Applied Game Theory and Optimal Mechanism Design

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Declaration

I herewith declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research. It includes nothing which is the outcome of collaboration work. Some of the work (Chapter 2) has been presented in two conferences: Warsaw International Economic Meeting 2013 and EEA 2013 Congresses. This thesis has not previously been presented in identical or similar form to any other examination board.

Abstract

This thesis applies game theory to study optimal toehold bidding strategies during takeover competition, the problem of optimal design of voting rules and the design of package bidding mechanism to implement the core allocations. It documents three different research questions that are all related to auction theory.

Chapter 2 develops a two-stage takeover game to explain toehold puzzle in the context of takeover. Potential bidders are allowed to acquire target shares in the open market, subject to some limitations. This pre-bid ownership is known as a toehold. Purchasing a toehold prior to making any takeover offer looks like a profitable strategy given substantial takeover premiums. However actual toehold bidding has decreased since 1980s and now is not common. Its time-series patter is centered on either zero or a large value.

Chapter 2 develops a two-stage takeover game. In the first stage of this two-stage game, each bidder simultaneously acquires a toehold. In the second stage, bidders observe acquired toehold sizes, and process this information to update their beliefs about rival's private valuation. Then each bidder competes to win the target under a sealed-bid second-price auction.

Different from previous toehold puzzle literature focusing on toehold bidding costs in the form of target managerial entrenchment, this chapter develops a two-stage takeover game and points another possible toehold bidding cost – the opportunity loss of a profitable resale.

Chapter 2 finds that, under some conditions, there exists a partial pooling Bayesian equilibrium, in which low-value bidders optimally avoid any

toehold, while high-value bidders pool their decisions at one size. The equilibrium toehold acquisition strategies coincide with the bimodal distribution of the actual toehold purchasing behavior.

Chapter 3 studies the problem of optimal design of voting rules when each agent faces binary choice. The designer is allowed to use any type of non-transferable penalty on individuals in order to elicit agents' private valuations. And each agent's private valuation is assumed to be independently distributed.

Early work showed that the simple majority rule has good normative properties in the situation of binary choice. However, their results relay on the assumption that agents' preferences have equal intensities. Chapter 3 shows that, under reasonable assumptions, the simple majority is the best voting mechanism in terms of utilitarian efficiency, even if voters' preferences are comparable and may have varying intensities.

At equilibrium, the mechanism optimally assigns zero penalty to every voter. In other words, the designer does not extract private information from any agent in the society, because the expected penalty cost of eliciting private information to select the better alternative is too high.

Chapter 4 presents a package bidding mechanism whose subgame perfect equilibrium outcomes coincide with the core of an underlying strictly convex transferable utility game. It adopts the concept of core as a competitive standard, which enables the mechanism to avoid the well-known weaknesses of VCG mechanism.

In this mechanism, only core allocations generate subgame perfect equilibrium payoffs, because non-core allocations provide arbitrage opportunities for some players. By the strict convexity assumption, the implementation of the core is achieved in terms of expectation.

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Contents

Chapter 1

Introduction

The thesis titled "Applied Game Theory and Optimal Mechanism Design" shows the outcome of PhD research training process. It focuses on game theory, in terms of auction theory, mechanism design and cooperative game theory. It includes three separate papers, documented in Chapter 2, 3, and 4.

Chapter 2 develops a two-stage takeover game to explain toehold puzzle in the context of takeover. Prior to making any takeover offer, potential bidders are allowed to purchase target shares in the open market, subject to some limitations. This pre-bid ownership is known as a toehold. It seems that toehold bidding is a profitable strategy given substantial takeover premiums. A bidder with a toehold may benefit by being a winner that only purchases remaining shares at the substantial takeover premium, or being a loser that sells out his shareholdings at a higher price. However, empirical literature has puzzling observation. They summarize that the actual toehold acquisition behavior follows a bimodal distribution – centered either on zero or a large size.

The two-stage takeover game has the following structure. In the first stage, each bidder simultaneously acquires a toehold. The pre-bid share price is normalized to zero. At the beginning of the second stage, bidders observe acquired toehold sizes, and they process this information to update their beliefs about rival's valuation. The second stage is structured as a sealed-bid second-price auction. Two bidders compete to win the target.

Three Bayesian equilibria – perfect separating Bayesian equilibrium, unrestricted partial pooling Bayesian equilibrium and restricted partial pooling Bayesian equilibrium

– are proposed and analyzed. At perfect separating Bayesian equilibrium, bidder's toehold acquisition strategy is strictly increasing or/and decreasing. At unrestricted partial pooling Bayesian equilibrium, low-value bidders pool their toehold decisions at θ_L, while high-value bidders pool at θ_H. The restricted partial pooling Bayesian equilibrium means that the lower toehold size θ_L is restricted to be zero. That is, low-value bidders acquire zero toehold, while high-value bidders acquire θ_H.

Among these equilibria, chapter 2 finds that, under some conditions, only the restricted partial pooling Bayesian equilibrium exists. Signal jamming occurs in equilibrium. When bidders play strict toehold acquisition strategies (no signal jamming) in the first stage of the game, they perfectly reveal their private information, and the second stage is the one under complete information, resulting in their rival's aggressively bidding behavior that wipes out winner's payoff. Then bidders attempt to keep their rival in doubt about their private information by playing non-strict toehold acquisition strategies at equilibrium.

Chapter 2 indicates that bidders, under this two-stage takeover game, may face a toehold bidding cost –the opportunity loss of a profitable resale. For instance, suppose bidder 1 is a low-value player. At equilibrium, bidder 1 chooses zero toehold, and truthfully bids in the second stage. Let bidder 1 deviate by acquiring θ_H. When she faces low-value rival, her deviation has no effect on rival's bidding strategy. That is, along the equilibrium path, her rival still processes truthfully bidding strategy. If she wins the target, she pays the rest of shares at rival's valuation. If she loses the target, she resells her toehold at a relatively lower price (bidder 1's valuation) and pays the resale cost d. Under some conditions, the low-value bidder's expected toehold cost is higher than its benefit. In addition, a high-value bidder's expected toehold benefit overweighs its cost. As a result, high-values optimally prefer toehold bidding.

Chapter 3 studies the problem of optimal design of voting rules when each voter faces binary choice. This chapter introduces a voting mechanism. The designer is allowed to use any type of non-transferable penalty on individuals in order to elicit agents' private valuations. Each agent's private valuation is assumed to be independently distributed.

Early studies indicate that the simple majority rule has good normative properties. However their results rely on the assumption that agents' preference has equal intensities. This chapter shows that, under reasonable assumptions, the simple majority is the best voting mechanism in terms of utilitarian efficiency, even if voters' preferences are comparable and may have varying intensities.

The mechanism, at equilibrium, works as follows. After all agents truthfully report their valuations to the mechanism, it produces a social decision and recommends a penalty scheme, in such a way that each agent has incentive to follow. At equilibrium, the mechanism optimally assigns zero penalty to every agent. The mechanism does not need actually to know all agents' valuations, but simply selects the alternative, which is preferred by the majority in the society. It may select a sub-efficient alternative, but can achieve a higher welfare. The reason is simply because the expected penalty cost is too high.

Chapter 4 introduces a package bidding mechanism. In many auction environments, bidders are more interested in the packages of items they win. Under the package auction, any bidder is allowed to bid directly for any non-trivial subset of items being sold. It is partially important when items are complements.

This chapter adopts the concept of core as a competitive standard, which enables the package bidding mechanism to avoid the well-known weakness (such as collusion, shill bidding) of VCG mechanism.

The mechanism has three stages. In stage 1, a player i is randomly selected as the first mover, and he or she proposes a payoff vector to the grand coalition. The proposed payoff vector is interpreted as the amounts must be paid by the first mover to remaining players for which they agree to cooperate with coalitional decision or action. In stage 2, the rest of players move sequentially to accept or reject the proposed payoff vector. In the final stage, trading with rejectors occurs.

The main result is that the subgame perfect equilibrium outcomes coincide with the core of an underlying strictly convex transferable utility game. Under any subgame that starts after player i has proposed a payoff π, the π is the subgame perfect equilibrium outcome if and only if it is a core allocation (Lemma 4.4.1 and Lemma 4.4.2). By the strict convexity assumption, the implementation of the core is achieved in terms of expectation. In addition, the first mover with monopoly power receives the best payoff in the core.

1. INTRODUCTION

Chapter 2

Toehold Puzzle and Two-stage Takeover Game

Purchasing a toehold prior to making any takeover offer looks like a profitable strategy given substantial takeover premiums. However, toehold bidding has decreased since 1980s and now is not common. Its time-series pattern is summarized as bimodal distribution, on average is centered on either zero or a large size. Chapter 2 develops a two-stage takeover game, in which acquired toehold size is regarded as a signal partially revealing each player's private information. Different from previous toehold puzzle literature focusing on toehold bidding costs in form of target managerial entrenchment, this Chapter points another possible toehold bidding cost – the opportunity loss of a profitable resale. Under some conditions, there exists a restricted partial pooling Bayesian equilibrium, in which low-values prefer zero toehold while high-values pool their decisions at one size.

2.1 Introduction

2.1.1 Toehold Puzzle

Before launching any bid, the bidding firm is allowed to purchase target shares in the open market, subject to some limitations. This pre-bid ownership is known as a toehold. It seems that buying a toehold is a profitable strategy given costly takeover premiums. Within a takeover battle, the bidding firm with a toehold may benefit by being a winner that only purchases remaining shares at the full takeover premium, or by being a loser that sells out his shareholdings at a higher price. There are substantial theoretical studies support this view. Burkart (1995) and Singh (1998) demonstrates that a bidder with a toehold bids aggressively at equilibrium. Walkling (1985), Betton & Eckbo (1995), Bulow *et al.* (1999) show that a toehold increases a bidder's probability of winning in a takeover battle. Under Bulow *et al.* (1999)'s common-value model[1], toehold bidders' aggressively bidding behavior increases winner's curse for a non-toehold bidder causing him to bid conservatively. As a result, toehold bidders bid more aggressively. They find that a toehold may reduce the price paid by the winner. Moreover, they show that, given the initial bidder holding a toehold, a new entrant attempts to purchase the similar amount. This result is consistent with the empirical evidence from Betton & Eckbo (2000).

Despite theoretical studies indicating the benefits of toehold bidding, empirical literature documents puzzling observations. Bradley *et al.* (1988) find that over half the bidders in their sample did not acquire any pre-bid ownership. More specifically, 236 successful tender offers were reported, but 155 of them did not hold any toehold over the period 1963 to 1986. Betton *et al.* (2009) document more than 10,000 initial acquirers bidding for publicly traded U.S. targets during 1973 to 2002. They find that the toehold bidding declined sharply since 1980s, and now is rare. Only 3% of initial bidders purchased short-term toeholds [2]. Once toehold bidding exists, the acquired amount were very large – around 20%. So they summarize the actual toehold acquisition behavior follows a bimodal distribution. That is, on average, bidders either process non toehold bidding or purchase a large amount of toehold.

2.1.2 Costs of Toehold Bidding

Given bidders are rational, there must exist some costs of toehold bidding that prevent many bidders from choosing it as their optimal strategy in the context of takeover battle. One possible cost is due to the mandatory information disclosure laws. Since 1968s Williams Act, toehold purchases of 5% or more than is required to file 13d with the Securities and Disclosure rules. It makes toeholds too costly since bidders have to

[1] In (almost) common value settings, Klemperer (1998) points the existence of the great effect of a small asymmetry (e.g. toehold) on the outcomes of standard auctions.

[2] The short-term toehold acquisition is occurred during the six months until the announcement of initial offer. This six-month period is defined as the actual bidding strategy being formulated.

reveal their intentions early in the takeover battle. However, Bulow *et al.* (1999) find that toehold bidding was common in the early 1980s. The passage of disclosure laws in the 1970s, therefore, cannot explain this time-series pattern of the entire sample period.

Another possible cost is the stock market illiquidity. Market illiquidity makes toehold bidding too costly, because toeholds cannot be exchanged or sold easily. Bulow *et al.* (1999) and Dasgupta & Tsui (2004), however, show that the declining in toehold bidding occurs when there is a steady increasing in stock market liquidity.

The last possible cost of toehold bidding is due to target managerial entrenchment. Goldman & Qian (2005) find out there exists a toehold bidding cost when entrenched target management successfully rejects the takeover offer. In their model, the degree of target entrenchment is an exogenous variable. Given successful resistance, there is a negative correlation between target share price and the rejected bidder's toehold size. Bidders trade off expected toehold benefits (higher success probability) with expected toehold cost (decreased share price). Thus, at equilibrium, some bidders optimally play non toehold bidding strategy. However, the empirical evidence from Betton *et al.* (2009) reject such negative correlation, and they regard target entrenchment degree as an endogenous variable.

In Betton *et al.* (2009)'s two-stage takeover game, an initial bidder approaches the target to negotiate a merger in which the initial bidder achieves a termination fee if the target withdraws from the negotiated agreement. In the second stage, the initial bidder competes with a public bidder without a toehold under a sealed-bid second-price auction.

In their model, toehold costs arise endogenously in the form of costs of resistance from entrenched management. Toehold bidding directly reduces management team's expected private payoff at equilibrium, causing the target to reject merger negotiations. And it in turn dictates initial bidder's equilibrium toehold acquisition strategy. Since fiduciary requirement, the target must consider any public bid in the interim period after concluding merger negotiations but before final shareholders' approval. The fact of this "fiduciary out" waiting period (the second stage in their model) contributes to the inclusion of provisions for target termination fees in takeover agreement[1]. They focus

[1]Because of fiduciary out clause, the winner has to compete with any public bidder before shareholders' approval. Since mid-1980s, the agreement includes a termination item, in which the winner will receive a breakup fee if target withdraws due to its fiduciary out clause, see Burch (2001), Officer (2003), Boone & Mulherin (March 2007).

on a particular rejection cost – the opportunity loss of termination fee. At equilibrium, some bidders optimally choose zero toehold to avoid this rejection cost, while some bidders purchase a toehold greater than a threshold, at where the expected toehold bidding benefit offsets expected rejection cost.

2.1.3 The General Takeover Process

Recall that in Betton *et al.* (2009), pre-public stage is modeled as a merger negotiation between the target and a single bidder (the initial bidder). The question is how this pre-public takeover process is actually carried out. Empirical evidence from Boone & Mulherin (March 2007) point the existence of pre-public, private takeover process. They find that it is a highly competitive market where half of the targets are auctioned among multiple bidders, while the remainder negotiates with a single bidder.

Actual corporative takeover is a very complex process, but follows some general characteristics described by Boone & Mulherin (2007), Boone & Mulherin (March 2007), Gorbenko & Malenko (2012) and Hansen (2001). It has two stages: private and public takeover stage. The private stage starts when the firm (i.e.the Board) decides to sell itself and contacts a group of potential bidders[1]. Those potential bidders are asked to sign confidentiality/standstill agreements to assess target's nonpublic information, and agree to stop trading any target shares. After learning that information, some of the potential bidders submit several rounds of bids, and the process is similar to an English auction. Then the winning bidder and the Board affirm and sign the takeover agreement, followed by a public announcement. It should be mentioned here, Betton *et al.* (2009) document the winner from private stage of takeover process as an initial bidder. At the public takeover stage, any public bid should be considered and the approval of previously announced takeover agreement also requires shareholders' vote. [2].

2.1.4 A Brief Introduction to the Two-stage Takeover Game

This chapter develops a two-stage takeover game. It is different from the one in Betton *et al.* (2009) in terms of the following points. First, this two-stage takeover game

[1] Firm Board analyzes a range of strategic alternatives, usually proposed by an investment bank, to enhance shareholders' value or provide greater liquidity for them.

[2] The final bid is formed when 126 trading days have passed without any other new bid.

models the pre-public takeover stage as a sealed-bid second-price auction, instead of the merger negotiation in Betton *et al.* (2009). Second, there is no target managerial entrenchment. As the general takeover process described in section 2.1.3, private stage is initiated by the target board. As a result, there is no target managerial entrenchment among invited bidders in private takeover stage. Finally, the open auction in Betton *et al.* (2009) disappears in this game. It is assumed that the target is the unique seller in the game, and aims to sell itself to the bidder with highest valuation. There is no principle-agent problem. Therefore, there is no any "fiduciary out" waiting period.

In the first stage of this two-stage game, each bidder simultaneously acquires a toehold. Same as in Betton *et al.* (2009), the pre-bid share price is normalized to zero. One may expect the target pre-bid share price appears in this model and expect this price impact to be monotonic in the stakes acquired, which would provide a natural deterrence to acquire a large toehold. However, there is the possibility that the number of shares acquired by a bidder may depend on his or her private valuation. In order to remove such pricing loop, this two-stage game assumes the pre-bid share price to be zero. Thereby, it aims to study what is the cost of toehold bidding except the costs of toehold purchasing and target managerial entrenchment.

In the second stage, bidders observe acquired toehold sizes, process this information to update their beliefs about rival's valuation. Then each bidder competes to win the target under a sealed-bid second-price auction. If a bidder holding a toehold wins the target, he has to pay the remaining shares at the second-highest bid price. While, if the bidder loses the game, he sells out his toehold at that price. Meanwhile, the resale generates a fixed cost d.

Chapter 2 finds, under some conditions, there exists a restricted partial pooling Bayesian equilibrium, in which low-value bidders optimally avoid any toehold while high-value bidders pool their decisions at one size. Signal jamming [1] occurs in the equilibrium. If bidders play perfect separating strategies in the first stage, they completely reveal their private information, and the second stage of the game becomes the one under complete information, resulting in fierce competition that reduces bidders' payoffs. Therefore, at equilibrium, bidders have incentives to conceal their valuations by playing partial pooling strategies.

[1] See Ding *et al.* (2010)

This chapter finds that bidders may face a toehold bidding cost –the opportunity loss of a profitable resale. Theorem 2.3.3 indicates if a low-value deviates by playing toehold bidding strategy, he triggers high value rival's aggressively bidding behavior in the second stage. With uncertainty about high-value rival's private valuation, if he overbids, he may win the target, but has to pay the remaining shares at the price much higher than his valuation, leading to a negative payoff. In this case, his best response is to lower his bid to loss the target with certainty. As a result, he has to sell his toehold at a relatively lower price and pays the resale cost d. Thus the low-value bidder's expected toehold bidding cost is higher than its benefit. At equilibrium, low-value bidders optimally choose zero toehold.

2.1.5 Other Related Literature

Grossman & Hart (1980) point out each shareholder may attempt to free ride on the raider's improvement of the corporation. And, they suggest the takeover bid mechanism to deal with such free-rider problem. Although an important paper, it deals with a different problem. Chapter 2 aims to explain the takeover puzzle by building up a two-stage model. And, it is interested in the private takeover stage introduced by Boone & Mulherin (2007). That is, the target is auctioned among invited bidders. And the process is not publicly announced until the takeover is agreed between two parties. While Grossman & Hart (1980)focus on public takeover stage – tender offer and study the free-rider problem during public takeover process. In this two-stage takeover game, a bidder may be able to sell out his or her toehold at a higher price. It seems like the bidder "free-ride" on the improvement of the competition in the second stage of the game. However, it is costly to resell a toehold. The model assumes that there exists a fixed resale cost d for each losing bidder.

Another related strand of literature refers to resale auction. Hafalir & Krishna (2008) study the effects of post-auction resale in a model with two private-value bidders. In their basic model, the first stage is modeled as a first-price auction followed by a resale via monopoly pricing. At equilibrium, the allocations in the first-price auction are inefficient, thereby bidders have incentives to join in post-auction resale. They find that a first-price auction with resale has a unique monotonic equilibrium, and the expected revenue of a first-price with resale exceeds that of a second-price auction. Garratt & Troger (2006) build up two-period interaction. In period 1, the good is offered through a first-price or second-price auction. In period 2, the winner makes a take-it-or-leave-it offer. They assume there is a speculator with zero valuation, which is commonly known. And the speculator does not make any profits in the first-price auctions. In Gupta & Lebrun (1999)'s two step model, one item is sold at a first price auction, which is followed by a resale stage. They assume there is a second stage where resale occurs in case of inefficient, and at the end of the auction, bidders' valuations are announced. Haile (2003) considers a two-stage, symmetric model in which an auction in the first stage is followed by a resale mechanism. Since bidders have only noisy information regarding their true valuations, the winner of the auction may receive the highest signal. But he or she may not have the highest true valuation. This provides the resale opportunities. Zhoucheng Zheng (2002) identifies a sufficient and necessary condition under which the optimal allocation characterized by Myerson(1981) can be achieved when resale is allowed.

Above resale auction literature studies the effects of post-auction resale when the allocations from an auction are inefficient. In general, they consider a two-stage model in which an auction in the first stage is followed by a resale. The resale is proceeded either via monopoly pricing or a mechanism. This strand of literature is different from chapter 2's two-stage game in terms of motivation and model setting up. Chapter 2 does not aim to study the inefficiency after an auction. In particular, in this two-stage takeover game, the appearance of the second-stage is not due to the inefficient outcome from the first stage. In chapter 2, the two-stage game (an open market followed by a second-price auction) is based on Boone & Mulherin (2007)'s private takeover process.

The rest of Chapter 2 is organized as follows. The next section introduces a two-stage takeover game and related assumptions. Section 2.3 characterizes the equilibrium, and finds that the signal jamming occurs at equilibrium. Section 2.4 concludes Chapter 2. Most of the proofs are relegated to Appendix A.

2.2 Two-Stage Takeover Game

There is a single object for sale – the target firm. There are two buyers, named 1 and 2, bidding for the target. Let two bidders be risk-neutral and symmetric, and they seek to maximize their expected payoffs. Two bidders are assumed to have different purpose and managing capability on the target. As a result, their valuations on the target are independent (private-value environment).

Bidder i ($i = 1, 2$) assigns a value v_i on the target, it is the maximum amount a bidder willing to pay for the target. While another bidder j ($j = 2, 1$) perceives v_i to be uniformly, independently and identically distributed on some interval $[0, 1]$ according to the increasing distribution function F. It is assumed that F admits a continuous and differentiable density function f and has full support. The distribution F is common knowledge to all bidders.

It is assumed that two bidders have no budget and liquidity constraints. Each of them is willing and able to pay up to his or her valuation.

The two-stage takeover game has the underlying structure. In the first stage (toehold acquisition stage), each bidder simultaneously acquires a toehold, and the pre-bid share price is normalized to zero. Acquired toehold size is measured by θ_i. Following Betton *et al.* (2009), I assume that θ_i is within $[0, \frac{1}{2}]$ for $i = 1, 2$. That is, $\frac{1}{2}$ is assumed to be the upper limit of the fraction of the target that can be acquired at zero price in the open market. If a bidder owns more than 50.1% outstanding shares, he or she can enact changes (e.g. takeover decisions) at target firm. To make target firm independent with bidders, it is assumed that no bidder is able to acquire over 50% ownership.

