly from the frequency averaged functions or from their estimates through model reduction.

6.1. General time analysis

The starting point is that, for a constant averaging width $a(\omega) = a$, the frequency averaging of the system's transfer function, as defined in equation (13), is its convolution by the weighting function, $p_a(\Delta\omega)$. As explained in [7], its inverse Fourier transform is then simply equal to the *exact* system's impulse response, scaled by the characteristic function of the averaging function, i.e.

$$\mathcal{IF} [\hat{g}(\omega)](t) = \underline{g}(t) \mathcal{IF} [p_a(\Delta\omega)](t) \quad (46)$$

where $\mathcal{IF} [v(\omega)](t)$ indicates the value at time t of the inverse Fourier transform of a function $v(\omega)$, as defined in equation (8). One can thus, in theory, retrieve the *exact* impulse response of the original system by the inverse scaling operation,

$$\underline{g}(t) = \mathcal{IF} [\hat{g}(\omega)](t) \{ \mathcal{IF} [p_a(\Delta\omega)](t) \}^{-1}. \quad (47)$$

Using the interpolated estimator, $\check{g}_{J+1}(\omega)$, instead of the exact frequency average, $\hat{g}(\omega)$, in this expression presents a very important characteristic in the case of a complex system: the impulse response can be evaluated with an economical form of equation (5). The key argument, as expressed in equation (41), is that the approximate average, $\check{g}_{J+1}(\omega)$, of the original system is the *exact average of the reduced model* defined by the matrices and vectors $\check{\mathbf{A}}$, $\check{\mathbf{B}}$, $\check{\mathbf{f}}$, and $\check{\mathbf{c}}$ for the same averaging function. Both equations (47) and (46) can therefore also be exploited on the latter system. Denoting $\check{\underline{g}}_{J+1}(t)$ the impulse response of the smaller system - which differs from the inverse Fourier transform of its frequency average -, one has *exactly*

$$\mathcal{IF} [\check{g}_{J+1}(\omega)](t) = \check{\underline{g}}_{J+1}(t) \mathcal{IF} [p_a(\Delta\omega)](t) \quad (48)$$

and

$$\check{\underline{g}}_{J+1}(t) = \mathcal{IF} [\check{g}_{J+1}(\omega)](t) \{ \mathcal{IF} [p_a(\Delta\omega)](t) \}^{-1}. \quad (49)$$

Considering $\check{g}_{J+1}(\omega) \approx \hat{g}(\omega)$ and both equations (49) and (47), one finds the approximation

$$\check{\underline{g}}_{J+1}(t) \approx \underline{g}(t). \quad (50)$$

One can thus work directly with the much smaller reduced model with the matrices $\check{\mathbf{A}}$, $\check{\mathbf{B}}$, and input and output vectors, $\check{\mathbf{f}}$ and $\check{\mathbf{c}}$, that were introduced in section (5) for a given set of interpolation points, poles, and rational orders. Denoting by m the smaller dimension of the reduced system, one can use exactly the same derivation as for the original system, with the main difference that one only needs the very limited modal information defined by

$$(\check{\mathbf{A}} - \check{\lambda}_j \check{\mathbf{B}}) \check{\phi}_j = \mathbf{0} \quad (51)$$

and $\check{\phi}_j \neq \mathbf{0}$ for $j = 1, \dots, m$. Storing as before the right-eigenvectors, $\check{\phi}_j$, in a matrix $\check{\Phi}$, the estimation of the impulse response of the original system has the following simple analytical *TRAX* expression, for the *Time Rational Analysis of Complex dynamic Systems*,

$$\underline{g}(t) \approx \check{\underline{g}}_{J+1}(t) = - \sum_{j=1, \dots, m} \left[\check{\mathbf{c}}^T \check{\phi}_j \mathbf{e}_j^T (\check{\mathbf{B}} \check{\Phi})^{-1} \check{\mathbf{f}} \mathbf{e}^{\check{\lambda}_j t} \right] \quad \text{for } t > 0. \quad (52)$$

Sources of approximations While theoretically exact, the proposed general strategy highlighted in equations (46) and (48) can, in finite precision arithmetic, only target specific regions of the time domain for a given averaging function. Indeed, the value of the scaling function $\mathcal{IF}[p_a(\Delta\omega)](t)$ must be large enough, otherwise the value of $\mathcal{IF}[\hat{g}(\omega)](t)$ may be so small that it is of the same order of magnitude or smaller than the accumulated numerical errors and therefore practically unusable. The physical or signal processing interpretation is thus that the averaging process focuses attention to those times at which $\mathcal{IF}[p_a(\Delta\omega)](t)$ is relatively large enough while the unnecessary details corresponding to the other times are smoothed out in the averaging process.

In practice, one must also be able to evaluate either the modes of the reduced model, or the frequency average and its inverse Fourier transform in manners that are both efficient and precise. For example, using the truncated functions $\hat{g}_{J+1}(\omega)$ instead of the exact $\hat{g}(\omega)$ evidently introduces some errors as does interpolating it by $\check{g}_{J+1}(\omega)$. The proposed approach is however fully functional, in particular when using the approach illustrated by equation (52), as is demonstrated in section (8).

6.2. Gaussian time analysis and TRAX-G algorithm

In the particular Gaussian case, it was detailed in [7] that $\mathcal{IF}[\hat{g}^{(g)}(\omega)](t) = \underline{g}(t)e^{-\frac{t^2 a^2}{2}}$ and the exact impulse response is theoretically available by the inverse scaling operation, $\underline{g}(t) = \mathcal{IF}[\hat{g}^{(g)}(\omega)](t)e^{\frac{t^2 a^2}{2}}$. The following expression of the impulse response is thus exact for constant averaging width, a ,

$$\underline{g}(t) = \frac{e^{(t^2 a^2)/2}}{2\pi} \int_{-\infty}^{\infty} \hat{g}^{(g)}(\omega) e^{i\omega t} d\omega \quad (53)$$

and the Gaussian version, *TRAX-G*, of the *TRAX* expression (52), is simply

$$\underline{g}(t) \approx \check{\underline{g}}_R^{(g)}(t) = - \sum_{j=1, \dots, m} \left[\check{\mathbf{c}}^T \check{\boldsymbol{\phi}}_j \mathbf{e}_j^T (\check{\mathbf{B}} \check{\boldsymbol{\Phi}})^{-1} \check{\mathbf{f}} e^{\check{\lambda}_j t} \right] \quad \text{for } t > 0 \quad (54)$$

where the reduced matrices are defined as in algorithm (2).