Let bidder i's toehold acquisition strategy is defined as a function $\tau(v_i)$,a mapping from i's private valuation to her acquired toehold size, that is

$$\tau : [0, 1] \longrightarrow [0, \frac{1}{2}] \quad \text{where } i = 1, 2 \tag{2.1}$$

At the beginning of the second stage, a pair of acquired toehold sizes becomes publicly observable. This assumption is consistent with takeover regulations that require bidders no disclose their shares. According to the Bayes' rule, two bidders update their beliefs about each other's private valuation on the target, conditional on the observed (θ_1, θ_2).

The second stage (takeover bidding stage) is structured as a sealed-bid second-price auction. A bidder with highest bid wins, and pays the second-highest bid. Let ϵ denote a slight amount of bid. It is assumed that there is no feasible bid price between b_i and $(b_i + \epsilon)$, for $i = 1, 2$. Bidder i's bidding strategy is represented by a function $\beta_i\big(v_i, \theta_i\big)$, for $i = 1, 2$, so that

$$\beta_i : [0, 1] \times [0, \frac{1}{2}] \longrightarrow R_+ \tag{2.2}$$

In the second stage of the game, each bidder submits a sealed bid of b_i. If bidders tie, the winner is selected by flipping a fair coin. If $b_i > b_j$, bidder i wins the target, and purchases the rest of shares $(1-\theta_i)$ at the second-highest bid price b_j. While, when $b_i < b_j$ occurs, i loses the target and sells out his shareholdings θ_i at his bid b_i. At the same time, it is assumed that the resale of ownership generates a fixed cost d for a losing bidder. The activities in private takeover process involve opportunity costs in terms of time and money. In this two-stage takeover game, d can be understood as the opportunity cost of resale a toehold, or being a "speculator".

Given bidders' bids, bidder i's payoff is:

$$\Pi_i = \begin{cases} v_i - (1-\theta_i)b_j & \text{if } b_i > b_j \\ \theta_i b_i - d & \text{if } b_i < b_j \end{cases} \quad \text{where } \; i,j = 1,2, \text{ and } i \neq j \tag{2.3}$$

The next section analyzes bidders' toehold acquisition behavior under this two-stage takeover game. Three Bayesian equilibria – perfect separating Bayesian equilibrium, unrestricted partial pooling Bayesian equilibrium and restricted partial pooling Bayesian equilibrium – are proposed and analyzed. Backward induction is adopted in Section 2.3. Each subsection starts with a proposed equilibrium toehold acquisition strategy. Propositions show bidders' optimal bidding strategies given their updated Bayesian believe at equilibrium. Section 2.3 finds that equilibrium toehold acquisition behavior is consistent with the bimodal distribution. Moreover, there exists a restricted partial pooling Bayesian equilibrium, in which toehold size is centered either on zero or a large value.

2.3 Optimal toehold acquisition strategies

This section proposes three Bayesian equilibria: perfect separating Bayesian equilibrium, unrestricted partial pooling Bayesian equilibrium and restricted partial pooling Bayesian equilibrium. At perfect separating Bayesian equilibrium, there exists a strictly increasing or/and decreasing toehold acquisition strategy. Each bidder perfectly identifies his or her private valuation in the first stage. At unrestricted partial pooling Bayesian equilibrium, some types of the bidder acquire a relatively lower and non-zero toehold size θ_L , while some others acquire a higher toehold size θ_H. The restricted partial pooling Bayesian equilibrium means the lower toehold size θ_L is restricted to be zero. That is, low-value bidders pool at zero, while high-value ones pool at θ_H.

2.3.1 Perfect Separating Bayesian Equilibrium

This section shows there can be no perfect separating Bayesian equilibrium. Suppose there exists a strictly increasing or/and decreasing, continuous toehold acquisition strategy τ in equilibrium.

At the beginning of the second stage, each bidder observes the signal (θ_1, θ_2), which can perfectly identify bidders' private valuations on the target through

$$\tau^{-1}(\theta_i) = v_i \quad \text{where } i = 1, 2 \tag{2.4}$$

After Bayesian updating, the second-stage becomes a complete information game. That is, when bidder 2 sees the acquired toehold size of bidder 1, he will assign probability 1 to bidder 1's true valuation v_1.

Suppose bidder 1 is a high-value player. Under complete information environment and given the allocation rule of second-price auction, bidder 1 is selected as a winner. She optimally bids her true valuation v_1 plus the fixed resale cost d at equilibrium. Bidder 2, as a loser, prefers to push up his bid price to the highest. At the same time, this bid price guarantees his losing position. So that bidder 2's best response is to bid just slightly less than $(v_1 + d)$. Two bidders' equilibrium bidding strategies are shown in Proposition 2.3.1.

Proposition 2.3.1. *Given the strictly increasing or/and decreasing toehold acquisition strategy and suppose bidder 1 has a higher private valuation, $v_1 > v_2$, bidder 1 wins the target at the second-stage of the takeover game. Two bidders' optimal bidding strategies are:*

$$\begin{cases} \beta_1(v_1,\ \theta_1) = v_1 + d \\ \beta_2(v_2,\ \theta_2) = v_1 + d - \epsilon \end{cases} \tag{2.5}$$

Proof: See Appendix A.1

Since bidder 2's beliefs are Bayesian by construction, and his bidding strategy at the second-stage is a best response given those beliefs, that is equilibrium if and only if bidder 1 has no incentive to deviate from this one-to-one toehold acquisition strategy at the first stage. Bidder 1 prefers to truthfully reveal her private value as long as the payoff it yields is at least as high as the one she gets if she deviates.

Let bidder 1 deviate and report $\underline{r}_1$, where $\underline{r}_1 < v_1$, given bidder 2 along the equilibrium path. Bidder 2 believes, upon observing $\tau(\underline{r}_1)$, the true value of bidder 1 is $\underline{r}_1$ with probability 1, then he follows the bidding strategy assigned in Proposition 2.3.1. Let $\underline{E\Pi}_1$ and $\overline{E\Pi}_1$ be bidder 1's out-of-equilibrium expected payoffs by reporting a lower value $\underline{r}_1$ and a higher value $\overline{r}_1$, respectively.

Lemma 2.3.1. *Given bidder 2 along the equilibrium path, if bidder 1 reports his value as $\underline{r}_1$ ($\underline{r}_1 < v_1$), her off-equilibrium expected payoff is the following:*

$$\underline{E\Pi}_1 = \int_0^{\underline{r_1}} \Big[v_1 - \big(1 - \tau(\underline{r_1})\big)(\underline{r_1} + d - \epsilon)\Big] dv_2 + \int_{\underline{r_1}}^{v_1} \Big[v_1 - \Big(1 - \tau(\underline{r_1})\Big)(v_2 + d)\Big] dv_2 + \int_{v_1}^{1} \Big[\tau(\underline{r_1})(v_2 + d - \epsilon) - d\Big] dv_2 \tag{2.6}$$

Similarly, when bidder 1 reports $\overline{r}_1$, where $\overline{r}_1 > v_1$, her off–equilibrium expected payoff is:

$$\overline{E\Pi}_1 = \int_0^{\overline{r}_1} \Big[\tau(\overline{r}_1)(\overline{r}_1 + d - 2\epsilon) - d\Big] dv_2 + \int_{\overline{r}_1}^{1} \Big[\tau(\overline{r}_1)(v_2 + d - \epsilon) - d\Big] dv_2 \tag{2.7}$$

Proof: See Appendix A.2

Theorem 2.3.1 shows that deviation yields a higher payoff for bidder 1, given another bidder along the equilibrium. The intuition is straightforward. Suppose toehold acquisition strategy is strictly increasing and $v_1 < v_2$, bidder 1 has incentive to report $\overline{r}_1$ ($\overline{r}_1 > v_2$), instead of her true value v_1. Given bidder 2's belief along the equilibrium, he regards $\overline{r}_1$ as 1's true value. So that his best response is to bid $(\overline{r}_1 + d - \epsilon)$ at the second-stage. Therefore, as a losing bidder, bidder 1 achieves a higher selling price for her shareholdings than the one yielded at equilibrium. This argument can be extended to show that there always exist some values of bidders optimally prefer to deviate from the strictly increasing or/and decreasing toehold acquisition strategy.

Theorem 2.3.1. *Under the two-stage takeover game and if all assumptions are satisfied, there is no perfect separating Bayesian equilibrium.*

Proof: See Appendix A.3

If a perfect separating Bayesian equilibrium exists, each bidder with different value chooses a different toehold size. Theorem 2.3.1 indicates toehold acquisition behavior is not continuously distributed and the signal pair (θ_1, θ_2) cannot completely reveal each bidder's private valuation.

2.3.2 Unrestricted Partial Pooling Bayesian Nash Equilibrium

This section starts to analyze the existence of an unrestricted partial pooling Bayesian Equilibrium. Suppose there are only two non-zero toehold sizes available for two bidders to choose at the first stage. Let θ_L be the relatively lower toehold size, and let θ_H be the relatively higher one. That is

$$0 < \theta_L < \theta_H \leq \frac{1}{2} \tag{2.8}$$

Let $\hat{v}$ within the range $(0, 1)$ be a valuation threshold. It classifies players into two groups: low value bidders with valuations less than $\hat{v}$ and high value bidders with valuations more than $\hat{v}$.

At equilibrium, high-values prefer θ_H, while low-values choose θ_L. If a bidder's valuation is $\hat{v}$, he acquires either θ_L or θ_H. He is indifferent by choosing these two toehold sizes in terms of expected payoff. For any i, Bidder i's toehold acquisition strategy is expressed as:

$$\begin{cases} \tau(v_i) = \theta_L & \text{if } v_i \in [0, \hat{v}) \\ \tau(v_i) = \theta_L \text{ or } \theta_H & \text{if } v_i = \hat{v} \\ \tau(v_i) = \theta_H & \text{if } v_i \in (\hat{v}, 1] \end{cases} \tag{2.9}$$

At the beginning of the second stage, two bidders may observe three possible pairs of toehold size. In one situation, both of them acquire a smaller size θ_L. Another opposite possibility is they all acquire a larger toehold size θ_H. The last situation is one of them chooses θ_H and another bidder selects θ_L instead[1]. As a result of the toehold acquisition strategy in (2.9), signal (θ_1, θ_2) only partially (imperfectly) reveals each bidder's private value– identifies the group of each bidder.

In the second stage, given the observed signal, each bidder optimally chooses bidding strategy by maximizing his or her expected payoff. Proposition 2.3.2 shows two bidders' optimal bidding strategies associated with three possible pairs of toehold size. A guessed linear bidding strategy is imposed, by inversing and substituting, to form a transformed expected payoff function. Bidding strategies (2.10) and (2.11) are achieved by maximizing the transformed expected payoff function with respect to bidder's bid price. In addition, under (θ_L, θ_H) situation, bidder 1 (2) is identified as a low value

[1] It is assumed two bidders are symmetric leading to two symmetric signals $(\theta_H, \theta_L)$and (θ_L, θ_H). This section discuses one of them to avoid reduplicative analysis.

(high value) player. So that bidder 2 wins the target and purchases the rest of shares at 1's bid price. Bidder 1 has incentive to increases her selling price as close as 2's bid. By knowing that, bidder 2's best response is to pick up the lowest bid ($\hat{v} + d$), which can still guarantee his winning position.

Proposition 2.3.2. *At the second-stage of this takeover game, given bidder's toehold acquisition strategy in (2.9) and a pair of observed toehold size* (θ_L, θ_L), *bidder i's optimal bidding strategy is*

$$\beta(v_i, \theta_L) = \frac{v_i}{1+\theta_L} + \frac{\theta_L \hat{v}}{1+\theta_L} + d \qquad \textit{where } i = 1, 2 \tag{2.10}$$

And if they observe (θ_H, θ_H), *bidder i's optimal bidding strategy is:*

$$\beta(v_i, \theta_H) = \frac{v_i}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} + d \qquad \textit{where } i = 1, 2 \tag{2.11}$$

If they observe (θ_L, θ_H), *bidder 2 wins the target and two bidders' optimal bidding strategies are the following:*

$$\begin{cases} \beta_1(v_1, \theta_L) = \hat{v} + d - \epsilon \\ \beta_2(v_2, \theta_H) = \hat{v} + d \end{cases} \tag{2.12}$$

Proof: See Appendix A.4

Given bidder 2's Bayesian constructed beliefs and his optimal bidding strategies based on those beliefs, that is an equilibrium if and only if neither low-value nor high-value of bidder 1 has incentive to deviate from the toehold acquisition strategy described in (2.9). At equilibrium, the valuation threshold $\hat{v}$ classifies bidders into two groups. At the same time, two groups of bidders optimally prefer to stick on the toehold acquisition choices assigned by (2.9). However, Theorem 2.3.2 shows that a bidder with zero valuation always has incentive to deviate by acquiring a higher toehold size θ_H.

Theorem 2.3.2. *Under the two-stage takeover game and if all assumptions are satisfied, there is no such unrestricted partial pooling Bayesian equilibrium, in which low value bidders always select* θ_L *while high value bidders always choose* θ_H.

Proof: See Appendix A.5

The intuition behind Theorem 2.3.2 is the following. Suppose two bidders are low-value players. Given bidder 2 along the equilibrium path, bidder 1 has incentive to pretend to be high-value bidder by purchasing θ_H. By observing $\theta_1 = \theta_H$, along the equilibrium, bidder 2 will overbid – $(\hat{v}+d-\epsilon)$. Then bidder 1 profitably prefers to lose the game by selling her toehold at such high price – $(\hat{v}+d-\epsilon)$, even there exists the resale cost d.

2.3.3 Restricted Partial Pooling Bayesian Equilibrium

This section proposes another partial pooling Bayesian equilibrium. Throughout this chapter, to distinguish the previous one, it is named as restricted partial pooling Bayesian equilibrium. Unlike the unrestricted partial pooling Bayesian equilibrium discussed in section 2.3.2, this equilibrium restricts θ_L to be zero. At restricted partial pooling Bayesian equilibrium, each bidder is able to choose either zero toehold or θ_H. And, a bidder with valuation $\hat{v}$, he or she is indifferent between zero toehold and θ_H. At equilibrium, low-values prefer zero toehold, while high-values choose θ_H. Bidder i's equilibrium toehold acquisition strategy is the following:

$$\begin{cases} \tau(v_i) = 0 & \text{if } v_i \in [0, \hat{v}) \\ \tau(v_i) = 0 \text{ or } \theta_H & \text{if } v_i = \hat{v} \\ \tau(v_i) = \theta_H & \text{if } v_i \in (\hat{v}, 1] \end{cases} \qquad \text{where } i = 1, 2, \text{ and } \theta_H \in (0, \frac{1}{2}] \tag{2.13}$$

At the beginning of the second-stage, each bidder may observe three possible pairs of toehold size. On possibility is none of them acquire any toehold at the first stage. As a result, in the second stage of the game, it is well-known that truthful bidding is a weakly dominant strategy[1]. When two bidders observe (θ_H, θ_H), their optimal bidding strategies are shown by (2.11) in Proposition 2.3.2. Another possible situation is $(0, \theta_H)$. After Bayesian updating, bidder 1 (2) is identified as a low value (high value) player. So that bidder 1 (2) loses (wins) the target at the second stage. For bidder 1 without any toehold, truthful bidding is still her weakly dominant strategy. Given 1's strategy, bidder 2 optimally submit a bid higher than $\hat{v}$.

Proposition 2.3.3. *At the second-stage of the takeover game, given bidder's toehold acquisition strategy in (2.13) and the pair of observed toehold size* $(0, \theta_H)$*, two bidders' optimal bidding strategies are*

$$\begin{cases} \beta_1(v_1) = v_1 \\ \beta_2(v_2, \theta_H) = b_2 \qquad \textit{where } b_2 > \hat{v} \end{cases} \tag{2.14}$$

[1]See Proposition 2.1 in Krishna (2009)

Proof: See Appendix A.6

Similarly as previous equilibrium analysis, the restricted toehold acquisition strategy (2.13) is stable if and only if both low value and high value bidders have no incentive to deviate from it. Theorem 2.3.3 finds that if the resale cost d and valuation threshold $\hat{v}$ satisfies the underlying conditions, then the restricted partial pooling Bayesian equilibrium exists.

Theorem 2.3.3. *Under this two-stage takeover game, if the fixed resale cost d is no more than $\frac{\theta_H^2}{1-\theta_H^2}$ and the valuation threshold $\hat{v}$ is within $(0, \frac{d+\theta_H-\sqrt{2d^2+\theta_H^2}}{\theta_H}]$, there exist a restricted partial pooling Bayesian equilibrium, in which low value bidders acquire nil toehold while high value bidders pool their toehold acquisition decisions at θ_H.*

Proof: See Appendix A.7

Signal jamming occurs in equilibrium. If bidders play strict toehold acquisition strategies in the first stage of the game, they perfectly reveal their private information, and the second stage is the one under complete information, resulting in their rival's aggressively bidding behavior that wipes out winner's payoff. In the second-stage of the game, under complete information environment, a losing bidder attempts to push up his or her bid as close as the winner's bid, to ensure he or she achieves the highest resale price. The winner has to pay the highest price for the remaining shares of the target. Thus bidders attempt to keep their rival in doubt about their valuations by using non-strict toehold acquisition strategies at equilibrium.

This section points a possible toehold bidding cost –the opportunity loss of a profitable resale. Instead of finding all possible equilibria, this chapter interests in a pooling equilibrium, which is consistent with the actual toehold purchasing behavior (bimodal distribution). By analyzing equilibrium toehold acquisition strategies, we are able to find out the cost of toehold bidding.

The existence of the restricted partial pooling Bayesian equilibrium (Theorem 2.3.3) shows that, for low-value bidders, deviation by acquiring a toehold θ_H cannot generate a relatively high toehold resale price. Let bidder 1 be the low-value player. At equilibrium, given bidder 2 is low-value, two bidders select zero toehold $(0, 0)$, and they will truthfully bid at the second stage. Suppose bidder 1 deviate by acquiring θ_H. Along the equilibrium path, bidder 2's best response is still v_2. Recall the analysis on unrestricted partial pooling Bayesian equilibrium, bidder 1's deviation leads bidder 2 to overbid, and then bidder 1 resells her toehold at this overbid price. However, in this case, bidder 1's toehold decision has no effect on bidder 2's bidding strategy. If bidder 1 wins the game, she only has to pay the rest of shares at 2's valuation. Her payoff becomes $v_1 - (1 - \theta_H)v_2$. While, when bidder 1 loses the game, she resells her toehold at her private valuation, v_1, and has to pay the resale cost d. Her payoff in this case is $\theta_H v_1 - d$. Let bidder 2 be the high-value, instead. To lose the game, bidder 1's best response is to bid a relatively lower price $\hat{v}$ and generate the resale cost d. Bidder 1's payoff becomes $\theta_H \hat{v} - d$. Theorem 2.3.3 indicates that, under some conditions, if a low-value deviates, his or her expected toehold benefit (only need to pay remaining shares at the full takeover premium) is lower than the expected toehold cost (unprofitable resale the toehold).

Bidders trade off expected toehold bidding benefit with expected toehold bidding cost. At equilibrium, a bidder with zero valuation obviously acquires zero toehold. This does not imply that non toehold bidding can only comes from a bidder with zero valuation. In this restricted partial pooling Bayesian equilibrium, a bidder with valuation $v < \hat{v}$ may also acquire zero toehold, because his toehold bidding cost is higher than its benefit.

When the expected benefit of toehold bidding overweighs its expected cost, a bidder (e.g. bidder 1) with valuation $v_1 > \hat{v}$ prefers to acquire a toehold at equilibrium. Given bidder 2 is a high-value player, by acquiring a toehold, bidder 1 triggers 2's aggressively bidding behavior with certainty. When bidder 1 wins the game, and only has to pay the remaining shares, $(1 - \theta_H)$, at full takeover premium (i.e. 2's bid). While, if bidder 1 loses the the target, she benefits from the resale of her shareholdings at such high price (2's over bidding price), although it generates resale fixed cost d. Given bidder 2 is a low-value player, at equilibrium, truthfully bidding is 2's weakly dominant strategy in the second stage. In this case, bidder 1 wins the target and pays the rest of shares at 2's private valuation. Overall, bidder 1 benefits from toehold bidding in terms of expectation. The existence of the restricted partial pooling Bayesian equilibrium indicates that the opportunity loss of a profitable resale can be regarded as another possible cost of toehold bidding to explain the toehold puzzle.

Theorem 2.3.3 assumes there is only one non-zero toehold size (the largest) available for bidders to choose. To robust theorem 2.3.3, we assume there exist another non-zero toehold size, θ_s, which is relatively smaller than θ_H.

Let θ_s be another feasible and positive toehold size θ_s, where $0 < \theta_s < \theta_H$. Before the second stage, if bidders observe rival's θ_s, they process this information using Bayes' rule to update their beliefs concerning their rival's valuation. That is their rival's valuations are uniformly, independently and identically distribution on the interval $(\hat{v}, 1]$ according to the increasing distribution function F. And, it is assumed that its density function f is continuous and differentiable.

At the beginning of the second stage, there are three more pairs of toehold sizes might occur: (θ_s, θ_s), $(0, \theta_s)$ and (θ_s, θ_H). When signal (θ_s, θ_s) is observed, two bidders' optimal bidding strategies can be achieved by repeating the proof of (2.11) in Proposition 2.3.2 and replacing θ_H by θ_s. Thus their optimal bidding strategies can be expressed as:

$$\beta(v_i, \theta_s) = \frac{v_i}{1 + \theta_s} + \frac{\theta_s}{1 + \theta_s} + d \qquad \text{where } i = 1, 2 \tag{2.15}$$

When signal $(0, \theta_s)$ is observed, bidders' optimal bidding strategies are same as in Proposition 2.3.3. The last possibility is (θ_s, θ_H), by repeating the proof of (2.11) in Proposition 2.3.2 and using θ_s instead of θ_H, we have bidder 1's optimal bidding strategy. That is

$$\beta_1(v_1, \theta_s) = \frac{v_1}{1 + \theta_s} + \frac{\theta_s}{1 + \theta_s} + d \tag{2.16}$$

Meanwhile, bidder 2's optimal bidding strategy is the same as (2.11) in Proposition 2.3.2. So that

$$\beta_2(v_2, \theta_H) = \frac{v_2}{1 + \theta_H} + \frac{\theta_H}{1 + \theta_H} + d \tag{2.17}$$

Theorem 2.3.4. *Suppose there exists a smaller positive toehold size θ_s than θ_H. And when θ_s is observed, the bidder believes his or her rival's private valuation is an independent uniform distribution F within the support $(\hat{v}, 1]$. Two bidders have no incentives to deviate from the restricted partial pooling Bayesian equilibrium.*

Proof: See Appendix A.8

Theorem 2.3.4 shows that low-values still optimally prefer non-toehold bidding while high-values have no incentives to deviate a smaller toehold size θ_s. Low value bidders choose non toehold bidding strategies at equilibrium, since their expected toehold bidding cost overweighs the expected toehold benefit. To enlarge the benefits of toehold bidding and to maximize their expected payoffs, high-values optimally pool their toehold acquisition decisions at the largest size θ_H.

The intuition is straightforward. Suppose bidder 1 is a high-value bidder. Given his or her rival is a high-value bidder, by acquiring a toehold at the first stage, bidder 1 triggers rival's aggressive bidding response in the second stage. One possibility is that bidder 1 loses the target. By holding a toehold θ_H, bidder 1 is able to profitably sell out the largest amount of toehold. Another possibility is that bidder 1 wins the target. Then he only has to buy the smallest amount of shares at the full takeover premium.

2.4 Conclusion

It seems profitable to purchase a toehold prior to making any takeover bid given substantial control premium in corporate takeover. A bidder with a toehold can gain as a winner that only pays the rest of shares at the substantial takeover premium, or as a loser that sells out his shareholdings at such high price. Although some theoretical studies support this argument, empirical literature documents puzzling observations. They find that acquired toehold sizes are on average centered either on zero or a large size – it is a bimodal distribution. Some literature argues that bidder toehold benefits are not withstanding. Betton *et al.* (2009)'s two-stage takeover model finds some bidders optimally choose non toehold bidding due to rejection cost – the opportunity loss of a target termination agreement. They model actual takeover process as two stages: private merger negotiation followed by a sealed-bid second-price auction. Private merger negotiation happens between the target board and a single bidder, and the board decides to accept or reject the proposed offer. The open auction is modeled as a sealed-bid second-price auction. It takes account the fiduciary requirement, and most importantly the initial bidder with merger agreement can achieve the target termination fees if a public offer is finally approved by shareholders. At their equilibrium, toehold bidding reduces target board private expected profits, causing a rejection of negotiation. It in turn results in the toehold costs. The target decision, hence dictates an equilibrium toehold acquisition strategy for the initial bidder in their paper.

This chapter develops a two-stage takeover game, which is different from Betton *et al.* (2009)'s in terms of two major points. First, this takeover game models private takeover stage as an auction, instead of a merger negotiation, since half of the targets are auctioned among multiple bidders. Second, there is no managerial entrenchment at the first stage of this game. The general takeover process usually starts when the target board contacts a group of potential bidders. The private takeover stage is initiated by the target board. In other words, there is no threat of target managerial resistance among invited potential bidders. Hence this game assumes there is an unique seller – the target. There are no different interests between target board and target shareholders (principle-agent problem). The target aims to sell itself to a bidder with highest valuation. And therefore the open auction in Betton *et al.* (2009) disappears in this game. These two differences help to extend the toehold puzzle story, and answer the question what is toehold bidding cost when there is no target managerial resistance.

Chapter 2's two-stage takeover game includes the toehold acquisition stage and takeover bidding auction. In the first stage, each bidder is allowed to freely acquire a toehold in the target. The observable toehold sizes are regarded as signals partially revealing each bidder's private information. The second stage is modeled as a sealed-bid second-price auction. Two bidders compete to buy the target, and the winner pays the loser's bid.

It is shown that, under some conditions, there exists a restricted partial pooling Bayesian equilibrium (Theorem 2.3.3), in which low-value bidders optimally choose zero toehold while high-value bidders pool their decisions to one size. The equilibrium toehold acquisition strategy is consistent with bimodal distribution of actual toehold purchasing. Signal jamming occurs at the equilibrium. That is, bidders optimally play partial pooling toehold acquisition strategy in the first stage to conceal their private information. When bidders play perfect separating toehold acquisition strategy, they completely reveal their private information, and the following auction becomes the one under complete information, resulting fierce bidding competition (Proposition 2.3.1), that reduces the bidder's payoff.

This chapter points a possible toehold bidding cost – the opportunity loss of a profitable resale. At equilibrium, a low-value bidder, whose valuation is less than a threshold $\hat{v}$, chooses non toehold bidding, since his or her expected toehold bidding cost overweighs the expected toehold benefit. On the other hand, a high-value bidder with valuation more than $\hat{v}$ acquires a toehold. Since his toehold bidding benefit exceeds the cost of toehold bidding in terms of expectation, he prefers to marks himself as high-value bidder. To maximize the benefit brought from toehold bidding, a bidder optimally reveals his valuation group (high-value group) by acquiring the highest toehold size.

Chapter 3

Majority Rule and Mechanism Design

This chapter studies the problem of optimal design of voting rules when each agent faces binary choice. The designer is allowed to use any type of non-transferable penalty on individuals in order to elicit agents' private valuations. And each agent's private valuation is independently distributed. The main result is that under reasonable assumptions, the society can do no better in terms of utilitarian efficiency, than to follow a simple majority rule.

3.1 Introduction

Arrow's impossibility theorem attracted a vast literature on majority rule. The story can trace back to Arrow (1951) who shows the impossibility of formulating a social welfare function thoroughly satisfying desired general democratic. Arrow tried to build up a consistent, fair voting system that would lead to transitive social preferences over more than two outcomes. But he proved that this was impossible. Arrow expressed a consistent and fair voting system in terms of transitivity, independence of irrelevant alternatives, unanimity and no-dictators. Arrow's impossibility theorem says that it is impossible to create a voting system that satisfies these four conditions when choosing among more than two outcomes. And the theorem states that, demanding the transitive social preferences, the first three conditions imply a dictatorship. While in game-theoretic terms, the equilibrium (known as the core) under majority rule is unlikely to exist. More specifically, the existence of cyclical majorities in a multi-person voting situation often implies that any proposed resolution to the underlying choice problem can be blocked by some coalition of voters, and hence the core of the voting game is empty.

A large part of literature studies the conditions that ensure the existence of majority rule equilibrium. Early work by Plott (1967) shows that under very restricted conditions, such as the preferences can be represented by smooth utility functions and the choice space is Euclidean, there exists a majority rule equilibrium. Similar results can be found from Rubinstein (1979) and Greenberg (1979). Rubinstein (1979) shows that the set of continuous preference profile has a non-empty core in the Kannai topology. Greenberg (1979) shows that when the alternative set is a convex and compact subset of Euclidean space with low dimension, for convex and continuous preference, there exists a majority rule equilibrium. May (1952) and Rae (1969) indicates that the simple majority has good normative properties in the situation of binary choice. However, their results relay on the assumption that agents' preferences have equal intensities. Chakravarty & Kaplan (2010) compare the surplus between two mechanisms: majority voting and a contest (who shouts the loudest chooses the outcome). They assume that agents have private valuations over the two alternatives, and it is costly to the voter when shouting acts as a signal. They find that if the number of voters is large and the value of each voter is bounded, the majority voting is optimal. And for any n, the superior mechanism is depends on the order statistics of the distribution of values. Kleiner & Drexl (2013) solve for the social choice function maximizing utilitarian welfare. They assume that agents have private valuations following distribution function F and have quasi-linear utilities. In their model, monetary transfers are feasible. Their main result is that if F has monotone hazard rates, the optimal social choice function is implementable by qualified majority voting[1] and it is optimal to exclude monetary transfers. Casella (2005) proposes a simple voting mechanism for players to meet repeatedly over time. It is assumed that players can store their votes and shift them intertemporally. As a result, the players cast more votes when preferences are more intense. It is found that the voting mechanism does not achieve full efficiency, but it can lead to a higher ex ante welfare. This chapter shows that the simple majority is

[1] Qualified majority voting means any decision rule that requires more than a simple majority of the votes to ratify a decision.

the best voting mechanism in terms of utilitarian efficiency, even if voters' preferences are comparable and may have varying intensities.

To understand it, consider the following example. There are two alternatives A and B, and three agents in the society. Agent 1 prefers A over B, and he values £10 on alternative A. While both agent 2 and 3 prefers B with £2 and £1 valuations on B, respectively. A should be selected, in terms of utilitarian efficiency, but B holds more votes. Since the details of environment are not specified, it is not clear what choice should be used. This chapter assumes that the strengths of voters' preferences are independently and identically distributed and privately known. In such an environment, the designer faces two voting mechanisms. The first voting mechanism generates a costly stand-off (e.g. individual cost of time, expert advice). It is possible to set up the penalty scheme in such a way that the revelation of private information occurs. Therefore the mechanism may select the better alternative more often. Since incentive compatibility constraints are imposed, the voters would have to pay in terms of this penalty. The second voting mechanism achieves the agreement early, but does not extract private information from the voters. And therefore, it may not achieve the first best. The first mechanism generates the expected penalty cost of eliciting private information in order to select the better alternative. It turns out that – if private information is independently and identically distributed from one of the common distributions – the second mechanism is efficient. The configurations of preferences similar to the above example can occur, and simple majority would indeed select sub-efficient B, because the expected efficiency cost of eliciting information to select the better alternative – A is too high.

In this chapter, a voting mechanism is introduced. Each agent's private preference is assumed to be independently distributed, in the situation of binary choice. The number of agents in each group, in which all agents prefer the same alternative, is known by the designer. After the agents have reported their valuations to the mechanism, the designer decides a collective decision over two alternatives and the penalty scheme to maximize ex-ante social expected payoff. The designer is allowed to use any type of non-transferable penalty on individuals, in order to elicit agents' private valuations. It finds that simple majority rule is the best in terms of utilitarian efficiency. At equilibrium, the mechanism optimally assigns zero penalty to every agent. In other words, the designer does not extract private information from any agent in the society, because the expected penalty cost of eliciting private information to select the better alternative is too high.

The mathematical technique used here builds on the algorithms in McAfee & McMillan (1992). Their analysis explains two commonly observed forms of cartel organizations during bidding process. One of them is defined as weak cartel, in which members are not allowed to make any transfer payments among themselves. According to bidders' reports, the mechanism assigns each bidder's bid and decides the non-transferable payments to maximize total expected profits. Meanwhile, the seller's action is passive. That is, the seller announces a reserve price and sells at the highest bid. To solve the maximization problem, they form a bound on profits, and then find out the implementation of the bound. They show that in weak cartels all bidders propose exactly the same bid when their private valuations are more than announced reserve price. The intuition behind it is, in the absence of transfers among bidders, incentive compatibility constraints require the item is awarded with equal probability to the bidders whose valuations are larger than the minimum price.

The organization of the Chapter 3 is as follows. Section 3.2 introduces the voting mechanism and related assumptions. Section 3.3 shows the main results. Section 4 summarizes this chapter. Most of the proofs are relegated to Appendix B.

3.2 The Voting Mechanism

Let $\Omega = \{A, B\}$ be a set of alternatives. There are n risk-neutral agents in the society, numbered $1, 2, ..., n$. Let N be this society, so that

$$N = \{1, ..., n\} \tag{3.1}$$

The society N is defined as a set of agents that are willing to play this voting game. That is, each agent within N chooses one of the alternatives, and assigns private value on his or her preferred alternative. Throughout Chapter 3, i and j represents typical agent in N.

Within the society N, all agents are categorized into two groups: N_A and N_B. Group N_A contains all agents preferring alternative A to B, and the number of agents within this group is denoted as $|N_A|$. Group N_B contains all agents preferring B to A, and the number of agents within this group is denoted as $|N_B|$. where $N_B = N \setminus N_A$. Let agent i and j present the agent in group N_A and N_B, respectively.

Chapter 3 models asymmetric information in the following structure. It is assumed that the number of agents in each group is known by the designer. For any $m \in M$, agent m assigns his or her valuation v_m on his or her preferred alternative, while all other agents perceive v_m to be independently distributed on some interval $[0, \bar{v}]$ according to the cumulative distribution function F. It is assumed that F admits a continuous and differentiable density function f and has full support, so that

$$F(v_m) = \int_0^{\bar{v}} f(v_m) dv_m \tag{3.2}$$

Let v represent a vector of all agents' valuations, $v = (v_1, v_2, ..., v_n)$. And let v_{-i} be a value vector of all agents except v_i, $v_{-i} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_n)$. The joint density function of v can be written as

$$f(v) = \prod_{m \in N} f(v_m) \tag{3.3}$$

Suppose, for each group, agents are ordered according to their valuation – from high to low. Let $H(v_m)$ be the difference between the valuation of agent m and the next smaller valuation. By the property of order statistics, we know that

$$H(v_m) = \frac{[1 - F(v_m)]}{f(v_m)} \tag{3.4}$$

McAfee & McMillan (1987) defines the expected $H(v)$ as a winning bidder's expected payoff within a second-price sealed-bid auction. This chapter assumes that $H'(v_m)$ has the characteristic (3.5),which is satisfied by most common distributions.

$$H'(v_m) < 0 \tag{3.5}$$

The voting mechanism works as follows. After the agents have reported their valuations to the mechanism, one of the alternatives is selected to maximize ex-ante social expected payoff. Meanwhile, the mechanism also recommends the optimal penalty scheme. For all $i \in N_A$, let $P_A(v_i, v_{-i})$ be the probability that alternative A is selected by the mechanism. And, for all $j \in N_B$, let $P_B(v_j, v_{-j})$ be the probability that alternative B is selected. The sum of these two probabilities is one.

For all agent $m \in N$, let t_m be a penalty scheme. The penalty scheme associates a real number $t_m(v)$ of agent m with each vector v. That is

$$t_m : [0, \bar{v}]^{|N|} \longrightarrow \mathbf{R} \tag{3.6}$$

It is assumed that $t_m(v)$ is non-negative and non-transferable among agents. To understand it, t_m could be individual waiting cost, or cost of negotiations, expert advice.

The utility function of each agent from two groups N_A and N_B is:

$$U_m(k_m, v_m) = k_m v_m - t_m(v), \qquad \forall m \in N;\ m = i, j;\ i \neq j \tag{3.7}$$

where

$$k_i = \begin{cases} 1 & \text{if A is selected} \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in N_A \tag{3.8}$$

and

$$k_j = \begin{cases} 1 & \text{if B is selected} \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in N_B \tag{3.9}$$

Suppose agent i has value v_i but reports v_i' given others' truthful reports, agent i's expected payoff is expressed as the following:

$$\tilde{\pi}_i(v_i, v_i') = E_{v_{-i}}[v_i P_A(v_i', v_{-i}) - t_i(v_i', v_{-i})] \quad \text{where } i \in N_A \tag{3.10}$$

$E_{v_{-i}}$ is the expectation over the vector v_{-i}, and $t_i(v_i', v_{-i})$ represents individual i's penalty when his valuation is v_i but reports v_i'. $P_A(v_i', v_{-i})$ is the probability that A is eventually selected, and v_{-i} represents the truthfully reported valuation vector of all agents except agent i, for $i \in N_A$.

And let agent j's expected payoff be $\tilde{\pi}_j(v_j, v_j')$, when he cheats his value but others truthfully report. That is

$$\tilde{\pi}_j(v_j, v_j') = E_{v_{-j}}[v_j P_B(v_j', v_{-j}) - t_j(v_j', v_{-j})] \quad \text{where } j \in N_B \tag{3.11}$$

In addition, when all agents in the society N truthfully report their private valuations, agent i's expected payoff is

$$\pi_i(v_i) = E_{v_{-i}}[v_i P_A(v_i, v_{-i}) - t_i(v_i, v_{-i})] \quad \text{where } i \in N_A \tag{3.12}$$

and agent j's truthful-revealing expected payoff is

$$\pi_j(v_j) = E_{v_{-j}}[v_j P_B(v_j, v_{-j}) - t_j(v_j, v_{-j})] \quad \text{where } j \in N_B \tag{3.13}$$

Next section shows Bayesian incentive compatibility conditions under this voting mechanism. It describes the maximization problem of the designer, subject to several constraints. The main result is the optimal social decision follows simple majority with zero penalties, under reasonable assumptions.

3.3 Simple Majority Rule with Zero Penalty

The revelation principle [1] states that under weak conditions any mechanism can be mimicked by a direct-revelation and incentive-compatible mechanism. In a direct and incentive-compatible mechanism, all players simultaneously and confidentially report their private valuations, and they have incentive to truthfully report. The revelation principle tells us that, without loss of generality, we can restrict attention to a direct and incentive compatible mechanism. Under this voting mechanism, each agent is asked to report his or her valuation to the mechanism. Proposition 3.3.1 shows an alternative form of expression of Bayesian incentive compatibility.

Proposition 3.3.1. *The Bayesian incentive compatibility of the voting mechanism is equivalent to:*

$$\begin{cases} (i) & \frac{d}{dv_i}\pi_i(v_i) = E_{v_{-i}} P_A(v_i, v_{-i}) \\ (ii) & \frac{\partial}{\partial v_i} E_{v_{-i}} P_A(v_i, v_{-i}) \geq 0 \end{cases} \quad \textit{where } i \in N_A \tag{3.14}$$

$$\begin{cases} (i') & \frac{d}{dv_j}\pi_j(v_j) = E_{v_{-j}} P_B(v_j, v_{-j}) \\ (ii') & \frac{\partial}{\partial v_j} E_{v_{-j}} P_B(v_j, v_{-j}) \geq 0 \end{cases} \quad \textit{where } j \in N_B \tag{3.15}$$

Proof: See Appendix B.1

After each agent reporting his or her valuation, the mechanism then decides which alternative and what sort of penalty scheme should be selected to maximize ex-ante social expected payoff. The ex-ante social expected payoff is the sum of each group's total ex-ante expected payoffs. Lemma 3.3.1 shows the ex-ante expected payoffs of agent i and j respectively. The incentive compatibility conditions (i), (i') in Proposition 3.3.1 are imposed to achieve the results.

Lemma 3.3.1. *Under the voting mechanism, agent i's ex-ante expected payoff is:*

$$E\pi_i = E\big[H(v_i)\ P_A(v_i, v_{-i})\big] + \pi_i(0) \qquad \forall i \in N_A \tag{3.16}$$

[1] See Myerson (1985)

and agent j's ex-ante expected payoff is:

$$E\pi_j = E\big[H(v_j)\ P_B(v_j, v_{-j})\big] + \pi_j(0) \qquad \forall j \in N_B \tag{3.17}$$

Proof: See Appendix B.2

By Lemma 3.3.1, we know that the ex-ante social expected payoff is the following:

$$E\Pi = \sum_{i \in N_A} E\pi_i + \sum_{j \in N_B} E\pi_j \tag{3.18}$$

The voting mechanism aims to maximize (3.18) by choosing appropriate collective rule and penalty scheme. The maximization problem is subject to five constraints. The first two are incentive-compatible conditions (ii) and (ii') in Proposition 3.3.1. The rest of constraints are standard characteristics of probability. Both P_A and P_B are non-negative, and sum of them must be one.

To solve the maximization problem, three boundaries ((I), (II) and (III)) on the ex-ante social expected payoff (3.18) are introduced sequentially (from low to high). Bound (I) is formed by letting the expected payoff of any agent with zero valuation be zero. By the assumption of decreasing function of $H(v_i)$ and Proposition 3.3.1, Bound (II) is formed. The last bound is developed by assuming there is a majority group, say N_A. The Theorem 3.3.1 claims that, to implement the Bound (III), the voting mechanism uses zero penalty schemes for all agents in the society. Meanwhile, the collective decision follows simple majority rule.

Theorem 3.3.1. *Under the direct and incentive compatible voting mechanism, and if all assumptions in section 3.2 are satisfied, the optimal social decision follows simple majority rule, and there is no penalty on each agent.*

Proof: See Appendix B.3

The mechanism in Theorem 3.3.1 works as follows. After all agents truthfully report their valuations to mechanism, the mechanism produces a social decision and recommends a penalty scheme, in such a way that each agent has incentive to follow.

The proof of Theorem 3.3.1 allows the penalty scheme t_n to be a function of all reported values of every agent in society N. However, Theorem 3.3.1 indicates that it is not needed because the penalty on any agent is zero[1]. In other words, the mechanism

[1] Chakravarty & Kaplan (2013) prove that a lottery is optimal if the H' is decreasing and the cost function for revealing private valuation does on depend on agents' valuations.

does not need actually to know all agents' valuations, but simply selects the alternative, which is preferred by the majority in the society.

Two examples are listed below: case $H'(v_m) < 0$ and case $H'(v_m) > 0$. These two examples aim to compare the net welfare of two mechanisms –a simple majority and an alternative mechanism. An alternative mechanism is proposed with decision rule that alternative A is selected if and only if $\sum v_i > \sum v_j$, where agent i prefers A and agent j prefers B. The alternative mechanism is allocation-efficient, but it does not necessary to be net efficient. The working and calculation is processed through Matlab, and the codes can be found in Appendix B.4.

Example 1: It is assumed that each agent's private value follows Gamma distribution with shape and scale being 5 and 1 respectively – $G(5,1)$. In this case, the inverse hazard rate H is decreasing ($H'(v_m) < 0$). There are 94 agents, which 49 of them prefer alternative A and the remaining agents prefer B[1]. Simple majority mechanism selects alternative A, while the alternative mechanism selects B since $\sum v_i > \sum v_j$. The net welfare of simple majority mechanism is $|N_A|E[H(v_i)P_A(v_i, v_{-i})] = 49 \times 2.5574 \approx 125.31$. The net welfare of the alternative mechanism is $|N_B|E[H(v_j)P_B(v_j, v_{-j})] = 45 \times 1.0349 \approx 46.57$. Thereby the simple majority mechanism is better than this alternative mechanism in terms of net welfare.

Example 2: Now let us consider another Gamma distribution, $G(0.1, 1)$. Then the inverse hazard rate H is increasing ($H'(v_m) > 0$). Again, simple majority and the alternative mechanism selects alternative A and B, respectively. The net welfare of simple majority mechanism is $|N_A|E[H(v_i)P_A(v_i, v_{-i})] = 49 \times 0.0551 \approx 2.7$. The net welfare of this alternative mechanism is $|N_B|E[H(v_j)P_B(v_j, v_{-j})] = 45 \times 0.0997 \approx 4.5$. In this case, simple majority mechanism becomes inferior.

3.4 Conclusion

This chapter studies the optimal design of voting rules when agents face binary choice. The agents report their private valuations to the mechanism, then the designer selects one of these alternatives and sets up the penalty scheme, in order to maximize ex-ante

[1] Agents' valuations are randomly created through $G(5,1)$ via Matlab. Without loss of generality, among these agents, no one is indifference between A and B – an agent with zero valuation $\notin$ N

social expected payoff. Theorem 3.3.1 shows the society cannot do better in terms of utilitarian efficiency, than to follow a simple majority rule with zero penalty scheme.

The designer may choose a voting mechanism which allows costly stand-off (such as individual cost of time, or monetary or mental cost of negotiations) and then sets up the penalty scheme in such a way that every agent has incentive to truthfully reveal their private information. The mechanism elicits private information, and therefore the better alternative may be selected more often. But, it generates the expected penalty cost for eliciting such private information. Chapter 3's voting mechanism achieves the agreement early, but does not extract private information from agents. And therefore it may not achieve the first best. Theorem 3.3.1 indicates the simply majority mechanism without any penalty may select a sub-efficient alternative, but can achieve a higher welfare. The reason is simply because the expected penalty cost is too high.