Sources of approximations Since the reduced values of the scaling Gaussian function $e^{-(t^2 a^2)/2}$ occur at larger time values of $a^2 t^2$, the predicted impulse response through Gaussian averaging is naturally geared towards lower times after impulse where the effect of finite precision arithmetic and accumulated errors is lesser. Other sources of approximation are the truncation of the rational approximations up to order R and, arguably, less importantly the errors introduced during the evaluation of this $\check{\underline{g}}_R^{(g)}$ function through Krylov model reduction, as described in theorem (2) and *FRADS-G* algorithm (1). The last important reason of differences compared to the exact expressions is when this approximation is itself interpolated as described in theorem (4) and in the *FRAX-G* algorithm (2). These differences depend notably on the choice of interpolation points and on the order of the Krylov subspaces built at each of them. In practice, one may recommend to use the *TRAX-G* approximation (54), only for times at which $e^{-(t^2 a^2)/2}$ is large enough. For example, for a threshold $e^{-(t^2 a^2)/2} \geq \epsilon$, one would only consider the estimation at times smaller than the following critical time T_ϵ ,

$$t \leq T_\epsilon = \sqrt{-2 \log(\epsilon)/(a^2)}. \quad (55)$$

This leads to the following algorithmic approach.

Algorithmic steps for Gaussian time rational analysis of complex systems The operations to obtain the estimated impulse response via rational semi-analytical interpolation of the frequency average through Krylov projection, as described in equation (54), are summarised in the *TRAX-G* algorithm (3).

Algorithm 3: *TRAX-G*: Time rational analysis of complex systems, in the Gaussian case, using the frequency interpolated approximation of equation (54).

Data: the system matrices and vectors, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{f}$, as defined in equation (2). A maximum time of interest, T .

Result: the estimated impulse response, $\underline{g}(t) \approx \check{\underline{g}}_R^{(g)}(t)$, obtained through interpolated rational estimation of a Gaussian frequency average.

- 1 choose a constant frequency dependent averaging width, a , so that $\sqrt{-2 \log(\varepsilon)/(a^2)} \geq T$ for some threshold ε ;
- 2 choose a rational order function, $R(\omega)$, and a set of interpolation points, $\sigma^{(p)}$, for $p = 1, \dots, P$;
- 3 denote the values of R at $\sigma^{(p)}$ by $R^{(p)} = R(\sigma^{(p)})$;
- 4 **for** $p \leftarrow 1$ **to** P **do**
- 5 choose the poles' shift by $L^{(p)} = 2^{-1/4}(R^{(p)} - 1)^{1/2}$, so that
 $\mathbf{A}_L^{(p)} = [(\mathbf{A} - L^{(p)}\sqrt{2}a\mathbf{B}) - (i\sigma^{(p)})\mathbf{B}]$;
- 6 produce right- and left-projection matrices, $\mathbf{V}^{(p)}$ and $\mathbf{W}^{(p)}$, for $R_f^{(p)} + R_c^{(p)} \geq R^{(p)}$, so that,
 $\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}^{(p)})$,
 $\bigcup_{r=1}^{R_c^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-T} \mathbf{B}^T, \left(\mathbf{A}_L^{(p)} \right)^{-T} \mathbf{c} \right\} \subseteq \text{colsp}(\mathbf{W}^{(p)})$;
- 7 **end**
- 8 produce right- and left-projection matrices, $\mathbf{V}$ and $\mathbf{W}$, so that, $\text{colsp}(\mathbf{V}^{(p)}) \subseteq \text{colsp}(\mathbf{V})$ and $\text{colsp}(\mathbf{W}^{(p)}) \subseteq \text{colsp}(\mathbf{W})$, for $p = 1, \dots, P$;
- 9 define $\check{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{V}$, $\check{\mathbf{B}} = \mathbf{W}^T \mathbf{B} \mathbf{V}$, $\check{\mathbf{f}} = \mathbf{W}^T \mathbf{f}$, and $\check{\mathbf{c}} = \mathbf{V}^T \mathbf{c}$;
// Alternatively, use other projection methods based on the information in $\mathbf{V}$ and $\mathbf{W}$, e.g. interpolatory methods if the system of equation (2) originates from a structured system
- 10 evaluate all $\check{\lambda}_j$, $\check{\phi}_j \neq \mathbf{0}$ defined by $(\check{\mathbf{A}} - \check{\lambda}_j \check{\mathbf{B}})\check{\phi}_j = \mathbf{0}$;
- 11 store the right-eigenvectors, $\check{\phi}_j$, in a matrix $\check{\Phi}$;
- 12 **for** each time, t , of interest **do**
- 13 evaluate $\check{\underline{g}}_R^{(g)}(t) = - \sum_{j=1, \dots, m} \left[\check{\mathbf{c}}^T \check{\phi}_j \mathbf{e}_j^T (\check{\mathbf{B}} \check{\Phi})^{-1} \check{\mathbf{f}} \mathbf{e}^{\check{\lambda}_j t} \right]$
- 14 **end**

Again, this pseudo-algorithm is essentially presenting the general steps of a particular occurrence of the general approach proposed in this paper. In terms of applicability, the interest of the algorithm mostly lies in situations where the original system is much more complex than what is necessary for the evaluation of its time response. In such a situation, the frequency average is significantly simpler than the original transfer function and can therefore be precisely represented by a much smaller reduced model that has only a few apparent resonances. The choice of interpolation points and rational orders at each of these points must capture these main features of the averaged transfer function. At this stage and in the absence of non-heuristic criteria to choose them, one may imagine a schema in which those parameters are progressively increased until the estimated frequency average has converged.

As before, the operations could generally be evaluated within an iterative procedure in which the reduced models and approximations are updated at each iteration. Different choices are also possible for the practical evaluation of the time response in step (13), for example time-stepping with the smaller system if desired or via an explicit estimation of the frequency average, again based on the smaller system.

A main advantage of the proposed theory and algorithm is that the time response of a complex system can be evaluated without having to evaluate its modes or having to run a time-stepping simulation with it.

7. FURTHER PRESERVED STABILITY PROPERTIES OF THE REDUCED MODELS

A possibly subtle but certainly important point in the practical case of finite precision arithmetic is that the reduced models might be more unstable than the original systems. More specifically, in the current presentation, it was assumed that the original system was damped, i.e. stable in the sense that all its eigenvalues had a positive imaginary part, $\Im(\omega_j) > 0$ and $\Re(\lambda_j) < 0$ for $j = 1, \dots, N$. While the goal of this assumption was to facilitate the discussion and introduction of the matrix form of the rational expressions of the average, this stability condition further assures that any error in the modal expressions or evaluation of equation (5) dies off when time increases. The stability of the smaller models obtained through model reduction is, in general, not assured, even if the original system is stable. This fact is not really problematic from a theoretical point of view as the expression (47) is exact, independently of the characteristics of the eigenvalues of the reduced model. From a practical point of view, some care must however be taken as numerical or approximation errors may be amplified at higher times by exponential terms such as $e^{\tilde{\lambda}_j t}$ if the reduced model happens to be unstable.