It should be mentioned here, some collective decision mechanisms require explicit support of some super-majority. Instead of the immediacy of simply majority, these mechanisms may generate individual waiting cost. Under some environments, these alternative mechanisms are better. When waiting cost can be controlled or ignored, then super-majority that selects a better alternative with higher probability, is better than a simple majority.

Chapter 4

A Package Bidding Mechanism to Implement the Core

This chapter adopts the concept of core as a competitive standard, which enables the package bidding mechanism to avoid the well-known weaknesses of VCG mechanism, when gross substitutes condition fails. A more practical procedure – package bidding is introduced to implement the core. It is proved that subgame perfect equilibrium outcomes yielded from the package bidding mechanism coincide with the core of an underlying strictly convex game.

4.1 Introduction

The cooperative game theory takes an abstract view of individual interaction. The characteristic function form of a game expresses the set of payoffs to a coalition. Although this approach has a clear advantage in terms of robustness [1], it lacks formal investigation describing the process that strategic players finally agreed on the outcomes.

Nash (1953) sets a new entire research agenda referred as Nash program for cooperative games. The Nash program tries to link the cooperative and non-cooperative game theory. It aims to develop non-cooperative procedures that yield cooperative solutions, such as the core, as their equilibrium outcomes. The concept of core, pioneered by

[1]Solution concepts are independent of the unimportant details of different procedures that underlie the same set of feasible payoffs

Edgeworth (1881), is probably the most widely used cooperative solution concept. It is the set of feasible payoff allocations that cannot be improved upon by any subset of the players.

A large number of papers contributes to the Nash program that explores non-cooperative outcomes related to the core. Banks & Duggan (2000) model the decision making process as the form of coalitional bargaining. They show, in certain environment, stationary equilibrium outcomes coincide with the core. Chatterjee *et al.* (1993) build a model of n-person coalitional bargaining with discounting and transferable utility (TU). They introduce a fixed protocol describing the order of proposers and respondents, and they show that in strictly superadditive TU game, as the discount factor tends to one, subgame perfect equilibrium outcomes converge to core outcomes. Perry & Reny (1994) provide a non-cooperative implementation of the core, and they consider a dynamic bargaining game in continuous time without a fixed order of moves. They prove that every stationary equilibrium of their game leads to the payoffs in the core. Benny & Eyal (1995) study non–transferable utility game without discounting, and they prove an equivalence between the core and stationary order independent equilibrium outcomes.

Serrano (1995) is closest to ours in spirit. Given a strictly convex TU coalitional function, Serrano (1995) constructs a game that resembles an asset market. The core is supported in subgame perfect equilibrium and obtained as those outcomes in which every arbitrary opportunity has vanished from the market. Unlike the literature mentioned above, the rules of the game form do not require complete knowledge of the coalitional function [1]. Serrano (1995) describes an asset market with randomly selected broker to centralize the trade. Each player initially owns one asset. In the first stage, the broker proposes an asset price vector, at which he wants to buy the assets from other players. In the second stage, the remaining players decide to either accept or reject this deal sequentially. If a player accepts the deal, he sells his asset to the broker at the proposed rate. While, when a player rejects it, he proposes a portfolio of assets that he wants to purchase at broker's proposed price. The broker's payoff is the worth of the final portfolio of assets he holds, plus the net monetary transfers that he received. When the transferable utility (TU) game is strictly convex, Shapley (1971) theorem can be used to prove that the implementation of the core is achieved in terms

[1] See Bergin & Duggan (1999)

of expectation. The core allocation, in Serrano (1995), is viewed as a situation in which a player has no arbitrage opportunities to buy underpriced assets from others.

This chapter contributes to the Nash program. It presents an extensive form, and tends to show a connection between subgame equilibria and the core. Different from Serrano (1995)'s design, it resembles a package bidding mechanism, in which the seller initially owns all items and rest of players are bidders competing to win the item(s) that they are interested in. The design works for substitute goods as well as complementary goods. One of the players is assumed to be randomly selected as the first mover, who invites others to form a grand coalition by proposing a payoff vector. By renting other players to achieve the joint benefit, the proposed payoff vector is interpreted as the amount that the first mover must pay to these players. In this mechanism, only core allocations generate subgame perfect equilibrium payoffs, because non-core allocations provide arbitrage opportunities for some players. In addition, the first mover with monopoly power receives the maximum core payoff in the subgame. By the strict convexity assumption, the implementation of the core is achieved in terms of expectation.

This chapter also contributes to the design of package auction when the gross substitutes condition[1] fails. In many auction environments bidders are more interested in the packages of items they win. Under a package auction (also known as combinatorial auction), any bidder is allowed to bid directly for non-trivial subsets (package) of items being sold. It is partially important when items are complements. Then bidders can more fully express their preferences, resulting in improved economic efficiency (allocating a package of items to the bidder who values it most) and greater auction revenues. Vickrey (1961) provides a mechanism in which it is a dominant strategy for bidders to report their values truthfully. The mechanism assigns item(s) efficiently and the bidders pay the opportunity cost of the item(s) won. For multiple identical items, in Vickrey's original setting, each bidder is assumed to have the diminishing marginal value of the item. Clarke (1971) and Groves (1973) extend Vickrey (1961)'s design. Their auction design does not require nonincreasing marginal values for bidders. The outcomes are still efficient and the bidders still pay the opportunity cost of the item(s)

[1]The gross substitutes condition requires that an increase in the price of an item (or a package of items) causes an increase in demand for other items. Under the setting with multiple identical objects and declining marginal values, this condition is obviously satisfied.

won. The extended Vickrey mechanism, throughout this chapter, is called the Vickrey-Clarke-Groves (VCG) mechanism.Ausubel & Milgrom (2002) prove that when gross substitutes condition is satisfied, the VCG payoffs lie in the core.

When the gross substitutes condition fails, VCG mechanism suffers from several practical drawbacks. Ausubel & Milgrom (2006) study the environment where items are complementary, and they list the possible drawbacks of using VCG mechanism. Under VCG pricing rule, the seller revenue can be very low (possibly even zero) and is non-monotonicity in both the number of bidders and their values. Collusion becomes feasible and easier among losing bidders. As a result, bidders may find shill bidding profitable.

Ausubel & Milgrom (2006) adopt the concept of the core as a competitive standard to ensure the outcomes of their package auction design do not have those practical drawbacks. The promise of a core allocation is that collusion among losing bidders becomes unprofitable. The basic idea behind in the core is similar as that behind a Nash equilibrium, in which an allocation is stable if no player has incentive to deviate. If an allocation is in the core, there is no tendency for a coalition to form and upset it. In terms of auction theory, the non-core allocations are unstable, in that some bidders are willing to pay more than the winner's payment.

The organization of chapter 4 is as follows. The next section defines the core of a strictly convex transferable utility game, and it introduces the definition of Davis & Maschler (1965)'s (DM) reduced game. Section 4.3 develops the package bidding mechanism and related assumptions. Section 4.4 presents and discusses the main results Section 4.5 concludes this chapter. And all proofs are relegated to Appendix C.

4.2 Core, Strict Convexity, and DM Reduced Game

4.2.1 Core

Let a transferable utility (TU) game be a pair (N, v) where N is a coalition (grand coalition) and v is the characteristic function of the game. A coalition S is defined to be a subset of N, $S \subset N$. The characteristic function v associates a real number $v(S)$ with each subset S of N. For any $S \in N$, $v(S)$ is called the worth of S. It is interpreted as the maximum value of S can create as a group. The pair (N, v), where v assigns $v(S)$ to each coalition S is defined as a game in characteristic function. Suppose X is the set of feasible allocations. Let the set of players be L with player 0 being the seller. In general, the characteristic function can be defined for coalition S as follows:

$$v(S) = \begin{cases} \max_{x \in X} \sum_{l \in S} v_l(x_l), & 0 \in S \\ 0, & 0 \notin S \end{cases} \tag{4.1}$$

It should be mentioned here, in this chapter, the rules of the game do not require that the designer has complete knowledge of the characteristic function, such as (4.1).

It is always assumed that $v(\emptyset) = 0$. Let R^N (here R denotes the real numbers) be the set of all payoff vectors of N. For any $\pi \in R^N$ and $S \in N$, it is denoted that

$$\pi(S) = \sum_{i \in S} \pi_i \tag{4.2}$$

We now can state the following definition. Let (N, v) be a TU game. The core of (N, v) is defined by:

$$C(N, v) = \Big\{ \pi : v(N) = \pi(N), v(S) \leq \pi(S) \text{ for all } S \in N \Big\} \tag{4.3}$$

The condition that $v(N) = \pi(N)$ tells us that the vector π is feasible, and the condition that $v(S) \leq \pi(S)$ tells us that there is no tendency for coalition S to form and upset the π because coalition S cannot guarantee each of its members receive more than they could gain from π. Thus the core of (N, v) is the set of feasible (for the whole coalition) payoff allocations that cannot be improved upon by any subset.

4.2.2 Strict Convexity

A game (N, v) is a strictly convex game if for all coalitions S and T we have

$$v(S \cup T) > v(S) + v(T) - v(S \cap T) \tag{4.4}$$

This condition arises when each player provides positive marginal contribution to the worth of the coalition. It also arises when each player holds a unique object and the objects are complementary (Topkis (1987)). It follows immediately that every strictly convex game is strictly superadditive[1]. It should be mentioned here, since the game is strictly superadditive, $v(N)$ is the largest total value received by any disjoint set of coalitions. Players have incentive to form the grand coalition for joint benefit.

[1]Superadditivity says that for all $S, T \in 2^N$ and $S \cap T = \emptyset$, then $v(S) + v(T) \leq v(S \cup T)$.

Shapley (1971) shows that the core of a strictly convex game is nonempty, and the extreme points (marginal contribution or "greedy value allocations") can be computed by the greedy algorithm. By listing the players in some order, player i's marginal contribution (or "greedy value allocations") to the coalition N of preceding players is the following:

$$v(N) - v(N \setminus \{i\}) \tag{4.5}$$

Shapley's theorem says that for any ordering of the players all greedy value allocations are in the core.

Assumption 1: It is assumed that the TU game (N, v) is strictly convex in chapter 4.

4.2.3 DM Reduced Game

This chapter uses the definition of Davis & Maschler (1965)'s (DM) reduced game. For any nonempty subset S and for any $\pi \in R^N$, the DM reduced game $(S, v_{\pi s})$ has the following characteristics:

$$v_{\pi s} = v\big(N\big) - \pi\big(N \setminus S\big) \tag{4.6}$$

and

$$v_{\pi s}\big(T\big) = \max_{Z \subseteq N \setminus S} \{v(T \cup Z) - \pi(Z)\} \qquad \forall T \subset S \tag{4.7}$$

Serrano (1995) interprets this reduced game as follows. The broker (first mover) proposes a price vector π. The reduced game $(S, v_{\pi s})$ is formed by paying all players in $N \setminus S$ at price π. The new worth of the total coalition S is defined as the coalition value after paying each member in $N \setminus S$ at π. Let T be any non-empty subset of S. The definition of $v_{\pi s}(T)$ allows T to consider the best deal from any group of players in $N \setminus S$.

One interpretation of $v_{\pi s}\big(T\big)$ is that T maximizes its return by renting group $Z \subset N \setminus S$ to join it and together create the joint benefit (worth) $v(T \cup Z)$. Meanwhile, T must pay all players in Z at price π.

To understand better the concept of core in a DM reduced game, consider the following example explaining core allocations under DM reduced game: Suppose there is a TU game (N, v), where $N = \{0, 1, 2\}$. The seller, labeled as 0, owns an indivisible object. Two bidders, labeled as 1 and 2, compete to win the object. Let the worth of total coalition be one, $v(N) = 1$. And, for any S consisting of a seller and one bidder, let the worth of any nonempty subset S be 0.6. If the seller is not included or if there is only the seller included in the coalition, the worth is zero. The proposed price vector π is the following

$$\pi = (0.4,\ 0.5,\ 0.1) \tag{4.8}$$

The definition of core tells us that π is not a core allocation, because the seller and player 2 can form a coalition S with $v(S) > \pi(S)$. That is, there exists the coalition $S = \{0, 2\}$ that can improve upon π.

Alternatively, let the core of reduced game $(S, v_{\pi s})$ be $C(S, v_{\pi s})$, let the projection of π be π^s. Peleg (1986) proved that, for balanced games, the core is the only solution that satisfies non-emptiness, rationality, superadditivity, and consistency. A solution is consistent if it is "independent" to the number of players. For instance, there is a game with n players and let $(S, v_{\pi s})$ be its reduced game. The solution of this game is consistent, if the projection of the original solution is still a solution in the reduced game $(S, v_{\pi s})$. The Peleg (1986)'s theorem tells us that if there exists a reduced game $(S, v_{\pi s})$ and its core allocation does not include π^s, then π is not a core allocation of game (N, v). In the example, there is a reduced game $(S, v_{\pi s})$, where $S = (0, 1)$ with the following characteristics:

$$v_{\pi s} = v(N) - \pi(N \setminus S) = 1 - 0.1 = 0.9 \tag{4.9}$$

and

$$v_{\pi s}\big(\{0\}\big) = \max\{v(\{0,2\}) - \pi_2, 0\} = \max\{0.6 - 0.1, 0\} = 0.5 \tag{4.10}$$

The corresponding projection of the given π is $\pi^s = (\pi_0,\ \pi_1)$, where $\pi^s = (0.4,\ 0.5)$. Since $v_{\pi s}\{0\} > \pi_0$, the seller has arbitrage opportunities. The seller has opportunity to rent an "underpriced" player 2. By the definition of core allocations, π^s is not in the $C(S, v_{\pi s})$. By Peleg (1986)'s theorem, we know that $\pi \notin C(N, v)$.

By using the definition of DM reduced game and Peleg's theorem, we can interpret that core allocations are those where the players have no opportunities to rent "underpriced" agent(s). That is, for some i and for some S, $i \in S$, we have

$$v_{\pi s}(\{i\}) \leq \pi_i \tag{4.11}$$

As the final part of this section, interpret the strictly convex TU game (N, v) as the following package bidding mechanism. Let $n \geq 2$ denote the number of players in the game, numbered from 1 to n, and let N be a finite set of players $N = (1, ..., n)$. There is only one seller within N, the rest of players are bidders. The seller owns some indivisible items. Bidders holding no items compete to win their preferred "baskets". And each player owns an amount of a perfectly divisible composite commodity called money. The worth of coalition S is the value that S can create. If the seller is not included in the coalition, the worth is zero.

4.3 Package Bidding Mechanism

This section introduces a package bidding mechanism in which one of the players is randomly selected as the first mover. Let player i be selected as the first mover with probability p_i. And let a vector $p = (p_1, ..., p_n)$ be the probability that every player is selected as the first mover. The worth of any coalition of the TU game (N, v) indicates the joint value of cooperation. The subgame where player i is the first mover has three stages.

The rules of this mechanism do not depend on the characteristic function. Otherwise, there exists some possibilities that a non-cooperative game whose equilibrium payoffs can be achieved by the trivial mechanism. [1]

In stage 1, player i proposes a payoff vector $\pi = (\pi_1, \pi_2, ..., \pi_n)$, where $\pi(N) = v(N)$. If the grand coalition is formed, each player in N receives a payoff at the rate of π, with the understand that player j receives π_j. By renting the players in $N \setminus \{i\}$ to form a coalition and create the joint benefit, the proposed π is interpreted as the amounts must be paid by player i to each player in $N \setminus \{i\}$. In addition, the mechanism requires the proposed payoff vector π to be individually rational. That is, for all $j \in N$

$$\pi_j \geq v(\{j\}) \tag{4.12}$$

And each element of the proposed π is non-negative. That is, for all $j \in N$, we have

$$\pi_j \geq 0 \tag{4.13}$$

Player i can be either the seller or one of the bidders. Meanwhile, Bidders are able to bid any item or items they are interested in. In other words, the mechanism works when items are complementary and/or substitutes.

[1]See Bergin & Duggan (1999) Proposition 1

In stage 2, players in $N \setminus \{i\}$ move sequentially according to a fixed and arbitrary protocol P. The fix protocol P describes the order of the first movers and respondents. They decide to accept or reject the deal at π. If any player $j \in N \setminus \{i\}$ rejects π, then j proposes a new coalition $T \subset N$ at π, where $i \notin T$. The action space of player $j \in N \setminus \{i\}$ is (α_j, b_j). If player j accepts π, α_j displays "1". Then b_j is the set of item(s) that j wants to buy from player i at π. For instance, player j's the action space $(1, \emptyset)$ indicates j accepts π and j is the seller. If player j rejects π, α_j displays "0". Then b_j is a set of players $T \in N \setminus \{i\}$, with whom j prefers to trade at π. If any player j accepts π, he is guaranteed to receive π_j at the end of trading. Player j receives π_j not because there is a trade between the first mover i and him, but because he agrees to join and contribute to the joint return at the proposed π.

In the final stage, trading with rejectors occurs. Let $M \in N \setminus \{i\}$ be the set of rejectors. Let player g be a rejector, where $g \in M$. Recall above definition of action space, g's action space is $(0, T)$, where $g \in T$ and $i \notin T$. T is a coalition that g proposes and prefers to trade with. The mechanism requires that these invited players $T \setminus \{g\}$ must agree to join. And player g must pay them at the rate of π. When the item(s) demanded by g is available in T, the trading takes place; otherwise there is no trading.

Assumption 2[1]: In chapter 4, it is assumed that any player in $N \setminus \{i\}$ is willing to accept the π when he or she is indifferent between acceptance and rejection.

The following section introduces player i's payoff function in the subgame where player i is selected as the first mover. Let Ω_i be player i's payoff function. If player i is the seller, his payoff function Ω_i is the following:

$$\Omega_i = \begin{cases} v(N) - \pi(N \setminus \{i\}) & \text{if all players in } N \setminus \{i\} \text{ accept } \pi \\ v(N \setminus M) - \pi(N \setminus \{M \cup \{i\}\}) & \text{if a set } M \text{ of players reject } \pi \end{cases} \tag{4.14}$$

If player i is not the seller, his payoff function Ω_i is the following:

$$\Omega_i = \begin{cases} v(N) - \pi(N \setminus \{i\}) & \text{if all } N \setminus \{i\} \text{ accept } \pi \\ v(N \setminus M) - \pi(N \setminus \{M \cup \{i\}\}) & \text{if the seller} \notin M \\ -\pi(N \setminus \{M \cup \{i\}\}) & \text{if the seller} \in M \end{cases} \tag{4.15}$$

In any subgame where player i is the first mover, i invites the rest of players to form a grand coalition by proposing a return vector π. If all players in $N \setminus \{i\}$ accept

[1] Same as Serrano (1995), all players take participant cost and the cost of time/waiting into account. A player will be willing to trade at rate of current π, when both decisions bring her same payoff.

the deal, player i must pay them at the rate of π. Then player i's payoff is the total value that the grand coalition N can create minus the promised payment to $N \setminus \{i\}$. Suppose there exist a set of rejectors M at the end of stage 2, and the seller is not included in M. Player i achieves the worth of coalition $N \setminus M$ by paying all players in $N \setminus \{M \cup \{i\}\}$ at π. When the seller is one of the rejectors in M, the worth of coalition $v(N \setminus M)$ is zero. But player i still has to pay all members in $N \setminus \{M \cup \{i\}\}$ at π.

4.4 Results: Subgame Perfect Equilibrium and Core Allocations

This section focuses on the connection between subgame perfect equilibrium (SPE) payoffs of the package bidding mechanism and core allocations of the strictly convex TU game (N, v). Let player i be selected as the first mover. Consider any subgame that starts immediately after player i has proposed a payoff vector π. By the interpretation of core allocation (shown in 4.11), we know that if the proposed payoff vector π is a point in the $C(N, v)$, then any player in N has no opportunity to rent "underpriced" players to join in a coalition at π (no arbitrage opportunities). Therefore, all players have no incentive to reject π. In addition, if π is not a core allocation, Lemma 4.4.1 indicates that there exists at least one player in $N \setminus \{i\}$ who could profitably reject π and propose a new coalition.

Lemma 4.4.1. *Consider any subgame that starts after player i has proposed a payoff vector π. Suppose all assumptions mentioned above are satisfied, then the proposed π is a core allocation if and only if all players within $N \setminus \{i\}$ accept the π.*

Proof: See Appendix C.1

Consider any subgame which starts after player i has proposed a vector π with $\pi_i \geq v(\{i\})$. Let Q_i be the set of core allocations where player i can gain his maximum payoff. That is

$$Q_i = \{\pi \in C(N, v), \text{such that } \forall x \in C(N, v), \pi_i \geq x_i\} \tag{4.16}$$

By Assumption 1 (strictly convex game) and Shapley (1971)'s theorem, we get $Q_i \neq \emptyset$ and

$$\pi_i = v(N) - v(N \setminus \{i\}), \quad \text{for } \pi \in Q_i \tag{4.17}$$

By moving firstly, player i has incentive to propose the vector π, where i can receive his or her maximum payoff π_i, shown in (4.17). Lemma 4.4.2 indicates that, for player i, to receive his or her maximum payoff, the proposed payoff vector π must be accepted by all players in $N \setminus \{i\}$.

Lemma 4.4.2. *Consider any subgame that starts after player i has proposed a payoff vector π and all assumptions mentioned above are satisfied. If π is rejected by at least one players in $N \setminus \{i\}$, player i's payoff Ω_i from proposing π is strictly less than $v(N) - v(N \setminus \{i\})$.*

Proof: See Appendix C.2

Lemma 4.4.2 indicates the first mover i has incentive to propose a payoff vector, which is accepted by all remaining players. Meanwhile, first mover i achieves his or her maximum payoff π_i. Lemma 4.4.1 and 4.4.2 implies that, in any subgame where player i is the first mover, π is a SPE payoff if and only if $\pi \in Q_i$.

Let SPE_p be the subgame perfect equilibrium of the package bidding mechanism with a fixed p (recall the definition of p in section 4.3: the probability that every player is selected as the first mover). According the strict convexity assumption (Assumption 1), the implementation of the core is achieved in terms of expectation.

Theorem 4.4.1. *Let the underlying TU game (N, v) be strictly convex, and let p be the probability each player is selected as the firs mover. Then under Assumption 2, $\pi \in C(N, v)$ if and only if π is the subgame perfect equilibrium outcome for any p.*

Proof: See Appendix C.3

This result is related to Serrano (1995), whose mechanism resembles an asset market with a broker. Each player owns an indivisible asset. One of the players is randomly selected as the first mover, denoted as a broker to centralize trade. The subgame starts immediately after a broker has proposed an asset price vector. In Serrano (1995), it is shown that any subgame perfect equilibrium outcomes coincide with the core allocation of the TU convex game.