7.1. Preservation of stability in particular case

For several practical situations, it is possible to assure that the reduced models are stable if the original model is. One has for example the case of theorem (5) and, in general, it is recommended to similarly choose the combination of matrices $\mathbf{W}$ and $\mathbf{V}$ in such a way that stability is preserved.

Theorem 5

If $\mathbf{A}$ is real symmetric, $\mathbf{B} = \mathbf{I}$ is the identity matrix, and the system of equation (1) is stable, then the reduced model of equation (7) is also stable if the left- and right-projectors are the same real orthogonal matrix, i.e. if $\mathbf{W} = \mathbf{V}$ and $\mathbf{V}^T \mathbf{V} = \mathbf{I}$.

Proof

The reduced matrix $\mathbf{V}^T \mathbf{B} \mathbf{V}$ is the identity, so that the original and reduced eigenvalues are the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{V}^T \mathbf{A} \mathbf{V}$ respectively. Stability is equivalent to positive definiteness, i.e., to the fact that, say for $\mathbf{A}$, for any real vector $\mathbf{v}$, $\mathbf{v}^T \mathbf{A} \mathbf{v} > 0$. Since $\mathbf{A}$ is positive definite, $\mathbf{V}^T \mathbf{A} \mathbf{V}$ is also positive definite: for any vector $\mathbf{y}$, $\mathbf{y}^T (\mathbf{V}^T \mathbf{A} \mathbf{V}) \mathbf{y} = (\mathbf{V} \mathbf{y})^T \mathbf{A} (\mathbf{V} \mathbf{y}) > 0$ $\square$

Four alternatives of projection choices are now described and it is shown that one of them assures stability of the reduced models for the two applications of section (8).

7.2. Four options of projection matrices and their properties

Two important aspects have to be kept in mind when choosing the projection matrices that assure interpolation: first, the conditions of inclusion of the Krylov subspaces in the column spaces of the projector matrices, $\mathbf{V}$ and $\mathbf{W}$, are *sufficient* conditions, and, second, when the $\mathbf{A}$ and $\mathbf{B}$ matrices and associated vectors come from a structured state-space formulation of a smaller dimensional problem, as in equation (58), interpolatory model reduction [25] can lead to the same interpolation property. These two points are now further developed and four projection alternatives are proposed that will be used in the example sections.

Complex state-space option The apparently simplest choice for the inclusion of the Krylov subspaces, such as $\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}^{(p)})$ in step (4) of algorithm (2) is to consider the equalities as $\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} = \text{colsp}(\mathbf{V}^{(p)})$. This is referred to as the *complex state-space* projection choice. The term *state-space* indicates that the full matrices and vectors, such as $\mathbf{A}$, are directly projected and the term *complex* refers to the fact that the projection matrices and therefore the reduced matrices and vectors are generally complex, even if the original matrices and vectors are real.

Real state-space option A *real state-space* alternative consists in using projection matrices whose column spaces are exactly equal to the union of the real and imaginary parts of the Krylov subspaces: $\Re \left(\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} \right) \cup \Im \left(\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} \right) = \text{colsp}(\mathbf{V}^{(p)})$. With this choice, any real original matrices and vectors will be projected into real reduced matrices and vectors.

Complex interpolatory option The *complex interpolatory* option corresponds to the case where the structures of the matrices $\mathbf{A}$ and $\mathbf{B}$ are considered and interpolatory model reduction [25, theorem 1] is used. In a nutshell, interpolatory model reduction assures interpolation by projecting the original matrices of the system, such as the mass, damping and stiffness matrices, in subspaces that contain the successive derivatives of the frequency response. Considering, say, equation (58) and theorem (3), one knows that the first half of the rows of Krylov subspaces, i.e. the subspaces defined as the column space of the matrix $\mathbf{V}_{\text{int}}^{(p)} = [\mathbf{I} \ \mathbf{0}] \mathbf{V}^{(p)}$ where $\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} \subseteq \text{colsp}(\mathbf{V}^{(p)})$, contains the frequency derivatives of the system's transfer functions of orders 0 to $R_f^{(p)} - 1$. The dual fact is true for the left-projectors, i.e. with $\mathbf{W}_{\text{int}}^{(p)} = [\mathbf{I} \ \mathbf{0}] \mathbf{W}^{(p)}$. The lower dimensional subspaces extracted from the Krylov subspaces can therefore be used to condense directly the system matrices and vectors. Presenting this explicitly, again for the particular case of equation (58), in the three algorithms, one has that step (5) of the *FRADS-G* algorithm (1) can be substituted by the steps presented in algorithm (4).

Algorithm 4: Complex interpolatory alternative to step (5) of *FRADS-G* algorithm (1) for the system of equation (58).

- 5.1 define the interpolatory projection matrices, $\mathbf{V}_{\text{int}} = [\mathbf{I} \ \mathbf{0}] \mathbf{V}$, $\mathbf{W}_{\text{int}} = [\mathbf{I} \ \mathbf{0}] \mathbf{W}$;
- 5.2 reduce the original system $\hat{\mathbf{K}}_{\text{int}} = \mathbf{W}_{\text{int}}^T \mathbf{K} \mathbf{V}_{\text{int}}$, $\hat{\mathbf{C}}_{\text{int}} = \mathbf{W}_{\text{int}}^T \mathbf{C} \mathbf{V}_{\text{int}}$, $\hat{\mathbf{M}}_{\text{int}} = \mathbf{W}_{\text{int}}^T \mathbf{M} \mathbf{V}_{\text{int}}$, $\hat{\mathbf{f}}_{\text{int}} = [\mathbf{W}_{\text{int}}^T \ \mathbf{0}] \mathbf{f}$, and $\hat{\mathbf{c}}_{\text{int}} = [\mathbf{V}_{\text{int}}^T \ \mathbf{0}] \mathbf{c}$;
- 5.3 define $\hat{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{K}}_{\text{int}} & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{M}}_{\text{int}} \end{bmatrix}$, $\hat{\mathbf{B}} = \begin{bmatrix} -\hat{\mathbf{C}}_{\text{int}} & -\hat{\mathbf{M}}_{\text{int}} \\ -\hat{\mathbf{M}}_{\text{int}} & \mathbf{0} \end{bmatrix}$, $\hat{\mathbf{A}}_L = [(\hat{\mathbf{A}} - L\sqrt{2}a\hat{\mathbf{B}}) - (i\omega)\hat{\mathbf{B}}]$, $\hat{\mathbf{f}} = \begin{bmatrix} \hat{\mathbf{f}}_{\text{int}} \\ \mathbf{0} \end{bmatrix}$, and $\hat{\mathbf{c}} = \begin{bmatrix} \hat{\mathbf{c}}_{\text{int}} \\ \mathbf{0} \end{bmatrix}$.

The step (6) of the *FRADS-G* algorithm is then unchanged but uses the alternative definition of the reduced matrices and vectors. *In theory*, exactly the same function $\hat{g}_R^{(g)}(\omega)$ is evaluated, independently of which alternative is used in step (5). In practice, the various evaluations however exhibit different numerical properties as will be seen in section (8), notably in figure (4).

As discussed before, the actual particular choice of the matrices $\mathbf{V}$ and $\mathbf{W}$ is relatively arbitrary as long as the inclusion conditions are satisfied. The equalities as $\bigcup_{r=1}^{R_f^{(p)}} \mathcal{K}_r \left\{ \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{B}, \left(\mathbf{A}_L^{(p)} \right)^{-1} \mathbf{f} \right\} = \text{colsp}(\mathbf{V}^{(p)})$ are also considered in the applications of section (8).

The *Complex interpolatory* alternative can also be used in the *FRAX-G* and *FRAX-G* algorithms (2) and (2), as presented in algorithm (5)

Algorithm 5: Complex interpolatory alternative to step (7) of *FRAX-G* algorithm (2) or step (9) of *TRAX-G* algorithm (3) for the system of equation (58).

- 7/9.1 define the interpolatory projection matrices, $\mathbf{V}_{\text{int}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{V}$, $\mathbf{W}_{\text{int}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{W}$;
 7/9.2 reduce the original system $\tilde{\mathbf{K}}_{\text{int}} = \mathbf{W}_{\text{int}}^T \mathbf{K} \mathbf{V}_{\text{int}}$, $\tilde{\mathbf{C}}_{\text{int}} = \mathbf{W}_{\text{int}}^T \mathbf{C} \mathbf{V}_{\text{int}}$, $\tilde{\mathbf{M}}_{\text{int}} = \mathbf{W}_{\text{int}}^T \mathbf{M} \mathbf{V}_{\text{int}}$,
 $\tilde{\mathbf{f}}_{\text{int}} = \begin{bmatrix} \mathbf{W}_{\text{int}}^T & \mathbf{0} \end{bmatrix} \mathbf{f}$, and $\tilde{\mathbf{c}}_{\text{int}} = \begin{bmatrix} \mathbf{V}_{\text{int}}^T & \mathbf{0} \end{bmatrix} \mathbf{c}$;
 7/9.3 define $\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{K}}_{\text{int}} & \mathbf{0} \\ \mathbf{0} & -\tilde{\mathbf{M}}_{\text{int}} \end{bmatrix}$, $\tilde{\mathbf{B}} = \begin{bmatrix} -\tilde{\mathbf{C}}_{\text{int}} & -\tilde{\mathbf{M}}_{\text{int}} \\ -\tilde{\mathbf{M}}_{\text{int}} & \mathbf{0} \end{bmatrix}$, $\tilde{\mathbf{f}} = \begin{bmatrix} \tilde{\mathbf{f}}_{\text{int}} \\ \mathbf{0} \end{bmatrix}$, and $\tilde{\mathbf{c}} = \begin{bmatrix} \tilde{\mathbf{c}}_{\text{int}} \\ \mathbf{0} \end{bmatrix}$.

Now, while the value of the estimated function $\tilde{g}_R^{(g)}(\omega)$ is, in theory, identical at the interpolation points, independently of the projection method chosen, its value *between* the interpolation points and therefore the time response $\tilde{g}_R^{(g)}(t)$, may both differ significantly depending on which choice is made.

Real interpolatory option Similar to the real state-space option, a *real interpolatory* alternative consists in using projection matrices whose column spaces are exactly equal to the union of the real and imaginary parts of the column space of the projection matrices of the complex interpolatory option. In practice, this corresponds to working with the the real projection matrices in the first sub-steps of algorithms (4) and (5), as detailed in algorithm (6). Note also that one needs to make sure that the left- and right-projection matrices have the same dimension.

Algorithm 6: Real interpolatory alternative to the first substeps of algorithms (4) and (5).

- 5/7/9.1 define the interpolatory matrices $\mathbf{V}_{\text{int}}$ and $\mathbf{W}_{\text{int}}$ such that
 $\text{colsp}(\mathbf{V}_{\text{int}}) = \text{colsp}(\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \Re\{\mathbf{V}\}) \cup \text{colsp}(\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \Im\{\mathbf{V}\})$ and
 $\text{colsp}(\mathbf{W}_{\text{int}}) = \text{colsp}(\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \Re\{\mathbf{W}\}) \cup \text{colsp}(\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \Im\{\mathbf{W}\})$

With this choice and as in the real state-space case, any real original matrices and vectors are projected into real reduced matrices and vectors. Additional properties of the original system may also be preserved such as stability, as explained in theorem (6), for the situation encountered in the two applications of section (8). The interpolation property is, of course, also retained at all interpolation points while the frequency response at other points and the impulse response will differ in general from their corresponding functions obtained with the alternative interpolation choices.

Theorem 6

Consider a viscously damped system whose dynamic stiffness matrix is $[\mathbf{K}(1 + i\omega\eta) - \omega^2\mathbf{M}]$ where $\mathbf{M}$ is real symmetric positive definite and $\mathbf{K}$ is real symmetric semi-positive definite, i.e. where $\mathbf{M} = \mathbf{M}^T$, $\mathbf{K} = \mathbf{K}^T$ and where, for any vector $\mathbf{z}$, $\mathbf{z}^T \mathbf{M} \mathbf{z} > 0$ and $\mathbf{z}^T \mathbf{K} \mathbf{z} \geq 0$. Assume further that η is a positive real number so that the system is stable, i.e. so that all eigenvalues defined by the zero determinant, $\det[\mathbf{K}(1 + i\omega\eta) - \omega^2\mathbf{M}]_{\omega=\omega_j} = 0$ have a positive imaginary part $\Im(\omega_j) > 0$.

For any full-rank real matrix $\mathbf{V}$, the reduced model defined by the projected dynamic stiffness matrix $\mathbf{V}^T \mathbf{K} \mathbf{V}(1 + i\omega\eta) - \omega^2 \mathbf{V}^T \mathbf{M} \mathbf{V}$ is then also stable, i.e. all its eigenvalues $\omega = \mu_j$ such that $\det[\mathbf{V}^T \mathbf{K} \mathbf{V}(1 + i\omega\eta) - \omega^2 \mathbf{V}^T \mathbf{M} \mathbf{V}]_{\omega=\mu_j} = 0$ have also a positive imaginary part $\Im(\mu_j) > 0$.