The broker has monopoly power since he firstly proposes his most preferred prices, while other players have to compete to win the assets they are interested in. As a result, the monopoly power of a broker enables him to gain his top-ranked payoff in the core. At equilibrium, the broker has incentive to propose the π, where the all remaining players accept (no arbitrage opportunities). Again, by the strict convexity assumption, the implementation of the core is achieved in terms of expectation.

At equilibrium, the mechanism works in the following way. Suppose there is one seller, denoted as S; three bidders, denoted as B_1, B_2, B_3. The seller S has four items (I_1, I_2, I_3, I_4), others hold nothing but compete to bid their preferred item(s).

Example 1: Given $P = (S, B_1, B_2, B_3)$, in the first stage, the seller S proposes a payoff vector $\pi(4) = v(4)$,where $\pi(4) = (\pi_s, \pi_{B_1}, \pi_{B_2}, \pi_{B_3})$. And, the proposed π satisfies condition (4.12) and (4.13). In the second stage, at equilibrium, all bidders accept the π. Their action spaces, for example, are the following: $B_1 : \big(1, (I_1)\big)$, $B_2 : \big(1, (I_1, I_2)\big)$, $B_3 : \big(1, (I_3, I_4)\big)$. The trading with accepters happens in this stage according to the order of P. The resulting package allocations and payoffs are: S receives π_s with zero items; B_1 gets item I_1 and payoff π_{B_1}; B_2 only wins item I_2 but receives compensation π_{B_2}; B_3 wins the package (I_3, I_4) and payoff π_{B_3}.

Example 2: Given $P = (B_3, S, B_1, B_2)$, in the first stage, the bidder B_3 proposes a payoff vector $\pi(4) = v(4)$,where $\pi(4) = (\pi_{B_3}, \pi_s, \pi_{B_1}, \pi_{B_2})$. And, the proposed π satisfies condition (4.12) and (4.13). In the second stage, at equilibrium, all players accept the offer. Again, their action spaces, for instance, can be the following: $S : (1, 0)$, $B_1 : \big(1, (I_1)\big)$, $B_2 : \big(1, (I_1, I_2)\big)$. The trading with accepters happens in this stage according to the order of P. The resulting package allocations and payoffs are: B_3 receives package (I_3, I_4) and payoff π_{B_3}; S still holds items I_1, I_2 and earns payoff π_s; B_1 and B_2 both wins nothing but gets compensation π_{B_1} and π_{B_2}, respectively.

Chapter 4 presents a mechanism to resemble a package auction. The seller owns all items and bidders compete to win the item(s) they are interested in. One of the players is randomly selected as the first mover to propose a payoff vector π. The first mover (either a bidder or the seller) has incentive to invite the rest of players to form the grand coalition N, because each player has positive marginal contribution to a coalition. At the same time, the first mover must make a payment to each player in $N \setminus \{i\}$ at the proposed rate π. Similar in Serrano (1995), Theorem 4.4.1 indicates that the first mover has monopoly power to achieve his or her best core payoff, $v(N) - v(N \setminus \{i\})$.

4.5 Conclusion

This chapter presents a package bidding mechanism to implement the core. The package bidding process has three stages, and the anonymity of the procedure stems from the random selection of first moving player. In the first stage, a randomly selected player announces a payoff vector. In the second stage, remaining players decide to accept or reject this deal sequentially. Finally, trading with rejectors takes place. The rules of the mechanism are independent of the characteristic function. The key assumption here is the strictly convexity of the TU game (N, v). As a result, Shapley (1971)' theorem can be applied to prove the main results. That is, the implementation of the core can then be achieved in terms of expectation

With the promise of core allocations, this package bidding mechanism avoids the practical drawbacks (such as collusion, shill bidding) of VCG mechanism when items are complementary. It is shown the subgame perfect equilibrium outcomes of this mechanism coincide with the core of a strictly convex transferable utility game. The monopoly power of the first mover enables him to achieve his top-ranked payoff in the core.

Chapter 5

Conclusion

The thesis applies game theory to analyze optimal toehold bidding strategies in the context of takeover bidding competition (chapter 2), to study optimal design of voting rules (chapter 3), and to develop a package bidding mechanism (chapter 4).

Chapter 2 introduces a two-stage takeover game and points a possible toehold bidding cost. It looks like profitable to purchase a toehold prior to making any takeover bid, given substantial control premiums. However the actual purchased toehold sizes follow a bimodal distribution. Some literature argues that toehold benefits are not withstanding. They find that there exists a toehold cost generated by target managerial entrenchment – the opportunity loss of a target termination agreement. Toehold bidding, in equilibrium, reduces toehold board expected private profits, result in a rejection of negotiation. Chapter 2's two-stage takeover game models private takeover stage as a sealed-bid second-price auction. Moreover, there is no managerial entrenchment at the first stage. Since, according to the general takeover process, the private takeover stage is initiated by the target board. In other words, there is no threat of target managerial resistance among invited potential bidders. At the beginning of the second stage of this game, acquired toehold sizes become publicly known, and therefore they are regarded as signals partially revealing each bidder's willingness to bid.

The main result in chapter 2 is that, under some conditions, there exists a restricted partial pooling Bayesian equilibrium, in which low-value bidders choose zero toehold while high-value bidders pool their toehold acquisition decisions at one size. Signal jamming occurs in equilibrium. At equilibrium, bidders play non-strict toehold acquisition strategies, pretending to be other bidders with some probability, in order to avoid fierce competition in the second stage. It is shown that, under this two-stage takeover model, bidders may face a toehold bidding cost – the opportunity loss of a profitable resale. As a low-value bidder, by acquiring a toehold (deviate from equilibrium), his toehold decision has no effect on low-value rival's bidding strategy. That is, along the equilibrium path, the low-value rival still bids her valuation. The low-value bidder with some probabilities loses the target. He sells out his toehold at a relatively low price and has to pay the resale cost. In terms of expectation, low-value's toehold bidding cost is higher than its benefit. As a result, at equilibrium, low-values optimally prefer zero toehold.

Chapter 3 studies the optimal design of voting rules when each agent faces binary choice. Each agent's preference is assumed to be independently distributed. The designer is able to use any non-transferable penalty on agents. After the agents report their private valuations to the mechanism, the designer selects one alternative and a penalty scheme to maximize ex-ante expected social payoff.

On one hand, the designer can choose a mechanism that allows costly stand-off, including individual cost of time, any monetary or mental cost of negotiations, etc. And then he sets up the penalty scheme to ensure each agent has incentive to truthfully report his or her private valuation. In this case, the mechanism elicits private information, and a better alternative would be selected more often. The mechanism also generates the expected penalty cost of eliciting private information. On the other hand, the designer can choose a mechanism that achieves the agreement early, but does not extract private information from agents. Therefore, it may not select the first best more often. And the mechanism does not generate the expected penalty cost.

Chapter 3 finds that, under reasonable assumptions, the society cannot do better in terms of utilitarian efficiency, than to follow a simple majority rule with zero penalty on each voter. In this case, the simple majority may select a sub-efficient alternative, but can achieve a higher welfare. The reason is simply because the expected penalty cost is too high.

Chapter 4 presents a package bidding mechanism whose subgame perfect equilibrium outcomes coincide with the core of an underlying strictly convex transferable utility (TU) game. This chapter adopts the concept of the core as a competitive standard to ensure the outcomes of the mechanism do not have the well-known practical drawbacks of VCG mechanism. Lemma 4.4.1 and 4.4.2 implies that, for any subgame that starts after player i has proposed a payoff vector π, the proposed π is the outcome of subgame perfect equilibrium if and only if it is a core allocation. Since the strict convexity assumption, the implementation of the core can be achieved in terms of expectation. Moreover, the first mover with monopoly power achieves the top-ranked payoff in the core.

Appendix A

Proofs in Chapter 2

A.1 Proof of Proposition 2.3.1

Let Π_i $(i = 1, 2)$ denote bidder i's equilibrium payoff, and let $v_1 > v_2$. At equilibrium, bidder 1 wins the target and she pays $v_1 + d - \epsilon$. So that

$$\Pi_1 = v_1 - (1 - \theta_1)(v_1 + d - \epsilon) \tag{A.1}$$

Meanwhile, bidder 2 loses the target and sells out his shareholdings at his bid price. Then his equilibrium payoff can be expressed as:

$$\Pi_2 = \theta_2(v_1 + d - \epsilon) - d \tag{A.2}$$

Let bidder 1 deviate by bidding higher than $(v_1 + d)$, given bidder 2 along the equilibrium path. That is

$$\bar{b}_1 > v_1 + d \tag{A.3}$$

Bidder 1 still wins the target at price $v_1 + d - \epsilon$. Let $\bar{\Pi}_1$ be bidder 1's off-equilibrium payoff when she deviates by increasing her bid. Then

$$\bar{\Pi}_1 = v_1 - (1 - \theta_1)(v_1 + d - \epsilon) \tag{A.4}$$

(A.1) and (A.4) tells us that

$$\Pi_1 = \bar{\Pi}_1 \tag{A.5}$$

Let bidder 1 deviate by bidding lower than $(v_1 + d - \epsilon)$, given bidder 2 along the equilibrium path. That is

$$\underline{b}_1 < v_1 + d - \epsilon \tag{A.6}$$

Then bidder 1 loses the target and sells out her shareholdings at her bid price. Let $\underline{\Pi}_1$ be bidder 1's off-equilibrium payoff when she deviates by reducing her bid. So that

$$\underline{\Pi}_1 = \theta_1 \underline{b}_1 - d \tag{A.7}$$

Then

$$\Pi_1 - \underline{\Pi}_1 = v_1 - (1-\theta_1)(v_1 + d - \epsilon) - \theta_1 \underline{b}_1 + d \tag{A.8}$$

That is

$$\Pi_1 - \underline{\Pi}_1 = \theta_1 \big(v_1 + d - \epsilon - \underline{b}_1\big) + \epsilon \tag{A.9}$$

Since (A.6), we have

$$\Pi_1 - \underline{\Pi}_1 > 0 \tag{A.10}$$

Let bidder 2 deviate by bidding lower than $(v_1 + d - \epsilon)$, given bidder 1 along the equilibrium path. That is

$$\underline{b}_2 < v_1 + d - \epsilon \tag{A.11}$$

Bidder 2 loses the target and sells out his shareholdings at his bid price. Let $\underline{\Pi}_2$ be bidder 2's off-equilibrium payoff when he deviates by reducing bid price. So that

$$\underline{\Pi}_2 = \theta_2 \underline{b}_2 - d \tag{A.12}$$

Then

$$\Pi_2 - \underline{\Pi}_2 = \theta_2 (v_1 + d - \epsilon) - d - \theta_2 \underline{b}_2 + d \tag{A.13}$$

That is

$$\Pi_2 - \underline{\Pi}_2 = \theta_2 (v_1 + d - \epsilon - \underline{b}_2) \tag{A.14}$$

By (A.11), we have

$$\Pi_2 - \underline{\Pi}_2 > 0 \tag{A.15}$$

Let bidder 2 deviate by bidding higher than $(v_1 + d)$, given bidder 1 along the equilibrium path. That is

$$\bar{b}_2 > v_1 + d \tag{A.16}$$

Bidder 2 wins the target at the price $(v_1 + d)$. Let $\bar{\Pi}_2$ be bidder 2's off-equilibrium payoff when he deviates by increasing his bid. So that

$$\bar{\Pi}_2 = v_2 - (1-\theta_2)(v_1 + d) \tag{A.17}$$

Then

$$\Pi_2 - \bar{\Pi}_2 = \theta_2(v_1 + d - \epsilon) - d - v_2 + (1 - \theta_2)(v_1 + d) \tag{A.18}$$

By the definition of ϵ, let ϵ be zero. We have

$$\Pi_2 - \bar{\Pi}_2 = v_1 - v_2 \tag{A.19}$$

Since $v_1 > v_2$, it follows that

$$\Pi_2 - \bar{\Pi}_2 > 0 \tag{A.20}$$

Therefore, we have shown that two bidders have no incentives to deviated from the equilibrium bidding strategies shown in Proposition 2.3.1.

A.2 Proof of Lemma 2.3.1

Bidder 1's true private valuation is v_1. Let $\underline{r_1}$ and $\bar{r}_1$ be bidder 1's announcement on her valuation. That is

$$\underline{r_1} < v_1 \tag{A.21}$$

and

$$\bar{r}_1 > v_1 \tag{A.22}$$

Let $\underline{E\Pi_1}$ and $\overline{E\Pi_1}$ denote bidder 1's off-equilibrium expected payoffs when he announces $\underline{r_1}$ and $\bar{r}_1$ respectively.

Given bidder 2 along the equilibrium path, three scenarios are considered to compute $\underline{E\Pi_1}$ by reporting $\underline{r_1}$.

(i) $v_2 \in [0, \underline{r_1}]$: Bidder 2' equilibrium bid is $(\underline{r_1} + d - \epsilon)$. Bidder 1 wins the target at price $(\underline{r_1} + d - \epsilon)$. Under this scenario, bidder 1's expected payoff is the following:

$$\int_0^{\underline{r_1}} \left[v_1 - \Big(1 - \tau(\underline{r_1})\Big)(\underline{r_1} + d - \epsilon)\right] f(v_2)dv_2 \tag{A.23}$$

(ii) $v_2 \in [\underline{r_1}, v_1]$: Bidder 2's equilibrium bid is $(v_2 + d)$. Bidder 1 has incentive to win the target at $(v_2 + d)$. Then bidder 1's expected payoff is:

$$\int_{\underline{r_1}}^{v_1} \left[v_1 - \Big(1 - \tau(\underline{r_1})\Big)(v_2 + d)\right] f(v_2)dv_2 \tag{A.24}$$

(iii) $v_2 \in [v_1, 1]$: Bidder 2's equilibrium bid is $(v_2 + d)$. Bidder 1 has incentive to lose the target at his bid price $(v_2 + d - \epsilon)$. Then bidder 1's expected payoff is:

$$\int_{v_1}^{1} \Big[\tau(\underline{r_1})(v_2 + d - \epsilon) - d\Big] f(v_2) dv_2 \tag{A.25}$$

By the uniform distribution of v_2, we have

$$\underline{E\Pi}_1 = \int_0^{\underline{r_1}} \Big[v_1 - \big(1 - \tau(\underline{r_1})\big)(\underline{r_1} + d - \epsilon)\Big] dv_2 + \int_{\underline{r_1}}^{v_1} \Big[v_1 - \Big(1 - \tau(\underline{r_1})\Big)(v_2 + d)\Big] dv_2 + \int_{v_1}^{1} \Big[\tau(\underline{r_1})(v_2 + d - \epsilon) - d\Big] dv_2 \tag{A.26}$$

□

Given bidder 2 along the equilibrium path, three scenarios are considered to compute $\overline{E\Pi}_1$ by reporting $\bar{r}_1$.

(i) $v_2 \in [0, v_1]$: Bidder 2's equilibrium bid price is $(\bar{r}_1 + d - \epsilon)$. Bidder 1 has incentive to lose the target by bidding $(\bar{r}_1 + d - 2\epsilon)$. Then her expected payoff is

$$\int_0^{v_1} \Big[\tau(\bar{r}_1)(\bar{r}_1 + d - 2\epsilon) - d\Big] f(v_2) dv_2 \tag{A.27}$$

(ii) $v_2 \in [v_1, \bar{r}_1]$: Bidder 2's equilibrium bid price is $(\bar{r}_1 + d - \epsilon)$. Bidder 1 has incentive to lose the target by bidding $(\bar{r}_1 + d - 2\epsilon)$. Then her expected payoff is:

$$\int_{v_1}^{\bar{r}_1} \Big[\tau(\bar{r}_1)(\bar{r}_1 + d - 2\epsilon) - d\Big] f(v_2) dv_2 \tag{A.28}$$

(iii) $v_2 \in [\bar{r}_1, 1]$: Bidder 2's equilibrium bid is $(v_2 + d)$. Bidder 1's best response is to lose with her bid price $(v_2 + d - \epsilon)$. Then her expected payoff is:

$$\int_{\bar{r}_1}^{1} \Big[\tau(\bar{r}_1)(v_2 + d - \epsilon) - d\Big] f(v_2) dv_2 \tag{A.29}$$

By the uniform distribution of v_2, we have

$$\overline{E\Pi}_1 = \int_0^{\bar{r}_1} \Big[\tau(\bar{r}_1)(\bar{r}_1 + d - 2\epsilon) - d\Big] dv_2 + \int_{\bar{r}_1}^{1} \Big[\tau(\bar{r}_1)(v_2 + d - \epsilon) - d\Big] dv_2 \tag{A.30}$$

□

A.3 Proof of Theorem 2.3.1

By the definition of ϵ and Lemma 2.3.1, we have

$$\overline{E\Pi}_1 = \int_0^{\overline{r}_1} \Big[\tau(\overline{r}_1)(\overline{r}_1 + d) - d\Big] dv_2 + \int_{\overline{r}_1}^1 \Big[\tau(\overline{r}_1)(v_2 + d) - d\Big] dv_2 \tag{A.31}$$

Then

$$\overline{E\Pi}_1 = \Big[\tau(\overline{r}_1)(\overline{r}_1 + d) - d\Big]\overline{r}_1 + \int_{\overline{r}_1}^1 \tau(\overline{r}_1) v_2 dv_2 + \int_{\overline{r}_1}^1 \Big[\tau(\overline{r}_1)d - d\Big] dv_2 \tag{A.32}$$

That is

$$\overline{E\Pi}_1 = \tau(\overline{r}_1)(\overline{r}_1 + d)\overline{r}_1 - d\overline{r}_1 + \frac{1}{2}\tau(\overline{r}_1)(1 - \overline{r}_1^2) + \Big[\tau(\overline{r}_1)d - d\Big](1 - \overline{r}_1) \tag{A.33}$$

Then

$$\begin{aligned}\overline{E\Pi}_1 = \tau(\overline{r}_1)\overline{r}_1^2 + \tau(\overline{r}_1)\overline{r}_1 d - d\overline{r}_1 + \frac{1}{2}\tau(\overline{r}_1) - \frac{1}{2}\tau(\overline{r}_1)\overline{r}_1^2 + \tau(\overline{r}_1)d \\ -\tau(\overline{r}_1)\overline{r}_1 d - d + d\overline{r}_1\end{aligned} \tag{A.34}$$

That is

$$\overline{E\Pi}_1 = \frac{1}{2}\tau(\overline{r}_1) + \frac{1}{2}\tau(\overline{r}_1)\overline{r}_1^2 + \tau(\overline{r}_1)d - d \tag{A.35}$$

By finding the partial derivative of the function $\overline{E\Pi}_1$ with respect to $\overline{r}_1$, we have

$$\frac{\partial \overline{E\Pi}_1}{\partial \overline{r}_1} = \tau(\overline{r}_1)\overline{r}_1 + \frac{1}{2}\tau'(\overline{r}_1) + \frac{1}{2}\tau'(\overline{r}_1)\overline{r}_1^2 + \tau'(\overline{r}_1)d \tag{A.36}$$

By evaluating $\frac{\partial \overline{E\Pi}_1}{\partial \overline{r}_1}$ at v_1, we have

$$\left.\frac{\partial \overline{E\Pi}_1}{\partial \overline{r}_1}\right|_{\overline{r}_1 = v_1} = \tau(v_1)v_1 + \frac{1}{2}\tau'(v_1) + \frac{1}{2}\tau'(v_1)v_1^2 + \tau'(v_1)d \tag{A.37}$$

By the definition of ϵ and Lemma 2.3.1, we have

$$\begin{aligned}\underline{E\Pi}_1 = \int_0^{\underline{r_1}} \Big[v_1 - \big(1 - \tau(\underline{r_1})\big)(\underline{r_1} + d)\Big] dv_2 + \int_{\underline{r_1}}^{v_1} \Big[v_1 - \Big(1 - \tau(\underline{r_1})\Big)(v_2 + d)\Big] dv_2 \\ + \int_{v_1}^1 \Big[\tau(\underline{r_1})(v_2 + d) - d\Big] dv_2\end{aligned} \tag{A.38}$$

Then

$$\underline{E\Pi}_1 = v_1\underline{r_1} - \Big[1-\tau(\underline{r_1})\Big](\underline{r_1}^2 + d\underline{r_1}) + \int_{\underline{r_1}}^{v_1}\Big[v_1 - \Big(1-\tau(\underline{r_1})\Big)(v_2+d)\Big]dv_2 + \int_{v_1}^{1}\Big[\tau(\underline{r_1})(v_2+d) - d\Big]dv_2 \qquad \text{(A.39)}$$

That is

$$\underline{E\Pi}_1 = v_1\underline{r_1} - \underline{r_1}^2 - d\underline{r_1} + d\underline{r_1}\tau(\underline{r_1}) + \tau(\underline{r_1})\underline{r_1}^2 + \int_{\underline{r_1}}^{v_1}\Big[v_1 - \Big(1-\tau(\underline{r_1})\Big)(v_2+d)\Big]dv_2 + \int_{v_1}^{1}\Big[\tau(\underline{r_1})(v_2+d) - d\Big]dv_2 \qquad \text{(A.40)}$$

By finding the partial derivative of the function $\underline{E\Pi}_1$ with respect to $\underline{r_1}$, we have

$$\frac{\partial\underline{E\Pi}_1}{\partial\underline{r_1}} = v_1 + \tau'(\underline{r}_1)\underline{r}_1^2 + 2\tau(\underline{r}_1)\underline{r}_1 + \tau'(\underline{r}_1)\underline{r}_1 d + \tau(\underline{r}_1)d - 2\underline{r}_1 - d - \Big[v_1 - \big(1-\tau(\underline{r}_1)\big)(\underline{r}_1 + d)\Big] + \int_{\underline{r}_1}^{v_1}\frac{\partial v_1 - (v_2+d) + (v_2+d)\tau(\underline{r}_1)}{\partial\underline{r}_1}dv_2 + \int_{v_1}^{1}\frac{\partial(v_2+d)\tau(\underline{r}_1) - d}{\partial\underline{r}_1}dv_2 \qquad \text{(A.41)}$$

Then

$$\frac{\partial\underline{E\Pi}_1}{\partial\underline{r_1}} = v_1 + \tau'(\underline{r}_1)\underline{r}_1^2 + 2\tau(\underline{r}_1)\underline{r}_1 + \tau'(\underline{r}_1)\underline{r}_1 d + \tau(\underline{r}_1)d - 2\underline{r}_1 - d - v_1 + \underline{r}_1 + d - \tau(\underline{r}_1)\underline{r}_1 - \tau(\underline{r}_1)d + \frac{1}{2}\tau'(\underline{r}_1)v_1^2 - \frac{1}{2}\tau'(\underline{r}_1)\underline{r}_1^2 + d(v_1 - \underline{r}_1)\tau'(\underline{r}_1) + \frac{1}{2}\tau'(\underline{r}_1) - \frac{1}{2}\tau'(\underline{r}_1)v_1^2 + d\tau'(\underline{r}_1)(1-v_1) \qquad \text{(A.42)}$$

Then

$$\frac{\partial\underline{E\Pi}_1}{\partial\underline{r_1}} = \frac{1}{2}\tau'(\underline{r}_1)\underline{r}_1^2 + \tau'(\underline{r}_1)\underline{r}_1 d + \underline{r_1}\big(\tau(\underline{r}_1) - 1\big) + \frac{1}{2}\tau'(\underline{r}_1) + d\tau'(\underline{r}_1)(1-\underline{r}_1) \qquad \text{(A.43)}$$

By evaluating $\frac{\partial\underline{E\Pi}_1}{\partial\underline{r_1}}$ at v_1, we have

$$\left.\frac{\partial\underline{E\Pi}_1}{\partial\underline{r_1}}\right|_{\underline{r}_1=v_1} = \frac{1}{2}\tau'(v_1)v_1^2 + \tau'(v_1)v_1 d + v_1\big(\tau(v_1) - 1\big) + \frac{1}{2}\tau'(v_1) + d\tau'(v_1)(1-v_1) \qquad \text{(A.44)}$$

If either $\left.\frac{\partial\overline{E\Pi}_1}{\partial\bar{r}_1}\right|_{\bar{r}_1=v_1} > 0$ or $\left.\frac{\partial\underline{E\Pi}_1}{\partial\underline{r_1}}\right|_{\underline{r_1}=v_1} < 0$ is found, bidder 1 has incentive to deviate from the perfect separating Bayesian equilibrium.