Proof

It is well known that for this combination of properties of the matrices, $\mathbf{K}$ and $\mathbf{M}$, the eigenvalues $\omega = \alpha_j$ of the undamped system $\mathbf{K} - \omega^2\mathbf{M}$ are real positive or zero [26]. The justification goes like this: since $\mathbf{M}$ is real symmetric, there is equivalence between its positive definiteness and the fact that all of its eigenvalues, $\beta_j > 0$, are strictly positive. It can furthermore be diagonalised by a unitary transformation, so that $\mathbf{R}^T \mathbf{B} \mathbf{R} = \text{diag}_j(\beta_j)$ where $\mathbf{R}$ is a real orthogonal matrix, i.e. $\mathbf{R}^T \mathbf{R} = \mathbf{I}$. Multiplying both sides of $\mathbf{K}$ and $\mathbf{M}$ by $\mathbf{R}^T$ and $\mathbf{R}$, one therefore has that the square

of the eigenvalues α_j are the eigenvalues of the matrix $\mathbf{K}$ divided by β_j since they need to cancel the determinant $\det(\mathbf{R}^T \mathbf{K} \mathbf{R} - \omega^2 \beta_j)$ at $\omega = \alpha_j$. Since $\mathbf{R}^T \mathbf{K} \mathbf{R}$ is itself real, symmetric, and semi-positive definite, all its eigenvalues $\alpha_j^2 \beta_j$ are real positive or zero. So are thus the eigenvalues α_j^2 of the matrix pair $\mathbf{K}$ and $\mathbf{M}$. For each of those there are two real opposite eigenvalues in terms of omega, i.e. $\omega = \alpha_j$ and $-\alpha_j$. For the actual damped system, one therefore has that the eigenvalues are defined by $\det([\mathbf{K} - \omega^2/(1 + i\omega\eta)\mathbf{M}]) = 0$ and that they happen at $\omega^2/(1 + i\omega\eta) = \alpha_j^2$ for the same real α_j .

One then has that each eigenvalue satisfies a quadratic equation in terms of ω , namely $a(1 + i\omega\eta) - \omega^2 = 0$ for some positive or zero $a = \alpha_j^2 \geq 0$. The eigenvalues thus come in pairs, $\omega_{j+,j-} = [(ian) \pm \sqrt{(-a^2\eta^2) + 4a}]/2 = [i(a\eta \pm \sqrt{a^2\eta^2 - 4a})]/2$. Since a and η are positive, the real part of the square root is smaller than $a\eta$ and the imaginary part of each of the two eigenvalues of the pair is positive (or zero if $a = \alpha_j = 0$).

The reduced matrices $\mathbf{V}^T \mathbf{K} \mathbf{V}$ and $\mathbf{V}^T \mathbf{M} \mathbf{V}$ are also real, symmetric, and respectively positive and semi-positive definite. Following the same discussion as for $\mathbf{K}$ and $\mathbf{M}$, one finds that the reduced system is stable. $\square$

Other choices and properties There are many other choices of actual projection matrices that also assure interpolation and other properties. For example, one could project the system, either in its state-space or its original form, with left- and right-projection matrices that are complex conjugate of each other. If left- and right-projectors, say $\mathbf{V}_{\text{int}}$ and $\mathbf{W}_{\text{int}}$, already exist that assure interpolation properties, the complex conjugate projectors, $\mathbf{V}$ and $\mathbf{W} = \mathbf{V}^*$, can be chosen to be such that the columns of the full-rank matrix $\mathbf{V}$ span the subspace defined by the columns of both $\mathbf{V}_{\text{int}}$ and $\mathbf{W}_{\text{int}}^*$.

An important difference between the *complex* and *real* choices presented in the current section is that the reduced models in the complex case do not necessarily generate real time responses. For having such a real response, the poles of a dynamic system indeed need to come in pairs $\omega_{j,+}$ and $\omega_{j,-}$ that have the same imaginary part and equal and opposite real parts. In this case, values of the transfer function at positive and negative frequencies are complex conjugate of each other and the impulse response is consequently real. Situations where the poles do not come in such appropriate pairs may happen when interpolation points are not chosen in such symmetry compared to the imaginary axis, i.e. when they do not come in pairs $\sigma_{j,+}$ and $\sigma_{j,-}$ where $\Re(\sigma_{j,+}) = -\Re(\sigma_{j,-})$ and $\Im(\sigma_{j,+}) = \Im(\sigma_{j,-})$. An example would be if the interpolation points are only chosen at frequencies with real positive part, as in order to particularly well approximate the system's transfer function at positive frequencies. Such systems can still be used to approximate time responses provided that sufficient care is taken in their use. For example, one may consider that the original system's transfer function at negative frequencies should be the complex conjugate of the one approximated by the reduced system's transfer function.

In practice, the *real* choices considered here are equivalent to forcing the use of complex projectors with the *appropriate* conditions given above, i.e. with interpolation points that come in pairs having opposite real parts and equal imaginary parts. A subspace spanned by the real and imaginary parts of the vectors is indeed equal to the subspace spanned by the vectors and their complex conjugate.

8. APPLICATIONS

The use of the three algorithms, i.e. the evaluation of the frequency average frequency per frequency (*FRADS-G*) or with interpolation (*FRAX-G*), and the time impulse response (*TRAX-G*), are illustrated on two complex systems. The first one is a benchmark made of a big mass with several attachments, while the second one corresponds to a model used to evaluate the response of a nuclear building to seismic excitation. The four alternatives proposed in section (7.2) for the choice of projectors are illustrated and discussed.

Figure 1. Illustration of the complex system benchmark.

Figure 2. Transfer function (a) and impulse response (b) of the complex system benchmark.