When $\tau'(v_1) > 0$, bidder 1 has incentive to deviate from the perfect separating Bayesian equilibrium (PSBE), since

$$\frac{\partial \overline{E\Pi}_1}{\partial \overline{r}_1}\Big|_{\overline{r}_1 = v_1} > 0 \tag{A.45}$$

When $\tau'(v_1) < 0$, bidder 1 has incentive to deviate from the PSBE, since

$$\frac{\partial \underline{E\Pi}_1}{\partial \underline{r_1}}\Big|_{\underline{r_1} = v_1} < 0 \tag{A.46}$$

Therefore we have shown that bidder 1 has incentive to deviate from the perfect separating Bayesian equilibrium (PSBE), given bidder 2 along the equilibrium path.

A.4 Proof of Proposition 2.3.2

Suppose there is a symmetric, increasing and differentiable equilibrium bidding strategy. That is

$$\beta(v_i, \theta_i) = b_i \qquad where \;\; i = 1, 2 \tag{A.47}$$

Then

$$\beta^{-1}(b_i, \theta_i) = v_i \qquad where \;\; i = 1, 2 \tag{A.48}$$

A guessed function (linear with two unknown parameters z and w) of equilibrium bidding strategy $\beta(\cdot)$ is set up, that is:

$$\beta(v_i, \theta_i) = zv_i + w \tag{A.49}$$

This section proves bidder's equilibrium bidding strategies under three pairs of (θ_1, θ_2). At the beginning of the second stage of the game, bidders observe a pair of (θ_1, θ_2) with three possibilities: (I) (θ_L, θ_L), (II) (θ_H, θ_H), (III) (θ_L, θ_H).

(I) (θ_L, θ_L): After Bayesian updating, it is known that $v_2 \in [0, \hat{v})$. By the uniform distribution of v_2, bidder 1's expected payoff is the following:

$$E\Pi_1 = \int_0^{v_1} \Big[v_1 - \big(1 - \theta_L\big)\beta(v_2, \theta_L)\Big] dv_2 + \int_{v_1}^{\hat{v}} \Big[\theta_L b_1 - d\Big] dv_2 \tag{A.50}$$

By imposing equation (A.48), we have

$$E\Pi_1 = \int_0^{\beta^{-1}(b_1, \theta_L)} \Big[v_1 - \big(1 - \theta_L\big)\beta(v_2, \theta_L)\Big] dv_2 + \int_{\beta^{-1}(b_1, \theta_L)}^{\hat{v}} \Big[\theta_L b_1 - d\Big] dv_2 \tag{A.51}$$

That is

$$E\Pi_1 = \int_0^{\beta^{-1}(b_1,\theta_L)} \Big[v_1 - \big(1-\theta_L\big)\beta(v_2,\theta_L)\Big]dv_2 + (\theta_L b_1 - d)\Big[\hat{v} - \beta^{-1}(b_1,\theta_L)\Big] \quad (A.52)$$

Then

$$E\Pi_1 = \int_0^{\beta^{-1}(b_1,\theta_L)} \Big[v_1 - \big(1-\theta_L\big)\beta(v_2,\theta_L)\Big]dv_2 + \theta_L b_1 \hat{v} \\ -\theta_L b_1 \beta^{-1}(b_1,\theta_L) - \hat{v}d + \beta^{-1}(b_1,\theta_L)d \quad (A.53)$$

Thus the first order condition is

$$\Big[v_1 - (1-\theta_L)\beta\Big(\beta^{-1}(b_1,\theta_L),\theta_L\Big)\Big]\frac{1}{\beta'\big(\beta^{-1}(b_1,\theta_L)\big)} + \theta_L\hat{v} \\ -\Big[\theta_L v_1 + \theta_L b_1 \frac{1}{\beta'\big(\beta^{-1}(b_1,\theta_L)\big)}\Big] + d\frac{1}{\beta'\big(\beta^{-1}(b_1,\theta_L)\big)} = 0 \quad (A.54)$$

That is

$$\Big[v_1 - (1-\theta_L)\beta(v_1,\theta_L)\Big]\frac{1}{\beta'(v_1)} + \theta_L\hat{v} - \Big[\theta_L v_1 + \theta_L b_1\frac{1}{\beta'(v_1)}\Big] + d\frac{1}{\beta'(v_1)} = 0 \quad (A.55)$$

The (A.49) tells us that

$$\beta'(v_1) = z \quad (A.56)$$

Thus the first order condition becomes

$$\Big[v_1 - (1-\theta_L)\beta(v_1,\theta_L)\Big]\frac{1}{z} + \theta_L\hat{v} - \Big[\theta_L v_1 + \theta_L b_1\frac{1}{z}\Big] + d\frac{1}{z} = 0 \quad (A.57)$$

Then

$$b_1 = (1 - z\theta_L)v_1 + z\theta_L\hat{v} + d \quad (A.58)$$

By (A.49), we have

$$z = 1 - z\theta_L \quad (A.59)$$

Hence

$$z = \frac{1}{1+\theta_L} \quad (A.60)$$

By (A.49) and (A.59), we get

$$w = z\theta_L\hat{v} + d \quad (A.61)$$

Since (A.60), it follows that

$$w = \frac{\theta_L\hat{v}}{1+\theta_L} + d \quad (A.62)$$

Thus bidder 1's optimal bidding strategy is the following:

$$\beta(v_1, \theta_L) = \frac{1}{1+\theta_L} v_1 + \frac{\theta_L \hat{v}}{1+\theta_L} + d \tag{A.63}$$

By repeating this process, we can get bidder 2's optimal bidding strategy. That is

$$\beta(v_2, \theta_L) = \frac{1}{1+\theta_L} v_2 + \frac{\theta_L \hat{v}}{1+\theta_L} + d \tag{A.64}$$

□

(II) (θ_H, θ_H): After Bayesian updating, it is known that $v_2 \in (\hat{v}, 1]$. By the uniform distribution of v_2, bidder 1's expected payoff is the following:

$$E\Pi_1 = \int_{\hat{v}}^{v_1} \Big[v_1 - \big(1-\theta_H\big)\beta(v_2, \theta_H)\Big] dv_2 + \int_{v_1}^{1} \Big[\theta_H b_1 - d\Big] dv_2 \tag{A.65}$$

By imposing inverse function (A.48), we have

$$E\Pi_1 = \int_{\hat{v}}^{\beta^{-1}(b_1,\theta_H)} \Big[v_1 - \big(1-\theta_H\big)\beta(v_2, \theta_H)\Big] dv_2 + \int_{\beta^{-1}(b_1,\theta_H)}^{1} \Big[\theta_H b_1 - d\Big] dv_2 \tag{A.66}$$

That is

$$E\Pi_1 = \int_{\hat{v}}^{\beta^{-1}(b_1,\theta_H)} \Big[v_1 - \big(1-\theta_H\big)\beta(v_2, \theta_H)\Big] dv_2 + (\theta_H b_1 - d)\Big[1 - \beta^{-1}(b_1, \theta_H)\Big] \tag{A.67}$$

Then

$$E\Pi_1 = \int_{\hat{v}}^{\beta^{-1}(b_1,\theta_H)} \Big[v_1 - \big(1-\theta_H\big)\beta(v_2, \theta_H)\Big] dv_2 + \theta_H b_1 - \theta_H b_1 \beta^{-1}(b_1, \theta_H) - d + d\beta^{-1}(b_1, \theta_H) \tag{A.68}$$

Then the first order condition is the following:

$$\Big[v_1 - (1-\theta_H)\beta\Big(\beta^{-1}(b_1,\theta_H), \theta_H\Big)\Big] \frac{1}{\beta'\big(\beta^{-1}(b_1,\theta_H)\big)} + \theta_H - \theta_H \beta^{-1}(b_1, \theta_H) - \theta_H b_1 \frac{1}{\beta'\big(\beta^{-1}(b_1,\theta_H)\big)} + d\frac{1}{\beta'\big(\beta^{-1}(b_1,\theta_H)\big)} = 0 \tag{A.69}$$

That is

$$\Big[v_1 - (1-\theta_H)\beta(v_1, \theta_H)\Big] \frac{1}{\beta'(v_1)} + \theta_H - \theta_H \beta^{-1}(b_1, \theta_H) - \theta_H b_1 \frac{1}{\beta'(v_1)} + d\frac{1}{\beta'(v_1)} = 0 \tag{A.70}$$

By applying (A.48) and (A.49), we have

$$\Big[v_1 - (1-\theta_H)b_1\Big]\frac{1}{z} + \theta_H(1-v_1) - \theta_H b_1\frac{1}{z} + d\frac{1}{z} = 0 \tag{A.71}$$

Then

$$b_1 = (1 - z\theta_H)v_1 + z\theta_H + d \tag{A.72}$$

By (A.49), we have

$$z = 1 - z\theta_H \tag{A.73}$$

Then

$$z = \frac{1}{1+\theta_H} \tag{A.74}$$

And by (A.49), we have

$$w = z\theta_H + d \tag{A.75}$$

Since (A.74), it follows that

$$w = \frac{\theta_H}{1+\theta_H} + d \tag{A.76}$$

Therefore, bidder 1's optimal bidding strategy is:

$$\beta(v_1, \theta_H) = \frac{1}{1+\theta_H}v_1 + \frac{\theta_H}{1+\theta_H} + d \tag{A.77}$$

By repeating this process, we can get bidder 2's optimal bidding strategy. That is

$$\beta(v_2, \theta_H) = \frac{1}{1+\theta_H}v_2 + \frac{\theta_H}{1+\theta_H} + d \tag{A.78}$$

□

(III) (θ_L, θ_H): Let Π_i $(i = 1, 2)$ denote each bidder's equilibrium payoff. At equilibrium, bidder 1 loses the target and sells out her shareholdings at her bid $(\hat{v} + d - \epsilon)$. Meanwhile, bidder 2 wins the target at $(\hat{v} + d - \epsilon)$ per share. So that

$$\begin{cases} \Pi_1 = \theta_L(\hat{v} + d - \epsilon) - d \\ \Pi_2 = v_2 - (1-\theta_H)(\hat{v} + d - \epsilon) \end{cases} \tag{A.79}$$

Let bidder 2 deviate by bidding higher than $(\hat{v} + d)$, given bidder 1 along the equilibrium path. That is

$$\overline{b}_2 > \hat{v} + d \tag{A.80}$$

Bidder 2 is still the winner. Thus his off-equilibrium payoff is the following

$$\overline{\Pi}_2 = v_2 - (1-\theta_H)(\hat{v} + d - \epsilon) \tag{A.81}$$

Then (A.79), (A.81) tells us that

$$\Pi_2 = \overline{\Pi}_2 \tag{A.82}$$

Let bidder 2 deviate by reducing his bid, given bidder 1 along the equilibrium path. That is

$$\underline{b}_2 < \hat{v} + d - \epsilon \tag{A.83}$$

Bidder 2 loses the target and then sells out his shareholdings at $\underline{b}_2$. Thus his off-equilibrium payoff is

$$\underline{\Pi}_2 = \theta_H \underline{b}_2 - d \tag{A.84}$$

Then, we have

$$\Pi_2 - \underline{\Pi}_2 = v_2 + (\theta_H - 1)(\hat{v} + d - \epsilon) - \theta_H \underline{b}_2 + d \tag{A.85}$$

That is

$$\Pi_2 - \underline{\Pi}_2 = (v_2 - \hat{v}) + \theta_H(\hat{v} + d - \epsilon - \underline{b}_2) + \epsilon \tag{A.86}$$

Since $v_2 > \hat{v}$ and (A.83), we get

$$\Pi_2 - \underline{\Pi}_2 > 0 \tag{A.87}$$

Let bidder 1 deviate by reducing her bid, given bidder 2 along the equilibrium path. That is

$$\underline{b}_1 < \hat{v} + d - \epsilon \tag{A.88}$$

Bidder 1 is still the loser. Thus his off-equilibrium payoff is

$$\underline{\Pi}_1 = \theta_L \underline{b}_1 - d \tag{A.89}$$

Then

$$\Pi_1 - \underline{\Pi}_1 = \theta_L(\hat{v} + d - \epsilon) - d - \theta_L \underline{b}_1 + d \tag{A.90}$$

That is

$$\Pi_1 - \underline{\Pi}_1 = \theta_L(\hat{v} + d - \epsilon - \underline{b}_1) \tag{A.91}$$

Since (A.88), we have

$$\Pi_1 - \underline{\Pi}_1 > 0 \tag{A.92}$$

Let bidder 1 deviate by increasing her bid, given bidder 2 along the equilibrium path. That is

$$\overline{b}_1 > \hat{v} + d \tag{A.93}$$

Bidder 1 becomes the winner. Then her off-equilibrium payoff is

$$\overline{\Pi}_1 = v_1 - (1-\theta_L)(\hat{v}+d) \tag{A.94}$$

By the definition of ϵ, we have

$$\Pi_1 - \overline{\Pi}_1 = \theta_L(\hat{v}+d) - d - v_1 + (1-\theta_L)(\hat{v}+d) \tag{A.95}$$

That is

$$\Pi_1 - \overline{\Pi}_1 = \hat{v} - v_1 \tag{A.96}$$

Since $v_1 < \hat{v}$, we have

$$\Pi_1 - \overline{\Pi}_1 > 0 \tag{A.97}$$

□

Therefore we have shown that each bidder's optimal bidding strategies are expressed in Proposition 2.3.2.

A.5 Proof of Theorem 2.3.2

Let $E\Pi_1^{truth}$ and $E\Pi_1^{cheat}$ denote bidder 1's expected payoffs when she truthfully and fraudulently claims her value, respectively.

Given bidder 2 along the equilibrium path and suppose $v_1 \in [0, \hat{v})$, by the optimal bidding strategies in Proposition 2.3.2 and uniform distribution of v_2, bidder 1's expected payoff at equilibrium is the following:

$$\begin{aligned} E\Pi_1^{truth} &= \int_0^{v_1} \Big[v_1 - \big(1-\theta_L\big)\big(\frac{v_2}{1+\theta_L} + \frac{\theta_L \hat{v}}{1+\theta_L} + d\big)\Big] dv_2 \\ &+ \int_{v_1}^{\hat{v}} \Big[\theta_L\big(\frac{v_1}{1+\theta_L} + \frac{\theta_L \hat{v}}{1+\theta_L} + d\big) - d\Big] dv_2 + \int_{\hat{v}}^{1} \Big[\theta_L(\hat{v} + d - \epsilon) - d\Big] dv_2 \end{aligned} \tag{A.98}$$

By the definition of ϵ, let ϵ be zero. Then by simplifying (A.98), we have

$$\begin{aligned} E\Pi_1^{truth} &= \Big[v_1 - \frac{\theta_L(1-\theta_L)}{1+\theta_L}\hat{v} - (1-\theta_L)d\Big] v_1 + \frac{1-\theta_L}{2(1+\theta_L)} v_1^2 + \Big[\frac{\theta_L}{1+\theta_L} v_1 \\ &+ \frac{\theta_L^2}{1+\theta_L}\hat{v} + \theta_L d\Big](\hat{v} - v_1) - d(\hat{v} - v_1) + (\theta_L \hat{v} + \theta_L d)(1-\hat{v}) - d(1-\hat{v}) \end{aligned} \tag{A.99}$$

Then

$$E\Pi_1^{truth} = \Big[1 - \frac{1-\theta_L}{2(1+\theta_L)} - \frac{\theta_L}{1+\theta_L}\Big]v_1^2 + \Big[-\frac{\theta_L(1-\theta_L)}{1+\theta_L} + \frac{\theta_L}{1+\theta_L} - \frac{\theta_L^2}{1+\theta_L}\Big]\hat{v}v_1 + \Big[-v_1(1-\theta_L) + \theta_L(\hat{v}-v_1) - (\hat{v}-v_1) + \theta_L(1-\hat{v}) - (1-\hat{v})\Big]d + \theta_L\hat{v} \tag{A.100}$$

Thus

$$E\Pi_1^{truth} = \frac{1}{2}v_1^2 + \Big[\frac{-\theta_L}{1+\theta_L}\hat{v}^2 + \theta_L\hat{v} + (\theta_L - 1)d\Big] \tag{A.101}$$

Let bidder 1 fraudulently claim his private value through $\theta_1 = \theta_H$, given bidder 2 along the equilibrium path. After Bayesian updating, bidder 2 believes that $v_1 \in (\hat{v}, 1]$. If $v_2 \in [0, \hat{v})$, bidder 2's optimal bid is $(\hat{v} + d - \epsilon)$. By foreseeing this, bidder 1 prefers to lose the target at $(\hat{v} + d - 2\epsilon)$. If $v_2 \in (\hat{v}, 1]$, bidder 2's optimal bid is $\frac{v_2}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} + d$, which is higher than $(\hat{v} + d)$. For foreseeing this, bidder 1 prefers to lose the target at $(\hat{v} + d)$. Then, by the uniform distribution of v_2, bidder 1's off-equilibrium expected payoff denoted by $E\Pi_1^{cheat}$ can be expressed as:

$$E\Pi_1^{cheat} = \int_0^{\hat{v}} \Big[\theta_H(\hat{v} + d - 2\epsilon) - d\Big]dv_2 + \int_{\hat{v}}^1 \Big[\theta_H(\hat{v} + d) - d\Big]dv_2 \tag{A.102}$$

By the definition of ϵ, we have

$$E\Pi_1^{cheat} = \theta_H\hat{v} + d(\theta_H - 1) \tag{A.103}$$

Then

$$E\Pi_1^{truth} - E\Pi_1^{cheat} = \frac{1}{2}v_1^2 + \Big[\frac{-\theta_L}{1+\theta_L}\hat{v}^2 + \theta_L\hat{v} + (\theta_L - 1)d\Big] - \theta_H\hat{v} - d(\theta_H - 1) \tag{A.104}$$

That is

$$E\Pi_1^{truth} - E\Pi_1^{cheat} = \frac{1}{2}v_1^2 + \Big[\frac{-\theta_L}{1+\theta_L}\hat{v}^2 - (\theta_H - \theta_L)\hat{v} + (\theta_L - \theta_H)d\Big] \tag{A.105}$$

When $v_1 = 0$, let y be $E\Pi_1^{truth} - E\Pi_1^{cheat}$. Then the quadratic function y can be expressed as the following:

$$y = \frac{-\theta_L}{1+\theta_L}\hat{v}^2 - (\theta_H - \theta_L)\hat{v} + (\theta_L - \theta_H)d \tag{A.106}$$

The axis of symmetry of y, hence, is:

$$\frac{-(\theta_H - \theta_L)}{-2\frac{-\theta_L}{1+\theta_L}} < 0 \tag{A.107}$$

When the discriminant of quadratic function y is positive, together with $(A.107)$, quadratic polynomial y has two negative roots, since

$$\frac{\theta_L - \theta_H}{\frac{-\theta_H}{1+\theta_L}} > 0 \tag{A.108}$$

According to above analysis on y's graph, it is obvious that, for $v_1 = 0$, we have

$$E\Pi_1^{truth} - E\Pi_1^{cheat} < 0 \tag{A.109}$$

When the discriminant of quadratic function y is non-positive, for $v_1 = 0$, we have

$$E\Pi_1^{truth} - E\Pi_1^{cheat} < 0 \tag{A.110}$$

Therefore, bidder 1 with zero valuation always has incentive to deviate from this partial pooling equilibrium.

A.6 Proof of Proposition 2.3.3

Let Π_i $(i = 1, 2)$ denote bidder i's equilibrium payoff. Let $\underline{b_2}$ be bidder 2's bid price, where $\underline{b_2} < b_2$. Let $\underline{\Pi_2}$ be bidder 2's off-equilibrium payoff. For bidder 1 without any toehold, it is known that truthful bidding is weakly dominate strategy. By observing $(0, \theta_H)$, bidder 1 loses the target. So that

$$\Pi_1 = 0 \tag{A.111}$$

Given bidder 1 along the equilibrium path, bidder 2 is the winner at bid price v_1. So that

$$\Pi_2 = v_2 + (1 - \theta_H)v_1 \tag{A.112}$$

Suppose bidder 2 deviate by reducing his bid, given bidder 1 along the equilibrium path. That is

$$\underline{b_2} < b_2 \tag{A.113}$$

By the range of b_2, we know that

$$\underline{b_2} < \hat{v} \tag{A.114}$$

If $\underline{b_2} < v_1$, bidder 2 loses the target and his payoff becomes

$$\underline{\Pi_2} = \theta_H \underline{b_2} \tag{A.115}$$

Then

$$\Pi_2 - \underline{\Pi}_2 = v_2 - (1-\theta_H)v_1 - \theta_H \underline{b_2} \tag{A.116}$$

That is

$$\Pi_2 - \underline{\Pi}_2 = (v_2 - v_1) + \theta_H(v_1 - \underline{b_2}) \tag{A.117}$$

So that

$$\Pi_2 - \underline{\Pi}_2 > 0 \tag{A.118}$$

If $\underline{b_2} > v_1$, bidder 2 is still the winner and his payoff is

$$\underline{\Pi}_2 = v_2 + (1-\theta_H)v_1 \tag{A.119}$$

So that

$$\Pi_2 = \underline{\Pi}_2 \tag{A.120}$$

Then, by reducing bid price, bidder 2 reduces his payoff with some probabilities. Therefore, bidder 2 has no incentive to deviate from the equilibrium bidding strategy.

A.7 Proof of Theorem 2.3.3

Let $E\Pi_1^{truth}$ and $E\Pi_1^{cheat}$ denote bidder 1's expected payoffs when she truthfully and fraudulently claims her value, respectively. This section has two parts indexed as (i) and (ii) . Part (i) proves that, given bidder 2 along the equilibrium path and $v_1 \in (\hat{v}, 1]$, there is no incentive for bidder 1 to deviate from the equilibrium. Part (ii) proves that, given bidder 2 along the equilibrium path and $v_1 \in [0, \hat{v})$, bidder 1 has no incentive to deviate.

(i): Given bidder 2 along the equilibrium path and suppose $v_1 \in (\hat{v}, 1]$, by the uniform distribution of v_2, bidder 1's expected payoff at equilibrium is the following:

$$\begin{aligned} E\Pi_1^{truth} = &\int_0^{\hat{v}} \Big[v_1 - (1-\theta_H)v_2\Big]dv_2 + \int_{\hat{v}}^{v_1} \Big[v_1 - (1-\theta_H)\big(\frac{v_2}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} + d\big)\Big]dv_2 \\ &+ \int_{v_1}^{1} \Big[\theta_H\big(\frac{v_1}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} + d\big) - d\Big]dv_2 \end{aligned} \tag{A.121}$$

Then

$$\begin{aligned} E\Pi_1^{truth} &= \hat{v}v_1 - (\frac{1-\theta_H}{2})\hat{v}^2 + \Big[v_1 - \frac{\theta_H(1-\theta_H)}{1+\theta_H} - (1-\theta_H)d\Big](v_1 - \hat{v}) \\ -(\frac{1-\theta_H}{2(1+\theta_H)})(v_1^2 - \hat{v}^2) &+ \Big[\frac{\theta_H}{1+\theta_H}v_1 + \frac{\theta_H^2}{1+\theta_H} + \theta_H d\Big](1-v_1) - d(1-v_1) \end{aligned} \tag{A.122}$$

That is

$$E\Pi_1^{truth} = \Big[1 - \frac{1-\theta_H}{2(1+\theta_H)} - \frac{\theta_H}{1+\theta_H}\Big]v_1^2 + \Big[-\frac{1-\theta_H}{2} + \frac{1-\theta_H}{2(1+\theta_H)}\Big]\hat{v}^2 + \Big[-\frac{\theta_H(1-\theta_H)}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} - \frac{\theta_H^2}{1+\theta_H}\Big]v_1 + \Big[-(1-\theta_H)(v_1-\hat{v}) + \theta_H(1-v_1) - (1-v_1)\Big]d + \frac{\theta_H^2}{1+\theta_H} + \frac{\theta_H(1-\theta_H)}{1+\theta_H}\hat{v} \tag{A.123}$$

Then

$$E\Pi_1^{truth} = \frac{1}{2}v_1^2 + \Big[\frac{\theta_H^2-\theta_H}{2(1+\theta_H)}\Big]\hat{v}^2 + \Big[\frac{\theta_H(1-\theta_H)}{1+\theta_H} + (1-\theta_H)d\Big]\hat{v} + \Big[\frac{\theta_H^2}{1+\theta_H} + d(\theta_H-1)\Big] \tag{A.124}$$

Let bidder 1 fraudulently claim her private value through $\theta_1 = 0$, given bidder 2 along the equilibrium path. After Bayesian updating, bidder 2 believes that $v_1 \in [0, \hat{v})$. If $v_2 \in [0, \hat{v})$, bidder 2's optimal bid is v_2. By foreseeing this, bidder 1 prefers to win the target at v_2. If $v_2 \in (\hat{v}, 1]$, by simplifying calculation, Proposition 2.3.3 tells us that bidder 1's best response can be v_2. Then by the uniform distribution of v_2, bidder 1's off-equilibrium expected payoff denoted by $E\Pi_1^{cheat}$ can be expressed as:

$$E\Pi_1^{cheat} = \int_0^{\hat{v}} \Big[v_1 - v_2\Big]dv_2 + \int_{\hat{v}}^{v_1} \Big[v_1 - v_2\Big]dv_2 \tag{A.125}$$

Then

$$E\Pi_1^{cheat} = \frac{1}{2}v_1^2 \tag{A.126}$$

Let y be $E\Pi_1^{truth} - E\Pi_1^{cheat}$. So that

$$y = \Big[\frac{\theta_H^2-\theta_H}{2(1+\theta_H)}\Big]\hat{v}^2 + \Big[\frac{\theta_H(1-\theta_H)}{1+\theta_H} + (1-\theta_H)d\Big]\hat{v} + \Big[\frac{\theta_H^2}{1+\theta_H} + d(\theta_H-1)\Big] \tag{A.127}$$

Let

$$d \leq \frac{\theta_H^2}{1-\theta_H^2} \tag{A.128}$$

That is

$$\theta_H^2 \geq d(1-\theta_H^2) \tag{A.129}$$

Then, we have

$$\theta_H^2 \geq d(1+\theta_H)(1-\theta_H) \tag{A.130}$$

Then

$$\frac{\theta_H^2}{1+\theta_H} + d(\theta_H - 1) \geq 0 \tag{A.131}$$

Thus discriminant of quadratic function y is positive, since

$$\Delta = \Big[\frac{\theta_H(1-\theta_H)}{1+\theta_H} + (1-\theta_H)d\Big]^2 + \frac{2(\theta_H - \theta_H^2)}{1+\theta_H}\Big[\frac{\theta_H^2}{1+\theta_H} + d(\theta_H - 1)\Big] > 0 \tag{A.132}$$

The axis of symmetry of function y is

$$-\frac{\frac{\theta_H(1-\theta_H)}{1+\theta_H} + (1-\theta_H)d}{2\frac{\theta_H^2-\theta_H}{2(1+\theta_H)}} > 0 \tag{A.133}$$

In addition, the quadratic polynomial y has two real roots – one positive and one negative, since

$$\frac{\frac{\theta_H^2}{1+\theta_H} + d(\theta_H - 1)}{\frac{\theta_H^2-\theta_H}{2(1+\theta_H)}} < 0 \tag{A.134}$$

Based on above analysis on quadratic function y's graph, it is obvious that $(E\Pi_1^{truth} - E\Pi_1^{cheat})$ is positive, given the condition (A.128).

(ii): Given bidder 2 along the equilibrium path and suppose $v_1 \in [0, \hat{v})$, by the uniform distribution of v_2, bidder 1's equilibrium expected payoff is the following:

$$E\Pi_1^{truth} = \int_0^{v_1} (v_1 - v_2)dv_2 \tag{A.135}$$

That is

$$E\Pi_1^{truth} = \frac{1}{2}v_1^2 \tag{A.136}$$

Let bidder 1 fraudulently claim her private value through $\theta_1 = \theta_H$, given bidder 2 along the equilibrium path. After Bayesian updating, bidder 2 believes that $v_1 \in (\hat{v}, 1]$. If $v_2 \in [0, v_1]$, bidder 2's optimal bid is v_2. By foreseeing this, bidder 1 prefers to win the target at price v_2. If $v_2 \in [v_1, \hat{v})$, bidder 2's optimal bid is still v_2. But bidder 1 has incentive to lose the target by bidding at v_1. If $v_2 \in (\hat{v}, 1]$, bidder 2's optimal bid is $\frac{v_2}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} + d$, which is higher than $\hat{v}$. By foreseeing this, bidder 1 has no incentive to win the target. That is, bidder 1 bids at $\hat{v}$ to lose the game with certainty. Thus, by the uniform distribution of v_2, bidder 1's off-equilibrium expected payoff denoted by $E\Pi_1^{cheat}$ can be expressed as:

$$E\Pi_1^{cheat} = \int_0^{v_1} \Big[v_1 - (1-\theta_H)v_2\Big]dv_2 + \int_{v_1}^{\hat{v}} \Big[\theta_H v_1 - d\Big]dv_2 + \int_{\hat{v}}^{1} \Big[\theta_H \hat{v} - d\Big]dv_2 \tag{A.137}$$

Then

$$E\Pi_1^{cheat} = v_1^2 - \frac{1-\theta_H}{2}v_1^2 + \theta_H v_1 \hat{v} - \theta_H v_1^2 - d(\hat{v} - v_1) + \theta_H \hat{v} - \theta_H \hat{v}^2 - d(1-\hat{v}) \tag{A.138}$$

That is

$$E\Pi_1^{cheat} = \big(\frac{1-\theta_H}{2}\big)v_1^2 + \big(\theta_H \hat{v} + d\big)v_1 - \theta_H \hat{v}^2 + \theta_H \hat{v} - d \tag{A.139}$$

Let y be $E\Pi_1^{truth} - E\Pi_1^{cheat}$. Then we have

$$y = \frac{\theta_H}{2}v_1^2 - (\theta_H \hat{v} + d)v_1 + \big(\theta_H \hat{v}^2 - \theta_H \hat{v} + d\big) \tag{A.140}$$

The axis of symmetry of quadratic function y is positive, since

$$-\frac{-(\theta_H \hat{v} + d)}{\theta_H} > 0 \tag{A.141}$$

Then, for any $v_1 \in [0,1]$, if $y(v_1)$ is positive, its discriminant Δ must be non-positive. That is

$$\Delta = (\theta_H \hat{v} + d)^2 - 2\theta_H(\theta_H \hat{v}^2 - \theta_H \hat{v} + d) \leq 0 \tag{A.142}$$

Then

$$\Delta = -\theta_H^2 \hat{v}^2 + \big(2\theta_H d + 2\theta_H^2\big)\hat{v} + d^2 - 2\theta_H d \leq 0 \tag{A.143}$$

Let D be the discriminant of function Δ. That is

$$D = (2\theta_H d + 2\theta_H^2)^2 1 + 4\theta_H^2(d^2 - 2\theta_H d) \tag{A.144}$$

Then

$$D = 4\theta_H^2(2d^2 + \theta_H^2) \tag{A.145}$$

In addition, we claim that the smaller root $\hat{v}_s$ of Δ is less than 1. The proof of this claim is the following. It is obvious that

$$d^2 + \theta_H^2 > 0 \tag{A.146}$$

That is

$$2d^2 + \theta_H^2 > d^2 \tag{A.147}$$

Then

$$\sqrt{2d + \theta_H^2} > d \tag{A.148}$$

Then

$$d + \theta_H - \sqrt{2d + \theta_H^2} < \theta_H \tag{A.149}$$

That is

$$\frac{d + \theta_H - \sqrt{2d + \theta_H^2}}{\theta_H} < 1 \tag{A.150}$$

Thus, we have

$$\frac{2\theta_H(d + \theta_H) - 2\theta_H\sqrt{2d^2 + \theta_H}}{2\theta_H^2} < 1 \tag{A.151}$$

Then

$$\frac{-2\theta_H(d + \theta_H) + 2\theta_H\sqrt{2d^2 + \theta_H}}{-2\theta_H^2} < 1 \tag{A.152}$$

By quadratic formula, we have

$$\hat{v}_s < 1 \tag{A.153}$$

Thus, if $\hat{v}$ satisfies the following condition, then the output of D is always negative.

$$\hat{v} \leq \hat{v}_s \tag{A.154}$$

That is

$$\hat{v} \leq \frac{d + \theta_H - \sqrt{2d^2 + \theta_H^2}}{\theta_H} \tag{A.155}$$

Then, when condition (A.155) is satisfied, $(E\Pi_1^{truth} - E\Pi_1^{cheat})$ is positive. Since two bidders are symmetric, then we have shown that, under conditions (A.128) and (A.155), bidders have no incentive to deviate from this partial pooling Bayesian equilibrium.

A.8 Proof of Theorem 2.3.4

Let $E\Pi_1^{\theta_s}$ denote bidder 1's off-equilibrium expected payoff by purchasing θ_s, where $\theta_s < \theta_H$. This section has two parts: (*i*) and (*ii*). Part (*i*) shows that bidder 1 with valuation v_1 within the range $(\hat{v}, 1]$ has no incentive to acquire θ_s, given bidder 2 along the equilibrium path. Part (*ii*) shows that bidder 1 with valuation v_1 within the range $[0, \hat{v})$ has no incentive to acquire θ_s, given bidder 2 along the equilibrium path.

(i) Let bidder 1 with $v_1 \in (\hat{v}, 1]$ deviate from the equilibrium by acquiring θ_s. Bidder 1's off-equilibrium expected payoff is expressed as:

$$E\Pi_1^{\theta_s} = \int_0^{\hat{v}} \Big[v_1 - (1-\theta_s)v_2\Big]dv_2 + \int_{\hat{v}}^{v_1} \Big[v_1 - (1-\theta_s)\big(\frac{1}{1+\theta_H}v_2 + \frac{\theta_H}{1+\theta_H} + d\big)\Big]dv_2 + \int_{v_1}^{1} \Big[\theta_s\big(\frac{1}{1+\theta_H}v_1 + \frac{\theta_H}{1+\theta_H} + d\big) - d\Big]dv_2 \tag{A.156}$$

Recall bidder 1's equilibrium expected payoff when her valuation v_1 within $(\hat{v}, 1]$, we have

$$E\Pi_1^{truth} = \int_0^{\hat{v}} \Big[v_1 - (1-\theta_H)v_2\Big]dv_2 + \int_{\hat{v}}^{v_1} \Big[v_1 - (1-\theta_H)\big(\frac{v_2}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} + d\big)\Big]dv_2 + \int_{v_1}^{1} \Big[\theta_H\big(\frac{v_1}{1+\theta_H} + \frac{\theta_H}{1+\theta_H} + d\big) - d\Big]dv_2 \tag{A.157}$$

By comparing with the integrals of $E\Pi_1^{\theta_s}$ and $E\Pi_1^{truth}$, it is obvious that

$$E\Pi_1^{truth} > E\Pi_1^{\theta_s} \tag{A.158}$$

Then bidder 1 has no incentive to acquire θ_s, given bidder 2 along the equilibrium path.

(ii) Let bidder 1 with $v_1 \in [0, \hat{v})$ deviate from the equilibrium by acquiring θ_s. Bidder 1's off-equilibrium expected payoff is expressed as:

$$E\Pi_1^{\theta_s} = \int_0^{v_1} \Big[v_1 - (1-\theta_s)v_2\Big]dv_2 + \int_{v_1}^{\hat{v}} \Big[\theta_s v_1 - d\Big]dv_2 + \int_{\hat{v}}^{1} \Big[\theta_s \hat{v} - d\Big]dv_2 \tag{A.159}$$

Recall bidder 1's off-equilibrium expected payoff when her valuation v_1 within $[0, \hat{v})$, we have

$$E\Pi_1^{cheat} = \int_0^{v_1} \Big[v_1 - (1-\theta_H)v_2\Big]dv_2 + \int_{v_1}^{\hat{v}} \Big[\theta_H v_1 - d\Big]dv_2 + \int_{\hat{v}}^{1} \Big[\theta_H \hat{v} - d\Big]dv_2 \tag{A.160}$$

By comparing with integrals of $E\Pi_1^{\theta_s}$ and $E\Pi_1^{cheat}$, it is obvious that

$$E\Pi_1^{\theta_s} < E\Pi_1^{cheat} \tag{A.161}$$

Theorem 2.3.3 tells us that, for $v_1 \in [0, \hat{v})$, if condition (A.155) satisfied, $E\Pi_1^{truth}$ is strictly higher than $E\Pi_1^{cheat}$. Thus, we have

$$E\Pi_1^{\theta_s} < E\Pi_1^{cheat} < E\Pi_1^{truth} \tag{A.162}$$

Therefore, we have shown that bidder 1 has no incentive to deviate from the equilibrium by acquiring θ_s, given bidder 2 along the equilibrium path. Because two bidders are symmetric, two bidders have no incentives to deviate from the equilibrium.

Appendix B

Proofs in Chapter 3

B.1 Proof of Proposition 3.3.1

This section shows the voting mechanism is Bayesian incentive compatible if and only if condition (3.14) and (3.15) are satisfied. The proof has two parts: (I) and (II). (I) shows that if the voting mechanism is Bayesian incentive compatible, then condition (3.14) and (3.15) are satisfied. (II) shows that if condition (3.14) and (3.15) are satisfied, then the voting mechanism is Bayesian incentive compatible.

(I) Recall the definition expected payoffs of agent $i \in N_A$: $\pi_i(v_i)$ and $\tilde{\pi}_i(v_i, v_i')$. That is

$$\pi_i(v_i) = E_{v_{-i}}[v_i P_A(v_i, v_{-i}) - t_i(v_i, v_{-i})] \tag{B.1}$$

$$\tilde{\pi}_i(v_i, v_i') = E_{v_{-i}}[v_i P_A(v_i', v_{-i}) - t_i(v_i', v_{-i})] \tag{B.2}$$

Let $\pi_i(v_i')$ be i's expected payoff when he truthfully reports his valuation v_i', given remaining players' truthful announcements. That is

$$\pi_i(v_i') = E_{v_{-i}}[v_i' P_A(v_i', v_{-i}) - t_i(v_i', v_{-i})] \tag{B.3}$$

And let $\tilde{\pi}_i(v_i', v_i)$ be i's expected payoff when he cheats his valuation by reporting v_i, given remaining players' truthful announcements.

$$\tilde{\pi}_i(v_i', v_i) = E_{v_{-i}}[v_i' P_A(v_i, v_{-i}) - t_i(v_i, v_{-i})] \tag{B.4}$$

We can change the express of $\tilde{\pi}_i(v_i, v_i')$ by adding and subtracting item $E_{v_{-i}} v_i' P_A(v_i', v_{-i})$. That is

$$\tilde{\pi}_i(v_i, v_i') = E_{v_{-i}}[v_i' P_A(v_i', v_{-i}) - t_i(v_i', v_{-i}) + v_i P_A(v_i', v_{-i}) - v_i' P_A(v_i', v_{-i})] \tag{B.5}$$

Then

$$\tilde{\pi}_i(v_i, v_i') = \pi_i(v_i') + (v_i - v_i') E_{v_{-i}} P_A(v_i', v_{-i}) \tag{B.6}$$

Similarly, we can change the express of $\tilde{\pi}_i(v_i', v_i)$ by adding and subtracting item $E_{v_{-i}} v_i P_A(v_i, v_{-i})$. Then we get

$$\tilde{\pi}_i(v_i', v_i) = \pi_i(v_i) + (v_i - v_i') E_{v_{-i}} P_A(v_i, v_{-i}) \tag{B.7}$$

By Bayesian incentive compatibility, we have

$$\pi_i(v_i) \geq \tilde{\pi}_i(v_i, v_i') \tag{B.8}$$

and

$$\pi_i(v_i') \geq \tilde{\pi}_i(v_i', v_i) \tag{B.9}$$

(B.6) and (B.8) imply that

$$\pi_i(v_i) \geq \pi_i(v_i') + (v_i - v_i') E_{v_{-i}} P_A(v_i', v_{-i}) \tag{B.10}$$

Then

$$E_{v_{-i}} P_A(v_i', v_{-i}) \geq \frac{\pi_i(v_i') - \pi_i(v_i)}{v_i' - v_i} \tag{B.11}$$

(B.7) and (B.9) imply that

$$\pi_i(v_i') \geq \pi_i(v_i) + (v_i - v_i') E_{v_{-i}} P_A(v_i, v_{-i}) \tag{B.12}$$

Then

$$\frac{\pi_i(v_i') - \pi_i(v_i)}{v_i' - v_i} \geq E_{v_{-i}} P_A(v_i, v_{-i}) \tag{B.13}$$

Then (B.11) and (B.13) together implies that

$$E_{v_{-i}} P_A(v_i', v_{-i}) \geq \frac{\pi_i(v_i') - \pi_i(v_i)}{v_i' - v_i} \geq E_{v_{-i}} P_A(v_i, v_{-i}) \tag{B.14}$$

Suppose $v_i' > v_i$. Then (B.14) indicates $E_{v_{-i}} P_A(v, v_{-i})$ is a nondecreasing function. That is

$$\frac{\partial}{\partial v_i} E_{v_{-i}} P_A(v_i, v_{-i}) \geq 0 \tag{B.15}$$

In addition, letting $v_i' \to v_i$, (B.14) also implies that

$$\frac{d}{dv_i}\pi_i(v_i) = E_{v_{-i}} P_A(v_i, v_{-i}) \tag{B.16}$$

Therefore, we have shown that the Bayesian incentive compatibility implies (3.14). We can duplicate above proof for agent j in group N_B.

(II) Let agent i's true valuation be v_i, but he announces v_i'. Suppose $v_i > v_i'$. It is obvious that

$$\pi_i(v_i) = \pi_i(0) + \pi_i(s)\Big|_0^{v_i} \tag{B.17}$$

and

$$\pi_i(v_i') = \pi_i(0) + \pi_i(s)\Big|_0^{v_i'} \tag{B.18}$$

That is

$$\pi_i(v_i) = \pi_i(0) + \int_0^{v_i} \frac{d}{ds}\pi_i(s)ds \tag{B.19}$$

and

$$\pi_i(v_i') = \pi_i(0) + \int_0^{v_i'} \frac{d}{ds}\pi_i(s)ds \tag{B.20}$$

In (3.14), (i) implies that

$$\pi_i(v_i) = \pi_i(0) + \int_0^{v_i} E_{v_{-i}} P_A(s, v_{-i})ds \tag{B.21}$$

and

$$\pi_i(v_i') = \pi_i(0) + \int_0^{v_i'} E_{v_{-i}} P_A(s, v_{-i})\, ds \tag{B.22}$$

Given $v_i > v_i'$, therefore, we have

$$\pi_i(v_i) - \pi_i(v_i') = \int_{v_i'}^{v_i} E_{v_{-i}} P_A(s, v_{-i})ds \tag{B.23}$$

Since $E_{v_{-i}} P_A(v, v_{-i})$ is a nondecreasing function (shown by (ii) in 3.14)

$$\pi_i(v_i) - \pi_i(v_i') \geq \int_{v_i'}^{v_i} E_{v_{-i}} P_A(v_i', v_{-i})ds \tag{B.24}$$

That is

$$\pi_i(v_i) - \pi_i(v_i') \geq E_{v_{-i}} P_A(v_i', v_{-i}) \int_{v_i'}^{v_i} 1ds \tag{B.25}$$

Then

$$\pi_i(v_i) - \pi_i(v_i') \geq (v_i - v_i') E_{v_{-i}} P_A(v_i', v_{-i}) \tag{B.26}$$

Thus

$$\pi_i(v_i) \geq \pi_i(v_i') + (v_i - v_i')E_{v_{-i}}P_A(v_i', v_{-i}) \tag{B.27}$$

By the definition of $\tilde{\pi}_i(v_i, v_i')$ in (B.6), we have

$$\pi_i(v_i') = \tilde{\pi}_i(v_i, v_i') - (v_i - v_i')E_{v_{-i}}P_A(v_i', v_{-i}) \tag{B.28}$$

By substituting (B.28) into (B.27),

$$\pi_i(v_i) \geq \tilde{\pi}_i(v_i, v_i') \tag{B.29}$$

Then (B.29) implies the Bayesian incentive compatibility. We can duplicate the proof in (II) for agent j in group N_B.