8.1. Complex system benchmark

The first model illustrated in figure (1) and referred to as the *complex system benchmark* is the same spring-mass-damper with attachments that was described in [7]. This type of systems has been used as benchmarks and case problems to study and illustrate the properties of complex systems. One of the first occurrences was the model proposed by Weaver in [27] when discussing an ensemble average alternative to other fuzzy and complex structures approaches that had been previously proposed as in [1] and [2]. As mentioned in [7], the current model can be seen as an adaptation of that in [3, section 7.1]. There is a main mass $m_1 = 2$ attached to a fixed point with a spring with stiffness $k_1 = 2$. Attached to the main mass are a further set of $n = 150$ randomly sampled spring-mass pairs, m_{j+1} and $k_{j+1} = m_{j+1}\mu_{j+1}^2$ for $j = 1, \dots, n$. The actual values, m_{j+1} and μ_{j+1} , can be found on the j -th line of the Electronic Supplementary Material files 'ESM_mn' and 'ESM_omegan' of [7]. Slight damping is also present in the form of viscous damping with magnitude $c_j = \eta k_j$ being added to each spring. The equations of motion of the system in the time domain, for a unit impulse excitation and response at the first mass, are then

$$\mathbf{M} \frac{d^2 \mathbf{z}(t)}{dt^2} + \mathbf{C} \frac{d \mathbf{z}(t)}{dt} + \mathbf{K} \mathbf{z}(t) = \mathbf{e}_1 \delta(t) \quad \text{and} \quad \underline{g}(t) = \mathbf{e}_1^T \mathbf{z}(t) \quad (56)$$

where the mass matrix $\mathbf{M} = \text{diag}_j(m_j)$ is the diagonal matrix with mass m_j in its j -th diagonal location, the stiffness matrix $\mathbf{K}$ is a symmetric matrix with non-zero coefficients only on its diagonal and first row and column, the damping matrix is proportional to that, $\mathbf{C} = \eta \mathbf{K}$, and $\mathbf{e}_1$ is the unit vector with 1 at its first coefficient. The coefficient in first row and column of $\mathbf{K}$ is the sum of all the stiffness factors $k_{1,1} = \sum_{j=1, \dots, n+1} k_j$, the rest of the first column, i.e. the vector at rows 2 to $n+1$, is $-\begin{bmatrix} k_2 & \dots & k_{n+1} \end{bmatrix}^T$, and the diagonal coefficients on the same rows 2 to $n+1$ are $k_2, \dots, k_{n+1}$. Using the second equation (8), the equations of motion in the frequency domain are

$$[\mathbf{K} + i\omega \mathbf{C} - \omega^2 \mathbf{M}] \mathbf{z}(\omega) = \mathbf{e}_1 \quad \text{and} \quad g(\omega) = \mathbf{e}_1^T \mathbf{z}(\omega). \quad (57)$$

This system can be written in the form of equation (9), for example by choosing here

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -\mathbf{C} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} \mathbf{z}(\omega) \\ i\omega \mathbf{z}(\omega) \end{bmatrix}, \mathbf{f} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{0} \end{bmatrix}. \quad (58)$$

With this particular choice, both matrices $\mathbf{A}$ and $\mathbf{B}$, are symmetric.

Figure 3. Illustration on the complex system benchmark of the relative error on $\hat{g}_R^{(g)}$ of equation (36) compared to $\hat{g}_{32}^{(g)}$ for $a = 0.1$, $R_f = R_c = 1, \dots, 15$. The real interpolatory option of the *FRADS-G* algorithm is used as detailed in algorithms (1), (4) and (6). The reference average function, $\hat{g}_{32}^{(g)}$ is presented in the inset.

The (non-averaged) transfer function and impulse response of the system are presented in figure (2). The three algorithms, *FRADS-G*, *FRAX-G*, *TRAX-G* are now successively applied and discussed.

8.2. Application of the *FRADS-G* averaging algorithm on the complex system benchmark

The frequency average *FRADS-G* algorithm (1) is first used to evaluate the average $\hat{g}_R^{(g)}(\omega)$ for a fixed averaging width, $a = 0.1$. The precision of the evaluations is presented in figure (3) for rational orders varying from $R_f = R_c = 1$ to 15. It is seen that the relative error (compared to the reference function obtained with $R_f = R_c = 16$) is small even for the coarser estimate obtained with $R_f = R_c = 1$ in which it is about 10% or less. Increasing the order of the reduced models and rational approximations leads steadily to better approximations as the absolute relative error decreases by about an order of magnitude per each rational order increase.

The rational average, $\hat{g}_R^{(g)}(\omega)$, was evaluated with the real interpolatory option in this example. In principle, all four options of the *FRADS-G* algorithm should give exactly the same evaluation of this function. While this is essentially the case in practice as illustrated in figure (4), one can see in the same figure that for just a very few discrete combinations of frequencies and rational orders, the evaluation using the three *complex* or *state-space* options are tainted with significant error. The reason for these large discrepancies can be found in the fact that the reduced models can be undamped when these options are used, as illustrated in figure (5) for $R_f = 8$ and $\omega = 3.83$ [rad./s]. One of the eigenvalues, $\hat{\lambda}_j$, of the resulting reduced model, such that $(\hat{\mathbf{A}} - \hat{\lambda}_j \hat{\mathbf{B}})\hat{\phi}_j \neq \mathbf{0}$ for some $\hat{\phi}_j \neq \mathbf{0}$, is very close in the *complex state-space* case to the interpolation frequency $\lambda = i\omega + L\sqrt{2}a$ at which the $\hat{\mathbf{A}}_L$ and $\hat{\mathbf{A}}_L$ matrices are evaluated. This leads to numerical issues when inverting $\hat{\mathbf{A}}_L$ as in equation (36) and in step (6) of algorithm (1), and consequently to the very sharp error seen in figure (4). These issues might be circumvented by adequate evaluation of equation (36) and step (6) of the algorithm. This topic is however not studied further here, notably since the resulting reduced model is assured to be stable and therefore free of these numerical issues in the *real interpolatory* case.

8.3. Application of the *FRAX-G* interpolated averaging algorithm on the complex system benchmark

The frequency average of the same transfer function is now evaluated by the *FRAX-G* algorithm, using six interpolation points at $\omega = 0.5, 1.5, \dots, 5.5$ [rad./s] whose set is called ‘Points A’. The

Figure 4. Illustration, on the complex system benchmark, of the error on $\hat{g}_R^{(g)}$ compared to $\hat{g}_{32}^{(g)}$ for $a = 0.1$, $R_f = R_c = 1, \dots, 13$, and for the four choices of projection techniques proposed in section (7.2). In each case, the plots for which discrepancy visibly occurs due to an undamped reduced system are highlighted in black and labelled by their corresponding value of R_f . The reduced models in the 'Real interpolatory' case are damped for any value of R_f .

Figure 5. Illustration, on the complex system benchmark, of the location (indicated as circles) of poles, $\lambda = \hat{\lambda}_j$, of the reduced models for the four options of projection techniques proposed in section (7.2). All reduced models have been constructed at $\omega = 3.83$ [rad./s] for $R_f = R_c = 8$. The location of the interpolation frequency $\lambda = i\omega + L\sqrt{2}a$ is also indicated by a cross.

resulting estimations are presented in figure (6) for the four choices of projection approaches and three values of the rational orders, $R_c = R_f = 1, 4, 7$, used at all six interpolation points. While they are identical at all interpolation points, the predictions obtained by the four choices vary between these points, particularly for $R_f = 1$. An increase of the rational order has again the effect of reducing the order of magnitude of the relative error, but also to reduce the difference between the approximations obtained through the four choices. The errors are visibly indistinguishable for $R_f = 7$, except for the *complex* predictions that vary significantly at frequencies below the lowest interpolation point, i.e. at $\omega < 0.5$ [rad./s].

This shows, first, that it is possible to obtain very good approximation of the frequency average on this example from responses that are evaluated at only a limited number of interpolation frequencies and, second, that it is important to assess how the model is actually reduced, in order to notably

Figure 6. Illustration on the complex system benchmark of the relative error on $\tilde{g}_R^{(g)}$ of equation (45) compared to $\hat{g}_{32}^{(g)}$ (itself defined in equation (36)) for $a = 0.1$, $R_f = R_c = 1, \dots, 15$, and for the four choices of section (7.2). The reference function, $\hat{g}_{32}^{(g)}$ is evaluated with the real interpolatory option of the *FRADS-G* algorithm.

Figure 7. Illustration of the effect of interpolation on the precision of the evaluated average of the complex system benchmark's transfer function. The relative error introduced by the interpolation is presented in (a) for the initial set of interpolation points (*Points A*) and for $R_f = R_c = 4$. The maximum absolute relative error in the frequency band is compared for the three sets of points and values of $R_f = R_c = 1, \dots, 10$ in (b). The relative interpolatory option is used in all cases.

preserve properties of the system. It would be worth further studying and trying to optimise in future work the number and location of interpolation points, as well as the dimension of the individual Krylov subspaces at each of these points. Attention is restricted to the real interpolatory projection method in the rest of the paper. The effect of the interpolation on the quality of the frequency average for a given rational order is presented in figure (7). It is seen that for the relative absolute error for $R_f = R_c = 4$ is smaller than one tenth of a percent in the whole frequency range and that this maximum relative absolute value decreases by about one order of magnitude per each rational order increase. The same is also true when the interpolation point set is somewhat modified, specifically into the alternative sets, '*Points B*' = {0.5, 1.5, 2.5, 3.5, 4.5, 1.8} [rad./s] with a shifted last point, and '*Points C*' = {0.5, 1.4, 2.3, 3.2, 4.1, 5.0} [rad./s] with a smaller distance between points.

8.4. Application of the TRAX-G time algorithm on the complex system benchmark

Based on the fact that the frequency averages of this system can be estimated with good precision by the rational interpolations, i.e. the *FRAX-G* approximations, one can expect that its time response is also precisely estimated by the *TRAX-G* algorithm. This is indeed the case as shown in figure (8) where the impulse response is presented for several values of the rational order and the same averaging width, $a = 0.1$ [rad./s], and the first nominal interpolation point set '*Points A*'. For this value of the averaging width, the critical times, T_{ε} , of equation (55) are respectively, $T_{0.001} = 37.17$, $T_{10^{-6}} = 52.57$, $T_{10^{-10}} = 67.86$, and $T_{10^{-15}} = 83.11$ [s] for $\varepsilon = 0.001, 10^{-6}, 10^{-10}, 10^{-15}$. One can

Figure 8. Estimated impulse response, $\check{g}_R^{(g)}$, of the complex system benchmark and its relative error for $a = 0.1$ and $R_f = R_c = 1, 3, 5, 8$. The ‘Points A’ set of interpolation points is used in all cases together with the real interpolatory projection choice. The relative error is defined as $\text{abs}\left[\left(\check{g}_R^{(g)}(t) - \underline{g}(t)\right) / \underline{g}(t)\right]$.

see that the precision of the predicted impulse responses is globally consistent with these critical values and the precision of the frequency average that has been presented in figure (7). One can see that the response of the system at shorter times after impulse is decently predicted, even with only $R_f = R_c = 1$, i.e. when only evaluating six response functions. As a quantitative measure of this, the time and value of the maximum absolute value of the impulse response, $\underline{g}_{\text{peak}} \approx \underline{g}(2.265[\text{s}]) \approx 0.2935$, is approximated by $\left[\check{g}_2^{(g)}\right]_{\text{peak}} \approx \check{g}_2^{(g)}(2.265[\text{s}]) \approx 0.2752$. Thus, the time of the peak response, $t_{\text{peak}} \approx 2.265$ [s], is correctly predicted while the magnitude of the peak is estimated within 6.25% of its actual value. These values are not distinguishable (their exact and approximate values are within 0.1% relative distance from each other) for $R_f = R_c = 3$, and neither are the values of the time and value of the smallest time response after impact, $t_{\text{low}} \approx 7.8000$ [s] and $\underline{g}_{\text{low}} = -0.2278$. The general trend is that increasing the rational order of the approximations improves the quality of the predicted time response at a specific time and provides the response within a given precision level up to larger times. For example, using $R_f = R_c = 5$ provides a precise response up to about $t = 50$ [s]. It is stressed again that the number, location and rational order of the interpolation points could be studied and optimised, while no real effort is made in this sense in this paper. The performance of the time response approach proposed in this paper is now assessed on another benchmark system.

8.5. Nuclear station benchmark

The second model, called here the *nuclear station benchmark* and illustrated in figure (9), originates from the Harwell-Boeing collection [28] of matrices via the stiffness and diagonal mass matrices of the BCSST26 problem that can be found in the matrix market [29]. These matrices have dimension 1922 and are both positive definite. The resulting system models the containment floor of a nuclear reactor and has been selected as it is a relatively challenging problem when analysing the reactor’s seismic response. One aspect is that the seismic analysis has been reported to necessitate at least 197 eigenvalues [28]. The evaluation of these eigenvalues furthermore requires at least about 500 solutions when using a shifted eigenproblem solver algorithm [30]. The interest here is in assessing if and how the proposed TRAX approach is able to deal with such complicated problem.

Slight viscous damping is again considered, now with a damping matrix $\mathbf{C} = \eta \mathbf{K}$ with $\eta = 10^{-8}$, and the input and output vectors are both equal to the vector whose coefficients are all one so that all degrees of freedom are equally excited and measured. The equations of motion in

Figure 9. Illustration of the nuclear station benchmark.

the time and frequency domains are thus now respectively, $\mathbf{M} \frac{d^2 \mathbf{z}(t)}{dt^2} + \mathbf{C} \frac{d\mathbf{z}(t)}{dt} + \mathbf{K} \mathbf{z}(t) = \mathbf{1} \delta(t)$, $\underline{g}(t) = \mathbf{1}^T \mathbf{z}(t)$ and $[\mathbf{K} + i\omega \mathbf{C} - \omega^2 \mathbf{M}] \mathbf{z}(\omega) = \mathbf{1}$, $g(\omega) = \mathbf{1}^T \mathbf{z}(\omega)$ for $\mathbf{1} = [1 \ \dots \ 1]^T$. The configuration similar to that of equation (58) is chosen (with appropriate matrices and vectors just defined and with $\mathbf{f} = [\mathbf{1}^T \ \mathbf{0}]^T$).