Therefore we have shown that the voting mechanism is Bayesian incentive compatible if and only if for (3.14) and (3.15) are satisfied for $i \in N_A$ and $j \in N_B$.

B.2 Proof of Lemma 3.3.1

Let E_{v_i} be the expectation over the distribution of v_i. For all $i \in N_A$, agent i's ex-ante expected payoff is:

$$E_{v_i}\pi_i = \int_0^{\bar{v}} \pi_i(v_i)f(v_i)\ dv_i \tag{B.30}$$

Then

$$E_{v_i}\pi_i = -\int_0^{\bar{v}} \pi_i(v_i)\ \big(-f(v_i)\big)dv_i \tag{B.31}$$

By integration by parts, we have

$$E_{v_i}\pi_i = -\pi_i(v_i)[1-F(v_i)]\Big|_0^{\bar{v}} + \int_0^{\bar{v}} \frac{d\pi_i(v_i)}{dv_i}\ [1-F(v_i)]\ dv_i \tag{B.32}$$

That is

$$E_{v_i}\pi_i = \pi_i(0) + \int_0^{\bar{v}} \frac{d\pi_i(v_i)}{dv_i}\ [1-F(v_i)]\ dv_i \tag{B.33}$$

The Bayesian incentive compatibility condition (i) in Proposition 3.3.1 tells us that

$$E_{v_i}\pi_i = \int_0^{\bar{v}} [1-F(v_i)]E_{v_{-i}}P_A(v_i, v_{-i})dv_i + \pi_i(0) \tag{B.34}$$

Then

$$E_{v_i}\pi_i = \int_0^{\bar{v}} \frac{[1-F(v_i)]}{f(v_i)}E_{v_{-i}}P_A(v_i, v_{-i})f(v_i)dv_i + \pi_i(0) \tag{B.35}$$

By the definition of $H(v_i)$, we get

$$E_{v_i}\pi_i = \int_0^{\bar{v}} H(v_i) E_{v_{-i}} P_A(v_i, v_{-i}) f(v_i)\ dv_i + \pi_i(0) \tag{B.36}$$

Let E be the expectation over the distribution of v Then

$$E\pi_i = \underbrace{\int_0^{\bar{v}} \ldots \int_0^{\bar{v}}}_{n} H(v_i) P_A(v_i, v_{-i}) f(v)\ dv + \pi_i(0) \tag{B.37}$$

where $dv = dv_1 \cdots dv_n$. Thus

$$E\pi_i = E[H(v_i)P_A(v_i, v_{-i})] + \pi_i(0) \tag{B.38}$$

By duplicating above proof for agent $j \in N_B$, we have

$$E\pi_j = E[H(v_j)P_B(v_j, v_{-j})] + \pi_j(0) \tag{B.39}$$

Therefore, we have shown equations (3.16) and (3.17) in Lemma 3.4.1

B.3 Proof of Theorem 3.3.1

This section solves the maximization problem in chapter 3. By the definition of social ex-ante expected payoff, for all $i \in N_A$ and all $j \in N_B$, we know that

$$E\Pi = \sum_{i \in N_A} E[H(v_i)P_A(v_i, v_{-i})] + \sum_{j \in N_B} E[H(v_j)P_B(v_j, v_{-j})] + \sum_{m \in N} \pi_m(0) \tag{B.40}$$

subject to:

$$\frac{\partial}{\partial v_i} E_{v_{-i}} P_A(v_i, v_{-i}) \geq 0 \tag{B.41}$$

$$\frac{\partial}{\partial v_j} E_{v_{-j}} P_B(v_j, v_{-j}) \geq 0 \tag{B.42}$$

$$P_A(v_i, v_{-i}) \geq 0 \tag{B.43}$$

$$P_B(v_j, v_{-j}) \geq 0 \tag{B.44}$$

$$P_A(v_i, v_{-i}) + P_B(v_j, v_{-j}) = 1 \tag{B.45}$$

By Lemma 3.3.1, we have

$$E\Pi = \sum_{i \in N_A} \int_0^{\bar{v}} H(v_i) E_{v_{-i}} P_A(v_i, v_{-i}) f(v_i) dv_i + \sum_{j \in N_B} \int_0^{\bar{v}} H(v_j) E_{v_{-j}} P_B(v_j, v_{-j}) f(v_j)\ dv_j + \sum_{m \in N} \pi_m(0) \tag{B.46}$$

For all $m \in N$, $\pi_m(0)$ is non-positive, since

$$\pi_m(0) = -E_{v_{-m}} t_m(0, v_{-m}) \tag{B.47}$$

Now suppose $\pi_m(0)$ is zero. That is, for all $m \in N$

$$t_m(0, v_{-m}) = 0 \tag{B.48}$$

Then we have the bound on $E\Pi$, indexed by bound (I). So that

$$E\Pi \leq \sum_{i \in N_A} \int_0^{\bar{v}} H(v_i) E_{v_{-i}} P_A(v_i, v_{-i}) f(v_i) dv_i + \sum_{j \in N_B} \int_0^{\bar{v}} H(v_j) E_{v_{-j}} P_B(v_j, v_{-j}) f(v_j) dv_j \tag{B.49}$$

Since function $H(v_i)$ is a decreasing and $E_{v_{-i}} P_A(v_i, v_{-i})$ is a nondecreasing, then the expected value of the product is no more than the product of the expected values. Thus we get the second bound on $E\Pi$, indexed as bound (II), where $I \leq II$. Then

$$\begin{aligned} E\Pi \leq & \sum_{i \in N_A} \int_0^{\bar{v}} \frac{1 - F(v_i)}{f(v_i)} f(v_i)\ dv_i \int_0^{\bar{v}} E_{v_{-i}} P_A(v_i, v_{-i}) f(v_i) dv_i \\ & + \sum_{j \in N_B} \int_0^{\bar{v}} \frac{1 - F(v_j)}{f(v_j)} f(v_j) dv_j \int_0^{\bar{v}} E_{v_{-j}} P_B(v_j, v_{-j}) dv_j \end{aligned} \tag{B.50}$$

Thus

$$\begin{aligned} E\Pi \leq & \sum_{i \in N_A} \int_0^{\bar{v}} [1 - F(v_i)] dv_i \int_0^{\bar{v}} E_{v_{-i}} P_A(v_i, v_{-i}) f(v_i) dv_i \\ & + \sum_{j \in N_B} \int_0^{\bar{v}} [1 - F(v_j)] dv_j \int_0^{\bar{v}} E_{v_{-j}} P_B(v_j, v_{-j}) dv_j \end{aligned} \tag{B.51}$$

That is

$$E\Pi \leq \int_0^{\bar{v}} [1 - F(v)] dv \Big(\sum_{i \in N_A} \int_0^{\bar{v}} E_{v_{-i}} P_A(v_i, v_{-i}) f(v_i) dv_i + \sum_{j \in N_B} \int_0^{\bar{v}} E_{v_{-j}} P_B(v_j, v_{-j}) dv_j \Big) \tag{B.52}$$

Alternatively, we have

$$E\Pi \leq \int_0^{\bar{v}} [1 - F(v)]dv\Big(E \sum_{i\in N_A} P_A(v_i, v_{-i}) + E \sum_{j\in N_B} P_B(v_j, v_{-j})\Big) \tag{B.53}$$

By the definition of $P_A(v_i, v_{-i})$ and $P_B(v_j, v_{-j})$, we have

$$E\Pi \leq \int_0^{\bar{v}} [1 - F(v)]dv\Big(|N_A|EP_A(v_i, v_{-i}) + (|N| - |N_A|)EP_B(v_j, v_{-j})\Big) \tag{B.54}$$

Then

$$E\Pi \leq \int_0^{\bar{v}} [1-F(v)]dv\Big(|N_A|E\big[P_A(v_i, v_{-i})+P_B(v_j, v_{-j})\big]+(|N|-2|N_A|)E\big[P_B(v_j, v_{-j})\big]\Big) \tag{B.55}$$

By equation (B.45), we get

$$E\Pi \leq \int_0^{\bar{v}} [1 - F(v)]dv\Big(|N_A| + (|N| - 2|N_A|)E\big[P_B(v_j, v_{-j})\big]\Big) \tag{B.56}$$

Suppose N_A is the majority group, that is, $|N_A| > \frac{|N|}{2}$, then

$$|N| - 2|N_A| < 0 \tag{B.57}$$

Given N_A is the majority group, by (B.44), we know that the term $[|N|-2|N_A|]EP_B(v_j, v_{-j})$ is negative. By letting $P_B(v_j, v_{-j})$ be zero (or $P_A(v_i, v_{-i}) = 1$), we have the third bound on $E\Pi$, indexed as bound III, where $II \leq III$. That is

$$E\Pi \leq |N_A| \int_0^{\bar{v}} [1 - F(v)]dv \tag{B.58}$$

We claim that, to implement the bound III of $E\Pi$, the voting mechanism should use zero penalty scheme for any agent $m \in N$. That is, for all $m \in N$,

$$t_m(v_m) = 0 \tag{B.59}$$

Meanwhile, the collective decision must base on simple majority rule.

Given $|N_A| \geq \frac{|N|}{2}$, by using simple majority rule, the voting mechanism assigns $P_A(v_i, v_{-i})$ and $P_B(v_i, v_{-i})$ to one and zero, respectively. Moreover, the penalty on each agent is zero. Then, for all $i \in N_A$ and $j \in N_B$, we know that $k_i = 1$ and $k_j = 0$. Thus, for all $i \in N_A$, agent i's ex-ante expected payoff is the following

$$E_{v_i}\pi_i = \int_o^{\bar{v}} v_i f(v_i)dv_i \tag{B.60}$$

The zero penalty scheme implies each individual's expected payoff is zero in group N_B. Therefore the ex-ante social expected payoff becomes

$$E\Pi = \sum_{i \in N_A} \int_0^{\bar{v}} v_i f(v_i) dv_i \tag{B.61}$$

That is

$$E\Pi = \sum_{i \in N_A} - \int_0^{\bar{v}} v_i [-f(v_i)] dv_i \tag{B.62}$$

By using integration by parts, we have

$$E\Pi = \sum_{i \in N_A} \Big(- v_i [1 - F(v_i)] \Big|_0^{\bar{v}} + \int_0^{\bar{v}} [1 - F(v_i)] dv_i \Big) \tag{B.63}$$

Thus, we social ex-ante expected payoff reaches the bound III. That is

$$E\Pi = |N_A| \Big(\int_0^{\bar{v}} [1 - F(v)] dv \Big) \tag{B.64}$$

Therefore we have shown that, under this direct and incentive compatible voting mechanism, the optimal social decision follows majority rule and there is no penalty on any individual.

B.4 Matlab Codes

(1) distrib.m sets up the function, which takes a vector of random variables ux uniformly distributed on $[-1, 1]$, where negative value indicates an agent preferring A and positive one indicates an agent preferring B. It returns x from a Gamma distribution and the inverse hazard rate H evaluated at x.

Codes:

function $[x, H] = distrib(ux)$

global shape scale N

$x = zeros(N, 1); H = x;$

for

$i1 = 1 : length(ux)$

$p = (ux(i1) + 1)/2;$

$si = sign(ux(i1));$

$xx = gaminv(p, shape, scale);$

$F = gamcdf(xx, shape, scale);$

$f = gampdf(xx, shape, scale);$

$x(i1) = si * xx;$

$H(i1) = si * (1 - F)/f;$

end

(2) $Net\_Wel.m$ calculates the "net welfare" for these two mechanism. In it, the net welfare nw_sm1 and nw_VCG from simple majority and the alternative mechanism is calculated based on Lemma 3.3.1.

Codes:

function $[nw\_sm1, nw\_VCG] = Net\_Wel(ux)$

global N

$z = zeros(N, 1);$

$[x, H] = distrib(ux);$

$x = x(isfinite(x));$

$x = x(x \ \ = 0);$

$H = H(isfinite(H));$

$H = H(H \ \ = 0);$

$sig\_x = sign(x);$

$ind\_x = sign(sum(sig\_x));$

$a\_sm = (ind\_x + 1)/2$

$sum\_x = sum(x);$

$a\_VCG = (sign(sum\_x) + 1)/2$

$HA = min(H, z);$

$HB = max(H, z);$

NetWelfareA= $-sum(HA)/N;$

NetWelfareB= $sum(HB)/N;$

$xA = min(x, z);$

$xB = max(x, z);$

AllocWelfareA= $-sum(xA)/N;$

AllocWelfareB= $sum(xB)/N;$

$nw\_sm1 = (1 - a\_sm) * AllocWelfareA + a_sm * AllocWelfareB;$

$nw\_VCG = (1 - a_VCG) * NetWelfareA + a_VCG * NetWelfareB;$

(3) *Mean_Net_Wel.m* calculates the mean of "net welfare", which approximates the expectation in Lemma 3.3.1.

Codes:

```
function [mean_nw_sm1, mean_nw_VCG] = Mean_Net_Wel
global N    r
tic
for
r1 = 1 : r x = rand(N, 1) * 2 − 1;
[nw_sm1(r1), nw_VCG(r1)] = Net_Wel(x);
end
toc
mean_nw_sm1 = mean(nw_sm1);
mean_nw_VCG = mean(nw_VCG);
```

(4)*shape_is_five_majority_better.m* can be run directly. It should be mentioned here, this file sets up the parameters of the model for Example 1. N is the number of agents in the society; r is the number of repetitions to get the mean. And the shape and scale are parameters of Gamma distribution $G(5, 1)$.

Codes:

```
clear all
global N   r shape scale
N = 94;
r = 10000;
shape = 5
scale = 1
[x, H] = distrib(−1 : .02 : 0.9)
plot(x, H)
H = H(H  = 0);
x = x(x  = 0);
x = x(isfinite(x));
H = H(isfinite(H));
[nw_sm1, nw_VCG] = Net_Wel(−1 : .02 : 0.9)
[mean_nw_sm1, mean_nw_VCG] = Mean_Net_Wel
```

(5) $shape_lessthanone_VCG_better.m$again can be run directly. However, it sets up shape and scale of Gamma distribution to be 0.1 and 1, respectively.

Codes:

clear all

global N r shape scale

$N = 94;$

$r = 10000;$

$shape = 0.1$

$scale = 1$

$[x, H] = distrib(-1 : .02 : 0.9)$

$plot(x, H)$

$H = H(H \ \ = 0);$

$x = x(x \ \ = 0);$

$x = x(isfinite(x));$

$H = H(isfinite(H));$

$[nw\_sm1, nw\_VCG] = Net\_Wel(-1 : .02 : 0.9)$

$[mean\_nw\_sm1, mean\_nw\_VCG] = Mean\_Net\_Wel$

Appendix C

Proofs in Chapter 4

C.1 Proof of Lemma 4.4.1

Let player i be the first mover. Consider any subgame which starts after player i has proposed a vector π with $\pi_i \geq v(\{i\})$. Recall that Q_i is the set of core allocations where player i can gain his maximum payoff. That is

$$Q_i = \left\{\pi \in C(N, v), \text{such that } \forall x \in C(N, v), \pi_i \geq x_i\right\} \tag{C.1}$$

By Assumption 1 (strictly convex game) and Shapley (1971)'s theorem, we get $Q_i \neq \emptyset$ and

$$\pi_i = v(N) - v(N \setminus \{i\}), \quad \text{for } \pi \in Q_i \tag{C.2}$$

The proof of Lemma 4.3.1 has two parts. Part (I) shows that, if π is a core allocation, then all players within $N\setminus\{i\}$ accept it. Part (II) shows that, if all players within $N\setminus\{i\}$ accept π, then π is a core allocation.

Part (I): Suppose π is a core allocation. By Peleg's theorem,

$$\pi_j \geq v_{\pi s}(\{j\}) \quad \text{for } S \subseteq N,\ j \in N \text{ and } j \in S \tag{C.3}$$

Then the Assumption 2 tells us that

$$\pi_j > v_{\pi s}(\{j\}) \quad \text{for } S \subseteq N,\ j \in N \text{ and } j \in S \tag{C.4}$$

In words, for all $j \in N$, there are no opportunities to rent "underpriced " players at π. Then all players within $N \setminus \{i\}$ have no incentive to reject π.

Part (II): Suppose all players within $N \setminus \{i\}$ accept π. If π is not a core allocation, the argument uses contradiction. Let T be the last blocking coalition, and j be the last member (according to P) in T. Peleg (1986)'s theorem tells us that π^s is not in the core of DM reduced game $(S, v_{\pi s})$. That is, for j and for some S, where $S \cap T = j$, we have

$$v_{\pi s}(\{j\}) > \pi_j \tag{C.5}$$

Then the definition of DM reduced game tells us that

$$v_{\pi s}(\{j\}) = \max_{Z \subseteq N \setminus S} \big[v(\{j\} \cup Z) - \pi(Z)\big] \tag{C.6}$$

Since $j \in T$, $S \cap T = j$, it follows that

$$T \setminus \{j\} \subseteq N \setminus S \tag{C.7}$$

Thus

$$v_{\pi s}(\{j\}) \geq v\big(\{j\} \cup T \setminus \{j\}\big) - \pi(T \setminus \{j\}) \tag{C.8}$$

That is

$$v_{\pi s}(\{j\}) \geq v(T) - \pi(T \setminus \{j\}) \tag{C.9}$$

Since T is the last blocking coalition, we have

$$v(T) > \pi(T) \tag{C.10}$$

That is

$$v(T) > \pi(T \setminus \{j\}) + \pi_j \tag{C.11}$$

Then

$$v(T) - \pi(T \setminus \{j\}) > \pi_j \tag{C.12}$$

Thus player j has incentive to reject the π and propose a coalition T and pay all players in $T \setminus \{j\}$ at π. In other words, if π is not a core allocation, at least one player must reject π. Therefore we have proved that if all players within $N \setminus \{i\}$ accept π, then π is a core allocation.

Based on the proofs in part (I) and (II), we can conclude that, π is a core allocation ($\pi \in C(N, v)$) if and only if all players within $N \setminus \{i\}$ accept π.

C.2 Proof of Lemma 4.4.2

Again let player i be the first mover. Recall the definition of Q_i in (4.16). Player i proposes π, where $\pi \in Q_i$ and $\pi_i = v(N) - v(N \setminus \{i\})$. Consider any subgame where i is the first mover.

Firstly, suppose player i is the seller. Let M be the set of rejectors, $(i \notin M)$. Player i's payoff when π is rejected by at least one player in $N \setminus \{i\}$ is

$$\Omega_i = v(N \setminus M) - \pi(N \setminus \{M \cup \{i\}\}) \tag{C.13}$$

Since for all elements of π are non-negative, then

$$\pi(N \setminus \{M \cup \{i\}\}) \geq 0 \tag{C.14}$$

By adding $v(N)$ on both sides, we have

$$\pi(N \setminus \{M \cup \{i\}\}) + v(N) \geq v(N) \tag{C.15}$$

If the seller is not included in a coalition, the value of the coalition is zero. Since $i \notin M$, then $v(M) = 0$. By subtracting $v(M)$ from (C.13), we have

$$\pi(N \setminus \{M \cup \{i\}\}) + v(N) \geq v(N) - v(M) \tag{C.16}$$

By Assumption 1(strictly convex game) [1], we have

$$v(N) - v(M) > v(N \setminus M) \tag{C.17}$$

Then

$$\pi(N \setminus \{M \cup \{i\}\}) + v(N) > v(N \setminus M) \tag{C.18}$$

That is

$$v(N) - \big[v(N \setminus M) - \pi(N \setminus \{M \cup \{i\}\})\big] > 0 \tag{C.19}$$

Since $v(N \setminus \{i\}) = 0$, we have

$$v(N) - v(N \setminus \{i\}) - \big[v(N \setminus M) - \pi(N \setminus \{M \cup \{i\}\})\big] > 0 \tag{C.20}$$

Therefore we have

$$\Omega_i < v(N) - v(N \setminus \{i\}) \tag{C.21}$$

[1] superadditivity

Secondly, suppose the player i is not the seller, and the seller is not included in M (recall above definition of M: the set of rejectors in the subgame where i is the first mover). We note that

$$v(N \setminus \{M \cup \{i\}\}) - \pi(N \setminus \{M \cup \{i\}\}) \leq 0 \tag{C.22}$$

Otherwise, for any player $j \in N \backslash M$ has incentive to change his decision from acceptance to rejection. And then j proposes a new coalition $N \setminus \{M \cup \{i\}\}$ with payment to all players in $N \setminus \{M \cup \{i, j\}\}$.

Since (C. 22), it follows that

$$-v(N \setminus \{M \cup \{i\}\}) \geq -\pi(N \setminus \{M \cup \{i\}\}) \tag{C.23}$$

By adding $v(N \setminus M)$ on both sides, we get

$$v(N \setminus M) - v(N \setminus \{M \cup \{i\}\}) \geq v(N \setminus M) - \pi(N \setminus \{M \cup \{i\}\}) \tag{C.24}$$

By Assumption 1 (strictly convex game), we have

$$v(N) - v(N \setminus \{i\}) > v(N \setminus M) - v(N \setminus \{M \cup \{i\}\}) \tag{C.25}$$

Then

$$v(N) - v(N \setminus \{i\}) > v(N \setminus M) - \pi(N \setminus \{M \cup \{i\}\}) \tag{C.26}$$

That is

$$v(N) - v(N \setminus \{i\}) > \Omega_i \tag{C.27}$$

Finally, consider the player i is not the seller, and the seller is included in M. Then player i's payoff is the following:

$$\Omega_i = -\pi(N \setminus \{M \cup \{i\}\}) \tag{C.28}$$

Since all elements in π are non-negative, we have

$$\Omega_i \leq 0 \tag{C.29}$$

The strict strictly convexity of a game (Assumption 1) tells us that each player has positive marginal contribution to the worth of the coalition. It implies that

$$v(N) - v(N \setminus \{i\}) > 0 \tag{C.30}$$

Therefore, we have

$$v(N) - v(N \setminus \{i\}) > \Omega_i \tag{C.31}$$

C.3 Proof of Theorem 4.4.1

Let the package bidding mechanism with a fixed p (recall the definition of p in section 4.3: the vector that every player is selected as the first mover) be M_p. Let SEP_p be the subgame game perfect equilibrium of M_p.

Under the subgame where player i is the first mover, Lemma 4.4.1 and 4.4.2 imply that, in any subgame where player i is the first mover, π is a SPE payoff if and only if $\pi \in Q_i$.

Since p is fixed in M_p and $C(N, v)$ is a strictly convex set, then any outcome $x \in SPE_p$, must be in the core. That is

$$x \in C(N, v) \tag{C.32}$$

By Shapley (1971)'s theorem, if (N, v) is convex, $C(N, v)$ is the convex hull of all x for any p. Then, given $x \in C(N, v)$, there exists a p such that $x \in SPE_p$.

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