8.6. Application of the TRAX-G time algorithm on the nuclear station benchmark

The focus is on the system's time response and its approximation via *TRAX-G*. Both the frequency transfer function and the impulse response of the original system are very complicated as can be seen in figure (10). An ad-hoc list of 70 interpolation points has been chosen with higher concentration in the frequency range $[0, 10^6]$ [rad./s] and maximum value 2×10^7 [rad./s] which provides a precise enough approximation $\tilde{g}_R^{(g)}(\omega)$ of the averaged transfer function, $\hat{g}^{(g)}(\omega)$, for $a = 250$, $R_f = R_c = 5$ and the real interpolatory projection matrices, $\mathbf{W} = \mathbf{V}$. This particular choice corresponds to 350 solutions which is less the 500 solutions mentioned for the shifted eigenproblem solver [30]. Considering the same choice and therefore the same resulting reduced model, it is seen that the time response is precisely approximated up to about $t = 0.035$ [s] where differences start to be visible between the exact and estimated impulse responses. The critical times, T_ε , of equation (55) are for this value of the averaging width, $T_{0.001} = 0.0149$ [s], $T_{10^{-6}} = 0.0210$ [s], $T_{10^{-16}} = 0.0343$ [s]. This shows that, even for this complex problem, the relative errors on the impulse response, not considering the use of finite precision arithmetic, remain limited to the order of about one percent of the general response magnitude. That is about the same ratio as that for the error on the transfer function. The response of the structure to an impulse or any time combination that simulates an earthquake can therefore be analysed. As in the first benchmark application, the maximum and minimum values of the displacements after impulse are correctly approximated. Other characteristics of the response such as, for example, the maximum acceleration or the number of oscillations above a given magnitude can similarly be estimated.

Again, the choice of interpolation points could be optimised in order to reduce the number of solutions and the dimension of the reduced model. Also, if the interest is only in shorter times after impulse, one can average with a larger averaging width which may lead to an even smaller number of interpolation points and solutions. The estimations of the impulse response when using a wider averaging width are presented in figure (11) where one can see the expected trend: the impulse response is retrieved for smaller times while errors, mostly due to finite precision arithmetic, dominate at larger times.

9. CONCLUSION

The proposed theoretical approach to evaluate frequency averages and impulse responses of complex systems has been presented and shown to be applicable in practice. Three algorithms based on theory, namely *FRADS* to evaluate the average through rational approximation, *FRAX* to interpolate the same frequency average, and *TRAX* to estimate the impulse response of the system have been detailed. The focus was particularly on the Gaussian frequency average or Gaussian filter

Figure 10. Illustration of the exact and interpolated frequency average of the *nuclear station* benchmark (a) and their corresponding impulse responses (b). The absolute errors on the evaluation of the two functions are presented in (c) and (d). The inset in (b) zooms on the shorter time just after impulse.

Figure 11. Comparison of the exact impulse response of the *nuclear station* benchmark with its evaluation through the *TRAX-G* algorithm (3) for varying width (a) $a = 250$, (c) $a = 2000$, and (e) $a = 10000$. The absolute errors on the evaluation of the impulse response are presented in (b), (d), and (f).

for which case the system's time response is particularly well approximated at small times, since the averaging acts as a low time filter.

In practice, the average can be evaluated precisely at a particular frequency with just a few solutions of the system with slight additional damping. The rational approximations are evaluated in reduced models obtained through projection in rational Krylov subspaces. In particular, the interpolated average combines the subspaces from a limited set of interpolation frequencies. The impulse of the original system is then simply estimated by that of the reduced interpolated system. By doing so, there is no need for modal analysis of the original system or for time-stepping, when evaluating the time response.

Several choices of projection methods and their properties are discussed. While they all provide interpolation of the average, their values between interpolation points as well as their properties differ in general. It is explained that the stability of an original system may be preserved in the reduced model, notably if an interpolatory real projection method is chosen that reduces the mass,

stiffness, and damping matrices of the original system with the real and imaginary parts of the projection subspaces.

The theory and algorithms have been applied and discussed on two complex systems whose time responses have been shown to be challenging in the literature, namely a complicated system made of a master vibrating element with several attachments and a model of a nuclear station containment floor. The algorithms performed well on both problems and it was shown that the time response of the original systems could be retrieved from the much smaller reduced models.

Among future areas of study, the analysis of the choice of interpolation points and the order of the Krylov subspaces at these points would be particularly useful in order to optimise the dimension of the reduced models. An automatic iterative algorithm that would progressively decrease the averaging width and choose additional interpolation points in regions where the modal density or the transfer function magnitude are larger is also suggested here.

A. PROOF OF THE MODAL EXPRESSION OF THE IMPULSE RESPONSE

One knows from the second equation (2) and the modal expansion that $\underline{g}(t) = \sum_{j=1,\dots,N} \mathbf{c}_j^T \phi_j \underline{y}_j(t)$. The value of the modal impulse responses, $\underline{y}_j(t)$ are now formally derived from equation (4). One first notes that the system is homogeneous for times strictly before and strictly after impulse, i.e.

$$\frac{d\underline{y}_j(t)}{dt} = \lambda_j \underline{y}_j(t) \quad \text{for} \quad t < 0 \quad \text{and} \quad t > 0. \quad (59)$$

One has therefore that $\underline{y}_j(t) = c_- e^{\lambda_j t}$ for $t < 0$ and $\underline{y}_j(t) = c_+ e^{\lambda_j t}$ for $t > 0$ where c_- and c_+ are two yet to be determined constant scalars. Considering again equation (4) and integrating on a small time interval of length ϵ that starts just before and finishes just after $t = 0$, one finds that

$$c_+ - c_- = -\mathbf{e}_j^T (\mathbf{B}\Phi)^{-1} \mathbf{f}. \quad (60)$$

Because of the stability condition $\Re(\lambda_j) < 0$, the response would have a non-physical infinite value when t tends to minus infinity if c_- was non zero. One therefore finds $c_- = 0$, $c_+ = -\mathbf{e}_j^T (\mathbf{B}\Phi)^{-1} \mathbf{f}$, $\underline{g}(t) = 0$ for $t < 0$ and equation (5).

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